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Lower Bound Analysis of L_{p+} Induced Norm for LTI Systems

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Abstract: In this paper, we focus on the lower bounds of the L_p $(p \in [1, \infty),\ p = \infty)$ induced norms of continuous-time LTI systems where input signals are restricted to be nonnegative. This induced norm, called the L_{p+} induced norm, is particularly useful for the stability analysis of nonlinear feedback systems constructed from linear systems and static nonlinearities where the nonlinearities provide only nonnegative signals for the case $p = 2$. To have deeper understanding on the L_{p+} induced norm, we analyze its lower bounds with respect to the standard L_p induced norm in this paper. As the main result, we show that the L_{p+} induced norm of an LTI system cannot be smaller than the L_p induced norm scaled by $2^{(1-p)/p}$ for $\in [1, \infty)$ (scaled by 2^{-1} for $p = \infty$). On the other hand, in the case where $p = 2$, we further propose a method to compute better (larger) lower bounds for single-input systems via reduction of the lower bound analysis problem into a semi-infinite programming problem. The effectiveness of the lower bound computation method, together with an upper bound computation method proposed in our preceding paper, is illustrated by numerical examples.

Keywords: nonnegative signals, L_{p+} induced norms, lower bounds, semi-infinite programming problems

1. INTRODUCTION

Recently, there has been a growing attention on control theoretic approaches for the analysis and synthesis of optimization algorithms (Lessard et al. [2016]) and neural networks (Revay et al. [2021], Yin et al. [2022], Fazlyab et al. [2022], Scherer [2022]). By capturing the input-output behavior of nonlinearities in the algorithms or neural networks via quadratic constraints, we can cast the analysis and synthesis problems into numerically tractable semidefinite programming problems. Along this stream, in Ebihara et al. [2021a,b] and Motooka and Ebihara [2022], we dealt with the stability analysis of recurrent neural networks (RNNs) with activation functions being rectified linear units (ReLUs). In particular, by focusing on the fact that the ReLUs return only nonnegative signals, we derived nonnegativity-based small-gain theorem for the stability analysis of RNNs.

Motivated by our preceding results based on signal nonnegativity, in this paper, we explore the analysis of the L_p $(p \in [1, \infty),\ p = \infty)$ induced norms of continuous-time LTI systems where input signals are restricted to be nonnegative. This induced norm is referred to as the L_{p+} induced norm in this paper. By definition, the L_{p+} induced norm of an LTI system is smaller than or equal to its standard L_p induced norm. To have deeper understanding on the L_{p+} induced norm, we analyze its lower bounds with respect to the L_p induced norm. As the main result, we show that the L_{p+} induced norm of an LTI system cannot be smaller than its L_p induced norm scaled by $2^{(1-p)/p}$ for $\in [1, \infty)$ (scaled by 2^{-1} for $p = \infty$). Moreover, we concretely construct (infinite-dimensional) LTI systems that attain these lower bounds. On the other hand, in the case where $p = 2$, we further propose a method to compute better (larger) lower bounds for single-input systems via reduction of the lower bound analysis problem into a semi-infinite programming problem. This reduction allows us to see that the L_{2+} induced norm of a finite-dimensional single-input LTI system is strictly larger than its L_2 induced norm scaled by $2^{-1/2}$. On the basis of this result, for finite-dimensional single-input LTI systems, we also derive a method for lower bound computation of the L_{2+} induced norm in such a sound way that it enables us to obtain a lower bound that is strictly better (larger) than the L_2 induced norm scaled by $2^{-1/2}$. The effectiveness of the lower bound computation method, together with an

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upper bound computation method proposed in Ebihara et al. [2022], is illustrated by numerical examples. We finally note that the analysis of the L_{p+} induced norm is also motivated by recent advancement on the study of positive systems (Briat [2013], Tanaka and Langbort [2011], Rantzer [2016], Ebihara et al. [2017], Kato et al. [2020]), where the treatments of nonnegative signals are essentially important.

We use the following notation in this paper. The set of n-dimensional real vectors (with nonnegative entries) is denoted by $\mathbb{R}^n$ ($\mathbb{R}^n_+$), and the set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$. For $w_1, w_2 \in \mathbb{R}^n$, we define $w_{\max} = \max(w_1, w_2) \in \mathbb{R}^n$ by $w_{\max,i} = \max(w_{1,i}, w_{2,i})$ $(i = 1, \cdots, n)$. The set of natural numbers is denoted by $\mathbb{N}$.

2. PRELIMINARIES AND PROBLEM SETTINGS

2.1 Preliminaries: Signals and Norms

For $v \in \mathbb{R}^{n_v}$, we define

$$|v|_p := \left(\sum_{i=1}^{n_v} |v_i|^p\right)^{1/p} \ \ (p \in [1,\infty)),\ |v|_\infty := \max_{i=1,\cdots,n_v} |v_i|.$$

For a matrix $M \in \mathbb{R}^{n\times m}$, we define

$$\|M\|_p = \max_{v\in\mathbb{R}^m,\ |v|_p=1} |Mv|_p \ (p \in [1,\infty),\ p = \infty).$$

For a continuous-time signal w defined over the time interval $[0,\infty)$, we define

$$\|w\|_p := \left(\int_0^\infty |w(t)|_p^p dt\right)^{1/p} \ \ (p \in [1,\infty)),$$
$$\|w\|_\infty := \operatorname*{ess\,sup}_{0\le t<\infty} |w(t)|_\infty.$$

For $p \in [1,\infty)$ and $p = \infty$, we also define

$$L_p := \{w :\ \|w\|_p < \infty\},$$
$$L_{p+} := \{w :\ w \in L_p,\ w(t) \ge 0\ \forall t \in [0,\infty)\}.$$

For a linear operator

$$G:\ L_p \ni w \mapsto z \in L_p\ (p \in [1,\infty),\ p = \infty), \tag{1}$$

we define its (standard) L_p induced norm by

$$\|G\|_p := \sup_{w\in L_p,\ \|w\|_p=1} \|z\|_p.$$

We also define

$$\|G\|_{p+} := \sup_{w\in L_{p+},\ \|w\|_p=1} \|z\|_p.$$

This is a variant of the L_p induced norm and referred to as the L_{p+} induced norm in this paper. We can readily see that $\|G\|_{p+} \le \|G\|_p$.

2.2 Relevance of L_{2+} Induced Norm in Stability Analysis of Recurrent Neural Networks

Recently, control theoretic approaches for the analysis of neural networks (NNs) have attracted great attention, see, e.g., Revay et al. [2021], Yin et al. [2022], Fazlyab et al. [2022], and Scherer [2022]. Along this stream, in Ebihara et al. [2021a,b] and Motooka and Ebihara [2022], we dealt with the stability analysis of recurrent neural networks (RNNs) focusing on signal nonnegativity, where the L_{2+} induced norm becomes quite relevant. We note that signal-nonnegativity-based analysis of NNs has also been proposed in Grönqvist and Rantzer [2022].

To quickly review the relevance of the L_{2+} induced norm, let us consider the feedback system shown in Fig. 1. Here, G is an LTI system and $\Phi:\ \mathbb{R}^m \mapsto \mathbb{R}^m_+$ is a static nonlinear operator satisfying $\|\Phi\|_2 = 1$. We focus on the stability analysis of this feedback system. Here, note that we have assumed that Φ returns only nonnegative signals. This problem setting typically appears in the stability analysis of RNNs with activation functions being rectified linear units (ReLUs), see Ebihara et al. [2021a,b], Motooka and Ebihara [2022].

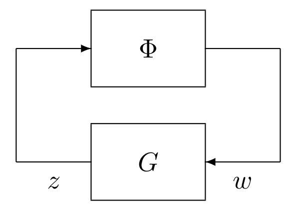

Fig. 1. Nonlinear Feedback System.

Then, from the standard L_2-induced-norm-based small-gain theorem (Khalil [2002]), we see that the feedback system shown in Fig. 1 is (well-posed and) globally stable if $\|G\|_2 < 1$. On the other hand, by actively using the nonnegative nature of Φ, it has been shown very recently in Motooka and Ebihara [2022] that the feedback system shown in Fig. 1 is (well-posed and) globally stable if $\|G\|_{2+} < 1$. As illustrated by this concrete example, the L_{2+}-induced-norm-based small-gain theorem has potential abilities for the stability analysis of feedback systems with nonnegative nonlinearities. This strongly motivates us to establish efficient methods for the computation of the L_{2+} induced norm of LTI systems. However, exact computation of the L_{2+} induced norm is inherently difficult. To get around this difficulty, effective methods for the upper bound computation of the L_{2+} induced norm have been proposed in Ebihara et al. [2021a, 2022]. In this paper, we focus on the lower bound analysis.

2.3 Problem Settings

As we have seen, $\|G\|_{p+} \le \|G\|_p$ does hold for $p \in [1,\infty)$ and $p = \infty$. Regarding this relationship, we are interested in how far $\|G\|_{p+}$ can be smaller than $\|G\|_p$. To clarify this point, let us consider the next problem.

Problem 1. (Uniform Lower Bound Analysis (ULBA)). For each $p \in [1,\infty)$ and $p = \infty$, find the uniform lower bound $\nu_p^\star$ defined by

$$\nu_p^\star := \sup\{\nu_p \in \mathbb{R} :\ \|G\|_{p+} \ge \nu_p \|G\|_p\ \forall G \in \mathcal{G}_{\rm LTI}\}. \tag{2}$$

Here, $\mathcal{G}_{\rm LTI}$ stands for the set of stable and causal LTI systems including infinite dimensional ones.

Remark 1. We can also characterize $\nu_p^\star$ by

$$\nu_p^\star = \inf_{G\in\mathcal{G}_{\rm LTI}} \frac{\|G\|_{p+}}{\|G\|_p}.$$

In our main result (Theorem 1), we concretely construct LTI systems that attain the above infimum.

On the other hand, as noticed above, we have already shown effective upper bound computation methods of the L_{2+} induced norm for (finite-dimensional) stable LTI systems (Ebihara et al. [2021a, 2022]). For the evaluation of the accuracy of the computed upper bounds, it is desirable that we can compute a lower bound of the L_{2+} induced norm that is as large as possible. Therefore we also consider the next problem.

Problem 2. (L_{2+} Lower Bound Analysis (LBA)). For a given stable LTI system G of the form (1), find a lower bound of $\|G\|_{2+}$ that is as large as possible.

3. MAIN RESULTS FOR ULBA PROBLEM

The next theorem provides our main results for Problem 1: the uniform lower bound analysis problem.

Theorem 1. For $p \in [1, \infty)$ and $p = \infty$, the uniform lower bounds $\nu_p^\star$ defined by (2) are given by

$$\nu_p^\star = 2^{\frac{1-p}{p}} \ (p \in [1, \infty)), \quad \nu_\infty^\star = \frac{1}{2}.$$

Moreover, the stable LTI system $G^\star(s) = 1 - e^{-Ls}$ ($L > 0$) attains

$$\|G^\star\|_{p+} = \nu_p^\star \|G^\star\|_p \ (p \in [1, \infty), \ p = \infty). \tag{3}$$

The rest of this section is devoted to the technical proof of Theorem 1. We start from showing the next result.

Lemma 1. For the uniform lower bounds $\nu_p^\star$ ($p \in [1, \infty), \ p = \infty$) defined by (2), we have

$$\nu_p^\star \geq 2^{\frac{1-p}{p}} \ (p \in [1, \infty)), \quad \nu_\infty^\star \geq \frac{1}{2}. \tag{4}$$

For the proof of this lemma, the next result is useful.

Lemma 2. For $p \in [1, \infty)$, suppose $x_1, x_2 \in \mathbb{R}_+$ satisfies $x_1^p + x_2^p = 1$. Then we have

$$x_1 + x_2 \leq 2^{\frac{p-1}{p}}. \tag{5}$$

Proof of Lemma 2: The function $f(x) = x^p$ ($p \in [1, \infty)$) is convex for $x \in \mathbb{R}_+$. Therefore we have

$$\left(\frac{x_1 + x_2}{2}\right)^p \leq \frac{x_1^p + x_2^p}{2} = \frac{1}{2}.$$

It follows that (5) holds. ■

Proof of Lemma 1: For $w \in L_p$ with $\|w\|_p = 1$, let us define $w_+, w_- \in L_{p+}$ such that $w = w_+ - w_-$ by $w_+(t) = \max(w(t), 0_n)$, $w_-(t) = \max(-w(t), 0_n)$ ($t \in [0, \infty)$). Then, for $p \in [1, \infty)$, we have

$$\begin{array}{rcl} \|Gw\|_p &=& \|Gw_+ - Gw_-\|_p \\ &\leq& \|Gw_+\|_p + \|Gw_-\|_p \\ &\leq& \|G\|_{p+}\|w_+\|_p + \|G\|_{p+}\|w_-\|_p \\ &=& \|G\|_{p+}(\|w_+\|_p + \|w_-\|_p). \end{array} \tag{6}$$

Here, from $\|w_+\|_p^p + \|w_-\|_p^p = \|w\|_p^p = 1$ and Lemma 2, we have

$$\|w_+\|_p + \|w_-\|_p \leq 2^{\frac{p-1}{p}}.$$

From this inequality and (6), we obtain

$$\sup_{w \in L_p, \ \|w\|_p = 1} \|Gw\|_p \leq \|G\|_{p+} 2^{\frac{p-1}{p}}.$$

or equivalently, $\|G\|_p \leq \|G\|_{p+} 2^{\frac{p-1}{p}}$. This clearly shows that $\nu_p^\star \geq 2^{\frac{1-p}{p}}$ ($p \in [1, \infty)$).

Similarly, for $w \in L_\infty$ with $\|w\|_\infty = 1$, we have

$$\begin{array}{rcl} \|Gw\|_\infty &=& \|Gw_+ - Gw_-\|_\infty \\ &\leq& \|Gw_+\|_\infty + \|Gw_-\|_\infty \\ &\leq& \|G\|_{\infty+}\|w_+\|_\infty + \|G\|_{\infty+}\|w_-\|_\infty \\ &=& \|G\|_{\infty+}(\|w_+\|_\infty + \|w_-\|_\infty) \\ &\leq& 2\|G\|_{\infty+}. \end{array}$$

From this inequality, we readily obtain

$$\sup_{w \in L_\infty, \ \|w\|_\infty = 1} \|Gw\|_\infty \leq 2\|G\|_{\infty+}$$

or equivalently, $\|G\|_\infty \leq 2\|G\|_{\infty+}$. This clearly shows $\nu_\infty^\star \geq \frac{1}{2}$. ■

We next prove (3). To this end, the next result is useful.

Lemma 3. For $w_1, w_2 \in L_{p+}$, we have

$$\begin{array}{l} \|w_1 - w_2\|_p \leq \left(\|w_1\|_p^p + \|w_2\|_p^p\right)^{\frac{1}{p}} \ (p \in [1, \infty)), \\ \|w_1 - w_2\|_\infty \leq \max(\|w_1\|_\infty, \|w_2\|_\infty). \end{array} \tag{7}$$

Proof of Lemma 3: Let us define $w_{\max} \in L_{p+}$ by

$$w_{\max}(t) := \max(w_1(t), w_2(t)) \ (t \in [0, \infty)).$$

Then, it is clear that

$$\|w_1 - w_2\|_p \leq \|w_{\max}\|_p \ (p \in [1, \infty)), \ \|w_1 - w_2\|_\infty \leq \|w_{\max}\|_\infty.$$

On the other hand, from the definition of the L_p norm of signals, it is clear that

$$\begin{array}{rcl} \|w_{\max}\|_p^p &\leq& \|w_1\|_p^p + \|w_2\|_p^p \ (p \in [1, \infty)), \\ \|w_{\max}\|_\infty &\leq& \max(\|w_1\|_\infty, \|w_2\|_\infty). \end{array}$$

It follows that (7) holds. ■

We now move on to the proof (3).

Proof of (3): We first prove that

$$\|G^\star\|_p = 2 \ (p \in [1, \infty), \ p = \infty). \tag{8}$$

To this end, note that

$$\|G^\star\|_p \leq \|1\|_p + \|e^{-Ls}\|_p = 2 \ (p \in [1, \infty), \ p = \infty). \tag{9}$$

To prove

$$\|G^\star\|_p \geq 2 \ (p \in [1, \infty), \ p = \infty), \tag{10}$$

let us consider the input signal $w_N^\star \in L_p$ ($p \in [1, \infty), \ p = \infty$) defined by

$$w_N^\star(t) = \begin{cases} 0 & t < 0, \\ 1 & 2mL \leq t < (2m+1)L, \\ -1 & (2m+1)L \leq t < 2(m+1)L, \\ 0 & 2NL \leq t. \end{cases}$$

where $N \in \mathbb{N}$ and $m = 0, \cdots, N-1$. Then, the corresponding output $z_N^\star \in L_p$ ($p \in [1, \infty), \ p = \infty$) of the system $G^\star$ is given by

$$z_N^\star(t) = \begin{cases} 0 & t < 0, \\ 1 & 0 \leq t < L, \\ 2w_N^\star & L \leq t < 2NL, \\ -1 & 2NL \leq t < (2N+1)L, \\ 0 & (2N+1)L \leq t. \end{cases}$$

For the signals $w_N^\star, z_N^\star \in L_p$ $(p \in [1,\infty),\ p=\infty)$, we readily see that

$$\|w_N^\star\|_p = (2NL)^{\frac{1}{p}}, \quad \|w_N^\star\|_\infty = 1,$$
$$\|z_N^\star\|_p = ((2N-1)L2^p + 2L)^{\frac{1}{p}}, \quad \|z_N^\star\|_\infty = 2.$$

From these results, it is obvious that $\|G^\star\|_\infty \geq 2$. In addition, by letting $N \to \infty$, we see $\|G^\star\|_p \geq 2$ $(p \in [1,\infty))$. It follows that (10) holds. From (9) and (10), we can readily conclude that (8) holds.

We are now in the right position to prove (3). To this end, let us consider any nonnegative input signal $w \in L_{p+}$ $(p \in [1,\infty),\ p=\infty)$ with $\|w\|_p = 1$ for $G^\star$. If we define $w_L \in L_{p+}$ $(p \in [1,\infty),\ p=\infty)$ with $\|w_L\|_p = 1$ by

$$w_L(t) = \begin{cases} 0 & 0 \leq t < L, \\ w(t-L) & L \leq t, \end{cases}$$

we see that the output z corresponding to the input w is given by $z = w - w_L$. By applying Lemma 3, we have

$$\|z\|_p \leq 2^{\frac{1}{p}},\ \|z\|_\infty \leq 1.$$

Namely, for any nonnegative signal $w \in L_{p+}$ $(p \in [1,\infty),\ p=\infty)$ with $\|w\|_p = 1$, the above inequalities hold. These results, together with, (8) lead us to

$$\begin{aligned} &\|G^\star\|_{p+} \leq 2^{\frac{1}{p}} = 2^{\frac{1-p}{p}}\|G\|_p = \nu_p^\star\|G\|_p\ (p \in [1,\infty)), \\ &\|G^\star\|_{\infty+} \leq 1 = \frac{1}{2}\|G\|_\infty = \nu_\infty^\star\|G\|_\infty. \end{aligned} \tag{11}$$

It follows from (4) and (11) that (3) holds. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1: Since we have verified (4) and (3), Theorem 1 has been validated. ■

Remark 2. In the proof of Lemma 1, we only rely on the linearity of underlying systems and the properties of L_p induced norms. It follows that Theorem 1 is valid even if we extend the set $\mathcal{G}_{\rm LTI}$ in Problem 1 to the set of linear, stable, time-varying, and noncausal systems including infinite dimensional ones.

4. MAIN RESULTS FOR L_{2+} LBA PROBLEM

From Theorem 1, we see that

$$\|G\|_{2+} \geq \frac{1}{\sqrt{2}}\|G\|_2\ \forall G \in \mathcal{G}_{\rm LTI}.$$

However, for each system $G \in \mathcal{G}_{\rm LTI}$, it is expected that $\|G\|_{2+}$ can be strictly larger than $\frac{1}{\sqrt{2}}\|G\|_2$. Regarding this issue, the main contributions of this section are as follows:

(i) For any finite-dimensional single-input LTI system G, we prove $\|G\|_{2+} > \frac{1}{\sqrt{2}}\|G\|_2$.
(ii) For a given finite-dimensional single-input LTI system G, we provide a method to compute a lower bound that is strictly larger than $\frac{1}{\sqrt{2}}\|G\|_2$.

We derive these results by using basics about frequency responses of LTI systems, and reducing the lower bound analysis problem into a semi-infinite programming problem (Shapiro [2009]). The treatment of multi-input systems is hard because we have to take phase-shift over inputs into consideration. Due to this reason, we focus on single-input systems in this paper. In the following, we denote by $\mathcal{G}_{\rm LTI,SI}$ the set of stable and single-input LTI systems including infinite dimensional ones.

4.1 Reduction to Semi-infinite Programming Problem

We first recall the next very basic result.

Lemma 4. For given $a_m \in \mathbb{R}$ $(m = 0,\cdots,N)$, $\phi_m \in \mathbb{R}$ $(m = 1,\cdots,N)$, $\omega > 0$, $T = 2\pi/\omega$, and $\mathcal{I} = [-T/2, T/2]$, we have

$$\frac{1}{T}\int_{\mathcal{I}}\left(a_0 + \sum_{m=1}^{N} a_m\cos(m\omega t + \phi_m)\right)^2 dt = a_0^2 + \frac{1}{2}\sum_{m=1}^{N} a_m^2.$$

For a given single-input LTI system G, let us inject the nonnegative input signal

$$w_\omega^{[N]}(t) := a_0 + \cos(\omega t) + \sum_{m=2}^{N} a_m\cos(m\omega t)\ (t \geq 0) \tag{12}$$

where we assume that a_0, a_m $(m = 2,\cdots,N)$ are chosen such that $w_\omega^{[N]}(t) \geq 0$ $(t \geq 0)$. If we denote by $z_{\omega,\infty}^{[N]}$ the corresponding steady-state output, we see from the steady-state analysis of the frequency response and Lemma 4 that

$$\begin{aligned} \|G\|_{2+} &\geq \frac{\sqrt{\frac{1}{T}\int_{\mathcal{I}} z_{\omega,\infty}^{[N]}(t)^T z_{\omega,\infty}^{[N]}(t)dt}}{\sqrt{\frac{1}{T}\int_{\mathcal{I}} w_\omega^{[N]}(t)^2 dt}} \\ &= \frac{\sqrt{a_0^2\|G(0)\|_2^2 + \frac{1}{2}\|G(j\omega)\|_2^2 + \frac{1}{2}\sum_{m=2}^{N} a_m^2\|G(jm\omega)\|_2^2}}{\sqrt{a_0^2 + \frac{1}{2} + \frac{1}{2}\sum_{m=2}^{N} a_m^2}} \\ &= \frac{\sqrt{2a_0^2\|G(0)\|_2^2 + \|G(j\omega)\|_2^2 + \sum_{m=2}^{N} a_m^2\|G(jm\omega)\|_2^2}}{\sqrt{2a_0^2 + 1 + \sum_{m=2}^{N} a_m^2}}. \end{aligned} \tag{13}$$

It follows that

$$\|G\|_{2+} \geq \frac{1}{\sqrt{2a_0^2 + 1 + \sum_{m=2}^{N} a_m^2}}\|G\|_2. \tag{14}$$

This result motivates us to consider the following semi-infinite programming problem:

$$\begin{aligned} &\gamma_N^\star := \inf_{a_0, a_2, \cdots, a_N} 2a_0^2 + 1 + \sum_{m=2}^{N} a_m^2 \quad \text{s.t.} \\ &w^{[N]}(t) := a_0 + \cos(t) + \sum_{m=2}^{N} a_m\cos(mt) \geq 0\ (\forall t \in \mathcal{I}), \\ &\mathcal{I} = [-\pi, \pi]. \end{aligned} \tag{15}$$

For each $\gamma_N^\star$, we readily see that $\|G\|_{2+} \geq \frac{1}{\sqrt{\gamma_N^\star}}\|G\|_2$ $(\forall G \in \mathcal{G}_{\rm LTI,SI})$. In addition, since $\gamma_N^\star$ is monotonically non-increasing with respect to $N \in \mathbb{N}$, and since $\gamma_N^\star \geq$

1 ($\forall N \in \mathbb{N}$), the sequence $\{\gamma_N^\star\}$ converges. If we define $\gamma^\star := \lim_{N\to\infty} \gamma_N^\star$, we readily obtain

$$\|G\|_{2+} \geq \frac{1}{\sqrt{\gamma^\star}}\|G\|_2 \ \forall G \in \mathcal{G}_{\mathrm{LTI,SI}}. \tag{16}$$

4.2 Effective Lower Bound Computation Methods

From Theorem 1, we know that $\|G^\star\|_{2+} = \frac{1}{\sqrt{2}}\|G^\star\|_2$ and hence $\gamma^\star \geq 2$ should hold in (16). Therefore, if we are able to construct $w(t)$ such that

$$\begin{aligned} &w(t) = a_0 + \cos(t) + \sum_{m=2}^{\infty} a_m \cos(mt) \geq 0 \ (\forall t \in \mathcal{I}),\ \mathcal{I} = [-\pi, \pi], \\ &2a_0^2 + 1 + \sum_{m=2}^{\infty} a_m^2 = 2, \end{aligned} \tag{17}$$

then this is an optimal solution for the semi-infinite programming problem (15) in the limit case $N \to \infty$. With this fact in mind, let us consider the nonnegative signal

$$w^\star(t) := \max(2\cos(t), 0) \ (t \geq 0) \tag{18}$$

whose Fourier series expansion is given by

$$w^\star(t) = \frac{2}{\pi} + \cos(t) + \frac{4}{\pi}\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{(2p+1)(2p-1)}\cos(2pt).$$

In (15), this corresponds to the case where

$$a_0^\star = \frac{2}{\pi},\ a_{2p}^\star = \frac{4}{\pi}\frac{(-1)^{p+1}}{(2p+1)(2p-1)},\ a_{2p+1}^\star = 0 \ (p \in \mathbb{N}).$$

From Parseval's identity, we readily see that

$$2a_0^{\star 2} + 1 + \sum_{m=2}^{\infty} a_m^{\star 2} = 2\frac{1}{2\pi}\int_{\mathcal{I}} w^\star(t)^2 dt = 2. \tag{19}$$

It follows that the signal $w^\star$ given by (18) satisfies the optimality condition (17) and hence is an optimal solution for the semi-infinite programming problem (15) in the limit case $N \to \infty$. From these results, we see that the next result holds for a given $G \in \mathcal{G}_{\mathrm{LTI,SI}}$:

$$\|G\|_{2+} \geq \sup_{\omega>0} \frac{\sqrt{2a_0^{\star 2}\|G(0)\|_2^2 + \|G(j\omega)\|_2^2 + \sum_{m=2}^{\infty} a_m^{\star 2}\|G(jm\omega)\|_2^2}}{\sqrt{2}}. \tag{20}$$

This expression leads us to the next results.

Theorem 2. Suppose $G \in \mathcal{G}_{\mathrm{LTI,SI}}$ is finite-dimensional. Then, we have $\|G\|_{2+} > \frac{1}{\sqrt{2}}\|G\|_2$. In particular, if $\|G\|_2 = \|G(0)\|_2$ holds, then $\|G\|_{2+} = \|G\|_2$ holds.

Proof of Theorem 2: We consider the following three cases: (i) $\|G\|_2$ is attained at the angular frequency $\omega = 0$, i.e., $\|G\|_2 = \|G(0)\|_2$; (ii) $\|G\|_2$ is given as $\|G\|_2 = \|G(j\infty)\|_2$ where $G(j\infty) := \lim_{\omega\to\infty} G(j\omega)$; (iii) $\|G\|_2$ is attained at $\omega = \omega^\star \in (0, \infty)$, i.e., $\|G\|_2 = \|G(j\omega^\star)\|_2$.

(i) Suppose $\|G\|_2 = \|G(0)\|_2$. Then, by letting $\omega \to 0$ in (20) and (19), we see that $\|G\|_{2+} \geq \|G(0)\|_2 = \|G\|_2$. Namely, $\|G\|_{2+} = \|G\|_2$ holds.

(ii) Suppose $\|G\|_2 = \|G(j\infty)\|_2$. Then, we see from (20) and (19) that

$$\begin{aligned} \|G\|_{2+} &\geq \frac{\sqrt{2a_0^{\star 2}\|G(0)\|_2^2 + (2 - 2a_0^{\star 2})\|G(j\infty)\|_2^2}}{\sqrt{2}} \\ &\geq \frac{\sqrt{(2 - 2a_0^{\star 2})}}{\sqrt{2}}\|G(j\infty)\|_2 \approx \frac{1.0906}{\sqrt{2}}\|G(j\infty)\|_2 \\ &> \frac{1}{\sqrt{2}}\|G\|_2. \end{aligned}$$

(iii) Suppose $\|G\|_2 = \|G(j\omega^\star)\|_2$ ($\omega^\star \in (0, \infty)$). Then, for $\|G\|_{2+} = \frac{1}{\sqrt{2}}\|G\|_2$ to hold, we see from (20) that the system G should satisfy the infinitely many interpolation constraints:

$$G(0) = 0,\ G(j2p\omega^\star) = 0 \ (p \in \mathbb{N}). \tag{21}$$

This is impossible for the finite-dimensional system G and hence $\|G\|_{2+} > \frac{1}{\sqrt{2}}\|G\|_2$. ■

Remark 3. In Theorem 1, we have shown that the infinite-dimensional system $G^\star(s) = 1 - e^{-Ls}$ ($L > 0$) satisfies $\|G^\star\|_{2+} = \frac{1}{\sqrt{2}}\|G^\star\|_2$. Therefore, from the proof of Theorem 2, the system $G^\star$ should satisfy the infinitely many interpolation constraints (21). Indeed, we see that $\omega^\star = \frac{1}{L}\pi$ for $G^\star$, and $G^\star$ does satisfy the interpolation constraints (21) since $G^\star(0) = 0$, and $G^\star(j2p\omega^\star) = 0$ ($p \in \mathbb{N}$).

For a given system $G \in \mathcal{G}_{\mathrm{LTI,SI}}$, we finally make active use of (20) for the lower bound computation of $\|G\|_{2+}$. By truncation of the infinite series, let us define

$$\upsilon_N(G) := \sup_{\omega>0} \frac{\sqrt{2a_0^{\star 2}\|G(0)\|_2^2 + \|G(j\omega)\|_2^2 + \sum_{m=2}^{N} a_m^{\star 2}\|G(jm\omega)\|_2^2}}{\sqrt{2}}. \tag{22}$$

Then, it is straightforward from Theorem 2 that the next results hold.

Theorem 3. Suppose $G \in \mathcal{G}_{\mathrm{LTI,SI}}$ is finite-dimensional and define $\upsilon_N(G)$ ($N \in \mathbb{N}$) by (22). Then, we have

$$\|G\|_{2+} \geq \upsilon_N(G) \geq \frac{1}{\sqrt{2}}\|G\|_2 \ (\forall N \in \mathbb{N}).$$

In particular, $\upsilon_N(G)$ is monotonically non-decreasing with respect to $N \in \mathbb{N}$, and for sufficiently large N we have $\upsilon_N(G) > \frac{1}{\sqrt{2}}\|G\|_2$.

We finally note that $\upsilon_N(G)$ can readily be computed since

$$\upsilon_N(G) = \|\widehat{G}_N\|_2 \ (\forall N \in \mathbb{N}),\ \widehat{G}_N(s) := \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}a_0^\star G(0) \\ G(s) \\ a_2^\star G(2s) \\ \vdots \\ a_N^\star G(Ns) \end{bmatrix}.$$

4.3 Numerical Examples

Let us consider the case where the system G in (1) is given by the state equation with coefficient matrices

$$A = \begin{bmatrix} -1.30 & 0.71 & -0.20 & 0.09 & 0.19 \\ -0.08 & -0.13 & 0.21 & -0.07 & 0.09 \\ 0.04 & 0.01 & 0.30 & -0.56 & -0.39 \\ -0.28 & 0.24 & 0.31 & -0.47 & -0.19 \\ -0.06 & -0.69 & 0.19 & 0.52 & -0.06 \end{bmatrix},\ B = \begin{bmatrix} -0.60 \\ -0.41 \\ -0.37 \\ 0.09 \\ 0.55 \end{bmatrix},$$

$C = [-0.41\ \ 0.09\ \ -0.06\ \ -0.13\ \ -0.06]\,,\ D = 0.58.$

In this case, it turned out that $\|G\|_2 = 2.3250$. On the other hand, by following the method in Ebihara et al. [2022], we computed upper bounds of $\|G\|_{2+}$. Then, the best (least) upper bound is 2.2324. With these facts in mind, we computed $\upsilon_N(G)$, the lower bounds of $\|G\|_{2+}$. The results are shown in Fig. 2. The best (largest) lower bound is 2.1888. This is indeed larger than $\frac{1}{\sqrt{2}}\|G\|_2 \approx 1.6440$ obtained from the uniform lower bound shown in Theorem 1. From these upper/lower bounds together with

$$\frac{2.2324 - 2.1888}{2.1888} \approx 0.0199,$$

we can conclude that the relative error between the upper bound 2.2324 and the true value of $\|G\|_{2+}$ is less than 2%.

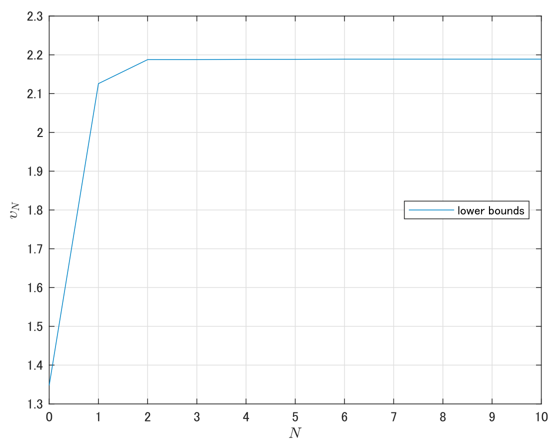

Fig. 2. Computed Lower Bounds $\upsilon_N(G)$.

5. CONCLUSION

In this paper, we introduced L_{p+} ($p \in [1,\infty), p = \infty$) induced norms for continuous time LTI systems, analyzed their lower bounds with respect to the standard L_p induced norms, and derived an effective method to compute lower bounds of the L_{2+} induced norm for single-input LTI systems. As the main results, we have shown that the L_{p+} induced norm of an LTI system cannot be smaller than the L_p induced norm scaled by $2^{(1-p)/p}$ for $\in [1,\infty)$ (scaled by 2^{-1} for $p = \infty$). For $p = 2$, we further clarified that the L_{2+} induced norm of a finite-dimensional single-input LTI system is strictly larger than its L_2 induced norm scaled by $2^{-1/2}$. For finite-dimensional single-input LTI systems, we also derived a method for lower bound computation of the L_{2+} induced norm in such a sound way that it enables us to obtain a lower bound that is strictly better (larger) than the L_2 induced norm scaled by $2^{-1/2}$. It is our important future issue to extend this method to multi-input systems.

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