Global singularity theory of generic differentiable maps

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Abstract.

One of the most effective and familiar methods to study the topological structure of a given differentiable manifold is to use Morse functions defined on it. Such functions can be regarded as generic differentiable maps into the real line. Then, what happens if we consider generic maps into general dimensional Euclidean spaces or manifolds? This might have been a motivation of Whitney or Thom around the middle of the 20th century for studying singularities of differentiable maps between manifolds. In this survey article, following such an idea, we overview some studies of structures of manifolds by using generic differentiable maps, and some global studies of generic differentiable maps with singularities themselves, including recent developments.

§1. Morse functions

In what follows, manifolds and maps between them are assumed to be smooth of class C^∞.

There are various methods for studying the topology of manifolds. Presumably the most important one is to use Morse functions. A function $f : M \to \mathbf{R}$ on a manifold M is said to be a *Morse function* if all its critical points are nondegenerate and all the critical values are distinct ([29]). The important properties of such functions are as follows.

(1) Morse functions exist (in abundance) on any manifold.
(2) By examining the critical points and the gradient (or gradient-like) vector fields, we can get information on the homology or the handlebody decomposition of the manifold.

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(3) Two arbitrary Morse functions on a given manifold can be deformed to each other by a finite number of certain "operations".

For the above reasons, Morse functions have played an important role in differential topology, especially in the mid to late 20th century. Reeb [32] showed that if there exists a Morse function on a closed manifold (compact manifold without boundary) with exactly two critical points (i.e., exactly one maxima and one minima), then the manifold is necessarily homeomorphic to the standard sphere. This theorem was used in the discovery of 7–dimensional exotic spheres by Milnor [27], although it may not be so well-known. Furthermore, Smale [52] showed that homotopy spheres of dimension greater than or equal to 5 are homeomorphic to the standard sphere by eliminating certain critical points of Morse functions by using the so-called Whitney's trick (see also [30]).

Also in lower dimensions, e.g. in dimension 3, the existence of Heegaard decompositions in the smooth category, can be shown by using the existence of Morse functions. Furthermore, Kirby Calculus [15] which is used to represent 3–manifolds by framed links is also based on the theory of Morse functions.

§2. Generic maps

At about the same time as those turbulent times in differential topology, Whitney [58] and Thom [55] began to work on "generic" maps to Euclidean spaces $\mathbf{R}^p$ of general dimension, or more generally to p–dimensional manifolds. Functions can be regarded as maps to 1–dimensional manifolds, and the idea of working on generic maps to general dimensional manifolds is a very natural generalization. Presumably Thom was the first to try to obtain deeper information about manifolds by using maps corresponding to Morse functions. On the other hand, Thom [56] proposed the so-called *catastrophe theory* and claimed that the study of such generic maps can contribute to various fields of science as well. However, computers and the technology to use them were not well developed at that time, and not much progress has been made since then.

Here, the definition of a "generic map" may vary depending on the situation. There are various possible definitions: however, basically we always have that *an arbitrary map can be approximated by a generic map*. In other words, in the space of all C^∞–maps, with an appropriate topology, the subset consisting of all generic maps is dense. (For the topology of the mapping space, see, for example, [10].) In particular, Thom shows that for a given pair of manifolds M, N, and an arbitrary submanifold W in the jet space $J^r(M, N)$, the set of maps $f : M \to N$

whose r–jet extension $j^r f : M \to J^r(M, N)$ is transverse to W is dense in the mapping space $C^\infty(M, N)$, where $r \geq 1$ is an integer. (This is called *Thom's jet transversality theorem.* For details, see [10], etc.) In fact, generic maps can often be formulated using such transversality. For example, the non-degeneracy of a critical point of a function can be formulated in these terms. Note that generic maps can also be formulated in terms of *stable maps* whose differential topological properties do not change even after a slight perturbation [10].

However, later, when dealing with maps, it turned out that it was surprisingly difficult to handle local singularities that correspond to non-degenerate critical points in the case of functions, and various problems arose. For smooth maps $f : M \to N$ between manifolds with $\dim M \geq \dim N$, as the dimension of the manifold N increases, the types of the singularities that appear generically are no longer finite, and even moduli can appear. (For details, see Mather's series of papers [21]–[26].)

Maybe because of this, little progress has been made since then in the study of manifolds using maps rather than functions. In the meantime, Burlet–de Rham [5] considered generic smooth maps of 3–dimensional closed manifolds into the plane $\mathbf{R}^2$. In this case, there appear three types of singularities, definite folds, indefinite folds, and cusps (see Definition 3.1). Those maps which have only definite fold singularities are called *special generic maps* (terminology introduced originally in French in [5]), and they completely determined 3–manifolds that allow such maps. This result did not seem to attract much attention at the time. More than a decade later, Kushner–Levine–Porto [18] began to study general generic maps of 3–manifolds into the plane, and Porto–Furuya [31] attempted to extend Burlet–de Rham's result to higher dimensions. The global study of such generic maps gradually began.

In this survey paper, we will consider the case of generic maps $f : M \to N$ between manifolds, with $\dim M \geq \dim N$. In the case where $\dim M < \dim N$, it should be noted that the global study of such maps has been developed by Szűcs and his group (see [54], etc.).

§3. Special generic maps

First, let us introduce some of the singularities that will appear frequently in this article.

Definition 3.1. For a map $f : M \to N$ between manifolds with $n = \dim M \geq \dim N = p \geq 1$, a point $q \in M$ is called a *fold* if f can be represented as

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, \pm x_p^2 \pm \cdots \pm x_n^2)$$

by certain local coordinates around q and $f(q)$. Furthermore, if f can be represented as

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, x_p^3 - x_{p-1}x_p \pm x_{p+1}^2 \pm \cdots \pm x_n^2),$$

then it is called a *cusp* (see Fig. 1).

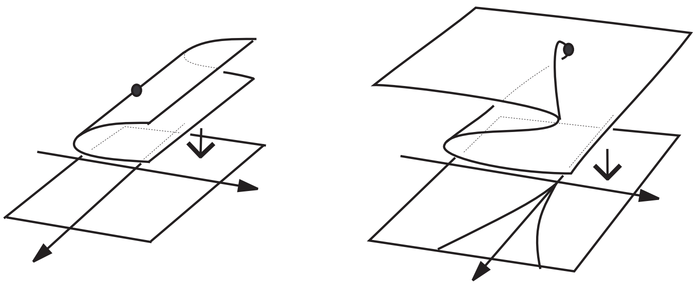

Fig. 1. Fold and cusp for the case of $n = p = 2$

Furthermore, in the quadratic form of the final component function appearing in the normal form of a fold, the smaller of the number of $+$ signs and the number of $-$ signs is called the *absolute index*. When it is equal to 0, the point is called a *definite fold*, and otherwise, it is called an *indefinite fold*.

The special generic maps defined by Burlet–de Rham as above are naturally extended to general dimensions. A map $f : M \to \mathbf{R}^p$ of a manifold with $\dim M \geq p$ is a *special generic map* if it has only definite folds as its singular points. When the target dimension is $p = 1$, f is a function, and in this case it is a Morse function which has only maxima and minima as its critical points. For example, it is easy to see that the standard n–dimensional sphere S^n has a special generic map into $\mathbf{R}^p$ for each $1 \leq p \leq n$. In fact, the following is known.

Theorem 3.2 ([35]). *For a connected closed manifold M of dimension $n \geq 1$, there exists a special generic map $M \to \mathbf{R}^p$ for all p with $1 \leq p \leq n$ if and only if M is diffeomorphic to the standard sphere S^n.*

By the Reeb theorem [32] mentioned in §1, for a given closed connected manifold, if we have a special generic map (or more precisely, function) into $\mathbf{R}$, then we know that the manifold is homeomorphic to S^n. However, since higher dimensional exotic spheres also admit such functions, we cannot identify the differentiable structures only by the existence of such functions. The above theorem implies that by increasing the target dimension, the relevant differentiable structure of S^n can be identified.

Subsequent studies have shown that the existence of a special generic map very often reflects the differentiable structure of its source manifold. For example, in dimension 4, there have been found a lot examples of pairs of mutually homeomorphic manifolds such that there is a special generic map into $\mathbf{R}^3$ on one of them and that no such map exists on the other (see, for example, [45, 46]). Moreover, even for noncompact manifolds, by considering proper special generic maps, it is possible to characterize the differentiable structure of the standard $\mathbf{R}^4$ ([41]).

Remark 3.3. As far as the author knows, special generic maps first appeared in the literature in Calabi's paper [6], where such maps have been used to get a result about the differentiable structure of a Riemannian manifold with pinched curvatures.

Remark 3.4. Cobordism group of special generic maps have been introduced in [36]. Then, the author [38] showed that for dimensions greater than or equal to 6, the cobordism group of special generic functions is isomorphic to the h-cobordism group of homotopy spheres of the same dimension.

Remark 3.5. The h-cobordism group of homotopy spheres has a natural filtration by subgroups introduced by Gromoll [11]. On the other hand, Wrazidlo [59] introduced the notion of *standard* special generic maps on homotopy spheres and defined another filtration of the h-cobordism group of homotopy spheres using the existence of such standard special generic maps. He also showed that the Gromoll filtration is included in such a filtration. Finally, the author [43] showed that in fact the two filtrations coincide with each other.

Remarks 3.4 and 3.5 reveal even stronger relationships between the special generic maps and differentiable structures on manifolds.

§4. Case of dimension 3

Consider a generic map $f : M \to \mathbf{R}^2$ of a closed orientable 3–dimensional manifold M into the plane $\mathbf{R}^2$. As mentioned above, the singularities are definite folds, indefinite folds and cusps. By the cusp elimination theorem of Levine [19], we see that f can be deformed homotopically so that it has no cusps. On the other hand, the singular point set $S(f)$ of f is a closed 1–dimensional submanifold of M (or a link in M), and if f has no cusps, then $f|_{S(f)}$ is an immersion. We may further assume that it is an immersion with normal crossings. Since it has at most double points, for each $y \in f(S(f))$, $f^{-1}(y) \cap S(f)$ consists of one or two points. The generic map is said to be *simple* if each component of $f^{-1}(y)$ contains at most one singular point for every $y \in f(S(f))$ [20]. For example, if $f|_{S(f)}$ is an embedding, then f is simple. Note also that a special generic map introduced above is simple. Then, the following holds.

Theorem 4.1 ([37]). *A closed orientable 3–dimensional manifold admits a simple generic map into $\mathbf{R}^2$ without cusps if and only if it is a graph manifold.*

Remark 4.2. It should be noted that the above result holds for generic maps which may possibly have cusps, although in [37], this is not proved. In fact, if we start with a simple generic map, by arranging Levine's cusp elimination process appropriately, we can get a generic map without cusps which is simple.

Here, a closed orientable 3–dimensional manifold is a *graph manifold* if it is the union of some S^1–bundles over compact surfaces attached along their torus boundaries. Recently Kitazawa and the author [16] have shown that on such a graph manifold M, there exists a generic map $f : M \to \mathbf{R}^2$ without cusps such that $f|_{S(f)}$ is an embedding onto a concentric family of circles centered at the origin. (See [17] for an extension of this result to higher dimensions. In addition, in [17, 37] some classification results of such maps by certain natural equivalence relations are obtained.)

Theorem 4.1 has since been developed into several directions of study: in particular, some deep results have been obtained concerning the relationship between the geometric structures of 3–manifolds and the complexity of generic maps into $\mathbf{R}^2$ (see [7, 12, 14]).

Another important aspect of generic maps on 3–manifolds is that the quantum invariants of 3–manifolds first defined by Reshetikhin and Turaev [33] have since been shown to be formulated using generic maps of 3–manifolds into $\mathbf{R}^2$ (so-called "shadows") (see, for example, [57]).

§5. Case of dimension 4

In this section, let M be a closed connected orientable 4–dimensional smooth manifold, and consider a generic map $f : M \to \mathbf{R}^3$. The only singularities that appear in this case are definite folds, indefinite folds, cusps, and swallowtails. Here, although we do not give a definition of a swallowtail (for details, see [40, 49], for example), please note that they appear discretely. Ando [1] showed that the swallowtails can always be eliminated by homotopy.

In general, for a "singularity type" Σ, let $\Sigma(f)$ be the subset of M of all singular points of f of type Σ. In many cases, its closure $\overline{\Sigma(f)}$ represents a homology class and its Poincaré dual $[\Sigma(f)]^* \in H^*(M)$ with appropriate coefficient gives a cohomology class of M. Furthermore, it is known that it does not depend on a particular map f. It is also known that it can be written in terms of polynomials, called *Thom polynomials* (see, for example, [13]), of characteristic classes of M. Thus, if the cohomology class does not vanish, then the singularity of that type cannot be eliminated. On the other hand, for a generic map $f : M \to \mathbf{R}^3$ on a 4–dimensional manifold the Thom polynomial for cusps is a 3–dimensional cohomology class, which is known to always vanish (for example, see [47]). In this case, the following is known.

Theorem 5.1 ([34, 39])**.** *Let M be a closed connected orientable 4–dimensional manifold. Then, there exists a generic map $f : M \to \mathbf{R}^3$ that has no cusps if and only if its cohomology ring $H^*(M; \mathbf{Z})$ is not isomorphic to $H^*(\pm\mathbf{C}P^2; \mathbf{Z})$ or $H^*(\pm(\mathbf{C}P^2 \sharp \mathbf{C}P^2); \mathbf{Z})$.*

Thus, for $\pm\mathbf{C}P^2$ and $\pm(\mathbf{C}P^2 \sharp \mathbf{C}P^2)$, the cusp singularities cannot be eliminated even though the Thom polynomial vanishes. In fact, denoting by $C(f)$ the set of cusp singular points of a generic map $f : M \to \mathbf{R}^3$ as above, we have that the Thom polynomial for $[C(f)]^* \in H^3(M; \mathbf{Z}_2)$ coincides with $w_3(M)$, the 3rd Stiefel–Whitney class of M, and $w_3(M)$ always vanishes for closed orientable 4–dimensional manifolds (for details, see [50, 55]).

Note that a generic map without a cusp has only folds as its singularities. In general, a map with only folds as singularities is called a *fold map*. The existence of such maps has been studied by Eliashberg [8] and Ando [2] based on the homotopy principle. The proof of the existence of the above theorem is based on such results.

As a corollary to the above theorem, we have the following, for example.

Corollary 5.2. *For an arbitrary closed connected orientable* 4*–dimensional manifold* M*, the connected sums*

$$M\sharp(S^2 \times S^2) \text{ and } M\sharp\mathbf{C}P^2\sharp(-\mathbf{C}P^2)$$

always admit generic maps into $\mathbf{R}^3$ *without cusps.*

Now, let us return to generic maps $f : M \to \mathbf{R}^3$ in general. As we have seen above, it is important to know not only the types of singularities of a given map, but also the position of the singular point set in the source manifold. From a somewhat different point of view, we can also consider the types of $f^{-1}(y)$ for a point $y \in \mathbf{R}^3$ in the target. If the chosen point $y \in \mathbf{R}^3$ is a regular value, then $f^{-1}(y)$ is diffeomorphic to a finite disjoint union of circles. What happens if it is a singular value? In such a case, $f^{-1}(y)$ is called a *singular fiber*, and it is classified by a certain equivalence relation [40]. (It is a bit surprizing to see that this theory has also been applied to data visualization together with some implementation. For details, see [51].) Furthermore, it is known that a special singular fiber among them gives information on the signature of a 4–dimensional manifold as follows.

Theorem 5.3 ([42, 49]). *For a closed oriented* 4*–dimensional manifold* M *and a generic map* $f : M \to \mathbf{R}^3$*, for each* III^8*–type singular fiber (see Fig.* 2*), we can define the sign* ± 1*, and the sum of their signs over all* III^8*–type singular fibers of* f *is equal to the signature* $\sigma(M)$ *of* M*, defined by using the intersection form.*

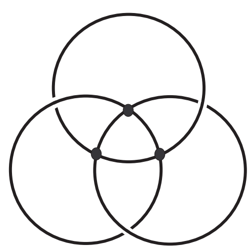

Fig. 2. III^8–type singular fiber (the black dots correspond to three fold points in the fiber)

For the definition of the signature $\sigma(M)$ of a closed oriented 4–dimensional manifold M, the reader is referred to [28], for example.

Related to the signature of a closed oriented 4–dimensional manifold is the existence problem of a generic map of such a 4–dimensional manifold into $\mathbf{R}^4$ with only fold and cusp singularities (see [48]).

Theorem 5.4. *Let M be a closed connected oriented 4–dimensional manifold. Then, there exists a smooth map $f : M \to \mathbf{R}^4$ with only fold and cusp singularities if and only if $\sigma(M) = 0$.*

The above theorem means that for elimination of singularities other than fold and cusps, the Thom polynomials are the unique obstructions. For details, see [48].

Let us go back to generic maps into $\mathbf{R}^3$. If we consider a generic map $f : M \to \mathbf{R}^3$ on a closed oriented 4–dimensional manifold M whose signature $\sigma(M)$ does not vanish, then its singular point set $S(f)$ is a closed surface in M and the map $f|_{S(f)}$ always has a triple point. This implies that the complexity of a manifold is, to some extent, reflected in the complexity of the map defined on it. In fact, Theorem 4.1 asserts something similar. For a generic map $f : M \to \mathbf{R}^2$ on a closed orientable 3–manifold M which is not a graph manifold, such as a hyperbolic 3–manifold, $f|_{S(f)}$ always has double points (see [12, 14] for more detailed results).

This raises the question of when $f|_{S(f)}$ can be an embedding for a generic map f. A smooth map f such that $f|_{S(f)}$ is a topological embedding is said to be *image simple*. In this case, the following is known. Here, a *Lefschetz singular point* of a smooth map of an oriented 4–dimensional manifold into an oriented surface is a singular point around which the map is expressed as $(z, w) \mapsto zw$ using C^∞ complex coordinates compatible with the orientations. (In fact, a smooth map that has only Lefschetz singular points is called a *Lefschetz fibration*, and a map that has only Lefschetz singular points and indefinite fold singularities is called a *broken Lefschetz fibration*, both of which play an important role in 4–dimensional manifold theory.)

Theorem 5.5 ([3, 4])**.** *Let M be a closed connected orientable 4–dimensional manifold. Then, there always exist an* image simple *smooth map $f : M \to S^2$ and an* image simple *generic map $g : M \to \mathbf{R}^2$ with the following properties.*

(1) *The singularities of f are indefinite folds and Lefschetz singularities and the image of the singular point set is as shown in Fig. 3.*

(2) *The image of the singular point set $S(g)$ of g is as shown in Fig. 4. (This is the image of a 4–dimensional manifold. The map gives a so-called* simplified trisection *of the 4–dimensional manifold.)*

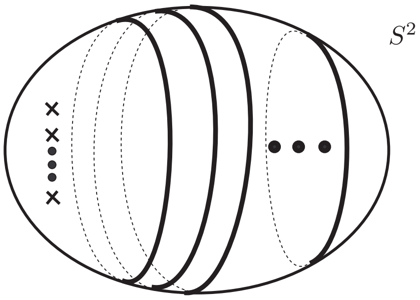

Fig. 3. The singular value set of f. The sign $\times$ depicts the image of a Lefschetz singular point.

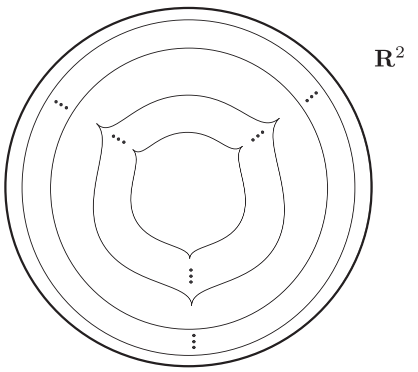

Fig. 4. The singular value set of g. The outermost circle is the image of definite folds, and the others are the images of indefinite folds and cusps.

§6. Case of general dimensions

In the previous section, we found that it is possible to simplify the singular value set in the case of maps of 4–dimensional manifolds into surfaces. By generalizing the method, we have recently shown the following [44].

Theorem 6.1. *Suppose that the dimension n of a closed manifold M is at least 4. In this case, an arbitrary map $M \to S^2$ can be homotoped to a generic map with the following properties.*

(1) *When $n = \dim M$ is odd, f has only folds of absolute index $(n-1)/2$.*

(2) *When $n = \dim M$ is even, f has only folds of absolute index $(n-2)/2$, and at most one cusp. Moreover, f is* image simple.

Note that the above theorem is an assertion about maps into S^2. For maps into $\mathbf{R}^2$, we have the following.

Theorem 6.2. *If the dimension of a closed manifold M is an even integer greater than or equal to 4, then it always admits an* image simple *generic map $f : M \to \mathbf{R}^2$ which has at most one cusp.*

Smale [53] has shown that if a closed manifold M is of dimension at least 6 and if it is k–connected (i.e., if the homotopy groups all vanish up to dimension k) with $k \geq 1$, then M admits a Morse function which has no critical points of index i with $1 \leq i \leq k$. On the other hand, for example, for a generic map $f : M \to \mathbf{R}^2$, it is known that if it does not have folds of absolute index 1, then the fundamental group of M is necessarily a free group. It is not known whether the converse is true or not. Thus, it is not clear how the topology of a manifold affects the absolute indices of folds that appear for generic maps defined on the manifold. This might be a major problem for the future. See also Remark 1.6 of [9]. (A related study can be found in [60].)

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