Maximum Altitude : An Invariant Imbedding

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Maximum Altitude
— An Invariant Imbedding —

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Abstract

In this paper we solve a continuous non-optimization dynamic programming problem: *Maximum Altitude* problem. The problem has been originated by R. Bellman. We consider three maximum altitude problems. A complete solution through an invariant imbedding is derived. Each problem accompanies a few typical examples.

1 Introduction

R. Bellman has originated dynamic programming [1] and applied it to a huge scientific and engineering areas [5]. He claims that invariant imbedding is a non-optimization dynamic programming [6]. In this paper we consider a typical invariant imbedding problem: *Maximum Altitude* problem, raised by Bellman [3] (see also [12, 13]). We give a complete solution and derive some related results.

In Section 2, we discuss a famous problem in physics — the upward throw against the gravity —through an invariant imbedding. We derive a pair of differential equations (de's) for maximum altitude and for maximum time. These are functions of initial velocity. Our derivation is based on the mean value theorem.

In Section 3, we consider the maximum altitude of material point governed by a nonhomogeneous second-order de. We illustrates the linear system and a quadratic one with typical examples.

In Section 4, we consider a general nonhomogeneous second-order de. We derive a pair of partial differential equations (pde's) for maximum altitude and for maximum time. These are functions of both initial position and initial velocity. We classify the second-order linear system into three typical cases. Each case accompanies an illustrative example.

2 Upward Throw

In this section we consider the maximum altitude when a Mass Point (MP) is upward thrown against the gravity g $(= 9.8m/sec^2)$.

2.1 Against Gravity

Let $h(v)$ be the maximum altitude when MP is upward thrown with an initial velocity v (≥ 0) and $t(v)$ be the time MP attains the maximum altitude. Then the *maximum altitude function* $h : [0,\infty) \to [0,\infty)$ and the *maximum-altitude time function* $t : [0,\infty) \to [0,\infty)$ satisfy the following differential equations (de's).

Theorem 2.1

$$\begin{aligned} \text{(i)}' \quad h'(v) &= \frac{v}{g}, \qquad h(0) = 0 \\ \text{(ii)}' \quad t'(v) &= \frac{1}{g}, \qquad t(0) = 0. \end{aligned}$$

Proof　A rigorous proof based upon an invariant imbedding is given in proof of Lemma 2.1.

Thus we have the solution

$$\text{(i)} \quad h(v) = \frac{v^2}{2g} \qquad 0 \leq v < \infty \tag{1}$$

$$\text{(ii)} \quad t(v) = \frac{v}{g} \qquad 0 \leq v < \infty. \tag{2}$$

For further analysis, we describe the trajectory of Mass Point (MP) by differential equations. Let $x(t)$ be the position (*altitude* or ground height) of MP at time t. Then we have

$$\mathcal{G}(v): \quad \ddot{x} = -g, \quad \dot{x}(0) = v, \quad x(0) = 0$$

That is,

$$\dot{x}(t) = v - gt, \quad x(0) = 0$$

or

$$x(t) = vt - \frac{1}{2}gt^2.$$

We note that $\mathcal{G}(v)$ denotes that MP is upward thrown against the gravity with an initial velocity v at time 0. The *initial position 0* is implied and omitted in $\mathcal{G}(v)$.

Lemma 2.1 (1) *It holds that for any small* $\Delta > 0$

$$\underline{\text{(i)}} \quad h(v) = x(\Delta) + h(\dot{x}(\Delta)) \tag{3}$$

$$\underline{\text{(ii)}} \quad t(v) = \Delta + t(\dot{x}(\Delta)). \tag{4}$$

(2) *Further, Eqs. (3) and (4) imply (1) and (2), respectively.*

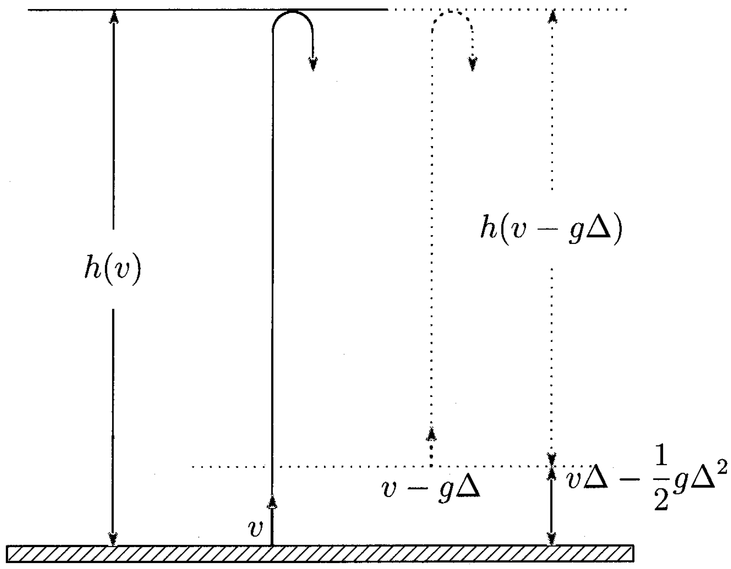

Figure 1: "Maximum Altitude"

Proof of Lemma 2.1 (1) We consider the process $x = \{x(\cdot)\}_{[0,\infty)}$ described by

$$\mathcal{G}(v): \quad \ddot{x} = -g, \quad \dot{x}(0) = v, \quad x(0) = 0.$$

Let τ be a maximizer for

$$\max_{0 \le t < \infty} x(t).$$

(In fact, there exists the first and unique maximizer.) Then we have

$$h(v) \;=\; x(\tau) \qquad \text{and} \qquad t(v) \;=\; \tau.$$

Take any small $\Delta > 0$ and define

$$X(t) \;:=\; x(t + \Delta) - x(\Delta) \qquad t \in [0, \infty).$$

Then the process $X = \{X(\cdot)\}_{[0,\infty)}$ satisfies

$$\mathcal{G}(\dot{x}(\Delta)): \quad \ddot{X} = -g, \quad \dot{X}(0) = \dot{x}(\Delta), \quad X(0) = 0.$$

We have

$$X(\tau - \Delta) \;=\; x(\tau) - x(\Delta).$$

Further we see that $\tau - \Delta$ is a maximizer for

$$\max_{0 \le t < \infty} X(t).$$

In fact, assume that some $\hat{\tau} \in (0, \infty)$ yields a value greater than $X(\tau - \Delta)$:

$$X(\hat{\tau}) \;>\; X(\tau - \Delta).$$

This in turn implies that

$$x(\hat{\tau}+\Delta) \;>\; x(\tau),$$

which contradicts the maximality of $x(\tau)$. Therefore the assertion is valid. Thus we have

$$h(\dot{x}(\Delta)) \;=\; X(\tau-\Delta) \qquad \text{and} \qquad t(\dot{x}(\Delta)) \;=\; \tau-\Delta.$$

Therefore we obtain

$$\begin{aligned} h(v) &= x(\Delta)+h(\dot{x}(\Delta)), \\ t(v) &= \Delta+t(\dot{x}(\Delta)). \end{aligned}$$

(2) Let us show that Eq.(3) implies (1). First take any velocity v (>0) and any small Δ (>0). Let us define

$$h_\Delta := h(\dot{x}(\Delta)) - h(v).$$

Then Eq.(3) reads as follows:

$$h_\Delta \;=\; -x(\Delta).$$

Applying the mean value theorem, we have

$$\begin{aligned} x(\Delta) &= \dot{x}(\theta\Delta)\Delta \qquad (0<\theta<1) \\ h_\Delta &= h'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta)\Delta \quad (0<\xi<1). \end{aligned}$$

Second, substituting these terms and dividing by Δ, we have

$$h'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta) \;=\; -\dot{x}(\theta\Delta).$$

Finally, letting Δ tend to zero, we have

$$h'(v) \;=\; \frac{v}{g}.$$

Similarly it is shown that (4) implies (2). This completes the proof of Lemma 2.1 .

Theorem 2.2

$$\text{(iii)} \qquad h'(v) = vt'(v) \qquad 0 \le c, v < \infty \tag{5}$$

Lemma 2.2 (1) *It holds that*

$$\underline{\text{(iii)}}\;\; h(v)-h(\dot{x}(\Delta)) \;=\; x(t_\Delta) \qquad \forall \textit{ small } \Delta>0 \tag{6}$$

where

$$t_\Delta \;=\; t(v)-t(\dot{x}(\Delta)). \tag{7}$$

(2) *Further, Eq. (6) together with (7) implies (5).*

Proof of Lemma 2.2 (1) It is easily shown that (iii) is a direct combination of (i) and (ii).

(2) First take any velocity v (> 0) and any small Δ (> 0). Let

$$h_\Delta = h(v) - h(\dot{x}(\Delta)).$$

Then (6) reads

$$h_\Delta = x(t_\Delta).$$

From the mean value theorem, we have

$$\begin{aligned} h_\Delta &= -h'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta)\Delta \qquad (0 < \xi < 1) \\ x(t_\Delta) &= \dot{x}(\theta t_\Delta)t_\Delta \qquad (0 < \theta < 1) \\ t_\Delta &= -t'(\dot{x}(\eta\Delta))\ddot{x}(\eta\Delta)\Delta \qquad (0 < \eta < 1) \end{aligned}$$

Second, substituting these terms and dividing by Δ , we have

$$h'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta) = \dot{x}(\theta t_\Delta)t'(\dot{x}(\eta\Delta))\ddot{x}(\eta\Delta).$$

Finally letting Δ tend to zero and dividing by $-g$, we have

$$h'(v) = vt'(v).$$

This completes the proof of Lemma 2.2.

3 Homogeneous System

We consider a homogeneous system

$$\mathcal{H}(v): \quad \ddot{x} = -g - h(\dot{x}) \text{ on } [0, \infty), \quad \dot{x}(0) = v, \quad x(0) = 0$$

where

$$h(v) \geq 0 \quad 0 \leq v < \infty, \qquad h(0) = 0. \tag{8}$$

Let $H(v)$ be the maximum altitude when MP follows the homogeneous system with an initial velocity v (≥ 0) and $T(v)$ be the time MP attains the maximum altitude.

Then the *maximum altitude function* $H : [0, \infty) \to [0, \infty)$ and the *maximum-altitude time function* $T : [0, \infty) \to [0, \infty)$ satisfy the following de's.

Theorem 3.1

$$\text{(i)} \quad H'(v) = \frac{v}{g + h(v)} \quad 0 \leq v < \infty, \qquad H(0) = 0 \tag{9}$$

$$\text{(ii)} \quad T'(v) = \frac{1}{g + h(v)} \quad 0 \leq v < \infty, \qquad T(0) = 0. \tag{10}$$

Lemma 3.1 (1) *It holds that for any small* $\Delta > 0$

$$\underline{\text{(i)}}\ \ H(v) = x(\Delta) + H(\dot{x}(\Delta)) \tag{11}$$

$$\underline{\text{(ii)}}\ \ T(v) = \Delta + T(\dot{x}(\Delta)). \tag{12}$$

(2) *Further, Eqs. (11) and (12) imply (9) and (10), respectively.*

Theorem 3.2

$$\text{(iii)} \qquad H'(v) = vT'(v) \qquad 0 \leq c, v < \infty \tag{13}$$

Lemma 3.2 (1) *It holds that for any small* $\Delta > 0$

$$\underline{\text{(iii)}}\ \ H(v) - H(\dot{x}(\Delta)) = x(T_\Delta) \tag{14}$$

where

$$T_\Delta = T(v) - T(\dot{x}(\Delta)). \tag{15}$$

(2) *Further, Eq. (14) together with (15) implies (13).*

Proof of Lemma 3.2 (1) It is easily shown that <u>(iii)</u> is a direct consequence of <u>(i)</u> and <u>(ii)</u>.

(2) First take any velocity v (> 0) and any small Δ (> 0). From the mean value theorem, we have

$$H_\Delta = \dot{x}(\theta T_\Delta)T_\Delta \qquad (0 < \theta < 1)$$

where

$$H_\Delta = H(v) - H(\dot{x}(\Delta)), \qquad T_\Delta = T(v) - T(\dot{x}(\Delta)).$$

Again by applying the mean value theorem, we have

$$\begin{aligned} H_\Delta &= -H'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta)\Delta \quad (0 < \xi < 1) \\ T_\Delta &= -T'(\dot{x}(\eta\Delta))\ddot{x}(\eta\Delta)\Delta \quad (0 < \eta < 1) \end{aligned}$$

Second, substituting these terms and dividing by Δ, we have

$$H'(\dot{x}(\xi\Delta))\ddot{x}(\xi\Delta) = \dot{x}(\theta\Delta)T'(\dot{x}(\eta\Delta))\ddot{x}(\eta\Delta).$$

Finally letting Δ tend to zero, we have

$$H'(v)(-g + h(v)) = vT'(v)(-g + h(v)).$$

Thus we obtain

$$H'(v) = vT'(v).$$

This completes the proof of Lemma 3.2.

3.1 Linear and Quadratic Dynamics

Let us consider the linear case $h(v) := bv$. Then de's

$$\begin{aligned}
\text{(i)}\quad H'(v) &= \frac{v}{bv+g}, & H(0) &= 0\\
\text{(ii)}\quad T'(v) &= \frac{1}{bv+g}, & T(0) &= 0
\end{aligned}$$

have the solutions

$$\begin{aligned}
\text{(i)}\quad H(v) &= \frac{v}{b} - \frac{g}{b^2}\log\frac{bv+g}{g} & 0 \le v < \infty\\
\text{(ii)}\quad T(v) &= \frac{1}{b}\log\frac{bv+g}{g} & 0 \le v < \infty
\end{aligned}$$

, respectively. The second-order linear de

$$\ddot{x} = -g - b\dot{x}, \quad \dot{x}(0) = v, \quad x(0) = 0$$

has the solution

$$x(t) = \frac{1}{b}\left(v + \frac{g}{b}\right)(1 - e^{-bt}) - \frac{g}{b}t$$

with

$$\dot{x}(t) = \left(v + \frac{g}{b}\right)e^{-bt} - \frac{g}{b}, \qquad \ddot{x}(t) = -(bv+g)e^{-bt}.$$

Let us consider a quadratic case $h(v) := \frac{1}{2}v^2$. Then de's

$$\begin{aligned}
\text{(i)}\quad H'(v) &= \frac{2v}{v^2+2g}, & H(0) &= 0\\
\text{(ii)}\quad T'(v) &= \frac{2}{v^2+2g}, & T(0) &= 0
\end{aligned}$$

have the solutions

$$\begin{aligned}
\text{(i)}\quad H(v) &= \log\frac{v^2+2g}{2g} & 0 \le v < \infty\\
\text{(ii)}\quad T(v) &= \sqrt{\frac{2}{g}}\tan^{-1}\frac{v}{\sqrt{2g}} & 0 \le v < \infty
\end{aligned}$$

, respectively. We consider the second-order nonlinear de

$$\ddot{x} = -g - \frac{1}{2}\dot{x}^2, \quad \dot{x}(0) = v, \quad x(0) = 0.$$

Letting

$$y(t) := \dot{x}(t),$$

we have

$$\dot{y} = -g - \frac{1}{2}y^2, \qquad y(0) = v.$$

This reduces

$$\frac{2dy}{y^2 + 2g} \quad = \quad -dt.$$

Integrating both sides, we get

$$\sqrt{\frac{2}{g}}\tan^{-1}\frac{y}{\sqrt{2g}} \quad = \quad -t + C.$$

where

$$C = \sqrt{\frac{2}{g}}\tan^{-1}\frac{v}{\sqrt{2g}}.$$

Thus we have

$$y(t) \quad = \quad \sqrt{2g}\tan\Big(\sqrt{\frac{g}{2}}(C-t)\Big).$$

Again operating $\displaystyle\int_0^t \quad ds$ for

$$\dot{x}(s) \quad = \quad \sqrt{2g}\tan\Big(\sqrt{\frac{g}{2}}(C-s)\Big),$$

we get

$$x(t) \quad = \quad 2\Big[\log\cos\Big(\sqrt{\frac{g}{2}}(C-t)\Big) - \log\cos\Big(\sqrt{\frac{g}{2}}C\Big)\Big].$$

Thus we have

$$x(t) \quad = \quad 2\log\frac{\cos\big(\sqrt{g/2}(C-t)\big)}{\cos\big(\sqrt{g/2}C\big)}.$$

3.2 Two Examples

Case 1 Let us consider the case $h(v) = v$. Then de's

$$\text{(i)} \quad H'(v) \quad = \quad \frac{v}{g+v}, \qquad H(0) \quad = \quad 0$$

$$\text{(ii)} \quad T'(v) \quad = \quad \frac{1}{g+v}, \qquad T(0) \quad = \quad 0$$

have the solutions

$$\text{(i)}\quad H(v) = v - g\log\frac{g+v}{g} \qquad 0 \le v < \infty$$

$$\text{(ii)}\quad T(v) = \log\frac{g+v}{g} \qquad 0 \le v < \infty.$$

Then the resulting second-order de

$$\ddot{x} = -g - \dot{x}, \quad \dot{x}(0) = v, \quad x(0) = 0$$

has the solution

$$x(t) = (v+g)(1-e^{-t}) - gt$$

with

$$\dot{x}(t) = (v+g)e^{-t} - g, \qquad \ddot{x}(t) = -(v+g)e^{-t}.$$

Case 2 Let us take $h(v) := gv$. Then de's

$$\text{(i)}\quad H'(v) = \frac{v}{g(1+v)}, \qquad H(0) = 0$$

$$\text{(ii)}\quad T'(v) = \frac{1}{g(1+v)}, \qquad T(0) = 0$$

have the solutions

$$\text{(i)}\quad H(v) = \frac{v - \log(1+v)}{g} \qquad 0 \le v < \infty$$

$$\text{(ii)}\quad T(v) = \frac{\log(1+v)}{g} \qquad 0 \le v < \infty$$

, respectively. The corresponding de

$$\ddot{x} = -g - g\dot{x}, \quad \dot{x}(0) = v, \quad x(0) = 0$$

has the solution

$$x(t) = \frac{1+v}{g}(1-e^{-gt}) - t$$

with

$$\dot{x}(t) = (1+v)e^{-gt} - 1, \qquad \ddot{x}(t) = -g(1+v)e^{-gt}.$$

4 Inhomogeneous System

We consider an inhomogeneous system ([9])

$$\mathcal{IH}(c,v): \quad \ddot{x} = h(x,\dot{x}) \ \text{ on } \ [0,\infty), \quad \dot{x}(0) = v, \quad x(0) = c$$

where

$$h(c,v) \leq -g \quad 0 \leq c, v < \infty, \qquad h(0,0) = -g. \tag{16}$$

We assume that the inhomogeneous system $x = \{x(t)\}_{[0,\infty)}$ attains a unique maximum value.

Let $H(c,v)$ be the maximum altitude in *relative-altitude* when MP follows this inhomogeneous system with an initial velocity v (≥ 0) at an initial position c (≥ 0) and $T(c,v)$ be the time it attains the maximum altitude. Then the *maximum relative-altitude function* $H : [0,\infty)^2 \to [0,\infty)$ and the *maximum relative-altitude time function* $T : [0,\infty)^2 \to [0,\infty)$ satisfy the following partial differential equations (pde's).

Theorem 4.1

$$\text{(i)} \qquad vH_c + h(c,v)H_v \ = \ -v \qquad 0 \leq c, v < \infty \tag{17}$$

$$H(c,0) = 0 \qquad 0 \leq c < \infty,$$

$$\text{(ii)} \qquad vT_c + h(c,v)T_v \ = \ -1 \qquad 0 \leq c, v < \infty \tag{18}$$

$$T(c,0) = 0 \qquad 0 \leq c < \infty.$$

Lemma 4.1 (1) *It holds that for any small* $\Delta > 0$

$$\underline{\text{(i)}} \ \ H(c,v) \ = \ x(\Delta) - c + H(x(\Delta), \dot{x}(\Delta)) \tag{19}$$

$$\underline{\text{(ii)}} \ \ T(c,v) \ = \ \Delta + T(x(\Delta), \dot{x}(\Delta)). \tag{20}$$

(2) *Further, Eqs. (19) and (20) imply (17) and (18), respectively.*

Proof (1) We consider the process $x = \{x(\cdot)\}_{[0,\infty)}$ described by

$$\mathcal{IH}(c,v): \quad \ddot{x} = h(x,\dot{x}), \quad \dot{x}(0) = v, \quad x(0) = c.$$

Let T be a maximizer for

$$\max_{0 \leq t < \infty} [x(t) - c].$$

Then we have

$$H(c,v) \ = \ x(T) - c \qquad \text{and} \qquad T(c,v) \ = \ T.$$

Take any small $\Delta > 0$ and define

$$X(t) \ := \ x(t + \Delta) \qquad t \in [0,\infty).$$

Then the process $X = \{X(\cdot)\}_{[0,\infty)}$ satisfies

$$\mathcal{IH}(x(\Delta), \dot{x}(\Delta)) : \quad \ddot{X} = h(X, \dot{X}), \quad \dot{X}(0) = \dot{x}(\Delta), \quad X(0) = x(\Delta).$$

We have

$$X(T - \Delta) \;=\; x(T) \;=\; c + H(c, v).$$

Further we see that $T - \Delta$ is a maximizer for

$$\max_{0 \le t < \infty} [X(t) - X(0)].$$

In fact, assume that some $\hat{T} \in (0, \infty)$ yields a value greater than $X(T - \Delta) - X(0)$:

$$X(\hat{T}) - X(0) \;>\; X(T - \Delta) - X(0).$$

This in turn implies that

$$x(\hat{T} + \Delta) - c \;>\; x(T) - c,$$

which contradicts the maximality of $x(T) - c$. Therefore the assertion is valid. Thus we have

$$H(x(\Delta), \dot{x}(\Delta)) \;=\; X(T - \Delta) - X(0) \qquad \text{and} \qquad T(x(\Delta), \dot{x}(\Delta)) \;=\; T - \Delta.$$

Therefore we obtain

$$\begin{aligned} H(c, v) &= x(\Delta) - c + H(x(\Delta), \dot{x}(\Delta)), \\ T(c, v) &= \Delta + T(x(\Delta), \dot{x}(\Delta)). \end{aligned}$$

(2) Let us show that Eq.(19) implies (17). First take any velocity $v (> 0)$, any position c (> 0) and any small Δ (> 0). Let us define

$$H_\Delta := H(x(\Delta), \dot{x}(\Delta)) - H(c, v).$$

Then Eq.(19) reads as follows:

$$H_\Delta \;=\; -[x(\Delta) - c].$$

Applying the mean value theorem, we have

$$\begin{aligned} x(\Delta) - c &= \dot{x}(\theta\Delta)\Delta \qquad (0 < \theta < 1) \\ H_\Delta &= [H_c \dot{x}(\xi\Delta) + H_v \ddot{x}(\xi\Delta)]\Delta \qquad (0 < \xi < 1) \end{aligned}$$

where

$$H_c = H_c(x(\xi\Delta), \dot{x}(\xi\Delta)), \qquad H_v = H_v(x(\xi\Delta), \dot{x}(\xi\Delta)).$$

Second, substituting these terms and dividing by Δ, we have

$$H_c\dot{x}(\xi\Delta) + H_v\ddot{x}(\xi\Delta) \;=\; -\dot{x}(\theta\Delta).$$

Finally, letting Δ tend to zero, we have

$$vH_c + h(c,v)H_v \;=\; -v.$$

Similarly it is shown that (20) implies (18). This completes the proof of Lemma 4.1.

Theorem 4.2

$$\text{(iii)} \qquad vH_c + h(c,v)H_v \;=\; v(vT_c + h(c,v)T_v) \qquad 0 \le c, v < \infty \tag{21}$$

Lemma 4.2 (1) *It holds that*

$$\underline{\text{(iii)}}\;\; H(c,v) - H(x(\Delta),\dot{x}(\Delta)) \;=\; x(T_\Delta) - c \qquad \forall\ \textit{small}\ \Delta > 0 \tag{22}$$

where

$$T_\Delta \;=\; T(c,v) - T(x(\Delta),\dot{x}(\Delta)). \tag{23}$$

(2) *Further, Eq. (22) together with (23) implies (21).*

Proof of Lemma 4.2 (1) It is easily shown that <u>(iii)</u> is a direct consequence of <u>(i)</u> and <u>(ii)</u>.

(2) First take any velocity v (>0), any position c (>0) and any small Δ (>0). Letting

$$H_\Delta := H(x(\Delta),\dot{x}(\Delta)) - H(c,v)$$
$$T_\Delta := T(x(\Delta),\dot{x}(\Delta)) - T(c,v),$$

we have from the mean value theorem

$$H_\Delta \;=\; \dot{x}(\theta T_\Delta)T_\Delta \qquad (0 < \theta < 1).$$

Again by applying the mean value theorem, we get

$$\begin{aligned} H_\Delta &= [H_c\dot{x}(\xi\Delta) + H_v\ddot{x}(\xi\Delta)]\Delta \quad (0 < \xi < 1) \\ T_\Delta &= [T_c\dot{x}(\eta\Delta) + T_v\ddot{x}(\eta\Delta)]\Delta \quad (0 < \eta < 1) \end{aligned}$$

where

$$H_c = H_c(x(\xi\Delta),\dot{x}(\xi\Delta)), \qquad H_v = H_v(x(\xi\Delta),\dot{x}(\xi\Delta))$$
$$T_c = T_c(x(\eta\Delta),\dot{x}(\eta\Delta)), \qquad T_v = T_v(x(\eta\Delta),\dot{x}(\eta\Delta)).$$

Second, substituting these terms and dividing by Δ, we have

$$H_c\dot{x}(\xi\Delta) + H_v\ddot{x}(\xi\Delta) \;=\; \dot{x}(\theta T_\Delta)(T_c\dot{x}(\eta\Delta) + T_v\ddot{x}(\eta\Delta)).$$

Finally, letting Δ tend to zero, we have

$$vH_c + h(c,v)H_v \;=\; v(vT_c + h(c,v)T_v).$$

This completes the proof of Lemma 4.2.

4.1 Linear Dynamics

Let us consider the linear system $h(x, \dot{x}) := -a\dot{x} - bx - g$. Then the characteristic equation is quadratic:

$$\lambda^2 + a\lambda + b = 0.$$

According as the duplicity of solution we consider the following three cases:
Case I: Two different negative solutions : $0 > \alpha > \beta$.
Case II: A common negative solution : $0 > \alpha = \beta$.
Case III: Complex solutions with a negative real part : $\lambda_1 = \alpha + i\beta,\ \ \lambda_2 = \alpha - i\beta;\ \ \alpha < 0$.

4.1.1 Different Negative Solutions

Case I We have the factorization

$$\begin{aligned} \lambda^2 + a\lambda + b &= (\lambda - \alpha)(\lambda - \beta) \\ &= \lambda^2 - (\alpha + \beta)\lambda + \alpha\beta, \end{aligned}$$

where

$$\begin{aligned} \alpha + \beta &= -a, \qquad \alpha\beta = b; \\ \alpha &= \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad \beta = \frac{-a - \sqrt{a^2 - 4b}}{2}. \end{aligned}$$

Then pde's

$$\begin{aligned} &\text{(i)} \quad vH_c - (g + bc + av)H_v = -v, \qquad H(c, 0) = 0 \\ &\text{(ii)} \quad vT_c - (g + bc + av)T_v = -1, \qquad T(c, 0) = 0 \end{aligned}$$

have the solutions

$$\begin{aligned} &\text{(i)} \quad H(c, v) = \frac{1}{\alpha\beta} \frac{(g + \alpha\beta c - \beta v)^{\alpha/(\alpha-\beta)}}{(g + \alpha\beta c - \alpha v)^{\beta/(\alpha-\beta)}} - \left(\frac{g}{\alpha\beta} + c\right) \qquad 0 \le c, v < \infty \\ &\text{(ii)} \quad T(c, v) = \frac{1}{\alpha - \beta} \log \frac{g + \alpha\beta c - \beta v}{g + \alpha\beta c - \alpha v} \qquad 0 \le c, v < \infty \end{aligned}$$

, respectively. In fact, the system

$$\ddot{x} = -g - bx - a\dot{x}, \quad \dot{x}(0) = v, \quad x(0) = c$$

has the solution

$$x(t) = \frac{-1}{\alpha(\alpha - \beta)}(g + \alpha\beta c - \alpha v)e^{\alpha t} + \frac{1}{\beta(\alpha - \beta)}(g + \alpha\beta c - \beta v)e^{\beta t} - \frac{g}{\alpha\beta}.$$

Then we have

$$\begin{aligned} \dot{x}(t) &= \frac{-1}{\alpha - \beta}(g + \alpha\beta c - \alpha v)e^{\alpha t} + \frac{1}{\alpha - \beta}(g + \alpha\beta c - \beta v)e^{\beta t} \\ \ddot{x}(t) &= \frac{-\alpha}{\alpha - \beta}(g + \alpha\beta c - \alpha v)e^{\alpha t} + \frac{\beta}{\alpha - \beta}(g + \alpha\beta c - \beta v)e^{\beta t}. \end{aligned}$$

4.1.2 Duplicate Negative Solution

Case II Then we have the factorization

$$\begin{aligned}\lambda^2 + a\lambda + b &= (\lambda - \alpha)(\lambda - \alpha)\\ &= \lambda^2 - 2\alpha\lambda + \alpha^2,\end{aligned}$$

where

$$2\alpha = -a, \qquad \alpha^2 = b, \qquad \alpha = \frac{-a}{2}.$$

Then pde's

$$\begin{aligned}&\text{(i)} \quad vH_c - (g + bc + av)H_v = -v, \qquad H(c,0) = 0\\ &\text{(ii)} \quad vT_c - (g + bc + av)T_v = -1, \qquad T(c,0) = 0\end{aligned}$$

have the solutions

$$\begin{aligned}&\text{(i)} \quad H(c,v) = \frac{g + \alpha^2 c - \alpha v}{\alpha^2} e^{\alpha v/(g+\alpha^2 c - \alpha v)} - \Big(\frac{g}{\alpha^2} + c\Big) \qquad 0 \le c, v < \infty\\ &\text{(ii)} \quad T(c,v) = \frac{v}{g + \alpha^2 c - \alpha v} \qquad 0 \le c, v < \infty\end{aligned}$$

, respectively. In Case II, the system

$$\ddot{x} = -g - bx - a\dot{x}, \quad \dot{x}(0) = v, \quad x(0) = c$$

has the solution

$$x(t) = \Big\{\Big(\frac{g + \alpha^2 c}{\alpha^2}\Big) - \Big(\frac{g + \alpha^2 c - \alpha v}{\alpha} t\Big)\Big\} e^{\alpha t} - \frac{g}{\alpha^2}.$$

Then we have

$$\begin{aligned}\dot{x}(t) &= \{v - (g + \alpha^2 c - \alpha v)t\} e^{\alpha t}\\ \ddot{x}(t) &= -\{(g + \alpha^2 c - 2\alpha v) + \alpha(g + \alpha^2 c - \alpha v)t\} e^{\alpha t}.\end{aligned}$$

4.1.3 Complex Solutions with Negative Imaginary Part

Case III Then we have the factorization

$$\begin{aligned}\lambda^2 + a\lambda + b &= (\lambda - \lambda_1)(\lambda - \lambda_2)\\ &= (\lambda - (\alpha + i\beta))(\lambda - (\alpha - i\beta))\\ &= \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2,\end{aligned}$$

where

$$\begin{aligned}&\alpha = -\frac{a}{2}, \qquad \beta = \frac{\sqrt{4b - a^2}}{2}, \qquad D := a^2 - 4b < 0,\\ &\lambda_1 + \lambda_2 = 2\alpha = -a, \qquad \lambda_1\lambda_2 = \alpha^2 + \beta^2 = b.\end{aligned}$$

Then pde's

$$
\begin{aligned}
\text{(i)}\quad vH_c-(g+bc+av)H_v &= -v, \qquad H(c,0)=0\\
\text{(ii)}\quad vT_c-(g+bc+av)T_v &= -1, \qquad T(c,0)=0
\end{aligned}
$$

have the solutions

$$
\begin{aligned}
\text{(i)}\quad H(c,v) &= \frac{1}{b}\sqrt{bv^2-2\alpha(bc+g)v+(bc+g)^2}\exp\Big(\frac{\alpha}{\beta}\tan^{-1}\frac{\beta v}{-\alpha v+bc+g}\Big)\\
&\qquad -\Big(\frac{g}{b}+c\Big)\\
\text{(ii)}\quad T(c,v) &= \frac{1}{\beta}\tan^{-1}\frac{\beta v}{-\alpha v+bc+g} \qquad\qquad 0\le c,v<\infty
\end{aligned}
$$

, respectively. In Case III, the system

$$
\ddot{x}=-g-bx-a\dot{x}, \quad \dot{x}(0)=v, \quad x(0)=c
$$

has the solution

$$
x(t) = e^{\alpha t}\Big[\Big(c+\frac{g}{b}\Big)\cos\beta t+\Big\{\frac{v}{\beta}-\frac{\alpha}{\beta}\Big(c+\frac{g}{b}\Big)\Big\}\sin\beta t\Big]-\frac{g}{b}.
$$

Then we have

$$
\begin{aligned}
\dot{x}(t) &= e^{\alpha t}\Big[v\cos\beta t+\Big\{\frac{\alpha v}{\beta}-\frac{b}{\beta}\Big(c+\frac{g}{b}\Big)\Big\}\sin\beta t\Big]\\
\ddot{x}(t) &= e^{\alpha t}\Big[\Big\{2\alpha v-b\Big(c+\frac{g}{b}\Big)\Big\}\cos\beta t+\Big\{\frac{\alpha^2-\beta^2}{\beta}v-\frac{\alpha b}{\beta}\Big(c+\frac{g}{b}\Big)\Big\}\sin\beta t\Big].
\end{aligned}
$$

4.2 Three Examples

Case 3 Let us consider $h(x,\dot{x}):=-g-2x-3\dot{x}$. Then pde's

$$
\begin{aligned}
\text{(i)}\quad vH_c-(g+2c+3v)H_v &= -v, \qquad H(c,0)=0\\
\text{(ii)}\quad vT_c-(g+2c+3v)T_v &= -1, \qquad T(c,0)=0
\end{aligned}
$$

have the solutions

$$
\begin{aligned}
\text{(i)}\quad H(c,v) &= \frac{(g+2c+v)^2}{2(g+2c+2v)}-\Big(c+\frac{g}{2}\Big) \qquad 0\le c,v<\infty\\
\text{(ii)}\quad T(c,v) &= \log\frac{g+2c+2v}{g+2c+v} \qquad 0\le c,v<\infty
\end{aligned}
$$

, respectively. In fact, the system

$$
\ddot{x}=-g-2x-3\dot{x}, \quad \dot{x}(0)=v, \quad x(0)=c
$$

has the solution

$$x(t) \;=\; (g+2c+v)e^{-t} - \Big(\frac{g}{2}+c+v\Big)e^{-2t} - \frac{g}{2}.$$

Then we have

$$\begin{aligned}\dot{x}(t) &= -(g+2c+v)e^{-t} + (g+2c+2v)e^{-2t}\\ \ddot{x}(t) &= (g+2c+v)e^{-t} - 2(g+2c+2v)e^{-2t}.\end{aligned}$$

Case 4 Let us consider $h(x,\dot{x}) := -g-x-2\dot{x}$. Then pde's

$$\begin{aligned}&\text{(i)}\quad vH_c-(g+c+2v)H_v \;=\; -v, \qquad H(c,0)=0\\ &\text{(ii)}\quad vT_c-(g+c+2v)T_v \;=\; -1, \qquad T(c,0)=0\end{aligned}$$

have the solutions

$$\begin{aligned}&\text{(i)}\quad H(c,v) \;=\; (g+c+v)e^{-v/(g+c+v)}-(g+c) \qquad 0\le c,v<\infty\\ &\text{(ii)}\quad T(c,v) \;=\; \frac{v}{g+c+v} \qquad 0\le c,v<\infty\end{aligned}$$

, respectively. In fact, the system

$$\ddot{x}=-g-x-2\dot{x}, \quad \dot{x}(0)=v, \quad x(0)=c$$

has the solution

$$x(t) \;=\; \{(g+c)+(g+c+v)t\}e^{-t}-g.$$

Then we have

$$\begin{aligned}\dot{x}(t) &= \{v-(g+c+v)t\}e^{-t}\\ \ddot{x}(t) &= \{-(g+c+2v)+(g+c+v)t\}e^{-t}.\end{aligned}$$

Case 5 Let us consider $h(x,\dot{x}) := -g-2x-2\dot{x}$. Then pde's

$$\begin{aligned}&\text{(i)}\quad vH_c-(g+2c+2v)H_v \;=\; -v, \qquad H(c,0)=0\\ &\text{(ii)}\quad vT_c-(g+2c+2v)T_v \;=\; -1, \qquad T(c,0)=0\end{aligned}$$

have the solutions

$$\begin{aligned}\text{(i)}\quad H(c,v) \;&=\; \frac{1}{2}\sqrt{2v^2+2(2c+g)v+(2c+g)^2}\exp\Big(-\tan^{-1}\frac{v}{v+2c+g}\Big)-\\ &\qquad -\Big(c+\frac{g}{2}\Big) \qquad 0\le c,v<\infty\\ \text{(ii)}\quad T(c,v) \;&=\; \tan^{-1}\frac{v}{v+2c+g} \qquad 0\le c,v<\infty\end{aligned}$$

, respectively. In fact, the system

$$\ddot{x} = -g - 2x - 2\dot{x}, \quad \dot{x}(0) = v, \quad x(0) = c$$

has the solution

$$x(t) = e^{-t}\Big[\Big(c + \frac{g}{2}\Big)\cos t + \Big\{v + \Big(c + \frac{g}{2}\Big)\Big\}\sin t\Big] - \frac{g}{2}.$$

Then we have

$$\begin{aligned}\dot{x}(t) &= e^{-t}\{v\cos t - (v + 2c + g)\sin t\}\\ \ddot{x}(t) &= e^{-t}\{-(2v + 2c + g)\cos t + (2c + g)\sin t\}.\end{aligned}$$

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