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A characterization of the existence of a pure-strategy Nash equilibrium. *

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Abstract

In this paper, we consider a non-cooperative n-person game in the strategic form. As is well known, the game has a mixed-strategy Nash equilibrium. While, it does not always have a pure-strategy Nash equilibrium. Wherein, Iimura (2003), and Sato and Kawasaki (to appear in Taiwanese J Math) provided a sufficient condition for the game to have a pure-strategy Nash equilibrium. This paper has two aims. The first is to extend the authors' sufficient condition. The second is to give a necessary condition for the existence of a pure-strategy Nash equilibrium in the case of two persons. In both sections, monotonicity of the best response correspondences plays the central role.

Keywords: pure-strategy, Nash equilibrium, non-cooperative n-person game, bimatirix game, fixed point theorem

1 Introduction

In this paper, we consider the non-cooperative n-person game $G = \{N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N}\}$, where

- $N := \{1, \ldots, n\}$ is the set of players.
- For any $i \in N$, S_i denotes the finite set, with a total order $\leqq_i$, of player i's pure strategies. An element of this set is denoted by s_i.
- $p_i : S := \prod_{j=1}^{n} S_j \to \mathbb{R}$ denotes the payoff function of player i.

It is well known that we can prove the existence of a mixed-strategy Nash equilibrium, originally introduced by Nash (1950, 1951), applying Kakutani's fixed point theorem (Kakutani

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1941) to the best response correspondence.

On the other hand, there are few unified results proving the existence of a pure-strategy Nash equilibrium. Iimura (2003) provided a discrete fixed point theorem based on discrete convex analysis, originally proposed by Murota (2003), and Brouwer's fixed point theorem (Brouwer 1912). As an application thereof, he defined a class of non-cooperative n-person games that certainly have a pure-strategy Nash equilibrium. Sato and Kawasaki (to appear) have provided a discrete fixed point theorem based on monotonicity of the mapping, and have given a class of non-cooperative n-person games that also certainly have a pure-strategy Nash equilibrium. However, these results are concerned with only sufficiency for the existence of a pure-strategy Nash equilibrium.

This paper has two aims. The first is to extend the class of non-cooperative n-person games provided in (Sato and Kawasaki to appear), that certainly have a pure-strategy Nash equilibrium. We introduce "partial monotonicity" in Section 3. The second aim is to show that the partial monotonicity is necessary for the existence of a pure-strategy Nash equilibrium in a bimatrix game. This is discussed in Section 4. In order to achieve our goal, we use a directed graphic representation of set-valued mappings.

2 Preliminaries

Since S is the product of finite sets S_i's, it is also finite, say, $S = \{s^1, \ldots, s^m\}$. For any non-empty set-valued mapping F from S to itself, we define a directed graph $D_F = (S, A_F)$ by $A_F = \{(s^i, s^j)\colon s^j \in F(s^i),\ s^i, s^j \in S\}$. For any selection f of F, that is, $f(s) \in F(s)$ for all $s \in S$, we similarly define a directed graph D_f. For any $s \in S$, we denote by $\mathrm{od}(s)$ and $\mathrm{id}(s)$ the outdegree and indegree of s, respectively. Then, $\mathrm{od}(s) \geq 1$ for D_F, and $\mathrm{od}(s) = 1$ for D_f.

Definition 2.1 (Cycle of length l) We say a set-valued mapping F has a directed cycle of length l if there exists l and distinct points $\{s^{i_1}, s^{i_2}, \ldots, s^{i_l}\}$ of S such that $s^{i_1} \in F(s^{i_l})$ and $s^{i_{k+1}} \in F(s^{i_k})$ for all $k \in \{1, \ldots, l-1\}$.

Example 2.1 Take $S = \{s^1, \ldots, s^9\}$ and define a non-empty set-valued mapping F by the following directed graph. For example, $F(s^6) = \{s^2, s^3, s^5, s^8\}$, $F(s^7) = \{s^7, s^8\}$ and etc. It is clear that $\{s^7\}$, $\{s^3, s^6\}$, $\{s^6, s^8, s^9\}$ and $\{s^1, s^4, s^5, s^2\}$ are directed cycles of length 1, 2, 3 and 4, respectively.

We now prove the following lemma required later:

Lemma 2.1 *If D_f is connected in the sense of the undirected graph, then f has only one directed cycle.*

Proof. We start with an arbitrary $s \in S$. Since S is finite, there exist $0 \leq k < l$ such that $f^k(s) = f^l(s)$, where f^k is the k-time composition of f. Hence, $\{f^k(s), f^{k+1}(s), \ldots, f^{l-1}(s)\}$

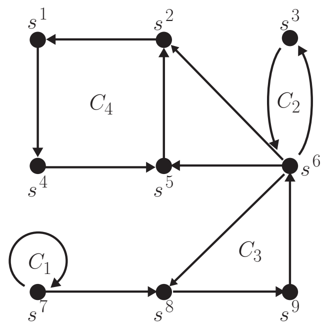

Fig. 1 The graph D_F has directed cycles of length 1, 2, 3 and 4.

is a directed cycle. Next, suppose that there are two distinct directed cycles C_1 and C_2. Since D_f is connected, there exists a path $\pi = \{s^{i_1}, \dots, s^{i_j}\}$ joining C_1 and C_2, where $s^{i_1} \in C_1$ and $s^{i_j} \in C_2$. Further, since any directed cycle has no outward arc, we obtain $s^{i_1} = f(s^{i_2})$, $s^{i_2} = f(s^{i_3}), \dots, s^{i_{j-1}} = f(s^{i_j})$, which contradicts that C_2 has no outward arc. $\square$

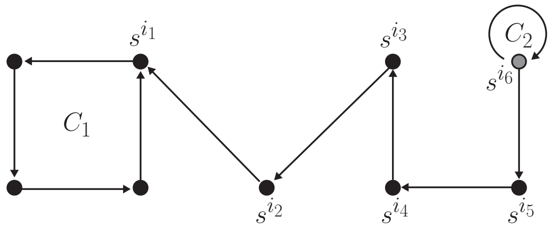

Fig. 2 The graph D_f has two cycles, and $\mathrm{od}(s^{i_6}) = 2$.

3 A sufficient condition for the existence of a pure-strategy Nash equilibrium

In this section, we present a class of non-cooperative n-person games that have a pure-strategy Nash equilibrium. We use the following notation:

For any $s \in S$, we set $s_{-i} := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$, and $S_{-i} := \prod_{j \neq i}^n S_j$. For any given $s_{-i} \in S_{-i}$, we denote by $F_i(s_{-i})$ the set of best responses of player i, that is,

$$F_i(s_{-i}) := \left\{ s_i \in S_i \colon p_i(s_i, s_{-i}) = \max_{t_i \in S_i} p_i(t_i, s_{-i}) \right\}.$$

We set $F(s) := \prod_{j \neq i}^n F_j(s_{-j})$ and $f(s) := (f_1(s_{-1}), \dots, f_n(s_{-n}))$, where f_i is a selection of F_i.

An element s^* of S is called a pure-strategy Nash equilibrium if

$$p_i(s_i^*, s_{-i}^*) \geq p_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i \quad (\forall i \in N).$$

Therefore, any pure-strategy Nash equilibrium is characterized by a fixed point of the best response correspondence F, that is, $s^* \in F(s^*)$. In other words, D_F has a cycle of length 1.

Our sufficient condition is based on monotonicity of a selection f. In order to define monotonicity, we need several kinds of orders.

Let T_i be a non-empty subset of S_i. For any bijection $\sigma_i : T_i \to T_i$, we define a total order $s_i \leqq_{\sigma_i} t_i$ on T_i by $\sigma_i(s_i) \leqq_i \sigma_i(t_i)$, where $\leqq_i$ is the total order on S_i. We denote by T_{σ_i} the ordered set $(T_i, \leqq_{\sigma_i})$. Further, $s_i <_{\sigma_i} t_i$ means $s_i \leqq_{\sigma_i} t_i$ and $s_i \neq t_i$.

We set $T := \prod_{i=1}^n T_i$ and $T_{-i} := \prod_{j\neq i}^n T_j$. For any $\sigma := (\sigma_1, \ldots, \sigma_n)$, T_σ denotes the partially ordered set $(T, \trianglelefteq_\sigma)$ such that $s \trianglelefteq_\sigma t$ if $s_i \leqq_{\sigma_i} t_i$ for all $i \in N$. The symbol $s \preceq_\sigma t$ means $s \trianglelefteq_\sigma t$ and $s \neq t$. T_{-i} is also equipped with the component-wise order $\trianglelefteq_{\sigma_{-i}}$, and the partially ordered set is denoted by $T_{\sigma_{-i}}$.

Definition 3.1 We say G is a partially monotone game if there exist a selection f of F, non-empty subsets $T_i \subset S_i$, and bijections σ_i from T_i into itself $(i \in N)$ such that at least one of T_i's has two or more elements, $f(T) \subset T$, and

$$s_{-i} \preceq t_{-i} \;\Rightarrow\; f_i(s_{-i}) \leqq_{\sigma_i} f_i(t_{-i}) \tag{1}$$

for any $i \in N$.

Theorem 3.1 *Any partially monotone non-cooperative n-person game has a pure-strategy Nash equilibrium.*

Proof. Since T_σ is the product of totally ordered sets, it has a minimum element, say t^0. Then $t^0 \trianglelefteq_\sigma f(t^0) =: t^1$. If $t^0 = t^1$, t^0 is a fixed point. If $t^0 \neq t^1$, set

$$N_1 := \{i \in N \colon t^0_{-i} = t^1_{-i}\}, \quad N_2 := \{i \in N \colon t^0_{-i} \preceq_{\sigma_{-i}} t^1_{-i}\}.$$

Then $t^0 \trianglelefteq_\sigma t^1$, $0 \leq |N_1| \leq 1$, and N is a disjoint union of N_1 and N_2. Next, take

$$t^2_i := \begin{cases} t^1_i, & i \in N_1, \\ f_i(t^1_{-i}), & i \in N_2. \end{cases}$$

Then, by partial monotonicity, we have $t^1_i = f_i(t^0_{-i}) \leqq_{\sigma_i} f_i(t^1_{-i}) = t^2_i$ for any $i \in N_2$. Therefore, $t^1 \trianglelefteq_\sigma t^2$. Since T is finite, this procedure stops in finite steps, and we get a fixed point, which is a pure-strategy Nash equilibrium. □

Here we recall the term "monotonicity" introduced by Sato and Kawasaki (to appear).

Definition 3.2 (Sato and Kawasaki to appear, Definition 3.1) We say G is a monotone game if $\varepsilon_i = 1$ or -1 is allocated to each $i \in N$, and

$$s^0_{-i} \preceq s^1_{-i},\; t^1_i \in F_i(s^0_{-i}) \;\Rightarrow\; \exists t^2_i \in F_i(s^1_{-i}) \text{ such that } \varepsilon_i t^1_i \leqq \varepsilon_i t^2_i$$

for any $i \in N$.

When G is a monotone game, by taking $T_i = S_i$, $\sigma_i = id$ and

$$f_i(s_{-i}) := \begin{cases} \text{maximum element of } F_i(s_{-i}), & \text{if } \varepsilon_i = 1, \\ \text{minimum element of } F_i(s_{-i}), & \text{if } \varepsilon_i = -1, \end{cases}$$

we see that G is a partially monotone game.

As a specific example, let us consider the following bimatrix game:

- $A = (a_{ij})$ is a payoff matrix of player 1 (P1), that is, $p_1(i,j) = a_{ij}$.
- $B = (b_{ij})$ is a payoff matrix of player 2 (P2), that is, $p_2(i,j) = b_{ij}$.
- $S_1 := \{1, \ldots, m_1\}$ is the set of pure strategies of P1, where $m_1 \in \mathbb{N}$.
- $S_2 := \{1, \ldots, m_2\}$ is the set of pure strategies of P2, where $m_2 \in \mathbb{N}$.
- For any $j \in S_2$, $F_1(j) := \{i^* \in S_1 : a_{i^*j} = \max_{i \in S_1} a_{ij}\}$ is the set of best responses of P1.
- For any $i \in S_1$, $F_2(i) := \{j^* \in S_2 : b_{ij^*} = \max_{j \in S_2} b_{ij}\}$ is the set of best responses of P2.
- $F(i,j) := F_1(j) \times F_2(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$.
- A pair (i^*, j^*) is a pure-strategy Nash equilibrium if $(i^*, j^*) \in F(i^*, j^*)$.

Example 3.1 Let $S_1 = S_2 = \{1,2,3\}$. The following is not a monotone bimatrix game.

$$A = \left(\begin{array}{c|c|c} ④ & 2 & 3 \\ 2 & ⑤ & ④ \\ 3 & 1 & ④ \end{array}\right), \quad B = \left(\begin{array}{ccc} 2 & 1 & ③ \\ \hline 1 & ④ & 2 \\ \hline ③ & ③ & 2 \end{array}\right).$$

Now we exchange the second and third columns. A and B are transformed into A' and B', respectively, as given below:

$$A' = \left(\begin{array}{c|c|c} ④ & 3 & 2 \\ 2 & ④ & ⑤ \\ 3 & ④ & 1 \end{array}\right), \quad B' = \left(\begin{array}{ccc} 2 & ③ & 1 \\ \hline 1 & 2 & ④ \\ \hline ③ & 2 & ③ \end{array}\right).$$

However, the bimatrix game defined by A' and B' is not a monotone game. Next, we remove the third row. Then A' and B' are transformed into A'' and B'', respectively:

$$A'' = \left(\begin{array}{c|c|c} ④ & 3 & 2 \\ 2 & ④ & ⑤ \end{array}\right), \quad B'' = \left(\begin{array}{ccc} 2 & ③ & 1 \\ \hline 1 & 2 & ④ \end{array}\right).$$

The bimatrix game defined by A'' and B'' is now a monotone game for $(\varepsilon_1, \varepsilon_2) = (1,1)$, and have a pure-strategy Nash equilibrium $(3,3)$. In the original bimatrix game, the equilibrium is $(2,2)$.

The above procedure is equivalent to taking $T_1 := \{1,2\} \subset S_1$, $\sigma_1 := \mathrm{id}$, $T_2 := S_2$ and σ_2 permutation $(2,3)$ in Definition 3.1. Therefore, the original game is a partially monotone game.

Moreover, in Figure 3 left, we plot the directed graph D_F corresponding to the best responses F of the original bimatrix game. Note that in the figure, we set ① $= (1,1)$, ② $= (1,2)$, ③ $= (1,3)$, ④ $= (2,1)$, ⑤ $= (2,2)$, ⑥ $= (2,3)$, ⑦ $= (3,1)$, ⑧ $= (3,2)$ and ⑨ $= (3,3)$. In Figure 3 right, we plot the directed graph corresponding to the best responses of the bimatrix game after the above procedure. It is clear that the directed graph has only one cycle of length 1.

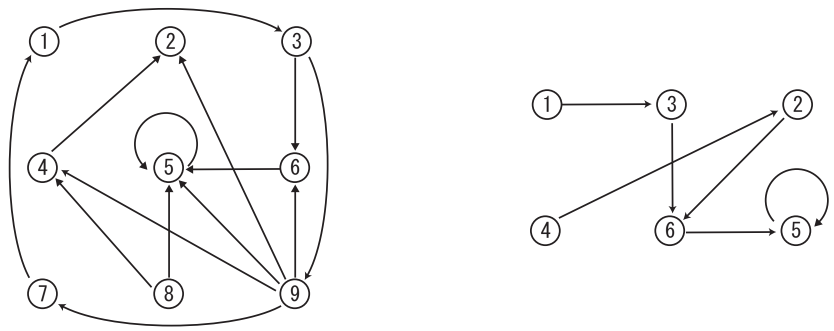

Fig. 3 Left: The directed graph defined by A and B. Right: The directed graph defined by A'' and B''.

4 A necessary condition for the existence of a pure-strategy Nash equilibrium

In this section, we consider the bimatrix game, and show that partial monotonicity is necessary for the existence of a pure-strategy Nash equilibrium.

Theorem 4.1 *Assume that a bimatrix game has a pure-strategy Nash equilibrium s^*. If a sequence $s^1, \dots, s^m = s^*$ in S satisfies $s^{k+1} \in F(s^k)$ and $F(s^k)$ is a singleton for all $k = 1, \dots, m-1$, then there exist non-empty subsets T_i $(i = 1,2)$ and bijections σ_i $(i = 1,2)$ from T_i into itself such that*

$$s^1 \preceq_\sigma s^2 \preceq_\sigma \cdots \preceq_\sigma s^m = s^*. \tag{2}$$

In particular, the bimatrix game is a partially monotone game or s^* is isolated.

Proof. Since $F(s^k)$ is singleton, we use $f(s^k)$ instead of $F(s^k)$. First, $s^1, s^2, \dots, s^m$ are different from each other. Indeed, if $s^k = s^l$ for some $1 \le k < l \le m$, then $\{s^k, s^{k+1}, \dots, s^l\}$ is a directed cycle. Since s^m is a fixed point of f, it is another directed cycle, which contradicts Lemma 2.1. Therefore, $s^1, s^2, \dots, s^m$ are different from each other. Next, assume that

$$s^{k-1} \not\preceq s^k \preceq s^{k+1} \preceq \cdots \preceq s^m \tag{3}$$

for some $2 \leq k \leq m$. In the case of $s_1^{k-1} > s_1^k$, s_1^{k-1} does not coincide with any s_1^l ($k \leq l \leq m$). Indeed, if $s_1^{k-1} = s_1^l$ for some $l > k$, $f_2(s_1^{k-1}) = f_2(s_1^l)$. Since $s_2^k = f_2(s_1^{k-1})$ and $s_2^{l+1} = f_2(s_1^l)$, we have $s_2^k = s_2^{l+1}$. Further, we see from (3) that

$$s_2^k = s_2^{k+1} = \cdots = s_2^l = s_2^{l+1}.$$

Thus, $f_1(s_2^k) = \cdots = f_1(s_2^{l+1})$, that is, $s_1^{k+1} = \cdots = s_1^{l+2}$, and we have $s^{k+1} = \cdots = s^{l+1}$, which contradicts (3). Therefore, s_1^{k+1} does not coincide with any s_1^l.

Hence, there exists $k \leq p \leq m-1$ such that $s_1^p < s_1^{k-1} < s_1^{p+1}$. Defining a bijection σ_1 from S_1 into itself by

$$\sigma_1(s_1) := \begin{cases} s_1^k, & \text{if } s_1 = s_1^{k-1} \\ s_1^{q+1}, & \text{if } s_1 = s_1^q \text{ for some } k \leq q \leq p-1 \\ s_1^{k-1}, & \text{if } s_1 = s_1^p \\ s_1, & \text{otherwise,} \end{cases} \tag{4}$$

we obtain

$$s_1^{k-1} \leqq_{\sigma_1} s_1^k \leqq_{\sigma_1} \cdots \leqq_{\sigma_1} s_1^m.$$

In the case of $s_2^{k-1} > s_2^k$, define a bijection σ_2 on S_2 as well as (4) and take $\sigma = (\sigma_1, \sigma_2)$. Then we get

$$s^{k-1} \preceq_\sigma s^k \preceq_\sigma \cdots \preceq_\sigma s^m. \tag{5}$$

Repeating this procedure, we obtain (2).

Finally, we set $T_i := \{s_i^k \colon 1 \leq k \leq m\}$, and take the restriction of σ_i on T_i. Then σ_i is a bijection from T_i into itself. Further, if $s_{-i} <_{\sigma_{-i}} t_{-i}$, there exist $1 \leq p \neq q \leq m$ such that $s_{-i} = s_{-i}^p$ and $t_{-i} = s_{-i}^q$. Since s^k's are ordered by σ, $p < q$. Therefore

$$f_i(s_{-i}) = f_i(s_{-i}^p) = s_i^{p+1} \leqq_{\sigma_i} s_i^{q+1} = f_i(s_{-i}^q) = f_i(t_{-i}).$$

When $m \geq 2$, this fact implies that the bimatrix game is a partially monotone game. When only $m = 1$ satisfies (2), we conclude that $\mathrm{id}(s^*) = \mathrm{od}(s^*) = 1$, that is, s^* is isolated. $\square$

If the number of players is three or more, then Theorem 4.1 fails.

Example 4.1 Let P1, P2 and P3 be players; let the player's strategies be $i \in \{1, 2\}$, $j \in \{1, 2\}$ and $k \in \{1, 2\}$, respectively; and let each player's best responses be the following:

P1	$k = 1$	$k = 2$
$j = 1$	$i = 2$	$i = 1$
$j = 2$	$i = 2$	$i = 2$

P2	$k = 1$	$k = 2$
$i = 1$	$j = 2$	$j = 2$
$i = 2$	$j = 1$	$j = 2$

P3	$j = 1$	$j = 2$
$i = 1$	$k = 2$	$k = 1$
$i = 2$	$k = 1$	$k = 2$

Then this game is not a partially monotone game. Indeed, there are only four combinations of two bijections on S_1 and S_2. The above table on P3 corresponds to $(\sigma_1, \sigma_2) = (id, id)$. Three tables below correspond to $((1,2), id)$, $(id, (1,2))$, and $((1,2),(1,2))$, respectively. In any case, the best response does not satisfy (1).

P3	$j=1$	$j=2$
$i=2$	$k=1$	$k=2$
$i=1$	$k=2$	$k=1$

P3	$j=2$	$j=1$
$i=1$	$k=1$	$k=2$
$i=2$	$k=2$	$k=1$

P3	$j=2$	$j=1$
$i=2$	$k=2$	$k=1$
$i=1$	$k=1$	$k=2$

On the other hand, since $(f_1(2,2), f_2(2,2), f_3(2,2)) = (2,2,2)$, $(i,j,k) = (2,2,2)$ is a pure-strategy Nash equilibrium, which is not isolated, see Figure 4.

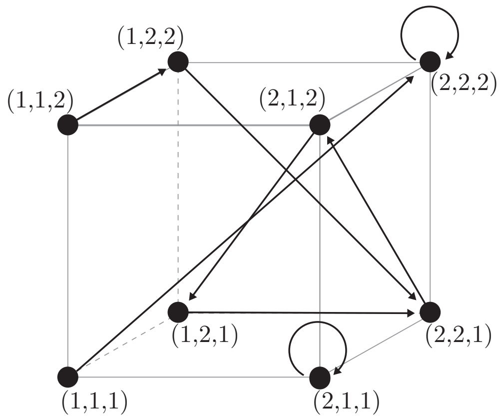

Fig. 4 Point $(2,2,2)$ is a pure-strategy Nash equilibrium, which is not isolated

5 Concluding remarks

In Section 3, we have extended the sufficient condition for the existence of a pure-strategy Nash equilibrium in two directions. One is taking a subgame and the other is reordering the pure-strategies of each player. By these extensions, we can deal with a wide range of non-cooperative n-person games. Furthermore, when $n = 2$, we have proved that our sufficient condition is very close to the necessary condition. In this sense, partial monotonicity of the best response characterizes the existence of a pure-strategy Nash equilibrium in the case of $n = 2$. However, when the number of players is three or more, there is still room for improvement.

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