

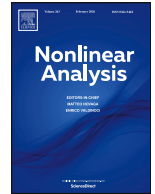
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Research Paper

Partial smoothing effects of local mild solutions of the Keller–Segel system with logistic growth in Besov spaces

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ABSTRACT

We consider the Keller–Segel system of parabolic-elliptic type with logistic growth $u - |u|^{k-l}u^l$ in the whole space $\mathbb{R}^n$, where $n \geq 3$, $1 < \kappa < 2$, and $l \in \{0, 1\}$. We show the existence and uniqueness of local mild solutions u for initial data $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ under the conditions $n/2 < p < n$, $1 \leq q \leq \infty$, and $2p/n < \kappa < 2$. In addition, partial smoothing effects of the mild solutions u are investigated. More precisely, we show that u satisfy the original system in a classical sense and have the property $u \in C_{loc}^{\kappa+1}((0, T]; L^\infty(\mathbb{R}^n)) \cap L_{loc}^\infty((0, T]; C^{\kappa+2}(\mathbb{R}^n))$. According to the regularities of the term $|u|^{k-l}u^l$ with the power nonlinearity, such regularities for u seem to be optimal under the general framework beyond the physical sense.

1. Introduction

Let us consider the initial value problem for the Keller–Segel system of parabolic-elliptic type with logistic growth in the whole space $\mathbb{R}^n$, $n \geq 3$;

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v) + \alpha u - \beta_0 |u|^{k-l} u^l, & t > 0, x \in \mathbb{R}^n, \\ 0 = \Delta v - \beta_1 v + u, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = a(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $u = u(t, x)$ and $v = v(t, x)$ are the unknown functions standing for the density of the cell and the concentration of the chemo-attractant, respectively. In addition, $a = a(x)$ is the given initial datum and $\alpha, \beta_0, \beta_1 \in [0, \infty)$, $1 < \kappa < 2$, and $l \in \{0, 1\}$ are the given constants.

In this paper, we show the existence and uniqueness of local mild solutions of (1.1) with initial data $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ under the conditions $n/2 < p < n$, $1 \leq q \leq \infty$, and $2p/n < \kappa < 2$, where $B_{p,q}^s(\mathbb{R}^n)$ denotes the Besov spaces defined in Section 2. In addition, we also show that the mild solution of (1.1) has partial smoothing effects so that the mild solution satisfies the original system (1.1) in a classical sense, even for the quite singular initial data.

In the case of $\alpha = \beta_0 = 0$, the corresponding system to (1.1) is known as the usual Keller–Segel system of parabolic-elliptic type [1]. This system is a mathematical model describing the phenomenon of chemotaxis, and thus there are various works on mathematical analysis. One of the typical mathematical topics is a construction of solutions with initial data a in various function spaces: In the case of $\alpha = \beta_0 = \beta_1 = 0$, we observe that the pair $(u_\lambda(t, x), v_\lambda(t, x)) := (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x))$ of functions satisfies the first and second equations of (1.1) for any $\lambda > 0$, provided that (u, v) is a global smooth solution of (1.1). Since the corresponding initial data necessarily satisfy $u_\lambda(0, x) = \lambda^2 u(0, \lambda x)$, we observe that $L^{n/2}(\mathbb{R}^n)$, $\dot{H}^{-2+n/p,p}(\mathbb{R}^n)$, and $\dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n)$ are scaling invariant spaces to (1.1).

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with $\alpha = \beta_0 = \beta_1 = 0$, where $\dot{H}^{s,p}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^s(\mathbb{R}^n)$ denote the homogeneous Sobolev and Besov spaces, respectively. For the previous works on the construction of solutions in the scaling invariant spaces framework, we refer to, e.g., [2–11]. We also mention the results in the case of the Keller–Segel system of parabolic-parabolic type such as [8,12–15].

Next we consider the system (1.1) with general $\alpha, \beta_0 \in (0, \infty)$ and $\kappa > 1$. The term $\alpha u - \beta_0 |u|^{\kappa-l} u^l$ is called logistic growth, which describes the population fluctuation due to the proliferation and death of cells. Mention that the system (1.1) is also considered and studied, as well as the original Keller–Segel system, as it is expected that the logistic growth prevents an excessive increase of the density of the cell u , i.e., blow-up phenomena; for the results, we refer to, e.g., [16–21]. Likewise, the case of parabolic-parabolic type is investigated by [22–27]; note that these results are concerned with the case of the Neumann boundary condition in a bounded smooth domain $\Omega \subset \mathbb{R}^n$. In addition, they often assume that the initial datum a has a suitable regularity. In this paper, we consider the case of $a \in \dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n)$ with $n/2 < p < n$; we handle the initial data a belonging to the Besov spaces with *negative* regularity indices. We also remark that the following relation

$$\dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n) = (\dot{B}_{p,q}^{-2+n/p} + L^p)(\mathbb{R}^n) := \left\{ \varphi = \varphi_0 + \varphi_1 \mid \varphi_0 \in \dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n), \varphi_1 \in L^p(\mathbb{R}^n) \right\}$$

holds under the condition $n/2 < p < \infty$ due to (2.1) below. In particular, we have $\dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n) \subset \dot{B}_{p,q}^{-2+n/p}(\mathbb{R}^n)$, and thus our function spaces are *larger* than the corresponding scaling invariant spaces. However, in this paper, we concentrate only on the local solvability of (1.1) as the system (1.1) with general $\alpha, \beta_0 \in (0, \infty)$ and $\kappa > 1$ does *not* have a scaling invariant structure, unlike the case of $\alpha = \beta_0 = \beta_1 = 0$.

The main part of this paper is to investigate the partial smoothing effects of solutions of (1.1); let us refer to the previous works on the smoothing effects of solutions to the Keller–Segel system of parabolic-parabolic type. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with a smooth boundary in what follows. Tao and Winkler [28] considered the case of $n = 3$ and showed that if the initial data $u_0 \in C(\bar{\Omega})$ and $\nabla v_0 \in (L^\infty(\Omega))^3$, then there exist $t_* > 0$ and a global solution that has sufficient regularities on $(t_*, \infty) \times \bar{\Omega}$ (implies C^2 in space and C^1 in time). Lankeit [29] treated the logistic growth $\alpha u - \beta_0 u^2$ in the case of $n = 3$ and showed that if the initial data $u_0 \in L^2(\Omega)$ and $v_0 \in H^{1,2}(\Omega)$, then there exist $t_* > 0$ and a global solution that has sufficient regularities on $(t_*, \infty) \times \bar{\Omega}$. Winkler [30] dealt with the case of $n = 1$ and proved that there exists a global solution (u, v) satisfying

$$u \in L_{\text{loc}}^2((0, \infty) \times \bar{\Omega}) \cap L_{\text{loc}}^1((0, \infty); L^1(\Omega)), \quad \partial_x u \in L_{\text{loc}}^1((0, \infty); L^1(\Omega)), \quad v \in L_{\text{loc}}^2((0, \infty); H^{1,2}(\Omega))$$

even if the initial data $u_0 \in (C(\bar{\Omega}))^*$ and $v_0 \in L^2(\Omega)$. Here $(C(\bar{\Omega}))^*$ denotes the dual space of $C(\bar{\Omega})$, which is identical to the space of the Radon measures. After that, Wang, Winkler, and Xiang [31] achieved treating the case of $n = 2$ by choosing the special domain $\Omega = B(0, R)$ and obtained global solutions that have sufficient regularities on $(0, \infty) \times \bar{\Omega}$ whenever the initial data $u_0 \in (C(\bar{\Omega}))^*$ and $v_0 \in L^2(\Omega)$ are radially symmetric. Lankeit [32] considered a similar problem to that of the previous work [29] in the 2D case and constructed solutions that have sufficient regularities on $(0, \infty) \times \bar{\Omega}$ for arbitrary initial data $u_0 \in L^1(\Omega)$ and $v_0 \in H^{1,2}(\Omega)$. We also refer to the result of Heihoff [33], who treated the case of parabolic-elliptic type and investigated the regularities of global solutions under a similar assumption to that of [30]. Remark that he [34] also has dealt with the smoothing effects of solutions to the attraction-repulsion Keller–Segel system.

Our motivation for investigating the smoothing effects for the problem (1.1) comes from the author's previous results [35,36] rather than the above results: The author [35,36] has investigated the smoothing effects of global solutions of the semilinear heat equation;

$$\begin{cases} \partial_t u = \Delta u + \lambda |u|^{\kappa-l} u^l, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = a(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 2$, $\kappa > 1 + 2/n$, $l \in \{0, 1\}$, and $\lambda \in \mathbb{R} \setminus \{0\}$. We usually rely on the bootstrap argument to obtain the regularities of solutions u . However, as the nonlinear term $|u|^{\kappa-l} u^l$ might *not* have sufficient regularities in the case of $\kappa \notin \mathbb{N}$, it is expected that there is a *threshold* for the regularity of u ; the author has shown that whereas $u \in C_w((0, \infty); C^{\kappa+2}(\mathbb{R}^n))$, there is $a \in C^\infty(\mathbb{R}^n)$ such that $u \notin C((0, \infty); C^{\kappa+2+\sigma}(\mathbb{R}^n))$ for any $\sigma > 0$. Here C_w denotes the space of weakly-star continuous functions. The present paper aims to extend such previous works to the Keller–Segel-type problem (1.1). To explain the expected regularities of solutions u of (1.1), we summarize the discussion of [35, Section 1]; suppose that the local solution u on $(0, T) \times \mathbb{R}^n$ of (1.1) with $\beta_0 > 0$, $1 < \kappa < 2$, and $l = 1$ is smooth. Letting $h(\tau) := |\tau|^{\kappa-1} \tau$ for $\tau \in \mathbb{R}$, we have $|u|^{\kappa-1} u = h(u)$. Since $\partial_\tau h = \kappa \cdot |u|^{\kappa-1} \in C_{\text{loc}}^{\kappa-1}(\mathbb{R})$, where C_{loc}^θ denotes the local θ -Hölder spaces, the chain rule for the composite functions yields $\partial_t(|u|^{\kappa-1} u) = \kappa |u|^{\kappa-1} \partial_t u$ and $\nabla(|u|^{\kappa-1} u) = \kappa |u|^{\kappa-1} \nabla u$. Therefore, we have

$$|u|^{\kappa-1} u \in C_{\text{loc}}^\kappa((0, T]; C(\mathbb{R}^n)) \cap C_{\text{loc}}^{\kappa-1}((0, T]; C^1(\mathbb{R}^n)) \cap C^1((0, T]; C_{\text{loc}}^{\kappa-1}(\mathbb{R}^n)) \cap C((0, T]; C_{\text{loc}}^\kappa(\mathbb{R}^n)).$$

As the terms $\partial_t u$ and Δu may have similar regularities to that of $|u|^{\kappa-1} u$ due to the parabolic smoothing effects, we obtain

$$\begin{aligned} u &\in C_{\text{loc}}^{\kappa+1}((0, T]; C(\mathbb{R}^n)) \cap C_{\text{loc}}^\kappa((0, T]; C^2(\mathbb{R}^n)) \cap C_{\text{loc}}^{\kappa-1}((0, T]; C^3(\mathbb{R}^n)) \\ &\cap C^2((0, T]; C_{\text{loc}}^{\kappa-1}(\mathbb{R}^n)) \cap C^1((0, T]; C_{\text{loc}}^{\kappa+1}(\mathbb{R}^n)) \cap C((0, T]; C_{\text{loc}}^{\kappa+2}(\mathbb{R}^n)) \end{aligned} \quad (1.2)$$

at least formally. In our result, we will obtain similar regularities to (1.2). Finally, we have to mention the optimality of the regularities; although the author has shown the optimality of the regularities of solutions to the semilinear heat equation in [35,36], we have to choose a *sign-changing* initial datum to derive such a result. As the unknown function u of the Keller–Segel system stands for the density of the cell, we should take the non-negative initial datum a in (1.1) to regard a as a proper physical quantity. Therefore, although we expect that there is a non-smooth solution u of (1.1), such a solution does not have a suitable physical sense due to the sign-changes; the optimality of the regularities is just a mathematical problem without the physical sense. To be more precise, in

general, the standard maximum principle of the parabolic equation ensures that the density u of the cell is strictly positive whenever the initial datum $a \not\equiv 0$ is non-negative (see, e.g., [22, Theorem 4.5], [18, Theorem 2.1], and [29, Theorem 1.1]) in the case of a bounded smooth domain). In such a case, as the nonlinear term becomes u^κ , whose higher-order derivatives are harmless, we expect that u is smooth. However, we should mention that our initial data $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ may be a distribution due to the negative regularity indices, and thus assuming a suitable property of a corresponding to the non-negativity without any additional regularity does not seem to be straightforward. Therefore, discussing further regularities is hard in our situation.

1.1. Main results

Before stating our main results, we simplify the system (1.1) by eliminating the unknown function v ; as the second equation of (1.1) yields $v = (\beta_1 I - \Delta)^{-1} u$ formally, we have

$$\begin{cases} \partial_t u - (\alpha I + \Delta)u = -\nabla \cdot (u \mathcal{K}_{\beta_1} u) - \beta_0 |u|^{\kappa-l} u^l & \text{in } (0, \infty) \times \mathbb{R}^n, \\ u(0) = a & \text{in } \mathbb{R}^n, \end{cases} \quad (1.3)$$

where $\mathcal{K}_{\beta_1} := \nabla(\beta_1 I - \Delta)^{-1}$ for simplicity. In what follows, we shall consider the equation (1.3). Moreover, let $L^{\gamma,q}$ denote the Lorentz spaces. Let $\{e^{t\Delta}\}_{t>0}$ denote the heat semigroup and set $e^{t(\alpha I + \Delta)} := e^{\alpha t} e^{t\Delta}$. The operator $\dot{\Delta}_j$ for $j \in \mathbb{Z}$ denotes the homogeneous dyadic block; the details of the notation will be given in Section 2. Our first result ensures the existence and uniqueness of local mild solutions of (1.3) with initial data $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$.

Theorem 1 (Mild solution). *Let $n \geq 3$, $\alpha, \beta_0, \beta_1 \in [0, \infty)$, $l \in \{0, 1\}$, $n/2 < p < n$, $1 \leq q \leq \infty$, and $2p/n < \kappa < 2$. Suppose that s_0 and s_1 satisfy*

$$\begin{cases} 2 - \frac{n}{p} < s_0 < \min \left\{ 1, 2 - \frac{1}{\kappa-1} \left(2 - \frac{n}{p} \right) \right\}, \\ \max \left\{ 2 - \frac{n}{p}, 2\kappa - n - (\kappa-1)s_0 \right\} < s_1 < \min \left\{ 3 - \frac{n}{p} - s_0, (\kappa-1)(2-s_0) \right\} \end{cases} \quad (1.4)$$

and suppose that the initial datum $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$. Then the following statements hold:

(i) In the case of $1 \leq q < \infty$, there exist $0 < T \leq 1$ and a solution u on $(0, T) \times \mathbb{R}^n$ of

$$u(t) = e^{t(\alpha I + \Delta)} a - \int_0^t e^{(t-\tau)(\alpha I + \Delta)} \left(\nabla \cdot (u \mathcal{K}_{\beta_1} u) + \beta_0 |u|^{\kappa-l} u^l \right) d\tau \quad (1.5)$$

satisfying

$$u \in BC([0, T]; B_{p,q}^{-2+n/p}(\mathbb{R}^n)), \quad t^{s_j/2} u \in BC([0, T]; B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n)), \quad u \in L^{2/s_j,q}((0, T); B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n)) \quad (1.6)$$

for $j \in \{0, 1\}$ and

$$\lim_{t \rightarrow +0} \left(\|u(t) - a\|_{B_{p,q}^{-2+n/p}(\mathbb{R}^n)} + \sum_{j=0}^1 t^{s_j/2} \|u(t)\|_{B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n)} \right) = 0. \quad (1.7)$$

In addition, it holds that

$$\|u\|_{L_T^\infty(B_{p,q}^{-2+n/p}(\mathbb{R}^n))} + \sum_{j=0}^1 \left(\|t^{s_j/2} u\|_{L_T^\infty(B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n))} + \|u\|_{L_T^{2/s_j,q}(B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n))} \right) \leq C e^\alpha \|a\|_{B_{p,q}^{-2+n/p}(\mathbb{R}^n)}, \quad (1.8)$$

where $C > 0$ is a constant independent of α , T , a , and u .

Concerning the uniqueness, there exists a constant $0 < \varepsilon_* < 1$ independently of a such that for every $0 < T_* \leq 1$, a solution u on $(0, T_*) \times \mathbb{R}^n$ of (1.5) satisfying

$$u \in L^{2/s_j,\infty}((0, T_*); H^{s_j-2+n/p,p}(\mathbb{R}^n)), \quad \limsup_{\lambda \rightarrow \infty} \lambda \mu \left(t \in (0, T_*) \mid \|u(t)\|_{H^{s_j-2+n/p,p}(\mathbb{R}^n)} > \lambda \right)^{s_j/2} \leq \varepsilon_* \quad (1.9)$$

for $j \in \{0, 1\}$ is unique, where μ denotes the usual Lebesgue measure on $(0, T_*)$.

(ii) In the case of $q = \infty$, there exists a constant $0 < \varepsilon < 1$ such that if $a \in B_{p,\infty}^{-2+n/p}(\mathbb{R}^n)$ satisfies

$$\limsup_{j \rightarrow \infty} 2^{(-2+n/p)j} \|\dot{\Delta}_j a\|_{L^p(\mathbb{R}^n)} \leq \varepsilon, \quad (1.10)$$

then there exist $0 < T \leq 1$ and a solution u on $(0, T) \times \mathbb{R}^n$ of (1.5) satisfying

$$u \in BC((0, T); B_{p,\infty}^{-2+n/p}(\mathbb{R}^n)), \quad t^{s_j/2} u \in BC((0, T); B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n))$$

for $j \in \{0, 1\}$ and

$$\lim_{t \rightarrow +0} \langle u(t) - a, \psi \rangle = 0 \quad (1.11)$$

for all $\psi \in B_{p',1}^{-2-n/p}(\mathbb{R}^n)$, where $p' := p/(p-1)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $B_{p,\infty}^{-2+n/p}(\mathbb{R}^n)$ and $B_{p',1}^{-2-n/p}(\mathbb{R}^n)$. In addition, it holds that (1.8) with q replaced by ∞ . The uniqueness assertion remains true in the same way as in (i).

(iii) Suppose that u and v are the solutions on $(0, T) \times \mathbb{R}^n$ of (1.5) with the initial data $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ and $b \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ obtained in (i) or (ii), respectively. Then it holds that

$$\|u - v\|_{L_T^\infty(B_{p,q}^{-2+n/p}(\mathbb{R}^n))} + \sum_{j=0}^1 \left(\|t^{s_j/2}(u - v)\|_{L_T^\infty(B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n))} + \|u - v\|_{L_T^{2/s_j,q}(B_{p,1}^{s_j-2+n/p}(\mathbb{R}^n))} \right) \leq C e^\alpha \|a - b\|_{B_{p,q}^{-2+n/p}(\mathbb{R}^n)},$$

where $C > 0$ is a constant independent of α , T , a , b , u , and v .

Remark 1. (i) Since we may choose $\alpha = \beta_0 = 0$, our result contains the case of the usual Keller–Segel system. However, remark that our result *excludes* the case of $n = 2$; concerning our spaces $B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ for the initial data, we observe that the regularity index becomes $-2 + 2/p$ if $n = 2$. While it is negative for any $p > 1$, the estimate for the nonlinear term $|u|^{\kappa-1}u'$ would be necessary. In the case of $\beta_0 = 0$, we handle only the nonlinear term $\nabla \cdot (u\mathcal{K}_{\beta_1}u)$ represented by the *divergence form*. Thus $n \geq 3$ has to be assumed unlike the case of $\beta_0 = 0$; see Proposition 7 and Lemma 2 for more details. We also mention that the assumption $\kappa < 2$ ensures the solvability in $B_{p,q}^{-2+n/p}(\mathbb{R}^n)$; if we consider the case of $\kappa > 2$, then choosing *different* function spaces would be required to show the solvability. In any case, the conditions $n \geq 3$ and $\kappa < 2$ are assumed in order to consider quite *large* function spaces $B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ for the initial data.

(ii) As mentioned before, our result only gives the local existence of solutions since the system (1.1) does *not* have a scaling invariant structure. However, even though, there are results [18,19] on the construction of global solutions by showing suitable a priori estimates with the aid of the logistic growth. Remark that our situation differs from such results in the sense that the domain $\Omega = \mathbb{R}^n$ is unbounded and the initial data a may be a distribution.

(iii) The properties (1.6) imply that the index $1 \leq q \leq \infty$ in the Besov spaces $B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ affects the regularity of solutions u in the time direction. More precisely, such a regularity is expressed as the Lorentz spaces. This is regarded as an improvement of the regularity of the solution due to a slight difference in the initial data a (cf. [37,38]). In the case of $q = \infty$, considering the time-Lorentz spaces does not make sense as such a regularity follows from the time-weighted spaces (cf. [39, Proposition 3.7]). Furthermore, the condition (1.9) is fulfilled for all solutions u satisfying the property (1.7); if u satisfies $t^{s/2}u \in L^\infty((0, T_*); X)$ with $\lim_{t \rightarrow +0} t^{s/2}\|u(t)\|_X = 0$ for some $0 < s < 2$ and Banach space X , then we see by [39, Proposition 3.7] that

$$\lim_{\lambda \rightarrow \infty} \lambda \mu(t \in (0, T_*) \mid \|u(t)\|_X > \lambda)^{s/2} = 0,$$

and thus the condition (1.9) for the uniqueness is *weaker* than that given by the standard Banach fixed point theorem in the case of $1 \leq q < \infty$.

(iv) To construct solutions u for the initial datum $a \in B_{p,\infty}^{-2+n/p}(\mathbb{R}^n)$, we have to assume the additional condition (1.10) as the function in $B_{p,\infty}^{-2+n/p}(\mathbb{R}^n)$ may *not* be approximated by the function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in general. We also remark that the initial datum $a \in B_{p,q}^{-2+n/p}(\mathbb{R}^n)$ for $1 \leq q < \infty$ necessarily satisfies $\lim_{j \rightarrow \infty} 2^{(-2+n/p)j} \|\Delta_j a\|_{L^p(\mathbb{R}^n)} = 0$.

Next we give our second result providing the partial smoothing effects of the solutions u of (1.5).

Theorem 2 (Partial smoothing effects). *In the statement of Theorem 1 (i) and (ii), the solution u of (1.5) also has the following regularities*

$$\begin{cases} u \in C_{\text{loc}}^{\kappa+1}((0, T]; (BUC \cap L^p)(\mathbb{R}^n)) \cap C_{\text{loc}}^{\kappa-j}((0, T]; (BUC^{2+j} \cap H^{2+j,p})(\mathbb{R}^n)), \\ \partial_t^2 u \in C_w((0, T]; C^{\kappa-1}(\mathbb{R}^n)), \quad \partial_t^j u \in C_w((0, T]; C^{\kappa+2-j}(\mathbb{R}^n)) \end{cases} \quad (1.12)$$

for $j \in \{0, 1\}$. In particular, it holds that

$$u \in C^1((0, T); BUC(\mathbb{R}^n)) \cap C((0, T); BUC^2(\mathbb{R}^n))$$

and u satisfies (1.3) pointwise in $(0, T) \times \mathbb{R}^n$.

Remark 2. (i) Although the regularities (1.12) of the solution u seem to be involved, it is expected that (1.12) is obtained naturally as a formal calculation yields (1.2). In particular, the regularities (1.12) provide that

$$u \in C_{\text{loc}}^{\kappa+1}((0, T]; BUC(\mathbb{R}^n)) \cap C_w((0, T]; C^{\kappa+2}(\mathbb{R}^n)),$$

which seems to be *optimal* in time and space, respectively, due to the combination of the C^κ -regularity of $|u|^{\kappa-1}u'$ and the parabolic smoothing effects (cf. [35,36]).

(ii) We have to note that the optimality mentioned above is meaningless in view of the physical model even though it holds true in the mathematical sense. The proof relies on choosing a sign-changing initial datum a as well as that in [35,36], which is unreasonable as a denotes the initial density. In addition, if we assume that $a \not\equiv 0$ is non-negative, then the standard maximum principle of the parabolic equation yields the positivity of u , which means that the proof of the optimality in [35,36] is no longer valid. Or rather, we expect that u is smooth. Nevertheless, the author suspects that a sufficient smoothness should be unknown as our framework of function spaces is too general to assume the “non-negativity” of a suitably.

1.2. Organizations of the paper

This paper is organized as follows: We give some notation and definitions of function spaces in the next section. We also recall the fundamental properties of the function spaces used in this paper and the heat semigroup in the Besov spaces framework. Section 3 is devoted to the construction of solutions of (1.5), namely, the proof of Theorem 1. To this end, we give several estimates of the nonlinear terms. In Section 4, we show the partial smoothing effects of the solution of (1.5); this implies the result of Theorem 2. We give further estimates for the nonlinear terms to show the desired regularity properties. In the appendix, we provide the detailed proofs of several propositions given in Section 2.

2. Preliminaries

2.1. Notation and function spaces

In this subsection, we give some notation and definitions of function spaces used in this paper: Let Ω denote a set of the Euclidean space. Let $BC(\Omega)$ and $BUC(\Omega)$ be the Banach spaces of bounded continuous and bounded uniformly continuous functions defined on Ω , respectively. In addition, for $k \in \mathbb{N} \cup \{0\}$, define $BUC^k(\Omega) := \{\varphi \in C^k(\Omega) \mid \partial^\alpha \varphi \in BUC(\Omega), |\alpha| \leq k\}$, where α denotes a multi-index. For $k \in \mathbb{N} \cup \{0\}$ and $0 < \theta < 1$, we also define the Hölder spaces $C^{k+\theta}(\Omega) := \{\varphi \in BUC^k(\Omega) \mid \|\varphi\|_{C^{k+\theta}(\Omega)} < \infty\}$ with the norm

$$\|\varphi\|_{C^{k+\theta}(\Omega)} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)| + \sum_{|\alpha|=k} \sup_{\substack{x_0, x_1 \in \Omega, \\ x_0 \neq x_1}} \frac{|\partial^\alpha \varphi(x_1) - \partial^\alpha \varphi(x_0)|}{|x_1 - x_0|^\theta}.$$

In addition, let $\varphi \in C_{\text{loc}}^{k+\theta}(\Omega)$ imply that $\varphi \in C^{k+\theta}(K)$ holds for an arbitrary compact subset $K \subset \Omega$. Moreover, if we consider the X -valued functions defined on the time interval $I \subset [0, \infty)$, then we use the notation like $BC(I; X)$, $C^{k+\theta}(I; X)$, and $C_{\text{loc}}^{k+\theta}(I; X)$ with slight modifications of the corresponding definitions. Let X^* denote the dual space of a Banach space X . Then $C_w(I; X^*)$ denotes the set of weakly-star continuous X^* -valued functions on I ; let $\varphi \in C_w(I; X^*)$ imply that

$$\lim_{t_* \rightarrow t} \langle \varphi(t_*) - \varphi(t), \psi \rangle = 0$$

for all $t \in I$ and $\psi \in X$ with the duality pairing $\langle \cdot, \cdot \rangle$ between X^* and X .

For $1 \leq \gamma \leq \infty$, let $L^\gamma(\Omega; X)$ denote the Bochner–Lebesgue spaces on Ω and let $\varphi \in L_{\text{loc}}^\gamma(\Omega; X)$ imply that $\varphi \in L^\gamma(K; X)$ holds for an arbitrary compact subset $K \subset \Omega$. For $1 < \gamma < \infty$ and $1 \leq q \leq \infty$, define the Lorentz spaces $L^{\gamma,q}(\Omega; X) := \{\varphi \in L_{\text{loc}}^1(\Omega; X) \mid \|\varphi\|_{L^{\gamma,q}(\Omega; X)} < \infty\}$ with the norm

$$\|\varphi\|_{L^{\gamma,q}(\Omega; X)} := \begin{cases} \left\{ \int_0^\infty (\tau^{1/\gamma} \varphi^*(\tau))^q \frac{d\tau}{\tau} \right\}^{1/q} & \text{if } 1 \leq q < \infty, \\ \sup_{\tau > 0} \tau^{1/\gamma} \varphi^*(\tau) & \text{if } q = \infty, \end{cases}$$

where φ^* denotes the rearrangement function of φ given by

$$\varphi^*(\tau) := \inf\{\lambda \in (0, \infty) \mid \mu(\{x \in \Omega \mid \|\varphi(x)\|_X > \lambda\}) \leq \tau\}, \quad \tau > 0$$

and μ denotes the usual Lebesgue measure on Ω . In particular, if $X = \mathbb{R}$, then we write $L^{\gamma,q}(\Omega; \mathbb{R}) = L^{\gamma,q}(\Omega)$ for simplicity. Remark that the Lorentz spaces $L^{\gamma,q}(\Omega; X)$ defined as above are *not* Banach spaces but quasi-Banach spaces. However, by a slight modification of the definition, we see that the Lorentz spaces become Banach spaces (cf. [40, Section 6.2]).

Next, let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. Then we set

$$(\lambda I - \Delta)^{\sigma/2} := \mathcal{F}^{-1}(\lambda + |\cdot|^2)^{\sigma/2} \mathcal{F}, \quad e^{t(\lambda I + \Delta)} := e^{\lambda t} \mathcal{F}^{-1} e^{-t|\cdot|^2} \mathcal{F}$$

for $\lambda, \sigma \in \mathbb{R}$ and $t > 0$. In addition, let $\chi \in \mathcal{S}(\mathbb{R}^n)$ be a function satisfying $\text{supp } \chi = \{\xi \in \mathbb{R}^n \mid 1/2 \leq |\xi| \leq 2\}$, $\chi(\xi) > 0$ for $1/2 < |\xi| < 2$, and $\sum_{j \in \mathbb{Z}} \chi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space. Then we put $\dot{\Delta}_j := \mathcal{F}^{-1} \chi(2^{-j}\cdot) \mathcal{F}$ and $\dot{S}_j := \mathcal{F}^{-1}(1 - \sum_{k \in \mathbb{N}} \chi(2^{-k-j}\cdot)) \mathcal{F}$ for $j \in \mathbb{Z}$. For $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $1 \leq q \leq \infty$, we define the Sobolev spaces $H^{s,p}(\mathbb{R}^n)$ and the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ and $\dot{B}_{p,q}^s(\mathbb{R}^n)$ as the spaces of functions φ in the dual space $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ such that the corresponding norms

$$\begin{aligned} \|\varphi\|_{H^{s,p}(\mathbb{R}^n)} &:= \|(I - \Delta)^{s/2} \varphi\|_{L^p(\mathbb{R}^n)}, \\ \|\varphi\|_{B_{p,q}^s(\mathbb{R}^n)} &:= \|\dot{S}_0 \varphi\|_{L^p(\mathbb{R}^n)} + \|\{2^{sj} \|\dot{\Delta}_j \varphi\|_{L^p(\mathbb{R}^n)}\}_{j \in \mathbb{N}}\|_{l^q(\mathbb{N})}, \\ \|\varphi\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &:= \|\{2^{sj} \|\dot{\Delta}_j \varphi\|_{L^p(\mathbb{R}^n)}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})} \end{aligned}$$

are finite. Note that although the spaces $H^{s,p}(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$ are Banach spaces for all $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $1 \leq q \leq \infty$, the spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ are Banach spaces provided $(s, q) \in (-\infty, n/p) \times [1, \infty]$ or $(s, q) = (n/p, 1)$. Even so, by considering the space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ modulo polynomials $P \in \mathcal{P}(\mathbb{R}^n)$, we may show that $\dot{B}_{p,q}^s(\mathbb{R}^n)$ become Banach spaces for any $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $1 \leq q \leq \infty$ (cf. [42, Theorem 2.33] and [41, Remark 2.26]). For more details on the Besov spaces, we refer to, e.g., [43, Chapter 6], [41, Chapter 2], and [42].

In what follows, we will abbreviate

$$\|\varphi\|_{L_T^r(X)} := \|\varphi\|_{L^r((0,T);X)}, \quad \|\varphi\|_{L_T^{r,q}(X)} := \|\varphi\|_{L^{r,q}((0,T);X)}, \quad \|\varphi\|_{X \cap Y} := \|\varphi\|_X + \|\varphi\|_Y$$

for simplicity. In addition, for a function space $X(\mathbb{R}^n)$ defined on the whole space $\mathbb{R}^n$, we will abbreviate $X := X(\mathbb{R}^n)$.

2.2. Fundamental properties of some function spaces

In this subsection, we recall the fundamental properties of the Besov and Lorentz spaces. Throughout this subsection, let $X \subset Y$ imply the continuous embedding, i.e., $\|\varphi\|_Y \leq C\|\varphi\|_X$ with some constant $C > 0$ independent of φ .

We begin with the properties of the Besov spaces; for $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, we note that

$$B_{p,q_0}^s \subset B_{p,q_1}^s \quad \text{for } 1 \leq q_0 \leq q_1 \leq \infty, \quad B_{p,\infty}^{s+\sigma} \subset B_{p,1}^s \quad \text{for } \sigma > 0,$$

$$B_{p,1}^s \subset H^{s,p} \subset B_{p,\infty}^s, \quad \dot{B}_{p,1}^0 \subset L^p \subset \dot{B}_{p,\infty}^0$$

due to [43, Theorems 6.2.4 and 6.3.1] and [42, Proposition 2.3]. We also note that [43, Theorem 6.5.1] and [42, Proposition 2.4] yield the Sobolev embedding results

$$B_{p_0,q}^s \subset B_{p_1,q}^{s-n(1/p_0-1/p_1)}, \quad \dot{B}_{p_0,q}^s \subset \dot{B}_{p_1,q}^{s-n(1/p_0-1/p_1)}$$

for $1 \leq p_0 \leq p_1 \leq \infty$, $s \in \mathbb{R}$, and $1 \leq q \leq \infty$ and $B_{p,1}^{k+n/p} \subset BUC^k$ for $1 \leq p \leq \infty$ and $k \in \mathbb{N} \cup \{0\}$. Moreover, it holds by [42, Theorem 2.7] that $B_{\infty,\infty}^{k+\theta} = C^{k+\theta}$ with the equivalence of norms for $k \in \mathbb{N} \cup \{0\}$ and $0 < \theta < 1$. For $1 \leq p \leq \infty$, $s > 0$, and $1 \leq q \leq \infty$, it holds that

$$B_{p,q}^s = L^p \cap \dot{B}_{p,q}^s, \quad B_{p,q}^{-s} = L^p + \dot{B}_{p,q}^{-s}. \quad (2.1)$$

In particular, we have $B_{p,q}^s \subset \dot{B}_{p,q}^s$ and $\dot{B}_{p,q}^{-s} \subset B_{p,q}^{-s}$, although the properties (2.1) are mentioned in [43, Theorem 6.3.2 and p.164], for the convenience of the readers, we will give the detailed proof in the appendix. Remark that [42, Theorems 2.2 and 2.28] yield

$$\|\nabla \varphi\|_{B_{p,q}^{s-1}} + \|(-\Delta)^{\sigma/2} \varphi\|_{B_{p,q}^{s-\sigma}} \leq C\|\varphi\|_{B_{p,q}^s}, \quad \|\nabla \varphi\|_{\dot{B}_{p,q}^{s-1}} + \|(-\Delta)^{\sigma/2} \varphi\|_{\dot{B}_{p,q}^{s-\sigma}} \leq C\|\varphi\|_{\dot{B}_{p,q}^s}$$

with a constant $C > 0$ independent of φ . We give the following proposition, whose proof will be attached in the appendix:

Proposition 1. *Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $j \in \mathbb{N}$. Then for every $a \in B_{p,\infty}^s$, it holds that $\dot{S}_j a \in B_{p,1}^{s+\sigma}$ for all $\sigma > 0$ with the properties*

$$\|\dot{S}_j a\|_{B_{p,1}^{s+\sigma}} \leq \frac{2^{\sigma(j+1)} C}{1-2^{-\sigma}} \|a\|_{B_{p,\infty}^s}, \quad \|(I - \dot{S}_j)a\|_{B_{p,\infty}^s} \leq C \sup_{k \geq j} 2^{sk} \|\dot{\Delta}_k a\|_{L^p},$$

where $C > 0$ is a constant independent of σ , j , and a .

Next, we summarize the properties of the Lorentz spaces; let $I \subset [0, \infty)$ and X be a time interval and a Banach space, respectively. It holds by [40, p.216 and Theorems 6.3 and 6.6] that

$$L^{\gamma,\gamma}(I; X) = L^\gamma(I; X), \quad L^{\gamma,q_0}(I; X) \subset L^{\gamma,q_1}(I; X)$$

for $1 < \gamma < \infty$ and $1 \leq q_0 \leq q_1 \leq \infty$. A direct computation with [40, Theorem 4.5 (h)] implies that

$$\left\| \|\varphi\|_X^\theta \right\|_{L^{\gamma/\theta,q}(\theta(I))} = \|\varphi\|_{L^{\gamma,q}(I;X)}^\theta \quad (2.2)$$

for $1 < \gamma < \infty$, $1 \leq q \leq \infty$, and $0 < \theta \leq 1$. We also give the following propositions:

Proposition 2. *Let $1 < \gamma < \infty$, $1 \leq q \leq \infty$, and $0 < T \leq \infty$. Then the following statements hold:*

(i) *Let $\gamma_0, \gamma_1 \in (1, \infty)$ and $q_0, q_1 \in [1, \infty]$ satisfy $1/\gamma = 1/\gamma_0 + 1/\gamma_1$ and $1/q = 1/q_0 + 1/q_1$, respectively. For every $\varphi_0 \in L^{\gamma_0,q_0}(0, T)$ and $\varphi_1 \in L^{\gamma_1,q_1}(0, T)$, it holds that $\varphi_0 \varphi_1 \in L^{\gamma,q}(0, T)$ with the estimate*

$$\|\varphi_0 \varphi_1\|_{L_T^{\gamma,q}} \leq 2^{1/\gamma} \|\varphi_0\|_{L_T^{\gamma_0,q_0}} \|\varphi_1\|_{L_T^{\gamma_1,q_1}}.$$

In particular, if $T < \infty$ and $\varphi_0 \in L^{\gamma_0,\infty}(0, T)$, then it holds that $\varphi_0 \in L^{\gamma,1}(0, T)$ with the estimate

$$\|\varphi_0\|_{L_T^{\gamma,1}} \leq \gamma_1 2^{1/\gamma} T^{1/\gamma_1} \|\varphi_0\|_{L_T^{\gamma_0,\infty}}.$$

(ii) *Assume that $0 < \lambda < 1 - 1/\gamma$. For $f \in L^{(1/\gamma+\lambda)^{-1},q}((0,T); X)$, define Ψ by setting*

$$\Psi(t) := \int_0^t (t-\tau)^{\lambda-1} \|f(\tau)\|_X d\tau, \quad 0 < t < T.$$

Then it holds that $\Psi \in L^{\gamma,q}(0, T)$ with the estimate

$$\|\Psi\|_{L_T^{\gamma,q}} \leq C \|f\|_{L_T^{(1/\gamma+\lambda)^{-1},q}(X)},$$

where $C > 0$ is a constant independent of T and f .

(iii) *Assume that $f \in L^{\gamma,\infty}((0,T); X)$ satisfies*

$$\limsup_{\lambda \rightarrow \infty} \lambda \mu(t \in (0, T) \mid \|f(t)\|_X > \lambda)^{1/\gamma} \leq \varepsilon$$

for some $\varepsilon > 0$. Then there exist a constant $M_\varepsilon > 0$ and a function $f_\varepsilon \in L^{\gamma,\infty}((0,T); X)$ such that

$$\|f_\varepsilon\|_{L_T^{\gamma,\infty}(X)} \leq 2\varepsilon, \quad \|f - f_\varepsilon\|_{L_T^\infty(X)} \leq M_\varepsilon.$$

The assertion of (i) implies the Hölder inequality in the Lorentz spaces. Combining [40, Theorem 4.11] and the usual Hölder inequality provides its proof; the detail is given in, e.g., [44, Proposition 2.1 (iii)]. In particular, by setting $\varphi_1 \equiv 1$, we have $\|1\|_{L_T^{\gamma_1,1}} = \gamma_1 T^{1/\gamma_1}$ from [40, p.216], and thus we obtain the latter inequality. The assertion of (ii) implies the Hardy–Littlewood–Sobolev inequality in the Lorentz spaces. The estimate is shown by the real interpolation method; for more details, we refer to [44, Proposition 4.1 (i)]. Furthermore, the proof of the assertion of (iii) is given in [39, Proposition 4.9].

2.3. Heat semigroup defined on Besov spaces

This subsection is devoted to the properties of the heat semigroup defined on Besov spaces. We will provide the proofs of all propositions in the appendix. First we verify the smoothing estimate and space-time estimate on the heat semigroup.

Proposition 3. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. Then the following statements hold:*

(i) *For every $a \in B_{p,\infty}^s$, there holds $e^{t\Delta}a \in C^\infty((0, \infty); B_{p,1}^{s+\sigma})$ for all $\sigma > 0$ with the estimate*

$$\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq C(1+t^{-\sigma/2})\|a\|_{B_{p,\infty}^s}$$

for all $t > 0$, where $C > 0$ is a constant independent of t and a . In particular, there holds

$$\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq Ct^{-\sigma/2}\|a\|_{B_{p,\infty}^s}$$

for all $0 < t \leq 1$.

(ii) *Let $1 \leq q \leq \infty$. For every $a \in B_{p,q}^s$, there holds $e^{t\Delta}a \in L^{2/\sigma,q}((0, T); B_{p,1}^{s+\sigma})$ for all $0 < \sigma < 2$ and $0 < T < \infty$ with the estimate*

$$\|e^{t\Delta}a\|_{L_T^{2/\sigma,q}(B_{p,1}^{s+\sigma})} \leq C(1+T^{\sigma/2})\|a\|_{B_{p,q}^s},$$

where $C > 0$ is a constant independent of T and a .

Combining Proposition 3 and the density and duality properties on Besov spaces, we may show the behavior of the heat semigroup as $t \rightarrow +0$.

Proposition 4. *Let $s \in \mathbb{R}$ and $\alpha \geq 0$. Then the following statements hold:*

(i) *Let $1 \leq p < \infty$, $1 \leq q < \infty$, and $\sigma > 0$. For every $a \in B_{p,q}^s$, there holds*

$$\lim_{t \rightarrow +0} t^{\sigma/2} \|e^{t(\alpha I + \Delta)}a\|_{B_{p,1}^{s+\sigma}} = 0.$$

(ii) *Let $1 < p \leq \infty$. For every $a \in B_{p,\infty}^s$, there holds*

$$\lim_{t \rightarrow +0} \langle e^{t(\alpha I + \Delta)}a - a, \psi \rangle = 0$$

for all $\psi \in B_{p',1}^{-s}$, where $p' := p/(p-1)$.

(iii) *Let $1 < p \leq \infty$, $s_0 \in \mathbb{R}$, $0 < \sigma < 2$, and $0 < T \leq 1$. If $f \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^{s_0})$ satisfies*

$$M := \sup_{0 < t < T} \left(t^{\sigma/2} \|f(t)\|_{B_{p,\infty}^{s_0}} + \left\| \int_0^t e^{(t-\tau)(\alpha I + \Delta)} f(\tau) d\tau \right\|_{B_{p,\infty}^s} \right) < \infty,$$

then there holds

$$\lim_{t \rightarrow +0} \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)} f(\tau) d\tau, \psi \right\rangle = 0$$

for all $\psi \in B_{p',1}^{-s}$.

Finally, we give the smoothing effects of mild solutions of the linear heat equation.

Proposition 5. *Let $1 \leq p_0 < \infty$, $p_0 \leq p \leq \infty$, $s_0, s \in \mathbb{R}$, $k \in \{0, 1\}$, $\alpha \geq 0$, and $0 < T < \infty$. For $f \in L^1((0, T); H^{s_0, p_0})$, define F by setting*

$$F(t) := \int_0^t e^{(t-\tau)(\alpha I + \Delta)} f(\tau) d\tau, \quad 0 < t < T. \quad (2.3)$$

Then the following statements hold:

(i) *Suppose that $f, \partial_t^k f \in L_{\text{loc}}^\infty((0, T); B_{p,\infty}^s)$. Then it holds that*

$$\partial_t^k F \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T); B_{p,1}^{s+2\theta}), \quad \partial_t^{k+1} F, \Delta \partial_t^k F \in L_{\text{loc}}^\infty((0, T); B_{p,\infty}^s).$$

(ii) *Let $0 < \gamma < 1$ and suppose that $f \in C_{\text{loc}}^{k+\gamma}((0, T); X^s)$. Then it holds that*

$$F \in C_{\text{loc}}^{k+1+\gamma}((0, T); X^s) \cap C_{\text{loc}}^{k+\gamma}((0, T); X^{s+2}),$$

where X^s denotes either H^{s, p_0} or BUC^s in the case of $s \in \mathbb{N} \cup \{0\}$.

3. Construction of mild solutions

In this section, we shall construct solutions of (1.5). To this end, we show the estimates of the nonlinear terms in (1.5). We begin with the estimate of the operator $\mathcal{K}_{\beta_1} = \nabla(\beta_1 I - \Delta)^{-1}$.

Proposition 6. *Let $1 < r < \infty$ and $\beta_1 \geq 0$. For every $\varphi \in L^{(1/r+1/n)^{-1}}$, there holds $\mathcal{K}_{\beta_1} \varphi \in (L^r)^n$ with the estimate*

$$\|\mathcal{K}_{\beta_1} \varphi\|_{L^r} \leq C \|\varphi\|_{L^{(1/r+1/n)^{-1}}},$$

where $C > 0$ is a constant independent of φ .

Proof. Since

$$\mathcal{K}_{\beta_1} = \nabla(-\Delta)^{-1}(\beta_1 I - \Delta - \beta_1 I)(\beta_1 I - \Delta)^{-1} = \nabla(-\Delta)^{-1}(I - \beta_1(\beta_1 I - \Delta)^{-1}) \quad (3.1)$$

for all $\beta_1 \geq 0$ and since $I - \beta_1(\beta_1 I - \Delta)^{-1}$ is a bounded operator from L^r onto itself due to $1 < r < \infty$, we have

$$\|\mathcal{K}_{\beta_1} \varphi\|_{L^r} \leq C \|\nabla(-\Delta)^{-1} \varphi\|_{L^r} \leq C \|\varphi\|_{\dot{H}^{-1,r}},$$

where $\dot{H}^{-1,r}$ denotes the homogeneous Sobolev spaces. Therefore, the desired estimate follows from the Sobolev embedding $L^{(1/r+1/n)^{-1}} \subset \dot{H}^{-1,r}$ (cf. [43, Theorem 6.5.1]). $\square$

Next we show the estimates of the nonlinear terms $\nabla \cdot (u \mathcal{K}_{\beta_1} u)$ and $|u|^{\kappa-l} u^l$.

Lemma 1. *Let $n/2 < p < n$, $1 < \kappa < 2$, $l \in \{0, 1\}$, and $\beta_1 \geq 0$. Suppose that s_0 , s_1 , r , and r_* satisfy*

$$\begin{cases} 2 - \frac{n}{p} < s_0 < 1, & \max \left\{ 2 - \frac{n}{p}, 2\kappa - n - (\kappa - 1)s_0 \right\} < s_1 < 2, \\ \frac{1}{r} = (\kappa - 1) \frac{2 - s_0}{n} + \frac{2 - s_1}{n}, & \frac{1}{r_*} = \frac{1 - s_0}{n} + \frac{2 - s_1}{n}. \end{cases} \quad (3.2)$$

Then there holds

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{H^{\eta-1,r_*}} \leq C \|\varphi_1\|_{H^{\eta+s_1-2+n/p,p}} \|\varphi_0\|_{H^{\eta+s_0-2+n/p,p}}$$

for all $\eta \geq 0$ and $\varphi_j \in H^{\eta+s_j-2+n/p,p}$ with $j \in \{0, 1\}$, where $C > 0$ is a constant independent of φ_0 and φ_1 . Moreover, there holds

$$\| |\varphi_1|^{\kappa-l} \varphi_1^l - |\varphi_0|^{\kappa-l} \varphi_0^l \|_{L^{(1/r-\sigma/n)^{-1}}} \leq C (\|\varphi_1\|_{H^{s_0-2+n/p,p}}^{\kappa-1} + \|\varphi_0\|_{H^{s_0-2+n/p,p}}^{\kappa-1}) \|\varphi_1 - \varphi_0\|_{H^{\sigma+s_1-2+n/p,p}}$$

for all $0 \leq \sigma < 2 - s_1$ and $\varphi_0, \varphi_1 \in H^{s_0-2+n/p,p} \cap H^{\sigma+s_1-2+n/p,p}$.

Proof. Note that

$$\frac{n}{1-s_0}, \frac{n}{2-s_1}, r, r_* \in (1, \infty)$$

due to the conditions (3.2). Since Proposition 6 yields

$$\|\mathcal{K}_{\beta_1} \varphi_0\|_{H^{\eta,n/(1-s_0)}} = \|\mathcal{K}_{\beta_1} (I - \Delta)^{\eta/2} \varphi_0\|_{L^{n/(1-s_0)}} \leq C \|\varphi_0\|_{H^{\eta,n/(2-s_0)}}$$

and since $H^{\eta+s_j-2+n/p,p} \subset H^{\eta,n/(2-s_j)}$ for $j \in \{0, 1\}$ from the Sobolev embeddings with (3.2), the fractional Leibniz rule in the Sobolev spaces (cf. [45, Theorem 1]) provides that

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{H^{\eta-1,r_*}} \leq C \|\varphi_1\|_{H^{\eta,n/(2-s_1)}} \|\mathcal{K}_{\beta_1} \varphi_0\|_{H^{\eta,n/(1-s_0)}} \leq C \|\varphi_1\|_{H^{\eta+s_1-2+n/p,p}} \|\varphi_0\|_{H^{\eta+s_0-2+n/p,p}}.$$

In addition, noting the Sobolev embeddings

$$H^{s_0-2+n/p,p} \subset L^{n/(2-s_0)}, \quad H^{\sigma+s_1-2+n/p,p} \subset L^{n/(2-s_1-\sigma)}$$

and the relation

$$\frac{1}{r} - \frac{\sigma}{n} = (\kappa - 1) \frac{2 - s_0}{n} + \frac{2 - s_1 - \sigma}{n},$$

we see by [35, Proposition 2.3 (iii)] that

$$\begin{aligned} \| |\varphi_1|^{\kappa-l} \varphi_1^l - |\varphi_0|^{\kappa-l} \varphi_0^l \|_{L^{(1/r-\sigma/n)^{-1}}} &\leq 2 (\|\varphi_1\|_{L^{n/(2-s_0)}}^{\kappa-1} + \|\varphi_0\|_{L^{n/(2-s_0)}}^{\kappa-1}) \|\varphi_1 - \varphi_0\|_{L^{n/(2-s_1-\sigma)}} \\ &\leq C (\|\varphi_1\|_{H^{s_0-2+n/p,p}}^{\kappa-1} + \|\varphi_0\|_{H^{s_0-2+n/p,p}}^{\kappa-1}) \|\varphi_1 - \varphi_0\|_{H^{\sigma+s_1-2+n/p,p}}, \end{aligned}$$

which completes the proof of Lemma 1. $\square$

As we have shown the estimates of the nonlinear terms in Lemma 1, it remains to estimate the nonlinear terms given by the integral forms. Before showing the estimates, we shall verify the conditions on the indices ensuring the desired nonlinear estimates.

Proposition 7. Let $n/2 < p < n$ and $2p/n < \kappa < 2$. Then there exist s_0 and s_1 such that (1.4). In addition, let r and r_* satisfy

$$\frac{1}{r} = (\kappa - 1) \frac{2 - s_0}{n} + \frac{2 - s_1}{n}, \quad \frac{1}{r_*} = \frac{1 - s_0}{n} + \frac{2 - s_1}{n},$$

respectively. Then it holds that

$$1 < r < \frac{n}{2}, \quad \frac{n}{3} < r_* < p, \quad 0 < n \left(\frac{1}{r} - \frac{1}{p} \right) < 2 - s_1.$$

Proof. As we have

$$1 - \left(2 - \frac{n}{p} \right) = \frac{n}{p} - 1 > 0, \\ 2 - \frac{1}{\kappa - 1} \left(2 - \frac{n}{p} \right) - \left(2 - \frac{n}{p} \right) = \frac{n}{p} - \frac{1}{\kappa - 1} \left(2 - \frac{n}{p} \right) > \frac{n}{p} - \frac{n}{2p - n} \cdot \left(2 - \frac{n}{p} \right) = 0,$$

we may take s_0 so that $2 - n/p < s_0 < \min\{1, 2 - (\kappa - 1)^{-1}(2 - n/p)\}$ holds. Moreover, we also deduce that

$$3 - \frac{n}{p} - s_0 - \left(2 - \frac{n}{p} \right) = 1 - s_0 > 0, \\ (\kappa - 1)(2 - s_0) - \left(2 - \frac{n}{p} \right) = (\kappa - 1) \left\{ 2 - \frac{1}{\kappa - 1} \left(2 - \frac{n}{p} \right) - s_0 \right\} > 0, \\ 3 - \frac{n}{p} - s_0 - \{2\kappa - n - (\kappa - 1)s_0\} = n + 3 - \frac{n}{p} - 2\kappa - (2 - \kappa)s_0 > n + 3 - 2 - 2\kappa - (2 - \kappa) = n - \kappa - 1 > 0, \\ (\kappa - 1)(2 - s_0) - \{2\kappa - n - (\kappa - 1)s_0\} = n - 2 > 0.$$

Hence, we may take s_1 having the desired property. In addition, we note that

$$n - 2 - \{(\kappa - 1)(2 - s_0) - s_1\} > n - 2 - (\kappa - 1)(2 - s_0) + 2\kappa - n - (\kappa - 1)s_0 = 0$$

and $(\kappa - 1)(2 - s_0) - s_1 > 0$ due to the conditions (1.4). Thus we observe that $0 < n/r - 2 < n - 2$ as r satisfies $(\kappa - 1)(2 - s_0) - s_1 = n/r - 2$. Concerning the condition for r_* , since there holds $0 < s_0 + s_1 < 3 - n/p$ from the conditions (1.4), we note $s_0 + s_1 = 3 - n/r_*$ to obtain $n/3 < r_* < p$. Moreover, as the conditions (1.4) yield

$$2 - s_1 - n \left(\frac{1}{r} - \frac{1}{p} \right) = 2 - s_1 - \{(\kappa - 1)(2 - s_0) + 2 - s_1\} + \frac{n}{p} = (\kappa - 1) \left\{ s_0 - 2 + \frac{n}{(\kappa - 1)p} \right\} > (\kappa - 1) \left(\frac{1}{\kappa - 1} - 1 \right) \frac{n}{p} > 0,$$

we have the desired property. $\square$

Let us show the nonlinear estimates under the conditions in Proposition 7.

Lemma 2. Let $n/2 < p < n$, $1 \leq q \leq \infty$, $2p/n < \kappa < 2$, $l \in \{0, 1\}$, $\alpha, \beta_l \in [0, \infty)$, and $0 < T \leq 1$. Suppose that s_0 and s_1 satisfy (1.4) and that

$$t^{s_j/2} u, t^{s_j/2} v \in L^\infty((0, T); H^{s_j - 2 + n/p, p}), \quad u, v \in L^{2/s_j, q}((0, T); H^{s_j - 2 + n/p, p})$$

for $j \in \{0, 1\}$. Define the functions $\mathcal{I}(u, v)$ and $\mathcal{J}(u, v)$ by setting

$$\begin{cases} \mathcal{I}(u, v)(t) := \int_0^t e^{(t-\tau)(\alpha I + \Delta)} \nabla \cdot (u \mathcal{K}_{\beta_l} v) d\tau, & 0 < t < T, \\ \mathcal{J}(u, v)(t) := \int_0^t e^{(t-\tau)(\alpha I + \Delta)} (|u|^{\kappa-l} u^l - |v|^{\kappa-l} v^l) d\tau, & 0 < t < T, \end{cases} \quad (3.3)$$

respectively. Then the following estimates

$$\|\mathcal{I}(u, v)(t)\|_{B_{p,1}^{s_* - 2 + n/p}} \leq C e^\alpha t^{-s_*/2} [u]_{X_t} [v]_{X_t}, \quad \|\mathcal{J}(u, v)(t)\|_{B_{p,1}^{s_* - 2 + n/p}} \leq C e^\alpha t^{-s_*/2} ([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1}) [u - v]_{X_t}$$

hold for all $0 < t < T$ and $s_* \in \{0, s_0, s_1\}$, where

$$[u]_{X_t} := \sum_{j=0}^1 \sup_{0 < \tau < t} \tau^{s_j/2} \|u(\tau)\|_{H^{s_j - 2 + n/p, p}}$$

and $C > 0$ is a constant independent of α , t , T , u , and v . Moreover, it holds that

$$\sum_{j=0}^1 \|\mathcal{I}(u, v)\|_{L_t^{2/s_j, q}(B_{p,1}^{s_j - 2 + n/p})} \leq C e^\alpha [u]_{Y_t} [v]_{Y_t}, \quad \sum_{j=0}^1 \|\mathcal{J}(u, v)\|_{L_t^{2/s_j, q}(B_{p,1}^{s_j - 2 + n/p})} \leq C e^\alpha ([u]_{Y_t}^{\kappa-1} + [v]_{Y_t}^{\kappa-1}) [u - v]_{Y_t}$$

for all $0 < t < T$ and $j \in \{0, 1\}$, where

$$[u]_{Y_t} := \sum_{j=0}^1 \|u\|_{L_t^{2/s_j, q}(H^{s_j - 2 + n/p, p})}.$$

Proof. First we consider the estimates of $I(u, v)$. Taking r_* so that $1/r_* = (1 - s_0)/n + (2 - s_1)/n$ holds, we have $n/3 < r_* < p < n$ from Proposition 7. Hence, the Sobolev embedding $B_{r_*,1}^{s_*-2+n/r_*} \subset B_{p,1}^{s_*-2+n/p}$ and Proposition 3 (i) imply that

$$\|e^{t(\alpha I + \Delta)} \nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{B_{p,1}^{s_*-2+n/p}} \leq C e^{\alpha t} \|e^{t\Delta} \nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{B_{r_*,1}^{s_*-2+n/r_*}} \leq C e^{\alpha t} t^{-(1/2)(s_*-1+n/r_*)} \|\nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{B_{r_*,\infty}^{-1}}$$

for all $0 < t < T$ and $s_* \in \{0, s_0, s_1\}$. Then we use $H^{-1,r_*} \subset B_{r_*,\infty}^{-1}$ and Lemma 1 to obtain

$$\|e^{t(\alpha I + \Delta)} \nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{B_{p,1}^{s_*-2+n/p}} \leq C e^{\alpha t} t^{-s_*/2-1+s_0/2+s_1/2} \|u\|_{H^{s_1-2+n/p,p}} \|v\|_{H^{s_0-2+n/p,p}}.$$

Since

$$\int_0^t (t-\tau)^{\lambda-1} \tau^{\mu-1} d\tau = B(\lambda, \mu) t^{\lambda+\mu-1} \quad (3.4)$$

for all $t > 0$ and $\lambda, \mu \in (0, \infty)$, where $B(\lambda, \mu) := \int_0^1 (1-\tau)^{\lambda-1} \tau^{\mu-1} d\tau < \infty$, we have

$$\|I(u, v)(t)\|_{B_{p,1}^{s_*-2+n/p}} \leq C e^{\alpha} [u]_{X_t} [v]_{X_t} \int_0^t (t-\tau)^{-s_*/2-1+s_0/2+s_1/2} \tau^{-s_1/2} \tau^{-s_0/2} d\tau \leq C e^{\alpha} t^{-s_*/2} [u]_{X_t} [v]_{X_t}$$

for all $0 < t < T$ due to $s_0 + s_1 < 2$. In addition, combining Proposition 2 (i) and (ii) yields

$$\sum_{j=0}^1 \|I(u, v)\|_{L_t^{2/s_j,q}(B_{p,1}^{s_j-2+n/p})} \leq C e^{\alpha} \|u\|_{H^{s_1-2+n/p,p}} \|v\|_{H^{s_0-2+n/p,p}} \|u\|_{L_t^{2/(s_0+s_1),q}} \leq C e^{\alpha} [u]_{Y_t} [v]_{Y_t}.$$

Next we consider the estimates of $J(u, v)$. Taking r so that $1/r = (\kappa - 1)(2 - s_0)/n + (2 - s_1)/n$ holds, we have $1 < r < n/2 < p$ due to Proposition 7. Hence, we combine $B_{r,1}^{s_*-2+n/r} \subset B_{p,1}^{s_*-2+n/p}$ and Proposition 3 (i) to obtain

$$\|e^{t(\alpha I + \Delta)} (|u|^{k-1} u^l - |v|^{k-1} v^l)\|_{B_{p,1}^{s_*-2+n/p}} \leq C e^{\alpha t} \|e^{t\Delta} (|u|^{k-1} u^l - |v|^{k-1} v^l)\|_{B_{r,1}^{s_*-2+n/r}} \leq C e^{\alpha t} t^{-(1/2)(s_*-2+n/r)} \| |u|^{k-1} u^l - |v|^{k-1} v^l \|_{B_{r,\infty}^0}$$

for all $0 < t < T$ and $s_* \in \{0, s_0, s_1\}$. Then we rely on $L^r \subset B_{r,\infty}^0$ and Lemma 1 to deduce that

$$\|e^{t(\alpha I + \Delta)} (|u|^{k-1} u^l - |v|^{k-1} v^l)\|_{B_{p,1}^{s_*-2+n/p}} \leq C e^{\alpha} t^{-s_*/2-\kappa+1+(\kappa-1)s_0/2+s_1/2} (\|u\|_{H^{s_0-2+n/p,p}}^{k-1} + \|v\|_{H^{s_0-2+n/p,p}}^{k-1}) \|u - v\|_{H^{s_1-2+n/p,p}}.$$

Thus we observe that

$$\begin{aligned} \|J(u, v)(t)\|_{B_{p,1}^{s_*-2+n/p}} &\leq C e^{\alpha} ([u]_{X_t}^{k-1} + [v]_{X_t}^{k-1}) [u - v]_{X_t} \int_0^t (t-\tau)^{-s_*/2-\kappa+1+(\kappa-1)s_0/2+s_1/2} \tau^{-(\kappa-1)s_0/2} \tau^{-s_1/2} d\tau \\ &\leq C e^{\alpha} t^{-s_*/2+(2-\kappa)} ([u]_{X_t}^{k-1} + [v]_{X_t}^{k-1}) [u - v]_{X_t} \end{aligned}$$

due to $\kappa < 2$ and $s_0 + s_1 < 2$. In addition, taking γ such that

$$\frac{1}{\gamma} = (\kappa - 1) \frac{s_0}{2} + \frac{s_1}{2} + (2 - \kappa),$$

we have $1 < \gamma < \infty$ from $s_1 < (\kappa - 1)(2 - s_0)$. As Proposition 2 (i) yields

$$\|\varphi_0 \varphi_1\|_{L_t^{\gamma,q}} \leq C t^{2-\kappa} \|\varphi_0 \varphi_1\|_{L_t^{((\kappa-1)s_0/2+s_1/2)-1,q}} \leq C \|\varphi_0\|_{L_t^{2/((\kappa-1)s_0),q}} \|\varphi_1\|_{L_t^{2/s_1,q}}$$

for all $\varphi_0 \in L^{2/((\kappa-1)s_0),q}(0, t)$ and $\varphi_1 \in L^{2/s_1,q}(0, t)$ provided $0 < t \leq 1$, by relying on (2.2) and Proposition 2 (ii), we observe that

$$\begin{aligned} \|J(u, v)\|_{L_t^{2/s_j,q}(B_{p,1}^{s_j-2+n/p})} &\leq C e^{\alpha} \left(\|u\|_{H^{s_0-2+n/p,p}}^{k-1} + \|v\|_{H^{s_0-2+n/p,p}}^{k-1} \|u - v\|_{H^{s_1-2+n/p,p}} \right) \|u - v\|_{L_t^{\gamma,q}} \\ &\leq C e^{\alpha} \left(\|u\|_{H^{s_0-2+n/p,p}}^{k-1} \|u - v\|_{L_t^{2/((\kappa-1)s_0),q}} + \|u\|_{H^{s_0-2+n/p,p}}^{k-1} \|u - v\|_{L_t^{2/((\kappa-1)s_0),q}} \right) \|u - v\|_{L_t^{2/s_1,q}(H^{s_1-2+n/p,p})} \\ &\leq C e^{\alpha} ([u]_{Y_t}^{k-1} + [v]_{Y_t}^{k-1}) [u - v]_{Y_t} \end{aligned}$$

for all $0 < t < T$, which completes the proof of Lemma 2. $\square$

Once we show Lemma 2, by applying the estimate of the heat semigroup in Proposition 3 and the usual Banach fixed point theorem, we may construct solutions of (1.5), i.e., show Theorem 1.

Proof of Theorem 1. (i) Let $0 < T \leq 1$ be arbitrary. Define the function space

$$Z_T := \left\{ u \left| \begin{array}{l} u \in BC([0, T]; B_{p,q}^{-2+n/p}), \quad t^{s_j/2} u \in BC([0, T]; B_{p,1}^{s_j-2+n/p}), \quad u \in L^{2/s_j,q}((0, T); B_{p,1}^{s_j-2+n/p}), \\ \lim_{t \rightarrow +0} t^{s_j/2} \|u(t)\|_{B_{p,1}^{s_j-2+n/p}} = 0, \quad j \in \{0, 1\} \end{array} \right. \right\} \quad (3.5)$$

with the norm $\|u\|_{Z_T} := \|u\|_{L_T^\infty(B_{p,q}^{-2+n/p})} + \|u\|_{X_T} + \|u\|_{Y_T}$, where

$$\|u\|_{X_t} := \sum_{j=0}^1 \sup_{0 < \tau < t} \tau^{s_j/2} \|u(\tau)\|_{B_{p,1}^{s_j-2+n/p}}, \quad \|u\|_{Y_t} := \sum_{j=0}^1 \|u\|_{L_t^{2/s_j,q}(B_{p,1}^{s_j-2+n/p})} \quad (3.6)$$

for $0 < t < T$. In addition, define the mapping Φ by setting

$$\Phi u(t) := e^{t(\alpha I + \Delta)} a - \int_0^t e^{(t-\tau)(\alpha I + \Delta)} \left(\nabla \cdot (u \mathcal{K}_{\beta_1} u) + \beta_0 |u|^{\kappa-1} u^l \right) d\tau = e^{t(\alpha I + \Delta)} a - \mathcal{I}(u, u)(t) - \beta_0 \mathcal{J}(u, 0)(t), \quad 0 < t < T \quad (3.7)$$

for $u \in Z_T$, where $\mathcal{I}(u, v)$ and $\mathcal{J}(u, v)$ are given by (3.3). As Lemma 2 implies that

$$\begin{aligned} \|\Phi u(t)\|_{B_{p,q}^{-2+n/p}} &\leq \|e^{t(\alpha I + \Delta)} a\|_{B_{p,q}^{-2+n/p}} + C e^\alpha (\|u\|_{X_t}^2 + \|u\|_{X_t}^\kappa), \\ \|\Phi u(t)\|_{B_{p,1}^{s_j-2+n/p}} &\leq \|e^{t(\alpha I + \Delta)} a\|_{B_{p,1}^{s_j-2+n/p}} + C e^\alpha t^{-s_j/2} (\|u\|_{X_t}^2 + \|u\|_{X_t}^\kappa) \end{aligned}$$

for all $u \in Z_T$, $0 < t < T$, and $j \in \{0, 1\}$, Proposition 4 (i) yields $\lim_{t \rightarrow +0} \|\Phi u\|_{X_t} = 0$ due to $1 \leq q < \infty$. Since Lemma 2 provides

$$\|\Phi u\|_{L_T^{2/s_j, q}(B_{p,1}^{s_j-2+n/p})} \leq C \|e^{t(\alpha I + \Delta)} a\|_{L_T^{2/s_j, q}(B_{p,1}^{s_j-2+n/p})} + C e^\alpha (\|u\|_{Y_T}^2 + \|u\|_{Y_T}^\kappa),$$

by combining the above estimates, we observe that

$$\begin{cases} \|\Phi u\|_{X_T \cap Y_T} \leq C e^\alpha \|e^{t\Delta} a\|_{X_T \cap Y_T} + C e^\alpha (\|u\|_{X_T \cap Y_T}^2 + \|u\|_{X_T \cap Y_T}^\kappa), \\ \|\Phi u\|_{Z_T} \leq C e^\alpha \|e^{t\Delta} a\|_{Z_T} + C e^\alpha (\|u\|_{X_T \cap Y_T}^2 + \|u\|_{X_T \cap Y_T}^\kappa). \end{cases} \quad (3.8)$$

Noting that Proposition 3 yields

$$\|e^{t\Delta} a\|_{Z_T} \leq C \|a\|_{B_{p,q}^{-2+n/p}}, \quad (3.9)$$

we obtain $\Phi u \in Z_T$ for all $u \in Z_T$. In addition, as there holds

$$\Phi u(t) - \Phi v(t) = -\mathcal{I}(u - v, u)(t) - \mathcal{I}(v, u - v)(t) - \beta_0 \mathcal{J}(u, v)(t)$$

for all $u, v \in Z_T$ and $0 < t < T$, we also see by Lemma 2 that

$$\|\Phi u - \Phi v\|_{Z_T} \leq C e^\alpha \left(\|u\|_{X_T \cap Y_T} + \|u\|_{X_T \cap Y_T}^{\kappa-1} + \|v\|_{X_T \cap Y_T} + \|v\|_{X_T \cap Y_T}^{\kappa-1} \right) \|u - v\|_{Z_T}. \quad (3.10)$$

Now we take $0 < T \leq 1$ sufficiently small such that

$$\|e^{t\Delta} a\|_{X_T \cap Y_T} \leq (8C e^\alpha)^{-\kappa/(\kappa-1)} \quad (3.11)$$

from Proposition 4 (i) with $1 \leq q < \infty$ and assume that $u, v \in Z_T$ satisfy

$$\max\{\|u\|_{X_T \cap Y_T}, \|v\|_{X_T \cap Y_T}\} \leq (8C e^\alpha)^{-1/(\kappa-1)}. \quad (3.12)$$

Then it holds by (3.8), (3.10), (3.11), and (3.12) that

$$\begin{aligned} \|\Phi u\|_{X_T \cap Y_T} &\leq C e^\alpha \cdot (8C e^\alpha)^{-\kappa/(\kappa-1)} + C e^\alpha \{ (8C e^\alpha)^{-2/(\kappa-1)} + (8C e^\alpha)^{-\kappa/(\kappa-1)} \} \\ &\leq \frac{1}{8} (8C e^\alpha)^{-1/(\kappa-1)} + C e^\alpha \cdot 2(8C e^\alpha)^{-\kappa/(\kappa-1)} \leq (8C e^\alpha)^{-1/(\kappa-1)}, \\ \|\Phi u - \Phi v\|_{Z_T} &\leq C e^\alpha \{ 2(8C e^\alpha)^{-1/(\kappa-1)} + 2(8C e^\alpha)^{-1} \} \|u - v\|_{Z_T} \\ &= \left\{ \frac{1}{4} (8C e^\alpha)^{-1/(\kappa-1)+1} + \frac{1}{4} \right\} \|u - v\|_{Z_T} \leq \frac{1}{2} \|u - v\|_{Z_T}. \end{aligned}$$

Therefore, we may apply the Banach fixed point theorem. Namely, there exists a unique function $u \in Z_T$ such that $\Phi u = u$ with the estimate $\|u\|_{X_T \cap Y_T} \leq (8C e^\alpha)^{-1/(\kappa-1)}$. Moreover, relying on the estimates (3.8) and (3.9), we observe that

$$\begin{aligned} \|u\|_{Z_T} &\leq C e^\alpha \|a\|_{B_{p,q}^{-2+n/p}} + C e^\alpha \{ (8C e^\alpha)^{-1/(\kappa-1)} + (8C e^\alpha)^{-1} \} \|u\|_{Z_T} \\ &= C e^\alpha \|a\|_{B_{p,q}^{-2+n/p}} + \left\{ \frac{1}{8} (8C e^\alpha)^{-1/(\kappa-1)+1} + \frac{1}{8} \right\} \|u\|_{Z_T} \\ &\leq C e^\alpha \|a\|_{B_{p,q}^{-2+n/p}} + \frac{1}{4} \|u\|_{Z_T}, \end{aligned}$$

which yields (1.8). From (3.7) and Lemma 2, we have

$$\|u(t) - a\|_{B_{p,q}^{-2+n/p}} \leq \|e^{t(\alpha I + \Delta)} a - a\|_{B_{p,q}^{-2+n/p}} + C e^\alpha (\|u\|_{X_t}^2 + \|u\|_{X_t}^\kappa)$$

for all $0 < t < T$. Since $\lim_{t \rightarrow +0} \|u\|_{X_t} = 0$ due to (3.5) and since $\{e^{t(\alpha I + \Delta)}\}_{t \geq 0}$ is a C_0 -semigroup on $B_{p,q}^{-2+n/p}$ due to $1 \leq q < \infty$, we obtain (1.7).

Finally, we show the uniqueness of solutions of (1.5). Suppose that u satisfies the condition (1.9). Then Proposition 2 (iii) implies that we may take $u_{\varepsilon_*}^{(j)}$ such that

$$\|u_{\varepsilon_*}^{(j)}\|_{L_{T_*}^{2/s_j, \infty}(H^{s_j-2+n/p, p})} \leq 2\varepsilon_*, \quad \|u - u_{\varepsilon_*}^{(j)}\|_{L_{T_*}^\infty(H^{s_j-2+n/p, p})} \leq M_{\varepsilon_*}$$

for $j \in \{0, 1\}$ with some $M_{\varepsilon_*} > 0$. Thus we observe that

$$\|u\|_{L^{2/s_j, \infty}((t, t+h); H^{s_j-2+n/p, p})} \leq C \left(\|u_{\varepsilon_*}^{(j)}\|_{L^{2/s_j, \infty}(H^{s_j-2+n/p, p})} + \|u - u_{\varepsilon_*}^{(j)}\|_{L^{2/s_j}((t, t+h); H^{s_j-2+n/p, p})} \right) \leq C(2\varepsilon_* + M_{\varepsilon_*} h^{s_j/2})$$

for all $t, h \in (0, T_*)$ such that $t + h < T_*$, which yield

$$\sum_{j=0}^1 \|u\|_{L^{2/s_j, \infty}((t, t+h); H^{s_j-2+n/p, p})} \leq C\varepsilon_* \quad (3.13)$$

for sufficiently small $0 < h < T_*$. Therefore, to show the uniqueness assertion, we may assume that (3.13). Now we suppose that u and v satisfy (1.5) with the corresponding condition to (3.13). We set

$$S(t) := \sum_{j=0}^1 \|u - v\|_{L_t^{2/s_j, \infty}(H^{s_j-2+n/p, p})}$$

for $0 < t < T_*$. Then, since it holds that

$$u(t) = e^{t(\alpha I + \Delta)} a - I(u, u)(t) - \beta_0 J(u, 0)(t), \quad v(t) = e^{t(\alpha I + \Delta)} a - I(v, v)(t) - \beta_0 J(v, 0)(t)$$

for all $0 < t < T_*$, we have

$$u(t) - v(t) = -I(u - v, u)(t) - I(v, u - v)(t) - \beta_0 J(u, v)(t).$$

Thus we see by Lemma 2 and (3.13) that $S(h) \leq C e^{\alpha(\varepsilon_* + \varepsilon_*^{\kappa-1})} S(h)$. By taking $0 < \varepsilon_* < 1$ sufficiently small, we deduce that $S(h) = 0$, which implies that $u = v$ on $(0, h)$. Hence, by repeating this argument, we may show that $u = v$ on $(0, T_*)$.

(ii) Let $0 < T \leq 1$ be arbitrary. Define the function space

$$\tilde{Z}_T := \left\{ u \mid u \in BC((0, T); B_{p, \infty}^{-2+n/p}), \quad t^{s_j/2} u \in BC((0, T); B_{p, 1}^{s_j-2+n/p}), \quad j \in \{0, 1\} \right\}$$

with the norm $\|u\|_{\tilde{Z}_T} := \|u\|_{L_T^\infty(B_{p, \infty}^{-2+n/p})} + \|u\|_{X_T}$, where X_T is given by (3.6). We consider Φ defined by (3.7) again. Then, in the same way as in the proof of (i), we obtain

$$\begin{aligned} \|\Phi u\|_{X_T} &\leq e^\alpha \|e^{t\Delta} a\|_{X_T} + C e^\alpha (\|u\|_{X_T}^2 + \|u\|_{X_T}^\kappa), \\ \|\Phi u\|_{\tilde{Z}_T} &\leq e^\alpha \|e^{t\Delta} a\|_{\tilde{Z}_T} + C e^\alpha (\|u\|_{X_T}^2 + \|u\|_{X_T}^\kappa), \\ \|\Phi u - \Phi v\|_{\tilde{Z}_T} &\leq C e^\alpha (\|u\|_{X_T} + \|u\|_{X_T}^{\kappa-1} + \|v\|_{X_T} + \|v\|_{X_T}^{\kappa-1}) \|u - v\|_{\tilde{Z}_T} \end{aligned}$$

for all $u, v \in \tilde{Z}_T$. Thus we see by (3.9) that $\Phi u \in \tilde{Z}_T$.

As $a \in B_{p, \infty}^{-2+n/p}$, we see by Proposition 1 that

$$\|\dot{S}_j a\|_{B_{p, 1}^{\sigma-2+n/p}} \leq \frac{2^{\sigma(j+1)} C}{1 - 2^{-\sigma}} \|a\|_{B_{p, \infty}^{-2+n/p}}, \quad \|(I - \dot{S}_j) a\|_{B_{p, \infty}^{-2+n/p}} \leq C \sup_{k \geq j} 2^{(-2+n/p)k} \|\dot{A}_k a\|_{L^p}$$

for all $j \in \mathbb{N}$ and $\sigma > 0$. Noting the condition (1.10), we may assume that $\|(I - \dot{S}_j) a\|_{B_{p, \infty}^{-2+n/p}} \leq C\varepsilon$ by taking $j \in \mathbb{N}$ sufficiently large.

Therefore, Proposition 3 (i) implies that

$$\begin{aligned} \|e^{t\Delta} a\|_{X_T} &\leq \|e^{t\Delta} (I - \dot{S}_j) a\|_{X_T} + \|e^{t\Delta} \dot{S}_j a\|_{X_T} \\ &\leq C \|(I - \dot{S}_j) a\|_{B_{p, \infty}^{-2+n/p}} + C(T^{(s_0-\sigma)/2} + T^{(s_1-\sigma)/2}) \|\dot{S}_j a\|_{B_{p, 1}^{\sigma-2+n/p}} \\ &\leq C\varepsilon + \frac{2^{\sigma(j+1)} C}{1 - 2^{-\sigma}} (T^{(s_0-\sigma)/2} + T^{(s_1-\sigma)/2}) \|a\|_{B_{p, \infty}^{-2+n/p}} \end{aligned}$$

provided $0 < \sigma < \min\{s_0, s_1\}$. Since we deduce that $\limsup_{T \rightarrow +0} \|e^{t\Delta} a\|_{X_T} \leq C\varepsilon$, we may assume that (3.11) by taking $0 < \varepsilon < 1$ and $0 < T \leq 1$ sufficiently small. Hence, in the same way as in the proof of (i), we obtain a solution $u \in \tilde{Z}_T$ of (1.5).

Concerning the weak-star continuity (1.11), we see by (3.7) that

$$\langle u(t) - a, \psi \rangle = \langle e^{t(\alpha I + \Delta)} a - a, \psi \rangle - \langle I(u, u)(t) + \beta_0 J(u, 0)(t), \psi \rangle$$

for all $\psi \in B_{p', 1}^{-2-n/p}$. Thus we apply Proposition 4 (ii) and (iii) to deduce that (1.11) holds.

(iii) Since u and v satisfy

$$u(t) = e^{t(\alpha I + \Delta)} a - I(u, u)(t) - \beta_0 J(u, 0)(t), \quad v(t) = e^{t(\alpha I + \Delta)} b - I(v, v)(t) - \beta_0 J(v, 0)(t)$$

for all $0 < t < T$, respectively, we obtain

$$u(t) - v(t) = e^{t(\alpha I + \Delta)} (a - b) - I(u - v, u)(t) - I(v, u - v)(t) - \beta_0 J(u, v)(t).$$

Therefore, noting the condition (3.12), we see by Proposition 3 and Lemma 2 that

$$\|u - v\|_{Z_T} \leq C e^\alpha \|a - b\|_{B_{p, q}^{-2+n/p}} + \frac{1}{2} \|u - v\|_{Z_T},$$

which yields the desired estimate. This completes the proof of Theorem 1. $\square$

4. Partial smoothing effects

4.1. First regularization result

In Theorem 1, we have constructed solutions of (1.5). In what follows, we shall show that the solution u of (1.5) has additional regularities in space and time, namely, we show Theorem 2. Consequently, it is verified that the solution u of (1.5) satisfies (1.3) pointwise in space and time. The desired assertion will be shown by the standard bootstrap arguments. We first remark that Proposition 3 (i) with the assumption $a \in B_{p,q}^{-2+n/p}$ yields $e^{t\Delta}a \in C^\infty((0, \infty); B_{p,1}^{s+\sigma})$ for all $\sigma > 0$, and hence the first term on the right-hand side of (1.5) has a sufficient regularity. We begin with the following smoothing property of the solution u of (1.5):

Lemma 3. *In the statement of Theorem 1 (i) and (ii), the solution u of (1.5) satisfies*

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{1+2\theta}).$$

Proof. Lemma 1 implies that the solution u of (1.5) satisfies

$$\begin{aligned} \|\nabla \cdot (u(t)\mathcal{K}_{\beta_1}u(t))\|_{H^{-1,r_*}} &\leq C\|u(t)\|_{H^{s_1-2+n/p,p}}\|u(t)\|_{H^{s_0-2+n/p,p}} \leq Ct^{-(s_0+s_1)/2}\|u\|_{X_T}^2, \\ \| |u(t)|^{\kappa-l}(u(t))^l \|_{L^r} &\leq C\|u(t)\|_{H^{s_0-2+n/p,p}}^{\kappa-1}\|u(t)\|_{H^{s_1-2+n/p,p}} \leq Ct^{-(\kappa-1)s_0/2-s_1/2}\|u\|_{X_T}^\kappa \end{aligned}$$

for all $0 < t < T$, where $\|u\|_{X_T}$ is given by (3.6) and $1/r = (\kappa - 1)(2 - s_0)/n + (2 - s_1)/n$ and $1/r_* = (1 - s_0)/n + (2 - s_1)/n$. As Proposition 7 yields $r, r_* \in (1, p)$, we see by the Sobolev embeddings that $L^r \subset H^{-n(1/r-1/p),p}$ and $H^{-1,r_*} \subset H^{-1-n(1/r_*-1/p),p}$, which provide

$$\begin{cases} \nabla \cdot (u\mathcal{K}_{\beta_1}u) \in (L^1 \cap L_{\text{loc}}^\infty)((0, T]; H^{-1-n(1/r_*-1/p),p}), \\ |u|^{\kappa-l}u^l \in (L^1 \cap L_{\text{loc}}^\infty)((0, T]; H^{-n(1/r-1/p),p}) \end{cases} \quad (4.1)$$

due to $s_0 + s_1 < 2$. We note that

$$\frac{n}{r} < 1 + \frac{n}{r_*} \quad (4.2)$$

from the definitions of r and r_* , which yields

$$H^{-n(1/r-1/p),p} \subset H^{-1-n(1/r_*-1/p),p} \subset B_{p,\infty}^{-1-n(1/r_*-1/p)}.$$

Therefore, applying Proposition 5 (i), we deduce that

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{-1-n(1/r_*-1/p)+2\theta}). \quad (4.3)$$

Since $\max\{s_0, s_1\} < s_0 + s_1 = 3 - n/r_*$, we have

$$\frac{1}{2} \left(\max\{s_0, s_1\} - 1 + \frac{n}{r_*} \right) < 1, \quad (4.4)$$

and thus we may take $0 < \eta < 1$ sufficiently small so that

$$0 < \frac{\eta}{2} + \frac{1}{2} \left(\max\{s_0, s_1\} - 1 + \frac{n}{r_*} \right) < 1$$

holds. Setting $\theta = \eta/2 + (1/2)(\max\{s_0, s_1\} - 1 + n/r_*)$ in (4.3), we deduce that

$$u \in L_{\text{loc}}^\infty((0, T]; B_{p,1}^{\eta+\max\{s_0, s_1\}-2+n/p}).$$

Then we apply Lemma 1 again to obtain

$$\|\nabla \cdot (u(t)\mathcal{K}_{\beta_1}u(t))\|_{H^{\eta-1,r_*}} \leq C\|u(t)\|_{B_{p,1}^{\eta+\max\{s_0, s_1\}-2+n/p}}^2 \quad (4.5)$$

for all $0 < t < T$, which yields

$$\nabla \cdot (u\mathcal{K}_{\beta_1}u) \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^{\eta-1-n(1/r_*-1/p)}) \quad (4.6)$$

due to $H^{\eta-1,r_*} \subset B_{p,\infty}^{\eta-1-n(1/r_*-1/p)}$. Therefore, combining the properties (4.1) and (4.6) with Proposition 5 (i) implies that

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{\min\{-n/r, \eta-1-n/r_*\}+n/p+2\theta}). \quad (4.7)$$

Here we assume that $\eta - 1 - n/r_* < -n/r$. Then there holds

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{\eta-1-n(1/r_*-1/p)+2\theta}).$$

Note that the space regularity of u is higher than that of (4.3). Hence, by repeating the above argument, we see that (4.7) holds with η replaced by 2η . Consequently, noting the condition (4.2), we may take $\eta = 1 + n(1/r_* - 1/r) > 0$ in (4.7). Namely, we may assume that

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{-n(1/r-1/p)+2\theta}). \quad (4.8)$$

Now we take $0 < \sigma < 1$ sufficiently small. Then the condition (4.4) yields

$$0 < \frac{\sigma}{2} + \frac{1}{2} \left(\max\{s_0, s_1\} - 1 + \frac{n}{r_*} \right) < 1, \quad (4.9)$$

and thus we set $\theta = \sigma/2 + (1/2)(\max\{s_0, s_1\} - 1 + n/r_*)$ in (4.8) to obtain

$$u \in L_{\text{loc}}^\infty((0, T]; B_{p,1}^{\sigma+\max\{s_0,s_1\}-1+n(1/r_*-1/r+1/p)}).$$

As (4.2) holds, by letting $\eta = \sigma + 1 + n(1/r_* - 1/r) > 0$ in (4.5), we observe that

$$\|\nabla \cdot (u(t) \mathcal{K}_{\beta_1} u(t))\|_{H^{\sigma+n(1/r_*-1/r), r_*}} \leq C \|u(t)\|_{B_{p,1}^{\sigma+\max\{s_0,s_1\}-1+n(1/r_*-1/r+1/p)}}^2$$

for all $0 < t < T$, which implies that

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^{\sigma-n(1/r-1/p)}) \quad (4.10)$$

due to $H^{\sigma+n(1/r_*-1/r), r_*} \subset B_{p,\infty}^{\sigma-n(1/r-1/p)}$. Moreover, by applying Lemma 1 again, we also obtain

$$\| |u(t)|^{\kappa-l} (u(t))^l \|_{L^{(1/r-\sigma/n)^{-1}}} \leq C \|u(t)\|_{H^{s_0-2+n/p,p}}^{\kappa-1} \|u(t)\|_{H^{\sigma+s_1-2+n/p,p}}$$

for all $0 < t < T$. Therefore, since

$$B_{p,1}^{\sigma+\max\{s_0,s_1\}-1+n(1/r_*-1/r+1/p)} \subset H^{s_0-2+n/p,p} \cap H^{\sigma+s_1-2+n/p,p}$$

from (4.2) and since $L^{(1/r-\sigma/n)^{-1}} \subset B_{p,\infty}^{\sigma-n(1/r-1/p)}$, we have

$$|u|^{\kappa-l} u^l \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^{\sigma-n(1/r-1/p)}). \quad (4.11)$$

Combining (4.10) and (4.11), we may apply Proposition 5 (i) to obtain

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{\sigma-n(1/r-1/p)+2\theta}), \quad (4.12)$$

and thus we see the gain of regularity in space compared with (4.8). By repeating this argument, we may consider a larger index $\sigma > 0$ in (4.12). As Proposition 7 yields $0 < n(1/r - 1/p) < 2 - s_1$, we observe that $\sigma = n(1/r - 1/p)$ satisfies the assumption of the estimate in Lemma 1. Therefore, we may take $\sigma = n(1/r - 1/p)$ in (4.10) and (4.11) to obtain

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u), |u|^{\kappa-l} u^l \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^0),$$

and thus Proposition 5 (i) provides that

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{2\theta}). \quad (4.13)$$

In addition, we take $0 < \sigma < 1$ so that (4.9) holds again. Since $n/p < 2$, we may take $1/2 < \theta < 1$ such that $\max\{n/p, \sigma + \max\{s_0, s_1\} - 1 + n/r_*\} < 2\theta$. Thus we see by (4.13) that

$$u \in L_{\text{loc}}^\infty((0, T]; B_{p,1}^{n/p} \cap B_{p,1}^{\sigma+\max\{s_0,s_1\}-1+n/r_*}).$$

Noting that $\nabla(|u|^{\kappa-l} u^l) = \kappa |u|^{\kappa-2+l} u^{1-l} \nabla u$, we deduce that

$$\|\nabla(|u|^{\kappa-l} (u(t))^l)\|_{L^p} \leq \kappa \|u(t)\|_{L^\infty}^{\kappa-1} \|\nabla u(t)\|_{L^p} \leq C \|u(t)\|_{B_{p,1}^{n/p}}^{\kappa-1} \|u(t)\|_{B_{p,1}^1}$$

for all $0 < t < T$ due to $B_{p,1}^{n/p} \subset L^\infty$ and $B_{p,1}^0 \subset L^p$, and thus we have

$$|u|^{\kappa-l} u^l \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^1) \quad (4.14)$$

from $H^{1,p} \subset B_{p,\infty}^1$. Letting $\eta = \sigma + 1 + n(1/r_* - 1/p)$ in (4.5), we observe that

$$\|\nabla \cdot (u(t) \mathcal{K}_{\beta_1} u(t))\|_{H^{\sigma+n(1/r_*-1/p), r_*}} \leq C \|u(t)\|_{B_{p,1}^{\sigma+\max\{s_0,s_1\}-1+n/r_*}}^2$$

for all $0 < t < T$, which yields

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in L_{\text{loc}}^\infty((0, T]; B_{p,\infty}^\sigma) \quad (4.15)$$

due to $H^{\sigma+n(1/r_*-1/p), r_*} \subset B_{p,\infty}^\sigma$. Therefore, by combining (4.14) and (4.15), we use Proposition 5 (i) to obtain

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{\sigma+2\theta}).$$

As the space regularity of u is higher than that of (4.13), by repeating this argument, we may take $\sigma = 1$ in (4.15). Hence, combining (4.14) and Proposition 5 (i) again, we have the desired regularity. This completes the proof of Lemma 3. $\square$

4.2. Nonlinear estimates in higher regularities

Here, we give the nonlinear estimates in higher regularities. By the standard bootstrap arguments with the aid of the nonlinear estimates, we may show the smoothing properties that are stronger than that in Lemma 3.

Lemma 4. (i) Let $n/2 < p < n$, $1 \leq p_* \leq \infty$, and $\beta_1 \geq 0$. In the case of $0 < \sigma < 2 - n/p$, suppose that $\varphi_1 \in B_{p_*,1}^\sigma$ and $\varphi_0 \in H^{1,p}$. Then there holds $\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0) \in B_{p_*,1}^{\sigma-1}$ with the estimate

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma-1}} \leq C \|\varphi_1\|_{B_{p_*,1}^\sigma} \|\varphi_0\|_{H^{1,p}},$$

where $C > 0$ is a constant independent of φ_0 and φ_1 . In the case of $\sigma > 1$, suppose additionally that $\varphi_0 \in B_{\infty,1}^{\sigma-1}$. Then there holds $\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0) \in B_{p_*,1}^{\sigma-1}$ with the estimate

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma-1}} \leq C \|\varphi_1\|_{B_{p_*,1}^\sigma} \|\varphi_0\|_{H^{1,p} \cap B_{\infty,1}^{\sigma-1}}.$$

(ii) Let $1 < \kappa < 2$, $l \in \{0, 1\}$, and $0 < \sigma < 1$. For every $\varphi \in C^\sigma$, there holds $|\varphi|^{\kappa-l} \varphi^l \in C^\sigma$ with the estimate

$$\|\varphi|^{\kappa-l} \varphi^l\|_{C^\sigma} \leq 5 \|\varphi\|_{L^\infty}^{\kappa-1} \|\varphi\|_{C^\sigma}.$$

In addition, for every $\varphi_0 \in B_{\infty,1}^1$ and $\varphi_1 \in C^{\kappa-1}$, there holds $|\varphi_0|^{\kappa-2+l} \varphi_0^{1-l} \varphi_1 \in C^{\kappa-1}$ with the estimate

$$\| |\varphi_0|^{\kappa-2+l} \varphi_0^{1-l} \varphi_1 \|_{C^{\kappa-1}} \leq C \|\varphi_0\|_{B_{\infty,1}^1}^{\kappa-1} \|\varphi_1\|_{C^{\kappa-1}}.$$

Lemma 5. Let $1 \leq p_* \leq \infty$, $0 < \gamma < 1$, and $0 < T < \infty$. Then the following statements hold:

(i) Let $n/2 < p < n$ and $\beta_1 \geq 0$. In the case of $0 < \sigma < 2 - n/p$, suppose that $u \in C_{\text{loc}}^\gamma((0, T]; B_{p_*,1}^\sigma)$ and $v \in C_{\text{loc}}^\gamma((0, T]; H^{1,p})$. Then there holds $\nabla \cdot (u \mathcal{K}_{\beta_1} v) \in C_{\text{loc}}^\gamma((0, T]; B_{p_*,1}^{\sigma-1})$ with the estimate

$$\|\nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{C_{\text{loc}}^\gamma((\varepsilon, T); B_{p_*,1}^{\sigma-1})} \leq C \|u\|_{C_{\text{loc}}^\gamma((\varepsilon, T); B_{p_*,1}^\sigma)} \|v\|_{C_{\text{loc}}^\gamma((\varepsilon, T); H^{1,p})}$$

for all $0 < \varepsilon < T$, where $C > 0$ is a constant independent of ε , T , u , and v . In the case of $\sigma > 1$, suppose additionally that $v \in C_{\text{loc}}^\gamma((0, T]; H^{1,p} \cap B_{\infty,1}^{\sigma-1})$. Then there holds $\nabla \cdot (u \mathcal{K}_{\beta_1} v) \in C_{\text{loc}}^\gamma((0, T]; B_{p_*,1}^{\sigma-1})$ with the estimate

$$\|\nabla \cdot (u \mathcal{K}_{\beta_1} v)\|_{C_{\text{loc}}^\gamma((\varepsilon, T); B_{p_*,1}^{\sigma-1})} \leq C \|u\|_{C_{\text{loc}}^\gamma((\varepsilon, T); B_{p_*,1}^\sigma)} \|v\|_{C_{\text{loc}}^\gamma((\varepsilon, T); H^{1,p} \cap B_{\infty,1}^{\sigma-1})}$$

for all $0 < \varepsilon < T$.

(ii) Let $1 < \kappa < 2$ and $l \in \{0, 1\}$. For every $u \in C_{\text{loc}}^\gamma((0, T]; L^\infty)$ and $v \in C_{\text{loc}}^{(\kappa-1)\gamma}((0, T]; L^{p_*})$, there holds $|u|^{\kappa-2+l} u^{1-l} v \in C_{\text{loc}}^{(\kappa-1)\gamma}((0, T]; L^{p_*})$ with the estimate

$$\| |u|^{\kappa-2+l} u^{1-l} v \|_{C^{(\kappa-1)\gamma}((\varepsilon, T); L^{p_*})} \leq 7 \|u\|_{C_{\text{loc}}^\gamma((\varepsilon, T); L^\infty)}^{\kappa-1} \|v\|_{C^{(\kappa-1)\gamma}((\varepsilon, T); L^{p_*})}$$

for all $0 < \varepsilon < T$. In addition, for every $u, \partial_t u \in L_{\text{loc}}^\infty((0, T]; L^\infty)$ and $v \in C_{\text{loc}}^{\kappa-1}((0, T]; L^{p_*})$, there holds $|u|^{\kappa-2+l} u^{1-l} v \in C_{\text{loc}}^{\kappa-1}((0, T]; L^{p_*})$ with the estimate

$$\| |u|^{\kappa-2+l} u^{1-l} v \|_{C^{\kappa-1}((\varepsilon, T); L^{p_*})} \leq \left(2 \|u\|_{L_{\text{loc}}^\infty((\varepsilon, T); L^\infty)}^{\kappa-1} + 5 \|\partial_t u\|_{L_{\text{loc}}^\infty((\varepsilon, T); L^\infty)}^{\kappa-1} \right) \|v\|_{C^{\kappa-1}((\varepsilon, T); L^{p_*})}$$

for all $0 < \varepsilon < T$.

To show the desired estimates, we recall the fractional Leibniz rules on the Besov spaces, whose proof is given by, e.g., Kaneko, Kozono, and Shimizu [46, Proposition 2.2 (ii)].

Proposition 8. Let $1 \leq p \leq \infty$, $s > 0$, and $1 \leq q \leq \infty$. Suppose that $p_0, p_1, r_0, r_1 \in [1, \infty]$ satisfy

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{r_0} + \frac{1}{r_1}.$$

For every $\varphi_0 \in \dot{B}_{p_0,q}^s \cap L^{r_0}$ and $\varphi_1 \in \dot{B}_{r_1,q}^s \cap L^{p_1}$, it holds that $\varphi_0 \varphi_1 \in \dot{B}_{p,q}^s$ with the estimate

$$\|\varphi_0 \varphi_1\|_{\dot{B}_{p,q}^s} \leq C (\|\varphi_0\|_{\dot{B}_{p_0,q}^s} \|\varphi_1\|_{L^{p_1}} + \|\varphi_0\|_{L^{r_0}} \|\varphi_1\|_{\dot{B}_{r_1,q}^s}),$$

where $C > 0$ is a constant independent of φ_0 and φ_1 .

Proof of Lemma 4. (i) We first suppose that $\psi \in \dot{B}_{p,1}^{\sigma_0-1+n/p}$ for $\sigma_0 \in \mathbb{R}$. Then we see by (3.1) that

$$\|\mathcal{K}_{\beta_1} \psi\|_{\dot{B}_{\infty,1}^{\sigma_0}} \leq C \|(I - \beta_1(I - \Delta)^{-1})\psi\|_{\dot{B}_{\infty,1}^{\sigma_0-1}}. \quad (4.16)$$

Since the Sobolev embedding yields $\dot{B}_{p,1}^{\sigma_0-1+n/p} \subset \dot{B}_{\infty,1}^{\sigma_0-1}$ and since $I - \beta_1(I - \Delta)^{-1}$ is a bounded operator from L^p onto itself due to $1 < p < \infty$, we observe that

$$\|\mathcal{K}_{\beta_1} \psi\|_{\dot{B}_{\infty,1}^{\sigma_0}} \leq C \|(I - \beta_1(I - \Delta)^{-1})\psi\|_{\dot{B}_{p,1}^{\sigma_0-1+n/p}} \leq C \|\psi\|_{\dot{B}_{p,1}^{\sigma_0-1+n/p}}. \quad (4.17)$$

Next we suppose that $\psi \in B_{\infty,1}^{\sigma_1-1}$ for $\sigma_1 > 1$. As (4.16) holds, by using $B_{\infty,1}^{\sigma_1-1} \subset \dot{B}_{\infty,1}^{\sigma_1-1}$ and the fact that $I - \beta_1(\beta_1 I - \Delta)^{-1}$ is bounded from $B_{\infty,1}^{\sigma_1-1}$ onto itself (cf. [42, Theorem 2.2]), we deduce that

$$\|\mathcal{K}_{\beta_1} \psi\|_{\dot{B}_{\infty,1}^{\sigma_1}} \leq C \|(I - \beta_1(\beta_1 I - \Delta)^{-1})\psi\|_{B_{\infty,1}^{\sigma_1-1}} \leq C \|\psi\|_{B_{\infty,1}^{\sigma_1-1}}. \quad (4.18)$$

In the following, we show the assertions. Let $0 < \sigma < 2 - n/p$. Then we have $\dot{B}_{p_*,1}^{\sigma-1} \subset B_{p_*,1}^{\sigma-1}$ due to $2 - n/p < 1$, and thus we obtain

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma-1}} \leq C \|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{\dot{B}_{p_*,1}^{\sigma-1}} \leq C \|\varphi_1 \mathcal{K}_{\beta_1} \varphi_0\|_{\dot{B}_{p_*,1}^{\sigma}}. \quad (4.19)$$

In addition, since $\dot{B}_{\infty,1}^0 \subset L^\infty$ and since

$$H^{1,p} \subset B_{p,1}^{\sigma-1+n/p} \subset \dot{B}_{p,1}^{-1+n/p} \cap \dot{B}_{p,1}^{\sigma-1+n/p}$$

from $0 < -1 + n/p < \sigma - 1 + n/p < 1$, it holds by (4.17) with $\sigma_0 = 0$ or $\sigma_0 = \sigma$ that

$$\|\mathcal{K}_{\beta_1} \varphi_0\|_{L^\infty} \leq C \|\mathcal{K}_{\beta_1} \varphi_0\|_{B_{\infty,1}^0} \leq C \|\varphi_0\|_{\dot{B}_{p,1}^{-1+n/p}} \leq C \|\varphi_0\|_{H^{1,p}}, \quad (4.20)$$

$$\|\mathcal{K}_{\beta_1} \varphi_0\|_{B_{\infty,1}^\sigma} \leq C \|\varphi_0\|_{\dot{B}_{p,1}^{\sigma-1+n/p}} \leq C \|\varphi_0\|_{H^{1,p}}. \quad (4.21)$$

Therefore, by applying Proposition 8 to the estimate (4.19) and recalling $B_{p_*,1}^\sigma = L^{p_*} \cap \dot{B}_{p_*,1}^\sigma$, we observe that

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma-1}} \leq C(\|\varphi_1\|_{\dot{B}_{p_*,1}^\sigma} \|\mathcal{K}_{\beta_1} \varphi_0\|_{L^\infty} + \|\varphi_1\|_{L^{p_*}} \|\mathcal{K}_{\beta_1} \varphi_0\|_{\dot{B}_{\infty,1}^\sigma}) \leq C \|\varphi_1\|_{B_{p_*,1}^\sigma} \|\varphi_0\|_{H^{1,p}}. \quad (4.22)$$

Next, let $\sigma > 1$. Then the definition of the Besov spaces yields

$$\begin{aligned} \|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma-1}} &= \|\dot{S}_0(\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0))\|_{L^{p_*}} + \sum_{j \in \mathbb{N}} 2^{(\sigma-1)j} \|\dot{\Delta}_j(\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0))\|_{L^{p_*}} \\ &\leq \|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma_*-1}} + \|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{\dot{B}_{p_*,1}^{\sigma-1}} \end{aligned}$$

for all $\sigma_* \in \mathbb{R}$. Concerning the first term, by taking σ_* such that $0 < \sigma_* < 2 - n/p$ and noting the estimate (4.22), we have

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{B_{p_*,1}^{\sigma_*-1}} \leq C \|\varphi_1\|_{B_{p_*,1}^{\sigma_*}} \|\varphi_0\|_{H^{1,p}} \leq C \|\varphi_1\|_{B_{p_*,1}^\sigma} \|\varphi_0\|_{H^{1,p}}$$

due to $\sigma > \sigma_*$. Moreover, by applying Proposition 8 and using (4.18) with $\sigma_1 = \sigma$ and (4.20), we observe that the second term may be estimated by

$$\|\nabla \cdot (\varphi_1 \mathcal{K}_{\beta_1} \varphi_0)\|_{\dot{B}_{p_*,1}^{\sigma-1}} \leq C \|\varphi_1 \mathcal{K}_{\beta_1} \varphi_0\|_{\dot{B}_{p_*,1}^\sigma} \leq C(\|\varphi_1\|_{B_{p_*,1}^\sigma} \|\mathcal{K}_{\beta_1} \varphi_0\|_{L^\infty} + \|\varphi_1\|_{L^{p_*}} \|\mathcal{K}_{\beta_1} \varphi_0\|_{\dot{B}_{\infty,1}^\sigma}) \leq C \|\varphi_1\|_{B_{p_*,1}^\sigma} \|\varphi_0\|_{H^{1,p} \cap B_{\infty,1}^{\sigma-1}},$$

which implies the desired estimate.

(ii) The first estimate is obtained by [35, Proposition 4.2 (i)], and thus we focus on the second estimate. We note $1 < \kappa < 2$ to observe that

$$\left| |\tau|^{k-2+l} \tau^{1-l} \tau_* - |\sigma|^{k-2+l} \sigma^{1-l} \sigma_* \right| \leq |\tau|^{k-1} |\tau_* - \sigma_*| + 5 |\tau - \sigma|^{k-1} |\sigma_*| \quad (4.23)$$

for all $\tau, \sigma, \tau_*, \sigma_* \in \mathbb{R}$. Indeed, as

$$\begin{aligned} \left| |\tau|^{k-2+l} \tau^{1-l} \tau_* - |\sigma|^{k-2+l} \sigma^{1-l} \sigma_* \right| &= \left| |\tau|^{k-2+l} \tau^{1-l} (\tau_* - \sigma_*) + (|\tau|^{k-2+l} \tau^{1-l} - |\sigma|^{k-2+l} \sigma^{1-l}) \sigma_* \right| \\ &\leq |\tau|^{k-1} |\tau_* - \sigma_*| + \left| |\tau|^{k-2+l} \tau^{1-l} - |\sigma|^{k-2+l} \sigma^{1-l} \right| |\sigma_*| \end{aligned}$$

holds, we obtain (4.23) from [35, Proposition 2.3 (i)]. Since it holds by (4.23) that

$$\begin{aligned} &\left| |\varphi_0(x_1)|^{k-2+l} (\varphi_0(x_1))^{1-l} \varphi_1(x_1) - |\varphi_0(x_0)|^{k-2+l} (\varphi_0(x_0))^{1-l} \varphi_1(x_0) \right| \\ &\leq |\varphi_0(x_1)|^{k-1} |\varphi_1(x_1) - \varphi_1(x_0)| + 5 |\varphi_0(x_1) - \varphi_0(x_0)|^{k-1} |\varphi_1(x_0)| \\ &\leq \|\varphi_0\|_{L^\infty}^{k-1} \|\varphi_1\|_{C^{k-1}} |x_1 - x_0|^{k-1} + 5 (\|\nabla \varphi_0\|_{L^\infty} |x_1 - x_0|)^{k-1} \|\varphi_1\|_{L^\infty} \end{aligned}$$

for all $x_0, x_1 \in \mathbb{R}^n$ with the aid of the mean value theorem, we note that $B_{\infty,1}^0 \subset L^\infty$ to deduce that

$$\| |\varphi_0|^{k-2+l} \varphi_0^{1-l} \varphi_1 \|_{C^{k-1}} \leq \| |\varphi_0|^{k-1} \varphi_1 \|_{L^\infty} + \|\varphi_0\|_{L^\infty}^{k-1} \|\varphi_1\|_{C^{k-1}} + 5 \|\nabla \varphi_0\|_{L^\infty}^{k-1} \|\varphi_1\|_{L^\infty} \leq C \|\varphi_0\|_{B_{\infty,1}^1}^{k-1} \|\varphi_1\|_{C^{k-1}}.$$

This completes the proof of Lemma 4. $\square$

Proof of Lemma 5. (i) Note that

$$\nabla \cdot (u(t_1) \mathcal{K}_{\beta_1} v(t_1)) - \nabla \cdot (u(t_0) \mathcal{K}_{\beta_1} v(t_0)) = \nabla \cdot ((u(t_1) - u(t_0)) \mathcal{K}_{\beta_1} v(t_1)) + \nabla \cdot (u(t_0) \mathcal{K}_{\beta_1} (v(t_1) - v(t_0)))$$

for all $t_0, t_1 \in (0, T)$. Take $0 < \varepsilon < T$ arbitrarily small. In the case of $0 < \sigma < 2 - n/p$, we use Lemma 4 (i) to obtain

$$\begin{aligned} & \|\nabla \cdot (u(t_1)\mathcal{K}_{\beta_1} v(t_1)) - \nabla \cdot (u(t_0)\mathcal{K}_{\beta_1} v(t_0))\|_{B_{p_*,1}^{\sigma-1}} \\ & \leq C \left(\|u(t_1) - u(t_0)\|_{B_{p_*,1}^{\sigma}} \|v(t_1)\|_{H^{1,p}} + \|u(t_0)\|_{B_{p_*,1}^{\sigma}} \|v(t_1) - v(t_0)\|_{H^{1,p}} \right) \\ & \leq C \left(\|u\|_{C^{\gamma}((\varepsilon,T);B_{p_*,1}^{\sigma})} |t_1 - t_0|^{\gamma} \cdot \|v(t_1)\|_{H^{1,p}} + \|u(t_0)\|_{B_{p_*,1}^{\sigma}} \cdot \|v\|_{C^{\gamma}((\varepsilon,T);H^{1,p})} |t_1 - t_0|^{\gamma} \right) \end{aligned}$$

for all $t_0, t_1 \in (\varepsilon, T)$, which implies that

$$\|\nabla \cdot (u\mathcal{K}_{\beta_1} v)\|_{C^{\gamma}((\varepsilon,T);B_{p_*,1}^{\sigma-1})} \leq C \|u\|_{C^{\gamma}((\varepsilon,T);B_{p_*,1}^{\sigma})} \|v\|_{C^{\gamma}((\varepsilon,T);H^{1,p})}.$$

In the case of $\sigma > 1$, we also obtain the desired estimate by applying Lemma 4 (i) again.

(ii) Take $0 < \varepsilon < T$ arbitrarily small. Relying on (4.23), we have

$$\begin{aligned} & \| |u(t_1)|^{k-2+l} (u(t_1))^{1-l} v(t_1) - |u(t_0)|^{k-2+l} (u(t_0))^{1-l} v(t_0) \|_{L^{p_*}} \\ & \leq \|u(t_1)\|_{L^{\infty}}^{k-1} \|v(t_1) - v(t_0)\|_{L^{p_*}} + 5 \|u(t_1) - u(t_0)\|_{L^{\infty}}^{k-1} \|v(t_0)\|_{L^{p_*}} \\ & \leq \|u(t_1)\|_{L^{\infty}}^{k-1} \cdot \|v\|_{C^{(k-1)\gamma}((\varepsilon,T);L^{p_*})} |t_1 - t_0|^{(k-1)\gamma} + 5 \left(\|u\|_{C^{\gamma}((\varepsilon,T);L^{\infty})} |t_1 - t_0|^{\gamma} \right)^{k-1} \cdot \|v(t_0)\|_{L^{p_*}} \end{aligned}$$

for all $t_0, t_1 \in (\varepsilon, T)$, which yields

$$\| |u|^{k-2+l} u^{1-l} v \|_{C^{(k-1)\gamma}((\varepsilon,T);L^{p_*})} \leq \| |u|^{k-1} v \|_{L^{\infty}((\varepsilon,T);L^{p_*})} + 6 \|u\|_{C^{\gamma}((\varepsilon,T);L^{\infty})}^{k-1} \|v\|_{C^{(k-1)\gamma}((\varepsilon,T);L^{p_*})} \leq 7 \|u\|_{C^{\gamma}((\varepsilon,T);L^{\infty})}^{k-1} \|v\|_{C^{(k-1)\gamma}((\varepsilon,T);L^{p_*})}.$$

Moreover, since we see by (4.23) and the mean value theorem that

$$\begin{aligned} & \| |u(t_1)|^{k-2+l} (u(t_1))^{1-l} v(t_1) - |u(t_0)|^{k-2+l} (u(t_0))^{1-l} v(t_0) \|_{L^{p_*}} \\ & \leq \|u(t_1)\|_{L^{\infty}}^{k-1} \|v(t_1) - v(t_0)\|_{L^{p_*}} + 5 \|u(t_1) - u(t_0)\|_{L^{\infty}}^{k-1} \|v(t_0)\|_{L^{p_*}} \\ & \leq \|u(t_1)\|_{L^{\infty}}^{k-1} \cdot \|v\|_{C^{k-1}((\varepsilon,T);L^{p_*})} |t_1 - t_0|^{k-1} + 5 \left(\|\partial_t u\|_{L^{\infty}((\varepsilon,T);L^{\infty})} |t_1 - t_0| \right)^{k-1} \cdot \|v(t_0)\|_{L^{p_*}} \end{aligned}$$

for all $t_0, t_1 \in (\varepsilon, T)$, we deduce that

$$\begin{aligned} \| |u|^{k-2+l} u^{1-l} v \|_{C^{k-1}((\varepsilon,T);L^{p_*})} & \leq \| |u|^{k-1} v \|_{L^{\infty}((\varepsilon,T);L^{p_*})} + \left(\|u\|_{L^{\infty}((\varepsilon,T);L^{\infty})}^{k-1} + 5 \|\partial_t u\|_{L^{\infty}((\varepsilon,T);L^{\infty})}^{k-1} \right) \|v\|_{C^{k-1}((\varepsilon,T);L^{p_*})} \\ & \leq \left(2 \|u\|_{L^{\infty}((\varepsilon,T);L^{\infty})}^{k-1} + 5 \|\partial_t u\|_{L^{\infty}((\varepsilon,T);L^{\infty})}^{k-1} \right) \|v\|_{C^{k-1}((\varepsilon,T);L^{p_*})}. \end{aligned}$$

This completes the proof of Lemma 5. $\square$

4.3. Further regularities

Let us show the following lemma, which yields the space regularities of the solution u :

Lemma 6. *In the statement of Theorem 1 (i) and (ii), the solution u of (1.5) satisfies*

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{\infty,1}^{k+2\theta}), \quad u \in C_w((0, T]; C^{k+2}).$$

Proof. Since $2 - n/p > 0$, we may take $0 < \theta < 1$ such that $0 < 2\theta - n/p < 1$. We also take $0 < \sigma < 1$ sufficiently small so that $0 < \sigma < 2\theta - n/p$ holds. Then we note that $B_{p,1}^{1+2\theta} \subset B_{\infty,1}^{1+2\theta-n/p} \subset B_{\infty,1}^{\sigma+1}$ from the Sobolev embedding to obtain

$$u \in L_{\text{loc}}^{\infty}((0, T); H^{1,p} \cap B_{\infty,1}^{\sigma+1})$$

due to Lemma 3. While Lemma 4 (i) yields

$$\nabla \cdot (u\mathcal{K}_{\beta_1} u) \in L_{\text{loc}}^{\infty}((0, T]; B_{\infty,1}^{\sigma}),$$

we see by $B_{\infty,1}^{\sigma+1} \subset B_{\infty,\infty}^{\sigma} = C^{\sigma}$ and Lemma 4 (ii) that

$$|u|^{k-l} u^l \in L_{\text{loc}}^{\infty}((0, T]; C^{\sigma}).$$

Hence, we note the properties (4.1) and Proposition 5 (i) to deduce that

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{\infty,1}^{\sigma+2\theta}) \subset C_{\text{loc}}^{1-\kappa/2}((0, T]; B_{\infty,1}^{k+\sigma}). \quad (4.24)$$

In addition, we combine the property (4.24) and Lemma 4 (i) to obtain

$$\nabla \cdot (u\mathcal{K}_{\beta_1} u) \in L_{\text{loc}}^{\infty}((0, T]; B_{\infty,1}^{k+\sigma-1}).$$

Since $\nabla(|u|^{k-l} u^l) = \kappa |u|^{k-2+l} u^{1-l} \nabla u$, it holds by Lemma 4 (ii) that

$$\|\nabla(|u|^{k-l} u^l)\|_{C^{k-1}} \leq C \|u(t)\|_{B_{\infty,1}^{k-1}}^{k-1} \|\nabla u(t)\|_{C^{k-1}} \leq C \|u(t)\|_{C^k}^k$$

for all $0 < t < T$, which implies that

$$|u|^{k-l} u^l \in L_{\text{loc}}^\infty((0, T]; C^k). \quad (4.25)$$

Therefore, combining the above relations with Proposition 5 (i), we have

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{\infty,1}^{k+\sigma-1+2\theta}) \subset C_{\text{loc}}^{\sigma/2}((0, T]; B_{\infty,1}^{k+1}).$$

Using Lemma 4 (i) again, we obtain

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in L_{\text{loc}}^\infty((0, T]; B_{\infty,1}^k).$$

Hence, by combining the above relation and (4.25), we may apply Proposition 5 (i) again to obtain

$$u \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{\infty,1}^{k+2\theta}), \quad \Delta u \in L_{\text{loc}}^\infty((0, T]; C^k)$$

with the aid of $B_{\infty,\infty}^k = C^k$. The weak-star continuity in the time direction follows from the standard duality argument (cf. [36, Proposition 3.1]). This completes the proof of Lemma 6. $\square$

The following lemma ensures that the solution u of (1.5) satisfies the equation (1.3) pointwise in space and time:

Lemma 7. *In the statement of Theorem 1 (i) and (ii), the solution u of (1.5) satisfies*

$$u \in C_{\text{loc}}^k((0, T]; BUC^1) \cap C_{\text{loc}}^{k-1}((0, T]; BUC^3).$$

In particular, the solution u satisfies (1.3) pointwise in $(0, T) \times \mathbb{R}^n$.

Proof. Letting $\theta = 1/2$, $\theta = (3 - \kappa)/2$, or $\theta = 2 - \kappa$ in the statement of Lemma 6, we have

$$u \in C_{\text{loc}}^{1/2}((0, T]; B_{\infty,1}^{k+1}) \cap C_{\text{loc}}^{(\kappa-1)/2}((0, T]; B_{\infty,1}^3) \cap C_{\text{loc}}^{k-1}((0, T]; B_{\infty,1}^{4-\kappa}). \quad (4.26)$$

Since Lemma 5 (i) yields

$$\begin{aligned} \|\nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{C^{(k-1)/2}((\epsilon, T); B_{\infty,1}^2)} &\leq C \|u\|_{C^{(k-1)/2}((\epsilon, T); B_{\infty,1}^3)} \|u\|_{C^{(k-1)/2}((\epsilon, T); H^{1,p} \cap B_{\infty,1}^2)}, \\ \|\nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{C^{k-1}((\epsilon, T); B_{\infty,1}^{3-\kappa})} &\leq C \|u\|_{C^{k-1}((\epsilon, T); B_{\infty,1}^{4-\kappa})} \|u\|_{C^{k-1}((\epsilon, T); H^{1,p} \cap B_{\infty,1}^{3-\kappa})} \end{aligned}$$

for all $0 < \epsilon < T$, we rely on (4.26) and Lemma 3 to deduce that

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in C_{\text{loc}}^{(k-1)/2}((0, T]; B_{\infty,1}^2) \cap C_{\text{loc}}^{k-1}((0, T]; B_{\infty,1}^{3-\kappa}). \quad (4.27)$$

As $|u|^{k-l} u^l = |u|^{k-2+(1-l)} u^{1-(1-l)} |u|$ and $\nabla(|u|^{k-l} u^l) = \kappa |u|^{k-2+l} u^{1-l} \nabla u$, we apply Lemma 5 (ii) to obtain

$$\|\nabla^j(|u|^{k-l} u^l)\|_{C^{(k-1)/2}((\epsilon, T); L^\infty)} \leq 7 \kappa^j \|u\|_{C^{1/2}((\epsilon, T); L^\infty)}^{k-1} \|\nabla^j u\|_{C^{(k-1)/2}((\epsilon, T); L^\infty)}$$

for all $0 < \epsilon < T$ and $j \in \{0, 1\}$, which yields

$$|u|^{k-l} u^l \in C_{\text{loc}}^{(k-1)/2}((0, T]; BUC^1) \quad (4.28)$$

due to (4.26). Hence, noting that (4.27) and (4.28), we rely on (4.1) and Proposition 5 (ii) to observe that

$$u \in C_{\text{loc}}^{(k+1)/2}((0, T]; BUC^1) \cap C_{\text{loc}}^{(k-1)/2}((0, T]; BUC^3)$$

as $B_{\infty,1}^{2-\kappa} \subset BUC^1$. In particular, we have $u \in C^1((0, T]; BUC^1)$. Similarly to the derivation of (4.28), we use Lemma 5 (ii) to deduce that

$$\|\nabla^j(|u|^{k-l} u^l)\|_{C^{k-1}((\epsilon, T); L^\infty)} \leq \kappa^j \left(2 \|u\|_{L^\infty((\epsilon, T); L^\infty)}^{k-1} + 5 \|\partial_t u\|_{L^\infty((\epsilon, T); L^\infty)}^{k-1} \right) \|\nabla^j u\|_{C^{k-1}((\epsilon, T); L^\infty)}$$

for all $0 < \epsilon < T$ and $j \in \{0, 1\}$, which yields

$$|u|^{k-l} u^l \in C_{\text{loc}}^{k-1}((0, T]; BUC^1).$$

Since $B_{\infty,1}^{3-\kappa} \subset BUC^1$, by combining the above regularity and (4.27) and applying Proposition 5 (ii), we obtain the desired result. This completes the proof of Lemma 7. $\square$

Finally, we show the additional regularities (1.12) of the solution u of (1.5).

Proof of Theorem 2. Note that Lemmas 6 and 7 imply that it remains to verify

$$u \in C_{\text{loc}}^{k-1}((0, T]; H^{3,p}), \quad (4.29)$$

$$u \in C_{\text{loc}}^{k+1}((0, T]; BUC \cap L^p) \cap C_{\text{loc}}^k((0, T]; BUC^2 \cap H^{2,p}), \quad (4.30)$$

$$\partial_t u \in C_w((0, T]; C^{k+1}), \quad \partial_t^2 u \in C_w((0, T]; C^{k-1}) \quad (4.31)$$

to obtain the desired regularities (1.12). First we shall show that (4.29). To this end, we take $0 < \sigma < 1$ sufficiently small. Applying Lemma 5 (i), we deduce that

$$\|\nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{C^{k-1}((\epsilon, T); B_{p,1}^\sigma)} \leq C \|u\|_{C^{k-1}((\epsilon, T); B_{p,1}^{\sigma+1})} \|u\|_{C^{k-1}((\epsilon, T); H^{1,p} \cap B_{\infty,1}^\sigma)}$$

for all $0 < \varepsilon < T$, which yields

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in C_{\text{loc}}^{k-1}((0, T]; H^{\sigma, p}) \quad (4.32)$$

from [Lemmas 3](#) and [7](#) with $B_{p,1}^\sigma \subset C^{k-1}$. In addition, similarly to the derivation of (4.28), we use [Lemma 5](#) (ii) to obtain

$$\|\nabla^j(|u|^{k-l} u^l)\|_{C^{k-1}((\varepsilon, T); L^p)} \leq \kappa^j \left(2\|u\|_{L^\infty((\varepsilon, T); L^\infty)}^{k-1} + \|\partial_t u\|_{L^\infty((\varepsilon, T); L^\infty)}^{k-1} \right) \|\nabla^j u\|_{C^{k-1}((\varepsilon, T); L^p)}$$

for all $0 < \varepsilon < T$ and $j \in \{0, 1\}$, and thus we see by [Lemmas 3](#) and [7](#) that

$$|u|^{k-l} u^l \in C_{\text{loc}}^{k-1}((0, T]; H^{1, p}). \quad (4.33)$$

Therefore, combining the relations (4.1), (4.32), and (4.33), we may apply [Proposition 5](#) (ii) to obtain

$$u \in C_{\text{loc}}^k((0, T]; H^{\sigma, p}) \cap C_{\text{loc}}^{k-1}((0, T]; H^{\sigma+2, p}).$$

As the estimate above (4.32) is still valid even for $\sigma = 1$, we also obtain (4.32) with $\sigma = 1$. Hence, we combine (4.33) and [Proposition 5](#) (ii) again to observe that

$$u \in C_{\text{loc}}^k((0, T]; H^{1, p}) \cap C_{\text{loc}}^{k-1}((0, T]; H^{3, p}), \quad (4.34)$$

which covers (4.29).

Next we prove (4.30). Take $0 < \sigma < 1$ sufficiently small so that $0 < \sigma < 1 - n/(2p)$ holds. Since

$$\partial_t \nabla \cdot (u \mathcal{K}_{\beta_1} u) = \nabla \cdot ((\partial_t u) \mathcal{K}_{\beta_1} u) + \nabla \cdot (u \mathcal{K}_{\beta_1} \partial_t u)$$

and since $\partial_t(|u|^{k-l} u^l) = \kappa|u|^{k-2+l} u^{1-l} \partial_t u$, [Lemma 5](#) provides that

$$\begin{aligned} \|\partial_t \nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{C^{k-1}((\varepsilon, T); B_{p,1}^{2\sigma-1})} &\leq C \left(\|\partial_t^j u\|_{C^{k-1}((\varepsilon, T); B_{p,1}^{2\sigma})} \|u\|_{C^{k-1}((\varepsilon, T); H^{1, p})} + \|u\|_{C^{k-1}((\varepsilon, T); B_{p,1}^{2\sigma})} \|\partial_t^j u\|_{C^{k-1}((\varepsilon, T); H^{1, p})} \right) \\ &\leq C \|u\|_{C^k((\varepsilon, T); B_{p,1}^{2\sigma})} \|u\|_{C^k((\varepsilon, T); H^{1, p})} \end{aligned}$$

and

$$\|\partial_t^j(|u|^{k-l} u^l)\|_{C^{k-1}((\varepsilon, T); L^{p_*})} \leq \kappa^j \left(2\|u\|_{L^\infty((\varepsilon, T); L^\infty)}^{k-1} + 5\|\partial_t u\|_{L^\infty((\varepsilon, T); L^\infty)}^{k-1} \right) \|\partial_t^j u\|_{C^{k-1}((\varepsilon, T); L^{p_*})}$$

for all $0 < \varepsilon < T$, $p \leq p_* \leq \infty$, and $j \in \{0, 1\}$. We note [Lemma 7](#) and (4.34) to obtain

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in C_{\text{loc}}^k((0, T]; B_{\infty,1}^{2\sigma-1} \cap B_{p,1}^{2\sigma-1}) \subset C_{\text{loc}}^k((0, T]; H^{2\sigma-1, p_0} \cap H^{2\sigma-1, p})$$

and

$$|u|^{k-l} u^l \in C_{\text{loc}}^k((0, T]; BUC \cap L^p) \subset C_{\text{loc}}^k((0, T]; L^{p_0} \cap L^p) \quad (4.35)$$

for $n/\sigma < p_0 < \infty$. Combining the above relations with [Proposition 5](#) (ii), we have

$$u \in C_{\text{loc}}^k((0, T]; H^{2\sigma+1, p_0} \cap H^{2\sigma+1, p}).$$

Note that the Sobolev embedding yields $H^{2\sigma+1, p_0} \subset B_{\infty, \infty}^{2\sigma+1-n/p_0} \subset B_{\infty,1}^{\sigma+1}$ due to $p_0 > n/\sigma$. Moreover, by applying [Lemma 5](#) (i) again, we have

$$\begin{aligned} \|\partial_t^j \nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{C^{k-1}((\varepsilon, T); B_{p,1}^\sigma)} &\leq C \left(\|\partial_t^j u\|_{C^{k-1}((\varepsilon, T); B_{p,1}^{\sigma+1})} \|u\|_{C^{k-1}((\varepsilon, T); H^{1, p} \cap B_{\infty,1}^\sigma)} + \|u\|_{C^{k-1}((\varepsilon, T); B_{p,1}^{\sigma+1})} \|\partial_t^j u\|_{C^{k-1}((\varepsilon, T); H^{1, p} \cap B_{\infty,1}^\sigma)} \right) \\ &\leq C \|u\|_{C^k((\varepsilon, T); B_{p,1}^{\sigma+1})} \|u\|_{C^k((\varepsilon, T); H^{1, p} \cap B_{\infty,1}^\sigma)} \end{aligned}$$

for all $0 < \varepsilon < T$, $p \leq p_* \leq \infty$, and $j \in \{0, 1\}$, which yields

$$\nabla \cdot (u \mathcal{K}_{\beta_1} u) \in C_{\text{loc}}^k((0, T]; B_{\infty,1}^\sigma \cap B_{p,1}^\sigma) \subset C_{\text{loc}}^k((0, T]; BUC \cap L^p).$$

Hence, combining the above relation and (4.35), we may apply [Proposition 5](#) (ii) again to obtain (4.30).

Finally, we show that (4.31). As we see by [Lemma 4](#) that

$$\|\partial_t^j \nabla \cdot (u \mathcal{K}_{\beta_1} u)\|_{L^\infty((\varepsilon, T); B_{\infty,1}^{k-1})} \leq C \left(\|\partial_t^j u\|_{L^\infty((\varepsilon, T); B_{\infty,1}^k)} \|u\|_{L^\infty((\varepsilon, T); H^{1, p} \cap B_{\infty,1}^{k-1})} + \|u\|_{L^\infty((\varepsilon, T); B_{\infty,1}^k)} \|\partial_t^j u\|_{L^\infty((\varepsilon, T); H^{1, p} \cap B_{\infty,1}^{k-1})} \right)$$

and

$$\|\partial_t^j(|u|^{k-l} u^l)\|_{L^\infty((\varepsilon, T); C^{k-1})} \leq C \|u\|_{L^\infty((\varepsilon, T); B_{\infty,1}^1)}^{k-1} \|\partial_t^j u\|_{L^\infty((\varepsilon, T); C^{k-1})}$$

for all $0 < \varepsilon < T$ and $j \in \{0, 1\}$, we have

$$\partial_t^j \nabla \cdot (u \mathcal{K}_{\beta_1} u), \partial_t^j(|u|^{k-l} u^l) \in L_{\text{loc}}^\infty((0, T]; B_{\infty,1}^{k-1})$$

due to $B_{\infty,1}^{k-1} = C^{k-1}$ and (4.30). Hence, by applying [Proposition 5](#) (i), we deduce that

$$\partial_t^2 u, \Delta \partial_t u \in L_{\text{loc}}^\infty((0, T]; C^{k-1}),$$

which leads to (4.31) from the standard duality argument (cf. [36, Proposition 3.1]). Therefore, we obtain the desired result. This completes the proof of [Theorem 2](#). $\square$

Data availability

No data was used for the research described in the article.

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Appendix A. Basic properties of the Besov spaces

In this appendix, we give the detailed proofs of the properties (2.1) and Proposition 1: Recall that $L^p + \dot{B}_{p,q}^{-s}$ is a Banach space equipped with the norm

$$\|\varphi\|_{L^p + \dot{B}_{p,q}^{-s}} := \inf_{\varphi = \varphi_0 + \varphi_1, \varphi_0 \in L^p, \varphi_1 \in \dot{B}_{p,q}^{-s}} (\|\varphi_0\|_{L^p} + \|\varphi_1\|_{\dot{B}_{p,q}^{-s}})$$

for $1 \leq p \leq \infty$, $s > 0$, and $1 \leq q \leq \infty$. Note that the operators $\dot{S}_j$ and $\dot{\Delta}_j$ defined in Section 2 are bounded from L^p onto itself for all $1 \leq p \leq \infty$, where the boundedness is uniform with respect to $j \in \mathbb{Z}$ (cf. [41, Remark 2.11]). Moreover, the conditions of χ yield $\dot{\Delta}_j \dot{\Delta}_k = 0$ for $|j - k| \geq 2$, $\dot{\Delta}_j \dot{S}_k = 0$ for $j - k \geq 2$, and $\dot{\Delta}_j \dot{S}_k = \dot{\Delta}_j$ for $j - k \leq -1$. In what follows, we show the desired assertions by relying on these properties.

Proof of the properties (2.1). Assume that $\varphi \in L^p \cap \dot{B}_{p,q}^{-s}$. Then the definitions of the Besov spaces imply that

$$\|\varphi\|_{B_{p,q}^s} \leq \|\dot{S}_0 \varphi\|_{L^p} + \|\varphi\|_{\dot{B}_{p,q}^s} \leq C \|\varphi\|_{L^p} + \|\varphi\|_{\dot{B}_{p,q}^s},$$

which yields $\varphi \in B_{p,q}^s$. Conversely, if $\varphi \in B_{p,q}^s$, then we note that $B_{p,q}^s \subset L^p$ due to $s > 0$. We decompose $\varphi = \dot{S}_0 \varphi + (I - \dot{S}_0) \varphi$ to obtain

$$\|\varphi\|_{\dot{B}_{p,q}^s} \leq \|\dot{S}_0 \varphi\|_{\dot{B}_{p,1}^s} + \|(I - \dot{S}_0) \varphi\|_{\dot{B}_{p,q}^s} \leq C \left(\|\dot{S}_0 \varphi\|_{L^p} \sum_{j \leq 1} 2^{sj} + \|\dot{\Delta}_0 \varphi\|_{L^p} + \|\varphi\|_{B_{p,q}^s} \right) \leq C \|\varphi\|_{B_{p,q}^s},$$

which implies that $\varphi \in L^p \cap \dot{B}_{p,q}^s$, and thus we have $B_{p,q}^s = L^p \cap \dot{B}_{p,q}^s$. In addition, assume that $\varphi \in B_{p,q}^{-s}$. Then, since $\|\dot{S}_0 \varphi\|_{L^p} \leq \|\varphi\|_{B_{p,q}^{-s}}$ holds and since $B_{p,1}^0 \subset L^p$ yields

$$\|(I - \dot{S}_0) \varphi\|_{\dot{B}_{p,q}^{-s}} \leq C (\|\dot{\Delta}_0 \varphi\|_{L^p} + \|\varphi\|_{B_{p,q}^{-s}}) \leq C (\|\dot{S}_0 (\dot{\Delta}_0 \varphi)\|_{L^p} + \|\dot{\Delta}_1 (\dot{\Delta}_0 \varphi)\|_{L^p} + \|\varphi\|_{B_{p,q}^{-s}}) \leq C \|\varphi\|_{B_{p,q}^{-s}},$$

we have $\varphi \in L^p + \dot{B}_{p,q}^{-s}$. Conversely, if $\varphi \in L^p + \dot{B}_{p,q}^{-s}$, then there holds $\varphi = \varphi_0 + \varphi_1$ with some $\varphi_0 \in L^p$ and $\varphi_1 \in \dot{B}_{p,q}^{-s}$. Since $\dot{B}_{p,1}^0 \subset L^p \subset B_{p,q}^{-s}$ from $s > 0$, we deduce that

$$\|\varphi\|_{B_{p,q}^{-s}} \leq C \|\varphi_0\|_{L^p} + \|\dot{S}_0 \varphi_1\|_{L^p} + \|\varphi_1\|_{\dot{B}_{p,q}^{-s}} \leq C \|\varphi_0\|_{L^p} + C \sum_{j \leq 1} 2^{sj} \cdot 2^{-sj} \|\dot{\Delta}_j \varphi_1\|_{L^p} + \|\varphi_1\|_{\dot{B}_{p,q}^{-s}} \leq C (\|\varphi_0\|_{L^p} + \|\varphi_1\|_{\dot{B}_{p,q}^{-s}}),$$

which implies that $\varphi \in B_{p,q}^{-s}$, and thus we obtain $B_{p,q}^{-s} = L^p + \dot{B}_{p,q}^{-s}$. This proves the properties (2.1). $\square$

Proof of Proposition 1. Note that $\dot{S}_0 \dot{S}_j a \equiv \dot{S}_0 a$ due to $j \in \mathbb{N}$. While the first inequality is given by

$$\|\dot{S}_j a\|_{B_{p,1}^{s+j}} = \|\dot{S}_0 (\dot{S}_j a)\|_{L^p} + \sum_{k=1}^{j+1} 2^{(\sigma+s)k} \|\dot{\Delta}_k (\dot{S}_j a)\|_{L^p} \leq \|\dot{S}_0 a\|_{L^p} + C \sup_{1 \leq k \leq j+1} 2^{sk} \|\dot{\Delta}_k a\|_{L^p} \sum_{k=1}^{j+1} 2^{\sigma k} \leq \frac{2^{\sigma(j+1)} C}{1 - 2^{-\sigma}} \|a\|_{B_{p,\infty}^s},$$

the second one follows from

$$\|(I - \dot{S}_j) a\|_{B_{p,\infty}^{s+j}} = \|\dot{S}_0 (I - \dot{S}_j) a\|_{L^p} + \sup_{k \geq j} 2^{sk} \|\dot{\Delta}_k (I - \dot{S}_j) a\|_{L^p} \leq C \sup_{k \geq j} 2^{sk} \|\dot{\Delta}_k a\|_{L^p},$$

which implies the desired result. $\square$

Appendix B. Several properties of the heat semigroup on Besov spaces

Here we shall show the propositions in Subsection 2.3. Before these proofs, we give some remarks on the respective assertions: Proposition 3 implies the smoothing effects for the heat semigroup on the Besov spaces. Whereas a similar assertion to that of (i) is shown by Kozono, Ogawa, and Taniuchi [47, Lemma 2.2], the assertion of (ii) is almost the same as that given by Kozono, Okada, and Shimizu [37, Lemma 2.5 (1)] (see also [38, Proposition 2.1]). Concerning Proposition 4, although the proof is almost the same as the case of the homogeneous Besov spaces, which is given by the author [39, Proposition 3.10 and Lemma 4.13], we attach the proof for the convenience of the readers. Finally, the proof of Proposition 5, which provides the smoothing effects of solutions of the linear heat equation, relies on the methods of Lunardi [48, Theorems 4.3.4 and 4.3.11] and the author [35, Proposition 4.1].

Proof of Proposition 3. (i) Assume that $a \in B_{p,\infty}^{s+\sigma}$. Then it holds by [47, Lemma 2.2 (i)] that $e^{t\Delta} a \in B_{p,1}^{s+\sigma}$ for all $\sigma > 0$ and $t > 0$. We also deduce that $e^{t\Delta} a \in C^\infty((0, \infty); B_{p,1}^{s+\sigma})$ due to $\partial_t e^{t\Delta} a = \Delta e^{t\Delta} a$. Next we fix $\sigma > 0$. Applying [47, Lemma 2.2 (i)] yields

$$\|e^{t\Delta} a\|_{B_{p,\infty}^{s+\sigma/2}} \leq C(1 + t^{-\sigma/4}) \|a\|_{B_{p,\infty}^{s+\sigma}}, \quad \|e^{t\Delta} a\|_{B_{p,\infty}^{s+3\sigma/2}} \leq C(1 + t^{-3\sigma/4}) \|a\|_{B_{p,\infty}^s}$$

for all $t > 0$. Since $(1/2)(s + \sigma/2) + (1/2)(s + 3\sigma/2) = s + \sigma$, the real interpolation of the Besov spaces (cf. [43, Theorem 6.4.5]) implies that $(B_{p,\infty}^{s+\sigma/2}, B_{p,\infty}^{s+3\sigma/2})_{1/2,1} = B_{p,1}^{s+\sigma}$, and thus the standard interpolation inequality (cf. [49, Corollary 1.7]) provides that

$$\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq C\|e^{t\Delta}a\|_{B_{p,\infty}^{s+\sigma/2}}^{1/2}\|e^{t\Delta}a\|_{B_{p,\infty}^{s+3\sigma/2}}^{1/2} \leq C(1+t^{-\sigma/4})^{1/2}\|a\|_{B_{p,\infty}^s}^{1/2} \cdot (1+t^{-3\sigma/4})^{1/2}\|a\|_{B_{p,\infty}^s}^{1/2} = C(1+t^{-\sigma/4})^{1/2}(1+t^{-3\sigma/4})^{1/2}\|a\|_{B_{p,\infty}^s}$$

for all $t > 0$. In the case of $0 < t < 1$, we have

$$\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq C(t^{-\sigma/4})^{1/2}(t^{-3\sigma/4})^{1/2}\|a\|_{B_{p,\infty}^s} = Ct^{-\sigma/2}\|a\|_{B_{p,\infty}^s},$$

while there holds $\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq C\|a\|_{B_{p,\infty}^s}$ in the case of $t > 1$. Thus we have the desired estimate.

(ii) Since $0 < \sigma < 2$, we may take σ_0 and σ_1 such that $0 < \sigma_0 < \sigma < \sigma_1 < 2$. Assume that $a \in B_{p,\infty}^{s+\sigma-\sigma_0}$. Then the assertion of (i) yields

$$\|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} \leq C(1+t^{-\sigma_j/2})\|a\|_{B_{p,\infty}^{s+\sigma-\sigma_j}} \leq C(T^{\sigma_j/2}+1)t^{-\sigma_j/2}\|a\|_{B_{p,\infty}^{s+\sigma-\sigma_j}}$$

for all $0 < t < T$ and $j \in \{0, 1\}$. Therefore, the characterization of the weak Lebesgue spaces (cf. [40, Theorem 6.6]) implies that

$$\begin{aligned} \|e^{t\Delta}a\|_{L_T^{2/\sigma_j,\infty}(B_{p,1}^{s+\sigma})} &= \sup_{\lambda>0} \lambda \mu \left(t \in (0, T) \mid \|e^{t\Delta}a\|_{B_{p,1}^{s+\sigma}} > \lambda \right)^{\sigma_j/2} \\ &\leq \sup_{\lambda>0} \lambda \mu \left(t \in (0, \infty) \mid C(T^{\sigma_j/2}+1)t^{-\sigma_j/2}\|a\|_{B_{p,\infty}^{s+\sigma-\sigma_j}} > \lambda \right)^{\sigma_j/2} \\ &= C(1+T^{\sigma_j/2})\|a\|_{B_{p,\infty}^{s+\sigma-\sigma_j}}. \end{aligned}$$

As $\sigma_0 < \sigma < \sigma_1$, we may take $0 < \theta < 1$ so that $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$ holds. Then, since

$$(1-\theta)(2/\sigma_0)^{-1} + \theta(2/\sigma_1)^{-1} = (2/\sigma)^{-1}, \quad (1-\theta)(s+\sigma-\sigma_0) + \theta(s+\sigma-\sigma_1) = s,$$

the real interpolation of the Lorentz and Besov spaces (cf. [43, Theorems 5.3.1 and 6.4.5]) yields

$$\begin{aligned} (L^{2/\sigma_0,\infty}((0, T); B_{p,1}^{s+\sigma}), L^{2/\sigma_1,\infty}((0, T); B_{p,1}^{s+\sigma}))_{\theta,q} &= L^{2/\sigma,q}((0, T); B_{p,1}^{s+\sigma}), \\ (B_{p,\infty}^{s+\sigma-\sigma_0}, B_{p,\infty}^{s+\sigma-\sigma_1})_{\theta,q} &= B_{p,q}^s. \end{aligned}$$

Applying the fundamental property of the real interpolation (cf. [49, Theorem 1.6]), we may show the desired estimate. This completes the proof of Proposition 3. $\square$

Proof of Proposition 4. (i) Noting that $p, q \in [1, \infty)$, we may take the sequence $\{a_j\}_{j \in \mathbb{N}} \subset \mathcal{S}$ of functions satisfying $\lim_{j \rightarrow \infty} \|a - a_j\|_{B_{p,q}^s} = 0$ due to [42, Proposition 2.10]. Proposition 3 (i) yields

$$t^{\sigma/2}\|e^{t(\alpha I + \Delta)}a\|_{B_{p,1}^{s+\sigma}} \leq t^{\sigma/2}\|e^{t(\alpha I + \Delta)}(a - a_j)\|_{B_{p,1}^{s+\sigma}} + t^{\sigma/2}\|e^{t(\alpha I + \Delta)}a_j\|_{B_{p,1}^{s+\sigma}} \leq Ce^\alpha\|a - a_j\|_{B_{p,q}^s} + Ce^\alpha t^{\sigma/4}\|a_j\|_{B_{p,1}^{s+\sigma/2}}$$

for all $\sigma > 0$ and $t > 0$. Therefore, letting $j \rightarrow \infty$ after taking $t \rightarrow +0$, we have the desired result.

(ii) Noting that $(B_{p',1}^{-s})^* = B_{p,\infty}^s$ from [42, Theorem 2.17] with $1 \leq p < \infty$, we observe that

$$\left| \langle e^{t(\alpha I + \Delta)}a - a, \psi \rangle \right| = \left| \langle a, e^{t(\alpha I + \Delta)}\psi - \psi \rangle \right| \leq C\|a\|_{B_{p,\infty}^s}\|e^{t(\alpha I + \Delta)}\psi - \psi\|_{B_{p',1}^{-s}}.$$

The right-hand side converges to 0 as $t \rightarrow +0$ from the continuity of the C_0 -semigroup $\{e^{t(\alpha I + \Delta)}\}_{t>0}$.

(iii) Note that $(B_{p',1}^{-s_0})^* = B_{p,\infty}^{s_0}$ due to [42, Theorem 2.17] with $1 \leq p < \infty$. Take $\{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{S}$ so that $\lim_{j \rightarrow \infty} \|\psi - \psi_j\|_{B_{p',1}^{-s}} = 0$ holds.

Recalling that (3.4) and applying Proposition 3 (i), we deduce that

$$\begin{aligned} \left| \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)}f(\tau)d\tau, \psi_j \right\rangle \right| &\leq \int_0^t \left| \langle f(\tau), e^{(t-\tau)(\alpha I + \Delta)}\psi_j \rangle \right| d\tau \\ &\leq C \int_0^t e^{\alpha(t-\tau)}\|f(\tau)\|_{B_{p,\infty}^{s_0}}\|e^{(t-\tau)\Delta}\psi_j\|_{B_{p',1}^{-s_0}} d\tau \\ &\leq Ce^\alpha \int_0^t M\tau^{-\sigma/2} \cdot (t-\tau)^{-(1/2-\sigma/4)}\|\psi_j\|_{B_{p',1}^{-s_0-1+\sigma/2}} d\tau \\ &\leq Ce^\alpha M t^{1/2-\sigma/4}\|\psi_j\|_{B_{p',1}^{-s_0-1+\sigma/2}}. \end{aligned}$$

Thus we see by $0 < \sigma < 2$ that the left-hand side of the above inequality converges to 0 as $t \rightarrow +0$. Since $(B_{p',1}^{-s})^* = B_{p,\infty}^s$ implies that

$$\begin{aligned} \left| \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)}f(\tau)d\tau, \psi \right\rangle \right| &\leq \left| \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)}f(\tau)d\tau, \psi - \psi_j \right\rangle \right| + \left| \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)}f(\tau)d\tau, \psi_j \right\rangle \right| \\ &\leq CM\|\psi - \psi_j\|_{B_{p',1}^{-s}} + \left| \left\langle \int_0^t e^{(t-\tau)(\alpha I + \Delta)}f(\tau)d\tau, \psi_j \right\rangle \right|, \end{aligned}$$

letting $j \rightarrow \infty$ after taking $t \rightarrow +0$, we have the desired result. This completes the proof of Proposition 4. $\square$

Before giving the proof of [Proposition 5](#), we verify the auxiliary assertion, which follows from the Lunardi's method [[48](#), Theorem 4.3.11] and the author's result [[35](#), Proposition B.1].

Proposition 9. *Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$, $\alpha \geq 0$, and $0 < T < \infty$. Suppose that $f \in L^\infty((0, T); B_{p,\infty}^s)$. Then it holds that*

$$F \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{s+2\theta}), \quad \partial_t F, \Delta F \in L^\infty((0, T); B_{p,\infty}^s),$$

where F is defined by (2.3).

Proof. Letting

$$g(t) := \begin{cases} e^{-\alpha t}(I - \Delta)^{(s-1)/2} f(t) & \text{for } t \in (0, T], \\ 0 & \text{for } t \in (T, \infty), \end{cases}$$

we have $g \in L^\infty((0, \infty); B_{p,\infty}^1)$ due to [[42](#), Theorem 2.2]. Hence, we apply [[35](#), Proposition B.1 (ii)] to deduce that

$$\int_0^t e^{(t-\tau)\Delta} g(\tau) d\tau \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, \infty); B_{p,1}^{1+2\theta}),$$

and thus the relation $e^{-\alpha t}(I - \Delta)^{(s-1)/2} F(t) = \int_0^t e^{(t-\tau)\Delta} g(\tau) d\tau$ for $0 < t < T$ yields

$$F \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{s+2\theta})$$

due to [[42](#), Theorem 2.2]. In addition, since $B_{p,\infty}^1 \subset \dot{B}_{p,\infty}^1$, by combining [[47](#), Lemma 2.2 (ii)] and [[42](#), Theorem 2.2], we observe that

$$\begin{aligned} \|e^{\lambda\Delta} \Delta^2 (I - \Delta)^{(s-1)/2} F(t)\|_{\dot{B}_{p,1}^0} &\leq C \int_0^t \|e^{\lambda\Delta} e^{(t-\tau)(\alpha I + \Delta)} (I - \Delta)^{(s-1)/2} f(\tau)\|_{\dot{B}_{p,1}^4} d\tau \\ &\leq C e^{\alpha t} \int_0^t (\lambda + t - \tau)^{-3/2} \|(I - \Delta)^{(s-1)/2} f(\tau)\|_{\dot{B}_{p,\infty}^1} d\tau \\ &\leq C e^{\alpha t} \|(I - \Delta)^{(s-1)/2} f\|_{L_T^\infty(B_{p,\infty}^1)} \left[(\lambda + t - \tau)^{-1/2} \right]_{\tau=0}^{\tau=t} \\ &\leq C e^{\alpha t} \lambda^{-1/2} \|f\|_{L_T^\infty(B_{p,\infty}^s)} \end{aligned}$$

for all $0 < t < T$ and $\lambda > 0$, which leads to

$$\sup_{\lambda > 0} \lambda^{1/2} \|e^{\lambda\Delta} \Delta^2 (I - \Delta)^{(s-1)/2} F(t)\|_{\dot{B}_{p,1}^0} \leq C e^{\alpha t} \|f\|_{L_T^\infty(B_{p,\infty}^s)}.$$

As the characterization of the homogeneous Besov spaces yields $\|\varphi\|_{\dot{B}_{p,\infty}^{-1}} \leq C \sup_{\lambda > 0} \lambda^{1/2} \|e^{\lambda\Delta} \varphi\|_{L^p}$ (cf. [[41](#), Theorem 2.34]), we note that $\dot{B}_{p,1}^0 \subset L^p$ and $\dot{B}_{p,\infty}^{-1} \subset B_{p,\infty}^{-1}$ to obtain

$$\|\Delta^2 (I - \Delta)^{(s-1)/2} F(t)\|_{B_{p,\infty}^{-1}} \leq C e^{\alpha t} \|f\|_{L_T^\infty(B_{p,\infty}^s)},$$

and hence we see by [[42](#), Theorem 2.2] that

$$\|\Delta F(t)\|_{B_{p,\infty}^s} \leq C(\|\Delta F(t)\|_{B_{p,\infty}^{s-2}} + \|\Delta^2 F(t)\|_{B_{p,\infty}^{s-2}}) \leq C(\|F(t)\|_{B_{p,\infty}^s} + \|\Delta^2 (I - \Delta)^{(s-1)/2} F(t)\|_{B_{p,\infty}^{-1}}),$$

which yields $\Delta F \in L^\infty((0, T); B_{p,\infty}^s)$ from $\|F(t)\|_{B_{p,\infty}^s} \leq e^{\alpha t} \|f\|_{L_T^\infty(B_{p,\infty}^s)}$. Therefore, the relation $(\alpha I + \Delta)F + f = \partial_t F$ provides the desired result. $\square$

Proof of Proposition 5. (i) Let $0 < \varepsilon < T$ be arbitrarily small and let $\chi_\varepsilon \in C^\infty(0, \infty)$ satisfy

$$0 \leq \chi_\varepsilon \leq 1, \quad \chi_\varepsilon(t) = \begin{cases} 0 & \text{for } t \in (0, \varepsilon/8), \\ 1 & \text{for } t \in (\varepsilon/4, \infty). \end{cases}$$

Since the present assumption yields $\chi_\varepsilon \partial_t^k f \in L^\infty((0, \infty); B_{p,\infty}^s)$, by setting

$$G_k(t) := \int_0^t e^{(t-\tau)(\alpha I + \Delta)} (\chi_\varepsilon(\tau) \partial_\tau^k f(\tau)) d\tau$$

for $0 < t < T$ and applying [Proposition 9](#), we have

$$G_k \in \bigcap_{0 < \theta < 1} C_{\text{loc}}^{1-\theta}((0, T]; B_{p,1}^{s+2\theta}), \quad \partial_t G_k, \Delta G_k \in L^\infty((0, T); B_{p,\infty}^s). \quad (\text{B.1})$$

In the case of $k = 0$, we use $f = (1 - \chi_\varepsilon)f + \chi_\varepsilon f$ to obtain the relation

$$F(t) = e^{(t-\varepsilon/2)(\alpha I + \Delta)} \int_0^{\varepsilon/4} e^{(\varepsilon/2-\tau)(\alpha I + \Delta)} ((1 - \chi_\varepsilon(\tau))f(\tau)) d\tau + G_0(t) \quad (\text{B.2})$$

for all $\varepsilon < t < T$. As $H^{s_0, p_0} \subset B_{p, \infty}^{s_0 - n(1/p_0 - 1/p)} \subset B_{p, 1}^{s_0 - n}$ from the Sobolev embedding with $p \geq p_0$, we have

$$\left\| \int_0^{\varepsilon/4} e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} ((1 - \chi_\varepsilon(\tau)) f(\tau)) d\tau \right\|_{B_{p, 1}^{s_0 - n}} \leq C e^\alpha \int_0^{\varepsilon/4} \|f(\tau)\|_{H^{s_0, p_0}} d\tau < \infty$$

due to the present assumption, and thus the first term on the right-hand side of (B.2) is smooth from Proposition 3 (i). Therefore, we see by (B.1) that the desired result holds. In the case of $k = 1$, we follow a similar argument to that in [35, p.870] to deduce that

$$\begin{aligned} (\alpha I + \Delta)F(t) &= e^{(t - \varepsilon/2)(\alpha I + \Delta)} \left\{ \int_0^{\varepsilon/2} (\alpha I + \Delta) e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} f(\tau) d\tau + f(\varepsilon/2) \right\} \\ &\quad - f(t) + G_1(t) - e^{(t - \varepsilon/2)(\alpha I + \Delta)} \int_{\varepsilon/8}^{\varepsilon/2} e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} (\chi_\varepsilon(\tau) \partial_\tau f(\tau)) d\tau \end{aligned}$$

for all $\varepsilon < t < T$. Since the result in the case of $k = 0$ ensures the equality $(\alpha I + \Delta)F + f = \partial_t F$, we have

$$\partial_t F(t) = e^{(t - \varepsilon/2)(\alpha I + \Delta)} \left\{ \int_0^{\varepsilon/2} (\alpha I + \Delta) e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} f(\tau) d\tau - \int_{\varepsilon/8}^{\varepsilon/2} e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} (\chi_\varepsilon(\tau) \partial_\tau f(\tau)) d\tau + f(\varepsilon/2) \right\} + G_1(t) \quad (\text{B.3})$$

for all $\varepsilon < t < T$. Similarly to the argument in the case of $k = 0$, we deduce that the first term on the right-hand side of (B.3) is smooth. Furthermore, as the present assumption yields

$$\left\| \int_{\varepsilon/8}^{\varepsilon/2} e^{(\varepsilon/2 - \tau)(\alpha I + \Delta)} (\chi_\varepsilon(\tau) \partial_\tau f(\tau)) d\tau \right\|_{B_{p, \infty}^s} \leq C e^\alpha \int_{\varepsilon/8}^{\varepsilon/2} \|\partial_\tau f(\tau)\|_{B_{p, \infty}^s} d\tau < \infty,$$

the second term is also smooth, and thus we rely on (B.1) to obtain the desired result.

(ii) As we have

$$G_k \in C_{\text{loc}}^{1+\gamma}((0, T]; X^s) \cap C_{\text{loc}}^\gamma((0, T]; X^{s+2})$$

due to [48, Theorem 4.3.1], similar calculations to the assertion of (i) provide the desired result. This completes the proof of Proposition 5. $\square$

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