A Single-Step Characteristic-Curve Finite Element Scheme of Second Order in Time for the Incompressible Navier-Stokes Equations

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A Characteristic-Curve Finite Element Scheme of Single Step and Second Order in Time Increment for the Navier-Stokes Equations

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Abstract

In this paper we present a new characteristic-curve finite element scheme of single step and second order in time increment for the nonstationary Navier-Stokes equations. After supplying correction terms in the variational formulation, we prove that the scheme is of second order in time increment. The convergence rate of the scheme is numerically recognized by computational results.

1 Introduction

In this paper we present a characteristic-curve finite element scheme of single step and second order in time increment for the Navier-Stokes equations. In devising numerical schemes for the Navier-Stokes equations, a key issue is how to approximate the nonlinear convection term. Especially it is crucial in the computation of high-Reynolds number problems, where the standard Galerkin approximation leads to disastrous oscillating results. Many kinds of approximations have been developed based on ideas such as upwinding, streamline diffusion, least square, characteristic curve, and so on. (See [1] , [3], [4] , [5] , [7] , [8] , [11] , [12] , [13], [14], [15], [16], [19] , [20] , [22] , [25] and references there in).

We focus on the approximation based on characteristic-curve method. The idea of the characteristic-curve method is to consider the trajectory of the fluid particle and discretize the material derivative term along the trajectory. The procedure is natural from the physical point of view. Furthermore, the matrix for the system of linear equations is symmetric, which leads to easy solvers for the system. Characteristic-curve finite element schemes for the Navier-Stokes equations of first order in time increment have been developed and analysed, see [14], [15], [19]. A scheme of second order in time increment has been presented and analysed by Boukir et al. [5]. They use a two-step method and approximate the material derivative by the values of two previous steps along the trajectory.

Here we propose a scheme of second order in time increment and of single step. The material derivative is approximated in the Crank-Nicolson way along the trajectory. The original idea of the approximation was developed in [17] for the convection-diffusion equations, which is extended to the Naveir-Stokes equations in this paper. As is pointed out in [17], in the Crank-Nicolson approximation on the trajectory, an additional correction term is indispensable to realize a second order accuracy. After supplying correction terms for the Navier-Stokes equations, the scheme is proved to be of second order in time increment. In the case of the Navier-Stokes equations the velocity is unknown and the obtained scheme becomes nonlinear. For the solution we present an internal iteration procedure which consists of solving a series of Stokes type equations. The scheme has such advantages that it is of second order in time increment and that every matrix to be treated is symmetric.

We use the function spaces $L^2(\Omega)$ and $H^1(\Omega)$, and their subspaces $H^1_0(\Omega)$ and

$$L^2_0(\Omega) \equiv \left\{ \phi \in L^2(\Omega);\ \int_\Omega \phi\ dx = 0 \right\}.$$

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We denote by $(\cdot,\cdot)$ the $L^2(\Omega)$-inner products in the scalar- and vector-valued function spaces. For any normed space X the norm is denoted by $\|\cdot\|_X$. When $X = L^2(\Omega)^d$ or $L^2(\Omega)$, we omit the subscript and denote it simply by $\|\cdot\|$. The dual pairing between X and the dual space X' is denoted by $\langle\cdot,\cdot\rangle$. The partial derivative $\partial u/\partial x_i$ of a function u is simply denoted by $u_{,i}$ and the Einstein convention $a_i b_i$ is used in place of $\sum_{i=1}^d a_i b_i$.

The outline of the paper is as follows. We present a characteristic-curve finite element scheme for the Navier-Stokes equations in Section 2. In Section 3 the consistency of the scheme is proved to be of second order in time increment. In the last section we show some numerical examples to observe the convergence rate.

2 A Characteristic-Curve Finite Element Scheme

In this section we present a characteristic-curve finite element scheme for the Navier-Stokes equations. It is of single step and second order in time increment.

Let $\Omega \subset \mathbb{R}^d (d = 2,3)$ be a bounded domain and T be a positive constant. We consider the nonstationary Navier-Stokes problem subject to the homogeneous Dirichlet boundary condition; find $(u, p) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nabla(2\nu D(u)) + \nabla p = f & \text{in } \Omega\times(0,T), \\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \Gamma\times(0,T), \\ u = u^0 & \text{in } \Omega, \text{ at } t=0, \end{cases} \tag{2.1}$$

where u is the velocity, p is the pressure, f is an exterior force, u^0 is an initial velocity, $\nu(>0)$ is a viscosity, $\Gamma \equiv \partial\Omega$ is the boundary of Ω, $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2}(u_{i,j} + u_{j,i}),$$

and

$$\big[\nabla(2\nu D(u))\big]_i \equiv 2\nu D_{ij,j}(u).$$

In order to present our scheme for (2.1) we prepare the following. For a time increment Δt and velocities u and $w : \Omega \to \mathbb{R}^d$, we define $X_1(u,\Delta t)$ and $X_2(u,w,\Delta t)$ by

$$X_1(u,\Delta t)(x) \equiv x - u(x)\Delta t,$$
$$X_2(u,w,\Delta t)(x) \equiv x - \big\{u(x) + w(x - w(x)\Delta t)\big\}\frac{\Delta t}{2},$$

respectively. We use the symbol $\circ$ to designate the composition of functions, e.g., for a function ψ defined in Ω

$$\Big(\psi\circ X_1(u,\Delta t)\Big)(x) \equiv \psi\big(x - u(x)\Delta t\big).$$

For $n \in \mathbb{N}\cup\{0\}$ we set

$$t^n \equiv \begin{cases} \Delta t_0 + (n-1)\Delta t & (n \geq 1), \\ 0 & (n = 0), \end{cases}$$

where Δt_0 is another time increment used only in the first step of the computation. Let $N_T \equiv [(T - \Delta t_0)/\Delta t] + 1$ be a total step number, $\mathscr{T}_h \equiv \{K\}$ be a triangulation of Ω. We define Ω_h by

$$\Omega_h \equiv \mathrm{int} \bigcup \{K;\ K \in \mathscr{T}_h\}$$

and $\Gamma_h \equiv \partial\Omega_h$. We set finite element spaces,

$$
\begin{aligned}
X_h &\equiv \left\{ v_h \in C^0(\overline{\Omega}_h)^d;\ v_h|_K \in P_2(K),\ \forall K \right\}, \quad V_h \equiv X_h \cap H^1_0(\Omega_h), \\
M_h &\equiv \left\{ q_h \in C^0(\overline{\Omega}_h);\ q_h|_K \in P_1(K),\ \forall K \right\}, \quad Q_h \equiv M_h \cap L^2_0(\Omega_h),
\end{aligned}
\tag{2.2}
$$

and define interpolation operators,

$$
\Pi_h^{(1)}:\ C^0(\overline{\Omega}_h) \to M_h, \quad \Pi_h^{(2)}:\ C^0(\overline{\Omega}_h)^d \to X_h.
$$

For u, w and $\zeta \in H^1_0(\Omega_h)^d$, p and $q \in H^1(\Omega_h)$ and r, f and $g \in L^2(\Omega_h)$ we define linear forms $\mathscr{A}_{h1}\,(u,w,r)$, $\mathscr{A}_{h2}(u,\zeta,w,p,q)$, $\mathscr{F}_{h1} f$ and $\mathscr{F}_{h2}(f,g,w)$ on V_h and $\mathscr{B}_h u$ on Q_h by

$$
\begin{aligned}
\mathscr{A}_{h1}(u,w,r) &\equiv \mathscr{M}_{h1}(u,w\,;\Delta t_0) + \mathscr{D}_{h1}u + \mathscr{C}_{h1}r, \\
\mathscr{A}_{h2}(u,\zeta,w,p,q) &\equiv \mathscr{M}_{h2}(u,\zeta,w\,;\Delta t) + \mathscr{D}_{h2}(u,w) + \mathscr{C}_{h2}(w,p,q), \\
\Big\langle \mathscr{B}_h u, q_h \Big\rangle &\equiv -\Big(\nabla\cdot u, q_h\Big), \quad \Big\langle \mathscr{F}_{h1} f, v_h \Big\rangle \equiv \Big(f, v_h\Big), \\
\Big\langle \mathscr{F}_{h2}(f,g,w), v_h \Big\rangle &\equiv \frac{1}{2}\Big(f + g\circ X_1(w,\Delta t), v_h\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\Big\langle \mathscr{M}_{h1}(u,w\,;\Delta t_0), v_h \Big\rangle &\equiv \Big(\frac{u - w\circ X_1(w,\Delta t_0)}{\Delta t_0},\ v_h\Big), \\
\Big\langle \mathscr{M}_{h2}(u,\zeta,w\,;\Delta t), v_h \Big\rangle &\equiv \Big(\frac{u - w\circ X_2(\zeta,w,\Delta t)}{\Delta t},\ v_h\Big), \\
\Big\langle \mathscr{D}_{h1}u, v_h \Big\rangle &\equiv 2\nu\Big(D(u), D(v_h)\Big), \quad \Big\langle \mathscr{C}_{h1}r, v_h \Big\rangle \equiv -\Big(\nabla\cdot v_h,\ r\Big), \\
\Big\langle \mathscr{D}_{h2}(u,w), v_h \Big\rangle &\equiv \nu\Big(D(u) + D(w)\circ X_1(w,\Delta t),\ D(v_h)\Big) + \nu\Delta t\Big(D_{ij}(w)w_{k,j},\ v_{hi,k}\Big), \\
\Big\langle \mathscr{C}_{h2}(w,p,q), v_h \Big\rangle &\equiv \frac{1}{2}\Big(\nabla p + \nabla q\circ X_1(w,\Delta t),\ v_h\Big).
\end{aligned}
$$

For $\{u^n\}_{n=0}^{N_T} \subset H^1_0(\Omega_h)^d$, $\{p^n\}_{n=1}^{N_T} \subset H^1(\Omega_h)$ and $\{f^n\}_{n=1}^{N_T} \subset L^2(\Omega_h)^d$, linear forms $\mathscr{A}_h^n(u,p)$ and $\mathscr{F}_h^n(f,u)$ on V_h are defined by

$$
\begin{aligned}
\mathscr{A}_h^n(u,p) &\equiv \begin{cases} \mathscr{A}_{h2}(u^n, u^n, u^{n-1}, p^n, p^{n-1}) & (n \ge 2), \\ \mathscr{A}_{h1}(u^1, u^0, p^1) & (n = 1), \end{cases} \\
\mathscr{F}_h^n(f,u) &\equiv \begin{cases} \mathscr{F}_{h2}(f^n, f^{n-1}, u^{n-1}) & (n \ge 2), \\ \mathscr{F}_{h1} f^1 & (n = 1). \end{cases}
\end{aligned}
$$

In order to unify the notation we put $\mathscr{B}_h^n u \equiv \mathscr{B}_h u^n$. For a given continuous function f we set $f_h^n \equiv \Pi_h^{(2)} f(t^n)$.

We now present the scheme for (2.1); find $\{(u_h^n,\ p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

$$
\begin{cases}
\mathscr{A}_h^n(u_h, p_h) = \mathscr{F}_h^n(f_h, u_h) & \text{in } V_h', \\
\mathscr{B}_h^n u_h = 0 & \text{in } Q_h', \\
u_h^0 = \Pi_h^{(2)} u^0. &
\end{cases}
\tag{2.3}
$$

For $n \geq 2$ this is equivalent to the equations,

$$\begin{cases} \Big(\dfrac{u_h^n - u_h^{n-1}\circ X_2(u_h^n, u_h^{n-1}, \Delta t)}{\Delta t},\ v_h\Big) \\ \quad + \nu\Big(D(u_h^n) + D(u_h^{n-1})\circ X_1(u_h^{n-1}, \Delta t),\ D(v_h)\Big) + \nu\Delta t\Big(D_{ij}(u_h^{n-1})u_{hk,j}^{n-1},\ v_{hi,k}\Big) \\ \quad + \dfrac{1}{2}\Big(\nabla p_h^n + \nabla p_h^{n-1}\circ X_1(u_h^{n-1}, \Delta t),\ v_h\Big) \\ \quad = \dfrac{1}{2}\Big(f_h^n + f_h^{n-1}\circ X_1(u_h^{n-1}, \Delta t),\ v_h\Big), \qquad \forall v_h \in V_h, \\ \Big(\nabla\cdot u_h^n,\ q_h\Big) = 0, \qquad \forall q_h \in Q_h. \end{cases}$$

In the next section the scheme is shown to be of second order in Δt for $n \geq 2$, and of first order in Δt_0 for $n = 1$. By taking $\Delta t_0 = O(\Delta t^2)$, the whole scheme becomes of second order in time increment Δt.

Remark 2.1 (i) For $v_h \in V_h$ and $q_h \in Q_h$ it holds that

$$\Big\langle \mathscr{C}_{h1} q_h, v_h \Big\rangle = \Big\langle \mathscr{B}_h v_h, q_h \Big\rangle,$$

i.e., $\mathscr{C}_{h1} = \mathscr{B}_h'$ on Q_h, though $\mathscr{C}_{h1}$ is defined on $L^2(\Omega_h)$.
(ii) In $\mathscr{A}_h^n$ $(n \geq 2)$, we need u^{n-1} and p^{n-1} to get u^n and p^n. If $\mathscr{A}_{h2}$ were used when $n = 1$, we would need p^0, which is not given as the initial condition in the Navier-Stokes equations. This is the reason why we use $\mathscr{A}_{h1}$ at $n = 1$. In the case of the convection-diffusion equation, such fact does not occur.

Since the scheme is nonlinear in u_h^n for $n \geq 2$, we prepare an *internal iteration procedure*. Let $\{(w_h^k, r_h^k)\}_{k=1}^\infty \subset V_h \times Q_h$ be the solution of

$$\begin{cases} \mathscr{A}_{h2}(w_h^k, w_h^{k-1}, u_h^{n-1}, r_h^k, p_h^{n-1}) = \mathscr{F}_{h2}(f_h^n, f_h^{n-1}, u_h^{n-1}) & \text{in } V_h', \\ \mathscr{B}_h w_h^k = 0 & \text{in } Q_h', \\ w_h^0 = u_h^{n-1}. \end{cases} \tag{2.4}$$

(u_h^n, p_h^n) is obtained as the limit of the sequence $\{(w_h^k, r_h^k)\}_{k=1}^\infty$. In the real computation if the convergence criterion,

$$\frac{\|w_h^k - w_h^{k-1}\|_{H^1(\Omega_h)^d} + \|r_h^k - r_h^{k-1}\|_{L^2(\Omega_h)}}{\|w_h^k\|_{H^1(\Omega_h)^d} + \|r_h^k\|_{L^2(\Omega_h)}} < \varepsilon_I \tag{2.5}$$

is satisfied for some k, we set $(u_h^n, p_h^n) \equiv (w_h^k, r_h^k)$. Here ε_I is a sufficiently small positive constant. We note that (2.4) is a linear problem in w_h^k and r_h^k whose matrix is symmetric.

Remark 2.2 One can choose other finite element spaces $V_h \times Q_h$ satisfying the inf-sup condition [6], [10] and $Q_h \subset H^1(\Omega_h)$.

Remark 2.3 Scheme (2.3) requires that Q_h is a subset of $H^1(\Omega_h)$, because the pressure term is written in a strong form. Using a weak form for the pressure, which requires only $Q_h \subset L^2(\Omega_h)$, we can derive a scheme,

$$\begin{cases} \tilde{\mathscr{A}}_h^n(u_h, p_h) = \mathscr{F}_h^n(f_h, u_h) & \text{in } V_h', \\ \mathscr{B}_h^n u_h = 0 & \text{in } Q_h', \\ u_h^0 = \Pi_h^{(2)} u^0, \end{cases} \tag{2.6}$$

where

$$\tilde{\mathscr{A}}_h^n(u, p) \equiv \begin{cases} \mathscr{M}_{h2}(u^n, u^n, u^{n-1}; \Delta t) + \mathscr{D}_{h2}(u^n, u^{n-1}) + \tilde{\mathscr{C}}_{h2}(u^{n-1}, p^n, p^{n-1}) & (n \geq 2), \\ \mathscr{M}_{h1}(u^1, u^0; \Delta t_0) + \mathscr{D}_{h1} u^1 + \mathscr{C}_{h1} p^1 & (n = 1), \end{cases}$$

$$\left\langle \tilde{\mathscr{C}}_{h2}(w,p,q), v_h \right\rangle \equiv -\frac{1}{2}\Big(\nabla\cdot v_h,\ p + q\circ X_1(w,\Delta t)\Big) - \frac{\Delta t}{2}\Big(q w_{i,j},\ v_{hj,i}\Big). \tag{2.7}$$

The last term of (2.7) is a correction term for second order accuracy in Δt. This scheme is proved to be of second order in Δt in a similar way to scheme (2.3) by using the analysis in the next section. Numerical experiments, however, show that scheme (2.6) is not so stable. In fact, we could not get solutions for $\nu \leq 10^{-2}$ in Example 1 of Section 4 because of oscillation. Hence we do not use this scheme.

3 Consistency of the Scheme

In this section we assume $\Omega_h = \Omega$ for the sake of simplicity. For an integer n $(2 \leq n \leq N_T)$, we set

$$t^{n-1/2} \equiv \frac{1}{2}(t^n + t^{n-1}).$$

For a function ψ on $\Omega\times(0,T)$ and $m \in \mathbb{N}\cup\{\mathbb{N}-1/2\}\cup\{0\}$ $(m \leq N_T)$, ψ^m means

$$\psi^m \equiv \psi(\cdot, t^m).$$

Proposition 3.1 (consistency) *Suppose that f is a sufficiently smooth function, (u,p) is the sufficiently smooth solution of (2.1) and $X_1(u^{n-1},\Delta t)(\Omega)$ and $X_2(u^n,u^{n-1},\Delta t)(\Omega) \subset \Omega$. Then for any $v_h \in V_h$ it holds that*

$$\left\langle \mathscr{A}_{h2}(u^n,u^n,u^{n-1},p^n,p^{n-1}) - \mathscr{F}_{h2}(f^n,f^{n-1},u^{n-1}),\ v_h \right\rangle = O(\Delta t^2)\|v_h\|, \tag{3.1a}$$

$$\left\langle \mathscr{A}_{h1}(u^1,u^0,p^1) - \mathscr{F}_{h1} f^1,\ v_h \right\rangle = O(\Delta t_0)\|v_h\|. \tag{3.1b}$$

□

We prepare some lemmas for the proof. The first one is trivial, but it is often used.

Lemma 3.1 *For a smooth function f it holds that*

$$\frac{1}{2}\Big(f(t) + f(t-\Delta t)\Big) = f(t - \frac{\Delta t}{2}) + O(\Delta t^2), \tag{3.2a}$$

$$\frac{f(t) - f(t-\Delta t)}{\Delta t} = f'(t - \frac{\Delta t}{2}) + O(\Delta t^2). \tag{3.2b}$$

□

Let $u : \Omega\times(0,T) \to \mathbb{R}^d$ be a smooth function. For a point $x \in \Omega$, we denote by $X(\,\cdot\,;x) : (0,T) \to \mathbb{R}^d$ the solution of the ordinary differential equation,

$$\begin{cases} X'(t) = u(X(t),t) \quad \text{in } (t^{n-1},t^n), \\ X(t^n) = x. \end{cases} \tag{3.3}$$

We denote the material derivation by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\cdot\nabla.$$

We note that the material derivative of a function $f : \Omega\times(0,T) \to \mathbb{R}$ is written as

$$\frac{Df}{Dt}\big(X(t),t\big) = \frac{d}{dt} f\big(X(t),t\big). \tag{3.4}$$

Setting

$$Y_1(u,\Delta t)(x) \equiv \frac{x + X_1(u,\Delta t)(x)}{2},$$

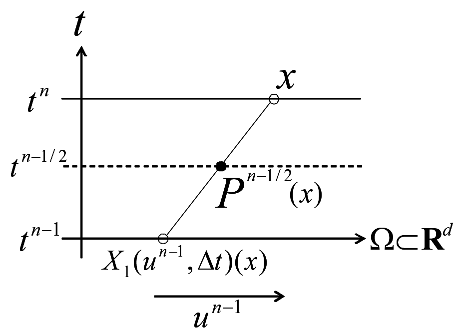

Fig. 1: The evaluation point for the consistency

we evaluate the equations at a point

$$P^{n-1/2}(x) \equiv \Big(Y_1(u^{n-1},\Delta t)(x),\ t^{n-1/2}\Big). \tag{3.5}$$

Using the approximation X_2 for $X(t^{n-1})$, we can construct a second order discretization of the material derivative as follows.

Lemma 3.2 *Let u be a sufficiently smooth function and $X_2\ (u^n,u^{n-1},\Delta t)(\Omega)\subset\Omega$. Then it holds that*

$$\frac{u^n(x)-u^{n-1}\circ X_2(u^n,u^{n-1},\Delta t)(x)}{\Delta t} = \frac{Du}{Dt}\big(P^{n-1/2}(x)\big)+O(\Delta t^2). \tag{3.6}$$

Proof Let X be the solution of (3.3). Substituting u into f in (3.4) and using (3.2b), we have

$$\frac{Du}{Dt}(X(t^{n-1/2}),t^{n-1/2}) = \frac{u^n(X(t^n))-u^{n-1}(X(t^{n-1}))}{\Delta t}+O(\Delta t^2). \tag{3.7}$$

Since the Heun method is of second order in time increment, we have

$$\begin{aligned} X(t^{n-1};x) &= x-\Big\{u^n(x)+u^{n-1}\big(x-u^n(x)\Delta t\big)\Big\}\frac{\Delta t}{2}+O(\Delta t^3) \\ &= x-\Big\{u^n(x)+u^{n-1}\big(x-u^{n-1}(x)\Delta t\big)\Big\}\frac{\Delta t}{2}+O(\Delta t^3) \\ &= X_2(u^n,u^{n-1},\Delta t)(x)+O(\Delta t^3). \end{aligned} \tag{3.8}$$

On the other hand, by (3.2a), it holds that

$$X(t^{n-1/2};x) = Y_1(u^{n-1},\Delta t)(x)+O(\Delta t^2). \tag{3.9}$$

Combining (3.8) and (3.9) with (3.7), we get (3.6). □

Lemma 3.3 *Suppose that $u,\ f:\ \Omega\times(0,T)\to\mathbb{R}^d$ and $p:\ \Omega\times(0,T)\to\mathbb{R}$ are sufficiently smooth functions and $X_1(u^{n-1},\Delta t)(\Omega)$ and $X_2(u^n,u^{n-1},\Delta t)(\Omega)\subset\Omega$. Then for any $x\in\Omega$ it holds that*

$$\begin{aligned} &\frac{u^n-u^{n-1}\circ X_2(u^n,u^{n-1},\Delta t)}{\Delta t}(x)-\nu\Big\{\nabla D(u^n)+\nabla D(u^{n-1})\circ X_1(u^{n-1},\Delta t)\Big\}(x) \\ &\quad+\frac{1}{2}\Big\{\nabla p^n+\nabla p^{n-1}\circ X_1(u^{n-1},\Delta t)\Big\}(x)-\frac{1}{2}\Big\{f^n+f^{n-1}\circ X_1(u^{n-1},\Delta t)\Big\}(x) \\ &=\Big(\frac{Du}{Dt}-2\nu\nabla D(u)+\nabla p-f\Big)\big(P^{n-1/2}(x)\big)+O(\Delta t^2), \end{aligned} \tag{3.10}$$

where $P^{n-1/2}(x)$ is a point defined by (3.5).

Proof Let $X(\cdot;x)$ be the solution of (3.3). Substituting $(-2\nu\nabla D(u)+\nabla p-f)(X(\cdot),\cdot)$ into f and t^n into t in (3.2a), using the relation

$$X(t^{n-1};x)=X_1(u^{n-1},\Delta t)(x)+O(\Delta t^2),$$

we have

$$\begin{aligned}
&-\nu\Big\{\nabla D(u^n)+\nabla D(u^{n-1})\circ X_1(u^{n-1},\Delta t)\Big\}(x)+\frac{1}{2}\Big\{\nabla p^n+\nabla p^{n-1}\circ X_1(u^{n-1},\Delta t)\Big\}(x)\\
&\quad-\frac{1}{2}\Big\{f^n+f^{n-1}\circ X_1(u^{n-1},\Delta t)\Big\}(x)\\
&=\Big\{-2\nu\nabla D(u)+\nabla p-f\Big\}\Big(P^{n-1/2}(x)\Big)+O(\Delta t^2).
\end{aligned}\tag{3.11}$$

Combining (3.11) with Lemma 3.2, we get the result. □

Lemma 3.4 *Let $u:\Omega\to\mathbb{R}^d$ be a sufficiently smooth function satisfying $\nabla\cdot u=0$ in Ω and $X_1(u,\Delta t)(\Omega)\subset\Omega$. Then for any $v_h\in V_h$ it holds that*

$$\begin{aligned}
&-\Big(\nabla D(u)\circ X_1(u,\Delta t),v_h\Big)\\
&\quad=\Big(D(u)\circ X_1(u,\Delta t),D(v_h)\Big)+\Delta t\Big(D_{ij}(u)u_{k,j},v_{hi,k}\Big)+O(\Delta t^2)\|v_h\|.
\end{aligned}\tag{3.12}$$

Proof Since $\nabla\cdot u=0$ in Ω, it holds that

$$\begin{aligned}
\Big(u_{i,j}\circ X_1(u,\Delta t),v_{hi,j}\Big)&=-\Big(\big(u_{i,j}\circ X_1(u,\Delta t)\big)_{,j},v_{hi}\Big)\\
&=-\Big(u_{i,jk}\circ X_1(u,\Delta t)(\delta_{kj}-u_{k,j}\Delta t),v_{hi}\Big)\\
&=-\Big(u_{i,jj}\circ X_1(u,\Delta t),v_{hi}\Big)+\Delta t\Big(u_{i,jk}\circ X_1(u,\Delta t)u_{k,j},v_{hi}\Big)\\
&=-\Big(u_{i,jj}\circ X_1(u,\Delta t),v_{hi}\Big)+\Delta t\Big(u_{i,jk}u_{k,j},v_{hi}\Big)+O(\Delta t^2)\|v_h\|\\
&=-\Big(u_{i,jj}\circ X_1(u,\Delta t),v_{hi}\Big)-\Delta t\Big(u_{i,j}u_{k,j},v_{hi,k}\Big)+O(\Delta t^2)\|v_h\|,
\end{aligned}$$

where δ_{kj} is Kronecker's delta. Similarly we have

$$\begin{aligned}
\Big(u_{i,j}\circ X_1(u,\Delta t),v_{hj,i}\Big)&=-\Big(\big(u_{i,j}\circ X_1(u,\Delta t)\big)_{,i},v_{hj}\Big)\\
&=-\Big(u_{i,ji}\circ X_1(u,\Delta t),v_{hj}\Big)-\Delta t\Big(u_{i,j}u_{k,i},v_{hj,k}\Big)+O(\Delta t^2)\|v_h\|\\
&=-\Big(u_{j,ij}\circ X_1(u,\Delta t),v_{hi}\Big)-\Delta t\Big(u_{j,i}u_{k,j},v_{hi,k}\Big)+O(\Delta t^2)\|v_h\|.
\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
\Big(D(u)\circ X_1(u,\Delta t),D(v_h)\Big)&=\frac{1}{2}\Big\{\Big(u_{i,j}\circ X_1(u,\Delta t),v_{hi,j}\Big)+\Big(u_{i,j}\circ X_1(u,\Delta t),v_{hj,i}\Big)\Big\}\\
&=-\Big(\nabla D(u)\circ X_1(u,\Delta t),v_h\Big)-\Delta t\Big(D_{ij}(u)u_{k,j},v_{hi,k}\Big)+O(\Delta t^2)\|v_h\|,
\end{aligned}$$

which completes the proof. □

Proof of Proposition 3.1 Substituting u^{n-1} into u in Lemma 3.4, we have

$$\begin{aligned}
&-\nu\Big(\nabla D(u^{n-1})\circ X_1(u^{n-1},\Delta t),v_h\Big)\\
&=\nu\Big\{\Big(D(u^{n-1})\circ X_1(u^{n-1},\Delta t),D(v_h)\Big)+\Delta t\Big(D_{ij}(u^{n-1})u^{n-1}_{k,j},v_{hi,k}\Big)\Big\}+O(\Delta t^2)\|v_h\|.
\end{aligned}\tag{3.13}$$

Obviously it holds that

$$-\nu\Big(\nabla D(u^n), v_h\Big) = \nu\Big(D(u^n), D(v_h)\Big). \tag{3.14}$$

Combining (3.13) and (3.14) with Lemma 3.3, we have

$$\begin{aligned}&\Big\langle \mathscr{A}_{h2}(u^n, u^n, u^{n-1}, p^n, p^{n-1}) - \mathscr{F}_h^n(f, u), v_h \Big\rangle \\ &= \Big(\big(\frac{Du}{Dt} - \nabla(2\nu D(u)) + \nabla p - f\big)^{n-1/2} \circ Y_1(u^{n-1}, \Delta t), v_h \Big) + O(\Delta t^2)\|v_h\|.\end{aligned}$$

Here we have used (3.2b) again. Since (u, p) is the solution of (2.1), we get (3.1a). The proof of (3.1b) is similar. □

4 Numerical Examples

In this section we show numerical results in $d = 2$ to observe the numerical convergence rate of the scheme. Since the matrix is symmetric, we use the CG method with ILU(0) preconditioner [2] for solving the system of linear equations. In the scheme we have to compute integrals of composite functions such as

$$\int_K u_h^{n-1} \circ X_2(w_h^{k-1}, u_h^{n-1}, \Delta t) v_h \, dx$$

on triangular elements K. The integrand

$$u_h^{n-1} \circ X_2(w_h^{k-1}, u_h^{n-1}, \Delta t) v_h$$

is not smooth on K. It is known that rough numerical integration causes oscillation even in the case that the stability is theoretically proved for a scheme with exact integration, see [21] and [23]. Hence, much attention should be paid to numerical integration of composite functions. Here we use a numerical integration formula of degree five on each triangle [18]. For a given series $\{\psi^n\}_{n=1}^{N_T}$ in a normed space X, we define

$$\|\psi\|_{l^2(X)} \equiv \left\{ \sum_{n=1}^{N_T} (t^n - t^{n-1}) \|\psi^n\|_X^2 \right\}^{\frac{1}{2}}.$$

Example 4.1 In (2.1) we take $\Omega = (0,1)^2$, $T = 1$, and five values of ν,

$$\nu = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}.$$

The functions f and u^0 are given so that the exact solution is

$$\begin{cases} u_1(x_1, x_2, t) = \{1 + \sin(\pi t)\} \sin^2(\pi x_1) \sin(2\pi x_2), \\ u_2(x_1, x_2, t) = -\{1 + \sin(\pi t)\} \sin^2(\pi x_2) \sin(2\pi x_1), \\ p(x_1, x_2, t) = \{1 + \sin(\pi t)\} \cos(\pi x_1) \cos(\pi x_2). \end{cases}$$

We used FreeFEM [9] for mesh generation. Let N_Ω be the division number of each side of Ω and $h \equiv 1/N_\Omega$ be the representative length of each mesh. Figure 2(left) shows a sample mesh $(N_\Omega = 8)$. We solve the problem by the scheme (2.3). Since the convergence rate of the backward Euler scheme of the P2/P1 Galerkin approximation is $O(\Delta t + h^2)$ for the Navier-Stokes equations, e.g., [24], we choose $\Delta t = h$. Furthermore we set $\Delta t_0 = h^2$ and $\varepsilon_I = 10^{-5}$. We calculated Err defined by

$$Err \equiv \frac{\|\Pi_h^{(2)} u - u_h\|_{l^2(H^1(\Omega)^2)} + \|\Pi_h^{(1)} p - p_h\|_{l^2(L^2(\Omega))}}{\|u_h\|_{l^2(H^1(\Omega)^2)} + \|p_h\|_{l^2(L^2(\Omega))}}.$$

Figure 2 (right) shows the graph of Err versus Δt in logarithmic scale for $N_\Omega = 8,\ 16,\ 32$ and 64, and the values of Err and the slopes are given in Table 1. We can observe a second order convergence in Δt.

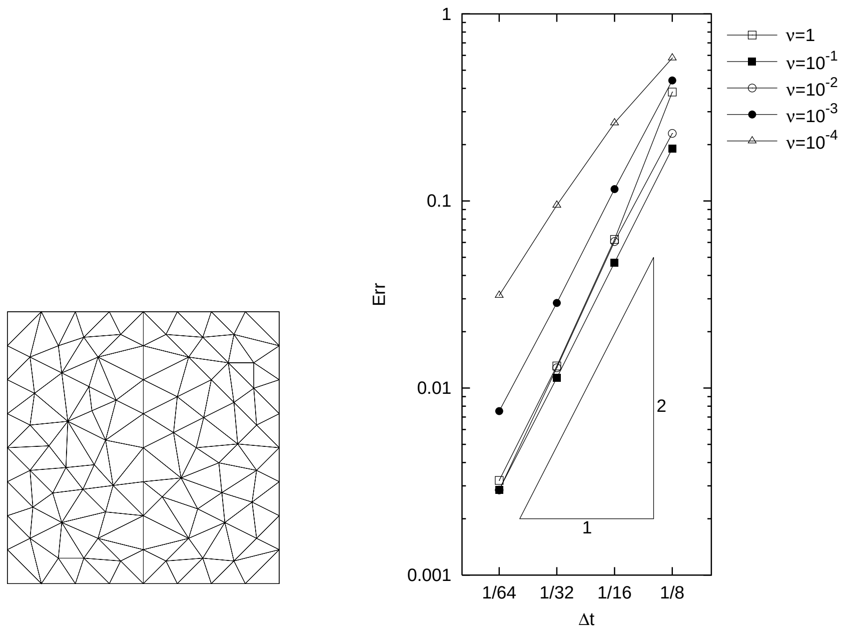

Fig. 2: A sample mesh $(N_\Omega = 8)$ and the graph of Err versus Δt in logarithmic scale

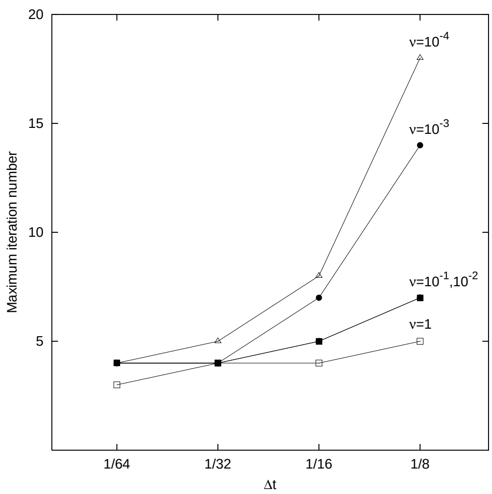

Fig. 3: The graph of maximum iteration number versus Δt for each ν

Figure 3 exhibits the graph of maximum internal iteration number versus Δt. It decreases as Δt becomes small and was equal to 3 or 4 for $\Delta t = 1/64$.

Now we examine the importance of the additional correction term

$$\nu \Delta t \Big(D_{ij}(u^{n-1}) u_{k,j}^{n-1},\ v_{hi,k} \Big) \tag{4.1}$$

in the definition of $\mathscr{D}_{h2}$. We compare results obtained by schemes with and without this term as well as the first order scheme. Dropping the term from the scheme (2.3), we get

$$\begin{cases} \hat{\mathscr{A}}_h^n(u_h, p_h) = \mathscr{F}_h^n(f_h, u_h) & \text{in } V_h', \\ \mathscr{B}_h^n u_h = 0 & \text{in } Q_h', \\ u_h^0 = \Pi_h^{(2)} u^0, & \end{cases} \tag{4.2}$$

where

$$\hat{\mathscr{A}}_h^n(u,p) \equiv \begin{cases} \hat{\mathscr{A}}_{h2}(u^n, u^n, u^{n-1}, p^n, p^{n-1}) & (n \geq 2), \\ \mathscr{A}_{h1}(u^1, u^0, p^1) & (n = 1), \end{cases}$$

$$\hat{\mathscr{A}}_{h2}(u, \zeta, w, p, q) \equiv \mathscr{M}_{h2}(u, \zeta, w\,; \Delta t) + \hat{\mathscr{D}}_{h2}(u,w) + \mathscr{C}_{h2}(w,p,q),$$

$$\left\langle \hat{\mathscr{D}}_{h2}(u,w), v_h \right\rangle \equiv \nu \Big(D(u) + D(w) \circ X_1(w, \Delta t),\ D(v_h) \Big).$$

The first order scheme is

$$\begin{cases} \mathscr{A}_{h1}^n(u_h, p_h) = \mathscr{F}_{h1}^n f_h & \text{in } V_h', \\ \mathscr{B}_h^n u_h = 0 & \text{in } Q_h', \\ u_h^0 = \Pi_h^{(2)} u^0, & \end{cases} \tag{4.3}$$

where

$$\mathscr{A}_{h1}^n(u,p) \equiv \begin{cases} \mathscr{M}_{h1}(u^n, u^{n-1}; \Delta t) + \mathscr{D}_{h1} u^n + \mathscr{C}_{h1} p^n & (n \geq 2), \\ \mathscr{M}_{h1}(u^1, u^0; \Delta t_0) + \mathscr{D}_{h1} u^1 + \mathscr{C}_{h1} p^1 & (n = 1). \end{cases}$$

$$\mathscr{F}_{h1}^n f \equiv \mathscr{F}_{h1} f^n.$$

In the first order scheme we do not need to use a first step with a small time increment Δt_0. For the comparison with other schemes, however, we use the first step with Δt_0. We solve Example 4.1 under the same condition. The results obtained from the three schemes are shown in Figure 4 and Table 1. These results exhibit the necessity of the additional correction term for second order in Δt. In the case of $\nu = 1$ the values of Err of the scheme (4.2) are worse than those of the scheme (4.3). In the case of $\nu = 10^{-1}$ the results of (4.2) is better than those of (4.3), but the slope of (4.2) is worse than that of the present scheme (2.3). In the cases $\nu = 10^{-2}, 10^{-3}$ and 10^{-4} there is no clear difference between the results by (4.2) and (2.3). These results are explained from the fact that the additional correction term (4.1) contains ν and is proportional to it.

5 Conclusions

We have presented a new characteristic-curve finite element scheme of single step and second order in Δt for the Navier-Stokes equations. We have shown an additional correction term for the scheme to be of second order in the time increment. Our approximation is based on the Crank-Nicolson method on the trajectory, which is the reason why the correction term is required. Since the scheme is nonlinear, we presented an internal iteration procedure. In each internal iteration the matrix is symmetric and identical. In our numerical examples the internal iteration has converged as few as three or four steps when the time increment is small. We have also given numerical results which confirm the second order accuracy in Δt and the importance of the additional correction term. To make a stable scheme in a weak form of the pressure is an open problem.

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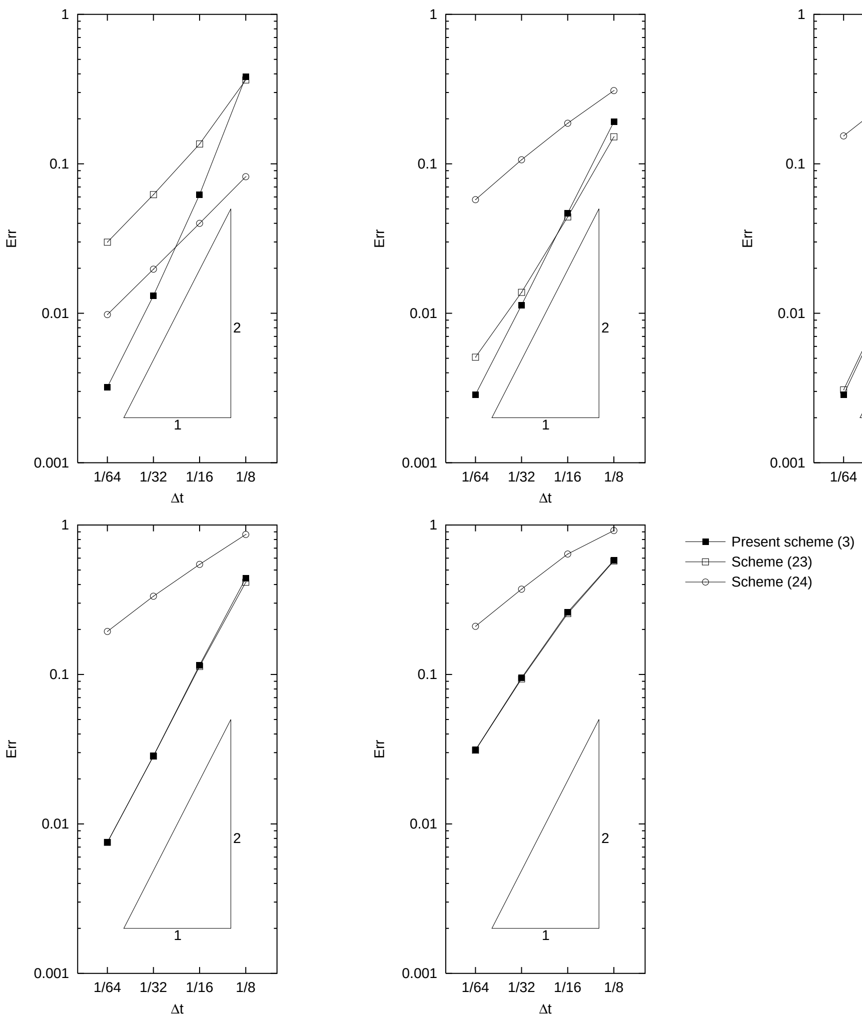

Fig. 4: Comparison of convergence order: $\nu = 1$ (top left), 10^{-1} (top center), 10^{-2} (top right), 10^{-3} (bottom left) and 10^{-4} (bottom right)

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Table 1: Values of Err and slopes of the graphs in Figures 2 and 4

	N_Ω	Present scheme (2.3) Err	slope	Scheme (4.2) Err	slope	Scheme (4.3) Err	slope
$\nu = 1:$	8	3.82×10^{-1}	—	3.64×10^{-1}	—	8.20×10^{-2}	—
	16	6.23×10^{-2}	2.62	1.36×10^{-1}	1.42	4.00×10^{-2}	1.04
	32	1.31×10^{-2}	2.25	6.22×10^{-2}	1.13	1.97×10^{-2}	1.02
	64	3.21×10^{-3}	2.03	2.99×10^{-2}	1.06	9.80×10^{-3}	1.01
$\nu = 10^{-1}:$	8	1.91×10^{-1}	—	1.51×10^{-1}	—	3.09×10^{-1}	—
	16	4.68×10^{-2}	2.03	4.41×10^{-2}	1.78	1.87×10^{-1}	0.73
	32	1.13×10^{-2}	2.05	1.38×10^{-2}	1.68	1.06×10^{-1}	0.81
	64	2.86×10^{-3}	1.99	5.08×10^{-3}	1.44	5.76×10^{-2}	0.89
$\nu = 10^{-2}:$	8	2.30×10^{-1}	—	2.26×10^{-1}	—	6.98×10^{-1}	—
	16	6.07×10^{-2}	1.92	6.26×10^{-2}	1.85	4.41×10^{-1}	0.66
	32	1.28×10^{-2}	2.24	1.35×10^{-2}	2.21	2.68×10^{-1}	0.72
	64	2.85×10^{-3}	2.17	3.07×10^{-3}	2.14	1.54×10^{-1}	0.80
$\nu = 10^{-3}:$	8	4.41×10^{-1}	—	4.14×10^{-1}	—	8.65×10^{-1}	—
	16	1.16×10^{-1}	1.93	1.13×10^{-1}	1.87	5.45×10^{-1}	0.67
	32	2.85×10^{-2}	2.02	2.84×10^{-2}	2.00	3.34×10^{-1}	0.71
	64	7.53×10^{-3}	1.92	7.53×10^{-3}	1.92	1.94×10^{-1}	0.78
$\nu = 10^{-4}:$	8	5.81×10^{-1}	—	5.75×10^{-1}	—	9.18×10^{-1}	—
	16	2.61×10^{-1}	1.15	2.56×10^{-1}	1.17	6.39×10^{-1}	0.52
	32	9.48×10^{-2}	1.46	9.35×10^{-2}	1.45	3.72×10^{-1}	0.78
	64	3.13×10^{-2}	1.60	3.11×10^{-2}	1.59	2.10×10^{-1}	0.83

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