Dog Chases Rabbit : An Invariant Imbedding

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Dog Chases Rabbit
– An Invariant Imbedding –

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Abstract

In this paper we solve a continuous non-optimization dynamic programming problem: *Dog Chases Rabbit* problem. The problem has been originated by R. Bellman. We give a complete solution through an invariant imbedding approach. Some related results are derived.

1 Introduction

R. Bellman has originated dynamic programming [1] and applied it to a huge scientific and engineering areas [5]. He claims that invariant imbedding is a non-optimization dynamic programming [6]. In this paper we consider a typical invariant imbedding problem; *Dog Chases Rabbit* problem, raised by Bellman [3] (see also [12]). We give a complete solution and derive further related relations.

In Section 2, we derive a underlying time-parametric differential equation (de) which governs both Dog's movement and Rabbit's one.

In Section 3, we analyze *Dog Chases Rabbit* problem through an invariant imbedding [7][8][13]. We derive two partial differential equations (pde's) which describe the time and position when and where Dog chases Rabbit, respectively. The analysis also gives a rigorous proof for the pde's.

In Section 4, we give a non-time description of Dog chasing Rabbit. And we show that the non-time de is derived from the time-parametric de.

In Section 5, we discuss the discrete-time problem. We give both the underlying difference equation and two difference equations which describe the catch-position and catch-time.

In the last section, we comment on a similar but not the same problem; Dog Follows Master problem, which is one of the classical geometrical problems. We give a few inter-related transformations among four descriptive equations.

2 Dog chases rabbit

2.1 System of differential dquations

There are two animals, a Dog and a Rabbit, on (x, y)-plane. The Dog always chases the Rabbit. The Dog is trying to catch the Rabbit, and the Rabbit is getting out of it.

Along the x-axis, the Rabbit begins to run from a designated position r (≥ 0) at time $t = 0$ with a constant velocity (speed) v_R (> 0). On the other hand, at a designated position d (≥ 0) on the y-axis, the Dog begins to *chase* the Rabbit at $t = 0$ with a constant velocity v_D $(> v_R)$. The Dog always runs the Rabbit's way. At any moment, the Dog directs its course toward the Rabbit.

Then our problem is when and where Dog catches Rabbit.

Let us solve this problem through an invariant imbedding approach. First we describe the trajectory of Dog and Rabbit by differential equations. Let $(x(t), y(t))$ be the position of Rabbit at time $t \in [0, \infty)$. Then we have

$$\mathcal{R}(r): \quad \begin{array}{lll} \text{(i)}_R & \dot{x} = v_R, & x(0) = r \\ \text{(ii)}_R & \dot{y} = 0, & y(0) = 0. \end{array}$$

That is,

$$\begin{array}{ll} \text{(i)}_R & x(t) = r + v_R t \\ \text{(ii)}_R & y(t) = 0 \end{array} \qquad 0 \leq t < \infty. \tag{1}$$

On the other hand, let $(X(t), Y(t))$ be the position of Dog at time t. Then we have

$$\mathcal{D}(d): \quad \begin{array}{ll} \text{(i)}_D & \dfrac{\dot{Y}(t)}{\dot{X}(t)} = \dfrac{Y(t)}{X(t) - x(t)} \\ \text{(ii)}_D & \dot{X}^2(t) + \dot{Y}^2(t) = v_D^2 \\ \text{(iii)}_D & (X(0), Y(0)) = (0, d) \end{array} \qquad 0 \leq t < \infty. \tag{2}$$

We remark that $\mathcal{R}(r)$ denotes that Rabbit starts from r at time 0 on x-axis. Similarly, $\mathcal{D}(d)$ denotes that Dog at d on y-axis at time 0 pursues Rabbit there on (x, y)-plane.

2.2 When and where does dog catch rabbit?

Now, let us solve *Dog Chases Rabbit* problem. We are interested in when and where Dog catches Rabbit. Unifying both the when-problem and the where-problem, we solve the two simultaneously through invariant imbedding.

Let $f(r, d)$ be the first time Dog catches Rabbit and $g(r, d)$ be the position in y-axis. Then the *catch-position function* $g = g(r, d) : [0, \infty)^2 \to [0, \infty)$ and the *catch-time function* $f = f(r, d) : [0, \infty)^2 \to [0, \infty)$ satisfy the following first-order linear pde's:

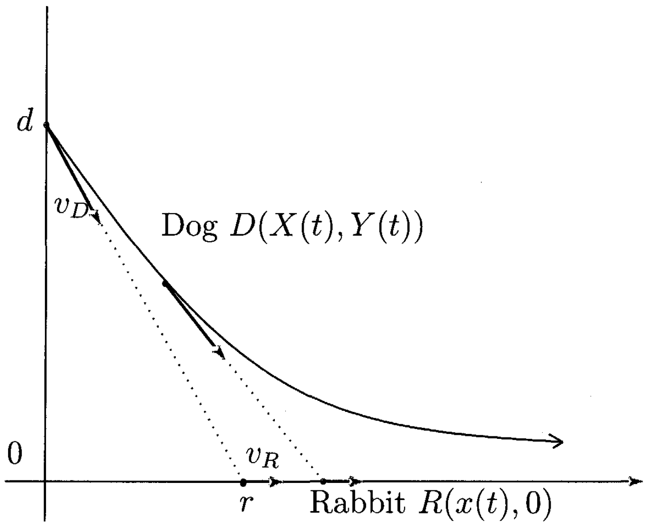

Figure 1: Position and Time "Dog Chases Rabbit"

Theorem 2.1

$$\text{(i)} \quad \left(v_R - \frac{r}{\sqrt{r^2+d^2}}v_D\right)g_r - \frac{d}{\sqrt{r^2+d^2}}v_D g_d = -\frac{r}{\sqrt{r^2+d^2}}v_D \qquad 0 \le r, d < \infty \tag{3}$$

$$g(r,0) = \frac{v_D}{v_D - v_R}r \qquad 0 \le r < \infty, \tag{4}$$

$$\text{(ii)} \quad \left(v_R - \frac{r}{\sqrt{r^2+d^2}}v_D\right)f_r - \frac{d}{\sqrt{r^2+d^2}}v_D f_d = -1 \qquad 0 \le r, d < \infty \tag{5}$$

$$f(r,0) = \frac{r}{v_D - v_R} \qquad 0 \le r < \infty. \tag{6}$$

Proof Both initial conditions (4) and (6) are straightforward. Both equations (3) and (5) are shown through an invariant imbedding in the proof of Lemma 2.1. The following lemma takes a fundamental role in deriving the pde's stated above.

Lemma 2.1 (1) *It holds that for any small* $\Delta > 0$

$$\underline{\text{(i)}} \quad g(r,d) = X(\Delta) + g(x(\Delta) - X(\Delta), Y(\Delta)), \tag{7}$$

$$\underline{\text{(ii)}} \quad f(r,d) = \Delta + f(x(\Delta) - X(\Delta), Y(\Delta)). \tag{8}$$

(2) *Further, Eqs. (7) and (8) imply (3) and (5), respectively.*

Proof of Lemma 2.1 is shown in the next section 3.

A combination of both catch functions satisfies the following first-order linear homogeneous equation:

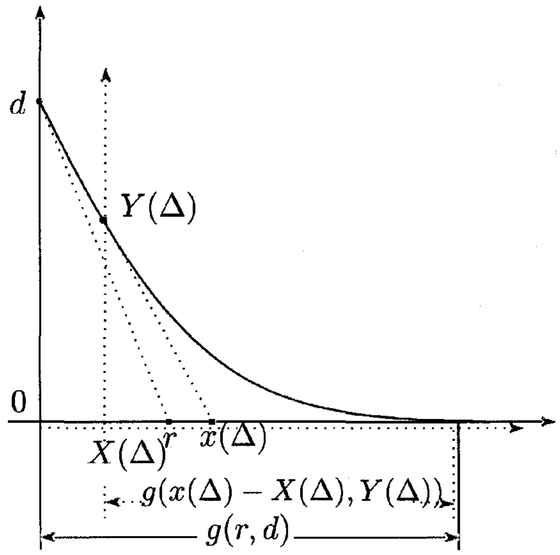

Figure 2: After small $\Delta(>0)$ seconds

Theorem 2.2

$$\text{(iii)} \quad \Big(v_R - \frac{r}{\sqrt{r^2+d^2}} v_D\Big)\Big(g_r - \frac{r}{\sqrt{r^2+d^2}} v_D f_r\Big) - \frac{d}{\sqrt{r^2+d^2}}\Big(g_d - \frac{r}{\sqrt{r^2+d^2}} v_D f_d\Big) = 0$$

$$0 \le r, d < \infty. \tag{9}$$

Proof is given in proof of Lemma 2.2 (2).

Lemma 2.2 (1) *It holds that*

$$\underline{\text{(iii)}} \quad g(r,d) - g(x(\Delta) - X(\Delta), Y(\Delta)) = X(f_\Delta) \qquad \forall \textit{ small } \Delta > 0 \tag{10}$$

where

$$f_\Delta = f(r,d) - f(x(\Delta) - X(\Delta), Y(\Delta)). \tag{11}$$

(2) *Further, Eq. (10) together with (11) implies (9).*

Proof (1) It is easily shown that $\underline{\text{(iii)}}$ is a direct consequence of $\underline{\text{(i)}}$ and $\underline{\text{(ii)}}$.
(2) First, we note that

$$X(0) = 0, \qquad Y(0) = d \ (>0), \qquad x(0) = r \ (>0)$$
$$\dot{X}(0) = v_D \frac{r}{\sqrt{r^2+d^2}}, \qquad \dot{Y}(0) = -v_D \frac{d}{\sqrt{r^2+d^2}}, \qquad \dot{x}(0) = v_R.$$

Take any small time $\Delta\ (>0)$. Letting

$$g_\Delta := g(r,d) - g(x(\Delta) - X(\Delta), Y(\Delta))$$

$$f_\Delta := f(r,d) - f(x(\Delta) - X(\Delta),\, Y(\Delta)),$$

we have from the mean value theorem

$$g_\Delta \;=\; \dot{X}(\theta f_\Delta) f_\Delta \qquad (0 < \theta < 1).$$

Again by applying the mean value theorem, we get

$$\begin{aligned} g_\Delta &= -[g_r(\dot{x}(\xi\Delta) - \dot{X}(\xi\Delta)) + g_d\dot{Y}(\xi\Delta)]\Delta \quad (0 < \xi < 1) \\ f_\Delta &= -[f_r(\dot{x}(\eta\Delta) - \dot{X}(\eta\Delta)) + f_d\dot{Y}(\xi\Delta)]\Delta \quad (0 < \eta < 1) \end{aligned}$$

where

$$\begin{aligned} g_r &= g_r(x(\xi\Delta) - X(\xi\Delta), Y(\xi\Delta)), \qquad g_d = g_d(x(\xi\Delta) - X(\xi\Delta), Y(\xi\Delta)) \\ f_r &= f_r(x(\eta\Delta) - X(\eta\Delta), Y(\eta\Delta)), \qquad f_d = f_d(x(\eta\Delta) - X(\eta\Delta), Y(\eta\Delta)). \end{aligned}$$

Second, substituting these terms and dividing by Δ, we have

$$g_r(\dot{x}(\xi\Delta) - \dot{X}(\xi\Delta)) + g_d\dot{Y}(\xi\Delta) \;=\; \dot{X}(\theta f_\Delta)[f_r(\dot{x}(\eta\Delta) - \dot{X}(\eta\Delta)) + f_d\dot{Y}(\xi\Delta)].$$

Finally, letting Δ tend to zero, we have

$$g_r(\dot{x}(0) - \dot{X}(0)) + g_d\dot{Y}(0) \;=\; \dot{X}(0)[f_r(\dot{x}(0) - \dot{X}(0)) + f_d\dot{Y}(0)],$$

where

$$g_r = g_r(r,d), \qquad g_d = g_d(r,d); \qquad f_r = f_r(r,d), \qquad f_d = f_d(r,d).$$

Therefore, we have

$$\Big(v_R - \frac{r}{\sqrt{r^2+d^2}}v_D\Big)\Big(g_r - \frac{r}{\sqrt{r^2+d^2}}v_D f_r\Big) - \frac{d}{\sqrt{r^2+d^2}}\Big(g_d - \frac{r}{\sqrt{r^2+d^2}}v_D f_d\Big) \;=\; 0.$$

This completes the proof of Lemma 2.2.

3 Invariant imbedding

Now, let us analyze *Dog Chases Rabbit* problem through invariant imbedding. We derive pde's which govern when and where Dog catches Rabbit. The analysis also gives a rigorous proof of Lemma 2.1.

We consider the coupled process $< (X,Y); (x,y) >= \{< (X(\cdot), Y(\cdot)); (x(\cdot), y(\cdot)) > \}_{[0,\infty)}$ described by

$$\mathcal{D}(d): \quad \begin{array}{ll} \text{(i)}_D & \dfrac{\dot{Y}}{\dot{X}} = \dfrac{Y}{X - x} \\ \text{(ii)}_D & \dot{X}^2 + \dot{Y}^2 = v_D^2 \\ \text{(iii)}_D & (X(0), Y(0)) = (0, d) \end{array} \qquad \text{on } [0,\infty)$$

and

$$\mathcal{R}(r): \quad \begin{array}{ll} (\mathrm{i})_R & \dot{x} = v_R \\ (\mathrm{ii})_R & \dot{y} = 0 \end{array} \quad \text{on} \quad [0, \infty) \quad \begin{array}{l} x(0) = r \\ y(0) = 0. \end{array}$$

For the sake of simplicity, we write $\mathcal{DR}(d; r)$ for the pair of $\mathcal{D}(d)$ and $\mathcal{R}(r)$. We note that $f(r, d)$ is the first time t (≥ 0) satisfying

$$x(t) \;\leq\; X(t)$$

and that in this case

$$g(r, d) \;=\; X(t).$$

Now let us assume that

$$f(r, d) \;=\; T. \tag{12}$$

Then we have

$$x(T) \;\leq\; X(T) \qquad \text{and} \qquad g(r, d) \;=\; X(T). \tag{13}$$

Take any small $\Delta > 0$ and define

$$\left.\begin{array}{l} \tilde{X}(t) := X(t+\Delta) - X(\Delta) \\ \tilde{Y}(t) := Y(t+\Delta) \\ \tilde{x}(t) := x(t+\Delta) - X(\Delta) \\ \tilde{y}(t) := y(t+\Delta) \end{array}\right. \qquad t \in [0, \infty). \tag{14}$$

Then the coupled process $< (\tilde{X}, \tilde{Y}); (\tilde{x}, \tilde{y}) >= \{< (\tilde{X}(\cdot), \tilde{Y}(\cdot)); (\tilde{x}(\cdot), \tilde{y}(\cdot)) >\}_{[0,\infty)}$ satisfies $\mathcal{DR}(Y(\Delta); x(\Delta) - X(\Delta))$. In fact, it is easily shown that $(\tilde{X}, \tilde{Y}) = \{(\tilde{X}(\cdot), \tilde{Y}(\cdot))\}_{[0,\infty)}$ satisfies $\mathcal{D}(Y(\Delta))$ and $(\tilde{x}, \tilde{y}) = \{(\tilde{x}(\cdot), \tilde{y}(\cdot))\}_{[0,\infty)}$ does $\mathcal{R}(x(\Delta) - X(\Delta))$. From (13),(14), we have

$$\tilde{x}(T - \Delta) \;\leq\; \tilde{X}(T - \Delta) \tag{15}$$

and

$$\tilde{X}(T - \Delta) \;=\; X(T) - X(\Delta). \tag{16}$$

Further we see that

$$\tilde{x}(t) \;>\; \tilde{X}(t) \qquad \forall\, t \in (0, T - \Delta). \tag{17}$$

In fact, assume that

$$\tilde{x}(\hat{t}) \;\leq\; \tilde{X}(\hat{t}) \qquad \text{for some } \hat{t} \in (0, T - \Delta).$$

Then this together with (14) in turn implies that

$$x(\hat{t}+\Delta) \;\leq\; X(\hat{t}+\Delta) \qquad \hat{t}+\Delta \in (0,T),$$

which contradicts (*the first* catch in) the definition of $f(r,d)=T$. Thus (17) holds true. Therefore, we have from (15)

$$f(x(\Delta)-X(\Delta)\,,\,Y(\Delta)) \;=\; T-\Delta. \tag{18}$$

At the same time, we obtain from (16)

$$\begin{aligned} g(x(\Delta)-X(\Delta)\,,\,Y(\Delta)) \;&=\; \tilde{X}(T-\Delta) \\ &=\; X(T)-X(\Delta). \end{aligned} \tag{19}$$

Finally combining (12),(13),(18) and (19) we have

$$\begin{aligned} f(r,d) \;&=\; \Delta + f(x(\Delta)-X(\Delta)\,,\,Y(\Delta)) \\ g(r,d) \;&=\; X(\Delta) + g(x(\Delta)-X(\Delta)\,,\,Y(\Delta)). \end{aligned}$$

Now let us show the implication (7) $\Longrightarrow$ (3). First, we note that

$$\begin{gathered} X(0)=0, \qquad Y(0)=d\ (>0), \qquad x(0)=r\ (>0) \\ \dot{X}(0)=v_D\frac{r}{\sqrt{r^2+d^2}}, \qquad \dot{Y}(0)=-v_D\frac{d}{\sqrt{r^2+d^2}}, \qquad \dot{x}(0)=v_R. \end{gathered}$$

Let us define

$$g_\Delta \;:=\; g(r,d)-g(x(\Delta)-X(\Delta)\,,\,Y(\Delta)).$$

Then (7) reads as follows:

$$g_\Delta \;=\; X(\Delta).$$

Applying the mean value theorem, we have

$$\begin{aligned} X(\Delta) \;&=\; \dot{X}(\theta\Delta)\Delta \qquad (0<\theta<1), \\ g_\Delta \;&=\; -[g_r(\dot{x}(\xi\Delta)-\dot{X}(\xi\Delta))+g_d\dot{Y}(\xi\Delta)]\Delta \quad (0<\xi<1) \end{aligned}$$

where

$$g_r=g_r(x(\xi\Delta)-X(\xi\Delta),Y(\xi\Delta)), \qquad g_d=g_d(x(\xi\Delta)-X(\xi\Delta),Y(\xi\Delta)).$$

Second, substituting these terms and dividing by Δ, we have

$$g_r(\dot{x}(\xi\Delta)-\dot{X}(\xi\Delta))+g_d\dot{Y}(\xi\Delta) \;=\; -\dot{X}(\theta\Delta).$$

Finally, letting Δ tend to zero, we have

$$g_r(\dot{x}(0)-\dot{X}(0))+g_d\dot{Y}(0) \;=\; -\dot{X}(0),$$

where

$$g_r = g_r(r,d), \qquad g_d = g_d(r,d).$$

Therefore, we obtain

$$\Big(v_R - \frac{r}{\sqrt{r^2+d^2}}v_D\Big)g_r - \frac{d}{\sqrt{r^2+d^2}}v_D g_d \;=\; -\frac{r}{\sqrt{r^2+d^2}}v_D.$$

Similarly, the implication (8) $\Longrightarrow$ (5) is shown. This completes the proof of Lemma 2.1.

3.1 Differential equation approach

In this subsection we consider a differential equation which describes the trajectory of Dog for *Dog Chases Rabbit* problem. Let, for any x (≥ 0) on x-axis, $(x, h(x))$ denote the position of Dog chasing Rabbit in its direct way on (x,y)-plane. Then the *position function* $h = h(\cdot)$ satisfies the second-order de as follows.

Theorem 3.1

$$\text{(i)} \qquad h(0) = d \tag{20}$$

$$\text{(ii)} \qquad h'(0) = -\frac{d}{r} \tag{21}$$

$$\text{(iii)} \qquad v_R h'^2\sqrt{1+h'^2} = v_D h h'' \qquad \textit{on} \;\; [0,\infty). \tag{22}$$

Proof Both initial conditions (20) and (21) are trivial. Eq (22) is shown in the following. First we keep track of two movements for the duration $[t, t+T]$. While Dog runs from $(x, h(x))$ to $(y, h(y))$ along the curve $h = h(\cdot)$, Rabbit runs from $\Big(x - \dfrac{h(x)}{h'(x)},\ 0\Big)$ to $\Big(y - \dfrac{h(y)}{h'(y)},\ 0\Big)$ on x-axis. This implies

$$\text{(i)}_{\mathrm{R}} \qquad y - \frac{h(y)}{h'(y)} - \Big(x - \frac{h(x)}{h'(x)}\Big) = v_R T \tag{23}$$

$$0 \leq x \leq y.$$

$$\text{(ii)}_{\mathrm{D}} \qquad \int_x^y \sqrt{1+h'^2(z)}dz = v_D T \tag{24}$$

Combining (23) and (24), we get

$$\frac{1}{v_R}\Big[y - \frac{h(y)}{h'(y)} - \Big(x - \frac{h(x)}{h'(x)}\Big)\Big] \;=\; \frac{1}{v_D}\int_x^y \sqrt{1+h'^2(z)}dz \qquad 0 \leq x \leq y.$$

Further substituting $(x,y) := (0,x)$, we have

$$\frac{1}{v_R}\Big(x - \frac{h(x)}{h'(x)} - r\Big) \;=\; \frac{1}{v_D}\int_0^x \sqrt{1+h'^2(y)}dy \qquad x \geq 0.$$

Thus the differentiation of both sides yields

$$\frac{h(x)h''(x)}{v_R h'^2(x)} = \frac{\sqrt{1+h'^2(x)}}{v_D} \qquad x \geq 0.$$

This implies the desired equality (22).

Remark The position function h is convex on $\{x|h(x) > 0\}$, because $h''(x) > 0$ on $[0, \infty)$. Further, $h'(x) = 0$ is equivalent to $h(x) = 0$. Let $x_1 > 0$ satisfy $h'(x_1) = h(x_1) = 0$. If h is a solution on $[0, x_1]$, then $g = g(y) := h(2x_1 - y)$ is a solution on $[x_1, 2x_1]$.

4 From time-parametric to non-parametric

Now we derive the non-parametric system of de's (20)-(22) from the time-parametric system of de's (1),(2):

$$\begin{array}{lll}
(\mathrm{i})_D & \dfrac{\dot{Y}(t)}{\dot{X}(t)} = \dfrac{Y(t)}{X(t) - x(t)} & \\
(\mathrm{ii})_D & \dot{X}^2(t) + \dot{Y}^2(t) = v_D^2 & 0 \leq t < \infty. \\
(\mathrm{iii})_D & (X(0), Y(0)) = (0, d) &
\end{array}$$

where

$$\begin{array}{lll}
(\mathrm{i})_R & x(t) = r + v_R t & \\
(\mathrm{ii})_R & y(t) = 0 & 0 \leq t < \infty.
\end{array}$$

Since $h(X(t)) = Y(t)$, we see that

$$h'(X) = \frac{dY}{dX} = \frac{\dot{Y}(t)}{\dot{X}(t)}$$

$$h''(X) = \frac{d^2Y}{d^2X} = \frac{1}{\dot{X}(t)} \frac{\ddot{Y}(t)\dot{X}(t) - \dot{Y}(t)\ddot{X}(t)}{\dot{X}^2(t)}.$$

Letting $t := 0$ in $(\mathrm{i})_D$, $(\mathrm{iii})_D$ and $(\mathrm{i})_R$, we have

$$h(0) = d$$

$$h'(0) = -\frac{d}{r}.$$

This implies Eqs (20),(21). Further Eq (22) reads as follows (hereafter, (t) is omitted):

$$v_R \frac{\dot{Y}^2}{\dot{X}^2} \sqrt{1 + \frac{\dot{Y}^2}{\dot{X}^2}} = v_D Y \frac{\ddot{Y}\dot{X} - \dot{Y}\ddot{X}}{\dot{X}^3}. \tag{25}$$

Now let us derive (25) in the following. First, from $(\mathrm{i})_D$ and $(\mathrm{ii})_D$, we note

$$(\mathrm{i})'_D \quad \frac{1}{X-x}=\frac{\dot{Y}}{\dot{X}Y}$$

$$(\mathrm{ii})'_D \quad \sqrt{1+\frac{\dot{Y}^2}{\dot{X}^2}}=\frac{v_D}{\dot{X}}.$$

Secondly, differentiating both sides in $(\mathrm{i})_D$ and substituting $(\mathrm{i})'_D$, we get

$$\begin{aligned}\frac{\ddot{Y}\dot{X}-Y\ddot{X}}{\dot{X}^2} &= \frac{\dot{Y}(X-x)-Y(\dot{X}-v_R)}{(X-x)^2}\\ &= \frac{\dot{Y}}{X-x}-\frac{Y(\dot{X}-v_R)}{(X-x)^2}\\ &= \frac{\dot{Y}^2}{\dot{X}Y}-\frac{\dot{Y}^2(\dot{X}-v_R)}{\dot{X}^2Y}\\ &= \frac{v_R\dot{Y}^2}{\dot{X}^2Y}.\end{aligned} \tag{26}$$

Thirdly, multiplying both most-left hand side and most-right in (26) by $\frac{v_DY}{\dot{X}}$ and substituting $(\mathrm{ii})'_D$, we obtain

$$\begin{aligned}v_DY\frac{\ddot{Y}\dot{X}-Y\ddot{X}}{\dot{X}^3} &= \frac{v_D}{\dot{X}}Y\frac{v_R\dot{Y}^2}{\dot{X}^2Y}\\ &= \sqrt{1+\frac{\dot{Y}^2}{\dot{X}^2}}Y\frac{v_R\dot{Y}^2}{\dot{X}^2Y}\\ &= v_R\frac{\dot{Y}^2}{\dot{X}^2}\sqrt{1+\frac{\dot{Y}^2}{\dot{X}^2}}.\end{aligned}$$

Therefore, Eq (25) is derived.

5 Discretion

Let us consider a discrete form of *Dog Chases Rabbit* problem. We derive the discrete time version of the continuous system of de's (1),(2):

$$\begin{aligned}&(\mathrm{i})_R \quad x(t)=r+v_Rt\\ &(\mathrm{i})_D \quad \frac{\dot{Y}(t)}{\dot{X}(t)}=\frac{Y(t)}{X(t)-x(t)} \qquad 0\le t<\infty.\end{aligned}$$

$$
\begin{aligned}
&(\mathrm{ii})_D \quad \dot{X}^2(t) + \dot{Y}^2(t) = v_D^2 \\
&(\mathrm{iii})_D \quad (X(0), Y(0)) = (0, d)
\end{aligned}
\tag{27}
$$

Let us take an arbitrarily small Δ (> 0). We define the sequence $\{(x_n, X_n, Y_n)\}_{n \geq 0}$ by

$$
x_n := x(n\Delta), \quad X_n := X(n\Delta), \quad Y_n := Y(n\Delta) \qquad n = 0, 1, \ldots.
$$

Take any t (> 0). Then we choose an integer n satisfying $n\Delta \leq t < (n+1)\Delta$ and approximate $(x(t), X(t), Y(t), \dot{X}(t), \dot{Y}(t))$ by

$$
\begin{aligned}
&x(t) \approx x_{n+1}, \quad X(t) \approx X_n, \quad Y(t) \approx Y_n, \\
&\dot{X}(t) \approx \frac{X_{n+1} - X_n}{\Delta}, \quad \dot{Y}(t) \approx \frac{Y_{n+1} - Y_n}{\Delta}.
\end{aligned}
$$

We have approximated $x(t)$ to $x(n+1)$ in stead of $x(n)$, because of the foresight of Dog for Rabbit's movement. Thus, from (27), we get the following approximate form:

$$
\mathcal{DR}(d; r): \quad
\begin{aligned}
&x_0 = r, \quad X_0 = 0, \quad Y_0 = d; \\
&x_{n+1} = x_n + \Delta v_R \\
&X_{n+1} = X_n + \Delta v_D \frac{x_{n+1} - X_n}{\sqrt{(x_{n+1} - X_n)^2 + Y_n^2}} \qquad n = 0, 1, \ldots. \\
&Y_{n+1} = Y_n - \Delta v_D \frac{Y_n}{\sqrt{(x_{n+1} - X_n)^2 + Y_n^2}}
\end{aligned}
\tag{28}
$$

When a stopping criterion ϵ (> 0) is given, stop the first n satisfying the relation

$$
(x_{n+1} - X_n)^2 + Y_n^2 \leq \epsilon. \tag{29}
$$

Thus we have discrete trajectories $\{(X_n, Y_n)\}_{0,1,\ldots,n}$ and $\{x_n\}_{0,1,\ldots,n}$, where the final distance is less than or equal to ϵ.

Now we consider the coupled discrete process $\mathcal{DR}(d; r)$ described by (28). Let $g(r, d)$ be the position of Dog where Dog exceeds Rabbit on X-axis and $f(r, d)$ be the time when Dog does Rabbit. Then we have

Theorem 5.1

$$
\begin{aligned}
(\mathrm{i}) \qquad g(r, d) &= X_1 + g(x_1 - X_1, Y_1) \qquad 0 \leq r, d < \infty \\
g(r, 0) &= n\Delta v_D \qquad 0 \leq r < \infty, \\
(\mathrm{ii}) \qquad f(r, d) &= 1 + f(x_1 - X_1, Y_1) \qquad 0 \leq r, d < \infty \\
f(r, 0) &= n \qquad 0 \leq r < \infty,
\end{aligned}
$$

where n is the smallest integer satisfying

$$\frac{r}{\Delta(v_D - v_R)} \leq n$$

and

$$x_1 = r + \Delta v_R, \quad X_1 = \Delta v_D \frac{x_1}{\sqrt{x_1^2 + d^2}}, \quad Y_1 = d - \Delta v_D \frac{d}{\sqrt{x_1^2 + d^2}}.$$

6 Dog follows master

6.1 System of differential equations

A Man takes a Dog for a walk on (x, y)-plane. The Dog always follows its Master with a cord of a *constant* length a (> 0).

Along the x-axis, Man begins to take a walk from the origin $(0, 0)$ at time $t = 0$ with a constant speed (velocity) v_M (> 0). On the other hand, at a position a (> 0) on the y-axis, Dog begins to *follow* Master at $t = 0$, keeping the constant length a. Dog always follows Master's way. At any moment, Dog directs its course toward Master.

The problem is where Dog traces him.

First we describe the trajectories of Master and Dog by differential equations. Let $(x(t), y(t))$ be the position of Master at time t. Then we have

$$\mathcal{M}: \quad \begin{array}{ll} (\mathrm{i})_M & \dot{x} = v_M \\ (\mathrm{ii})_M & \dot{y} = 0 \end{array} \quad \text{on } [0, \infty) \quad \begin{array}{l} x(0) = 0 \\ y(0) = 0. \end{array}$$

That is,

$$\begin{array}{ll} (\mathrm{i})_M & x(t) = v_R t \\ (\mathrm{ii})_M & y(t) = 0 \end{array} \qquad 0 \leq t < \infty.$$

On the other hand, let $(X(t), Y(t))$ be the position of Dog at time t. Then we have

$$\mathcal{D}_{tim}: \quad \begin{array}{ll} (\mathrm{i})_D & \dfrac{\dot{Y}}{\dot{X}} = \dfrac{Y}{X - x} \\ (\mathrm{ii})_D & (X - x)^2 + Y^2 = a^2 \\ (\mathrm{iii})_D & (X(0), Y(0)) = (0, a) \end{array} \qquad \text{on } [0, \infty).$$

We remark that $\mathcal{D}_{tim}$ denotes Dog's behavior in time-parameter.

6.2 Where does dog follow master?

Now, let us solve *Dog Follows Master* problem. We are interested in Dog's trajectory. In this subsection we consider a differential equation which describes the trajectory of Dog. Let, for any x (≥ 0) on x-axis, $(x, h(x))$ denote the position of Dog following Master in its direct way. Then the *position function* $h = h(\cdot)$ satisfies the first-order de as follows.

Theorem 6.1

$$\mathcal{D}_{de}: \quad \begin{aligned} &\text{(i)} \quad h(0) = a \\ &\text{(ii)} \quad \left|\frac{h}{h'}\right|\sqrt{1+h'^2} = a \qquad on \ [0, \infty). \end{aligned}$$

Proof The initial condition (i) is trivial. The Dog proceeds from $(x, h(x))$ along the curve $h = h(\cdot)$ the very moment Master moves from $\left(x - \dfrac{h(x)}{h'(x)},\ 0\right)$ on x-axis. Thus we have

$$\left(\frac{h(x)}{h'(x)}\right)^2 + h^2(x) = a^2$$

, which implies (ii).

6.2.1 From time-parametric to non-parametric

Now we derive the non-parametric system $\mathcal{D}_{de}$ from the time-parametric system $\mathcal{D}_{tim}$.

Since $h(X(t)) = Y(t)$, we see that

$$h'(X) = \frac{dY}{dX} = \frac{\dot{Y}(t)}{\dot{X}(t)}.$$

First, letting $t := 0$ in $(\text{iii})_D$, we have

$$h(0) = d.$$

Second, from $(\text{i})_D$, we get

$$\frac{h}{h'} = \frac{\dot{X}Y}{\dot{Y}} = X - x.$$

Third, from $(\text{ii})_D$, we obtain

$$\frac{h^2}{h'^2} + h^2 = (X - x)^2 + Y^2 = a^2.$$

Therefore, (ii) is derived.

6.2.2 Angle-parametric expression

The tractrix has an angle-parametric expression as follows.

$$\mathcal{D}_{ang}: \quad \begin{aligned} x(\alpha) &= a(\log\tan\frac{\alpha}{2} + \cos\alpha) \\ y(\alpha) &= a\sin\alpha. \end{aligned}$$

The transformation

$$t = t(\alpha) := \frac{a}{v}\log\tan\frac{\alpha}{2} \quad (v = v_M)$$

or its inverse transformation

$$\alpha = \alpha(t) = 2\tan^{-1} e^{vt/a}$$

reduces the time-parametric system $(X, Y) = \{(X(t), Y(t)) : -\infty < t < \infty\}$ to the angle-parametric system $(x, y) = \{x(\alpha), y(\alpha)) : 0 < t < \pi\}$, and vice versa:

$$x(\alpha) = X(t(\alpha)), \quad y(\alpha) = Y(t(\alpha))$$
$$X(t) = x(\alpha(t)), \quad Y(t) = y(\alpha(t)),$$

where

$$\frac{dt}{d\alpha} = \frac{a}{v\sin\alpha}, \qquad \frac{d\alpha}{dt} = \frac{v\sin\alpha}{a}.$$

Thus Dog's speed

$$v_D(t) = \sqrt{\dot{X}^2(t) + \dot{X}^2(t)}$$

is expressed as follows:

$$\begin{aligned} v_D(t) &= \frac{a}{vt - X(t)}\dot{X}(t) \\ &= v\cos\alpha. \end{aligned}$$

Thus, Dog follows Master with increasing speed up to Master's speed v_M:

$$\lim_{t\to\infty} v_D(t) = \lim_{\alpha\to\pi} v\cos\alpha = v_M.$$

6.2.3 Implicit function expression

The tractrix is expressed by an implicit function as follows.

$$\mathcal{D}_{imp}: \quad x + \sqrt{a^2 - y^2} = a\log\frac{a + \sqrt{a^2 - y^2}}{y}.$$

A simple calculation shows that $\mathcal{D}_{imp}$ is equivalent to $\mathcal{D}_{de}$.

References

[1] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

[2] R.E. Bellman, *Perturbation Techniques in Mathematics, Physics and Engineering*, Holt, Rinehart and Winston, Inc, NY, 1964.

[3] R.E. Bellman, *Some Vistas of Modern Mathematics*, University of Kentuky Press, Lexington, KY, 1968.

[4] R.E. Bellman, *Modern Elementary Differential Equations*, Addison-Wesley, Mass, 1968.

[5] List of Publications: Richard Bellman, IEEE Transactions on Automatic Control, **AC-26**(1981), No.5(Oct.), 1213-1223.

[6] R.E Bellman, *Eye of the Hurricane: an Autobiography*, World Scientific, Singapore, 1984.

[7] R.E. Bellman and E.D. Denman, *Invariant Imbedding*, Lect. Notes in Operation Research and Mathematical Systems, Vol. 52, Springer-Verlag, Berlin, 1971.

[8] R.E. Bellman and E.D. Denman, *Invariant Imbedding : Proceedings of the Summer Workshop on Invariant Imbedding Held at the University of Southern California, June - August 1970*, Lect. Notes in Operation Research and Mathematical Systems, Vol. 52, Springer-Verlag, Berlin, 1971.

[9] R.E. Bellman and R. Kalaba, *Dynamic Programming and Modern Control Theory*, Academic Press, NY, 1965.

[10] R.E. Bellman and G.M. Wing, *An Introduction To Invariant Imbedding*, Wiley, NY, 1975.

[11] J. Casti and R. Kalaba, *Imbedding Methods in Applied Mathematics*, Addison-Wesly, London, 1973.

[12] S. Iwamoto, *Theory of Dynamic Program: Japanese*, Kyushu Univ. Press, Fukuoka, 1987.

[13] E.S. Lee, *Quasilinearization and Invariant Imbedding*, Academic Press, New York, 1968.

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