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AN APPLICATION OF BORSUK'S ANTIPODAL THEOREM TO SET-EVALUATION

HIDEFUMI KAWASAKI

Abstract. When evaluating two students using two types of tests, assigning weights to their scores can be a useful approach. Consider the following scenarios: Suppose student a_1's scores are (40, 80) and student a_2's scores are (60, 60). With equal weights of 1:1, their overall evaluations are the same. With weights of 2:1, student a_2 outperforms student a_1. With weights of 1:2, the conclusion is reversed. This observation indicates that one can control the overall evaluation by weights. This paper shows the same is true for not only individuals but also groups (sets). The mathematical foundation for this approach is based on Borsuk ' s antipodal theorem ([1]).

1. Introduction

Topology has long been applied to protein structure analysis and data analysis. While fixed-point theorems are essential tools in optimization theory and game theory, it cannot be said that topology has been fully utilized in these fields. The author has been working on applying Borsuk's antipodal theorem to optimization theory ([3]-[6]). Borsuk's antipodal theorem states that for any continuous mapping φ from the n-sphere S^n to the n-dimensional Euclidean space $\mathbb{R}^n$, there exists a point $\boldsymbol{u} \in S^n$ such that $\varphi(\boldsymbol{u}) = \varphi(-\boldsymbol{u})$. This paper shows Borsuk's antipodal theorem is useful for evaluating sets.

In Section 2, we introduce a family of parametric optimization problems with parameter $\boldsymbol{u} \in S^n$, and present an antipodal theorem for it. In Section 3, we provide a set evaluation theorem as a corollary of our antipodal theorem. In Section 4, we will deal with a positive weight.

2. Antipodal theorem for parametric optimization problem

For any point $\boldsymbol{u} = (u_1, \ldots, u_{n+1})$ of the n-sphere $S^n = \{\boldsymbol{u} \mid u_1^2 + \cdots + u_{n+1}^2 = 1\}$, we write $\boldsymbol{u} = (u, u_{n+1})$. D^n denotes the n-disk $\{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$. Throughout this paper, we assume for any $i = 1, \ldots, n$ that $A_i \subset \mathbb{R}^n$ is a nonempty compact set in $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \times D^n \to \mathbb{R}$ is a continuous function that is antipodal

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w.r.t. $u \in D^n$, that is, $f_i(x, -u) = -f_i(x, u)$ for any $(x, u) \in \mathbb{R}^n \times D^n$. Then, it holds that $f_i(x, \mathbf{0}) = 0$.

We define an optimal-value function κ_i with a shift term by

$$\kappa_i(\boldsymbol{u}) := \max_{x \in A_i} f_i(x, u) - \gamma_i u_{n+1}, \tag{2.1}$$

where γ_i is a given real constant. Then, κ_i is continuous. Applying Borsuk's antipodal theorem to $\kappa := (\kappa_1, \ldots, \kappa_n)$, we obtain the following.

Theorem 2.1. (1) *For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$, there exists a point $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that*

$$\max_{x \in A_i} f_i(x, u) + \min_{x \in A_i} f_i(x, u) = 2\gamma_i u_{n+1} \quad (i = 1, \ldots, n). \tag{2.2}$$

In particular, when γ_i is nonzero for some i, u is a nonzero vector.

(2) *There exists a nonzero $u \in D^n$ such that $\max_{x \in A_i} f_i(x, u) + \min_{x \in A_i} f_i(x, u)$ does not depend on $i = 1, \ldots, n$.*

(3) *If there is no nonzero $v \in D^n$ satisfying*

$$\max_{x \in A_i} f_i(x, v) + \min_{x \in A_i} f_i(x, v) = 0 \ (i = 1, \ldots, n), \tag{2.3}$$

then for any $\gamma \in \mathbb{R}^n$, there exists an interior point u of D^n such that $\max_{x \in A_i} f_i(x, u) + \min_{x \in A_i} f_i(x, u)$ $(i = 1, \ldots, n)$ are proportionate to γ_i $(i = 1, \ldots, n)$.

Proof. By Borsuk's antipodal theorem, there exists a point $\boldsymbol{u} \in S^n$ such that $\kappa(\boldsymbol{u}) = \kappa(-\boldsymbol{u})$. Since $f_i(x, -u) = -f_i(x, u)$, we have

$$\max_{x \in A_i} f_i(x, u) - \gamma_i u_{n+1} = \max_{x \in A_i} f_i(x, -u) + \gamma_i u_{n+1} = -\min_{x \in A_i} f_i(x, u) + \gamma_i u_{n+1},$$

which implies (2.2). Suppose that $u = \mathbf{0}$, then it follows from (2.2) that

$$2\gamma_i u_{n+1} = \max_{x \in A_i} f_i(x, \mathbf{0}) + \min_{x \in A_i} f_i(x, \mathbf{0}) = 0.$$

Since $\gamma_i \neq 0$ for some i, we have $u_{n+1} = 0$, which contradicts $(u, u_{n+1}) \neq (\mathbf{0}, 0)$. Therefore, $u \neq \mathbf{0}$. (2) follows from (1) by taking $\gamma_i = 1$ for every i. (3) It follows from (1) and the assumption on (2.3) that $u_{n+1} \neq 0$. Hence, we obtain the desired proportional relation. □

3. Set evaluation theorem

When we take the inner product $\langle u, x \rangle = u_1 x_1 + \cdots + u_n x_n$ as $f_i(x, u)$, we have

$$\kappa_i(\boldsymbol{u}) = \max_{x \in A_i} f_i(x, u) - \gamma_i u_{n+1} = \delta^*(u \mid A_i) - \gamma_i u_{n+1},$$

where $\delta^*(u \mid A_i)$ is the support function of A_i. Further, (2.2) reduces to

$$c_i(u) := \frac{\delta^*(u \mid A_i) + \delta_*(u \mid A_i)}{2} = \gamma_i u_{n+1}, \tag{3.1}$$

where $\delta_*(u \mid A_i) := \min_{x \in A_i} \langle u, x \rangle$. Figure 1 indicates that $c_i(u)$ is a kind of center of A_i in the direction u.

Theorem 3.1 is a direct consequence of Theorem 2.1.

Theorem 3.1. *(Set evaluation theorem)*

(1) *For any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists a point $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that*

$$c_i(u) = \gamma_i u_{n+1} \ (i = 1, \ldots, n). \tag{3.2}$$

In particular when γ_i is nonzero for some i, u is a nonzero vector.

(2) *There exists a nonzero $u \in D^n$ such that $c_i(u)$ does not depend on $i = 1, \ldots, n$.*

(3) *If there is no nonzero $v \in D^n$ such that $c_i(u) = 0$ for any i, then for any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists an interior point u of D^n such that*

$$c_1(u) : \cdots : c_n(u) = \gamma_1 : \cdots : \gamma_n. \tag{3.3}$$

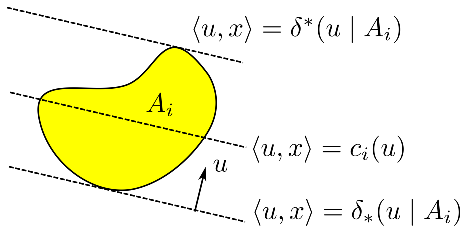

Figure 1. The hyperplane $\langle u, x\rangle = c_i(u)$ equally divides the width of A_i in the direction u.

Theorem 3.1 (3) demonstrates the evaluation method with weights is applicable to sets. In Figures 2 and 3, there is no line passing the origin that equally divides the width of A_1 and A_2 in any direction. Hence, it follows from (3) that for any γ_1 and γ_2, there exists a weight $u \in \mathbb{R}^2$ such that $c_1(u) : c_2(u) = \gamma_1 : \gamma_2$.

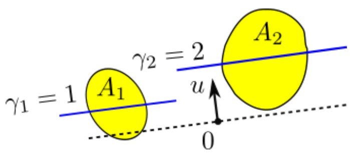

Figure 2. $c_1 : c_2 = 1 : 2$

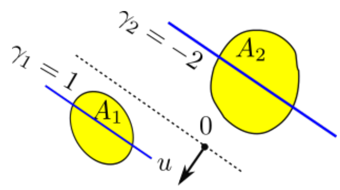

Figure 3. $c_1 : c_2 = 1 : -2$

In particular when A_i is symmetric w.r.t. a point p_i, the weight $\boldsymbol{u}$ in Theorem 3.1 can be computed by a linear equation.

Lemma 3.2. *If A_i is symmetric w.r.t. a point $p_i \in \mathbb{R}^n$, then $c_i(u) = \langle p_i, u\rangle$. So, (3.2) reduces to the linear equation*

$$P^T u = u_{n+1}\gamma, \tag{3.4}$$

where $P := (p_1 \ \dots \ p_n)$*. Further, there is no nonzero* $u \in \mathbb{R}^n$ *satisfying* $c_i(u) = 0$ $(i = 1, \dots, n)$ *if and only if* $p_1, \dots, p_n$ *are linearly independent. In such a case, the solution* $(u, u_{n+1}) \in S^n$ *of (3.4) is obtained by solving*

$$P^T u = \gamma, \tag{3.5}$$

and normalizing $(u, 1)$*.*

Proof. . Since $C_i := A_i - p_i = \{x - p_i \mid x \in A_i\}$ is symmetric w.r.t. the origin, $x \in C_i$ is equivalent to $-x \in C_i$. Hence

$$\delta_*(u \mid C_i) = \min_{x \in C_i} \langle u, x\rangle = -\max_{x \in C_i} \langle u, -x\rangle = -\max_{-x \in C_i} \langle u, -x\rangle = -\delta^*(u \mid C_i).$$

Thus,

$$\begin{aligned} 2c_i(u) &= \delta^*(u \mid A_i) + \delta_*(u \mid A_i) \\ &= \delta^*(u \mid C_i) + \langle u, p_i\rangle + \delta_*(u \mid C_i) + \langle u, p_i\rangle \\ &= 2\langle p_i, u\rangle. \end{aligned}$$

Therefore, $c_i(u) = \langle p_i, u\rangle$, and (3.2) reduces to $p_i^T u = u_{n+1}\gamma_i$ $(i = 1, \dots, n)$.

Further, $c_i(u) = 0$ $(i = 1, \dots, n)$ has no nonzero solution u if and only if $(p_1, \dots, p_n)^T u = \mathbf{0}$ has no nonzero solution, which is equivalent to $p_1, \dots, p_n$ are linearly independent. Then, (3.5) has a unique solution. □

4. Positive weight

As we have seen in Figures 2 and 3, some components of the weight u may be negative. When each A_i is symmetric w.r.t. a point p_i, the weight u is a solution of the linear equation $P^T u = \gamma$. Hence, the positive (nonnegative) weight reduces to the positive (nonnegative) solution of the linear equation.

For any $n \times n$ matrix Q, The (i, j)-minor of Q, denoted $|Q_{ij}|$, is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting i-th row and j-th column of Q. The (i, j) cofactor of Q is defined by $\widetilde{q}_{ij} := (-1)^{i+j}|Q_{ij}|$, and the cofactor matrix of Q is defined by $\widetilde{Q} := (\widetilde{q}_{ij})^T$.

When P is nonsingular, the solution of $P^T u = \gamma$ is expressed as

$$u = (P^T)^{-1}\gamma = \frac{1}{|P^T|}\widetilde{P^T}\gamma = \frac{1}{|P|}\widetilde{P}^T\gamma. \tag{4.1}$$

Theorem 4.1. *Assume that* A_i *is symmetric w.r.t.* p_i *for any* $i = 1, \dots, n$ *and* $p_1, \dots, p_n$ *are linearly independent. Let* $\widetilde{p}_{ij}$ *be the* (i, j) *cofactor of* P*. Then, for any* $\gamma > \mathbf{0}$*, there exists a weight* $u > \mathbf{0}$ *such that (3.4) if and only if*

$$|P|\,\widetilde{p}_{ij} > 0 \ (1 \le i, j \le n). \tag{4.2}$$

For any $\gamma \ge \mathbf{0}$*, there exists a weight* $u \ge \mathbf{0}$ *such that (3.4) if and only if*

$$|P|\,\widetilde{p}_{ij} \ge 0 \ (1 \le i, j \le n). \tag{4.3}$$

Proof. First, we note that (3.4) has a positive solution (u, u_{n+1}) if and only if $P^T u = \gamma$ has a positive solution u. It follows from (4.1) that $P^T u = \gamma$ has a positive solution u for any $\gamma > \mathbf{0}$ if and only if (4.2) holds. The latter part can be proven in the same way. □

Next, let us change our approach and find the set of all γ that are achieved by a positive (nonnegative) weight u:

$$\Gamma_{++} := \{P^T u \mid u > \mathbf{0}\}, \quad \Gamma_{+} := \{P^T u \mid u \geq \mathbf{0}\}.$$

For comparison, we set $\Gamma := \{P^T u \mid u \in \mathbb{R}^n\}$. Theorem 4.2 is easily proven.

Theorem 4.2. *Assume that A_i is symmetric w.r.t. p_i* [1] *for any $i = 1, \dots, n$, and set $P^T = (q_1, \dots, q_n)$. Then,*

(1) $\Gamma_{++} = \{\sum_{i=1}^n u_i q_i \mid u_i > 0 \ (i = 1, \dots, n)\}$.
(2) Γ_{+} *equals to the convex cone generated by $q_1, \dots, q_n$.*
(3) *For any $\gamma > \mathbf{0}$ there exists a weight $u > \mathbf{0}$ such that $P^T = \gamma$ if and only if the interior of the convex cone generated by $q_1, \dots, q_n$ contains the positive orthant of $\mathbb{R}^n$. Then, $p_1, \dots, p_n$ have to be linearly independent.*
(4) *For any $\gamma \geq \mathbf{0}$, there exists a weight $u \geq \mathbf{0}$ such that (3.4) if and only if the convex cone generated by $q_1, \dots, q_n$ contains the nonnegative orthant of $\mathbb{R}^n$. Then, $p_1, \dots, p_n$ have to be linearly independent.*

Proof. Both (1) and (2) are trivial. (3) The former part is a direct consequence of (1). Then, the rank of P^T is equal to n. (4) can be proven in the same way. □

Example 4.3. In Figure 4, the centers of A_1 and A_2 are $p_1 = (60, 60)$ and $p_2 = (40, 80)$, respectively. Then, Γ_+ is the convex cone generated by $q_1 = (40, 60)$ and $q_2 = (80, 60)$. If weights are allowed to be negative, then any $\gamma \in \mathbb{R}^2$ is achieved by a weight vector. In Figure 5, the centers of A_1 and A_2 are $p_1 = (60, 60)$ and $p_2 = (80, 80)$, respectively. Then, Γ_+ is the ray extending from the origin to the point $q_1 = (60, 80)$. Namely, only $\gamma_1 : \gamma_2 = 3 : 4$ is achieved by positive weights. On the other hand, Γ equals to the line generated by $q_1 = (60, 80)$.

5. Concluding remarks

The set evaluation theorem is applicable to any nonempty compact sets $A_1, \dots, A_n$. In particular when every A_i is symmetric, the weight $u \in \mathbb{R}^n$ is obtained by solving linear equation $P^T u = \gamma$. When A_i is not symmetric, we need some alternative to p_i. For example, the geometrical center and the center point ([2]) are candidates. This is a matter for future work.

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[1] $p_1, \dots, p_n$ can be linearly dependent.

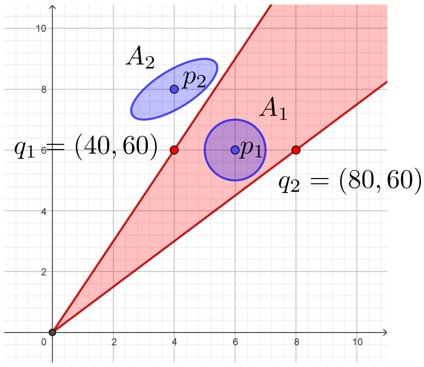

FIGURE 4. Γ_+ is the convex cone generated by $q_1 = (40, 60)$ and $(80, 60)$.

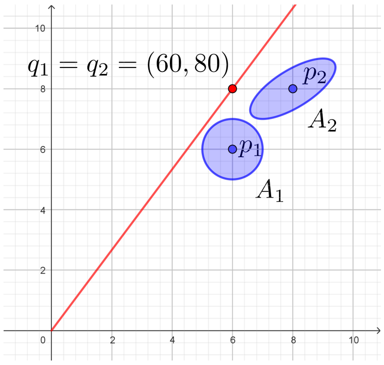

FIGURE 5. $\gamma_1 : \gamma_2 = 3 : 4$

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