# NONPARAMETRIC ESTIMATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY HERMITE PROCESSES WITH RANDOM EFFECTS

B. L. S. Prakasa Rao
CR Rao Advanced Institute of Mathematics, Statistics and Computer Science



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# NONPARAMETRIC ESTIMATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY HERMITE PROCESSES WITH RANDOM EFFECTS

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**Abstract**

We discuss nonparametric estimation of the probability density function of the random effects in models governed by stochastic differential equations driven by Hermite processes.

*Key Words and Phrases:* Hermite process; Kernel method; Nonparametric estimation; Random effects; Stochastic differential equation.

## 1. Introduction

Statistical inference for fractional diffusion processes satisfying stochastic differential equations driven by a fractional Brownian motion (fBm) has been studied earlier and a comprehensive survey of various methods is given in Mishura (2008) and in Prakasa Rao (2010). There has been a recent interest to study similar problems for stochastic processes driven by $\alpha$-stable noises, fractional Levy processes and general Gaussian processes. Prakasa Rao (2024) investigated nonparametric estimation of linear multiplier in SDEs driven by general Gaussian processes. Prakasa Rao (2026) discusses the recent advances in statistical inference for processes driven by various fractional processes such as sub-fractional Brownian motion, mixed fractional Brownian motion, fractional Levy processes and some general Gaussian processes. Hermite processes are another class of processes which can be used for modeling long range dependence in a non-Gaussian environment. Stochastic differential equation models with random effects are used in the biomedical field for the study of repeated measurements collected on a series of individuals/subjects. For instance, see Antic et al. (2009), Delattre et al. (2012), Ditlevsen and De Gaetano (2005), Nie and Yang (2005), Nie (2006, 2007), Picchini et al. (2010) and Picchini and Ditlevsen (2011) among others. Nonparametric estimation for fractional diffusion processes with random effects has been investigated in El Omari et al. (2019) and Prakasa Rao (2021). They study the properties of kernel and histogram estimators for estimation of the probability density function of the random effects.

Our aim in this paper is to study nonparametric estimation when the process is governed by a stochastic differential equation with random effects driven by a Hermite process following the ideas of density function estimation and regression function estimation in classical statistical inference. Several methods are present for nonparametric

* CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India. tel +91-9949185041 blsprao@gmail.com

function estimation as described in Prakasa Rao (1983). The method of kernels is widely used for the estimation of a density function or a regression function and it is known that the properties of such an estimator do not depend on the choice of the kernel in general but on the choice of the bandwidth. Properties of the estimators of a density function and a regression function, using the method of kernels, are described in Prakasa Rao (1983). Our aim is to propose a kernel type estimator for the probability density function of the random effects and study its properties. We will show that the kernel type estimator is consistent and obtain the asymptotic distribution of the estimator.

Hermite processes are a class of non-Gaussian processes which have been proposed as the driving forces for modeling stochastic phenomenon with long range dependence in a non-Gaussian environment (cf. Taqqu (1978); Lawrance and Kottegoda (1977)). Statistical inference for Vasicek-type model driven by Hermite process was studied in Nourdin and Diu Tran (2017) (cf. Diu Tran (2018)). Coupek and Kriz (2025) discuss parametric and nonparametric inference for nonlinear stochastic differential equations driven by Hermite processes. Our aim in this paper is to study nonparametric estimation of the probability density function of the random effects when the process is governed by a stochastic differential equation driven by a Hermite process.

## 2. Hermite Processes

As the definition and properties of Hermite processes are not widely known, we now give a short review of the properties of Hermite processes for completeness following Diu Tran (2018) and Tudor (2013).

Let $W = \{W(h), h \in L^2(R)\}$ be a Brownian field defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, that is, a centered Gaussian family of random variables satisfying

$$E[W(h)W(g)] = \langle h, g\rangle_{L^2(R)}$$

for any $h, g \in L^2(R)$. For every $q \geq 1$, the $q$-th Wiener chaos $\mathcal{H}_q$ is defined as the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables $\{H_q(W(h)), h \in L^2(R), \|h\|_{L^2(R)} = 1\}$, where $H_q$ is the $q$-th Hermite polynomial. Recall that $H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x, \ldots$. The mapping $I_q^W(h) = H_q(W(h))$ can be extended to a linear isometry between $L_s^2(R^q)$, the space of symmetric square integrable functions on $R^q$ equipped with the modified norm $\sqrt{q!}\|.\|_{L^2(R^q)}$, and the $q$-th Wiener chaos $\mathcal{H}_q$. If $f \in L_s^2(R^q)$, then the random variable $I_q^W(f)$,defined below, is called the *multiple Wiener integral* of the function $f$ of order $q$ and it is denoted by

$$I_q^W(f) = \int_{R^q} f(\psi_1, \ldots, \psi_q) dW_{\psi_1} \ldots dW_{\psi_q}.$$

For the existence and the properties of such multiple Wiener integrals, see Nualart (2006) and Major (2014).

We now define the Hermite process and discuss its properties following Diu Tran (2018) and Tudor (2013).

**Definition 2.1:** The *Hermite process* $\{Z_t^{q,H}, t \geq 0\}$ of order $q \geq 1$ and the self-similarity parameter $H \in (\frac{1}{2}, 1)$ is defined as

$$Z_t^{q,H} = c(q,H) \int_{R^q} (\int_0^t \Pi_{j=1}^q (s-\psi_j)_+^{H_0-\frac{3}{2}} ds) dW_{\psi_1} \dots dW_{\psi_q} \tag{1}$$

where

$$c(q,H) = \sqrt{\frac{H(2H-1)}{q!(\beta(H_0-\frac{1}{2}, 2-2H_0))^q}} \tag{2}$$

and

$$H_0 = 1 + \frac{H-1}{q}. \tag{3}$$

Note that $H_0 \in (1-\frac{1}{2q}, 1)$. The integral in (1) is a multiple Wiener integral of order $q$. Observe that $Z_0^{q,H} = 0$ a.s. The constant $c(q,H)$ is chosen in (1) so that $E[(Z_1^{q,H})^2] = 1$. Hermite process of order $q = 1$ is the fractional Brownian motion. It is the only Hermite process that is Gaussian. The Hermite process of order $q = 2$ is also called the *Rosenblatt process*. The function $\beta(.,.)$ in the equation (2) is the standard beta function.

For any two processes $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$, we say that $\{X_t, t \geq 0\} \triangleq \{Y_t, t \geq 0\}$ if the finite dimensional distributions of the process $X$ are the same as the finite dimensional distributions of the process $Y$.

The following properties are satisfied by the Hermite processes.

**Theorem 2.1:** Suppose $Z = \{Z_t^{q,H}, t \geq 0\}$ is the Hermite process of order $q \geq 1$ and the parameter $H \in (\frac{1}{2}, 1)$. Then the process is centered and the following properties hold:

(i)the process $Z$ is self-similar, that is, for all $a > 0$,

$$\{Z_{at}^{q,H}, t \geq 0\} \triangleq \{a^H Z_t^{q,H}, t \geq 0\};$$

(ii) the process $Z$ has stationary increments, that is, for any $h > 0$,

$$\{Z_{t+h}^{q,H} - Z_h^{q,H}, t \geq 0\} \triangleq \{Z_t^{q,H}, t \geq 0\};$$

(iii) the covariance function of the process $Z$ is given by

$$E[Z_t^{q,H} Z_s^{q,H}] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), t \geq 0, s \geq 0;$$

(iv) the process $Z$ has long range dependence property, that is,

$$\sum_{n=0}^{\infty} E[Z_1^{q,H}(Z_{n+1}^{q,H} - Z_n^{q,H})] = \infty;$$

(v) for any $\alpha \in (0, H)$ and any interval $[0,T] \subset R_+$, the process $\{Z_t^{q,H}, 0 \leq t \leq T\}$ admits a version with Hölder continuous sample paths of order $\alpha$; and

(vi) for every $p \geq 1$, there exists a positive constant $C_{p,q,H}$ such that

$$E[|Z_t^{q,H}|^p] \leq C_{p,q,H} t^{pH}, t \geq 0.$$

For proof of this theorem, see Tudor (2013).

**Wiener integral with respect to the Hermite process:**

The Wiener integral of a non-random function $f$, with respect to the Hermite process $Z = \{Z_t^{q,H}, t \geq 0\}$, is studied in Maejima and Tudor (2007) and it is denoted by

$$E[\int_R f(u) dZ_u^{q,H}]$$

whenever it exists. It is known that the Wiener integral of the function $f$ with respect to the Hermite process of order $q$ and parameter $H$ exists if

$$\int_R \int_R |f(u)f(v)||u-v|^{2H-2}\, dudv < \infty. \tag{4}$$

Let $|\mathcal{H}|$ denote the class of functions $f$ satisfying the inequality the inequality (4). It can be shown that, for any $f, g \in |\mathcal{H}|$,

$$E[\int_R f(u) dZ_u^{q,H} \int_R g(u) dZ_u^{q,H}] = H(2H-1) \int_R \int_R f(u) g(v) |u-v|^{2H-2} dudv. \tag{5}$$

In particular,

$$E([\int_0^T f(u) dZ_u^{q,H}]^2) = H(2H-1) \int_0^T \int_0^T f(u) f(v) |u-v|^{2H-2} dudv \tag{6}$$

for any $T > 0$. Furthermore, the Wiener integral of $f$ with respect to the process $Z$ admits the representation

$$\int_R f(u) dZ_u^{q,H} = c(q,H) \int_{R^q} (\int_R f(u) \Pi_{j=1}^q (u-\psi_j)_+^{H_0 - \frac{3}{2}} du) dW_{\psi_1} \ldots dW_{\psi_q} \tag{7}$$

where $c(q,H)$ and $H_0$ are as defined by the equations (2) and (3) respectively.

## 3. Main Results

Let us consider a system of linear stochastic differential equations

$$dX_t^i = (S(X^i(t)) + \phi_i b(t))\, dt + d\tilde{Z}_t^{q,H,i}, X^i(0) = x^i, 0 \leq t \leq T, i = 1, \ldots, N \tag{8}$$

where the function $S(.)$ is unknown but the function $b(.)$ and the parameters $q$ and $H$ are known and $H > \frac{1}{2}$. Suppose the random effects $\phi_i, i = 1, \ldots, N$ are independent and identically distributed with the probability density function $f(.)$ which is unknown and the driving forces $\{\tilde{Z}^{q,H,i}, i = 1, \ldots, N\}$ are independent Hermite processes. The problem is to estimate the probability density function $f(.)$ based on the observations $\{X^i(t), 0 \leq t \leq T, i = 1, \ldots, N\}$ and study its asymptotic properties as $N$ and $T$ tend to infinity. One method of estimation of the density $f(.)$ is by the kernel method (cf. Prakasa Rao (1983)) and it is given by

$$\hat{f}_h(x) = \frac{1}{Nh}\Sigma_{i=1}^N K(\frac{x-\phi_i}{h}) \tag{9}$$

where $K(.)$ is a suitable kernel satisfying some regularity conditions and $h > 0$ is a smoothing parameter known as the band-width. Since the random effects $\phi_i, i = 1,\ldots,N$ are not observable, the estimator $\hat{f}_h(x)$ is not computable. We will now obtain an alternate estimator which is computable and prove that the estimator is consistent as $N$ and $T$ tend to infinity. Our results and their proofs are analogous to those in El Omari et al. (2019) and Prakasa Rao (2021) where they consider the driving force to be fractional Brownian motion and mixed fractional Brownian motion respectively. We have already indicated the reasons for studying inference for stochastic differential equations driven by a Hermite process in the introduction of this paper.

Define

$$J_t^{(1,i)} = \int_0^t b(s)dX^i(s), J_t^{(2)} = \int_0^t b^2(s)ds$$

and

$$G_t^{(i)} = \int_0^t S(X^i(s))b(s)ds, A_t^{(i)} = \int_0^t b(s)d\tilde{Z}_s^{q,H,i}$$

for $i = 1,\ldots,N$. The random variables $A^{(i)} = \{A_t^{(i)}, t \geq 0\}, i = 1,\ldots,N$ are Wiener integrals with respect to the Hermite process. Sufficient conditions for the integrals $A^{(i)}$ to be well defined are discussed above. We assume that the following conditions hold:
$(C_1)$ There exists $c_0 > 0$ and $c_1 > 0$ such that

$$c_0^2 \leq b^2(s) \leq c_1^2, s \in R_+.$$

$(C_2)$ For $i = 1,\ldots,N, M_i = E[\int_0^\infty S^2(X^i(s))ds]^2 < \infty$.

**Theorem 3.1:** *Suppose the conditions $(C_1)$ and $(C_2)$ hold and $H \in (\frac{1}{2},1)$. Let*

$$\hat{\phi}_{i,T} = \frac{J_T^{(1,i)}}{J_T^{(2)}}, i = 1,\ldots,N.$$

*Then*

$$E|\hat{\phi}_{i,T} - \phi_i|^2 \to 0 \text{ as } T \to \infty.$$

**Proof:** From the representation of the system (8), it follows that

$$J_t^{(1,i)} = G_t^{(i)} + \phi_i J_t^{(2)} + A_t^{(i)}, 0 \leq t \leq T, i = 1,\ldots,N.$$

Hence

$$E|\hat{\phi}_{i,T} - \phi_i|^2 \leq 2[E(\frac{G_T^{(i)}}{J_T^{(2)}})^2 + E(\frac{A_T^{(i)}}{J_T^{(2)}})^2], \tag{10}$$

and

$$
\begin{aligned}
E(\frac{A_T^{(i)}}{J_T^{(2)}})^2 &\leq \frac{1}{c_0^4 T^2} E(\int_0^T b(s) d\tilde{Z}_s^{q,H,i})^2 \\
&\leq \frac{1}{c_0^4 T^2} H(2H-1) \int_0^T \int_0^T b(u)b(v)|u-v|^{2H-2} du dv \\
&\leq \frac{H(2H-1)}{c_0^4 T^2} c_1^2 \int_0^T \int_0^T |u-v|^{2H-2} du dv \\
&= \frac{H(2H-1)}{c_0^4 T^2} c_1^2 \frac{T^{2H}}{H(2H-1)} \\
&= \frac{c_1^2}{c_0^4} T^{2H-2}.
\end{aligned} \tag{11}
$$

The last term tends to zero as $T \to \infty$ since $H < 1$. Using the inequality

$$
|uv| \leq \frac{1}{2}(\epsilon u^2 + \frac{v^2}{\epsilon})
$$

for all $u, v \in R$ and for all $\epsilon > 0$, it follows that

$$
\begin{aligned}
E(\frac{G_T^{(i)}}{J_T^{(2)}})^2 &\leq \frac{1}{4} E[\frac{\epsilon \int_0^T S^2(X^i(s))ds + \epsilon^{-1} J_T^{(2)}}{J_T^{(2)}}]^2 \\
&\leq \frac{1}{2}[\frac{1}{\epsilon^2} + \frac{\epsilon^2}{c_0^4 T^2} E(\int_0^\infty S^2(X^i(s))ds)^2].
\end{aligned} \tag{12}
$$

Choosing $\epsilon = T^{1/2}$, we get that the last term tends to zero as $T \to \infty$. This completes the proof of Theorem 3.1.

Substituting $\phi_i$ by its estimator $\hat{\phi}_{i,T} = \frac{J_T^{(1,i)}}{J_T^{(2)}}$, we obtain the kernel type estimator

$$
\hat{f}_h^{(1)}(x) = \frac{1}{Nh} \sum_{i=1}^N K(\frac{x - \hat{\phi}_{i,T}}{h}) \tag{13}
$$

as an estimator for the probability density function of the mixed random effects.

**Theorem 3.2:** *Suppose the conditions $(C_1)$ and $(C_2)$ hold. In addition, suppose that the kernel function $K(.)$ is continuously differentiable with derivative $K'(.)$ satisfying the condition*

$$
||K||^2 + ||K'||^2 < \infty
$$

*where $||f||^2 = \int_{-\infty}^\infty f^2(x)dx$. Then*

$$
E||\hat{f}_h^{(1)} - f||^2 \leq 2||f_h - f||^2 + \frac{||K||^2}{Nh} + \frac{||K'||^2}{Th^3}(1 + \frac{M_1}{c_0^4} + \frac{2c_1^2}{c_0^4} + \frac{2C_H^2 c_1^2}{c_0^4 T^{1-2H}}) \tag{14}
$$

*where*

$$f_h(x) = \frac{1}{h}\int_{-\infty}^{\infty} K(\frac{x-u}{h})f(u)du.$$

**Proof:** It is easy to check that

$$\begin{aligned} E||\hat{f}_h^{(1)} - f||^2 &= ||f - E(\hat{f}_h^{(1)})||^2 + E(||\hat{f}_h^{(1)} - E(\hat{f}_h^{(1)})||^2) \\ &\leq 2||f - f_h||^2 + 2||f_h - E(\hat{f}_h^{(1)})||^2 + E(||\hat{f}_h^{(1)} - E(\hat{f}_h^{(1)})||^2). \\ &= 2I_1 + 2I_2 + I_3 \ \ \text{(say)}. \end{aligned} \tag{15}$$

We will now obtain bounds on the last two terms in the inequality given above. Let

$$K_h(u) = \frac{1}{h}K(\frac{u}{h})$$

and define

$$\eta_{i,T}(x) = K_h(x - \hat{\phi}_{i,T}) - E(K_h(x - \hat{\phi}_{i,T})), i = 1, \ldots, N.$$

Note that the random variables $\eta_{i,T}(x), i = 1, \ldots, N$ are independent and identically distributed with mean zero and

$$\begin{aligned} \int_{-\infty}^{\infty} E(\eta_{i,T}(x))^2 dx &= \int_{-\infty}^{\infty} Var(K_h(x - \hat{\phi}_{i,T}))dx \\ &\leq \int_{-\infty}^{\infty} E(K_h(x - \hat{\phi}_{i,T}))^2 dx \\ &= \frac{1}{h^2}E[\int_{-\infty}^{\infty}(K(\frac{(x - \hat{\phi}_{i,T})}{h}))^2 dx] \\ &\leq \frac{1}{h}\int_{-\infty}^{\infty} K^2(y)dy. \end{aligned} \tag{16}$$

The last inequality is obtained by using the change of variable $y = \frac{x-\hat{\phi}_{i,T}}{h}$. Hence

$$\begin{aligned} I_3 &\equiv E(||\hat{f}_h^{(1)} - E(\hat{f}_h^{(1)})||^2) \\ &\leq E(\int_{-\infty}^{\infty}(\hat{f}_h^{(1)}(x) - E[\hat{f}_h^{(1)}(x)])^2 dx) \\ &= \frac{1}{N^2}E(\int_{-\infty}^{\infty}(\sum_{i=1}^{N}\eta_{i,T}(x))^2 dx) \\ &= \frac{1}{N}\int_{-\infty}^{\infty} E(\eta_{i,T}(x))^2 dx \\ &\leq \frac{||K||^2}{Nh}. \end{aligned} \tag{17}$$

We will now obtain an upper bound for the term $I_2$. Note that

$$\begin{aligned} f_h(x) - E(\hat{f}_h^{(1)}(x)) &= \int_{-\infty}^{\infty} f(y)K_h(x-y)dy - E(K_h(x-\hat{\phi}_{1,T})) \\ &= E[K_h(x-\phi_1) - K_h(x-\hat{\phi}_{1,T})]. \end{aligned} \tag{18}$$

Furthermore, it follows that

$$K_h(x-\hat{\phi}_{1,T}) - K_h(x-\phi_1) = (\frac{\phi_1 - \hat{\phi}_{1.T}}{h^2})\int_0^1 K'(\frac{x-\phi_1+u(\phi_1-\hat{\phi}_{1,T})}{h})du$$

by applying the Taylor's theorem with the integral form of the remainder term. Let

$$g(x,u) = K'(\frac{x-\phi_1+u(\phi_1-\hat{\phi}_{1,T})}{h}).$$

Combining the equations given above, it follows that

$$\begin{aligned} I_2 &\equiv ||f_h - E(\hat{f}_h^{(1)})||^2 \\ &= \int_{-\infty}^{\infty}[E(K_h(x-\phi_1) - K_h(x-\hat{\phi}_{1,T}))]^2 dx \\ &\leq E[\frac{(\phi_i-\hat{\phi}_{i,T})^2}{h^4}\int_{-\infty}^{\infty}(\int_0^1 g(x,u)du)^2 dx] \\ &\leq E[\frac{(\phi_i-\hat{\phi}_{i,T})^2}{h^4}(\int_0^1(\int_{-\infty}^{\infty} g^2(x,u)dx)^{1/2}du)^2]. \end{aligned} \tag{19}$$

The last inequality is a consequence of the generalized Minkowski inequality (cf. Tsybakov (2009), Lemma A.1, p.191)). Applying the transformation

$$y = \frac{1}{h}(x-\phi_1+u(\phi_1-\hat{\phi}_{1,T})),$$

it follows that

$$\int_{-\infty}^{\infty} g^2(x,u)du = ||K'||^2 h.$$

Therefore

$$\begin{aligned} I_2 &\equiv ||f_h - E(\hat{f}_h^{(1)})||^2 \\ &\leq \frac{||K'||^2}{h^3}E[(\phi_1-\hat{\phi}_{1,T})^2]. \end{aligned} \tag{20}$$

Applying the upper bounds derived in the inequalities (15)-(20), it follows that

$$\begin{aligned} E||\hat{f}_h^{(1)} - f||^2 &\leq 2||f_h - f||^2 + \frac{||K||^2}{Nh} \\ &\quad + \frac{||K'||^2}{Th^3}(1 + \frac{M_1}{c_0^4} + \frac{2c_1^2}{c_0^4} + \frac{2c_1^2}{c_0^4 T^{1-2H}}). \end{aligned} \tag{21}$$

This completes the proof of Theorem 3.2.

In addition to the conditions $(C_1)$ and $(C_2)$,suppose the following condition holds. $(C_3)$ Suppose the function $K(.)$ is a kernel of order $\ell \geq 1$, that is,

$$\int_{-\infty}^{\infty} K(u)du = 1; \int_{-\infty}^{\infty} u^j K(u)du = 0, j = 1, \ldots, \ell.$$

For $\gamma > 0$ and $L > 0$, let $D(\gamma, L)$ be the class of real-valued functions $r(.)$ defined on the real line such that the derivative $r^{(\ell)}$ of order $\ell = [\gamma]$ exists and satisfies the inequality

$$[\int_{-\infty}^{\infty} (r^{(\ell)}(x+t) - r^{(\ell)}(x))^2 dx]^{1/2} \leq L|t|^{\gamma-\ell}$$

where $[\gamma]$ is the integral part of $\gamma$.

It can be checked that the following theorem holds under the conditions $(C_1)-(C_3)$. We omit the details.

**Theorem 3.3:** *Suppose the conditions $(C_1)$ to $(C_3)$ hold and the density function $f \in D(\gamma, L)$ and the kernel $K(.)$ has order $\ell = [\gamma]$. Let $h = N^{-1/(2\gamma+1)}$ and $T = N^{(2\gamma+3)/(2\gamma+1)}$. Then*

$$E||\hat{f}_h^{(1)} - f||^2 \leq 2||f_h - f||^2 + O(N^{-2\gamma/(2\gamma+1)}). \tag{22}$$

**Remarks:** For results on kernel method for density estimation for independent and identically distributed random variables and bounds on the mean integrated squared error, see Theorems 2.1.5 - 2.1.7 in Prakasa Rao (1983). Results obtained here and proofs given here are similar to those in El Omari et al. (2019) and Prakasa Rao (2021) for kernel method of estimation for processes driven by a fractional Brownian motion and mixed fractional Brownian motion respectively. Our contribution is to indicate that similar results continue to hold for processes driven by a Hermite process.

We shall now study the limiting distribution if any of the estimator $\hat{f}_h^{(1)}(x)$ after suitable normalization. Suppose the kernel $K(.)$ satisfies the following conditions:

$(C_4)$ $(i)\sup_y |K(y)| < \infty, K(y) = K(-y), y \in R$,
(ii) $\int_{-\infty}^{\infty} |K(y)|dy < \infty, \int_{-\infty}^{\infty} K(y)dy = 1$, and
(iii) $\lim_{|y|\to\infty} |yK(y)| = 0$.
Further suppose that
$(C_5)$ $h_n \to 0, nh_n \to \infty$ as $n \to \infty$.
Observe that the random variables $\hat{\phi}_{iT}, 1 \leq i \leq n$ are independent and identically distributed random variables.
$(C_6)$ Suppose the random variable $\hat{\phi}_{1T}$ has the probability density $g_T(.)$ which is continuous at $x$.

The estimator $\hat{f}_{h_n}^{(1)}(x)$ can be written in the form

$$\hat{f}_{h_n}^{(1)}(x) = \frac{1}{n}\sum_{k=1}^{n} Z_{nk}$$

where

$$Z_{nk} = \frac{1}{h_n} K(\frac{(x - \hat{\phi}_{kT})}{h_n})$$

are independent and identically distributed random variables.

A necessary and sufficient condition for

$$\{\hat{f}^{(1)}_{h_n}(x) - E[\hat{f}^{(1)}_{h_n}(x)]\}\{Var[\hat{f}^{(1)}_{h_n}(x)]\}^{-1/2} \to N(0,1) \tag{23}$$

in distribution as $n \to \infty$ is that, for every $\epsilon > 0$,

$$nP[|(Z_{n1} - E(Z_{n1}))[Var(Z_{n1})]^{-1/2}| \geq \epsilon n^{1/2}] \to 0 \ \text{ as } \ n \to \infty$$

by Loeve (1963, p.316). A sufficient condition for (23) to hold is that, for some $\delta > 0$,

$$\frac{E|Z_{n1} - E(Z_{n1})|^{2+\delta}}{n^{\delta/2}[Var(Z_{n1})]^{1+\delta/2}} \to 0 \ \text{ as } \ n \to \infty. \tag{24}$$

It can be checked that

$$E|Z_{n1}|^{2+\delta} \simeq \frac{1}{h_n^{1+\delta}} g_T(x) \int_{-\infty}^{\infty} |K(y)|^{2+\delta} dy \tag{25}$$

and

$$Var(Z_{n1}) \simeq \frac{1}{h_n} \int_{-\infty}^{\infty} K^2(y) dy$$

by Theorem 2.1.1 in Prakasa Rao (1983). Using the $C_r$-inequality (cf. Loeve (1963)), it can be seen that the condition (25) holds. Hence the estimator $\hat{f}^{(1)}_{h_n}(x)$ is asymptotically normal after suitable scaling and we have the following result.

**Theorem 3.4:** Suppose the conditions $(C_4) - (C_6)$ hold. Then

$$\{\hat{f}^{(1)}_{h_n}(x) - E[\hat{f}^{(1)}_{h_n}(x)]\}\{Var[\hat{f}^{(1)}_{h_n}(x)]\}^{-1/2} \to N(0,1) \tag{26}$$

in distribution as $n \to \infty$.

The closeness of the approximation can be obtained by using the Berry-Esseen bound (Loeve (1963)). As a consequence of the Berry-Esseen bound, it follows that there exists an absolute constant $C$ such that

$$\sup_y |P(\frac{\hat{f}^{(1)}_{h_n}(x) - E[\hat{f}^{(1)}_{h_n}(x)]}{[Var(\hat{f}^{(1)}_{h_n}(x))]^{1/2}} \leq y) - \Phi(y)| \leq C \frac{E|Z_{n1}|^3}{n^{1/2}[Var(Z_{n1})]^{3/2}} \tag{27}$$

when $\delta = 1$ and the term on the right side of the above inequality is of the order

$$[nh_n g_T(x)]^{-1/2} (\int_{-\infty}^{\infty} |K(y)|^3 dy)(\int_{-\infty}^{\infty} K^2(y) dy)^{-3/2}.$$

Here the function $\Phi(.)$ denotes the standard normal distribution function. Uniform Berry-Esseen bounds can be obtained as in Theorem 2.1.19 of Prakasa Rao (1983), p. 62 under some additional conditions. We omit the details.

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