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出版情報：Proceedings of Czech-Japanese Seminar in Applied Mathematics 2004, pp.73-78, 2004-08

Proceedings of Czech–Japanese Seminar in Applied Mathematics 2004
August 4-7, 2004, Czech Technical University in Prague
http://geraldine.fjfi.cvut.cz
pp. 73–78

THE BOUNDEDNESS OF PROPAGATION SPEEDS OF DISTURBANCES FOR REACTION-DIFFUSION SYSTEMS

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Abstract. It is shown that propagation speeds of disturbances are bounded for a class of reaction-diffusion systems. It turns out that solutions for various initial states are confined by traveling waves. A new technique is presented for the construction of the comparison functions. The technique is based on the operator-splitting methodology, which is known as a numerical computation method. By using an exact solution of the Fisher equation we can make a simple proof. In this paper the outline of the proof is given.

Key words. reaction-diffusion system, comparison technique, operator-splitting method, traveling wave

AMS subject classifications. 35B05, 35K45, 35K57

1. Introduction.

We consider the propagation of disturbances for reaction-diffusion systems. Reaction-diffusion systems on the real line take the form

$$\begin{cases} \boldsymbol{u}_t(x,t) = D\boldsymbol{u}_{xx}(x,t) + \boldsymbol{f}(\boldsymbol{u}(x,t)), \quad x \in \mathbb{R}, \ \ t>0, \\ \qquad\qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ is $\mathbb{R}^n$-valued and D is a diagonal matrix:

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}, \qquad d_i \geq 0 \ \ (i=1,2,\ldots,n). \tag{1.2}$$

The reaction term is also $\mathbb{R}^n$-valued:

$$\boldsymbol{f}(\boldsymbol{u}) = (f_1(\boldsymbol{u}), \ldots, f_n(\boldsymbol{u})).$$

Note that the results obtained in this study can be extended to the multidimensional domain $\mathbb{R}^d$ $(d>1)$ straightforward.

In this paper we impose several conditions on the reaction term $\boldsymbol{f}(\boldsymbol{u})$. Denote by $\mathcal{R}$ a rectangle $[a_1,b_1] \times \cdots \times [a_n,b_n]$, where a_i and b_i are constants and $a_i < b_i$ $(i=1,\ldots,n)$. We assume the followings:

(A1) The function $\boldsymbol{f}$ is smooth from $\mathcal{R}$ to $\mathbb{R}^n$.

(A2) The function $\boldsymbol{f}$ vanishes at a point in $\mathcal{R}$:

$$\boldsymbol{f}(c_1,\ldots,c_n) = 0, \tag{1.3}$$

where c_i are constants and $a_i \leq c_i \leq b_i$ $(i=1,\ldots,n)$.

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(A3) The rectangle $\mathcal{R}$ is an invariant region:

$$\begin{aligned} f_i(\boldsymbol{u}) \geq 0 \quad \text{on } \{\boldsymbol{u} \in \partial\mathcal{R} \mid u_i = a_i\}, \\ f_i(\boldsymbol{u}) \leq 0 \quad \text{on } \{\boldsymbol{u} \in \partial\mathcal{R} \mid u_i = b_i\}. \end{aligned} \tag{1.4}$$

The above assumptions are satisfied by FitzHugh-Nagumo equations, Field-Noyes equations for the Belousov-Zhabotinskii reaction and a model for epidermal wound healing (See Chapter 8 of [8], Section 3, Chapter 9 of [9] and section B, Chapter 14 of [11]).

Now we recall some known results for scalar equations. Consider the scalar equation

$$u_t(x,t) = u_{xx}(x,t) + f(u(x,t)),$$

where f is C^1 function on $[0,1]$ and

$$\begin{cases} f(u) = 0, & u = 0, 1, \\ f(u) > 0, & 0 < u < 1. \end{cases}$$

Under the above condition, propagations of disturbances have been intensively investigated in several literatures. In particular a remarkable result was given by Aronson-Weinberger in [2] and [3]. They proved that disturbances propagate at the speed c^* which is determined by $f(u)$ for a wide class of initial states. Recently Lucia et al. [7] have shown how the propagation speed c^* is determined.

As for the case $n \geq 2$, significant studies have been acomplished on existences and stabilities of special solutions such as traveling waves and pulses (see [6], [10], [13] and references therein).

In this study we estimate propagation speeds of disturbances for (1.1) under the conditions (**A1**)–(**A3**). An advantage of our approach is that we can treat a wide class of initial states in the general setting. To this end we use a traveling wave solution of Fisher's equation

$$u_t = u_{xx} + u(1-u), \qquad x \in \mathbb{R}, \ \ t > 0, \tag{1.5}$$

where u is scalar-valued.

Let

$$\phi(z) = 1/\{1 + \exp(z/\sqrt{6})\}^2. \tag{1.6}$$

Note that $\phi(x - \theta_{\mathrm{e}} t)$, where $\theta_{\mathrm{e}} = 5/\sqrt{6}$, is a traveling wave solution to (1.5) (see the equation (9) in [1]). Here we introduce some notations. Constants γ_i^+, γ_i^-, ω are defined as

$$\gamma_i^+ = \begin{cases} \displaystyle\sup_{\substack{\boldsymbol{u}\in\partial\mathcal{R} \\ u_i = b_i}} \sup_{0\leq\sigma<\tau\leq 1} \frac{|f_i(\boldsymbol{c}+\tau(\boldsymbol{u}-\boldsymbol{c})) - f_i(\boldsymbol{c}+\sigma(\boldsymbol{u}-\boldsymbol{c}))|}{|(b_i - c_i)(\tau-\sigma)|} & \text{if } b_i \neq c_i, \\ 0 & \text{if } b_i = c_i, \end{cases}$$

$$\gamma_i^- = \begin{cases} \displaystyle\sup_{\substack{\boldsymbol{u}\in\partial\mathcal{R} \\ u_i = a_i}} \sup_{0\leq\sigma<\tau\leq 1} \frac{|f_i(\boldsymbol{c}+\tau(\boldsymbol{u}-\boldsymbol{c})) - f_i(\boldsymbol{c}+\sigma(\boldsymbol{u}-\boldsymbol{c}))|}{|(a_i - c_i)(\tau-\sigma)|} & \text{if } a_i \neq c_i, \\ 0 & \text{if } a_i = c_i, \end{cases}$$

$$\omega = \max_{1\le i\le n}\{\gamma_i^{\pm}\},$$

where $\boldsymbol{c} = (c_1, \dots, c_n)$. In addition we put

$$\theta_0 = 2\sqrt{6}\,\omega, \quad \theta_1 = \theta_{\mathrm{e}} \max_{1\le i\le n}\{d_i\}, \tag{1.7}$$

and

$$\theta = \theta_0 + \theta_1. \tag{1.8}$$

Now we are in a position to state our theorem.

Theorem 1.1. *Let $u_i(x,0) \in BUC^1(\mathbb{R})$ and $(a_i - c_i)\phi(x) \le u_i(x,0) - c_i \le (b_i - c_i)\phi(x)$ $(i = 1, 2, \dots, n)$. Suppose the reaction term $\boldsymbol{f}$ satisfies the conditions* (**A1**)–(**A3**). *Then solutions $\boldsymbol{u}$ to* (1.1) *satisfies*

$$(a_i - c_i)\phi(x - \theta t) \le u_i(x,t) - c_i \le (b_i - c_i)\phi(x - \theta t) \tag{1.9}$$

for all $x \in \mathbb{R}$ and $t > 0$ $(i = 1, 2, \dots, n)$.

Remark. *Even if reaction-diffusion systems have no invariant regions, we could apply the theorem. When solutions in question are bounded in L^∞ norm, we may cut off the reaction term $f(\boldsymbol{u})$ properly so that the systems have an invariant region.*

We must refer to Conway and Smoller's comparison theorem [5]. They constructed comparison functions that are spatially homogeneous, by which they proved asymptotic behaviors of some reaction-diffusion systems. Our result may be regarded as an extension of theirs.

2. A nonlinear Trotter product formula. In this section we introduce some notations and present a nonlinear Trotter product formula, which we use later.

Denote by $\mathcal{F}^t$ the flow generated by the vector field $\boldsymbol{f}(\boldsymbol{u})$. Thus, for $\boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{v}(t) = \mathcal{F}^t\boldsymbol{w}$ is a solution to

$$\begin{cases} \dfrac{d\boldsymbol{v}(t)}{dt} = \boldsymbol{f}(\boldsymbol{v}(t)), \\ \boldsymbol{v}(0) = \boldsymbol{w}. \end{cases}$$

We define V and W as follows:

$$V = \{(u_1, \dots, u_n) \mid u_i \in BUC^1(\mathbb{R}) \ \ (i = 1, \dots, n)\}, \tag{2.1}$$

$$W = \{(u_1, \dots, u_n) \mid u_i \in BUC(\mathbb{R}) \ \ (i = 1, \dots, n)\}. \tag{2.2}$$

The norms of V and W are given by

$$\|\boldsymbol{u}\|_V = \max_{1\le i\le n}\{\|u_i\|_{BUC^1(\mathbb{R})}\}, \tag{2.3}$$

$$\|\boldsymbol{u}\|_W = \max_{1\le i\le n}\{\|u_i\|_{BUC(\mathbb{R})}\}. \tag{2.4}$$

We denote by e^{tL} the evolution operator generated by the linear part of the system (1.1), that is,

$$\begin{cases} \partial_t e^{tL}\boldsymbol{u}_0 = \partial_x^2 D e^{tL}\boldsymbol{u}_0 \ \ (t > 0) \\ \lim\limits_{t\downarrow 0} \left\| e^{tL}\boldsymbol{u}_0 - \boldsymbol{u}_0 \right\|_W = 0, \end{cases}$$

where $\boldsymbol{u}_0 \in W$ and ∂_t, ${\partial_x}^2$ act on every component.

Following the notations of Section 15 in [12], we set

$$\boldsymbol{v}_k = \left(e^{(1/m)L}\mathcal{F}^{1/m}\right)^k \boldsymbol{u}_0, \tag{2.5}$$

$$\boldsymbol{v}(t) = e^{sL}\mathcal{F}^s \boldsymbol{v}_k \qquad \text{for } t = k/m + s,\ 0 \le s \le 1/m, \tag{2.6}$$

to use Proposition 5.4 in [12] (p.314). We state the proposition in the suitable form for our problem.

Theorem 2.1 (M. E. Taylor). *Let $\boldsymbol{u}_0$ be an element of V and $\boldsymbol{u}$ the solution to* (1.1). *Suppose $\boldsymbol{v}$ is defined as in* (2.6). *For any positive number T there exists a constant number $C = C(T, D, f)$ such that*

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_W \le Cm^{1/2} \quad \textit{for } 0 \le t \le T \tag{2.7}$$

For more information about the product formula, consult the survey paper written by Chorin et al. ([4]). In the paper we can find some applications of the formula to nonlinear problems such as the Navier-Stokes equations.

3. The Outline of Proof. The strategy of the proof is to approximate the solution of (1.1) by the nonlinear Trotter product formula and to estimate the reaction part and the diffusion part separately. In fact we can prove the following propositions.

Proposition 3.1. *Let $\boldsymbol{w}(x,t) = \mathcal{F}^t \boldsymbol{w}_0(x)$ for $\boldsymbol{w}_0(x) = (w_{0,1}(x), \dots, w_{0,n}(x)) \in W$. If $(a_i - c_i)\phi(x) \le w_{0,i}(x) - c_i \le (b_i - c_i)\phi(x)$ $(i = 1, 2, \dots, n)$, every component $w_i(x,t)$ of $\boldsymbol{w}(x,t)$ is estimated as*

$$(a_i - c_i)\phi(x - \theta_0 t) \le w_i(x,t) - c_i \le (b_i - c_i)\phi(x - \theta_0 t) \quad \textit{for } t \ge 0.$$

Proposition 3.2. *Let $\boldsymbol{w}(x,t) = e^{tL}\boldsymbol{w}_0(x)$ for $\boldsymbol{w}_0(x) \in W$. If $(a_i - c_i)\phi(x) \le w_{0,i}(x) - c_i \le (b_i - c_i)\phi(x)$ $(i = 1, 2, \dots, n)$,*

$$(a_i - c_i)\phi(x - \theta_1 t) \le w_i(x,t) - c_i \le (b_i - c_i)\phi(x - \theta_1 t) \quad \textit{for } t \ge 0.$$

From these propositions it follows

$$(a_i - c_i)\phi(x - (\theta_0 + \theta_1)t) \le v_i(x,t) - c_i \le (b_i - c_i)\phi(x - (\theta_0 + \theta_1)t),$$

where v_i is the ith component of $\boldsymbol{v}$ defined by (2.5), (2.6). Passing to infinity in m we obtain the inequalities (1.9).

For the proof of Proposition 3.1 we introduce a family of rectangles $\mathcal{R}(\sigma)$ defined by

$$\mathcal{R}(\sigma) := \{\boldsymbol{u} \in \mathbb{R}^n \mid c_i + \sigma(a_i - c_i) \le u_i \le c_i + \sigma(b_i - c_i)\ \ (i = 1, \dots, n)\}.$$

Then for each i we set

$$\mathcal{S}_i^+(\sigma) := \{\boldsymbol{u} \in \partial\mathcal{R}(\sigma) \mid u_i = c_i + \sigma(b_i - c_i)\},$$
$$\mathcal{S}_i^-(\sigma) := \{\boldsymbol{u} \in \partial\mathcal{R}(\sigma) \mid u_i = c_i + \sigma(a_i - c_i)\},$$

$$\mu_i^+(\sigma) = \begin{cases} \sup\limits_{\boldsymbol{u}\in\mathcal{S}_i^+(\sigma)} \dfrac{f_i(\boldsymbol{u})}{b_i - c_i} & \text{if } b_i \neq c_i, \\ 0 & \text{if } b_i = c_i, \end{cases}$$

$$\mu_i^-(\sigma) = \begin{cases} \sup\limits_{\boldsymbol{u}\in\mathcal{S}_i^-(\sigma)} \dfrac{f_i(\boldsymbol{u})}{a_i - c_i} & \text{if } a_i \neq c_i, \\ 0 & \text{if } a_i = c_i. \end{cases}$$

In addition we define $M(\sigma)$ by

$$M(\sigma) := \max_{1\leq i\leq n} \{\mu_i^\pm(\sigma)\}. \tag{3.1}$$

Some arguments yield that $M(\sigma)$ is Lipschitz continuous.

We consider the ordinary differential equation determined by $M(\sigma)$ to estimate the flow $\mathcal{F}^t$:

$$\begin{cases} \dfrac{d}{dt}\sigma = M(\sigma), \\ \sigma(0) = \sigma_0. \end{cases} \tag{3.2}$$

We can see that $\mathcal{F}^t u_0 \in \mathcal{R}(\sigma(t))$ for $u_0 \in \mathcal{R}(\sigma_0)$ $(t > 0)$. On the other hand we obtain the following lemma.

LEMMA 3.3. *Let $\sigma(t)$ be a solution of the ordinary differential equation* (3.2). *If $\phi(x) \geq \sigma_0$, then $\phi(x - \theta_0 t) \geq \sigma(t)$ for $t \geq 0$.*

By using this lemma we have Proposition 3.1.

To prove Proposition 3.2 we just apply the comparison principle for heat equations.

Remark. *If $d_1 = d_2 = \cdots = d_n$ we may use a family of convex sets such as balls instead of $\mathcal{R}(\sigma)$.*

Acknowledgement. This work is supported by Kyushu University 21st Century COE Program, Development of Dynamic Mathematics with High Functionality, of the Ministry of Education, Culture, Sports, Science and Technology of Japan. The author thanks Prof. Shimozawa for his stimulus. Without it the research would not be completed. He is also grateful to Prof. Maruno for informing him of the exact solution.

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