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Research Article

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Abstract: The non-stationary Navier–Stokes system in the whole space $\mathbb{R}^n$ ($n \geq 2$) is considered. Our first result provides the unique existence theorem of global strong solutions for small initial data and external forces in the scaling invariant homogeneous Besov spaces framework. In particular, our result is still applicable even for non-decaying external forces in time if $n \geq 3$. As non-decaying forces include stationary forces, our second result gives the stability of stationary solutions obtained by Kaneko, Kozono, and Shimizu (Indiana Univ. Math. J. **68** (2019), no. 3, 857–880). Although similar results have already been investigated, our method, which is based on the maximal regularity theorem, refines the previous works as perturbed external forces are allowed.

Keywords: Navier–Stokes system; homogeneous Besov spaces; maximal regularity; stability of stationary solutions

MSC 2020: Primary 35Q30; Secondary 35A01, 35B35, 35B40, 42B35, 76D05

1 Introduction

Consider the following initial value problem for the Navier–Stokes system in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} + \boldsymbol{f}, & t > 0,\ x \in \mathbb{R}^n, \\ \nabla \cdot \boldsymbol{u} = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \boldsymbol{u}(0, x) = \boldsymbol{a}(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $p : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ are the unknown functions standing for the velocity of the fluid and the pressure, respectively. In addition, $\boldsymbol{a} : \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{f} : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are the given functions denoting the initial velocity and the external force, respectively. If a solution $(\boldsymbol{u}, p)$ to (1.1) is smooth, then the condition $\nabla \cdot \boldsymbol{u} = 0$ implies that $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u})$, where $\otimes$ denotes the usual tensor product. We moreover introduce the Helmholtz projection operator P by setting $P := \mathcal{F}^{-1} M_P \mathcal{F}$, where $M_P(\xi) := I - |\xi|^{-2}(\xi \otimes \xi)$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ and where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. As $P\nabla p = 0$ and $P\boldsymbol{u} = \boldsymbol{u}$ due to $\nabla \cdot \boldsymbol{u} = 0$, we note that $P\Delta = \Delta P$ to observe that the system (1.1) is reduced to the following abstract form:

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$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} = P\boldsymbol{f} - P\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), & t > 0,\ x \in \mathbb{R}^n, \\ \nabla \cdot \boldsymbol{u} = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \boldsymbol{u}(0, x) = \boldsymbol{a}(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

The aim of this paper is to construct global solutions to the system (1.2) with a small initial datum $\boldsymbol{a} \in P\Big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n$ and some small external force $\boldsymbol{f}$, where $\dot{B}^s_{r,q}(\mathbb{R}^n)$ denotes the homogeneous Besov spaces defined in Section 2. More precisely, we assume that the external force $\boldsymbol{f}$ satisfies $t^\theta \boldsymbol{f} \in L^\infty\Big((0,\infty); \Big(\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n\Big)$ with $1 < r < n/(1-\theta)$. While we impose the condition $0 < \theta < 1$ in the case of $n = 2$, we deal with $0 \le \theta < 1$ in the case of $n \ge 3$. In other words, we show the existence of global solutions to (1.2) with *non-decaying* forces $\boldsymbol{f} \in L^\infty\Big((0,\infty); \Big(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n\Big)$, provided that $n \ge 3$. As an application, we also prove the stability of stationary solutions obtained by Kaneko, Kozono, and Shimizu [1].

1.1 Relation with the stationary problem

In what follows, we explain the role of this paper. We first recall the scaling invariance property for (1.2); if a function $\boldsymbol{u}$ on $(0,\infty) \times \mathbb{R}^n$ satisfies (1.2) with some $\boldsymbol{a}$ and $\boldsymbol{f}$, then $\boldsymbol{u}_\lambda(t,x) := \lambda \boldsymbol{u}(\lambda^2 t, \lambda x)$ also satisfies (1.2) for all $\lambda > 0$, provided that $\boldsymbol{a}$ and $\boldsymbol{f}$ are replaced with $\boldsymbol{a}_\lambda(x) := \lambda \boldsymbol{a}(\lambda x)$ and $\boldsymbol{f}_\lambda(t,x) := \lambda^3 \boldsymbol{f}(\lambda^2 t, \lambda x)$, respectively. We remark that the following quantities

$$\|\boldsymbol{a}_\lambda\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)}, \quad \sup_{t>0} t^\theta \|\boldsymbol{f}_\lambda(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)}$$

are essentially independent of $\lambda > 0$ due to [2, Remark 2.19]. Hence, our conditions for $\boldsymbol{a}$ and $\boldsymbol{f}$ are based on the scaling invariant spaces. After the pioneering works of Fujita and Kato [3, 4], various results have been obtained from the viewpoint of the construction of solutions to (1.2) in the scaling invariant spaces framework. Before summarizing the previous works, we go back to stating the role of this paper. As we may deal with non-decaying forces $\boldsymbol{f} \in L^\infty\Big((0,\infty); \Big(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n\Big)$ for $n \ge 3$, we may consider *stationary* forces $\boldsymbol{f}_* \in \Big(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n$ in particular. Such forces naturally provide the following stationary problem for the Navier–Stokes system:

$$\begin{cases} -\Delta \boldsymbol{u}_* = P\boldsymbol{f}_* - P\nabla \cdot (\boldsymbol{u}_* \otimes \boldsymbol{u}_*), & x \in \mathbb{R}^n, \\ \nabla \cdot \boldsymbol{u}_* = 0, & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

Although there are abundant results on the construction of solutions to the stationary problem (1.3) such as [5–15], we focus on the works in the homogeneous Besov spaces framework. Kaneko, Kozono, and Shimizu [1] first constructed solutions for small $\boldsymbol{f}_* \in \Big(\dot{B}^{-3+n/r}_{r,q}(\mathbb{R}^n)\Big)^n$ with $n \ge 3$, $1 \le r < n$, and $1 \le q \le \infty$. Here, the restrictions $n \ge 3$ and $r < n$ are essential; Tsurumi [16, 17] showed the discontinuity of the solution map from $\boldsymbol{f}_* \in \Big(\dot{B}^{-3+n/r}_{r,q}(\mathbb{R}^n)\Big)^n$ to $\boldsymbol{u}_* \in \Big(\dot{B}^{-1+n/r}_{r,q}(\mathbb{R}^n)\Big)^n$ for $n \ge 3$, $n < r \le \infty$, and $1 \le q \le \infty$. Fujii [18] achieved dealing with the case of $n = 2$ to prove the discontinuity of the solution map from $\boldsymbol{f}_* \in \Big(\dot{B}^{-3+2/r}_{r,1}(\mathbb{R}^2)\Big)^2$ to $\boldsymbol{u}_* \in \Big(\dot{B}^{-1+2/r}_{r,1}(\mathbb{R}^2)\Big)^2$ for $1 \le r \le 2$ (see also [19]). We refrain from the discussion of the delicate case $r = n \ge 3$ here, but it should be referred to, e.g., [17, 20, 21]. Therefore, the assumption of this paper, i.e., $t^\theta \boldsymbol{f} \in L^\infty\Big((0,\infty); \Big(\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)\Big)^n\Big)$ with $1 < r < n/(1-\theta)$, gives a natural connection to the stationary problem (1.3) because the corresponding case $\theta = 0$ yields the *same* restriction $r < n$. In addition, even though we have to assume that $0 < \theta < 1$ in the case of $n = 2$, this restriction also corresponds to the negative result of [18] as $\theta \ne 0$ implies that the stationary forces are *excluded*. We shall emphasize that although the dimension $n \in \mathbb{N}$ is discrete, the case of $n = 2$ treated in [18] gives a quite subtle problem as the marginal case $\theta = 0$ is just excluded while $\theta > 0$ is still allowed to choose. This seems to be a new insight obtained by considering the non-stationary problem (1.2) with time-weighted forces. We also note that our stability result extends the previous works in terms of the choices of external forces, but we postpone the description after the main results.

1.2 Further previous works

There are numerous previous works on the construction of solutions to (1.1) in the scaling invariant spaces framework. We shall recall some of them, mainly in the case of the whole space $\mathbb{R}^n$; under the assumption $\boldsymbol{f} \equiv 0$, there are well-known series of the results established by, e.g., Fujita and Kato [3, 4], Kato [22], Giga [23], Kozono and Yamazaki [24], Cannone [25, 26], Cannone and Planchon [27], Planchon [28], Amann [29], and Koch and Tataru [30]. It is also worth mentioning the negative consequences obtained by Bourgain and Pavlović [31], Yoneda [32], and Wang [33]. Concerning the case of $\boldsymbol{f} \not\equiv 0$, Fujita and Kato [3, 4] dealt with the homogeneous Sobolev spaces framework, i.e., the case of $\boldsymbol{a} \in P(\dot{H}^{-1+n/2,2}(\mathbb{R}^n))^n$ with $t^{3/4}\boldsymbol{f} \in L^\infty\left((0,\infty); (\dot{H}^{-(3-n)/2,2}(\mathbb{R}^n))^n\right)$ for $n \in \{2,3\}$. A similar situation in a smooth bounded domain was considered by Giga and Miyakawa [34]. Cannone and Planchon [35] handled the case of $n = 3$ and assumed that $\boldsymbol{f} = \nabla \cdot F$ with some matrix-valued function F. Then they considered the case of $\boldsymbol{a} \in P\left(\dot{B}^{-1+3/r}_{r,\infty}(\mathbb{R}^3)\right)^3$ with $t^{s/2}F \in L^\infty((0,\infty); (L^{3/(2-s)}(\mathbb{R}^3))^9)$ for $r > 3$ and $1 - 3/r < s \leq 1$. They also considered the case of $\boldsymbol{a} \in P\left(\dot{B}^{-1+3/r}_{r,2r/(r-3)}(\mathbb{R}^3)\right)^3$ with $F \in L^{2/s}((0,\infty); (L^{3/(2-s)}(\mathbb{R}^3))^9)$ for $3 < r < 9$ and $\max\{1/2, 1-3/r\} < s \leq 1$. Yamazaki [36] treated the weak Lebesgue spaces framework to assume that $\boldsymbol{a} \in P(L^{n,\infty}(\mathbb{R}^n))^n$ and $F \in BC\left((0,\infty); (L^{n/2,\infty}(\mathbb{R}^n))^{n^2}\right)$; remark that he considered not only the whole space $\mathbb{R}^n$ but also the half space $\mathbb{R}^n_+$ and a smooth exterior domain. Cannone and Karch [7] handled the case of $\boldsymbol{a} \in P(\mathcal{PM}^2(\mathbb{R}^3))^3$ and $\boldsymbol{f} \in L^\infty((0,\infty); (\mathcal{PM}^0(\mathbb{R}^3))^3)$, where $\mathcal{PM}^s(\mathbb{R}^3)$ denotes the pseudo-measure spaces. We also refer to their other work [37] dealing with somewhat involved external forces. Dao and Nguyen [38] used the notion of the $(\mathcal{H}_1, 2)$-capacity of a Borel set to handle singular external forces. Kozono and Shimizu [39] considered the case of $\boldsymbol{a} \in P(L^{n,\infty}(\mathbb{R}^n))^n$ and $\boldsymbol{f} \in L^{2/s,\infty}\left((0,\infty); (L^{n/(3-s)}(\mathbb{R}^n))^n\right)$ for $1 < s < 2$. In addition, they [40, 41] also considered the case of $\boldsymbol{a} \in P\left(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n$ and $t^{s/2}\boldsymbol{f} \in L^\infty\left((0,\infty); \left(\dot{B}^{s-3+n/r_0}_{r_0,\infty}(\mathbb{R}^n)\right)^n\right)$ for $r > n$ and suitable indices $3/n < r_0 \leq r$ and $0 < s < 2$ and the case of $\boldsymbol{a} \in P\left(\dot{B}^{-1+n/r}_{r,q}(\mathbb{R}^n)\right)^n$ and $\boldsymbol{f} \in L^{2/s,q}\left((0,\infty); \left(\dot{B}^{s-3+n/r_0}_{r_0,\infty}(\mathbb{R}^n)\right)^n\right)$ for $r > n$, $r_0 \geq r$, $2 - n/r_0 < s < 2$, and $1 \leq q \leq \infty$. We remark that while the former case is similar to that of Amann [29, Theorem 6.1 (i)] treating various domains, the latter case relies on the maximal regularity result shown by Giga and Sohr [42]. Lemarié-Rieusset [43] gave suitable forces so that the result [30] of Koch and Tataru may be extended in terms of considering external forces. The author [44] considered a similar problem to those of [40, 41] as well.

As for the results on the stability of stationary solutions to (1.3), after the establishment of the mathematical foundation by Heywood [12], Secchi [15] obtained a result under the conditions $\boldsymbol{f}_* \in (L^r(\mathbb{R}^n))^n$ and $\boldsymbol{u}_* \in ((L^n \cap L^r)(\mathbb{R}^n))^n$ and perturbations $\boldsymbol{a} \in P(L^r(\mathbb{R}^n))^n$, where $n \geq 3$ and $r > n$. Chen [8] considered the case of $\boldsymbol{f}_* \in \left(\left(\dot{H}^{-1,2n/5} \cap \dot{H}^{-1,2n/3}\right)(\mathbb{R}^n)\right)^n$ and $\boldsymbol{a} \in P(L^r(\mathbb{R}^n))^n$ for $n \geq 3$. Kozono and Yamazaki [13] used the Morrey spaces framework, namely, assumed that $(-\Delta)^{-1}\boldsymbol{f}_* \in (\mathcal{M}^{n,r}(\mathbb{R}^n))^n$ for $n \geq 3$ and $2 < r \leq n$ and considered suitable perturbations $\boldsymbol{a}$. The result of Cannone and Karch [7], which has been mentioned above, also includes the stability assertion. Bjorland and Schonbek [6] considered the case of $\boldsymbol{f}_* \in (H^{-1,2}(\mathbb{R}^3))^3$ and $\boldsymbol{a} \in P(L^2(\mathbb{R}^3))^3$ via the technique based on the energy structure. Bjorland, Brandolese, Iftimie, and Schonbek [5] assumed that $(-\Delta)^{-1}\boldsymbol{f}_* \in (L^{3,\infty}(\mathbb{R}^3))^3$ and $\boldsymbol{a} \in P(L^{3,\infty}(\mathbb{R}^3))^3$; whereas the assumption is restricted from that of [13], this result provides more detailed asymptotic properties in accordance with additional conditions of $\boldsymbol{f}_*$. Phan and Phuc [14] considered the case of $(-\Delta)^{-1}\boldsymbol{f}_* \in (\mathcal{V}^{1,2})^n$ and $\boldsymbol{a} \in P(\mathcal{V}^{1,2})^n$ by introducing the space $\mathcal{V}^{1,2}$ that is close to the Morrey space $\mathcal{M}^{n,2}(\mathbb{R}^n)$, namely, the endpoint case of [13]. The stability result of solutions obtained by Kaneko, Kozono, and Shimizu [1], which is most related to this paper, is given by Cunanan, Okabe, and Tsutsui [45] and Kozono and Shimizu [46]. Finally, we refer to the results of Gallagher, Iftimie, and Planchon [47] and Auscher, Dubois, and Tchamitchian [48] as the stability of global solutions to the non-stationary Navier–Stokes system without external forces; a noteworthy point of this result is that the smallness condition of the original solution is unnecessary even though such a global existence result is unknown.

1.3 Main results and comparison to the related works

We give the main results of this paper. As stated before, while our first result gives the global existence result for (1.2) with external forces that may not decay in time, our second result gives the stability of stationary solutions. We abbreviate

$$\|\partial_t \boldsymbol{u}(t,\cdot), \Delta \boldsymbol{v}(t,\cdot)\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)} := \|\partial_t \boldsymbol{u}(t,\cdot)\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)} + \|\Delta \boldsymbol{v}(t,\cdot)\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)}$$

in what follows.

Theorem 1.1 (Global solutions for the decaying and non-decaying forces)**.** *Let $n \geq 2$, $0 \leq \theta < 1$, and $1 < r < n/(1-\theta)$. In the case of $n = 2$, suppose additionally that $\theta > 0$. There exists a constant $\varepsilon > 0$ such that if the initial datum $\boldsymbol{a} \in P\big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n$ and the external force $\boldsymbol{f} \in L^\infty_{\mathrm{loc}}\Big((0,\infty); \big(\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big)$ satisfy*

$$\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)} \leq \varepsilon, \tag{1.4}$$

then the system (1.2) *has a unique global solution $\boldsymbol{u}$ such that*

$$\boldsymbol{u} \in L^\infty\Big((0,\infty); P\big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big), \quad t^\theta \boldsymbol{u} \in L^\infty\Big((0,\infty); P\big(\dot{B}^{2\theta-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big) \tag{1.5}$$

and

$$\sup_{t>0}\Big(\|\boldsymbol{u}(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + t^\theta \|\boldsymbol{u}(t,\cdot)\|_{\dot{B}^{2\theta-1+n/r}_{r,\infty}(\mathbb{R}^n)}\Big) \leq \sqrt{\varepsilon}.$$

In addition, it holds that

$$t^\theta \partial_t \boldsymbol{u} \in L^\infty\Big((0,\infty); P\big(\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big) \tag{1.6}$$

and

$$\lim_{t\to+0} \langle \boldsymbol{u}(t,\cdot) - \boldsymbol{a}, \boldsymbol{\psi}\rangle = 0 \tag{1.7}$$

for all $\boldsymbol{\psi} \in \big(\dot{B}^{1-n/r}_{r',1}(\mathbb{R}^n)\big)^n$, where $r' := r/(r-1)$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n$ and $\big(\dot{B}^{1-n/r}_{r',1}(\mathbb{R}^n)\big)^n$. The solution $\boldsymbol{u}$ satisfies the estimate

$$\begin{aligned}
&\sup_{t>0}\Big(\|\boldsymbol{u}(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + t^\theta \|\partial_t \boldsymbol{u}(t,\cdot), \Delta \boldsymbol{u}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)}\Big) \\
&\quad \leq C\Big(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)}\Big),
\end{aligned} \tag{1.8}$$

where $C > 0$ is a constant independent of ε, $\boldsymbol{a}$, f, and $\boldsymbol{u}$.

Remark 1.2. (i) Including the case of $\theta = 0$ implies that non-decaying forces may be chosen. As mentioned before, the condition $r < n/(1-\theta)$ gives a natural connection to the result of [1]. The exclusion of $\theta = 0$ in the case of $n = 2$ also seems a natural requirement in view of [18].

(ii) The properties (1.5) and (1.6) imply that

$$t^\theta \partial_t \boldsymbol{u}, t^\theta \Delta \boldsymbol{u} \in L^\infty\Big((0,\infty); P\big(\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big),$$

namely, the global solution $\boldsymbol{u}$ has the maximal regularity property. Therefore, although the statement of Theorem 1.1 might be ambiguous as there is no definition of the "solution" $\boldsymbol{u}$ to (1.2), we indeed see that $\boldsymbol{u}$ is a *strong solution* to (1.2) in the $\dot{B}^{2\theta-3+n/r}_{r,\infty}(\mathbb{R}^n)$-sense. In particular, if $\theta = 0$, then the second condition of $\boldsymbol{u}$ in (1.5) coincides with the first one; the maximal regularity class has to become a natural requirement $\boldsymbol{u} \in L^\infty\Big((0,\infty); P\big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big)$ for the initial datum $\boldsymbol{a} \in P\big(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n$. In other words, to include the case of $\theta = 0$, i.e., stationary forces, we have to rely on the maximal regularity property, which will be given in

Theorem 3.1 below. We should mention that such a maximal regularity result has been essentially shown by, e.g., Lunardi [49], Ogawa and Shimizu [50], and Nogayama and Sawano [51].

(iii) The property (1.7) implies the weak-star continuity since the duality property $\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n) = \left(\dot{B}^{1-n/r}_{r',1}(\mathbb{R}^n)\right)^*$ holds due to [2, Propositions 2.27 and 2.29]; we regard (1.7) as a satisfaction of the initial condition $\boldsymbol{u}(0,\cdot) = \boldsymbol{a}$ of the solution $\boldsymbol{u}$. Note that if $\theta \neq 0$, then the results of [52, Theorem 1.3 (ii)] and Theorem 3.1 below show that $\boldsymbol{u} \in BC\left((0,\infty); P\left(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n\right)$.

Theorem 1.3 (Stability of stationary solutions). *Let $n \geq 3$, $1 < r < n$, and $0 \leq \eta < 1/2$ and let $\boldsymbol{a} \in P\left(\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n$. Suppose that the external forces $\boldsymbol{f}_* \in \left(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n$ and $\boldsymbol{f} \in L^\infty\left((0,\infty); \left(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n\right)$ satisfy*

$$\sup_{t>0} t^\eta \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)} < \infty. \tag{1.9}$$

In addition, let $\boldsymbol{u}$ be a global solution to (1.2) *with the initial datum $\boldsymbol{a}$ and the external force f and let $\boldsymbol{u}_*$ be a solution to* (1.3) *with the external force f_*. There exists a constant $\varepsilon_0 > 0$ independent of $\boldsymbol{a}$, f_*, and f such that if*

$$\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + \|\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)} + \sup_{t>0}\|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)} \leq \varepsilon_0, \tag{1.10}$$

then it holds that

$$t^\eta \partial_t \boldsymbol{u}, t^\eta \Delta(\boldsymbol{u} - \boldsymbol{u}_*) \in L^\infty\left((0,\infty); P\left(\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)\right)^n\right)$$

with the estimates

$$\begin{aligned}
&\sup_{t>0}\|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} \\
&\quad \leq C\left(\|\boldsymbol{a} - \boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + \min_{\lambda\in\{0,\eta\}} \sup_{t>0} t^\lambda \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}(\mathbb{R}^n)}\right), \\
&\sup_{t>0} t^\eta \left\|\partial_t \boldsymbol{u}(t,\cdot), \Delta(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_*)\right\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)} \\
&\quad \leq C\left(\|\boldsymbol{a} - \boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} + \sup_{t>0} t^\eta \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)}\right),
\end{aligned}$$

where $C > 0$ is a constant independent of ε_0, $\boldsymbol{a}$, f_*, f, $\boldsymbol{u}_*$, and $\boldsymbol{u}$. Moreover, it holds that

$$\begin{aligned}
&\limsup_{t\to\infty}\|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} \\
&\quad \leq C\left([\![\boldsymbol{a} - \boldsymbol{u}_*]\!] + \min_{\lambda\in\{0,\eta\}} \limsup_{t\to\infty} t^\lambda \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}(\mathbb{R}^n)}\right), \\
&\limsup_{t\to\infty} t^\eta \left\|\partial_t \boldsymbol{u}(t,\cdot), \Delta(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_*)\right\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)} \\
&\quad \leq C\left([\![\boldsymbol{a} - \boldsymbol{u}_*]\!] + \limsup_{t\to\infty} t^\eta \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)}\right),
\end{aligned}$$

where

$$[\![\boldsymbol{a} - \boldsymbol{u}_*]\!] := \limsup_{j\to-\infty} 2^{(-1+n/r)j} \|\dot{\Delta}_j(\boldsymbol{a} - \boldsymbol{u}_*)\|_{L^r(\mathbb{R}^n)}$$

and $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ denotes the homogeneous dyadic blocks.

Remark 1.4. (i) Since the condition (1.10) is assumed, the results of Kaneno, Kozono, and Shimizu [1, Theorem 1.1] and Theorem 1.1 ensure the existence of unique solutions $\boldsymbol{u}_*$ and $\boldsymbol{u}$. We recall that the conditions $n \geq 3$ and $r < n$ are necessary from the viewpoint of the results [16–18] for the stationary problem (1.3). Remark that while we assume the smallness condition (1.10), the quantity (1.9) does *not* have to be small. We should mention that Fujii and Tsurumi [53] have fairly recently obtained a similar result to ours (see also Fujii's work [19] in the case of the time-periodic external force).

(ii) Theorem 1.3 gives the asymptotic stability in the sense that

$$\|\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_*\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}(\mathbb{R}^n)} = O(t^{-\eta}) \quad \text{as } t\to\infty$$

if $\eta \neq 0$. We note that the result also includes the case of $\eta = 0$. Such a case is regarded as a stability result under the *minimum* necessary assumptions of the forces f_* and f since the condition (1.9) is necessarily fulfilled. Although we do not know whether $\boldsymbol{u}(t,\cdot)$ tends to $\boldsymbol{u}_*$ if $\eta = 0$, we observe that the difference $\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_*$ is estimated by the differences $\boldsymbol{a}-\boldsymbol{u}_*$ and $f(t,\cdot)-f_*$.

(iii) Even if $\eta = 0$, we deduce that $\lim_{t\to\infty}\|\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)} = 0$ under the assumptions

$$[\![\boldsymbol{a}-\boldsymbol{u}_*]\!] = 0, \quad \limsup_{t\to\infty}\|\boldsymbol{f}(t,\cdot)-\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)} = 0.$$

Note that the quantity $[\![\boldsymbol{a}-\boldsymbol{u}_*]\!]$ implies the low-frequency part of the norm $\|\boldsymbol{a}-\boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}(\mathbb{R}^n)}$. For instance, if $\boldsymbol{a}-\boldsymbol{u}_* \in \left(\dot{B}^{-1+n/r}_{r,q}(\mathbb{R}^n)\right)^n$ with $1\le q<\infty$, then it necessarily holds that $[\![\boldsymbol{a}-\boldsymbol{u}_*]\!] = 0$. We remark that a similar consequence has been shown by Cannone and Karch [7, Theorem 5.1; 11, Theorem 2.2] and Eguchi and the author [54, Theorem 1.3 (ii)].

(iv) In the case of $\eta \neq 0$, the minimum of two quantities

$$\|\boldsymbol{f}(t,\cdot)-\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)}, \quad t^\eta\|\boldsymbol{f}(t,\cdot)-\boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}(\mathbb{R}^n)}$$

appears in the estimates since the external forces f_* and f satisfy the additional condition (1.9). While the standard assumption of forces is given by one scaling invariant space, we also assume another condition (1.9) based on the scaling invariant space. Hence, there is a *choice* of two quantities for the estimates; an advantage of choosing either one is given in the Appendix.

Let us give the comparison to closely related works; the readers might suspect that our results coincide with or are included in the known results stated previously because there are some similarities. To clearly distinguish between our results and the known results, we shall recall related works and give an advantage of our results (Table 1). Concerning Theorem 1.1, the assumption of the given data is the same as that of Kozono and Shimizu [40]. However, the essential achievement of our result is to include the case of $\theta = 0$ if $n\ge 3$; non-decaying forces may *not* be chosen in [40], unlike our result. Furthermore, even in the case of $\theta > 0$, our result refines [40] in terms of the establishment of maximal regularity estimates. By virtue of the maximal regularity property, we observe that our solutions to (1.2) are regarded as strong solutions instead of mild solutions, which mean solutions to the integral forms. Next, as for Theorem 1.3, we stated that the stability result has already been shown by Cunanan, Okabe, and Tsutsui [45] and Kozono and Shimizu [46]. An advantage of our result is that we may consider perturbations by *non-stationary forces* along with initial data. The results of [45, 46] dealt only with perturbations by initial data; if we assume that $\boldsymbol{f}(t,\cdot) = \boldsymbol{f}_*$ in (1.2), then the difference between (1.2) and (1.3) would lead to

$$\partial_t(\boldsymbol{u}-\boldsymbol{u}_*) - \Delta(\boldsymbol{u}-\boldsymbol{u}_*) = -P\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + P\nabla\cdot(\boldsymbol{u}_*\otimes\boldsymbol{u}_*), \tag{1.11}$$

Table 1: A comparison of the results.

	Similarity	**Advantage of our result**
Kozono–Shimizu [40]	Assumption of data	Non-decaying forces & maximal regularity
Cunanan–Okabe–Tsutsui [45]	Stability result for [1]	Perturbation by non-stationary forces
Kozono–Shimizu [46]		
Yamazaki [36]	Non-decaying forces	More singular forces
Cannone–Karch [7]		

which means that solving the system (1.11) allows us to avoid considering the problem (1.2) with the external force $\boldsymbol{f}_*$. We remark that the stationary problem (1.3) and the perturbed non-stationary problem (1.11) will be solved *without* using a maximal regularity property. However, if the force is also perturbed, then the term $P(\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*)$ appears in (1.11), and thus our case would be necessary to rely on the result of Theorem 1.1. Another advantage of this paper is that a proof of the stability result is more *straightforward*; whereas they [45, 46] had to analyze the linearized operator associated with (1.11), we give a simple proof by regarding the right-hand side of (1.11) as perturbations like the usual nonlinear terms. Establishing a maximal regularity property contributes to the reduction of the proof as well as the improvement of the preceding results. Although the above comparisons mean that dealing with non-stationary forces is a crucial achievement of this paper, Yamazaki [36] and Cannone and Karch [7] have also treated non-stationary forces. Therefore, finally, we compare them with our result; we observe that our assumption of forces is *weaker* than those of [7, 36]. More precisely, we may show that $\boldsymbol{f} \in L^\infty\Big((0,\infty); \big(\dot{B}^{-3+n/r}_{r,\infty}(\mathbb{R}^n)\big)^n\Big)$, provided that $\boldsymbol{f} = \nabla \cdot F$ with $F \in BC\Big((0,\infty); \big(L^{n/2,\infty}(\mathbb{R}^n)\big)^{n^2}\Big)$ or $\boldsymbol{f} \in L^\infty((0,\infty); (\mathcal{PM}^0(\mathbb{R}^3))^3)$. The detail is given in the Appendix.

1.4 Organization of the paper

This paper is organized as follows: In the next section, we give the notation and function spaces and recall several fundamental facts. Section 3 is devoted to the proof of the maximal regularity estimate that contributes to establishing our main results. A remark on the known results is also given. In Section 4, we show the main results. More precisely, after the preliminary estimate for the nonlinear terms, we give the proof of the global existence result for the non-stationary problem (1.2), that is, Theorem 1.1. Furthermore, we give an auxiliary estimate and complete the proof of the stability of stationary solutions obtained by Kaneko, Kozono, and Shimizu [1], namely, Theorem 1.3.

2 Preliminaries

In this section, we summarize the notation and function spaces used in this paper: For $1 \le r \le \infty$, let $L^r(\mathbb{R}^n)$ denote the usual Lebesgue spaces. In addition, let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space and let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \le |\xi| \le 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$, and $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. We then introduce the homogeneous dyadic blocks $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ by letting $\dot{\Delta}_j := \mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}$ for $j \in \mathbb{Z}$. For $1 \le r \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$, the homogeneous Besov spaces $\dot{B}^s_{r,q}(\mathbb{R}^n)$ are defined by

$$\dot{B}^s_{r,q}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \;\middle|\; \|f\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)} := \left\| \left\{ 2^{sj} \|\dot{\Delta}_j f\|_{L^r(\mathbb{R}^n)} \right\}_{j\in\mathbb{Z}} \right\|_{l^q(\mathbb{Z})} < \infty \right\},$$

where $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{P}(\mathbb{R}^n)$ denote the dual space of $\mathscr{S}(\mathbb{R}^n)$ and the subspace of $\mathscr{S}'(\mathbb{R}^n)$ consisting of polynomials, respectively. We remark that although an element f of $\dot{B}^s_{r,q}(\mathbb{R}^n)$ has to be regarded as that of not $\mathscr{S}'(\mathbb{R}^n)$ but $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ due to the convergence of the low-frequency part, we may modify the definition of $\dot{B}^s_{r,q}(\mathbb{R}^n)$ so that its element may be regarded as that of $\mathscr{S}'(\mathbb{R}^n)$, provided that $(s,q) \in (-\infty, n/r) \times [1,\infty]$ or $(s,q) = (n/r, 1)$. For more details, we refer to, e.g., [2, Remarks 2.24 and 2.26]. In this paper, we regard $\dot{B}^s_{r,q}(\mathbb{R}^n)$ as a subspace of $\mathscr{S}'(\mathbb{R}^n)$ under the above conditions. It is also worth mentioning that $\dot{B}^s_{r,q}(\mathbb{R}^n)$ has the following continuous embedding properties

$$\dot{B}^0_{r,1}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n) \subset \dot{B}^0_{r,\infty}(\mathbb{R}^n), \quad \dot{B}^{s+n/r_0}_{r_0,q_0}(\mathbb{R}^n) \subset \dot{B}^{s+n/r_1}_{r_1,q_1}(\mathbb{R}^n)$$

for all $1 \le r \le \infty$, $1 \le r_0 \le r_1 \le \infty$, $s \in \mathbb{R}$, and $1 \le q_0 \le q_1 \le \infty$ [2, Propositions 2.20 and 2.39]. For $1 \le r \le \infty$ and a Banach space E, let $L^r((0,\infty); E)$ denote the Bochner–Lebesgue spaces on $(0,\infty)$. In addition, let $f \in L^r_{\mathrm{loc}}((0,\infty); E)$ imply that $f \in L^r(I; E)$ for any compact subintervals $I \subset (0,\infty)$. The closed subspace of

$L^\infty((0,\infty);\ E)$ consisting of all bounded continuous E-valued functions on $(0,\infty)$ is denoted by $BC((0,\infty);\ E)$. As mentioned at the beginning of this paper, let P denote the Helmholtz projection operator. Note that P is a bounded operator from $\dot{B}^s_{r,q}(\mathbb{R}^n)$ onto itself for any $r, q \in [1,\infty]$ and $s \in \mathbb{R}$ due to [40, Proposition 2.1]; we frequently use this fact without mentioning in what follows. Define the fractional Laplacian $(-\Delta)^{s/2}$ and the heat semigroup $\{e^{t\Delta}\}_{t>0}$ by letting $(-\Delta)^{s/2} := \mathcal{F}^{-1}|\cdot|^s\mathcal{F}$ for $s \in \mathbb{R}$ and $e^{t\Delta} := \mathcal{F}^{-1}e^{-t|\cdot|^2}\mathcal{F}$ for $t > 0$, respectively. Concerning the heat semigroup defined on $\dot{B}^s_{r,q}(\mathbb{R}^n)$, we will make use of the following estimates shown by Kozono, Ogawa, and Taniuchi [55, Lemma 2.2]:

Proposition 2.1. *Let $1 \le r \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$. Suppose that $\psi \in \dot{B}^s_{r,q}(\mathbb{R}^n)$. Then it holds that $e^{t\Delta}\psi \in \left(\dot{B}^s_{r,q} \cap \dot{B}^{s+\sigma}_{r,1}\right)(\mathbb{R}^n)$ for all $t > 0$ and $\sigma > 0$ with the estimates*

$$\left\|e^{t\Delta}\psi\right\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)} \le C\|\psi\|_{\dot{B}^s_{r,q}(\mathbb{R}^n)}, \quad \left\|e^{t\Delta}\psi\right\|_{\dot{B}^{s+\sigma}_{r,1}(\mathbb{R}^n)} \le Ct^{-\sigma/2}\|\psi\|_{\dot{B}^s_{r,\infty}(\mathbb{R}^n)},$$

where $C > 0$ is a constant independent of t and ψ.

3 Maximal regularity estimate

We abbreviate $L^r := L^r(\mathbb{R}^n)$ and $\dot{B}^s_{r,q} := \dot{B}^s_{r,q}(\mathbb{R}^n)$ hereafter. In addition, we also use the notation $\|f\|_{L^\infty\left(\dot{B}^s_{r,q}\right)} := \sup_{t>0}\|f(t,\cdot)\|_{\dot{B}^s_{r,q}}$ for simplicity. We first prove the following maximal regularity property, which plays a key role in the proof of our main results:

Theorem 3.1. *Let $1 \le r \le \infty$, $s \in \mathbb{R}$, and $0 \le \theta < 1$. Suppose that $f \in L^\infty_{\mathrm{loc}}((0,\infty); \dot{B}^s_{r,\infty})$ satisfies*

$$\sup_{t>0} t^\theta\|f(t,\cdot)\|_{\dot{B}^s_{r,\infty}} < \infty. \tag{3.1}$$

Then it holds that

$$\|\partial_t[\mathcal{I}(t)]f, \Delta[\mathcal{I}(t)]f\|_{\dot{B}^s_{r,\infty}} \le Ct^{-\theta}\left(\delta^{1-\theta}\sup_{\tau>0}\tau^\theta\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}} + \sup_{\tau>\delta t}\tau^\theta\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\right) \tag{3.2}$$

for all $t > 0$ and $0 < \delta \le 1/2$, where

$$[\mathcal{I}(t)]f := \int_0^t e^{(t-\tau)\Delta}f(\tau,\cdot)\mathrm{d}\tau, \quad t > 0$$

and $C > 0$ is a constant independent of t, δ, and f. Moreover, if $\theta \ne 0$, then it holds that

$$[\mathcal{I}(\cdot)]f \in BC\left((0,\infty); \dot{B}^{s+2-2\theta}_{r,1}\right)$$

with the estimate

$$\|[\mathcal{I}(\cdot)]f\|_{L^\infty\left(\dot{B}^{s+2-2\theta}_{r,1}\right)} \le C\sup_{t>0} t^\theta\|f(t,\cdot)\|_{\dot{B}^s_{r,\infty}}.$$

Remark 3.2. Although Theorem 3.1 is quite crucial to prove our main results, we have to remark that such results have been essentially known: To the best of the author's knowledge, Lunardi [49, Theorems 4.3.8 and 4.3.12] has given the results in the abstract framework. However, the homogeneous Besov spaces do not satisfy the assumption of the abstract setting, in which a general Banach space X has to include the domain $D(A)$ of the operator $A : D(A) \to X$. Hence, we should verify that the corresponding result is still valid even though the method is almost the same. Concerning similar results in the homogeneous Besov spaces, Ogawa and Shimizu [50, Theorem 1.1] have shown our assertion in the case of $\theta = 0$ and $r = 1$; their result is also applicable for the L^ρ-framework in time, where $1 < \rho \le \infty$. Furthermore, Nogayama and Sawano [51, Theorem 1.1] have refined the result of [50, Theorem 1.1] to extend the assumption to $r, \rho \in [1, \infty]$. As the case of $\theta = 0$ allows us to choose non-decaying forces, the result of [51] essentially provides our desired result. A slight difference between Theorem 3.1 and [49–51] is that a *refined* estimate (3.2) is obtained through a new parameter δ; while letting $\delta = 1/2$ implies the standard consequence, taking $\delta \to +0$ after letting $t \to \infty$ yields

$$\limsup_{t\to\infty} t^\theta \|\partial_t[\mathcal{I}(t)]f, \Delta[\mathcal{I}(t)]f\|_{\dot{B}^s_{r,\infty}} \le C \limsup_{t\to\infty} t^\theta \|f(t,\cdot)\|_{\dot{B}^s_{r,\infty}}.$$

Such an estimate provides asymptotic results given in Theorem 1.3. We also mention that a similar result has been obtained by Cannone and Karch [7, Lemma 5.1] and Eguchi and the author [54, Lemma 3.1].

The proof of Theorem 3.1 is given by using the following respective estimates:

Lemma 3.3. *Let* $1 \le r \le \infty$, $s \in \mathbb{R}$, *and* $0 \le \theta < 1$. *Suppose that* $f \in L^\infty_{\mathrm{loc}}((0,\infty); \dot{B}^s_{r,\infty})$ *satisfies* (3.1). *Then it holds that*

$$\int_0^{\delta t} \left\| e^{(t-\tau)\Delta} f(\tau,\cdot)\right\|_{\dot{B}^{s+2}_{r,1}} \mathrm{d}\tau \le \frac{C\delta^{1-\theta}}{1-\delta} t^{-\theta} \sup_{0<\tau<t} \tau^\theta \|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}},$$

$$\left\| \int_{\delta t}^{t} e^{(t-\tau)\Delta} f(\tau,\cdot)\mathrm{d}\tau \right\|_{\dot{B}^{s+2}_{r,\infty}} \le C\delta^{-\theta} t^{-\theta} \sup_{\delta t<\tau<t} \tau^\theta \|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}$$

for all $t > 0$ *and* $0 < \delta < 1$, *where* $C > 0$ *is a constant independent of* t, δ, *and* f.

Proof. Since $0 \le \theta < 1$, we use Proposition 2.1 to observe that

$$\begin{aligned}\int_0^{\delta t} \left\| e^{(t-\tau)\Delta} f(\tau,\cdot)\right\|_{\dot{B}^{s+2}_{r,1}} \mathrm{d}\tau &\le C \int_0^{\delta t} (t-\tau)^{-1} \|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}} \mathrm{d}\tau \\ &\le \frac{C}{(1-\delta)t} \sup_{0<\tau<t} \tau^\theta \|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}} \int_0^{\delta t} \tau^{-\theta}\mathrm{d}\tau \\ &\le \frac{C\delta^{1-\theta}}{1-\delta} t^{-\theta} \sup_{0<\tau<t} \tau^\theta \|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\end{aligned}$$

for all $t > 0$ and $0 < \delta < 1$. In addition, we rely on the characterization of the homogeneous Besov spaces [2, Theorem 2.34] to deduce that

$$\|g\|_{\dot{B}^{s+2}_{r,\infty}} \le C \left\| (-\Delta)^{s/2+2} g \right\|_{\dot{B}^{-2}_{r,\infty}} \le C \sup_{\lambda>0} \lambda \left\| e^{\lambda\Delta} (-\Delta)^2 (-\Delta)^{s/2} g \right\|_{L^r},$$

and thus we see by Proposition 2.1 that

$$\begin{aligned}\left\| \int_{\delta t}^{t} e^{(t-\tau)\Delta} f(\tau,\cdot)\mathrm{d}\tau \right\|_{\dot{B}^{s+2}_{r,\infty}} &\le C \sup_{\lambda>0} \lambda \int_{\delta t}^{t} \left\| (-\Delta)^2 e^{(\lambda+t-\tau)\Delta} (-\Delta)^{s/2} f(\tau,\cdot)\right\|_{L^r} \mathrm{d}\tau \\ &\le C \sup_{\lambda>0} \lambda \int_{\delta t}^{t} (\lambda+t-\tau)^{-2} \left\| (-\Delta)^{s/2} f(\tau,\cdot)\right\|_{\dot{B}^0_{r,\infty}} \mathrm{d}\tau\end{aligned}$$

$$\leq C(\delta t)^{-\theta} \sup_{\delta t<\tau<t} \tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}} \sup_{\lambda>0}\lambda \int_{\delta t}^{t} (\lambda+t-\tau)^{-2}\mathrm{d}\tau$$

$$\leq C\delta^{-\theta}t^{-\theta} \sup_{\delta t<\tau<t} \tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}},$$

which completes the proof of Lemma 3.3. □

Proof of Theorem 3.1. As $0<\delta\leq 1/2$, the estimates in Lemma 3.3 imply that

$$\left\|\int_0^{\delta t} e^{(t-\tau)\Delta} f(\tau,\cdot)\mathrm{d}\tau\right\|_{\dot{B}^{s+2}_{r,1}} \leq C\delta^{1-\theta}t^{-\theta}\sup_{\tau>0}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}},$$

$$\left\|\int_{t/2}^{t} e^{(t-\tau)\Delta} f(\tau,\cdot)\mathrm{d}\tau\right\|_{\dot{B}^{s+2}_{r,\infty}} \leq Ct^{-\theta}\sup_{\tau>t/2}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}$$

for all $t>0$. In addition, since it holds by Proposition 2.1 that

$$\int_{\delta t}^{t/2} \left\|e^{(t-\tau)\Delta} f(\tau,\cdot)\right\|_{\dot{B}^{s+2}_{r,1}}\mathrm{d}\tau \leq C\int_{\delta t}^{t/2} (t-\tau)^{-1}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\mathrm{d}\tau$$

$$\leq C(t/2)^{-1}\sup_{\tau>\delta t}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\int_0^{t/2}\tau^{-\theta}\mathrm{d}\tau$$

$$\leq Ct^{-\theta}\sup_{\tau>\delta t}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}$$

for all $t>0$, we simply combine these estimates to observe that

$$\|[\mathcal{I}(t)]f\|_{\dot{B}^{s+2}_{r,\infty}} \leq Ct^{-\theta}\left(\delta^{1-\theta}\sup_{\tau>0}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}} + \sup_{\tau>\delta t}\tau^{\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\right),$$

which yields (3.2) due to the equality $\Delta[\mathcal{I}(\cdot)]f = \partial_t[\mathcal{I}(\cdot)]f - f$. The remaining assertion has been shown in [54, Proposition 2.5 (i)]. This completes the proof of Theorem 3.1. □

By following the proof of [56, Lemma 4.13] with the aid of the maximal regularity estimate in Theorem 3.1, we may show the weak-star continuity property even for the marginal case. Its proof will be given in the Appendix.

Lemma 3.4. *Let $1<r\leq\infty$, $s\in\mathbb{R}$, and $0\leq\theta<1$. Suppose that $f\in L^{\infty}_{\mathrm{loc}}((0,\infty);\dot{B}^s_{r,\infty})$ satisfies (3.1). Then it holds that*

$$\lim_{t\to+0}\langle[\mathcal{I}(t)]f,\psi\rangle = 0$$

for all $\psi\in\dot{B}^{-s-2+2\theta}_{r',1}$, where $r' := r/(r-1)$.

4 Proof of the main results

4.1 Construction of solutions: Proof of Theorem 1.1

Once we obtain the maximal regularity estimate by Theorem 3.1, we may obtain the desired solution from the standard fixed point argument. We emphasize that the assumptions for the nonlinear estimates allow us to reveal the relation of the solvability between the non-stationary and stationary problems. We recall the bilinear estimates in homogeneous Besov spaces established by Kaneko, Kozono, and Shimizu [1, Proposition 2.2 (i)].

Proposition 4.1. *Let $1 \le q \le \infty$ and $s, \lambda, \mu \in (0,\infty)$. Suppose that $q_0, q_1, r_0, r_1 \in [1,\infty]$ satisfy $1/q = 1/q_0 + 1/q_1 = 1/r_0 + 1/r_1$. Then for every $\psi_0 \in \dot{B}^{s+\lambda}_{q_0,\infty} \cap \dot{B}^{-\mu}_{r_0,\infty}$ and $\psi_1 \in \dot{B}^{s+\mu}_{r_1,\infty} \cap \dot{B}^{-\lambda}_{q_1,\infty}$, it holds that*

$$\|\psi_0\psi_1\|_{\dot{B}^s_{q,\infty}} \le C\Big(\|\psi_0\|_{\dot{B}^{s+\lambda}_{q_0,\infty}}\|\psi_1\|_{\dot{B}^{-\lambda}_{q_1,\infty}} + \|\psi_0\|_{\dot{B}^{-\mu}_{r_0,\infty}}\|\psi_1\|_{\dot{B}^{s+\mu}_{r_1,\infty}}\Big),$$

where $C > 0$ is a constant independent of ψ_0 and ψ_1.

Lemma 4.2. *Let $n \ge 2$, $0 \le \theta < 1$, and $1 < r < n/(1-\theta)$. In the case of $n = 2$, suppose additionally that $\theta > 0$. Then for every $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1 \in \left(\dot{B}^{-1+n/r}_{r,\infty} \cap \dot{B}^{2\theta-1+n/r}_{r,\infty}\right)^n$, it holds that*

$$\|\nabla\cdot(\boldsymbol{\psi}_0 \otimes \boldsymbol{\psi}_1)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}} \le C\Big(\|\boldsymbol{\psi}_0\|_{\dot{B}^{2\theta-1+n/r}_{r,\infty}}\|\boldsymbol{\psi}_1\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \|\boldsymbol{\psi}_0\|_{\dot{B}^{-1+n/r}_{r,\infty}}\|\boldsymbol{\psi}_1\|_{\dot{B}^{2\theta-1+n/r}_{r,\infty}}\Big),$$

where $C > 0$ is a constant independent of $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$.

Proof. We first verify that there exist $q_0, q_1 \in (1,\infty)$ such that

$$\max\left\{0, \frac{2-2\theta}{n} - \frac{1}{r}\right\} < \frac{1}{q_0} < \min\left\{\frac{1}{n}, \frac{1}{r}\right\}, \tag{4.1}$$

$$\max\left\{\frac{1}{r} - \frac{1}{q_0}, \frac{2-2\theta}{n} - \frac{1}{q_0}\right\} < \frac{1}{q_1} < \min\left\{\frac{1}{r}, 1 - \frac{1}{q_0}\right\}. \tag{4.2}$$

Indeed, as we have

$$\begin{aligned}
\frac{1}{n} - \left(\frac{2-2\theta}{n} - \frac{1}{r}\right) &= \frac{1}{r} + \frac{2\theta-1}{n} > \frac{1-\theta}{n} + \frac{2\theta-1}{n} = \frac{\theta}{n} \ge 0,\\
\frac{1}{r} - \left(\frac{2-2\theta}{n} - \frac{1}{r}\right) &= \frac{2}{r} - \frac{2-2\theta}{n} > \frac{2-2\theta}{n} - \frac{2-2\theta}{n} = 0
\end{aligned}$$

due to the present assumptions, we may take $1 < q_0 < \infty$ satisfying (4.1). We moreover rely on (4.1) to observe that

$$\begin{aligned}
&\frac{1}{r} - \left(\frac{1}{r} - \frac{1}{q_0}\right) = \frac{1}{q_0} > 0, \quad \frac{1}{r} - \left(\frac{2-2\theta}{n} - \frac{1}{q_0}\right) > 0,\\
&1 - \frac{1}{q_0} - \left(\frac{1}{r} - \frac{1}{q_0}\right) = 1 - \frac{1}{r} > 0.
\end{aligned}$$

We remark that the following quantity

$$1 - \frac{1}{q_0} - \left(\frac{2-2\theta}{n} - \frac{1}{q_0}\right) = \frac{1}{n}(n-2+2\theta)$$

becomes positive for any $\theta \ge 0$ provided that $n \ge 3$. In the case of $n = 2$, the condition $\theta > 0$ is necessary. Therefore, under the present assumptions, we may take $1 < q_1 < \infty$ satisfying (4.2).

In what follows, we show the desired nonlinear estimate. Let q satisfy $1/q = 1/q_0 + 1/q_1$ and let $\lambda := 1 - n/q_0$. Then the conditions (4.1) and (4.2) imply that $1 < q < r$ and $\lambda > 0$. In addition, as we have $q_0, q_1 \in (r,\infty)$ due to (4.1) and (4.2), the Sobolev embeddings lead to

$$\dot{B}^{2\theta-2+n/q}_{q,\infty} \subset \dot{B}^{2\theta-2+n/r}_{r,\infty}, \quad \dot{B}^{-1+n/r}_{r,\infty} \subset \dot{B}^{-\lambda}_{q_0,\infty}, \quad \dot{B}^{2\theta-1+n/r}_{r,\infty} \subset \dot{B}^{2\theta-1+n/q_1}_{q_1,\infty} = \dot{B}^{2\theta-2+n/q+\lambda}_{q_1,\infty}.$$

Hence, by noting that $2\theta - 2 + n/q > 0$ from (4.2) and applying Proposition 4.1, we deduce that

$$\begin{aligned}\|\nabla\cdot(\boldsymbol{\psi}_0\otimes\boldsymbol{\psi}_1)\|_{\dot B^{2\theta-3+n/r}_{r,\infty}} &\le C\|\boldsymbol{\psi}_0\otimes\boldsymbol{\psi}_1\|_{\dot B^{2\theta-2+n/q}_{q,\infty}}\\ &\le C\Big(\|\boldsymbol{\psi}_0\|_{\dot B^{2\theta-2+n/q+\lambda}_{q_1,\infty}}\|\boldsymbol{\psi}_1\|_{\dot B^{-\lambda}_{q_0,\infty}}+\|\boldsymbol{\psi}_0\|_{\dot B^{-\lambda}_{q_0,\infty}}\|\boldsymbol{\psi}_1\|_{\dot B^{2\theta-2+n/q+\lambda}_{q_1,\infty}}\Big)\\ &\le C\Big(\|\boldsymbol{\psi}_0\|_{\dot B^{2\theta-1+n/r}_{r,\infty}}\|\boldsymbol{\psi}_1\|_{\dot B^{-1+n/r}_{r,\infty}}+\|\boldsymbol{\psi}_0\|_{\dot B^{-1+n/r}_{r,\infty}}\|\boldsymbol{\psi}_1\|_{\dot B^{2\theta-1+n/r}_{r,\infty}}\Big)\end{aligned}$$

for all $t > 0$, which completes the proof of Lemma 4.2. □

The remaining step is just combining Theorem 3.1 with Lemma 4.2. Before the proof, we introduce the function space as follows;

$$X_\theta := \Big\{\boldsymbol{u}\in L^\infty\Big((0,\infty);P\Big(\dot B^{-1+n/r}_{r,\infty}\Big)^n\Big)\ \Big|\ \|\boldsymbol{u}\|_{X_\theta}<\infty\Big\}, \tag{4.3}$$
$$\|\boldsymbol{u}\|_{X_\theta} := \sup_{t>0}\Big(\|\boldsymbol{u}(t,\cdot)\|_{\dot B^{-1+n/r}_{r,\infty}} + t^\theta\|\boldsymbol{u}(t,\cdot)\|_{\dot B^{2\theta-1+n/r}_{r,\infty}}\Big),$$

where $P\Big(\dot B^s_{r,q}\Big)^n := \Big\{P\boldsymbol{\psi}\mid \boldsymbol{\psi}\in\Big(\dot B^s_{r,q}\Big)^n\Big\}$.

Proof of Theorem 1.1. Define the mapping $\mathbf{\Phi}: X_\theta\to X_\theta$ by setting

$$(\mathbf{\Phi u})(t) := e^{t\Delta}\boldsymbol{a} + \int_0^t e^{(t-\tau)\Delta}P\boldsymbol{f}(\tau,\cdot)\mathrm{d}\tau - \boldsymbol{N}[\boldsymbol{u},\boldsymbol{u}](t) \tag{4.4}$$

for $t > 0$ and $\boldsymbol{u}\in X_\theta$, where

$$\boldsymbol{N}[\boldsymbol{u},\boldsymbol{v}](t) := \int_0^t e^{(t-\tau)\Delta}P\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{v})(\tau,\cdot)\mathrm{d}\tau. \tag{4.5}$$

As Proposition 2.1 yields

$$\|e^{t\Delta}\boldsymbol{a}\|_{\dot B^{-1+n/r}_{r,\infty}} \le C\|\boldsymbol{a}\|_{\dot B^{-1+n/r}_{r,\infty}},\quad \|e^{t\Delta}\boldsymbol{a}\|_{\dot B^{2\theta-1+n/r}_{r,\infty}} \le Ct^{-\theta}\|\boldsymbol{a}\|_{\dot B^{-1+n/r}_{r,\infty}}$$

for all $t > 0$, we combine Theorem 3.1 and Lemma 4.2 to obtain

$$\begin{aligned}\|\mathbf{\Phi u}\|_{X_\theta} &= \sup_{t>0}\Big(\|(\mathbf{\Phi u})(t,\cdot)\|_{\dot B^{-1+n/r}_{r,\infty}} + t^\theta\|(\mathbf{\Phi u})(t,\cdot)\|_{\dot B^{2\theta-1+n/r}_{r,\infty}}\Big)\\ &\le C\|\boldsymbol{a}\|_{\dot B^{-1+n/r}_{r,\infty}} + C\sup_{t>0}\Big(t^\theta\|\boldsymbol{f}(t,\cdot)\|_{\dot B^{2\theta-3+n/r}_{r,\infty}} + t^\theta\|\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})(t,\cdot)\|_{\dot B^{2\theta-3+n/r}_{r,\infty}}\Big)\\ &\le C\Big(\|\boldsymbol{a}\|_{\dot B^{-1+n/r}_{r,\infty}} + \sup_{t>0}t^\theta\|\boldsymbol{f}(t,\cdot)\|_{\dot B^{2\theta-3+n/r}_{r,\infty}} + \|\boldsymbol{u}\|^2_{X_\theta}\Big).\end{aligned} \tag{4.6}$$

Similarly, we also observe that

$$\|\mathbf{\Phi u}-\mathbf{\Phi v}\|_{X_\theta} \le C(\|\boldsymbol{u}\|_{X_\theta}+\|\boldsymbol{v}\|_{X_\theta})\|\boldsymbol{u}-\boldsymbol{v}\|_{X_\theta}$$

for $\boldsymbol{u},\boldsymbol{v}\in X_\theta$. Let $\varepsilon>0$ satisfy $\sqrt{\varepsilon}\le 1/(4C)$ and assume that

$$\max\{\|\boldsymbol{u}\|_{X_\theta},\|\boldsymbol{v}\|_{X_\theta}\}\le\sqrt{\varepsilon}.$$

Then the condition (1.4) ensures that

$$\|\mathbf{\Phi u}\|_{X_\theta} \le C(\varepsilon+\varepsilon) \le C\cdot\frac{1}{2C}\cdot\sqrt{\varepsilon}\le\sqrt{\varepsilon},$$
$$\|\mathbf{\Phi u}-\mathbf{\Phi v}\|_{X_\theta} \le C\Big(\sqrt{\varepsilon}+\sqrt{\varepsilon}\Big)\|\boldsymbol{u}-\boldsymbol{v}\|_{X_\theta}\le\frac12\|\boldsymbol{u}-\boldsymbol{v}\|_{X_\theta},$$

which imply that we may apply the Banach fixed point theorem to obtain $\boldsymbol{u} \in X_\theta$ satisfying $\boldsymbol{\Phi u} = \boldsymbol{u}$. The weak continuity property (1.7) follows from the standard property of the heat semigroup [56, Proposition 3.10 (ii)] and Lemma 3.4. Furthermore, we see by (4.6) that

$$\begin{aligned}\|\boldsymbol{u}\|_{X_\theta} &\le C\left(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}} + \sqrt{\varepsilon}\|\boldsymbol{u}\|_{X_\theta}\right) \\ &\le C\left(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}}\right) + \frac{1}{4}\|\boldsymbol{u}\|_{X_\theta}.\end{aligned}$$

Consequently, applying Proposition 2.1 and Theorem 3.1 provide that

$$\begin{aligned}&\sup_{t>0} t^\theta \|\partial_t \boldsymbol{u}(t,\cdot), \Delta\boldsymbol{u}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}} \\ &\quad\le C\left(\sup_{t>0} t^\theta \|e^{t\Delta}\boldsymbol{a}\|_{\dot{B}^{2\theta-1+n/r}_{r,\infty}} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}} + \|\boldsymbol{u}\|_{X_\theta}^2\right) \\ &\quad\le C\left(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \sup_{t>0} t^\theta \|\boldsymbol{f}(t,\cdot)\|_{\dot{B}^{2\theta-3+n/r}_{r,\infty}}\right) + \frac{1}{4}\|\boldsymbol{u}\|_{X_\theta},\end{aligned}$$

which yields (1.8). □

4.2 Stability of stationary solutions: Proof of Theorem 1.3

Finally, we show Theorem 1.3. Although Theorem 1.3 imposes the additional assumption (1.9), we may *not* expect the additional regularity of the solution $\boldsymbol{u}_*$ to the stationary problem (1.3). Therefore, another nonlinear estimate is necessary instead of Lemma 4.2 to derive the additional properties.

Lemma 4.3. *Let $n \ge 3$, $0 \le \eta < 1/2$, and $1 < r < n$. Then for every $\boldsymbol{\psi}_0 \in \left(\dot{B}^{2\eta-1+n/r}_{r,\infty}\right)^n$ and $\boldsymbol{\psi}_1 \in \left(\dot{B}^{-1+n/r}_{r,\infty}\right)^n$, it holds that*

$$\|\nabla\cdot(\boldsymbol{\psi}_0 \otimes \boldsymbol{\psi}_1)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} + \|\nabla\cdot(\boldsymbol{\psi}_1 \otimes \boldsymbol{\psi}_0)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} \le C\|\boldsymbol{\psi}_0\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}} \|\boldsymbol{\psi}_1\|_{\dot{B}^{-1+n/r}_{r,\infty}},$$

where $C > 0$ is a constant independent of $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$.

Proof. Recall that we may take $q_0, q_1 \in (1,\infty)$ such that (4.1) and (4.2) with θ replaced by η. In addition, since $0 \le \eta < 1/2$ and since the condition $1 < r < n$ yields

$$\frac{1}{r} - \frac{1}{q_0} + \frac{1-2\eta}{n} - \left(\frac{2-2\eta}{n} - \frac{1}{q_0}\right) = \frac{1}{r} - \frac{1}{n} > 0,$$

there is $1 < q_1 < \infty$ satisfying

$$\max\left\{\frac{1}{r} - \frac{1}{q_0}, \frac{2-2\eta}{n} - \frac{1}{q_0}\right\} < \frac{1}{q_1} < \min\left\{\frac{1}{r} - \frac{1}{q_0} + \frac{1-2\eta}{n}, 1 - \frac{1}{q_0}\right\}.$$

Taking q such that $1/q = 1/q_0 + 1/q_1$ in a similar manner to the proof of Lemma 4.2, we observe that

$$\frac{1-2\eta}{n} - \left(\frac{1}{q} - \frac{1}{r}\right) = \frac{1}{r} - \frac{1}{q_0} + \frac{1-2\eta}{n} - \frac{1}{q_1} > 0,$$

which ensures the existence of $1 < r_0 < \infty$ satisfying

$$\frac{1}{q} - \frac{1}{r} < \frac{1}{r_0} < \frac{1-2\eta}{n}$$

due to $r > q$. Hence, choosing r_1 so that $1/r_1 = 1/q - 1/r_0$, we have $r_0, r_1 \in (r, \infty)$ from $r < n$. Let $\lambda := 1 - n/q_0$ and $\mu := 1 - 2\eta - n/r_0$. We then deduce that $\lambda,\ \mu \in (0, \infty)$ due to the above condition. Therefore, combining the following Sobolev embeddings

$$\dot{B}^{2\eta-1+n/r}_{r,\infty} \subset \dot{B}^{-\mu}_{r_0,\infty}, \quad \dot{B}^{-1+n/r}_{r,\infty} \subset \dot{B}^{-1+n/r_1}_{r_1,\infty} = \dot{B}^{2\eta-2+n/q+\mu}_{r_1,\infty}$$

with the same discussion as in the proof of Lemma 4.2, we obtain

$$\begin{aligned}\|\nabla \cdot (\boldsymbol{\psi}_0 \otimes \boldsymbol{\psi}_1)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} &\leq C\|\boldsymbol{\psi}_0 \otimes \boldsymbol{\psi}_1\|_{\dot{B}^{2\eta-2+n/q}_{q,\infty}} \\ &\leq C\Big(\|\boldsymbol{\psi}_0\|_{\dot{B}^{2\eta-2+n/q+\lambda}_{q_1,\infty}} \|\boldsymbol{\psi}_1\|_{\dot{B}^{-\lambda}_{q_0,\infty}} + \|\boldsymbol{\psi}_0\|_{\dot{B}^{-\mu}_{r_0,\infty}} \|\boldsymbol{\psi}_1\|_{\dot{B}^{2\eta-2+n/q+\mu}_{r_1,\infty}} \Big) \\ &\leq C\Big(\|\boldsymbol{\psi}_0\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}} \|\boldsymbol{\psi}_1\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \|\boldsymbol{\psi}_0\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}} \|\boldsymbol{\psi}_1\|_{\dot{B}^{-1+n/r}_{r,\infty}} \Big)\end{aligned}$$

for all $t > 0$, which completes the proof of Lemma 4.3. □

As we will have to discuss the cases of $\eta = 0$ and $\eta \neq 0$ separately, we first give the corresponding result as a simple proposition.

Proposition 4.4. *Let $1 \leq r \leq \infty$ and $0 \leq \eta < 1$. Suppose that $\boldsymbol{g} \in L^\infty((0,\infty); \big(\dot{B}^{-3+n/r}_{r,\infty}\big)^n)$ satisfies*

$$\sup_{t>0} t^\eta \|\boldsymbol{g}(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} < \infty.$$

Then it holds that

$$\begin{aligned}\|[\mathcal{I}(\cdot)]\boldsymbol{g}\|_{L^\infty\left(\dot{B}^{-1+n/r}_{r,\infty}\right)} &\leq C \min_{\lambda\in\{0,\eta\}} \|\boldsymbol{g}\|_{Y_\lambda(0)}, \\ \|[\mathcal{I}(t)]\boldsymbol{g}\|_{\dot{B}^{-1+n/r}_{r,\infty}} &\leq C\Big(\delta\|\boldsymbol{g}\|_{Y_0(0)} + \min_{\lambda\in\{0,\eta\}} \|\boldsymbol{g}\|_{Y_\lambda(\delta t)}\Big)\end{aligned}$$

for all $t > 0$ and $0 < \delta \leq 1/2$, where

$$\|\boldsymbol{g}\|_{Y_\lambda(t_*)} := \sup_{t>t_*} t^\lambda \|\boldsymbol{g}(t,\cdot)\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}} \tag{4.7}$$

for $t_* \geq 0$ and $\lambda \in \{0, \eta\}$ and where $C > 0$ is a constant independent of δ, t, and $\boldsymbol{g}$.

Proof. Since Theorem 3.1 yields

$$\|[\mathcal{I}(t)]\boldsymbol{g}\|_{\dot{B}^{-1+n/r}_{r,\infty}} \leq C(\delta\|\boldsymbol{g}\|_{Y_0(0)} + \|\boldsymbol{g}\|_{Y_0(\delta t)})$$

for all $t > 0$ and $0 < \delta \leq 1/2$, the desired estimates have been obtained in the case of $\eta = 0$. We then assume that $\eta \neq 0$. Applying Proposition 2.1 and the first estimate of Lemma 3.3 lead to

$$\begin{aligned}\|[\mathcal{I}(t)]\boldsymbol{g}\|_{\dot{B}^{-1+n/r}_{r,\infty}} &\leq C\int_0^t (t-\tau)^{-1+\eta} \|\boldsymbol{g}(\tau,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} \mathrm{d}\tau \\ &\leq C\|\boldsymbol{g}\|_{Y_\eta(0)} \int_0^t (t-\tau)^{-1+\eta}\tau^{-\eta}\mathrm{d}\tau \leq C\|\boldsymbol{g}\|_{Y_\eta(0)}\end{aligned}$$

and

$$\|[\mathcal{I}(t)]\boldsymbol{g}\|_{\dot{B}^{-1+n/r}_{r,\infty}} \le \int_0^{\delta t} \left\|e^{(t-\tau)\Delta}\boldsymbol{g}(\tau,\cdot)\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} d\tau + \int_{\delta t}^{t} \left\|e^{(t-\tau)\Delta}\boldsymbol{g}(\tau,\cdot)\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} d\tau$$

$$\le \frac{C\delta}{1-\delta}\|\boldsymbol{g}\|_{Y_0(0)} + C\int_{\delta t}^{t} (t-\tau)^{-1+\eta}\|\boldsymbol{g}(\tau,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} d\tau$$

$$\le C(\delta\|\boldsymbol{g}\|_{Y_0(0)} + \|\boldsymbol{g}\|_{Y_\eta(\delta t)})$$

for all $t > 0$, which yield the desired estimates. □

Proof of Theorem 1.3. We first note that the smallness condition (1.10) ensures the existence of solutions $\boldsymbol{u}_* \in P\left(\dot{B}^{-1+n/r}_{r,\infty}\right)^n$ to (1.3) and $\boldsymbol{u} \in X_0$ to (1.2) with the estimates

$$\|\boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}} \le C\|\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}}, \quad \|\boldsymbol{u}\|_{X_0} \le C(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \|\boldsymbol{f}\|_{Y_0(0)}) \tag{4.8}$$

due to the results of Kaneko, Kozono, and Shimizu [1, Theorem 1.1] and Theorem 1.1, where X_0 and $\|\cdot\|_{Y_0(0)}$ are defined by (4.3) and (4.7), respectively. Let $\overline{\boldsymbol{a}} := \boldsymbol{a} - \boldsymbol{u}_*$ and $\overline{\boldsymbol{f}} := \boldsymbol{f} - \boldsymbol{f}_*$. Define the mapping $\boldsymbol{\Psi} : X_\eta \to X_\eta$ by setting

$$(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t) := e^{t\Delta}\overline{\boldsymbol{a}} + \int_0^t e^{(t-\tau)\Delta}P\overline{\boldsymbol{f}}(\tau,\cdot)d\tau - \boldsymbol{N}[\boldsymbol{u},\overline{\boldsymbol{u}}](t) - \boldsymbol{N}[\overline{\boldsymbol{u}},\boldsymbol{u}_*](t)$$

for $t > 0$ and $\overline{\boldsymbol{u}} \in X_\eta$, where $\boldsymbol{N}$ is defined by (4.5). Then, since we have assumed that (1.10), we combine Lemma 4.3 and the estimates (4.8) to deduce that

$$\begin{aligned}\left\|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\right\|_{Y_\lambda(0)} + \left\|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\right\|_{Y_\lambda(0)} &\le C\left(\|\boldsymbol{u}\|_{X_0}\|\overline{\boldsymbol{u}}\|_{X_\lambda} + \|\overline{\boldsymbol{u}}\|_{X_\lambda}\|\boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}}\right)\\ &\le C\left(\|\boldsymbol{a}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \|\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}} + \|\boldsymbol{f}\|_{Y_0(0)}\right)\|\overline{\boldsymbol{u}}\|_{X_\lambda}\\ &\le C\varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_\lambda}\end{aligned}$$

for $\lambda \in \{0,\eta\}$ and $\overline{\boldsymbol{u}} \in X_\lambda$. Therefore, Propositions 2.1 and 4.4 provide that

$$\begin{aligned}\|\boldsymbol{\Psi}\overline{\boldsymbol{u}}\|_{X_0} &\le C\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + C\left(\min_{\lambda\in\{0,\eta\}}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\lambda(0)} + \left\|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\right\|_{Y_0(0)} + \left\|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\right\|_{Y_0(0)}\right)\\ &\le C\left(\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \min_{\lambda\in\{0,\eta\}}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\lambda(0)}\right) + C\varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_0}\end{aligned} \tag{4.9}$$

for $\overline{\boldsymbol{u}} \in X_0$ and

$$\begin{aligned}\|\boldsymbol{\Psi}\overline{\boldsymbol{u}}\|_{X_\eta} &\le C\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + C(\left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)} + \left\|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\right\|_{Y_\eta(0)} + \left\|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\right\|_{Y_\eta(0)})\\ &\le C\left(\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)}\right) + C\varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_\eta}\end{aligned} \tag{4.10}$$

for $\overline{\boldsymbol{u}} \in X_\eta$. Similarly, we also obtain

$$\|\boldsymbol{\Psi}\overline{\boldsymbol{u}} - \boldsymbol{\Psi}\overline{\boldsymbol{v}}\|_{X_\lambda} \le C\varepsilon_0\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{v}}\|_{X_\lambda} \tag{4.11}$$

for $\lambda \in \{0, \eta\}$ and $\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}} \in X_\lambda$. We use Lemma 4.3 and Proposition 4.4 again to observe that

$$\begin{aligned}
&\|(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}} \\
&\quad \le \left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} + C\Big(\delta\left\|\overline{\boldsymbol{f}}\right\|_{Y_0(0)} + \min_{\lambda\in\{0,\eta\}}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\lambda(\delta t)} + \delta\|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\|_{Y_0(0)} \\
&\qquad + \|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\|_{Y_0(\delta t)} + \delta\|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\|_{Y_0(0)} + \|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\|_{Y_0(\delta t)}\Big) \\
&\quad \le C\Big(\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \min_{\lambda\in\{0,\eta\}}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\lambda(\delta t)}\Big) + C\delta(\left\|\overline{\boldsymbol{f}}\right\|_{Y_0(0)} + \varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_0}) \\
&\qquad + C\varepsilon_0\sup_{\tau>\delta t}\|\overline{\boldsymbol{u}}(\tau,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}}
\end{aligned}$$

for all $t > 0$, $0 < \delta \le 1/2$, and $\overline{\boldsymbol{u}} \in X_0$. Hence, letting $\delta \to +0$ after the limit $t \to \infty$, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\|(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}} \\
&\quad \le C\Big(\limsup_{t\to\infty}\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \min_{\lambda\in\{0,\eta\}}\limsup_{t\to\infty}t^\lambda\left\|\overline{\boldsymbol{f}}(t,\cdot)\right\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}}\Big) \\
&\qquad + C\varepsilon_0\limsup_{t\to\infty}\|\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}}.
\end{aligned} \tag{4.12}$$

Moreover, applying Proposition 2.1 and Theorem 3.1 with a similar calculation, we obtain

$$\begin{aligned}
&t^\eta\|\partial_t(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot),\Delta(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} \\
&\quad \le C\Big(t^\eta\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}} + \delta^{1-\eta}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(\delta t)} + \delta^{1-\eta}\|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\|_{Y_\eta(0)} \\
&\qquad + \|\nabla\cdot(\boldsymbol{u}\otimes\overline{\boldsymbol{u}})\|_{Y_\eta(\delta t)} + \delta^{1-\eta}\|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\|_{Y_\eta(0)} + \|\nabla\cdot(\overline{\boldsymbol{u}}\otimes\boldsymbol{u}_*)\|_{Y_\eta(\delta t)}\Big) \\
&\quad \le C\Big(\left\|e^{(t/2)\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(\delta t)}\Big) + C\delta^{1-\eta}\Big(\left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)} + \varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_\eta}\Big) \\
&\qquad + C\varepsilon_0\sup_{\tau>\delta t}\tau^\eta\|\overline{\boldsymbol{u}}(\tau,\cdot)\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\limsup_{t\to\infty}t^\eta\|\partial_t(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot),\Delta(\boldsymbol{\Psi}\overline{\boldsymbol{u}})(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} \\
&\quad \le C\Big(\limsup_{t\to\infty}\left\|e^{(t/2)\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \limsup_{t\to\infty}t^\eta\left\|\overline{\boldsymbol{f}}(t,\cdot)\right\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}}\Big) \\
&\qquad + C\varepsilon_0\limsup_{t\to\infty}t^\eta\|\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}}.
\end{aligned} \tag{4.13}$$

Taking $\varepsilon_0 > 0$ such that $\varepsilon_0 \le 1/(2C)$ implies that the Banach fixed point theorem may be applied because of (4.11). Namely, there exists $\overline{\boldsymbol{u}} \in X_\eta$ such that $\boldsymbol{\Psi}\overline{\boldsymbol{u}} = \overline{\boldsymbol{u}}$.

Hence, the estimate (4.9) yields

$$\|\overline{\boldsymbol{u}}\|_{X_0} \le C\Big(\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \min_{\lambda\in\{0,\eta\}}\left\|\overline{\boldsymbol{f}}\right\|_{Y_\lambda(0)}\Big).$$

Similarly to the derivation of the estimate (4.10), we use Proposition 2.1 and Theorem 3.1 to obtain

$$\begin{aligned}
\sup_{t>0}t^\eta\|\partial_t\overline{\boldsymbol{u}}(t,\cdot),\Delta\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} &\le C\Big(\sup_{t>0}t^\eta\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{2\eta-1+n/r}_{r,\infty}} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)}\Big) + C\varepsilon_0\|\overline{\boldsymbol{u}}\|_{X_\eta} \\
&\le C\Big(\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)}\Big) + \frac{1}{2}\|\overline{\boldsymbol{u}}\|_{X_\eta},
\end{aligned}$$

which implies that

$$\sup_{t>0} t^{\eta}\|\partial_t \overline{\boldsymbol{u}}(t,\cdot), \Delta\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} \le C\left(\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \left\|\overline{\boldsymbol{f}}\right\|_{Y_\eta(0)}\right).$$

Furthermore, since the definition of the homogeneous Besov spaces and the estimate of the heat semigroup [52, Theorem 1.3 (i)] provide that

$$\begin{aligned}\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} &\le \sup_{j\ge j_0} 2^{(-1+n/r)j}\|\dot{\Delta}_j(e^{t\Delta}\overline{\boldsymbol{a}})\|_{L^r} + \sup_{j<j_0} 2^{(-1+n/r)j}\left\|\dot{\Delta}_j\overline{\boldsymbol{a}}\right\|_{L^r}\\ &\le C\exp\left(-2^{2j_0-5}t\right)\|\overline{\boldsymbol{a}}\|_{\dot{B}^{-1+n/r}_{r,\infty}} + \sup_{j<j_0} 2^{(-1+n/r)j}\left\|\dot{\Delta}_j\overline{\boldsymbol{a}}\right\|_{L^r}\end{aligned}$$

for all $t>0$ and $j_0 \in \mathbb{Z}$, we have

$$\limsup_{t\to\infty}\left\|e^{t\Delta}\overline{\boldsymbol{a}}\right\|_{\dot{B}^{-1+n/r}_{r,\infty}} \le \limsup_{j\to-\infty} 2^{(-1+n/r)j}\left\|\dot{\Delta}_j\overline{\boldsymbol{a}}\right\|_{L^r}.$$

Therefore, the estimates (4.12) and (4.13) imply that

$$\begin{aligned}&\limsup_{t\to\infty}\|\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{-1+n/r}_{r,\infty}}\\ &\quad\le C\left(\limsup_{j\to-\infty} 2^{(-1+n/r)j}\left\|\dot{\Delta}_j\overline{\boldsymbol{a}}\right\|_{L^r} + \min_{\lambda\in\{0,\eta\}}\limsup_{t\to\infty} t^{\lambda}\left\|\overline{\boldsymbol{f}}(t,\cdot)\right\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}}\right),\\ &\limsup_{t\to\infty} t^{\eta}\|\partial_t\overline{\boldsymbol{u}}(t,\cdot), \Delta\overline{\boldsymbol{u}}(t,\cdot)\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}}\\ &\quad\le C\left(\limsup_{j\to-\infty} 2^{(-1+n/r)j}\left\|\dot{\Delta}_j\overline{\boldsymbol{a}}\right\|_{L^r} + \limsup_{t\to\infty} t^{\eta}\left\|\overline{\boldsymbol{f}}(t,\cdot)\right\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}}\right).\end{aligned}$$

Finally, we verify that $\overline{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_*$. Since $\boldsymbol{u} \in X_0$ has been obtained as a fixed point of (4.4), we observe that

$$\begin{aligned}\big(\boldsymbol{\Psi}(\boldsymbol{u}-\boldsymbol{u}_*)\big)(t) &= e^{t\Delta}\overline{\boldsymbol{a}} + \int_0^t e^{(t-\tau)\Delta}P\overline{\boldsymbol{f}}(\tau,\cdot)\mathrm{d}\tau - \boldsymbol{N}[\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}_*](t) - \boldsymbol{N}[\boldsymbol{u}-\boldsymbol{u}_*, \boldsymbol{u}_*](t)\\ &= e^{t\Delta}(\boldsymbol{a}-\boldsymbol{u}_*) + \int_0^t e^{(t-\tau)\Delta}P(\boldsymbol{f}(\tau,\cdot)-\boldsymbol{f}_*)\mathrm{d}\tau - \boldsymbol{N}[\boldsymbol{u},\boldsymbol{u}](t) + \boldsymbol{N}[\boldsymbol{u}_*,\boldsymbol{u}_*](t)\\ &= \boldsymbol{u}(t,\cdot) - e^{t\Delta}\boldsymbol{u}_* - \int_0^t e^{(t-\tau)\Delta}P\boldsymbol{f}_*\,\mathrm{d}\tau + \boldsymbol{N}[\boldsymbol{u}_*,\boldsymbol{u}_*](t)\end{aligned}$$

for all $t>0$. We also note that $\boldsymbol{u}_* \in P\left(\dot{B}^{-1+n/r}_{r,\infty}\right)^n$ has been obtained as a solution to (1.3). Hence, we have

$$\begin{aligned}\boldsymbol{u}_* - e^{t\Delta}\boldsymbol{u}_* &= \int_0^t \partial_\tau(e^{(t-\tau)\Delta}\boldsymbol{u}_*)\mathrm{d}\tau = \int_0^t e^{(t-\tau)\Delta}(-\Delta\boldsymbol{u}_*)\mathrm{d}\tau\\ &= \int_0^t e^{(t-\tau)\Delta}P\boldsymbol{f}_*\,\mathrm{d}\tau - \boldsymbol{N}[\boldsymbol{u}_*,\boldsymbol{u}_*](t)\end{aligned}$$

for all $t>0$, which leads to $\boldsymbol{\Psi}(\boldsymbol{u}-\boldsymbol{u}_*) = \boldsymbol{u}-\boldsymbol{u}_*$. Namely, $\boldsymbol{u}-\boldsymbol{u}_* \in X_0$ is also a fixed point of $\boldsymbol{\Psi}$. As the estimate (4.11) with $\lambda = 0$ ensures the uniqueness of the fixed point even in a larger space X_0 than X_η, we deduce that $\overline{\boldsymbol{u}} = \boldsymbol{u}-\boldsymbol{u}_*$, which completes the proof of Theorem 1.3. □

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Appendix A. Remarks on the main results

In Remark 1.4 (iv), we mentioned that the case of $\eta \neq 0$ yields a choice of two quantities for the estimates. Let us give simple examples showing that the appearance of the minimum *refines* the stability result: Suppose that $\boldsymbol{f}_* \in \left(\dot{B}^{2\eta-3+n/r}_{r,\infty} \cap \dot{B}^{-3+n/r}_{r,\infty}\right)^n$ with $n \geq 3$, $1 < r < n$, and $0 < \eta < 1/2$. We then introduce the external forces $\boldsymbol{f}$ and $\boldsymbol{g}$ by letting

$$\boldsymbol{f}(t,x) := \boldsymbol{f}_*(x) + \frac{1}{(1+t)^\eta}\boldsymbol{f}_*(x), \quad \boldsymbol{g}(t,x) := \boldsymbol{f}_*(x) + \frac{1}{(1+t)^3}\boldsymbol{f}_*\left(\frac{x}{1+t}\right)$$

for $(t,x) \in (0,\infty) \times \mathbb{R}^n$. As we see by [2, Remark 2.19] that

$$\begin{aligned} t^\lambda \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}} &= \frac{t^\lambda}{(1+t)^\eta}\|\boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}}, \\ t^\lambda \|\boldsymbol{g}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}} &\sim \frac{t^\lambda}{(1+t)^{2\lambda}}\|\boldsymbol{f}_*\|_{\dot{B}^{2\lambda-3+n/r}_{r,\infty}} \end{aligned}$$

for all $t > 0$ and $\lambda \in \{0, \eta\}$, we deduce that both $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the assumption of Theorem 1.3, provided that $\|\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}} \ll 1$. Furthermore, the above properties imply that

$$\begin{aligned} &\lim_{t\to\infty}\|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}} = 0, \quad \lim_{t\to\infty} t^\eta \|\boldsymbol{f}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} = \|\boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}}, \\ &\lim_{t\to\infty} t^\eta \|\boldsymbol{g}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{2\eta-3+n/r}_{r,\infty}} = 0, \quad \liminf_{t\to\infty}\|\boldsymbol{g}(t,\cdot) - \boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}} \gtrsim \|\boldsymbol{f}_*\|_{\dot{B}^{-3+n/r}_{r,\infty}}. \end{aligned}$$

We assume that $\boldsymbol{a} = \boldsymbol{u}_*$ in the estimates of Theorem 1.3 for simplicity. Then such examples show that the force $\boldsymbol{f}$ provides the asymptotic stability $\lim_{t\to\infty}\|\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_*\|_{\dot{B}^{-1+n/r}_{r,\infty}} = 0$ by choosing the case of $\lambda = 0$, while the force $\boldsymbol{g}$ requires to choose the case of $\lambda = \eta$.

Next, we compare the previous works with our results; after giving the main results, we stated that our assumption of forces is weaker than those of Yamazaki [36] and Cannone and Karch [7]. In what follows, we verify that our assumption certainly covers the cases of [7, 36]; whereas the weak Lebesgue spaces $L^{r,\infty}$ might be well known, the pseudo-measure spaces $\mathcal{PM}^s$ do not seem to be so standard, and thus we begin with its definition. For $s \geq 0$, define

$$\mathcal{PM}^s := \left\{ f \in \mathscr{S}' \;\middle|\; \|f\|_{\mathcal{PM}^s} := \sup_{\xi\in\mathbb{R}^n}|\xi|^s|(\mathcal{F}f)(\xi)| < \infty \right\}.$$

To give comparisons, we shall show the following simple assertion, which might have already been mentioned somewhere (we refer to [7, Lemma 7.1; 14, Lemma 2.1; 46, Theorem 2]).

Proposition A.1. *The continuous embedding $L^{q,\infty} \subset \dot{B}^{-n(1/q-1/r)}_{r,\infty}$ holds for $1 < q < r \leq \infty$. In addition, the continuous embedding $\mathcal{PM}^s \subset \dot{B}^{s-n+n/r}_{r,\infty}$ holds for $2 \leq r \leq \infty$ and $s \geq 0$.*

Proof. The first assertion is given by [28, Theorem 2] and [45, Lemma 2.1]. To be more precise, we may choose q_0, q_1, and θ so that $1 < q_0 < q < q_1 < r$ and $q = (1-\theta)/q_0 + \theta/q_1$ from the assumption, where we note that $0 < \theta < 1$ due to $q_0 < q < q_1$. As the standard Sobolev embeddings provide that $L^{q_j} \subset \dot{B}^{-n(1/q_j - 1/r)}_{r,\infty}$ for $j \in \{0,1\}$, we rely on the real interpolation results $(L^{q_0}, L^{q_1})_{\theta,\infty} = L^{q,\infty}$ and $(\dot{B}^{-n(1/q_0-1/r)}_{r,\infty}, \dot{B}^{-n(1/q_1-1/r)}_{r,\infty})_{\theta,\infty} = \dot{B}^{-n(1/q-1/r)}_{r,\infty}$ to obtain the desired result.

As for the second assertion, we use the Plancherel theorem to deduce that

$$\begin{aligned}\|\dot{\Delta}_j f\|_{L^2} &\le C\|\varphi(2^{-j}\cdot)\mathcal{F}f\|_{L^2} \le C\||\cdot|^{-s}\varphi(2^{-j}\cdot)\|_{L^2}\||\cdot|^s\mathcal{F}f\|_{L^\infty}\\ &\le C2^{-sj}\cdot 2^{(n/2)j}\||\cdot|^{-s}\varphi\|_{L^2}\|f\|_{\mathcal{PM}^s}\end{aligned}$$

for all $j \in \mathbb{Z}$, which implies that $\mathcal{PM}^s \subset \dot{B}^{s-n/2}_{2,\infty} \subset \dot{B}^{s-n+n/r}_{r,\infty}$ for $2 \le r \le \infty$ with the aid of the Sobolev embedding. □

We consider the assumption of Yamazaki [36]; suppose the replesentation $\boldsymbol{f} = \nabla \cdot F$ and $F \in BC((0,\infty); (L^{n/2,\infty})^{n^2})$ with $n \ge 3$. We then note that $L^{n/2,\infty} \subset \dot{B}^{-2+n/r}_{r,\infty}$ for $r > n/2$ to observe that

$$\|\boldsymbol{f}\|_{L^\infty\left(\dot{B}^{-3+n/r}_{r,\infty}\right)} \le C\|F\|_{L^\infty\left(\dot{B}^{-2+n/r}_{r,\infty}\right)} \le C\|F\|_{L^\infty(L^{n/2,\infty})},$$

which means that $\boldsymbol{f} \in L^\infty((0,\infty); \left(\dot{B}^{-3+n/r}_{r,\infty}\right)^n)$ appearing in the assumption of Theorem 1.1. As for the case of Cannone and Karch [7], that is, the assumption $\boldsymbol{f} \in L^\infty((0,\infty); (\mathcal{PM}^0(\mathbb{R}^3))^3)$, we see by $\mathcal{PM}^0(\mathbb{R}^3) \subset \dot{B}^{-3+3/r}_{r,\infty}(\mathbb{R}^3)$ for $r \ge 2$ that $\boldsymbol{f} \in L^\infty((0,\infty); (\dot{B}^{-3+3/r}_{r,\infty}(\mathbb{R}^3))^3)$. Therefore, both cases are included in our assumption.

Appendix B. Proof of the weak-star continuity property

We give the proof of Lemma 3.4 here. Its proof is almost the same as that of [56, Lemma 4.13].

Proof of Lemma 3.4. In the case of $\theta \ne 0$, as Proposition 2.1 yields

$$\begin{aligned}\int_0^t \left\|e^{(t-\tau)\Delta} f(\tau,\cdot)\right\|_{\dot{B}^{s+2-2\theta}_{r,1}} \mathrm{d}\tau &\le C\int_0^t (t-\tau)^{-1+\theta}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\mathrm{d}\tau\\ &\le C\sup_{\tau>0}\tau^\theta\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\int_0^t (t-\tau)^{-1+\theta}\tau^{-\theta}\mathrm{d}\tau\\ &\le C\sup_{\tau>0}\tau^\theta\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\end{aligned}$$

for all $t > 0$, the desired result follows from [56, Lemma 4.13] with $\alpha = 2\theta$ and $\beta = 0$. In the case of $\theta = 0$, we apply Proposition 2.1 to obtain

$$\begin{aligned}\int_0^t \left\|e^{(t-\tau)\Delta} f(\tau,\cdot)\right\|_{\dot{B}^{s+1}_{r,1}} \mathrm{d}\tau &\le C\int_0^t (t-\tau)^{-1/2}\|f(\tau,\cdot)\|_{\dot{B}^s_{r,\infty}}\mathrm{d}\tau\\ &\le C\|f\|_{L^\infty\left(\dot{B}^s_{r,\infty}\right)}\int_0^t (t-\tau)^{-1/2}\mathrm{d}\tau\\ &\le Ct^{1/2}\|f\|_{L^\infty\left(\dot{B}^s_{r,\infty}\right)}\end{aligned}$$

for all $t > 0$. In addition, we note that there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{S}$ of functions such that

$$\lim_{j\to\infty} \|\psi - \psi_j\|_{\dot{B}^{-s-2}_{r',1}} = 0$$

due to [2, Proposition 2.27] and $r > 1$. Hence, we combine the above estimate and Theorem 3.1 to deduce that

$$\begin{aligned} |\langle [\mathcal{I}(t)]f, \psi\rangle| &\le \left|\left\langle [\mathcal{I}(t)]f, \psi - \psi_j\right\rangle\right| + \left|\left\langle [\mathcal{I}(t)]f, \psi_j\right\rangle\right| \\ &\le C\left(\|[\mathcal{I}(t)]f\|_{\dot{B}^{s+2}_{r,\infty}} \|\psi - \psi_j\|_{\dot{B}^{-s-2}_{r',1}} + \left\| \int_0^t e^{(t-\tau)\Delta} f(\tau, \cdot)\mathrm{d}\tau \right\|_{\dot{B}^{s+1}_{r,\infty}} \|\psi_j\|_{\dot{B}^{-s-1}_{r',1}} \right) \\ &\le C\|f\|_{L^\infty\left(\dot{B}^{s}_{r,\infty}\right)} \left(\|\psi - \psi_j\|_{\dot{B}^{-s-2}_{r',1}} + t^{1/2}\|\psi_j\|_{\dot{B}^{-s-1}_{r',1}} \right) \end{aligned}$$

for all $j \in \mathbb{N}$ and $t > 0$ because of the duality property $\dot{B}^{s_0}_{r,\infty} = \left(\dot{B}^{-s_0}_{r',1}\right)^*$ from [2, Propositions 2.27 and 2.29]. Therefore, taking the limit $j \to \infty$ after letting $t \to +0$ provides the desired result. □

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