Free resolution of the logarithmic derivation modules and its application

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出版情報：Kyushu University, 2025, 博士（数理学），課程博士
バージョン：
権利関係：

九州大学
KYUSHU UNIVERSITY

Kyushu University

Graduate School of Mathematics

Free resolution of the logarithmic derivation modules and its application

(対数微分加群の自由分解とその応用)

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PhD Thesis

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Date: 13 June 2025

Abstract

The study of hyperplane arrangements lies at the intersection of combinatorics, algebra, and geometry, with far-reaching implications across areas such as commutative algebra and data science. A hyperplane arrangement is a finite collection of hyperplanes in a vector space, and the associated algebraic structures—most notably the module of logarithmic derivations—encode deep information about both the geometry and the combinatorics of the arrangement. Much of the classical theory has focused on free arrangements, where the module of logarithmic derivations is free, leading to particularly tractable algebraic and topological properties. However, understanding the algebraic structure of arrangements that are close to free remains a challenging and open problem.

In the first part of this thesis, we introduce and study a new class of arrangements constructed by deleting two hyperplanes from a free arrangement. This operation typically destroys freeness, resulting in arrangements whose logarithmic derivation modules are no longer free and exhibit more intricate algebraic behavior. We analyze the minimal free resolutions of these modules—a central object in commutative algebra—as a tool to investigate their complexity. In particular, we establish lower bounds on the graded Betti numbers of these resolutions, shedding light on the relationship between algebraic invariants and the combinatorial structure of the arrangement. Focusing on the three-dimensional case, we explicitly determine the minimal free resolutions and provide concrete examples that illustrate how the deletion of hyperplanes affects the delicate interplay between algebra and combinatorics in arrangements that are close to free.

Motivated by both theoretical and applied considerations, the second part of the thesis turns to the problem of interpolating sparsely observed vector fields, which is fundamental in data science, computer graphics, and physics. Consider the task of reconstructing a vector field within a convex polyhedral domain in arbitrary dimensions, given a finite set of observations, such as measurements from simulations or physical experiments. In many practical scenarios, the vector field is subject to boundary conditions—for example, a no-penetration and slip condition, requiring the field to be tangent to the domain's boundary. We develop a novel algorithmic framework that, for a given integer k, constructs a degree k polynomial vector field interpolating the data in the least squares sense, while exactly satisfying the tangency condition along the boundary. The central theoretical advance enabling our scheme is the identification and characterization of the module of polynomial vector fields tangent to the boundary, through connections with the theory of hyperplane arrangements—specifically, by viewing the faces of the polyhedral domain as an arrangement of hyperplanes. This perspective not only provides a rigorous algebraic foundation for the interpolation

scheme, but also ties the problem back to the study of logarithmic derivation modules from the first part of the thesis.

Finally, we turn to a fundamental structural question related to Saito's criterion, a cornerstone of the theory of free arrangements. We discover that if an arrangement has projective dimension one, its logarithmic derivation module satisfies a relation reminiscent of Saito's criterion. This observation leads us to conjecture a generalization of Saito's criterion to the non-free setting. Pursuing this direction, we prove a theorem that constructs minimal generators for logarithmic derivation modules of arrangements with projective dimension one. As a corollary, this generalized version of Saito's criterion holds in the case of arrangements in dimension three, providing new insight into the structure of non-free arrangements and offering a promising avenue for further exploration.

Together, these three parts—exploring the algebraic structure of close to free arrangements, developing interpolation techniques grounded in hyperplane theory, and advancing a generalized version of Saito's criterion—underscore the deep and fruitful connections between pure mathematics and its applications. This thesis contributes to the foundational understanding of hyperplane arrangements while also demonstrating their utility in solving practical problems in modern computational and data-driven sciences.

Contents

Acknowledgments

As I come to the end of this chapter in my academic journey, I find myself reflecting on the many people, experiences, and moments that have shaped my time at Kyushu University. This journey has been filled with growth, discovery, and transformation—both as a researcher and as a person. None of it would have been possible without the generous support, encouragement, and companionship of those around me. I would like to take this opportunity to express my heartfelt gratitude to everyone who has walked alongside me during this unforgettable time.

I would like to begin by expressing my deepest gratitude to Professors Takuro Abe and Shizuo Kaji for their unwavering support and guidance throughout my time at Kyushu University. Their insightful advice and tireless dedication during our regular group meetings have been instrumental in shaping my academic development. Whenever I found myself doubting my abilities, they were always there with words of encouragement—often reminding me, "You’ve done well." These moments of reassurance, along with the wisdom they generously shared, gave me the confidence to stay the course and pursue my goals with renewed determination. I am especially grateful for the many opportunities they offered me to attend international conferences. These experiences not only broadened my academic horizons but also deepened my understanding of my research field. Their mentorship has been a cornerstone of my journey, and I will remain forever thankful.

Moving to Japan and settling into life at Kyushu University, I faced many challenges. However, I was incredibly fortunate to meet Paul Mücksch and Akiko Yazawa early on in my time here. Though you both left Kyushu University within less than a year, I will always treasure the time we spent together. You were there with me during my first Lantern Festival in Japan, and I’ll never forget enjoying my first hot pot meal with you. Those moments of warmth and camaraderie made my transition to life in a foreign country so much easier and gave me a sense of home when I needed it most.

I would also like to thank the Faculty of Mathematics for organizing the orientation events that helped me meet many wonderful people, including Edoardo Fabbrini, Professor Shizuo Kaji, and his wife, Mikiko Kaji. Your kindness and friendship have meant a lot to me during my time here. I am equally grateful to my Chinese friends, as well as Graiff Zurita Sebastían Elías, fellow advisee Matias de Jong van Lier and his wife Carolina Sayuri Takeda, and fellow student Edoardo Fabbrini and his girlfriend Sofia Fasolato. Your support helped me navigate difficult times, and I deeply appreciate the encouragement you’ve offered.

In addition, I would like to extend my heartfelt thanks to my friends in China. I am especially grateful to my two old friends, Panpan Wang and Yanfang Liu. Although we haven't seen each other for a long time, you continue to show your care and concern. Your messages of care have been a reminder that no matter how far apart we are, the bond of friendship remains strong.

A special thank you also goes to my cat, whose name is "慕暮." Her quiet companionship has brought me comfort and joy throughout this journey.

Finally, I would like to express my deepest appreciation to the China Scholarship Council for their generous financial support of this research. Without their assistance, this work would not have been possible.

In closing, I would like to express my profound gratitude to all the people I have met, both near and far, whose unwavering support has been a constant source of strength. Your encouragement, kindness, and belief in me have helped me overcome countless challenges, and I will always be grateful for the role each of you has played in my journey. Thank you from the bottom of my heart.

Chapter 1

Free resolution of the logarithmic derivation modules of close to free arrangements

This chapter studies the algebraic structure of a new class of hyperplane arrangement $\mathcal{A}$ obtained by deleting two hyperplanes from a free arrangement. We provide some information on the minimal free resolutions of the logarithmic derivation module of $\mathcal{A}$, which can be used to compute a lower bound for the graded Betti numbers of the resolution.

Specifically, for the three-dimensional case, we determine the minimal free resolution of the logarithmic derivation module of $\mathcal{A}$. We present illustrative examples [1] of our main theorems to provide insights into the relationship between algebraic and combinatorial properties for close-to-free arrangements.

The results in this section are based on [5].

1.1 Introduction

Let V be the ℓ-dimensional vector space $\mathbb{K}^\ell$ over a field $\mathbb{K}$. The coordinate ring $S = Sym(V^*) \cong \mathbb{K}[x_1, \dots, x_\ell]$ is equipped with the usual grading and its degree i homogeneous part is denoted by S_i. A (central) *hyperplane arrangement* $\mathcal{A}$ is a finite set of linear hyperplanes in V. For a hyperplane $H \in \mathcal{A}$, the defining linear form is denoted by $\alpha_H \in S_1$ with $H = \ker \alpha_H$. The *defining polynomial* $Q(\mathcal{A})$ of $\mathcal{A}$ is defined as $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$, and it is defined up to a scalar multiple. One of the most important algebraic invariants associated to an arrangement $\mathcal{A}$ is its *logarithmic derivation module* $D(\mathcal{A})$ defined by

$$D(\mathcal{A}) = \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H \text{ for any } H = \ker \alpha_H \in \mathcal{A}\},$$

where $\operatorname{Der} S$ is the free S-module of derivations generated by $\{\partial_{x_i} \mid 1 \leq i \leq \ell\}$. Given a derivation $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in D(\mathcal{A})$, we say it is homogeneous if all $f_i \in S_d$ for some

[1]We have written a program and used it to compute examples.
You can find the code at https://github.com/jcwjmz/LogarithmicDerivationModule.git

$d \in \mathbb{Z}_{\geq 0}$, and we write $\deg \theta = d$. Generally, $D(\mathcal{A})$ is a reflexive graded S-module [24] and is not always free. If $D(\mathcal{A})$ is free, then there exist homogeneous derivations $\theta_1, \dots, \theta_\ell \in D(\mathcal{A})$ such that $D(\mathcal{A}) = \bigoplus_{i=1}^{\ell} S\theta_i$. In this situation, we say that $\mathcal{A}$ is free with *exponents* $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$, where $d_i = \deg \theta_i$.

The study of $D(\mathcal{A})$ has focused primarily on the case when it is free (see [34] for a survey). Much remains unexplored when it is not free. In order to understand non-free cases, a natural approach is to look at their graded minimal free resolution. Some works consider the degrees (the Betti numbers) of the graded minimal free resolution [8, 18, 25]. In particular, [31, 32] study the *derivation degree sequence*, denoted as $\mathrm{DS}(\mathcal{A})$, which is defined as the unordered sequence of the degrees of the minimal homogeneous generators of $D(\mathcal{A})$. Moreover, we use $|\mathrm{DS}(\mathcal{A})|$ to represent the number of the minimal homogeneous generators of $D(\mathcal{A})$. In cases where $\mathcal{A}$ is free, we have $\mathrm{DS}(\mathcal{A}) = \exp(\mathcal{A})$. Although minimal homogeneous generators are not unique, their degrees do not depend on the choice since they are the degrees of $\mathrm{Tor}_0(\mathbb{K}, D(\mathcal{A}))$. One of the main challenges in the study of non-free arrangements is that the determination of these two algebraic properties of $D(\mathcal{A})$ is generally influenced not only by the combinatorics of $\mathcal{A}$ but also by its geometry (see Example 1.4.8).

To tackle this issue, our initial approach involves examining arrangements that are close to free arrangements. This is inspired by the *next-to-free minus (NT-free-1)* defined by Abe.

Definition 1.1.1 (Definition 1.3 and 6.1 in [1])**.** We say that $\mathcal{B}$ is next-to-free minus (NT-free-1) if there exist a free arrangement $\mathcal{A}$ and a hyperplane $H \in \mathcal{A}$ such that $\mathcal{B} = \mathcal{A} \setminus \{H\}$.

In this case, we say $\mathcal{A}$ is a **free addition** of $\mathcal{B}$.

Now we introduce a class of arrangement, which has a nice structure called *strictly plus-one generated (SPOG)*, closely related to the NT-free-1 arrangement.

Definition 1.1.2 (Definition 1.1 in [1])**.** An arrangement $\mathcal{B}$ is said to be *strictly plus-one generated (SPOG)* with exponents $\mathrm{POexp}(\mathcal{B}) = (d_1, \dots, d_\ell)$ and level d, if there exist $f_1, \dots, f_\ell, \alpha \in S$ with $\alpha \neq 0$ such that $D(\mathcal{B})$ has a minimal free resolution of the following form:

$$0 \to S[-d-1] \xrightarrow{(\alpha, f_1, \dots, f_\ell)} S[-d] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{B}) \to 0.$$

In particular, $\mathrm{DS}(\mathcal{B}) = (d_1, \dots, d_\ell, d) = (\mathrm{POexp}(\mathcal{B}), d)$.

Remark 1.1.3. In other words, $\mathcal{B}$ is SPOG if there is a minimal set of homogeneous generators $\theta_1, \theta_2, \dots, \theta_\ell$ and φ for $D(\mathcal{B})$ such that $\deg \theta_i = d_i$, $\deg \varphi = d$, and

$$\sum_{i=1}^{\ell} f_i \theta_i + \alpha \varphi = 0,$$

where $f_i \in S$ and $0 \neq \alpha \in S_1$. This α is called the level coefficient, and φ is a level element. Moreover, when $d = d_i$ for some i and $f_i \neq 0$, then θ_i also can be a level element. In other words, the choice of the level coefficient and element is not unique.

Abe [1] shows that an NT-free-1 arrangement $\mathcal{B}$ is either free or SPOG. If $\mathcal{B}$ is SPOG and NT-free-1, the level of $\mathcal{B}$ can be determined by the combinatorial properties of its free addition. To state his results, let us recall that the definition of the intersection lattice of $\mathcal{A}$, denoted by $L(\mathcal{A})$, as follows:

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A} \right\},$$

where $L(\mathcal{A})$ is equipped with a partial order induced by reverse inclusion. For a given $X \in L(\mathcal{A})$, the localization $\mathcal{A}_X$ of $\mathcal{A}$ at X is defined by

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\},$$

and the restriction $\mathcal{A}^X$ of $\mathcal{A}$ onto X is defined by

$$\mathcal{A}^X := \{H \cap X \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

The following is a significant theorem about NT-free-1 arrangements, which plays a crucial role in shaping our results.

Theorem 1.1.4 (Theorem 1.4 and Proposition 5.3 in [1])**.** *Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$ and $H \in \mathcal{A}$. Then $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is free, or SPOG with $\mathrm{POexp}(\mathcal{A}') = (d_1, \dots, d_\ell)$ and level $d = |\mathcal{A}'| - |\mathcal{A}^H|$. Moreover, if $\ell = 3$, then $d \geq \max\{d_1, d_2, d_3\}$.*

Remark 1.1.5. Suppose that $\mathcal{A}$ is free and $\mathcal{A}' = \mathcal{A} \setminus H$ is SPOG.

(1) Assume that $\theta_1, \dots, \theta_\ell$ is a basis of $D(\mathcal{A})$. Since $\mathrm{DS}(\mathcal{A}') = (\mathrm{POexp}(\mathcal{A}), d)$ and $\mathrm{POexp}(\mathcal{A}') = \exp(\mathcal{A})$, there exists a $\varphi \in D(\mathcal{A}')$ of degree d such that $\theta_1, \dots, \theta_\ell, \varphi$ generate $D(\mathcal{A}')$. Since $D(\mathcal{A}) \subsetneq D(\mathcal{A}')$, we have $D(\mathcal{A}') = D(\mathcal{A}) + S\varphi$, and $\varphi \notin D(\mathcal{A})$ as a level element of $D(\mathcal{A}')$.

(2) Let $\varphi \notin D(\mathcal{A})$ be a level element of $\mathcal{A}'$. Let $\alpha_H \in S_1$ be such that $\ker \alpha_H = H$. Since $\varphi \in D(\mathcal{A}') \setminus D(\mathcal{A})$, we have $\alpha_H \varphi \in D(\mathcal{A})$. Thus, we obtain a relationship between the minimal generators of $D(\mathcal{A}')$. Since SPOG arrangements have a unique relation among the minimal generators, we may always assume that α_H is the level coefficient.

In this chapter, we introduce a new class of possibly non-free arrangements obtained by deleting two hyperplanes from free arrangements.

Definition 1.1.6. We say that $\mathcal{B}$ is *next-to-free-minus-two* (NT-free-2) if there exist a free arrangement $\mathcal{A}$ and two hyperplanes $H_1, H_2 \in \mathcal{A}$ such that $\mathcal{B} = \mathcal{A} \setminus \{H_1, H_2\}$.

By analyzing the minimal free resolution, we deduce the following theorem:

Theorem 1.1.7. *Let $\mathcal{B}$ be an NT-free-2 arrangement. Then the projective dimension $pd_S(D(\mathcal{B})) \leq 1$ if and only if $|\mathrm{DS}(\mathcal{B})| \leq \ell + 2$.*

Let $\mathcal{A} = \{H_1, \dots, H_p \mid H_i = \ker \alpha_i\}$ be a free arrangement. We denote the NT-free-1 arrangement $\mathcal{A} \setminus \{H_j\}$ by $\mathcal{A}_j$ and the NT-free-2 arrangement $\mathcal{A} \setminus \{H_j, H_k\}$ by $\mathcal{A}_{j,k}$. We note that $\mathcal{A}_{j,k} = \mathcal{A}_{k,j}$. We also denote $\mathcal{A}^{H_j}$ by $\mathcal{A}^j$.

If $\mathcal{A}_1$ or $\mathcal{A}_2$ is free, then $\mathcal{A}_{1,2}$ is NT-free-1 and it is either free or SPOG by Theorem 1.1.4. Therefore, we focus on the case when none of $\mathcal{A}_1$ and $\mathcal{A}_2$ are free. We show that when

$|\operatorname{DS}(\mathcal{A}_{1,2})| \le \ell + 2$, the minimal free resolution of $D(\mathcal{A}_{1,2})$ assumes one of the forms listed in Theorem 1.3.8. When $|\operatorname{DS}(\mathcal{A}_{1,2})| > \ell + 2$, we obtain a lower bound for the Betti numbers of $D(\mathcal{A}_{1,2})$ using the information provided in Theorem 1.3.9.

Note that since the logarithmic derivation module is a reflexive graded S-module, its projective dimension is less than or equal to $\ell - 2$ (see, Lemma 1.2.10). Consequently, for a three-dimensional NT-free-2 arrangement $\mathcal{A}_{1,2}$, we can infer that $\operatorname{pd}_S(D(\mathcal{A}_{1,2})) \le 3-2 = 1$. Utilizing Theorem 1.1.7, we then establish that $|\operatorname{DS}(\mathcal{A}_{1,2})| \le \ell + 2 = 5$. Furthermore, we determine the precise form of the minimal free resolution of $D(\mathcal{A}_{1,2})$.

Theorem 1.1.8. *Assume $\ell = 3$. Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (d_1, d_2, d_3)$, and $\mathcal{A}_1$ and $\mathcal{A}_2$ be SPOG with levels c_1 and c_2, respectively. We may assume $c_1 \le c_2$. Then $\mathcal{A}_{1,2}$ can never be free in this set-up, and the module $D(\mathcal{A}_{1,2})$ has a minimal free resolution in one of the following forms:*

(1) Suppose that $|\mathcal{A}_{H_1 \cap H_2}| = 2$.

There exist $f_i, g_i \in S$ such that

$$0 \to S[-c_1-1] \oplus S[-c_2-1] \xrightarrow[]{\substack{(\alpha_1, 0, f_1, f_2, f_3)\\(0, \alpha_2, g_1, g_2, g_3)}} S[-c_1] \oplus S[-c_2] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right) \to D(\mathcal{A}_{1,2}) \to 0,$$

where $i = 1, 2, 3$.

(2) Suppose that $|\mathcal{A}_{H_1 \cap H_2}| > 2$.

(2.1) If $c_1 = c_2$, there exist $f_i \in S$ such that

$$0 \to S[-c_1-1] \xrightarrow{(\alpha_1\alpha_2, f_1, f_2, f_3)} S[-c_1+1] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right) \to D(\mathcal{A}_{1,2}) \to 0.$$

Moreover, $\mathcal{A}_{1,2}$ is SPOG if and only if $c_1 = c_2 = \max\{d_1, d_2, d_3\}$. If $\mathcal{A}_{1,2}$ is SPOG, let $d_2 = \max\{d_1, d_2, d_3\}$, then $f_2 \in S_1$.

(2.2) If $c_1 < c_2$, there exist $f_i, g_i, g \in S$ with $g \ne 0$ such that

$$0 \to S[-c_1-1] \oplus S[-c_2] \xrightarrow[]{\substack{(\alpha_1, 0, f_1, f_2, f_3)\\(g, \alpha_2, g_1, g_2, g_3)}} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right) \to D(\mathcal{A}_{1,2}) \to 0,$$

where $i = 1, 2, 3$.

1.2 Preliminaries

In this section, we will summarize several results and definitions. Let $\mathcal{A}$ be an arrangement and $L(\mathcal{A})$ be its intersection lattice.

Let $\theta_E = \sum_{i=1}^{\ell} x_i \partial_{x_i}$ be the *Euler derivation*, which is a homogeneous derivation of $\deg \theta_E = 1$ and always contained in $D(\mathcal{A})$. Recall that for every $H \in \mathcal{A}$, there exists a decomposition:

$$D(\mathcal{A}) \cong S\theta_E \oplus D_H(\mathcal{A}), \tag{1.2.1}$$

where $D_H(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0\}$ (see, for example, [21, Lemma 1.33]). This implies that $1 \in \mathrm{DS}(\mathcal{A})$ if $\mathcal{A}$ is not empty. Furthermore, if $\mathcal{A} \neq \emptyset$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, we may assume that $d_1 = \deg \theta_E = 1$.

The following has been well-known and frequently utilized by specialists.

Proposition 1.2.1 ([31]). *Let $H \in \mathcal{A}$. Then there is a polynomial B of degree $|\mathcal{A}| - |\mathcal{A}^H| - 1$ such that $\alpha_H \nmid B$, and $\theta(\alpha_H) \in (\alpha_H, B)$ for all $\theta \in D(\mathcal{A} \setminus \{H\})$.*

From the above proposition, we can easily derive the following proposition. We include it here since we frequently utilize it.

Proposition 1.2.2 (Corollary 3.3 in [1]). *Let $H \in \mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \setminus \{H\}$. Assume that there exists $\varphi \in D(\mathcal{A}')$ with $\deg \varphi = |\mathcal{A}'| - |\mathcal{A}^H|$ such that $\varphi \notin D(\mathcal{A})$. Then $D(\mathcal{A}') = D(\mathcal{A}) + S\varphi$.*

We now define the *Euler restriction map* $\rho : D(\mathcal{A}) \to D(\mathcal{A}^H)$ for an arrangement $\mathcal{A}$ by taking modulo α_H. Additionally, let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$. We have an exact sequence as follows:

Proposition 1.2.3 (Proposition 4.45 in [21]).

$$0 \to D(\mathcal{A}') \xrightarrow{\cdot \alpha_H} D(\mathcal{A}) \xrightarrow{\rho} D(\mathcal{A}^H).$$

Moving forward, let's delve into some results pertaining to the logarithmic derivation module $D(\mathcal{A})$ and its freeness.

Theorem 1.2.4 (Addition-Deletion Theorem. Removal Theorem in [31]). *Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' := \mathcal{A}^H$. Then two of the following imply the third:*

(1) $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.

(2) $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.

(3) $\mathcal{A}''$ is free with $\exp(\mathcal{A}'') = (d_1, \ldots, d_{\ell-1})$.

Moreover, all the three above hold true if $\mathcal{A}$ and $\mathcal{A}'$ are free.

Theorem 1.2.5 (Saito's criterion. [24]). *Let $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$. Then $\mathcal{A}$ is free with basis $\{\theta_1, \ldots, \theta_\ell\}$ if and only if one of the following conditions holds:*

(1) The $\theta_1, \ldots, \theta_\ell$ are homogenous and linearly independent over S, and

$$\sum_{i=1}^{\ell} \deg \theta_i = |\mathcal{A}|.$$

(2) $\det M[\theta_1, \ldots, \theta_\ell] \in \mathbb{K}Q(A) \setminus \{0\}$, where

$$\det M[\theta_1, \ldots, \theta_\ell] = \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_1(x_\ell) \\ \vdots & \cdots & \vdots \\ \theta_\ell(x_1) & \cdots & \theta_\ell(x_\ell) \end{bmatrix}.$$

Let us introduce the multiarrangement theory. A *multiarrangement* is a pair $(\mathcal{A}, m)$, where $\mathcal{A}$ is a hyperplane arrangement in V and multiplicity m is a map $m\colon \mathcal{A} \to \mathbb{Z}_{\geq 0}$. Define $|m| = \sum_{H\in\mathcal{A}} m(H)$. If $m(H) = 1$ for all $H \in \mathcal{A}$, we say that the multiarrangement $(\mathcal{A}, m)$ is a hyperplane arrangement, which is also called a simple arrangement. The *logarithmic derivation module* $D(\mathcal{A}, m)$ is defined as follows:

$$D(\mathcal{A}, m) = \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H^{m(H)} \text{ for any } H = \ker \alpha_H \in \mathcal{A}\}.$$

The module $D(\mathcal{A}, m)$ is also a reflexive graded S-module, which is not always free. We can define the concepts of freeness and exponents for $(\mathcal{A}, m)$ in the same way as for simple arrangements.

Definition 1.2.6 ([35])**.** For an arrangement $\mathcal{A}$ and $H \in \mathcal{A}$, define the Ziegler multiplicity $m^H : \mathcal{A}^H \to \mathbb{Z}_{\geq 0}$ by $m^H(X) := |\{L \in \mathcal{A} \setminus \{H\} \mid L \cap H = X\}|$ for $X \in \mathcal{A}^H$. The pair $(\mathcal{A}^H, m^H)$ is called the *Ziegler restriction* of $\mathcal{A}$ onto H. Also, there is a *Ziegler restriction map*:

$$\pi : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$$

by taking modulo α_H. In particular, there is an exact sequence:

$$0 \to D_H(\mathcal{A}) \xrightarrow{\cdot\alpha_H} D_H(\mathcal{A}) \xrightarrow{\pi} D(\mathcal{A}^H, m^H).$$

Theorem 1.2.7 (Theorem 11 in [35])**.** *Assume that $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$. Then for any $H \in \mathcal{A}$, the Ziegler restriction $(\mathcal{A}^H, m^H)$ is also free with $\exp(\mathcal{A}^H, m^H) = (d_2, \ldots, d_\ell)$. Explicitly, for the Ziegler restriction $\pi : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$, any basis $\theta_2, \ldots, \theta_\ell$ for $D_H(\mathcal{A})$, $\pi(\theta_2), \ldots, \pi(\theta_\ell)$ form a basis for $D(\mathcal{A}^H, m^H)$. In particular, π is surjective when $\mathcal{A}$ is free.*

Lemma 1.2.8 (Lemma 4.2 and Lemma 4.3 in [3])**.** *Let $\mathcal{A}$ be central line arrangement and let m, m' be multiplicities on $\mathcal{A}$ such that $|m| = |m'| + 1$ and $m(H) \geq m'(H)$ for any $H \in \mathcal{A}$. If $\exp(\mathcal{A}, m') = (a, b)$, then $\exp(\mathcal{A}, m) = (a + 1, b)$ or $(a, b + 1)$.*

Finally, we present the basics of free resolutions.

Definition 1.2.9. For arrangement $\mathcal{B}$, we denote the minimal free resolution of the module $D(\mathcal{B})$ by

$$0 \to M_k \xrightarrow{R_k} M_{k-1} \xrightarrow{R_{k-1}} \cdots \xrightarrow{R_2} M_1 \xrightarrow{R_1} M_0 \to D(\mathcal{B}) \to 0,$$

where $k = pd_S(D(\mathcal{B}))$ is the projective dimension of $D(\mathcal{B})$. Here, R_i $(i = 1, \ldots, k)$ are represented by matrices acting by multiplication from the left. We use $R_i(j)$ to denote the j-th row of R_i, and $R_i(j_1, j_2)$ to denote the (j_1, j_2) entry of R_i.

Let $JQ(\mathcal{A})$ be the Jacobian ideal of $Q(\mathcal{A})$ generated by $\frac{\partial Q(\mathcal{A})}{\partial x_i}$ for $i = 1, \ldots, \ell$. Since $D(\mathcal{A}) = \{\theta \in \operatorname{Der} S \mid \theta(Q(\mathcal{A})) \in SQ(\mathcal{A})\}$, we have a free resolution of the form

$$0 \to N_\ell \to \cdots \to N_3 \to D_H(\mathcal{A}) \to S^\ell \to S \to S/JQ(\mathcal{A}) \to 0 \tag{1.2.2}$$

by Hilbert's syzygy theorem. Since $pd_S(D(\mathcal{A})) = pd_S(D_H(\mathcal{A}))$ by (1.2.1), we have

Lemma 1.2.10. $pd_S(D(\mathcal{A})) \leq \ell - 2$.

From (1.2.2), we see $D(\mathcal{A})$ is reflexive since it is a second syzygy.

The following theorem is a simplified version of the result found in the reference [18].

Theorem 1.2.11 (Theorem 0.2 in [18]). *If the logarithmic derivation module $D(\mathcal{B})$ has a free resolution given by:*

$$0 \to \bigoplus_{i=1}^{r_k} S[-d_i^k] \to \cdots \to \bigoplus_{i=1}^{r_1} S[-d_i^1] \to \bigoplus_{i=1}^{r_0} S[-d_i^0] \to D(\mathcal{B}) \to 0,$$

then $|\mathcal{B}| = \sum_{j=0}^{k}(-1)^j \sum_{i=1}^{r_j} d_i^j$.

As a corollary, we have the following property for SPOG arrangements:

Proposition 1.2.12 (Proposition 4.1 in [1]). *Let $\mathcal{B}$ be SPOG with* $\mathrm{POexp}(\mathcal{B}) = (d_1, d_2, \ldots, d_\ell)$ *and level d. Then* $\sum_{i=1}^{\ell} d_i - 1 = |\mathcal{B}|$.

1.3 The Minimal Free Resolution of NT-Free-2 Arrangements

In this section, we introduce some notations. If U is a subset of $\operatorname{Der} S$, then $SU := \sum_{\xi \in U} S\xi$. Throughout this section, we assume that $\mathcal{A} = \{H_i \mid H_i : \alpha_i = 0\}$ is free, with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$, and $\mathcal{A}_j$ is SPOG with $\mathrm{POexp}(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$ and level c_j for $j = 1, 2$. Let $c_1 \le c_2$. By Proposition 1.2.3, we have Figure 1.1.

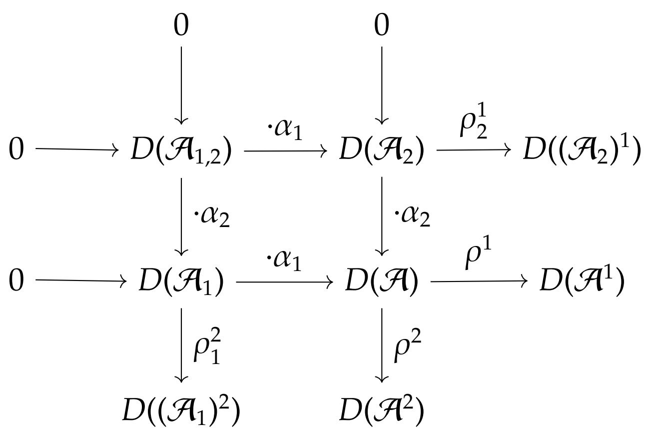

Figure 1.1: Exact Sequence Diagram

Remark 1.3.1.

(1) By Remark 1.1.5 Case (1), we may assume that the level element for $\mathcal{A}_j$ is not in $D(\mathcal{A})$.

(2) As there is no confusion, we can simplify by stating that $D(\mathcal{A}_2^1) := D((\mathcal{A}_2)^1)$ and $D(\mathcal{A}_1^2) := D((\mathcal{A}_1)^2)$.

Lemma 1.3.2. *Let $\mathcal{A}_i$ be SPOG, where $H_i \in \mathcal{A}$. For any homogeneous basis element θ for $D(\mathcal{A})$, we have $\theta \notin \ker \rho^i$.*

Proof. Assume that $\theta_1, \dots, \theta_\ell$ is a basis for $D(\mathcal{A})$ such that $\theta_1 \in \ker \rho^i$. By Proposition 1.2.3, there is an element $\varphi \in D(\mathcal{A}_i)$ such that $\theta_1 = \alpha_i \varphi$. Since $\varphi, \theta_2, \dots, \theta_\ell \in D(\mathcal{A}_i)$ are S-independent, by Theorem 1.2.5, we may get that $\mathcal{A}_i$ is free, which is a contradiction. □

Lemma 1.3.3. *We have $D(\mathcal{A}) + \ker \rho_i^j \subset D(\mathcal{A}_i)$, where $\{i, j\} = \{1, 2\}$. Moreover,*

(1) there exists an element $\varphi \in D(\mathcal{A}_{1,2})$ such that $\alpha_2\varphi$ is the level element for $\mathcal{A}_1$ if and only if $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$.

(2) there exists an element $\varphi \in D(\mathcal{A}_{1,2})$ such that $\alpha_1\varphi$ is the level element for $\mathcal{A}_2$ if and only if $D(\mathcal{A}_2) = D(\mathcal{A}) + \ker \rho_2^1$.

Proof. By Figure 1.1, it follows that $\ker \rho_i^j \subset D(\mathcal{A}_i)$. Note that $D(\mathcal{A}) \subset D(\mathcal{A}_i)$; hence, we have $D(\mathcal{A}) + \ker \rho_i^j \subset D(\mathcal{A}_i)$.

(1) By Proposition 1.2.3, we can deduce the following equivalences:

There exists an element $\varphi \in D(\mathcal{A}_{1,2})$ such that $\alpha_2\varphi$ serves as the level element for $\mathcal{A}_1$.
$\Longleftrightarrow$ There exists a level element $\theta \in D(\mathcal{A}_1)$ such that $\theta \in \ker \rho_1^2$.

By Remark 1.1.5, we can represent $D(\mathcal{A}_1)$ as $D(\mathcal{A}) + S\theta$. Consequently, we can state the following equivalences:

There exists a level element $\theta \in D(\mathcal{A}_1)$ such that $\theta \in \ker \rho_1^2$.
$\Longleftrightarrow$ $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$.

(2) This scenario bears resemblance to the one discussed in Case (1).

□

Lemma 1.3.4. *For every element $\varphi \in D(\mathcal{A}_{1,2})$, if $\alpha_j\varphi \in D(\mathcal{A})$ for some $j \in \{1, 2\}$, then $\varphi \in D(\mathcal{A}_j)$. Moreover, if both $\alpha_1\varphi$ and $\alpha_2\varphi$ are in $D(\mathcal{A})$, then φ itself is an element of $D(\mathcal{A})$.*

Proof. Let $\varphi \in D(\mathcal{A}_{1,2})$ be given, and assume that $\alpha_1\varphi \in D(\mathcal{A})$. By Proposition 1.2.3 and fig. 1.1, we have $\alpha_1\varphi \in \ker \rho^1 = \alpha_1 D(\mathcal{A}_1)$. Hence, we can conclude that $\varphi \in D(\mathcal{A}_1)$.

Similarly, if $\alpha_2\varphi \in D(\mathcal{A})$, we can deduce that $\varphi \in D(\mathcal{A}_2)$.

Moreover, if $\alpha_1\varphi \in D(\mathcal{A})$ and $\alpha_2\varphi \in D(\mathcal{A})$, then $\varphi \in D(\mathcal{A}_1) \cap D(\mathcal{A}_2) = D(\mathcal{A})$. □

Lemma 1.3.5. *If $|\mathcal{A}_{H_1 \cap H_2}| = 2$, then $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + D(\mathcal{A}_2)$.*

Proof. Let $\theta_{\ell,2}$ be a level element for $D(\mathcal{A}_2)$. Since $|\mathcal{A}_{H_1 \cap H_2}| = 2$, we can deduce that $|\mathcal{A}^2| = |\mathcal{A}_1^2| + 1$. Consequently, we have $\deg \theta_{\ell,2} = |\mathcal{A}_2| - |\mathcal{A}^2| = (1 + |\mathcal{A}_{1,2}|) - (|\mathcal{A}_1^2| + 1) = |\mathcal{A}_{1,2}| - |\mathcal{A}_1^2|$. Importantly, $\theta_{\ell,2} \in D(\mathcal{A}_{1,2}) \setminus D(\mathcal{A}_1)$. According to Proposition 1.2.2, this implies that $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + S\theta_{\ell,2}$. Since $\theta_{\ell,2} \in D(\mathcal{A}_2)$, we have $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + D(\mathcal{A}_2)$. □

Lemma 1.3.6. *If $D(\mathcal{A}) + \ker \rho_2^1 = D(\mathcal{A}_2)$, then $|\mathcal{A}_{H_1 \cap H_2}| > 2$.*

Proof. By Lemma 1.3.3 Case (2), we may assume that $\alpha_1\varphi$ is a level element for $D(\mathcal{A}_2)$. If $|\mathcal{A}_{H_1\cap H_2}| = 2$, it follows that $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + D(\mathcal{A}_2)$ by Lemma 1.3.5. This observation further implies that

$$\varphi \in D(\mathcal{A}_{1,2})_{<c_2} \subset D(\mathcal{A}_1)_{<c_2} + D(\mathcal{A}_2)_{<c_2} \subset D(\mathcal{A}_1) + D(\mathcal{A}) \subset D(\mathcal{A}_1).$$

Thus, $\alpha_1\varphi \in D(\mathcal{A})$, which is a contradiction, since $\alpha_1\varphi \notin D(\mathcal{A})$ is a level element for $D(\mathcal{A}_2)$. Hence, $|\mathcal{A}_{H_1\cap H_2}| > 2$. □

Proposition 1.3.7. *Let $\{\theta_1, \ldots, \theta_\ell\}$ be a basis for $D(\mathcal{A})$. There exist two level elements, $\theta_{\ell,1}$ for $D(\mathcal{A}_1)$ and $\theta_{\ell,2}$ for $D(\mathcal{A}_2)$, such that a minimal generator set for $D(\mathcal{A}_{1,2})$ falls into one of the following cases:*

(1) If $D(\mathcal{A})+\ker\rho_1^2 \subsetneq D(\mathcal{A}_1)$ and $D(\mathcal{A})+\ker\rho_2^1 = D(\mathcal{A}_2)$, then $|\mathcal{A}_{H_1\cap H_2}| > 2$, and a minimal generator set for $D(\mathcal{A}_{1,2})$ can be expressed as either $\{\theta_1, \ldots, \theta_\ell, \varphi_2, \theta_{\ell,1} \mid \theta_{\ell,2} = \alpha_1\varphi_2\}$ or $\{\theta_1, \ldots, \theta_{\ell-1}, \varphi_2, \theta_{\ell,1} \mid c_1 < c_2 = \deg\theta_\ell, \alpha_2\theta_{\ell,2} = \alpha_2\alpha_1\varphi_2 \in S\theta_1 + \cdots + S\theta_{\ell-1} + \mathbb{K}^*\alpha_1\theta_\ell\}$, where $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$.

(2) If $D(\mathcal{A}) + \ker\rho_1^2 = D(\mathcal{A}_1)$, then $c_1 = c_2$, $|\mathcal{A}_{H_1\cap H_2}| > 2$, $D(\mathcal{A}) + \ker\rho_2^1 = D(\mathcal{A}_2)$, and a minimal generator set for $D(\mathcal{A}_{1,2})$ can be expressed as $\{\theta_1, \ldots, \theta_\ell, \varphi_1 \mid \theta_{\ell,1} = \alpha_2\varphi_1\}$.

(3) If $D(\mathcal{A}) + \ker\rho_2^1 \subsetneq D(\mathcal{A}_2)$, then the set $\{\theta_1, \ldots, \theta_\ell, \theta_{\ell,1}, \theta_{\ell,2}\}$ forms, or can be extended to, a minimal generator set for $D(\mathcal{A}_{1,2})$.

Moreover, if $|\mathcal{A}_{H_1\cap H_2}| = 2$, the set $\{\theta_1, \ldots, \theta_\ell, \theta_{\ell,1}, \theta_{\ell,2}\}$ forms a minimal generator set for $D(\mathcal{A}_{1,2})$.

Proof. Note that $c_1 \le c_2$, which means this statement is not symmetric with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$. We prove this statement case by case.

(1) $D(\mathcal{A}) + \ker\rho_1^2 \subsetneq D(\mathcal{A}_1)$ and $D(\mathcal{A}) + \ker\rho_2^1 = D(\mathcal{A}_2)$.

By Lemma 1.3.6, we have $|\mathcal{A}_{H_1\cap H_2}| > 2$. Consequently, $|\mathcal{A}^2| = |\mathcal{A}_1^2|$. By Lemma 1.3.3 Case (2), we may assume that $\theta_{\ell,2} = \alpha_1\varphi_2$, where $\varphi_2 \in D(\mathcal{A}_{1,2})$. By Theorem 1.1.4, this implies that

$$\deg\varphi_2 = \deg\theta_{\ell,2} - 1 = (|\mathcal{A}_2| - |\mathcal{A}^2|) - 1 = |\mathcal{A}_{1,2}| - |\mathcal{A}_1^2|.$$

Note that $\varphi_2 \notin D(\mathcal{A}_1)$. If it were, then $\alpha_1\varphi_2 \in D(\mathcal{A})$, which, since it is the level element of $D(\mathcal{A}_2)$, is not in $D(\mathcal{A})$.

According to Proposition 1.2.2 and since $\varphi_2 \in D(\mathcal{A}_{1,2}) \setminus D(\mathcal{A}_1)$, this indicates that $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + S\varphi_2$. Hence, the set $T = \{\theta_1, \ldots, \theta_\ell, \theta_{\ell,1}, \varphi_2\}$ generates $D(\mathcal{A}_{1,2})$.

If T is not a minimal generating set, there exists an element $\gamma \in T$ such that γ is generated by $T\setminus\{\gamma\}$. If $\gamma = \theta_{\ell,1}$, then there is some $h \in S$ such that $\theta_{\ell,1} - h\varphi_2 \in D(\mathcal{A})$. Since $\theta_{\ell,1} \in D(\mathcal{A}_1)$ and $D(\mathcal{A}) \subset D(\mathcal{A}_1)$, we can deduce that $h\varphi_2 \in D(\mathcal{A}_1)$. However, we have claimed that $\varphi_2 \notin D(\mathcal{A}_1)$. Therefore, it follows that $\alpha_2 \mid h$, and thus $h\varphi_2 \in \ker\rho_1^2$. By $\theta_{\ell,1} - h\varphi_2 \in D(\mathcal{A})$ and $D(\mathcal{A}_1) = D(\mathcal{A}) + S\theta_{\ell,1}$, we have

$$D(\mathcal{A}_1) = D(\mathcal{A}) + Sh\varphi_2.$$

Since $h\varphi_2 \in \ker\rho_1^2$, we then have

$$D(\mathcal{A}_1) = D(\mathcal{A}) + \ker\rho_1^2,$$

which contradicts our assumption. If $\gamma = \varphi_2$, then $D(\mathcal{A}_{1,2}) = D(\mathcal{A}) + S\theta_{\ell,1} = D(\mathcal{A}_1)$, which contradicts $D(\mathcal{A}_1) \subsetneq D(\mathcal{A}_{1,2})$. If there is a $k \in \{1, \dots, \ell\}$ such that $\gamma = \theta_k$, then we may assume that

$$0 = p_1\theta_1 + \cdots + p_k\theta_k + \cdots + p_\ell\theta_\ell + u\theta_{\ell,1} + v\varphi_2, \tag{1.3.1}$$

where $p_k = 1$. It follows that $v\varphi_2 \in D(\mathcal{A}_1)$. Note that we claimed that $\varphi_2 \notin D(\mathcal{A}_1)$. Hence $\alpha_2 \mid v$, and $v = \alpha_2 v'$, where $v' \in S$. Since $p_k = 1$ and $\theta_1, \cdots, \theta_\ell, \theta_{\ell,1}$ form a minimal generating set of $D(\mathcal{A}_1)$, we have $v \neq 0$. Thus $\deg\theta_k = \deg v + \deg\varphi_2 \geq 1 + \deg\varphi_2 = c_2$. If $\deg\theta_k > c_2$, then $q_k = 0$ and $\alpha_2\varphi_2 \in S\theta_1 + \cdots + S\hat{\theta}_k + \cdots + S\theta_\ell + S\theta_{\ell,1}$. Since $v = \alpha_2 v'$, by Equation (1.3.1), it follows that $\theta_k \in S\theta_1 + \cdots + S\hat{\theta}_k + \cdots + S\theta_\ell + S\theta_{\ell,1}$, which is a contradiction. Thus, $\deg\theta_k = c_2$. We may let $v = \alpha_2$. If $u = 0$, we have $\alpha_2\varphi_2 \in D(\mathcal{A})$. Then $\varphi_2 \in D(\mathcal{A}_2)$, which contradicts the fact that $\alpha_1\varphi_2$ is a level element of $D(\mathcal{A}_2)$. Hence, $u \neq 0$. If $c_1 = c_2$, then we may assume that $u = 1$. Thus, $\alpha_2\varphi_2$ can be a level element for $D(\mathcal{A}_1)$. By Lemma 1.3.3 Case (1), we have $D(\mathcal{A}) + \ker\rho_1^2 = D(\mathcal{A}_1)$, which contradicts our assumption.

As a conclusion, if T is not a minimal generating set, we may set $k = \ell$. Then, we have $c_1 < c_2 = \deg\theta_\ell$, and $T \setminus \{\theta_\ell\}$ forms a minimal generating set for $D(\mathcal{A}_{1,2})$. Since $\alpha_1\theta_{\ell,1} \in D(\mathcal{A})$, we have $\alpha_1\alpha_2\varphi_2 \in S\theta_1 + \cdots + S\theta_{\ell-1} + \mathbb{K}^*\alpha_1\theta_\ell$.

(2) $D(\mathcal{A}) + \ker\rho_1^2 = D(\mathcal{A}_1)$.

By Lemma 1.3.3 Case (1), we may assume that $\theta_{\ell,1} = \alpha_2\varphi_1$, where $\varphi_1 \in D(\mathcal{A}_{1,2})$. This implies that $\deg\varphi_1 = c_1 - 1$. If $\alpha_1\varphi_1 \in D(\mathcal{A})$, we can conclude, based on Lemma 1.3.4, that $\varphi_1 \in D(\mathcal{A}_1)$, which contradicts the assertion that $\theta_{\ell,1} = \alpha_2\varphi_1$ is a level element for $D(\mathcal{A}_1)$. Therefore, we must conclude that $\alpha_1\varphi_1 \in D(\mathcal{A}_2)_{\leq c_1} \setminus D(\mathcal{A})$. It follows that $c_1 = c_2$. Hence, $\alpha_1\varphi_1$ can serve as a level element for $D(\mathcal{A}_2)$. According to Lemma 1.3.3 (2), this implies $D(\mathcal{A}) + \ker\rho_2^1 = D(\mathcal{A}_2)$.

Referring to Lemma 1.3.6, we establish that $|\mathcal{A}_{H_1\cap H_2}| > 2$. Analogous to the proof in Case (1), it holds that $D(\mathcal{A}_{1,2}) = D(\mathcal{A}_1) + S\varphi_2$ and $\theta_{\ell,2} = \alpha_1\varphi_2$ is a level element for $D(\mathcal{A}_2)$. Since both $\alpha_1\varphi_1$ and $\alpha_1\varphi_2$ can be level elements of $D(\mathcal{A}_2)$, it follows that there exists a $k \in \mathbb{K} \setminus \{0\}$ such that $k\alpha_1\varphi_1 - \alpha_1\varphi_2 \in D(\mathcal{A})$, which implies $k\varphi_1 - \varphi_2 \in D(\mathcal{A}_1)_{<c_1} \subset D(\mathcal{A})$. Thus, $\varphi_2 \in S\{\theta_1, \dots, \theta_\ell, \varphi_1\}$. Note that $D(\mathcal{A}_1) \subset S\{\theta_1, \dots, \theta_\ell, \varphi_1\}$. Consequently, the set $T = \{\theta_1, \dots, \theta_\ell, \varphi_1\}$ generates $D(\mathcal{A}_{1,2})$.

If T is not a minimal generating set, there exists an element $\gamma \in T$ such that γ is generated by $T \setminus \{\gamma\}$. Then $D(\mathcal{A}_{1,2})$ is free. Note that $\deg\theta_1 + \cdots + \deg\theta_\ell = |\mathcal{A}| > |\mathcal{A}_{1,2}|$. By Theorem 1.2.5, there is a $k \in \{1, \dots, \ell\}$ such that $\gamma = \theta_k$. Suppose

$$\theta_k = p_1\theta_1 + \cdots + p_{k-1}\theta_{k-1} + p_{k+1}\theta_{k+1} + \cdots + p_\ell\theta_\ell + p\varphi_1.$$

Note that φ_1 is not in either $D(\mathcal{A}_1)$ or $D(\mathcal{A}_2)$. It follows that $\alpha_1\alpha_2 \mid p$. Note that $\theta_{\ell,1} = \alpha_2\varphi_1$. Thus, $\theta_k \in S\theta_1 + \cdots + S\theta_{k-1} + S\theta_{k+1} + \cdots + S\theta_\ell + S\theta_{\ell,1}$, which contradicts the fact that $\theta_1, \dots, \theta_k, \dots, \theta_\ell, \theta_{\ell,1}$ form a minimal generating set of $D(\mathcal{A}_1)$.

(3) $D(\mathcal{A}) + \ker\rho_2^1 \subsetneq D(\mathcal{A}_2)$.

By the proof of Case (2), we can conclude that $D(\mathcal{A}) + \ker\rho_2^1 \subsetneq D(\mathcal{A}_2)$ implies $D(\mathcal{A}) + \ker\rho_1^2 \subsetneq D(\mathcal{A}_1)$. Obviously, the set $\{\theta_1, \dots, \theta_\ell, \theta_{\ell,1}, \theta_{\ell,2}\}$ forms, or can be extended to form, a generating set for $D(\mathcal{A}_{1,2})$. Suppose that the set $T =$

$\{\theta_i, \gamma_j, \theta_{\ell,1}, \theta_{\ell,2} \mid i = 1, \ldots, \ell, j = 1, \ldots, t\}$ forms a generating set for $D(\mathcal{A}_{1,2})$ such that $\gamma_j \notin S(T \setminus \{\gamma_j\})$. Thus, γ_j is not in either $D(\mathcal{A}_1)$ or $D(\mathcal{A}_2)$. This implies that $\deg \gamma_j \geq c_2$.

If T is not minimal, there exists an element $\gamma \in T$ such that γ is generated by $T \setminus \{\gamma\}$. Assume that

$$\gamma = p_1\theta_1 + \cdots + p_\ell\theta_\ell + p\theta_{\ell,1} + q\theta_{\ell,2} + q_1\gamma_1 + \cdots + q_t\gamma_t, \tag{1.3.2}$$

where $p_i, q_j, p, q \notin \mathbb{K}^*$. If $\gamma = \theta_{\ell,1}$, then we may assume that $p = 0$. Note that $\deg \gamma_j \geq c_2 \geq c_1 = \deg \theta_{\ell,1}$. Thus, $q_j \in \mathbb{K}$. It follows that $q_j = 0$ for all $j = 1, \ldots, t$. Therefore, $\theta_{\ell,1} \in D(\mathcal{A}_2)$, which is a contradiction.

Similarly, we can conclude that $\gamma \neq \theta_{\ell,2}$.

If there is a $k \in \{1, \ldots, \ell\}$ such that $\gamma = \theta_k$, then we may assume that $p_k = 0$. If $q_j = 0$ for all j, then $pq \neq 0$. By Equation (1.3.2), we have $\deg \theta_k = \deg q + \deg \theta_{\ell,2} > c_2$. Thus, $p_i = 0$ for any $\deg \theta_i \geq \deg \theta_k$. By the definition of an SPOG arrangement, both $\theta_{\ell,1}$ and $\theta_{\ell,2}$ are S-dependent with $\{\theta_i \mid \deg \theta_i \leq c_2\}$. It follows that θ_k is S-dependent with $\{\theta_i \mid \deg \theta_i < \deg \theta_k\}$, which is a contradiction. Thus, we may assume that q_1 has the smallest degree among the non-zero coefficient $\{q_j\}$. Therefore, $\deg \theta_k = \deg q_1 + \deg \gamma_1 > \deg \gamma_1$. Since $\alpha_1\gamma_j \in D(\mathcal{A}_1)$ and γ_j is not in $D(\mathcal{A}_1)$, it follows that γ_j is S-dependent with $\{\theta_{\ell,1}\} \cup \{\theta_i \mid \deg \theta_i \leq \deg \gamma_j\}$. Note that $\deg \gamma_j \geq c_2$ for all j, and $\theta_{\ell,1}$ is S-dependent with $\{\theta_i \mid \deg \theta_i \leq c_2\}$. It follows that θ_k is S-dependent with $\{\theta_i \mid \deg \theta_i \leq \deg \gamma_1\}$, which is a contradiction.

In conclusion, the set $\{\theta_1, \ldots, \theta_\ell, \theta_{\ell,1}, \theta_{\ell,2}\}$ forms, or can be extended to, a minimal generating set for $D(\mathcal{A}_{1,2})$. Furthermore, in the case where $|\mathcal{A}_{H_1 \cap H_2}| = 2$, as corroborated by Lemma 1.3.5, the set $\{\theta_1, \ldots, \theta_\ell, \theta_{\ell,1}, \theta_{\ell,2}\}$ serves as a minimal generating set for $D(\mathcal{A}_{1,2})$.

□

When constructing a graded free resolution, it is minimal if, and only if, at each step, we select a minimal homogeneous system of generators for the kernel of the differential. Refer to Construction 4.2 and Theorem 7.3 in [22] for details. Let us employ this approach to construct a minimal free resolution for $D(\mathcal{A}_{1,2})$.

Theorem 1.3.8. *If* $|\operatorname{DS}(\mathcal{A}_{1,2})| \leq \ell + 2$, *then* $D(\mathcal{A}_{1,2})$ *has a minimal free resolution of one of the following forms:*

(1) If $D(\mathcal{A}) + \ker \rho_1^2 = D(\mathcal{A}_1)$, *there exist* $f_1, \ldots, f_\ell \in S$ *such that*

$$0 \to S[-c_1 - 1] \xrightarrow{(\alpha_1\alpha_2, f_1, \ldots, f_\ell)} S[-c_1 + 1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

(2) If $D(\mathcal{A}) + \ker \rho_1^2 \subsetneq D(\mathcal{A}_1)$, $D(\mathcal{A}) + \ker \rho_2^1 = D(\mathcal{A}_2)$, *and* $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 2$, *there exist*

$f_i, g_i, g \in S$ with $g \neq 0$ such that

$$0 \to S[-c_1-1] \oplus S[-c_2] \xrightarrow{\substack{(\alpha_1, 0, f_1, \ldots, f_\ell) \\ (g, \alpha_2, g_1, \ldots, g_\ell)}} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

(3) *If $D(\mathcal{A}) + \ker \rho_1^2 \subsetneq D(\mathcal{A}_1)$, $D(\mathcal{A}) + \ker \rho_2^1 = D(\mathcal{A}_2)$, and $|\mathrm{DS}(\mathcal{A}_{1,2})| = \ell + 1$, there exist $f_1, \ldots, f_{\ell-1} \in S$ such that*

$$0 \to S[-c_1-1] \xrightarrow{(\alpha_1, 0, f_1, \ldots, f_{\ell-1})} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{\ell-1} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

Moreover, in this case, $\mathcal{A}_{1,2}$ is SPOG.

(4) *If $D(\mathcal{A}) + \ker \rho_2^1 \subsetneq D(\mathcal{A}_2)$, there exist $f_i, g_i \in S$ such that*

$$0 \to S[-c_1-1] \oplus S[-c_2-1] \xrightarrow{\substack{(\alpha_1, 0, f_1, \ldots, f_\ell) \\ (0, \alpha_2, g_1, \ldots, g_\ell)}} S[-c_1] \oplus S[-c_2] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

Proof. We retain the notation introduced in Proposition 1.3.7 and its proof. Given $|\mathrm{DS}(\mathcal{A}_{1,2})| \leq \ell + 2$ and utilizing Proposition 1.3.7, we analyze the proof case by case.

(1) $D(\mathcal{A}) + \ker \rho_1^2 = D(\mathcal{A}_1)$.

Note that $\{\varphi_1, \theta_i \mid i = 1, \ldots, \ell\}$ is a minimal generator set for $D(\mathcal{A}_{1,2})$. Since $\theta_{\ell,1} = \alpha_2 \varphi_1$ is a level element for $D(\mathcal{A}_1)$, we may assume that $\alpha_1(\alpha_2\varphi_1) + \sum_{i=1}^{\ell} f_i \theta_i = 0$. In other words, there exists a relation

$$(\alpha_1\alpha_2, f_1, \ldots, f_\ell) \tag{1.3.3}$$

between $\{\varphi_1, \theta_1, \ldots, \theta_\ell\}$.

Suppose there exists another S-independent relation, say:

$$(p, p_1, \ldots, p_\ell). \tag{1.3.4}$$

This implies that $p\varphi_1 \in D(\mathcal{A})$. Consequently, $p\varphi_1(\alpha_2) \in S\alpha_2$. If $\varphi_1(\alpha_2) \in S\alpha_2$, it implies that $\varphi_1 \in D(\mathcal{A}_1)$, which contradicts the statement that $\theta_{\ell,1} = \alpha_2\varphi_1$ is a level element for $D(\mathcal{A}_1)$. As a result, we conclude that $\alpha_2 | p$. Therefore Relation (1.3.4) is actually a relation amongst the minimal generators of the SPOG arrangement $\mathcal{A}_1$, of which there is only one by Definition 1.1.2. Consequently, there exist $f_1, \ldots, f_\ell \in S$ such that $D(\mathcal{A}_{1,2})$ has a minimal free resolution of the following forms:

$$0 \to S[-c_1-1] \xrightarrow{(\alpha_1\alpha_2, f_1, \ldots, f_\ell)} S[-c_1+1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

(2) $D(\mathcal{A}) + \ker \rho_1^2 \subsetneq D(\mathcal{A}_1)$, $D(\mathcal{A}) + \ker \rho_2^1 = D(\mathcal{A}_2)$ and $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 2$.

Note that $\{\theta_{\ell,1}, \varphi_2, \theta_1, \dots, \theta_\ell\}$ is a minimal generator set for $D(\mathcal{A}_{1,2})$. Since $\theta_{\ell,1}$ is a level element for $D(\mathcal{A}_1)$, we can assume that

$$\alpha_1\theta_{\ell,1} + \sum_{i=1}^{\ell} f_i\theta_i = 0.$$

In other words, there exists a relation

$$(\alpha_1, 0, f_1, \dots, f_\ell) \tag{1.3.5}$$

between $\{\theta_{\ell,1}, \varphi_2, \theta_1, \dots, \theta_\ell\}$.

Since $\alpha_2\varphi_2 \in D(\mathcal{A}_1) \setminus D(\mathcal{A})$, we may assume that

$$g\theta_{\ell,1} + \alpha_2\varphi_2 + \sum_{i=1}^{\ell} g_i\theta_i = 0,$$

with $g \neq 0$. This gives another relation

$$(g, \alpha_2, g_1, \dots, g_\ell) \tag{1.3.6}$$

between $\{\theta_{\ell,1}, \varphi_2, \theta_1, \dots, \theta_\ell\}$, and it is S-independent with Relation (1.3.5).

If there exist additional relations, denoted as $(p, q, p_1, \dots, p_\ell)$, the implication is as follows:

$$p\theta_{\ell,1} + q\varphi_2 + \sum_{i=1}^{\ell} p_i\theta_i = 0.$$

This implies that $q\varphi_2 \in D(\mathcal{A}_1)$. Consequently, $q\varphi_2(\alpha_2) \in S\alpha_2$. If $\varphi_2(\alpha_2) \in S\alpha_2$, it implies that $\alpha_1\varphi_2 \in D(\mathcal{A})$, which contradicts the assertion that the level element $\theta_{\ell,2} = \alpha_1\varphi_2$ for $D(\mathcal{A}_2)$ is not in $D(\mathcal{A})$. As a result, we conclude that $\alpha_2 | q$. We see $\frac{q}{\alpha_2}(g, \alpha_2, g_1, \dots, g_\ell) - (p, q, p_1, \dots, p_\ell)$ is a relation of the form (1.3.5), which is the unique $D(\mathcal{A}_1)$ relation. Thus, $(p, q, p_1, \dots, p_\ell)$ can be represented by the relations (1.3.5) and (1.3.6). That (1.3.5) and (1.3.6) are S-independent is clear since $\alpha_2 \neq 0$.

Thus $D(\mathcal{A}_{1,2})$ has a minimal free resolution of the following forms:

$$0 \to S[-c_1 - 1] \oplus S[-c_2] \xrightarrow[(g, \alpha_2, g_1, \dots, g_\ell)]{(\alpha_1, 0, f_1, \dots, f_\ell)} S[-c_1] \oplus S[-c_2 + 1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

(3) $D(\mathcal{A}) + \ker \rho_1^2 \subsetneq D(\mathcal{A}_1)$, $D(\mathcal{A}) + \ker \rho_2^1 = D(\mathcal{A}_2)$ and $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 1$.

Since $\mathcal{A}_1$ is SPOG and $D(\mathcal{A}_1)$ is generated by $\{\theta_{\ell,1}, \theta_1, \dots, \theta_\ell\}$, we may assume that

$$\alpha_1\theta_{\ell,1} + \sum_{i=1}^{\ell} f_i\theta_i = 0.$$

By Proposition 1.3.7 Case (1), we have $c_1 < c_2 = \deg\theta_\ell$. Thus, $f_\ell = 0$, and we obtain a relation

$$(\alpha_1, 0, f_1, \dots, f_{\ell-1})$$

between $\{\theta_{\ell,1}, \varphi_2, \theta_1, \dots, \theta_{\ell-1}\}$, the minimal generating set of $D(\mathcal{A}_{1,2})$.

If additional relations are exist, denoted as $(p, q, p_1, \dots, p_{\ell-1})$, the corresponding equation is expressed as follows:

$$p\theta_{\ell,1} + q\varphi_2 + \sum_{i=1}^{\ell-1} p_i\theta_i = 0.$$

Note that $\theta_{\ell,1}$ is S-dependent with $\theta_1, \dots, \theta_{\ell-1}$ by $f_\ell = 0$. By rearranging, we get:

$$-q\varphi_2 = p\theta_{\ell,1} + \sum_{i=1}^{\ell-1} p_i\theta_i,$$

which implies that $q\varphi_2$ is S-dependent with $\theta_1, \dots, \theta_{\ell-1}$. However, considering $q\alpha_2\theta_{\ell,2} = q\alpha_2\alpha_1\varphi_2 \in S\theta_1 + \cdots + S\theta_{\ell-1} + q\mathbb{K}^*\alpha_1\theta_\ell$, we deduce that $q = 0$. This gives a relation between the generators for the SPOG arrangement $\mathcal{A}_1$. Hence, we can conclude that $\{\theta_{\ell,1}, \varphi_2, \theta_1, \dots, \theta_{\ell-1}\}$ has a unique relation $(\alpha_1, 0, f_1, \dots, f_{\ell-1})$. Thus, $D(\mathcal{A}_{1,2})$ has a minimal free resolution of the following form:

$$0 \to S[-c_1 - 1] \xrightarrow{(\alpha_1, 0, f_1, \dots, f_{\ell-1})} S[-c_1] \oplus S[-c_2 + 1] \oplus \left(\bigoplus_{i=1}^{\ell-1} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

Moreover, based on the definition of SPOG, we deduce that $\mathcal{A}_{1,2}$ is SPOG.

(4) $D(\mathcal{A}) + \ker\rho_2^1 \subsetneq D(\mathcal{A}_2)$.

Note that $\{\theta_{\ell,1}, \theta_{\ell,2}, \theta_1, \dots, \theta_\ell\}$ is a minimal generator set for $D(\mathcal{A}_{1,2})$. Since $\theta_{\ell,j}$, where $j = 1, 2$, is a level element for $D(\mathcal{A}_j)$, we can assume that

$$\alpha_1\theta_{\ell,1} + \sum_{i=1}^{\ell} f_i\theta_i = 0,$$
$$\alpha_2\theta_{\ell,2} + \sum_{i=1}^{\ell} g_i\theta_i = 0.$$

In other words, there are two relations

$$(\alpha_1, 0, f_1, \dots, f_\ell) \tag{1.3.7}$$
$$(0, \alpha_2, g_1, \dots, g_\ell) \tag{1.3.8}$$

between $\{\theta_{\ell,1}, \theta_{\ell,2}, \theta_1, \dots, \theta_\ell\}$. If there exist additional relations, denoted as $(p, q, p_1, \dots, p_\ell)$, the implication is as follows:

$$p\theta_{\ell,1} + q\theta_{\ell,2} + \sum_{i=1}^{\ell} p_i\theta_i = 0.$$

This implies that $q\theta_{\ell,2} \in D(\mathcal{A}_1)$. Consequently, $q\theta_{\ell,2}(\alpha_2) \in S\alpha_2$. Considering that $\theta_{\ell,2}$ is a level element for $D(\mathcal{A}_2)$, it follows that $\theta_{\ell,2}(\alpha_2) \notin S\alpha_2$. As a result, we conclude that $\alpha_2 | q$. According to Relation (1.3.8), we have $q\theta_{\ell,2} \in D(\mathcal{A})$. This observation suggests that this relation establishes a connection among $\{\theta_{\ell,1}, \theta_1, \dots, \theta_\ell\}$. Given that Relations (1.3.7) and (1.3.8) uniquely characterize the relationship between sets $\{\theta_{\ell,1}, \theta_1, \dots, \theta_\ell\}$ and $\{\theta_{\ell,2}, \theta_1, \dots, \theta_\ell\}$, respectively, and taking into account the S-independence of $\theta_1, \dots, \theta_\ell$, we can deduce that Relation $(p, q, p_1, \dots, p_\ell)$ can be represented by Relations (1.3.7) and (1.3.8). That Relations (1.3.7) and (1.3.8) are S-independent is clear since $\alpha_1, \alpha_2 \neq 0$.

Thus $D(\mathcal{A}_{1,2})$ has a minimal free resolution of the following forms:

$$0 \to S[-c_1 - 1] \oplus S[-c_2 - 1] \xrightarrow{\substack{(\alpha_1, 0, f_1, \dots, f_\ell) \\ (0, \alpha_2, g_1, \dots, g_\ell)}} S[-c_1] \oplus S[-c_2] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0.$$

□

Theorem 1.3.9. *Let $\mathcal{B}$ in Definition 1.2.9 be defined as $\mathcal{A}_{1,2}$. Assume that $|\mathrm{DS}(\mathcal{A}_{1,2})| > \ell + 2$ with $\mathrm{DS}(\mathcal{A}_{1,2}) = (d_1, \dots, d_\ell, c_1, \dots, c_r)$. Then, the following statements hold:*

(1) We have $\bigoplus_{j=1}^{r} S[-c_j - 1] \subsetneq M_1$, and there exist $h_{ij}, h_j \in S$ with $h_j \neq 0$ such that

$$\begin{aligned} R_1(1) &= (h_{11}, \dots, h_{\ell 1}, \alpha_1, 0, \dots, 0), \\ R_1(2) &= (h_{12}, \dots, h_{\ell 2}, 0, \alpha_2, 0 \dots, 0), \\ R_1(j) &= (h_{1j}, \dots, h_{\ell j}, h_j, 0, \dots, 0, \alpha_2, 0, \dots, 0), \end{aligned}$$

where $R_1(j)$ denotes the j-th row of R_1, and $R_1(j)(\ell + j) = \alpha_2$ for $j = 3, \dots, r$.

(2) We write

$$\begin{aligned} R_1 &= (\psi_1, \dots, \psi_r, \phi_1, \dots, \phi_t)^T, \\ M_1 &= \left(\bigoplus_{j=1}^{r} S[-c_i - 1] \right) \oplus \left(\bigoplus_{i=1}^{t} S[-e_i] \right) \end{aligned}$$

with some $t > 0$. Here, ψ_j corresponds to $R_1(j)$ as defined in (1), where $j = 1, \cdots, r$.

Then, $\bigoplus_{i=1}^{t} S[-e_i - 1] \subset M_2$. In particular, $M_2 \neq 0$. Moreover, $R_2(i)$ has the form $(f_{1i}, \dots, f_{ri}, 0, \dots, 0, \alpha_2, 0, \dots, 0)$, where $f_{ji} \in S$ and $R_2(i)(r + i) = \alpha_2$ for $i = 1, \dots, t$. When $\bigoplus_{i=1}^{t} S[-e_i - 1] = M_2$, we have $t = r - 2$ and $M_3 = 0$.

Proof. (1) By Proposition 1.3.7, we may assume that the minimal generator set of $D(\mathcal{A}_{1,2})$ is $T = \{\theta_i, \varphi_j \mid \deg \theta_i = d_i, \deg \varphi_j = c_j, i = 1, \dots, \ell, j = 1, \dots, r\}$. Here $\{\theta_i \mid i = 1, \dots, \ell\}$ forms a basis for $D(\mathcal{A})$, and φ_1 and φ_2 represent the level elements of $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$, respectively. By Proposition 1.3.7, it holds that $c_i \geq c_2$ for $i > 2$, and

$$M_0 = \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \bigoplus \left(\bigoplus_{j=1}^{r} S[-c_i] \right).$$

Note that $\alpha_1\varphi_1, \alpha_2\varphi_2 \in D(\mathcal{A})$ and $\alpha_2\varphi_j \in D(\mathcal{A}_1) \setminus D(\mathcal{A})$ for $j > 2$. Hence we have the following relations:

$$\begin{aligned}
\psi_1 &:= (h_{11}, \dots, h_{\ell 1}, \alpha_1, 0, \dots, 0),\\
\psi_2 &:= (h_{12}, \dots, h_{\ell 2}, 0, \alpha_2, 0 \dots, 0),\\
\psi_j &:= (h_{1j}, \dots, h_{\ell j}, h_j, 0, \dots, 0, \alpha_2, 0, \dots, 0),
\end{aligned}$$

where $h_j \neq 0$ and $\alpha_2 = \psi_j(\ell + j)$.

Suppose there exists a $j_0 \in \{1, \dots, r\}$ such that ψ_{j_0} is not $\mathbb{K}$-spanned by $\{R_1(i)\}_i$. In this case, we can assume the existence of relations $R_1(1), \dots, R_1(w) \in R_1$ such that $\psi_{j_0} = \sum_{i=1}^{w} p_i R_1(i)$, where $p_i \in S_{\geq 1}$. It's noteworthy that p_i and $R_1(i)(j)$, where $i = 1, \dots, w$ and $j = 1, \dots, \ell + r$, are homogeneous polynomials in $S_{\geq 1}$.

If $j_0 = 1$, this implies that

$$\alpha_1 = \psi_1(\ell + 1) = \sum_{i=1}^{w} p_i R_1(i)(\ell + 1) \in S_{\geq 2},$$

leading to a contradiction. Hence, we deduce that $j_0 > 1$. Further, examining the expression

$$\alpha_2 = \psi_{j_0}(\ell + j_0) = \sum_{i=1}^{w} p_i R_1(i)(\ell + j_0) \in S_{\geq 2},$$

we again arrive at a contradiction. Consequently, we have that $R_1(1) = \psi_1, \dots, R_1(r) = \psi_r$ and $\bigoplus_{j=1}^{r} S[-c_i - 1] \subset M_1$.

Assume $\bigoplus_{j=1}^{r} S[-c_i - 1] = M_1$. Observe that $\psi_1(\ell+1, \dots, \ell+r), \dots, \psi_r(\ell+1, \dots, \ell+r)$ form a lower triangular matrix. It is evident that the relations $\psi_1, \dots, \psi_r$ are S-independent over S. As a consequence, we deduce that $M_2 = 0$. By referencing Theorem 1.2.11, we can deduce that $|\mathcal{A}_{1,2}| = (\sum_{i=1}^{\ell} d_i + \sum_{j=1}^{r} c_j) - (\sum_{j=1}^{r}(c_j + 1)) = \sum_{i=1}^{\ell} d_i - r$. Considering $|\mathcal{A}_{1,2}| = |\mathcal{A}| - 2 = \sum_{i=1}^{\ell} d_i - 2$, it follows that $r = 2$, contradicting the assumption $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + r > \ell + 2$. Hence, we conclude that $\bigoplus_{j=1}^{r} S[-c_i - 1] \subsetneq M_1$.

(2) We can assume that $\phi_i = (u_{1i}, \dots, u_{\ell i}, v_{1i}, \dots, v_{ri})$. We set ϕ as follows:

$$\begin{aligned}
\phi =& \alpha_2\phi_i - (v_{2i}\psi_2 + \cdots + v_{ri}\psi_r)\\
=& (\alpha_2 u_{1i} + \sum_{j=2}^{r} v_{ji}h_{1j}, \dots, \alpha_2 u_{\ell i} + \sum_{j=2}^{r} v_{ji}h_{\ell j}, \alpha_2 v_{1i} + \sum_{j=3}^{r} v_{ji}h_j, 0, \dots, 0).
\end{aligned}$$

If $\phi = 0$, let $f_{1i} = 0$ and $f_{ji} = -v_{ji}$ for $j = 2, \dots, r$. Then we have

$$R_{\phi_i} = (f_{1i}, \dots, f_{ri}, 0, \dots, 0, \alpha_2, 0, \dots, 0)$$

which represents a relation among the relations $R_1(1), \dots, R_1(r + t)$, where $\alpha_2 = R_{\phi_i}(r + i)$. Otherwise, ϕ forms a relation among $\theta_1, \dots, \theta_\ell, \varphi_1$. Since $\mathcal{A}_1$ is SPOG, ψ_1 is the unique relation among $\theta_1, \dots, \theta_\ell, \varphi_1$. Hence, we deduce the existence of some $v \in S$ such that $\phi - v\psi_1 = 0$. Let $f_{1i} = -v$ and $f_{ji} = -v_{ji}$ for $j = 2, \dots, r$. It follows that

$$R_{\phi_i} = (f_{1i}, \dots, f_{ri}, 0, \dots, 0, \alpha_2, 0, \dots, 0)$$

represents a relation among the relations $R_1(1), \ldots, R_1(r+t)$, where $\alpha_2 = R_{\phi_i}(r+i)$.

If there exists an $i_0 \in \{1, \ldots, t\}$ such that $R_{\phi_{i_0}}$ is not $\mathbb{K}$-spanned by $\{R_2(i)\}_i$, we can assume the existence of relations $R_2(1), \ldots, R_2(w)$ such that $R_{\phi_{i_0}} = \sum_{i=1}^{w} p_i R_2(i)$, where $p_i \in S_{\geq 1}$. It's worth noting that p_i and $R_2(i)(j)$, where $i = 1, \ldots, w$ and $j = 1, \ldots, r+t$, are homogeneous polynomials in $S_{\geq 1}$. This implies that

$$\alpha_2 = R_{\phi_{i_0}}(r+i_0) = \sum_{i=1}^{w} p_i R_2(i)(r+i_0) \in S_{\geq 2},$$

leading to a contradiction. Consequently, this reasoning implies that $R_2(1) = R_{\phi_1}, \ldots, R_2(t) = R_{\phi_t}$ and $\bigoplus_{j=1}^{t} S[-e_i-1] \subset M_2$.

Assume $M_2 = \bigoplus_{i=1}^{t} S[-e_i-1]$. Observe that $R_{\phi_1}(r+1, \ldots, r+t), \ldots, R_{\phi_t}(r+1, \ldots, r+t)$ form a lower triangular matrix, it is evident that the relations $R_{\phi_1}, \ldots, R_{\phi_t}$ are S-independent. As a consequence, we deduce that $M_3 = 0$. Since $(\sum_{i=1}^{\ell} d_i + \sum_{j=1}^{r} c_j) - (\sum_{j=1}^{r}(c_j+1) + \sum_{i=1}^{t} e_i) + (\sum_{i=1}^{t}(e_i+1)) = |\mathcal{A}_{1,2}|$ by Theorem 1.2.11, it follows that $r - t = 2$.

□

Proof of Theorem 1.1.7. By Theorems 1.1.4, 1.3.8 and 1.3.9, the conclusion follows immediately. □

We would like to show some examples.

Example 1.3.10. Let $\ell = 4$ and

$$\begin{aligned} Q(\mathcal{A}) = & x_1x_2x_3x_4(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4) \\ & (x_3-x_4)(x_2-x_3+x_4)(x_1-x_2+x_3-x_4). \end{aligned}$$

Then, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,3,4,4)$. The order of $H_i \in \mathcal{A}$ is consistent with its order of appearance in the polynomial.

By computer, $\mathcal{A}_1$ and $\mathcal{A}_2$ both are SPOG with level 4, and $\mathcal{A}_8$ and $\mathcal{A}_{10}$ both are SPOG with level 5. Moreover, we have:

(1) $\mathrm{DS}(\mathcal{A}_{1,2}) = (1,3,3,4,4)$. The minimal free resolution of $D(\mathcal{A}_{1,2})$ has the following forms,

$$0 \to S[-5] \to S[-4]^2 \oplus S[-3]^2 \oplus S[-1] \to D(\mathcal{A}_{1,2}) \to 0.$$

(2) $\mathrm{DS}(\mathcal{A}_{2,10}) = (1,3,4,4,4,4)$. The minimal free resolution of $D(\mathcal{A}_{2,10})$ has the following forms,

$$0 \to S[-5]^2 \to S[-4]^4 \oplus S[-3] \oplus S[-1] \to D(\mathcal{A}_{2,10}) \to 0.$$

(3) $\mathrm{DS}(\mathcal{A}_{1,8}) = (1,3,4,4,4,5)$. The minimal free resolution of $D(\mathcal{A}_{1,8})$ has the following forms,

$$0 \to S[-6] \oplus S[-5] \to S[-5] \oplus S[-4]^3 \oplus S[-3] \oplus S[-1] \to D(\mathcal{A}_{1,8}) \to 0.$$

Example 1.3.11. Let $\ell = 4$ and

$$Q(\mathcal{A}) = x_1x_2x_3x_4(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_3 - x_4)$$
$$(x_2 - x_3 + x_4)(x_1 - x_2 + x_3 - x_4).$$

Then, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 3, 3)$. The order of $H_i \in \mathcal{A}$ is consistent with its order of appearance in the polynomial.

Note that $\mathcal{A}^2$ is not free, serving as a counter-example to Orlik's Conjecture. This can be found in [9] or [20] by a coordinate change.

By computation, $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ are SPOG with level 3, and $|\mathrm{DS}(\mathcal{A}_{2,j})| = 5 < \ell + 2$ for any $H_j \in \mathcal{A}_2$. Additionally, for $\mathcal{A}^1$ and $\mathcal{A}^3$ are free, we found that $|\mathrm{DS}(\mathcal{A}_{1,3})| = 7 > \ell + 2$. The minimal free resolution of $D(\mathcal{A}_{1,3})$ has the following form:

$$0 \to S[-5] \to S[-4]^4 \to S[-3]^6 \oplus S[-1] \to D(\mathcal{A}_{1,3}) \to 0.$$

Remark 1.3.12. The counterexample to Orlik's conjecture mentioned above corresponds to the situation $|D(\mathcal{A}_{1,2})| > \ell + 2$. However, every NT-free-2 arrangement of Edelman and Reiner's counterexample, with dimension $\ell = 5$, contains at most $7 = \ell + 2$ minimal generators.

Applying our theorem allows us to recover the result in [4] and verify a part of Conjecture 4.4 from [1].

Corollary 1.3.13. *[4] If $\mathcal{A}$ and $\mathcal{A}_{1,2}$ are both free, then at least one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is free.*

Proof. This is Theorem 1.2 in [4]. Moreover, in this chapter, we can readily arrive at the same conclusion through Proposition 1.3.7. □

Corollary 1.3.14. *If $\mathcal{A}_{1,2}$ is SPOG, then $c_2 = d_j$ for some $j \in \{2, \dots, l\}$.*

Proof. By referring to Theorem 1.3.8, we establish its validity for Case (3) in Theorem 1.3.8, focusing solely on Theorem 1.3.8 Case (1).

Assuming $\mathrm{POexp}(\mathcal{A}_{1,2}) = (1, p_2, \dots, p_\ell)$, Proposition 1.2.12 implies $\sum_{i=2}^{\ell} p_i = |\mathcal{A}_{1,2}|$. Examining Theorem 1.3.8 Case (1), we find $\mathrm{DS}(\mathcal{A}_{1,2}) = (1, d_2, \dots, d_\ell, c_2 - 1)$. It's noteworthy that $1 + \sum_{i=2}^{\ell} d_i = |\mathcal{A}|$ and $\mathrm{POexp}(\mathcal{A}_{1,2}) \subset \mathrm{DS}(\mathcal{A}_{1,2})$. This implies the existence of a d_j such that $\sum_{i=2}^{\ell} d_i + (c_2 - 1) - d_j = |\mathcal{A}_{1,2}|$. Consequently, we deduce that $c_2 = d_j$.

□

1.4 Three Dimensional Case

In this section, we use the notation in the previous section. Furthermore, we fix $\ell = 3$.

If $|\mathcal{A}_{H_1 \cap H_2}| > 2$, there exists a plane $H \in \mathcal{A}_{H_1 \cap H_2}$ that is distinct from both H_1 and H_2. We can assume that $H = \ker x_2$, $H_1 = \ker x_1$, and $H_2 = \ker(x_1 - x_2)$.

Proposition 1.4.1. *If $|\mathcal{A}_{H_1\cap H_2}| > 2$, then for any basis θ_2, θ_3 for $D_H(\mathcal{A})$, there exist a level element $\theta_{3,i}$ for $D(\mathcal{A}_i)$ such that*

$$x_1\theta_{3,1} = f_2\theta_2 + f_3\theta_3, f_2, f_3 \in \mathbb{K}[x_2, x_3], \tag{1.4.1}$$
$$(x_1 - x_2)\theta_{3,2} = g_2\theta_2 + g_3\theta_3, g_2, g_3 \in \mathbb{K}[x_2, x_3]. \tag{1.4.2}$$

Proof. Suppose that $\theta'_{3,i} \in D_H(\mathcal{A}_i)$ is a level element for $D(\mathcal{A}_i)$, where $i = 1, 2$. Using Theorem 1.1.4, we may assume that:

$$x_1\theta'_{3,1} = f'_2\theta_2 + f'_3\theta_3, \tag{1.4.3}$$
$$(x_1 - x_2)\theta'_{3,2} = g'_2\theta_2 + g'_3\theta_3, \tag{1.4.4}$$

Since f'_i and g'_i are homogeneous polynomial in $S = \mathbb{K}[x_1, x_2, x_3]$, we may assume that:

$$f'_i = x_1 f''_i + f_i,\ i = 2, 3,\ f_i \in \mathbb{K}[x_2, x_3],$$
$$g'_i = (x_1 - x_2)g''_i + g_i,\ i = 2, 3,\ g_i \in \mathbb{K}[x_2, x_3].$$

Let $\theta_{3,1} = \theta'_{3,1} - f''_2\theta_2 - f''_3\theta_3$. Considering that $\theta_{3,1} \notin D(\mathcal{A})$ and $\deg\theta_{3,1} = \deg\theta'_{3,1}$, it is evident that $\theta_{3,1}$ qualifies as a level element for $D(\mathcal{A}_1)$. By substituting $\theta_{3,1}$ into Equation (1.4.3), we obtain Equation (1.4.1). A parallel application yields Equation (1.4.2). □

Lemma 1.4.2. *Consider the scenario where $|\mathcal{A}_{H_1\cap H_2}| > 2$, and let θ_2, θ_3 form a basis for $D_H(\mathcal{A})$. Assume that $\theta_{3,1}$ and $\theta_{3,2}$ are the level elements for $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$, respectively, satisfying Proposition 1.4.1. If*

$$\begin{cases} f_2 = x_2 f'_2 + k x_3^r,\ k \in \mathbb{K}^*, r \in \mathbb{Z}^+,\ f'_2 \in \mathbb{K}[x_2, x_3], \\ g_2 = x_2 g'_2 + k' x_3^{r'},\ \ k' \in \mathbb{K}^*, r' \in \mathbb{Z}^+,\ g'_2 \in \mathbb{K}[x_2, x_3], \end{cases} \quad \begin{cases} f_3 = x_2 f'_3,\ f'_3 \in \mathbb{K}[x_2, x_3], \\ g_3 = x_2 g'_3,\ g'_3 \in \mathbb{K}[x_2, x_3], \end{cases}$$

then

(1) $c_1 = c_2$ if and only if $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker\rho_1^2$. In this case, $c_1 = c_2 = \max\{d_1, d_2, d_3\}$ if and only if $\mathcal{A}_{1,2}$ is SPOG.

(2) $c_1 < c_2$ if and only if $D(\mathcal{A}_1) \subsetneq D(\mathcal{A}) + \ker\rho_1^2$ and $D(\mathcal{A}_2) = D(\mathcal{A}) + \ker\rho_2^1$.

Proof. Since $\deg\theta_{3,1} = c_1 \le \deg\theta_{3,2} = c_2$, we have $r' \ge r$. Assuming that $h = \frac{k'}{k}x_3^{r'-r}$, then we may get that:

$$\begin{aligned} x_1(\theta_{3,2} - h\theta_{3,1}) &= x_1\theta_{3,2} - hx_1\theta_{3,1} \\ &= (x_1 - x_2)\theta_{3,2} + x_2\theta_{3,2} - hx_1\theta_{3,1} \\ &= (g_2\theta_2 + g_3\theta_3) + x_2\theta_{3,2} - h(f_2\theta_2 + f_3\theta_3) \\ &= (g_2 - hf_2)\theta_2 + (g_3 - hf_3)\theta_3 + x_2\theta_{3,2} \\ &= x_2(g'_2 - hf'_2)\theta_2 + x_2(g'_3 - hf'_3)\theta_3 + x_2\theta_{3,2}. \end{aligned}$$

Consequently, $x_1(\theta_{3,2} - h\theta_{3,1}) \in x_2 D(\mathcal{A}_2)$, implying $\theta_{3,2} - h\theta_{3,1} \in x_2 D(\mathcal{A}_{1,2})$. We may assume that:

$$\theta_{3,2} - h\theta_{3,1} = x_2\varphi, \text{ where } \varphi \in D(\mathcal{A}_{1,2}).$$

This leads to:

$$\theta_{3,2} - x_1\varphi = h\theta_{3,1} - (x_1 - x_2)\varphi. \tag{1.4.5}$$

Since the left side belongs to $D(\mathcal{A}_2)$, and the right side belongs to $D(\mathcal{A}_1)$, this implies that $\theta_{3,2} - x_1\varphi \in D(\mathcal{A})$. Note that $\theta_{3,2}$ is a level element for $D(\mathcal{A}_2)$. They imply that $x_1\varphi$ is a level element for $D(\mathcal{A}_2)$. By Lemma 1.3.3, we know that $D(\mathcal{A}_2) = D(\mathcal{A}) + \ker \rho_2^1$.

Now, we prove the statement by case-by-case argument.

(1) If $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$, by Proposition 1.3.7 Case (2), we have $c_1 = c_2$. Conversely, assume that $h \in \mathbb{K}^*$. Let $h = 1$. By Equation (1.4.5), this implies that $\theta_{3,1} - (x_1 - x_2)\varphi \in D(\mathcal{A})$. Note that $\theta_{3,1}$ is a level element for $D(\mathcal{A}_1)$. They imply that $(x_1 - x_2)\varphi$ is a level element for $D(\mathcal{A}_1)$. By Lemma 1.3.3, leading to the deduction that $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$.

If $c_1 = c_2 = \max\{d_1, d_2, d_3\}$, then by Theorem 1.3.8 Case (1) and the definition of an SPOG arrangement, we have that $\mathcal{A}_{1,2}$ is SPOG with level $\max\{d_1, d_2, d_3\}$. Furthermore, if $\mathcal{A}_{1,2}$ is SPOG, by employing Corollary 1.3.14 and Theorem 1.1.4, it follows that $c_1 = c_2 = \max\{d_1, d_2, d_3\}$.

(2) Given that $c_1 < c_2$, by Proposition 1.3.7 Case (2), we establish that $D(\mathcal{A}_1) \subsetneq D(\mathcal{A}) + \ker \rho_1^2$.

If $D(\mathcal{A}_1) \subsetneq D(\mathcal{A}) + \ker \rho_1^2$ and $D(\mathcal{A}_2) = D(\mathcal{A}) + \ker \rho_2^1$ with $c_1 = c_2$, referring to the previous case, we encounter a contradiction.

□

Now, we are going to prove Theorem 1.1.8.

Proof. Let $|\mathcal{A}_{H_1 \cap H_2}| > 2$. Recall $S = \mathbb{K}[x_1, x_2, x_3]$ and $\exp(\mathcal{A}) = (1, d_2, d_3)$. We may assume that $S' = \mathbb{K}[x_1, x_3]$. Since $\mathcal{A}_1^H = \mathcal{A}_2^H = \mathcal{A}^H = C$, we may let the Ziegler restriction of $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ on H be $(C, m_0), (C, m_1)$ and (C, m_2), respectively. Note that $H_1 \cap H = H_2 \cap H$. Thus $(C, m_1) = (C, m_2)$.

Since $\mathcal{A}$ is free, by Theorem 1.2.7, we may get that (C, m_0) is free with $\exp(C, m_0) = (d_2, d_3)$. Note that we do not differentiate between d_2 and d_3. Since $|m_0| = |m_1| + 1$ and $m_0(L) \geq m_1(L)$ for any $L \in C$, by Lemma 1.2.8, we may assume that (C, m_1) is free with $\exp(C, m_1) = (d_2 - 1, d_3)$. Moreover, there exists a basis φ_1, φ_2 for $D(C, m_1)$ such that $x_1\varphi_1, \varphi_2$ forms a basis for $D(C, m_0)$. Let us look at the following exact sequences:

$$0 \longrightarrow D_H(\mathcal{A}) \xrightarrow{\cdot x_2} D_H(\mathcal{A}) \xrightarrow{\pi} D(C, m_0),$$

$$0 \longrightarrow D_H(\mathcal{A}_i) \xrightarrow{\cdot x_2} D_H(\mathcal{A}_i) \xrightarrow{\pi_i} D(C, m_1),$$

where $i = 1, 2$.

By Theorem 1.2.7, there exists a basis θ_1, θ_2 for $D_H(\mathcal{A})$ such that $\pi(\theta_1) = x_1\varphi_1$ and $\pi(\theta_2) = \varphi_2$.

Using Proposition 1.4.1, we may assume that $\theta_{3,i}$ is a level element for $D(\mathcal{A}_i)$ such that

$$x_1\theta_{3,1} = f_1\theta_1 + f_2\theta_2, f_1, f_2 \in \mathbb{K}[x_2, x_3]. \tag{1.4.6}$$
$$(x_1 - x_2)\theta_{3,2} = g_1\theta_1 + g_2\theta_2, g_1, g_2 \in \mathbb{K}[x_2, x_3]. \tag{1.4.7}$$

Thus

$$\pi(x_1\theta_{3,1}) = \overline{f_1}x_1\varphi_1 + \overline{f_2}\varphi_2 = x_1\pi_1(\theta_{3,1})$$
$$\pi((x_1 - x_2)\theta_{3,2}) = \overline{g_1}x_1\varphi + \overline{g_2}\varphi_2 = x_1\pi_2(\theta_{3,2})$$

Note that $\pi_i(\theta_{3,i}) \in D(C, m_1) = S'\varphi_1 + S'\varphi_2$. Thus $\overline{f_2}, \overline{g_2} \in x_1S'$. Recall that $f_j, g_j \in \mathbb{K}[x_2, x_3]$, where $j = 1, 2$, thus $\overline{f_2} = \overline{g_2} = 0$, i.e., $f_2, g_2 \in x_2\mathbb{K}[x_2, x_3]$.

By the definition and $\theta_{3,1} \in D_H(\mathcal{A}_1)$, we have $\pi_1(\theta_{3,1}) = \rho_1^2(\theta_{3,1})$. By Lemma 1.3.3, we have $\rho_1^2(\theta_{3,1}) \neq 0$. This implies that $\overline{f_1} \neq 0$. Since $f_1 \in \mathbb{K}[x_2, x_3]$, we may get that $f_1 \in \mathbb{K}[x_2, x_3] \setminus x_2\mathbb{K}[x_2, x_3]$. Similarly, we have $g_1 \in \mathbb{K}[x_2, x_3] \setminus x_2\mathbb{K}[x_2, x_3]$. Thus we may assume that

$$f_1 = x_2f_1' + kx_3^r, f_2 = x_2f_2'.$$
$$g_1 = x_2g_1' + k'x_3^{r'}, g_2 = x_2g_2'.$$

where $f_i', g_i' \in \mathbb{K}[x_2, x_3]$, $r, r' \in \mathbb{Z}^+$ and $k, k' \in \mathbb{K}^*$. Note that we assumed $|\mathcal{A}_{H_1\cap H_2}| > 2$. By Lemma 1.4.2, $c_1 = c_2$ if and only if $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$. By Lemma 1.4.2, if $c_1 < c_2$, then $D(\mathcal{A}_1) \subsetneq D(\mathcal{A}) + \ker \rho_1^2$ and $D(\mathcal{A}_2) = D(\mathcal{A}) + \ker \rho_2^1$. In this case, if $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 1$, by Proposition 1.3.7 Case (1), it is evident that $c_1 < c_2 = d_2$ or d_3. By Theorem 1.1.4, we have $\max\{d_2, d_3\} \leq c_1$, which is a contradiction. Thus, if $c_1 < c_2$ we have $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 2$.

As a conclusion, $|\mathcal{A}_{H_1\cap H_2}| > 2$ if and only if either $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$ and $c_1 = c_2$, or $|\operatorname{DS}(\mathcal{A}_{1,2})| = \ell + 2$ and $c_1 < c_2$.

By Proposition 1.3.7, we have $|\operatorname{DS}(\mathcal{A}_{1,2})| \leq \ell + 2$ when $\ell = 3$. Combining with Theorem 1.3.8, we successfully conclude the proof of the theorem. The rest part in Theorem 1.1.8 Case (2.1) is from Lemma 1.4.2 Case (1). □

Remark 1.4.3. As demonstrated in Theorem 1.1.8 Case (2.1), it is evident that $\mathcal{A}_{1,2}$ is not necessarily SPOG.

Now we would like to provide some examples to apply Theorem 1.1.8.

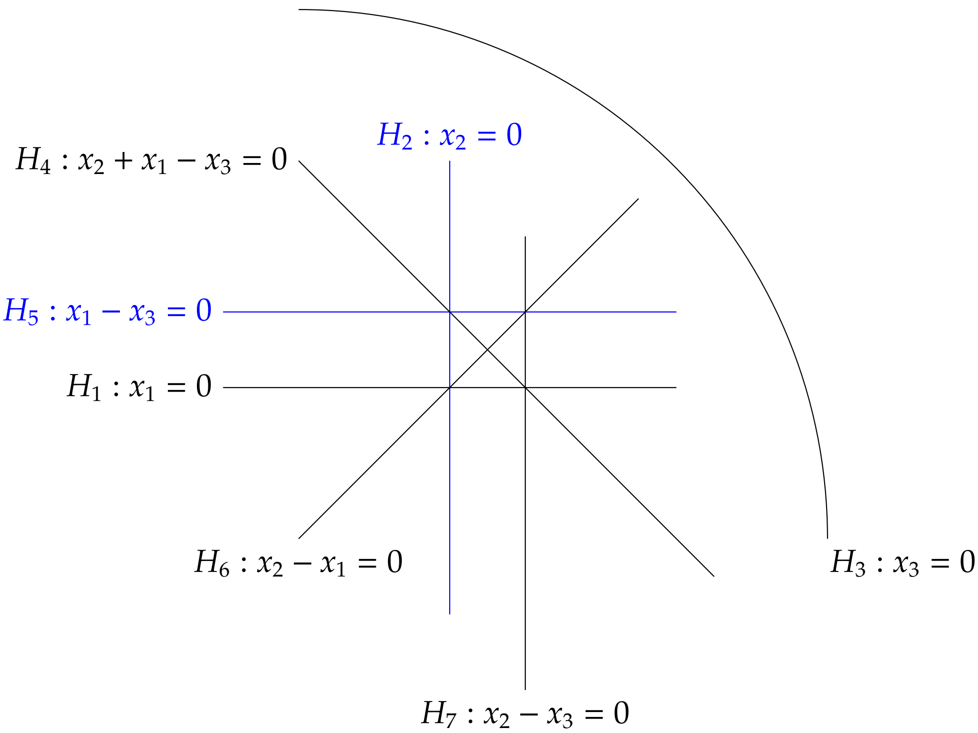

Figure 1.2: Free arrangement $\mathcal{A}$ in $\mathbb{P}^2$

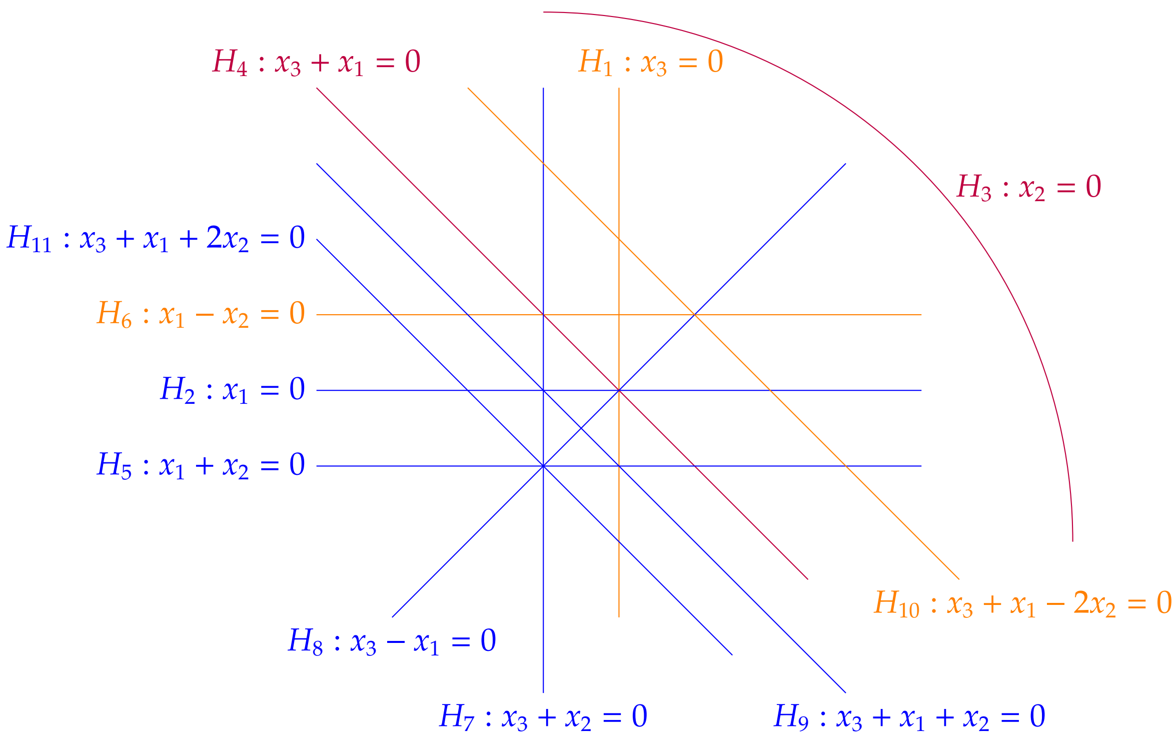

Figure 1.3: Free arrangement $\mathcal{A}$ in $\mathbb{P}^2$

Example 1.4.4. Let

$$Q(\mathcal{A}) = x_1x_2x_3(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 - x_3).$$

Then, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 3)$. See Figure 1.2.

By Theorem 1.1.4, it follows that $\mathcal{A}_2$ and $\mathcal{A}_5$ both are plus-one generated with exponents $(1,3,3)$ and level 3. By Theorem 1.1.8, we have $\mathcal{A}_{2,5}$ is plus-one generated with $\mathrm{POexp}(\mathcal{A}_{2,5}) = (1,2,3)$ and level 3.

Example 1.4.5. Let

$$\begin{aligned} Q(\mathcal{A}) =& x_1x_2x_3(x_1+x_2)(x_1-x_2)(x_1+x_3)(x_1-x_3)(x_2+x_3) \\ & (x_1+x_2+x_3)(x_1+2x_2+x_3)(x_1-2x_2+x_3). \end{aligned}$$

Then, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,5,5)$. See Figure 1.3.

By Theorem 1.1.4, it is easy to see that $\mathcal{A}_1$, $\mathcal{A}_6$ and $\mathcal{A}_{10}$ are free with exponent $(1,4,5)$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are SPOG with $\mathrm{POexp}(\mathcal{A}_3) = \mathrm{POexp}(\mathcal{A}_4) = (1,5,5)$ and level 6, and the remaining $\mathcal{A}_j$ are all SPOG with $\mathrm{POexp}(\mathcal{A}_j) = (1,5,5)$ and level 5. By Theorem 1.1.8, this implies the following results:

(1) Note that $\mathcal{A}_2$ and $\mathcal{A}_{11}$ are not free, and $|\mathcal{A}_{H_2\cap H_{11}}| = 2$, we may get that $\mathrm{DS}(\mathcal{A}_{2,11}) = (1,5,5,5,5)$, and the minimal free resolution of $D(\mathcal{A}_{2,11})$ has the following forms:

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{2,11}) \to 0.$$

(2) Note that $\mathcal{A}_3$ and $\mathcal{A}_4$ are SPOG with level 6, and $|\mathcal{A}_{H_3\cap H_4}| > 2$, we may get that $D(\mathcal{A}_{3,4})$ is not SPOG, $\mathrm{DS}(\mathcal{A}_{3,4}) = (1,5,5,5)$, and the minimal free resolution of $D(\mathcal{A}_{3,4})$ has the following forms:

$$0 \to S[-7] \to S[-5]^3 \oplus S[-1] \to D(\mathcal{A}_{3,4}) \to 0.$$

(3) Note that $\mathcal{A}_5$ and $\mathcal{A}_7$ are SPOG with level 5, and $|\mathcal{A}_{H_5\cap H_7}| > 2$, we may get that $\mathcal{A}_{5,7}$ is SPOG with $\mathrm{POexp}(\mathcal{A}_{5,7}) = (1,4,5)$ and level 5.

(4) Note that $\mathcal{A}_3$ is SPOG with level 6 and $\mathcal{A}_5$ is SPOG with level 5, and $|\mathcal{A}_{H_3\cap H_5}| > 2$, we may get that $\mathrm{DS}(\mathcal{A}_{3,5}) = (1,5,5,5,5)$, and the minimal free resolution of $D(\mathcal{A}_{3,5})$ has the following forms:

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{3,5}) \to 0.$$

Theorem 1.1.4 by Abe shows that NT-free-1 arrangements that are non-free are SPOG. It is known that the converse does not hold in general [15, 26]. Using our result, we give such an example. First, recall the following result:

Proposition 1.4.6 (Theorem 6.2 of [1]). *Let C be SPOG with a minimal set of homogeneous generators $\{\gamma_1,\dots,\gamma_\ell,\varphi\}$ for $D(C)$, where φ is a level element with a level coefficient α. Suppose there exists a free addition $\mathcal{B} = C \cup \{H\}$ of C. If $\deg\varphi > \deg\gamma_i$ for any $i \in \{1,2,\dots,\ell\}$, then $H = \ker\alpha$.*

The following is a SPOG arrangement that does not admit a free addition.

Example 1.4.7. Let

$$Q(\mathcal{A}) = x_1x_2x_3(x_1-x_2)(x_1+x_2)(x_1+2x_2)(2x_1+x_2)(3x_1+x_2)(x_2+x_3)(3x_1+x_2+x_3),$$

and let $H_1 = \{x_1 = 0\}$ and $H_2 = \{x_2 = 0\}$. By computer, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,3,6)$. We may assume that $\{\theta_1,\theta_2,\theta_3\}$ is a basis for $D(\mathcal{A})$. By Theorem 1.1.8, $\mathcal{A}_{1,2}$ is SPOG with $\mathrm{POexp}(\mathcal{A}_{1,2}) = (d_1 = 1, d_2 = 3, c_1 - 1 = 5)$ with level $d_3 = 6$. By Proposition 1.3.7, we have a minimal generator set for $D(\mathcal{A}_{1,2})$ as $\{\theta_1, \theta_2, \theta_3, \varphi\}$, where $\alpha_2\varphi$ is a level element for $D(\mathcal{A}_1)$. Note that θ_3 is the level element of $\mathcal{A}_{1,2}$ with the level coefficient $\alpha = x_2 + x_3$. Since $\ker \alpha \in \mathcal{A}_{1,2}$, by Proposition 1.4.6, it follows that $\mathcal{A}_{1,2}$ does not admit free addition.

Example 1.4.8. We give two arrangements $\mathcal{B}$ and C such that $L(\mathcal{B}) \cong L(C)$ but $DS(\mathcal{B}) \neq DS(C)$. Their defining polynomials are

$$Q_{\mathcal{B}} = x_1x_2x_3(x_2 - 3x_3)(x_2 + 3x_3)(x_1 - x_3)(x_1 + x_3)(x_1 + x_2)(x_1 + x_2 - 3x_3)(x_1 + x_2 + 3x_3)$$
$$Q_C = x_1x_2x_3(x_2 - 3x_3)(x_2 + 3x_3)(x_1 - x_3)(x_1 + x_3)(x_1 + x_2)(x_1 + x_2 - 4x_3)(x_1 + x_2 + 3x_3).$$

Their derivation degree sequences are

$$\mathrm{DS}(\mathcal{B}) = (1, 5, 6, 6)$$
$$\mathrm{DS}(C) = (1, 6, 6, 6, 6, 6).$$

Such a pair is now called the Ziegler pair and a kind of important object to study. This is one example of such pairs. It shows the intricate nature of the derivation degree sequence that depends not only on the combinatorics but also on the geometry of the arrangement.

1.5 Future Work

There remain several promising directions that naturally extend the present study. A longstanding open problem, articulated most recently by Abe, is to formulate an analogue of the classical Addition–Deletion Theorem for SPOG arrangements. Proving such a result would supply a recursive framework for constructing SPOG arrangements from smaller ones, just as the ordinary Addition–Deletion Theorem does for hyperplane arrangements. In turn, it would yield inductive formulas for the characteristic polynomial and other combinatorial invariants, offer a route toward a classification of SPOG arrangements, illuminate connections with Terao's conjecture and freeness criteria, and open further avenues of investigation.

By establishing an Addition–Deletion Theorem tailored to SPOG arrangements we expect to unlock a cascade of results—mirroring the rich interplay between combinatorics, topology, and algebra that the classical theorem has already provided for hyperplane arrangements—ultimately deepening our understanding of both theories and expanding the toolkit available for future research.

Chapter 2

Polynomial interpolation of a vector field on a convex polygonal domain

Although the theory of hyperplane arrangements has attracted considerable attention, its applications are still limited. In this chapter, we explore how the logarithmic derivation module can be used to solve a classical fitting problem in physical sciences and data analysis. Specifically, we introduce a method for interpolating a sampled vector field confined within a convex polyhedral domain of arbitrary dimension using a polynomial vector field, under a no-penetration (slip) boundary condition. For a prescribed integer k, our algorithm computes a degree-k polynomial vector field that best fits the samples in the least-squares sense while exactly respecting the boundary constraint. Notably, the defining condition of the logarithmic derivation module is precisely the required tangency, which naturally links the theory of arrangements to our problem. By leveraging known structural results on logarithmic derivation modules, we resolve the fitting task efficiently.

The results in this section are based on [7].

2.1 Introduction

The study of hyperplane arrangements has played a central role in algebraic geometry, combinatorics, and singularity theory. While much of the existing literature emphasizes the theoretical properties of such arrangements—ranging from their combinatorial invariants to the structure of associated derivation modules—applications of this theory, particularly in data-driven contexts, have remained comparatively underexplored.

In this chapter, we present a novel approach that leverages the theory of logarithmic derivation modules to solve a class of constrained vector field fitting problems. Specifically, given a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ and a finite set of vector-valued samples within it, our objective is to construct a polynomial vector field that (i) fits the given data, and (ii) satisfies a no-penetration (slip) boundary condition along the boundary of the polytope. Such a scenario is common in modeling the dynamics of fluidic materials or particles confined within a rigid container of shape $\mathcal{P}$, where only sparse velocity measurements are available. Enforcing the boundary condition exactly poses a significant challenge, yet it is crucial in applications where physical conservation laws—such as volume preservation—must be upheld.

A key insight of this work is the realization that the slip boundary condition is, in fact, equivalent to the tangency condition that defines the logarithmic derivation module associated with a hyperplane arrangement. Here, an arrangement arises naturally from the affine hyperplanes supporting the facets of the polytope. This equivalence allows us to reframe the original fitting problem in purely algebraic terms. More precisely, our method proceeds as follows:

1. Represent $\mathcal{P}$ as the intersection of finitely many half-spaces.
2. Homogenize the defining affine hyperplanes to obtain a central arrangement $\widehat{\mathcal{P}}$ in $\mathbb{R}^{d+1}$.
3. Establish an explicit bijection between tangential polynomial vector fields on $\mathcal{P}$ (degree $\leq k$) and homogeneous degree-k vector fields on $\mathbb{R}^{d+1}$ that are tangent to every hyperplane of $\widehat{\mathcal{P}}$ and parallel to the slice $x_0 = 0$.
4. Identify the latter space with the syzygy module of the Jacobian ideal of the defining polynomial of $\widehat{\mathcal{P}}$.
5. Use Gröbner-basis techniques (Schreyer's algorithm) to compute a basis and perform least-squares fitting.

From a practical standpoint, our method provides, to the best of our knowledge, the first algorithm that guarantees the exact satisfaction of the tangency boundary condition for polynomial vector field interpolation on convex polyhedral domains. From a theoretical perspective, our framework bridges seemingly disparate areas by employing algebro-geometric techniques to address a classical approximation problem, thus demonstrating the effectiveness of modern mathematical tools in applied settings.

2.2 Previous work

Vector field reconstruction in $\mathbb{R}^2$ was explored in [19] as a generalisation of scalar function interpolation using basis expansions. A method using a partition of unity to patch polynomial vector fields over planar domains is described in [17], while approaches employing radial basis functions are proposed in [29, 30]. These works do not consider boundary constraints.

Alternative formulations of the reconstruction problem have also been studied. For instance, [14, 33, 27] examine vector field reconstruction from streamlines in two and three dimensions. Reconstruction from scalar-valued observations sampled along trajectories is addressed in [13].

Beyond fluid dynamics, tangential vector fields on convex polytopes in $\mathbb{R}^3$ have been investigated from a topological perspective in [23], which classifies homotopy types of such fields with motivations stemming from the modelling of nematic liquid crystals.

To our knowledge, with the exception of [23] which focuses on classification rather than construction, there are no existing studies that address the reconstruction of vector fields subject to no-penetration (slip) boundary conditions on convex polyhedral domains.

2.3 Polynomial Interpolation of Vector Fields

In this section, we establish notation and formulate the main problem.

Definition 2.3.1. A (possibly non-compact) *convex polyhedral space* in $\mathbb{R}^d$ ($d \geq 1$), associated with face normals $\{\alpha_i \in \mathbb{R}^d \mid 1 \leq i \leq m\}$ and offsets $\{\ell_i \in \mathbb{R} \mid 1 \leq i \leq m\}$, is defined as the intersection of m half-spaces:

$$\mathcal{P} = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^d \mid \langle \alpha_i, x \rangle + \ell_i \leq 0\},$$

where $\langle \ , \ \rangle$ denotes the standard inner product on $\mathbb{R}^d$. The hyperplane $H_i = \{x \in \mathbb{R}^d \mid \langle \alpha_i, x \rangle + \ell_i = 0\}$ is called a supporting hyperplane of $\mathcal{P}$, and $F_i = H_i \cap \partial\mathcal{P}$ is called a facet (if it is $(d-1)$-dimensional). We denote by h_i the affine linear function

$$h_i(x) = \langle \alpha_i, x \rangle + \ell_i = \sum_{q=1}^{d} (\alpha_i)_q x_q + \ell_i,$$

which defines the hyperplane H_i.

We assume throughout that none of the m supporting hyperplanes are redundant; that is,

$$\mathcal{P} \subsetneq \bigcap_{i \neq j} \{x \in \mathbb{R}^d \mid \langle \alpha_i, x \rangle + \ell_i \leq 0\}$$

for any j. By abuse of notation, we often identify $\mathcal{P}$ with its set of supporting hyperplanes $\{H_i\}$, assuming the orientations of the half-spaces are implicitly specified.

We set $\mathcal{P} = \mathbb{R}^d$ when $m = 0$.

We now define the relevant spaces of polynomial vector fields on $\mathcal{P}$.

Definition 2.3.2. A homogeneous polynomial vector field of degree k on $\mathcal{P}$ is a map $\xi : \mathcal{P} \to \mathbb{R}^d$ represented by a tuple of homogeneous polynomials $(f_1, f_2, \ldots, f_d)$, where each $f_q \in \mathbb{R}[x_1, \ldots, x_d]$ has $\deg(f_q) = k$ for $1 \leq q \leq d$. Denote by $\mathrm{Poly}(\mathcal{P})_k$ the space of such vector fields. Similarly, denote by $\mathrm{Poly}(\mathcal{P})_{\leq k}$ the space of vector fields where each f_q is a polynomial of degree at most k.

The subspace of $\mathrm{Poly}(\mathcal{P})_k$ (respectively, $\mathrm{Poly}(\mathcal{P})_{\leq k}$) consisting of vector fields tangent to the boundary of $\mathcal{P}$ is defined by

$$\mathrm{Poly}_\partial(\mathcal{P})_k = \left\{\xi \in \mathrm{Poly}(\mathcal{P})_k \mid \langle \xi(x), \alpha_i \rangle = 0 \text{ for all } x \in F_i,\ 1 \leq i \leq m\right\}. \tag{2.3.1}$$

In other words, the direction of $\xi \in \mathrm{Poly}_\partial(\mathcal{P})_k$ at any point on a facet F_i is orthogonal to the normal vector α_i. The space $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ is defined analogously for non-homogeneous vector fields.

The space $\mathrm{Poly}_\partial(\mathcal{P}) = \bigcup_{k \geq 0} \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ of non-homogeneous polynomial tangential fields and the space $\mathrm{Poly}_\partial(\mathcal{P})_{hom} = \bigoplus_{k \geq 0} \mathrm{Poly}_\partial(\mathcal{P})_k$ of homogeneous polynomial tangential fields are both $\mathbb{R}[x_1, \ldots, x_d]$-modules under componentwise multiplication of vector fields by polynomials.

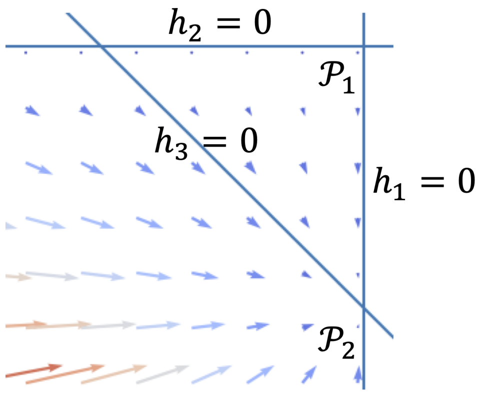

Figure 2.1: Examples of convex polyhedral spaces in $\mathbb{R}^2$ and a tangential vector field defined on them.

The following example illustrates the distinction between homogeneous and non-homogeneous polynomial tangential fields.

Example 2.3.3. Consider the case $d = 2$. Let $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$, and $\alpha_3 = (1, 1)$, with $\ell_1 = \ell_2 = 0$ and $\ell_3 = 1$, so that $h_1 = x_1$, $h_2 = x_2$, and $h_3 = x_1 + x_2 + 1$. Define

$$\mathcal{P}_1 = \{x \mid h_1(x) \le 0,\ h_2(x) \le 0,\ -h_3(x) \le 0\}, \quad \mathcal{P}_2 = \{x \mid h_1(x) \le 0,\ h_2(x) \le 0,\ h_3(x) \le 0\}.$$

Then $\mathcal{P}_1$ is compact, and $\mathcal{P}_2$ is non-compact. Both are convex polyhedral spaces in $\mathbb{R}^2$, as illustrated in Figure 2.1.

For either $\mathcal{P}_1$ or $\mathcal{P}_2$, the vector field $\xi = (x_1x_2, x_2^2 + x_2)$ is a non-homogeneous tangential field. We verify the tangency condition facet by facet:

- On F_1 (defined by $x_1 = 0$), $\xi(0, x_2) = (0, x_2^2 + x_2)$, and $\alpha_1 = (1, 0)$, so $\langle \xi, \alpha_1 \rangle = 0$.

- On F_2 (defined by $x_2 = 0$), $\xi(x_1, 0) = (0, 0)$, and $\alpha_2 = (0, 1)$, so $\langle \xi, \alpha_2 \rangle = 0$.

- On F_3 (defined by $x_1 + x_2 + 1 = 0$, or $x_1 = -1 - x_2$), the normal is $\alpha_3 = (1, 1)$, and

$$\xi(-1 - x_2, x_2) = ((-1 - x_2)x_2,\ x_2^2 + x_2), \quad \langle \xi, \alpha_3 \rangle = -x_2 - x_2^2 + x_2^2 + x_2 = 0.$$

Thus, ξ is tangent to all three facets.

Note that the degree-one component of ξ, namely $(0, x_2)$, is not tangential to F_3, since $\langle (0, x_2), (1, 1) \rangle = x_2 \neq 0$ in general. This demonstrates that $\bigoplus_{j=0}^{k} \mathrm{Poly}_\partial(\mathcal{P})_j \subsetneq \mathrm{Poly}_\partial(\mathcal{P})_{\le k}$ in general, making a degree-wise analysis of $\mathrm{Poly}_\partial(\mathcal{P})_{\le k}$ more intricate.

A vector field $\xi = (f_1, f_2, \ldots, f_d)$ acts on a polynomial $h \in \mathbb{R}[x_1, \ldots, x_d]$ via the Lie derivative:

$$\xi[h] = \langle \nabla h, \xi \rangle = \sum_{q=1}^{d} f_q \frac{\partial h}{\partial x_q}. \tag{2.3.2}$$

For brevity, we denote $\frac{\partial}{\partial x_q}$ by ∂_q.

We now formulate our main goal. Suppose we are given a set of *observations*

$$\left\{ (x_s, u_s) \in \mathcal{P} \times \mathbb{R}^d \mid s \in \mathcal{O} \right\},$$

indexed by a finite set $\mathcal{O}$. The central problem addressed in this chapter is the following:

Problem 2.3.4. Given a natural number k, find $\xi \in \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ that minimises the sum of squared errors:

$$\mathcal{E}(\xi, \mathcal{O}) = \sum_{s \in \mathcal{O}} |\xi(x_s) - u_s|^2 . \tag{2.3.3}$$

This problem seeks a polynomial vector field of degree at most k that best fits the observed data in the least-squares sense while satisfying the tangency boundary condition exactly. Note that the choice of k governs a trade-off between approximation accuracy and model complexity.

2.4 Identity Theorem for Polynomial Vector Fields

It is straightforward to verify that any polynomial vector field defined on $\mathcal{P}$ can be uniquely extended to a polynomial vector field on all of $\mathbb{R}^d$, so that $\mathrm{Poly}(\mathcal{P})_k = \mathrm{Poly}(\mathbb{R}^d)_k$. In this section, we establish an analogous result for tangential vector fields.

We begin by recalling a classical fact: a polynomial function is uniquely determined by its values on a sufficiently large finite set.

Lemma 2.4.1. *Let $k \in \mathbb{N}$ and $d \geq 1$, and let $f, g \in \mathbb{R}[x_1, \ldots, x_d]$ be polynomials of degree at most k. Suppose $X = C_1 \times \cdots \times C_d \subset \mathbb{R}^d$, where each $C_q \subset \mathbb{R}$ satisfies $|C_q| > k$. Then $f \equiv g$ if and only if $f(x) = g(x)$ for all $x \in X$. Consequently, two vector fields $\xi, \gamma \in \mathrm{Poly}(\mathbb{R}^d)_k$ coincide globally if and only if $\xi(x) = \gamma(x)$ for all $x \in X$.*

Proof. Replacing $f - g$ with f, it suffices to show that a polynomial f of degree at most k vanishes identically if $f(x) = 0$ for all $x \in X$.

We proceed by induction on d. The base case $d = 1$ follows from the classical fact that a univariate polynomial of degree at most k with more than k zeros must be identically zero. Assume the result holds for $d - 1$. For a fixed $(x_1, \ldots, x_{d-1}) \in C_1 \times \cdots \times C_{d-1}$, consider the univariate polynomial

$$p(t) = f(x_1, \ldots, x_{d-1}, t).$$

Since p vanishes on C_d, which contains more than $k \geq \deg p$ points, we conclude $p \equiv 0$. Thus, all coefficients of powers of x_d in f vanish as polynomials in $x_1, \ldots, x_{d-1}$. By the induction hypothesis, $f \equiv 0$. □

Corollary 2.4.2. *Let $H \subset \mathbb{R}^d$ be a hyperplane with normal vector α. A polynomial vector field $\xi \in \mathrm{Poly}(\mathbb{R}^d)_k$ is tangent to H if and only if $\langle \xi(x), \alpha \rangle = 0$ on any open subset of H.*

Consequently, every $\xi \in \mathrm{Poly}_\partial(\mathcal{P})_k$ admits a unique extension to a polynomial vector field in $\mathrm{Poly}(\mathbb{R}^d)_k$ that is tangent to all supporting hyperplanes of the polyhedral domain $\mathcal{P}$.

Proof. Apply Lemma Lemma 2.4.1 to the polynomial $\langle \xi(x), \alpha \rangle$. □

By Corollary 2.4.2, the space $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ can be identified with the solution space of a linear system. However, in the next section, we introduce a more sophisticated approach for computing $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$.

2.5 Space of Tangent Fields

Problem 2.3.4 reduces to a standard least-squares problem, provided a basis for $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ is available. Our key idea is to recast the boundary condition as a module membership problem in a polynomial ring, and to identify $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ with the syzygy module of a Jacobian ideal, which can be effectively computed using Gröbner basis techniques.

We begin with a simple observation that translates the tangency condition into an algebraic one.

Lemma 2.5.1. *Let $h \in \mathbb{R}[x_1, \dots, x_d]$ be a degree-one polynomial defining the hyperplane $H = \{x \in \mathbb{R}^d \mid h(x) = 0\}$. A polynomial vector field ξ on $\mathbb{R}^d$ is tangent to H if and only if $\xi[h] \in (h)$, where (h) is the principal ideal generated by h in the ring $\mathbb{R}[x_1, \dots, x_d]$.*

Proof. Write $h(x) = \langle \alpha, x \rangle + \ell$ so that $\nabla h = \alpha$. By (2.3.1) and (2.3.2), ξ is tangent to H if and only if

$$0 = \langle \alpha, \xi(x) \rangle = \langle \nabla h, \xi(x) \rangle = \xi[h](x) \quad \forall\, x \in H.$$

Hence

$$\xi \text{ tangent to } H \iff \xi[h]\big|_{V(h)} \equiv 0 \iff \xi[h] \in I\big(V(h)\big),$$

where we write H as the variety $V(h)$ defined by h, and $I\big(V(h)\big)$ is the vanishing ideal of the variety $V(h)$. Since h is a nonzero degree-one polynomial, the ideal (h) is prime. Therefore, the assertion follows by the Nullstellensatz $I\big(V(h)\big) = \sqrt{(h)} = (h)$. □

To utilise algebraic machinery, it is more convenient to work with linear subspaces than with affine subspaces H_i. For this, we embed $\mathcal{P}$ into one dimension higher space by appending one more spatial variable x_0.

Definition 2.5.2. For a convex polyhedral space $\mathcal{P} \subset \mathbb{R}^d$, its *cone* $\widehat{\mathcal{P}}$ is the (non-compact) convex polyhedral space in $\mathbb{R}^{d+1}$ defined by the following supporting hyperplanes

$$\widehat{H_i} = \left\{ \widehat{x} = (x_0, x_1, \dots, x_d) \in \mathbb{R}^{d+1} \mid \ell_i x_0 + \sum_{q=1}^{d} (\alpha_i)_q x_q = 0 \right\}.$$

We denote by $\widehat{h_i}$ the corresponding linear form. See fig. 2.2.

All the facets of $\widehat{\mathcal{P}}$ meet at the origin, and the original convex polyhedral space $\mathcal{P}$ is embedded in $\widehat{\mathcal{P}}$ at the slice $x_0 = 1$, hence the name "cone". We will see that there is also a correspondence between certain tangential polynomial vector fields of $\mathcal{P}$ and $\widehat{\mathcal{P}}$. Let $\overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$ be the subspace of $\mathrm{Poly}_\partial(\widehat{\mathcal{P}})_k$ consisting of those fields parallel to the plane defined by $x_0 = 0$. That is, the first coordinate of an element of $\overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$ is the zero polynomial.

Lemma 2.5.3. *We have an isomorphism of vector spaces*

$$\mathrm{Poly}_\partial(\mathcal{P})_{\leq k} \cong \overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k.$$

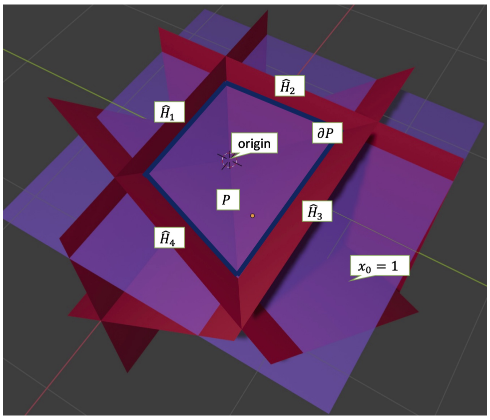

Figure 2.2: The cone $\widehat{\mathcal{P}}$ in $\mathbb{R}^3$ corresponding to a trapezoid $\mathcal{P}$ in $\mathbb{R}^2$. First, $\mathcal{P}$ is embedded into $\mathbb{R}^3$ on the plane $x_0 = 1$. By joining the origin with each facet of $\mathcal{P}$, we obtain hyperplanes in $\mathbb{R}^3$ that share the origin as the apex. An element of $\mathrm{Poly}_\partial(\mathcal{P})$ can be extended to a field tangent to $\widehat{\mathcal{P}}$ simply by scaling by the "height" x_0. The field thus obtained is parallel to the plane $x_0 = 0$. Conversely, any homogeneous tangential field to $\widehat{\mathcal{P}}$ can be made parallel to $x_0 = 0$ by subtracting an $\mathbb{R}[x_1, \ldots, x_d]$-multiple of $\xi_E = (x_1, \ldots, x_d)$, which is a "radial" vector field that is tangent to any plane going through the origin. By restricting the parallel field to $x_0 = 1$, we obtain an element of $\mathrm{Poly}_\partial(\mathcal{P})$.

Proof. The degree-k homogenenisation of $f \in \mathbb{R}[x_1, \ldots, x_d]$ with $\deg f \leq k$ is given by $\widehat{f} = x_0^k f(x_1/x_0, \ldots, x_d/x_0)$ with $\deg \widehat{f_i} = k$. Define a map Ψ from $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ to $\overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$ by

$$(f_1, f_2, \ldots, f_d) \mapsto (0, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_d}).$$

We will check the image of Ψ is indeed contained in $\overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$. Since $(f_1, f_2, \ldots, f_d)$ is tangent to every H_i for $1 \leq i \leq n$, we have

$$\sum_{q=1}^{d} (\alpha_i)_q f_q(x) = 0, \text{ for any } x \in H_i.$$

By Lemma 2.5.1, we may assume that $\sum_{q=1}^{d} (\alpha_i)_q f_q = h_i g$ for some $g \in \mathbb{R}[x_1, \ldots, x_d]$. It follows that

$$\begin{aligned}
\sum_{q=1}^{d} (\alpha_i)_q \widehat{f_q} &= \sum_{q=1}^{d} (\alpha_i)_q (x_0^k f_q(x_1/x_0, \ldots, x_d/x_0)) \\
&= x_0^k \left(\sum_{q=1}^{d} (\alpha_i)_q f_q(x_1/x_0, \ldots, x_d/x_0) \right) \\
&= x_0^k h_i(x_1/x_0, \ldots, x_d/x_0) g(x_1/x_0, \ldots, x_d/x_0) \\
&= \widehat{h_i}(x_0, \ldots, x_d) x_0^{k-1} g(x_1/x_0, \ldots, x_d/x_0).
\end{aligned}$$

Since $\deg g \leq k-1$, we deduce that $x_0^{k-1} g(x_1/x_0, \ldots, x_d/x_0)$ is a homogeneous polynomial of degree $k-1$. This implies that $\sum_{q=1}^{d} (\alpha_i)_q \widehat{f_q} \in (\widehat{h_i})$ and we have $(0, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_d}) \in \overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$.

Conversely, define a map Φ from $\overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k$ to $\mathrm{Poly}_\partial(P)_k$ by

$$(0, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_d}) \mapsto (f_1, f_2, \ldots, f_d),$$

where $\widehat{f_i}$ is homogenous polynomial in $\mathbb{R}[x_0, x_1, \ldots, x_d]$ with $\deg \widehat{f_i} = k$, and $f_i \in \mathbb{R}[x_1, \ldots, x_d]$ with $\deg f_i \le k$ are obtained by evaluating $\widehat{f_i}$ at $x_0 = 1$. We will check that $(f_1, f_2, \ldots, f_d) \in \mathrm{Poly}_\partial(\mathcal{P})_k$. Assume $(0, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_d})$ is tangent to every $\widehat{H_i} \in \widehat{\mathcal{P}}$ for $1 \le i \le n$. Since the normal vector of $\widehat{H_i}$ is $\widehat{\alpha_i} = (\ell_i, (\alpha_i)_1, \ldots, (\alpha_i)_d)$, it follows that

$$\sum_{q=1}^{d} (\alpha_i)_q \widehat{f_q}(\widehat{x}) = 0, \text{ for any } \widehat{x} \in \widehat{H_i}.$$

By restricting at $x_0 = 1$, we have

$$\sum_{q=1}^{d} (\alpha_i)_q f_q(x) = 0, \text{ for any } x \in H_i$$

and $(f_1, f_2, \ldots, f_d)$ is tangent to every H_i. Since both $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identities, the assertion follows. □

Let $S = \mathbb{R}[x_0, x_1, \ldots, x_d]$ be the polynomial ring with $d+1$ variables, and S_k be its degree k component. By abuse of notation, we use $\widehat{x}$ to denote both a point in $\mathbb{R}^{d+1}$ and the coordinate function. The space $\mathrm{Poly}(\mathbb{R}^{d+1})$ of the polynomial vector fields over $\mathbb{R}^{d+1}$ is endowed with the structure of a graded S-module.

We now introduce algebraic objects that will enable an efficient computation of $\mathrm{Poly}_\partial(\mathcal{P})_{\le k}$. A small but important trick is to add a special plane $\widehat{H_0} = \{x_0 = 0\}$, which plays a role in Theorem 2.5.9.

Definition 2.5.4. Let $\widehat{h_0} = x_0$ and

$$Q_{\widehat{\mathcal{P}}} = \prod_{i=0}^{m} \widehat{h_i} \in S$$

and define the graded S-module

$$\mathcal{D}(\widehat{\mathcal{P}}) = \left\{ \xi \in \mathrm{Poly}(\mathbb{R}^{d+1}) \mid \xi[Q_{\widehat{\mathcal{P}}}] \in (Q_{\widehat{\mathcal{P}}}) \right\}$$

called the *logarithmic derivation module*, where $(Q_{\widehat{\mathcal{P}}})$ is the principal ideal of S generated by $Q_{\widehat{\mathcal{P}}}$.

The submodule of $\mathcal{D}(\widehat{\mathcal{P}})$ consisting of those vector fields parallel to the hyperplane $x_0 = 0$ is denoted by $\overline{\mathcal{D}}(\widehat{\mathcal{P}})$: An element of its degree k component is represented by a tuple $(0, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_d})$ of homogeneous degree k polynomials $\widehat{f_i}$.

The simultaneous tangency to multiple hyperplanes $\{\widehat{H_i}\}$ is described as a module membership condition.

Proposition 2.5.5 ([21, Chap. 4]). *We have*

$$\mathcal{D}(\widehat{\mathcal{P}}) = \bigcap_{i=0}^{m} \left\{ \xi \in \mathrm{Poly}(\mathbb{R}^{d+1}) \mid \xi[\widehat{h_i}] \in (\widehat{h_i}) \right\}.$$

Proof. First, we see for any $\xi \in \mathcal{D}(\widehat{\mathcal{P}})$, it holds that $\xi[\widehat{h_i}] \in (\widehat{h_i})$ for any i. Since

$$\xi[Q_{\widehat{\mathcal{P}}}] = \xi\left(\widehat{h_i} \cdot \frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}}\right) = \xi[\widehat{h_i}]\,\frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}} + \xi\!\left(\frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}}\right)\widehat{h_i}.$$

and $\xi[Q_{\widehat{\mathcal{P}}}] \in (Q_{\widehat{\mathcal{P}}}) \subseteq (\widehat{h_i})$, we have $\xi\!\left(\frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}}\right)\widehat{h_i} \in (\widehat{h_i})$. As $(\widehat{h_i})$ is prime and $\frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}} \notin (\widehat{h_i})$, we have $\xi[\widehat{h_i}] \in (\widehat{h_i})$.

Conversely, we show $\xi[Q_{\widehat{\mathcal{P}}}] \in (Q_{\widehat{\mathcal{P}}})$ if $\xi[\widehat{h_i}] \in (\widehat{h_i})$ for any i. We have

$$\begin{aligned}
\xi[Q_{\widehat{\mathcal{P}}}] &= \xi[\widehat{h_0}\cdots\widehat{h_{m-1}}\,\widehat{h_m}] \\
&= \xi[\widehat{h_0}\cdots\widehat{h_{m-1}}]\,\widehat{h_m} + \xi[\widehat{h_m}]\,\widehat{h_0}\cdots\widehat{h_{m-1}} \\
&= \left(\xi[\widehat{h_0}\cdots\widehat{h_{m-2}}]\,\widehat{h_{m-1}}\widehat{h_m} + \xi[\widehat{h_{m-1}}]\,\widehat{h_0}\cdots\widehat{h_{m-2}}\widehat{h_m}\right) + \xi[\widehat{h_m}]\,\widehat{h_0}\cdots\widehat{h_{m-1}} \\
&= \sum_{i=0}^{m} \xi[\widehat{h_i}]\frac{Q_{\widehat{\mathcal{P}}}}{\widehat{h_i}} \in (Q_{\widehat{\mathcal{P}}})
\end{aligned}$$

as required. □

Combining Lemma 2.5.3 and Proposition 2.5.5, we obtain the following:

Corollary 2.5.6. *A vector field $\xi \in \mathrm{Poly}(\mathbb{R}^{d+1})$ is tangent to every $\widehat{H_i} \in \widehat{\mathcal{P}}$ for $i = 0, \ldots, m$ (with $\widehat{H_0} = \{x_0 = 0\}$) if and only if $\xi \in \mathcal{D}(\widehat{\mathcal{P}})$. That is, $\mathrm{Poly}_\partial(\widehat{\mathcal{P}}) = \mathcal{D}(\widehat{\mathcal{P}})$. Moreover, $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k} \cong \overline{\mathrm{Poly}}_\partial(\widehat{\mathcal{P}})_k = \overline{\mathcal{D}}(\widehat{\mathcal{P}})_k$.*

Let $J(Q_{\widehat{\mathcal{P}}})$ be the Jacobian ideal of $Q_{\widehat{\mathcal{P}}}$ in S, which is generated by the partial derivatives $\partial_q Q_{\widehat{\mathcal{P}}} = \dfrac{\partial Q_{\widehat{\mathcal{P}}}}{\partial x_q}$ for $0 \leq q \leq d$.

Lemma 2.5.7. *When $\bigcap_i \widehat{H_i} = \{0\}$, the set $\{\partial_q Q_{\widehat{\mathcal{P}}} \mid 0 \leq q \leq d\}$ forms a minimal generating set of $J(Q_{\widehat{\mathcal{P}}})$. By abuse of notation, we denote the tuple $\left(\partial_0 Q_{\widehat{\mathcal{P}}}, \partial_1 Q_{\widehat{\mathcal{P}}}, \ldots, \partial_d Q_{\widehat{\mathcal{P}}}\right)$ also by $J(Q_{\widehat{\mathcal{P}}})$.*

Proof. Assume otherwise. Since $\partial_q Q_{\widehat{\mathcal{P}}}$ have the same total degree for all q, they have a $\mathbb{R}$-linear relation $\sum_{q=0}^{d} c_q \partial_q Q_{\widehat{\mathcal{P}}} = 0$ for some $c \in \mathbb{R}^{d+1}$. This means the directional derivative of $Q_{\widehat{\mathcal{P}}}$ in c vanishes and $Q_{\widehat{\mathcal{P}}}$ is constant in the direction. Because $\bigcap_i \widehat{H_i} = \{0\}$, there must be some $\widehat{H_i}$ that does not contain c. As $Q_{\widehat{\mathcal{P}}}(x) = 0$ for any $x \in \widehat{H_i}$, and $\widehat{H_i}$ and c span the whole $\mathbb{R}^{d+1}$, we have $Q_{\widehat{\mathcal{P}}} \equiv 0$, which is a contradiction. □

Consider a graded S-free resolution (see [10, Chap. 1] for the basics of free resolutions)

$$0 \to N_{d+1} \xrightarrow{\varphi_{d+1}} \cdots \xrightarrow{\varphi_4} N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} S^{d+1} \xrightarrow{\varphi_1} S \to S/J(Q_{\widehat{\mathcal{P}}}) \to 0, \tag{2.5.1}$$

where φ_1 maps the generators of S^{d+1} to $\partial_q Q_{\widehat{\mathcal{P}}}$ for $0 \leq q \leq d$. Recall that $N_p = 0$ for $p > d+1$ by Hilbert's syzygy theorem and $\ker \varphi_1$ is called the syzygy module of $J(Q_{\widehat{\mathcal{P}}})$, and denoted by $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$. More concretely,

$$\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}})) = \left\{ [\widehat{f_0}, \dots, \widehat{f_d}] \in S^{d+1} \middle| [\widehat{f_0}, \dots, \widehat{f_d}] \cdot J(Q_{\widehat{\mathcal{P}}}) = \sum_{q=0}^{d} \widehat{f_q} \partial_q Q_{\widehat{\mathcal{P}}} = 0 \right\}. \tag{2.5.2}$$

Its degree-k component is denoted by $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))_k$.

There exists a "radial" field that is tangent to any cone. This special field plays a role in proving our main theorem.

Lemma 2.5.8. *Let $\xi_E = (x_0, x_1, \dots, x_d)$. We have*

$$\xi_E(\widehat{f}) = \deg(\widehat{f})\widehat{f}$$

for any homogeneous polynomial $\widehat{f} \in S$. In particular, $\xi_E \in \mathrm{Poly}_\partial(\widehat{\mathcal{P}})$ for any $\widehat{\mathcal{P}}$.

Proof. The first assertion is trivial for monomials, and the general case follows from linearity. The second assertion is a consequence of the first and Lemma 2.5.1. □

The following isomorphism is essential to our scheme.

Theorem 2.5.9. *We have an isomorphism of $\mathbb{R}$-vector spaces:*

$$\mathrm{Poly}_\partial(\mathcal{P})_{\leq k} \cong \mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))_k.$$

Proof. By Corollary 2.5.6, we have only to prove the map $\phi : \overline{\mathcal{D}}(\widehat{\mathcal{P}})_k \to \mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))_k$ defined by

$$\phi(\widehat{f}) = [0, \widehat{f_1}, \dots, \widehat{f_d}] - \frac{\widehat{f}(Q_{\widehat{\mathcal{P}}})}{\deg(Q_{\widehat{\mathcal{P}}})Q_{\widehat{\mathcal{P}}}}[x_0, x_1, \dots, x_d]$$

is an isomorphism of $\mathbb{R}$-vector spaces, where $\widehat{f} = (0, \widehat{f_1}, \dots, \widehat{f_d}) \in \overline{\mathcal{D}}(\widehat{\mathcal{P}})_k$. Note that we represent elements both in $\overline{\mathcal{D}}(\widehat{\mathcal{P}})$ and $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$ by $(d+1)$-tuples of polynomials in S, but with different parentheses.

By the definition of the action of a vector field (2.3.2), we have

$$\widehat{f}(Q_{\widehat{\mathcal{P}}}) = \sum_{q=1}^{d} \widehat{f_q} \partial_q Q_{\widehat{\mathcal{P}}},$$

which is divisible by $Q_{\widehat{\mathcal{P}}}$ by Definition 2.5.4. Also, we have

$$[0, \widehat{f_1}, \dots, \widehat{f_d}] \cdot J(Q_{\widehat{\mathcal{P}}}) = \sum_{q=1}^{d} \widehat{f_q} \partial_q Q_{\widehat{\mathcal{P}}} = \widehat{f}(Q_{\widehat{\mathcal{P}}}).$$

We have $[x_0, x_1, \dots, x_d] \cdot J(Q_{\widehat{\mathcal{P}}}) = \sum_{q=0}^{d} x_q \partial_q Q_{\widehat{\mathcal{P}}} = \deg(Q_{\widehat{\mathcal{P}}})Q_{\widehat{\mathcal{P}}}$ by Lemma 2.5.8. Therefore, $\phi(\widehat{f}) \in \mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))_k$ by (2.5.2), and the map ϕ is well-defined.

Conversely, define

$$\psi(\widehat{g}) = (\widehat{g}_0, \ldots, \widehat{g}_d) - \frac{\widehat{g}_0}{x_0}\xi_E$$

for $\widehat{g} = [\widehat{g}_0, \ldots, \widehat{g}_d] \in \mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}})_k$.

Since $\widehat{g} \in \mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$, we have

$$\widehat{g} \cdot J(Q_{\widehat{\mathcal{P}}}) = \sum_{q=0}^{d} \widehat{g}_q \partial_q Q_{\widehat{\mathcal{P}}} = 0.$$

This means, $(\widehat{g}_0, \ldots, \widehat{g}_d)(Q_{\widehat{\mathcal{P}}}) = \sum_{q=0}^{d} \widehat{g}_q \partial_q Q_{\widehat{\mathcal{P}}} = 0$ and $(\widehat{g}_0, \ldots, \widehat{g}_d) \in \mathcal{D}(\widehat{\mathcal{P}})$. We see $\widehat{g}_0 = (\widehat{g}_0, \ldots, \widehat{g}_d)(x_0) \in (x_0)$ by Proposition 2.5.5 so that $\frac{\widehat{g}_0}{x_0} \in S$. Therefore,

$$\psi(\widehat{g})(Q_{\widehat{\mathcal{P}}}) = (\widehat{g}_0, \ldots, \widehat{g}_d)(Q_{\widehat{\mathcal{P}}}) - \frac{\widehat{g}_0}{x_0} \deg(Q_{\widehat{\mathcal{P}}}) Q_{\widehat{\mathcal{P}}} \in (Q_{\widehat{\mathcal{P}}}).$$

Since the first coordinate of $\frac{\widehat{g}_0}{x_0}\xi_E$ is $\widehat{g}_0$, we have $\psi(\widehat{g})_0 = 0$ and $\psi(\widehat{g}) \in \overline{\mathcal{D}}(\widehat{\mathcal{P}})_k$.

Finally, we verify $\psi \circ \phi$ is the identity as follows:

$$\begin{aligned}
\psi \circ \phi(\widehat{f}) &= \psi\left([0, \widehat{f}_1, \ldots, \widehat{f}_d] - \frac{\widehat{f}(Q_{\widehat{\mathcal{P}}})}{\deg(Q_{\widehat{\mathcal{P}}}) Q_{\widehat{\mathcal{P}}}} [x_0, x_1, \ldots, x_d] \right) \\
&= \left((0, \widehat{f}_1, \ldots, \widehat{f}_d) - \frac{\widehat{f}(Q_{\widehat{\mathcal{P}}})}{\deg(Q_{\widehat{\mathcal{P}}}) Q_{\widehat{\mathcal{P}}}} (x_0, x_1, \ldots, x_d) \right) - \frac{-\widehat{f}(Q_{\widehat{\mathcal{P}}})}{\deg(Q_{\widehat{\mathcal{P}}}) Q_{\widehat{\mathcal{P}}} x_0} x_0 \xi_E \\
&= \widehat{f}.
\end{aligned}$$

Similarly, we can easily verify $\phi \circ \psi$ is the identity. □

This theorem identifies the space $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ with the syzygy module of the Jacobian ideal of $Q_{\widehat{\mathcal{P}}}$, which is computable using Gröbner basis by Schreyer's algorithm [28]. This enables us to obtain a basis of $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$. Note that the elements of $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ that are vector fields with degree *at most* k correspond to the elements of the *homogeneous* degree k component of $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$.

2.6 Algorithm

Our construction in Section 2.5 can be implemented as an explicit algorithm as in Algorithm 1. An accompanying implementation is available in SageMath at `https://github.com/jcwjmz/LogarithmicDerivationModule`.

We may also ask for the lowest possible degree polynomial vector field that meets a specified error bound.

Problem 2.6.1. Given an error tolerance $\epsilon \geq 0$, find a polynomial vector field $\xi \in \mathrm{Poly}_\partial(\mathcal{P})$ of minimum degree j such that $\sum_{s \in O} \|\xi(x_s) - u_s\|^2 < \epsilon$.

Algorithm 1 Find a polynomial vector field in $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ with minimal error

procedure FindPolyTangentialWithDegreeBound($k, \{(x_s, u_s) \mid s \in \mathcal{O}\}$)
Require: $k \geq 0$, observation data $\{(x_s, u_s)\}_{s \in \mathcal{O}}$.
    Compute a set of minimal homogeneous generators for the $\mathbb{R}[x_0, x_1, \ldots, x_d]$-module $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$.
    From these generators, construct an $\mathbb{R}$-basis $\{\hat{\phi}^j\}_{j=1}^N$ of degree up to k.
    Dehomogenise the basis by setting $x_0 = 1$: $\phi^j(x_1, \ldots, x_d) := \hat{\phi}^j(1, x_1, \ldots, x_d)$.
    Form the matrix A where $A_{s,j} = \phi^j(x_s)$.
    Form the vector b from u_s.
    Find coefficients $c = (c^j)$ that minimize $\|Ac - b\|^2$ (least-squares problem).
    Let the resulting vector field be $\xi(x) = \sum_{j=1}^N c_j \phi^j(x)$.
    return ξ;
end procedure

Estimating this minimum degree analytically is challenging and related to algebraic properties of the underlying polynomial modules (e.g., regularity), for which only limited results are known (e.g., [32, 5, 6]). However, since solving the least-squares problem for a fixed degree is often computationally cheaper, we can adopt a straightforward iterative strategy, as outlined in Algorithm 2.

Algorithm 2 Find a polynomial field in $\mathrm{Poly}_\partial(\mathcal{P})$ meeting a specified error bound

procedure FindPolyTangentialWithErrorBound($\epsilon, \{(x_s, u_s) \mid s \in \mathcal{O}\}$)
Require: Error tolerance $\epsilon \geq 0$, observation data $\{(x_s, u_s)\}_{s \in \mathcal{O}}$.
    $k \leftarrow 0$;
    Initialise $\xi \leftarrow \mathbf{0}$ (the zero vector field).
    Initialise 'error' $\leftarrow \sum_{s \in \mathcal{O}} \|u_s\|^2$ (error for $\xi = \mathbf{0}$).
    while 'error' $\geq \epsilon$ **do**
        Construct an $\mathbb{R}$-basis $\{\phi_j\}_{j=1}^{N_k}$ for $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$.
        Find $\xi_{LS} \in \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ that minimizes $\sum_{s \in \mathcal{O}} \|\xi_{LS}(x_s) - u_s\|^2$ using the basis $\{\phi_j\}$.
        'error' $\leftarrow \sum_{s \in \mathcal{O}} \|\xi_{LS}(x_s) - u_s\|^2$.
        $\xi \leftarrow \xi_{LS}$.
        $k \leftarrow k + 1$;
    end while
    return ξ;
end procedure

The following proposition ensures that Algorithm 2 terminates because for $\epsilon = 0$ (exact interpolation), a solution exists with a sufficiently high polynomial degree.

Proposition 2.6.2. *Let*

$$\{(x_s, u_s) \mid x_s \in \mathcal{P},\ u_s \in \mathbb{R}^d\}_{s \in \mathcal{O}}$$

be a finite set of observations, where each point x_s is distinct and belongs to the polytope $\mathcal{P}$, and each associated vector $u_s \in \mathbb{R}^d$. Suppose that for any x_s lying on a boundary facet F_j of $\mathcal{P}$, the vector u_s is tangent to F_j. Then, there exists a polynomial vector field $\xi \in \mathrm{Poly}_\partial(\mathcal{P})$ (i.e., $\xi \in \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ for some k) such that $\xi(x_s) = u_s$ for all $s \in \mathcal{O}$. Consequently, Problem Problem 2.6.1 has a solution for any $\epsilon \geq 0$.

Proof. The proof is constructive. We build the polynomial vector field ξ by first satisfying the interpolation conditions on the boundary facets and then addressing the conditions in the interior of the polytope $\mathcal{P}$.

Let the set of observation indices O be partitioned based on the location of the points x_s:

- Let $O_{int} = \{s \in O \mid x_s \in \text{int}(\mathcal{P})\}$.
- For each boundary facet F_j of $\mathcal{P}$ (where $j = 1, \dots, m$), let $O_j = \{s \in O \mid x_s \in F_j\}$.

For each facet F_j, we construct a polynomial vector field $\xi_j \in \text{Poly}_\partial(\mathcal{P})$ that interpolates the data in O_j and is tangent to all facets of $\mathcal{P}$.

Let $\{x_s\}_{s\in O_j}$ be the set of observation points on facet F_j. As these points are distinct, we can define Lagrange basis polynomials $L_s^{(j)}(x)$ for $s \in O_j$ on the affine subspace containing F_j, such that $L_s^{(j)}(x_t) = \delta_{st}$ for all $s, t \in O_j$.

To ensure that ξ_j is tangent to all other facets F_l for $l \neq j$, we define the polynomial $Q_j(x) = \prod_{l\neq j} h_l(x)$. This polynomial vanishes on every facet except F_j. We then define the vector field $\xi_j(x)$ as:

$$\xi_j(x) = Q_j(x) \sum_{s\in O_j} L_s^{(j)}(x) \frac{u_s}{Q_j(x_s)}$$

Note that for $x_s \in F_j$, we have $x_s \notin F_l$ for $l \neq j$, so $h_l(x_s) \neq 0$ and thus $Q_j(x_s) \neq 0$, making the expression well-defined.

The vector field ξ_j has the following properties:

- **Interpolation on F_j:** For any $t \in O_j$,

$$\xi_j(x_t) = Q_j(x_t) \sum_{s\in O_j} L_s^{(j)}(x_t) \frac{u_s}{Q_j(x_s)} = Q_j(x_t) \left(1 \cdot \frac{u_t}{Q_j(x_t)}\right) = u_t.$$

- **Tangency to F_l ($l \neq j$):** For any $x \in F_l$, $h_l(x) = 0$, which implies $Q_j(x) = 0$. Thus, $\xi_j(x) = 0$ on F_l, which ensures $\langle \xi_j(x), \alpha_l \rangle = 0$.
- **Tangency to F_j:** By hypothesis, $\langle u_s, \alpha_j \rangle = 0$ for all $s \in O_j$. Since $Q_j(x)$ and $L_s^{(j)}(x)$ are scalar polynomials, we have $\langle \xi_j(x), \alpha_j \rangle = Q_j(x) \sum_{s\in O_j} L_s^{(j)}(x) \frac{\langle u_s, \alpha_j \rangle}{Q_j(x_s)} = 0$.

Thus, ξ_j is tangent to all facets of $\mathcal{P}$.

Let $\xi_{\text{bnd}}(x) = \sum_{j=1}^m \xi_j(x)$. We now define new target vectors u_s' for the interior points $s \in O_{int}$ to account for the contribution of ξ_{bnd}:

$$u_s' = u_s - \xi_{\text{bnd}}(x_s) = u_s - \sum_{j=1}^m \xi_j(x_s)$$

We need to find a polynomial vector field ξ_{int} such that $\xi_{int}(x_s) = u_s'$ for all $s \in O_{int}$ and which is tangent to all boundary facets. To ensure tangency, we construct ξ_{int} to vanish on the entire boundary $\partial\mathcal{P}$. Let $H(x) = \prod_{j=1}^m h_j(x)$. $H(x)$ is zero on $\partial\mathcal{P}$.

Let $L_s^{int}(x)$ be the Lagrange basis polynomials for the interior points $\{x_s\}_{s\in O_{int}}$. We define $\xi_{int}(x)$ as:

$$\xi_{int}(x) = H(x) \sum_{s\in O_{int}} L_s^{int}(x) \frac{u'_s}{H(x_s)}$$

Since $x_s \in \text{int}(\mathcal{P})$, $H(x_s) \neq 0$, so this is well-defined. The field $\xi_{int}(x)$ vanishes on all facets F_j (since $H(x)|_{F_j} = 0$) and thus is tangent to the boundary. By construction, it interpolates the modified vectors: $\xi_{int}(x_s) = u'_s$ for all $s \in O_{int}$.

The final interpolating vector field $\xi(x)$ is the sum of the boundary and interior components:

$$\xi(x) = \xi_{int}(x) + \xi_{\text{bnd}}(x) = \xi_{int}(x) + \sum_{j=1}^{m} \xi_j(x)$$

We verify that $\xi(x)$ satisfies all interpolation conditions:

- For an interior point x_t with $t \in O_{int}$:

$$\xi(x_t) = \xi_{int}(x_t) + \xi_{\text{bnd}}(x_t) = u'_t + \xi_{\text{bnd}}(x_t) = (u_t - \xi_{\text{bnd}}(x_t)) + \xi_{\text{bnd}}(x_t) = u_t.$$

- For a boundary point x_t with $t \in O_k$ for some facet F_k: $\xi_{int}(x_t) = 0$ since $x_t \in \partial\mathcal{P}$. For any $j \neq k$, $\xi_j(x_t) = 0$ since $x_t \in F_k$ implies $h_k(x_t) = 0$, and $h_k(x)$ is a factor of $Q_j(x)$. Therefore:

$$\xi(x_t) = \xi_{int}(x_t) + \xi_k(x_t) + \sum_{j\neq k} \xi_j(x_t) = 0 + u_t + 0 = u_t.$$

The final field $\xi(x)$ is tangent to the boundary of $\mathcal{P}$ because it is a sum of vector fields (ξ_{int} and all ξ_j) that are each tangent to the boundary. This completes the construction. □

Remark 2.6.3. A natural question arises regarding whether a Weierstrass-type approximation theorem holds in this setting. Specifically, given a continuous vector field ξ_{true} on $\mathcal{P}$ and an error tolerance $\epsilon > 0$, can we always find a polynomial vector field $\xi_{\text{poly}} \in \text{Poly}_\partial(\mathcal{P})$ such that $|\xi_{\text{true}} - \xi_{\text{poly}}| < \epsilon$ under an appropriate norm? The choice of norm is crucial and depends on the geometric properties of $\mathcal{P}$. For compact domains, the $L^\infty(\mathcal{P})$ norm provides uniform approximation, while for non-compact settings, weighted $L^2(\mathcal{P})$ norms or Sobolev norms may be more appropriate.

2.7 Example

We illustrate our method with a regular pentagon as the domain $\mathcal{P} \subset \mathbb{R}^d$ with $d = 2$.

Let $\mathcal{P}$ be the convex hull of five points forming a regular pentagon centered at the origin:

$$\mathcal{P} = \text{conv}\left\{\left(\cos\left(\frac{(2i-1)\pi}{5}\right), \sin\left(\frac{(2i-1)\pi}{5}\right)\right) \mid i = 1,2,3,4,5\right\}.$$

Its $m = 5$ supporting hyperplanes are given by

$$H_i = \left\{ x \in \mathbb{R}^2 \mid \cos\frac{2i\pi}{5}\, x_1 + \sin\frac{2i\pi}{5}\, x_2 - \cos\frac{\pi}{5} = 0 \right\} \quad (i = 1, 2, 3, 4, 5).$$

By taking the cone,

$$\widehat{H}_i = \left\{ x \in \mathbb{R}^3 \mid \cos\frac{2i\pi}{5}\, x_1 + \sin\frac{2i\pi}{5}\, x_2 - \cos\frac{\pi}{5}\, x_0 = 0 \right\}$$

and by Definition 2.5.4,

$$Q_{\widehat{\mathcal{P}}} = x_0 \prod_{i=1}^{5} \widehat{h}_i = x_0 \prod_{i=1}^{5} \left(\cos\frac{2i\pi}{5}\, x_1 + \sin\frac{2i\pi}{5}\, x_2 - \cos\frac{\pi}{5}\, x_0 \right).$$

The syzygy module of the Jacobian ideal, $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$, can be computed by Schreyer's algorithm. In our codes[1], we rely on the implementation in a computer algebra system `Singular` through its Sage interface. In this example, we find $\mathrm{Syz}(J(Q_{\widehat{\mathcal{P}}}))$ is generated as an $\mathbb{R}[x_0, x_1, \ldots, x_d]$-module by five degree-four elements.

We compute the $\mathbb{R}$-basis of $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ for degree $k = 4$ and $k = 5$ using Theorem 2.5.9. In particular, we find

$$\dim(\mathrm{Poly}_\partial(\mathcal{P})_{\leq 4}) = 5, \quad \dim(\mathrm{Poly}_\partial(\mathcal{P})_{\leq 5}) = 12.$$

Consider the observations $\{(x_s, u_s)\} \subset \mathcal{P} \times \mathbb{R}^2$, with

$$\begin{aligned}
x_{s1} &= (-1/3,\ -0.7), & u_{s1} &= (3,\ 0),\\
x_{s2} &= (1/4,\ 1/10), & u_{s2} &= (0,\ 0),\\
x_{s3} &= (-0.8,\ 0), & u_{s3} &= (-2,\ 4),\\
x_{s4} &= (1/3,\ 0.7), & u_{s4} &= (2,\ 0),\\
x_{s5} &= (1/5,\ -0.5), & u_{s5} &= (2,\ -1),\\
x_{s6} &= (1/5,\ 0.5), & u_{s6} &= (0,\ 0).
\end{aligned}$$

Here, $(x_{s1}, u_{s1}) = ((-1/3,\ -0.7),\ (3,\ 0))$ means the observed value of the vector field at $(-1/3,\ -0.7) \in \mathcal{P}$ is $(3,\ 0)$. Note that $(x_{s2}, u_{s2}) = ((1/4, 1/10),\ (0, 0))$ is a singular point.

We solve the least-squares problem Problem 2.3.4 to find a best fitting tangential polynomial field of degree at most k; that is, to find an element in $\mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ which best interpolates the observations.

The figures below show the results for various choices of degree and number of observations.

[1] `https://github.com/jcwjmz/LogarithmicDerivationModule`

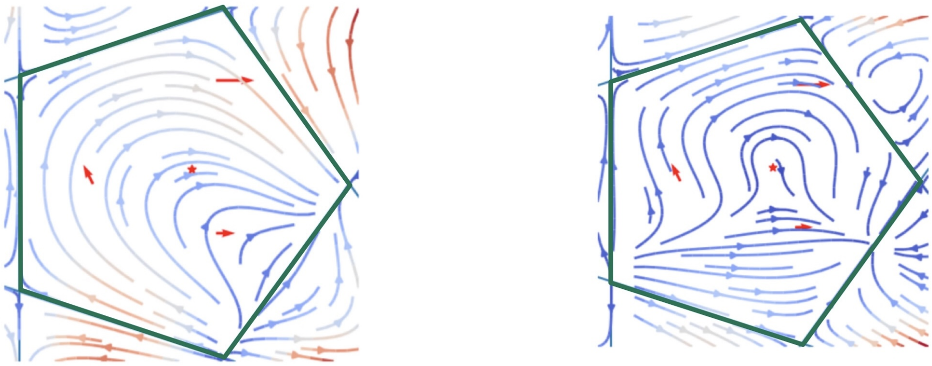

Figure 2.3: The best fitting vector field for the first four observations at $x_{s1}, \ldots, x_{s4}$ (indicated by red arrows), using degree 4 (left) and degree 5 (right). For degree 4, the fit exhibits noticeable deviation from the observations, as reflected in the higher error value (2.3.3) around 2.7. In contrast, with degree 5, the fitted vector field exactly interpolates the observations, resulting in zero error.

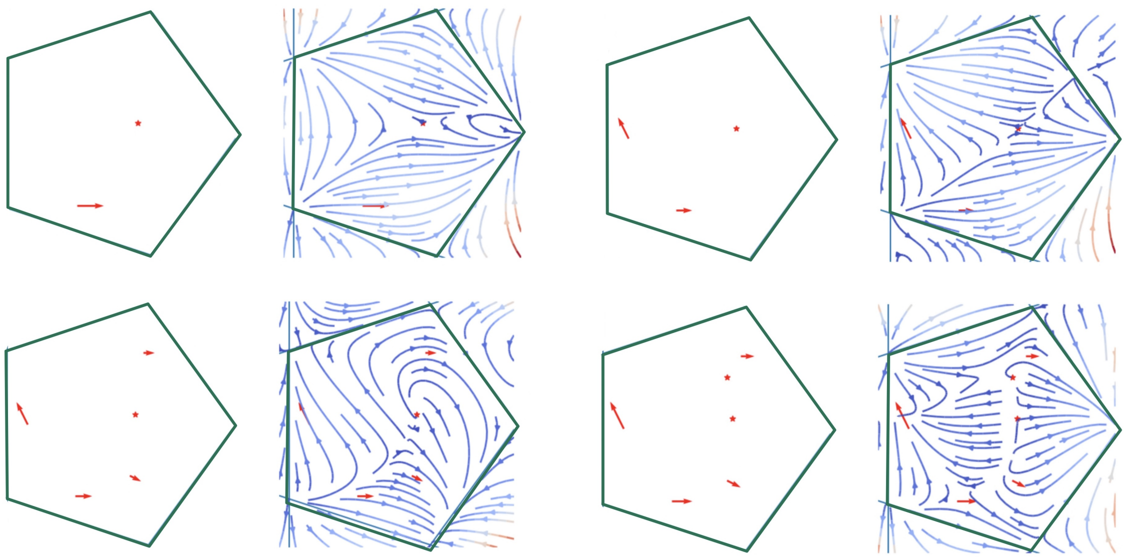

Figure 2.4: We gradually increase the number of observation points and compute the best fitting fields of degree 5. The error is zero for every case.

2.8 Future Work

The main problem Problem 2.3.4 can be extended to finding vector fields within specific $\mathbb{R}$-vector subspaces of $\mathrm{Poly}(\mathcal{P})_{\leq k}$ that also satisfy the tangency condition. Of particular importance are the following three classes, especially in relation to the Helmholtz decomposition.

Define:

$$\mathrm{DivFree}(\mathcal{P}) = \left\{ \xi = (f_1, \ldots, f_d) \in \mathrm{Poly}(\mathcal{P}) \,\middle|\, \nabla \cdot \xi = \sum_{q=1}^{d} \frac{\partial f_q}{\partial x_q} = 0 \right\}$$

as the space of divergence-free polynomial vector fields,

$$\begin{aligned}\mathrm{RotFree}(\mathcal{P}) &= \left\{ \xi = (f_1, \ldots, f_d) \in \mathrm{Poly}(\mathcal{P}) \,\middle|\, \frac{\partial f_q}{\partial x_p} - \frac{\partial f_p}{\partial x_q} = 0, \quad \forall\, 1 \leq p, q \leq d \right\} \\ &= \{ \xi \in \mathrm{Poly}(\mathcal{P}) \mid \text{there exists a polynomial } \Phi \text{ such that } \xi = \nabla \Phi \}\end{aligned}$$

as the space of rotation-free (curl-free or irrotational) polynomial vector fields, and

$$\mathrm{Harm}(\mathcal{P}) = \{ \xi = (f_1, \ldots, f_d) \in \mathrm{Poly}(\mathcal{P}) \mid \Delta \xi = (\Delta f_1, \ldots, \Delta f_d) = \mathbf{0} \}$$

as the space of harmonic polynomial vector fields, where $\Delta = \sum_{q=1}^{d} \frac{\partial^2}{\partial x_q^2}$ is the Laplacian operator.

Proposition 2.8.1 (Helmholtz Decomposition; c.f. [12]). *Let $\mathcal{P}$ be a convex polyhedral domain in $\mathbb{R}^d$. Any polynomial vector field $\xi \in \mathrm{Poly}(\mathcal{P})$ can be decomposed as*

$$\xi = \nabla \Phi + \mathbf{w},$$

where Φ is a polynomial scalar potential and $\mathbf{w} \in \mathrm{DivFree}(\mathcal{P})$ is a polynomial vector field that is divergence-free. If ξ has components of degree at most k, then Φ can be chosen as a polynomial of degree at most $k+1$, and the components of $\mathbf{w}$ will have degree at most k. This decomposition can be made unique by imposing an additional condition on Φ, such as $\Phi(x_0) = 0$ for some fixed $x_0 \in \mathcal{P}$.

Furthermore, by Poincaré's lemma,

1. *Since $\mathcal{P}$ is simply connected, a polynomial vector field $\xi \in \mathrm{Poly}(\mathcal{P})$ is in $\mathrm{RotFree}(\mathcal{P})$ (curl-free) if and only if it is the gradient of a polynomial scalar potential Φ.*
2. *Since $\mathcal{P}$ is contractible, a polynomial vector field $\xi \in \mathrm{DivFree}(\mathcal{P})$ (divergence-free) can be expressed using potentials:*
   - *For $d = 3$, $\xi = \nabla \times \mathbf{A}$ for some polynomial vector potential $\mathbf{A}$.*
   - *For $d = 2$, if $\xi = (f_1, f_2)$, then there exists a polynomial stream function Ψ such that $f_1 = \partial_2 \Psi$ and $f_2 = -\partial_1 \Psi$.*

The intersection $\mathrm{RotFree}(\mathcal{P}) \cap \mathrm{DivFree}(\mathcal{P})$ consists of vector fields $\xi = \nabla \Phi$ where Φ is a harmonic polynomial (i.e., $\Delta \Phi = 0$), so such fields ξ are also harmonic vector fields, i.e., $\xi \in \mathrm{Harm}(\mathcal{P})$.

For these classes, we define $\mathrm{DivFree}_\partial(\mathcal{P})_{\leq k} = \mathrm{DivFree}(\mathcal{P}) \cap \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$, $\mathrm{RotFree}_\partial(\mathcal{P})_{\leq k} = \mathrm{RotFree}(\mathcal{P}) \cap \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$, and $\mathrm{Harm}_\partial(\mathcal{P})_{\leq k} = \mathrm{Harm}(\mathcal{P}) \cap \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$.

Problem 2.8.2. Solve Problem 2.3.4 with the additional constraint that ξ must belong to one of the spaces $\mathrm{DivFree}_\partial(\mathcal{P})_{\leq k}$, $\mathrm{RotFree}_\partial(\mathcal{P})_{\leq k}$, or $\mathrm{Harm}_\partial(\mathcal{P})_{\leq k}$.

Our method straightforwardly extends to this setting (see Figure 2.5).

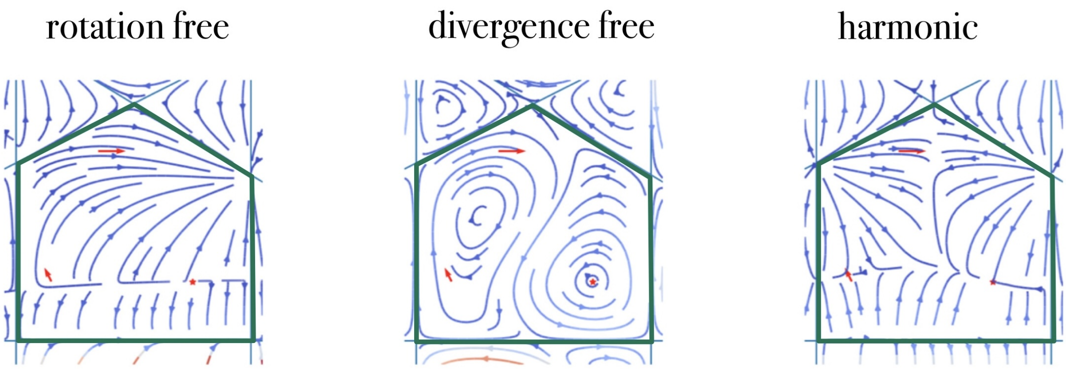

Figure 2.5: Example of Problem 2.8.2.

Observations need not be restricted to direct vector values; they can also involve other quantities derived from the vector fields. For example, one can consider reconstructing vector fields from sampled vorticity:

Problem 2.8.3. Given a natural number k, find $\xi = (f_1, \ldots, f_d) \in \mathrm{Poly}_\partial(\mathcal{P})_{\leq k}$ that minimises $\sum_{s \in O} \|(\nabla \times \xi)(x_s) - u_s\|^2$. Here, $(\nabla \times \xi)$ denotes the curl of ξ.

- For $d = 2$, $\nabla \times \xi$ is identified with the scalar field $\omega = \partial_1 f_2 - \partial_2 f_1$, and $u_s \in \mathbb{R}$. The norm $\|\cdot\|$ is the absolute value.
- For $d = 3$, $\nabla \times \xi = (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1)$ is a vector field, and $u_s \in \mathbb{R}^3$. The norm $\|\cdot\|$ is the standard Euclidean norm.
- For $d > 3$, $\nabla \times \xi$ can be defined as the bivector field $d\omega_\xi$, where ω_ξ is the 1-form associated with ξ. The observations u_s would then be bivectors.

Finally, there are important theoretical questions regarding the dimensions of the spaces $\mathrm{DivFree}(\mathcal{P})$, $\mathrm{RotFree}(\mathcal{P})$, and $\mathrm{Harm}(\mathcal{P})$.

Problem 2.8.4. Give a cohomological identification of the dimension of the space of divergence-free (or rotation-free, or harmonic) polynomial vector fields of degree at most k on $\mathcal{P}$.

Chapter 3

Generalization of Saito's criterion

Saito's criterion is a foundational result that offers a concrete test for determining when the module of logarithmic derivations along a divisor is free. In the context of hyperplane arrangements, the specialized version of this criterion (Theorem 1.2.5) has been especially influential, serving as a cornerstone in the study of freeness via the structure of the logarithmic derivation module.

Recent advances have extended Saito's original insight to broader settings. For instance, a generalized version of Saito's criterion for modules of higher-order $\mathcal{A}$-differential operators appears in Holm's thesis [16, Proposition III 5.8], with a detailed proof provided by Abe and Nakashima [2, Theorem 2.10]. Furthermore, Daniele, Marcos, and William [11, Theorem 4.5] develop a generalization for assessing the freeness of toric logarithmic sheaves, enriching the geometric scope of the theory.

Motivated by the study of non-free arrangements, this chapter proposes a conjecture that extends Saito's criterion to the non-free case. Following this line of inquiry, we establish a theorem that determines minimal generators for logarithmic derivation modules of arrangements with projective dimension one. As a corollary, we confirm that this generalized version of Saito's criterion holds for arrangements in dimension three. This result offers new insight into the structure of non-free arrangements and opens avenues for further investigation into their algebraic and combinatorial properties.

3.1 Preliminaries

In this section, we follow the notations and conventions established in Section 1.2. In particular, we adopt the definitions and results regarding hyperplane arrangements, logarithmic derivation modules, and projective dimension as previously introduced.

We first fix some additional notations.

1. $\mathcal{A}$ is a central arrangement in $\mathbb{K}^\ell$.
2. $Q = Q(\mathcal{A})$, and $\theta_i \in D(\mathcal{A})$ for $i = 2, \ldots, \ell + 1$.
3. $M = M[\theta_1, \ldots, \theta_{\ell+1}]$, where $M(i, j) = \theta_i(x_j)$, $i = 1, \ldots, \ell + 1$ and $j = 1, \ldots, \ell$.

4. $M_i = M[\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_{\ell+1}]$ for $i = 1, \ldots, \ell + 1$.

5. $\Delta_i = (-1)^i \det M_i$ for $i = 1, \ldots, \ell + 1$.

6. $\Delta = (\Delta_1, \ldots, \Delta_{\ell+1})$.

7. Since $\Delta_i \in QS$ for each i, we can write $\Delta_i = g_i Q$ for some $g_i \in S$, where $i = 1, \ldots, \ell+1$.

To simplify the discussion, we introduce the following definition.

Definition 3.1.1. We say that the relation in $D(\mathcal{A})$

$$f_1\theta_1 + \cdots + f_{\ell+1}\theta_{\ell+1} = 0$$

is of minimal degree (or primitive) if the polynomials $f_1, \ldots, f_{\ell+1}$ have no nonunit common divisor.

3.2 Main theorem

Before presenting the main theorem, we first establish several auxiliary results.

Lemma 3.2.1. *We have*

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0.$$

Proof. Note that M is a $(\ell + 1) \times \ell$ matrix over ring S. We may consider the following composition:

$$S^\ell \xrightarrow{M} S^{\ell+1} \xrightarrow{\Delta} S$$

The i-th entry of the composite map ΔM is $\sum_j \Delta_j M(j, i)$, that is, k times the Laplace expansion of the determinant of a $(\ell + 1) \times (\ell + 1)$ matrix obtained from M by repeating its i-th column. However, since any matrix with a repeated column has a determinant of zero, it follows that $\Delta M = 0$. (see Lemma 3.7 in [10]).

It follows that

$$\Delta_1\theta_1(x_i) + \cdots + \Delta_{\ell+1}\theta_{\ell+1}(x_i) = 0$$

for all $i = 1, \ldots, \ell$. Note that $\Delta_i = g_i Q$ for $i = 1, \ldots, \ell + 1$. Since S is a domain, we have

$$g_1\theta_1(x_i) + \cdots + g_{\ell+1}\theta_{\ell+1}(x_i) = 0.$$

For any $f \in S$ we have

$$\theta_i(f) = \big(\theta_i(x_1), \theta_i(x_2), \cdots, \theta_i(x_\ell)\big)\begin{pmatrix}\partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_\ell} f\end{pmatrix}.$$

Then

$$
\begin{aligned}
(g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1})(f) &= \left(g_1, g_2, \cdots, g_{\ell+1}\right)\begin{pmatrix} \theta_1(f) \\ \theta_2(f) \\ \vdots \\ \theta_{\ell+1}(f) \end{pmatrix} \\
&= \left(g_1, g_2, \cdots, g_{\ell+1}\right) M \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_\ell} f \end{pmatrix} \\
&= \left(\textstyle\sum_{j=1}^{\ell+1} g_j\theta_j(x_1), \sum_{j=1}^{\ell+1} g_j\theta_j(x_2), \cdots, \sum_{j=1}^{\ell+1} g_j\theta_j(x_\ell)\right)\begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_\ell} f \end{pmatrix} \\
&= 0.
\end{aligned}
$$

It follows that

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0.$$

□

As a consequence of Lemma 3.2.1, we obtain the following property.

Proposition 3.2.2. *If $g_{\ell+1} \neq 0$, then*

(1) $g_{\ell+1}\eta \in S\theta_1 + \cdots + S\theta_\ell$ for any $\eta \in D(\mathcal{A})$.

(2) Moreover, the coefficients $f_i \in S$ for $i = 1, \ldots, \ell$ in

$$f_1\theta_1 + \cdots + f_\ell\theta_\ell + g_{\ell+1}\eta = 0,$$

is uniquely determined by $\Gamma_i = f_i Q$, where $\Gamma_i = (-1)^i \det M[\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_\ell, \eta]$.

Proof. (1) Since

$$\theta_i = \sum_{j=1}^{\ell} \theta_i(x_j)\partial_{x_j},$$

then

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_\ell \end{pmatrix} = M_{\ell+1}\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_\ell} \end{pmatrix}.$$

By Cramer's rule, we have $(\det M_{\ell+1})\partial_{x_j} = \det M^j$, where M^j the $\ell \times \ell$ matrix obtained from $M_{\ell+1}$ by replacing its j-th column by $\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_\ell \end{pmatrix}$. It follows that $g_{\ell+1}Q\partial_{x_j} \in S\theta_1 + \cdots + S\theta_\ell$. Then for any $\eta \in D(\mathcal{A})$, there exists $h_i \in S$ such that

$$g_{\ell+1}Q\eta = h_1\theta_1 + \cdots + h_\ell\theta_\ell.$$

Since $\theta_1, \dots, \theta_\ell, \eta \in D(\mathcal{A})$, we have $\Gamma_i \in QS$. Since

$$\begin{aligned} g_{\ell+1}Q\Gamma_i &= (-1)^i g_{\ell+1}Q \det M[\theta_1, \dots, \hat{\theta}_i, \dots, \theta_\ell, \eta] \\ &= (-1)^i \det M[\theta_1, \dots, \hat{\theta}_i, \dots, \theta_\ell, g_{\ell+1}Q\eta] \\ &= (-1)^i \det M[\theta_1, \dots, \hat{\theta}_i, \dots, \theta_\ell, h_i\theta_i] \\ &= (-1)^\ell h_i \det M_{\ell+1} \\ &= -h_i g_{\ell+1} Q, \end{aligned}$$

this implies that $-h_i g_{\ell+1} Q = g_{\ell+1}Q\Gamma_i \in g_{\ell+1}Q^2 S$. Hence $Q | h_i$ for all i. This shows that $g_{\ell+1}\eta \in S\theta_1 + \cdots + S\theta_\ell$ for all $\eta \in D(\mathcal{A})$.

(2) Let $u_i = -h_i/Q \in S$. We have

$$u_1\theta_1 + \cdots + u_\ell\theta_\ell + g_{\ell+1}\eta = 0.$$

Since

$$f_1\theta_1 + \cdots + f_\ell\theta_\ell + g_{\ell+1}\eta = 0,$$

it follows that

$$(f_1 - u_1)\theta_1 + \cdots + (f_\ell - u_\ell)\theta_\ell = 0.$$

Since $\det M_{\ell+1} \neq 0$ and $\theta_1, \dots, \theta_\ell$ are S-independent, $f_i = u_i$ for all $i = 1, \dots, \ell$. Since $\Gamma_i = -h_i = u_i Q$, we obtain the assertion.

□

We say $f_1, \dots, f_p$ have a common divisor h modulo f when $f_i \in (h, f)$ for all i.

Proposition 3.2.3. *If $\theta_1, \dots, \theta_{\ell+1}$ form a minimal generating set for $D(\mathcal{A})$ and satisfy a unique relation*

$$f_1\theta_1 + \cdots + f_{\ell+1}\theta_{\ell+1} = 0 \tag{3.2.1}$$

of minimal degree, then there exists $c \in \mathbb{K}^*$ such that $\Delta_i = c f_i Q$ for all $i = 1, \dots, \ell + 1$.

Moreover, if $f_{\ell+1} \neq 0$, then $f_1, \dots, f_\ell$ have no nonunit common divisor modulo $f_{\ell+1}$.

Proof. By Lemma 3.2.1, we have

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0.$$

The first assertion follows from the uniqueness.

Suppose, for contradiction, that the second assertion is not true. Then we may assume that

$$f_i = f f_i' + f_{\ell+1} h_i$$

for some $f_i', h_i \in S$ and $f \in S_{>0} \setminus \{0\}$. Since Equation (3.2.1) is primitive, it follows that f and $f_{\ell+1}$ have no nonunit common divisor.

Let

$$\theta = \theta_{\ell+1} + \sum_{i \leq \ell} h_i\theta_i \in D(\mathcal{A}).$$

Substituting into Equation (3.2.1), we obtain

$$ff_1'\theta_1 + \cdots + ff_\ell'\theta_\ell + f_{\ell+1}\theta = 0. \tag{3.2.2}$$

Let $\theta = f\theta'$. Then both $f_{\ell+1}\theta'$ and $f\theta'$ belong to $D(\mathcal{A})$. Since f and $f_{\ell+1}$ have no nonunit common divisor, this implies that $\theta' \in D(\mathcal{A})$, this contradicts that θ_i's are minimal generators.

□

The following lemma only involves $\theta_1, \ldots, \theta_\ell$, but we retain the same notation for consistency.

Lemma 3.2.4. *Assume $g_{\ell+1} \in S_1 \setminus \{0\}$. If $\mathcal{A}$ is free, there exists $k \in \{1, \ldots, \ell\}$ and $\eta \in D(\mathcal{A})$ such that $g_{\ell+1}\eta \in \mathbb{K}^*\theta_k + \sum_{j\neq k, j\leq \ell} S\theta_j$ and*

$$\theta_1, \ldots, \theta_{k-1}, \eta, \theta_{k+1}, \ldots, \theta_\ell$$

form a basis of $D(\mathcal{A})$.

Proof. Assume that $\eta_1 = \theta_E, \eta_2, \ldots, \eta_\ell$ form a basis of $D(\mathcal{A})$. Let $\deg\theta_i = d_i$, $\deg\eta_i = e_i$ with $d_{i-1} \leq d_i$, $e_{i-1} \leq e_i$ for $i = 2, \ldots, \ell$. Then $e_1 + \cdots + e_\ell = |\mathcal{A}|$. Since $\det M_{\ell+1} = (-1)^{\ell+1}g_{\ell+1}Q$ is not zero and $g_{\ell+1} \in S_1$, we have

$$d_1 + \cdots + d_\ell = |\mathcal{A}| + 1 = e_1 + \cdots + e_\ell + 1. \tag{3.2.3}$$

Let $k = \min\{i : d_i \neq e_i\}$. Then $d_i = e_i$ for all $i < k$. Since $\theta_1, \ldots, \theta_\ell$ are S-independent since $\det M_{\ell+1} \neq 0$, we may assume that $\eta_i = \theta_i$ for all $i < k$ and $\theta_k \in S\eta_k + \cdots + S\eta_\ell$. But $d_k \neq e_k$, then $d_k > e_k$. By Equation (3.2.3), we have

$$d_{k+1} + \cdots + d_\ell \leq e_{k+1} + \cdots + e_\ell.$$

If $d_i = e_i$ for all $i > k$, then $d_k = e_k + 1$, and we may let $\alpha\eta_k = \theta_k$ and $\eta_i = \theta_i$ for some $\alpha \in S_1$ and $i = 1, \ldots, k-1, k+1, \ldots, \ell$. Thus

$$\theta_1, \ldots, \theta_{k-1}, \eta_k, \theta_{k+1}, \ldots, \theta_\ell$$

are S-independent. By Theorem 1.2.5, they form a basis of $D(\mathcal{A})$.

If $d_i \neq e_i$ for some $i > k$, let $k_1 = \min\{i : d_i > e_i, i > k\}$ and $k_2 = \min\{i : d_i < e_i, i > k\}$. Then we may assume that $\theta_i = \eta_i$ for $i \neq k, k_1, k_2$ and $i < \max\{k_1, k_2\}$. If $k_2 > k_1$, then $d_k \leq d_{k_1} \leq d_{k_2} < e_{k_2}$. We may assume that

$$\begin{aligned}
\theta_k &= \sum_{\substack{i<k_2 \\ i\neq k, k_1}} m_i\theta_i + m\eta_k + m'\eta_{k_1}, \\
\theta_{k_1} &= \sum_{\substack{i<k_2 \\ i\neq k, k_1}} n_i\theta_i + n\eta_k + n'\eta_{k_1}, \\
\theta_{k_2} &= \sum_{\substack{i<k_2 \\ i\neq k, k_1}} u_i\theta_i + u\eta_k + u'\eta_{k_1}.
\end{aligned}$$

It follows that $\theta_1, \ldots, \theta_{k_2}$ are S-dependent, which is a contradiction. Therefore, $k_2 < k_1$. Then $k_2 < k_1$ and $d_k \leq d_{k_2} < e_{k_2}$. We may assume that

$$\theta_k = \sum_{\substack{i<k_2 \\ i \neq k}} m_i \theta_i + m\eta_k,$$
$$\theta_{k_2} = \sum_{\substack{i<k_2 \\ i \neq k}} u_i \theta_i + u\eta_k.$$

It follows that $\theta_1, \ldots, \theta_{k_2}$ are S-dependent, which is also a contradiction.

□

If some $g_i \in \mathbb{K}^*$, then $\mathcal{A}$ is free by Theorem 1.2.5. We now consider the following lemma.

Lemma 3.2.5. *Assume that $g_{\ell+1} \in S_1 \setminus \{0\}$, and that $g_1, \ldots, g_\ell \in S_{>0}$ have no nonunit common divisor modulo $g_{\ell+1}$. Then $\mathcal{A}$ is not free.*

Proof. If $\mathcal{A}$ is free, then by Lemma 3.2.4, we may assume that $g_{\ell+1}\eta = \theta_\ell + \sum_{j<\ell} f_j \theta_j$, and that

$$\theta_1, \ldots, \theta_{\ell-1}, \eta$$

form a basis of $D(\mathcal{A})$. Let $\theta_{\ell+1} = f'\eta + \sum_{j<\ell} f'_j \theta_j$. By Lemma 3.2.1, we have a relation

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0. \tag{3.2.4}$$

Substituting the expressions for $\theta_{\ell+1}$ and θ_ℓ, we obtain

$$\sum_{j<\ell} (g_j - g_\ell f_j + g_{\ell+1} f'_j)\theta_j + g_{\ell+1}(g_\ell + f')\eta = 0.$$

Since $\theta_1, \ldots, \theta_{\ell-1}, \eta$ are S-independent, it follows that

$$g_j = g_\ell f_j - g_{\ell+1} f'_j \quad \text{for } j = 1, \ldots, \ell - 1,$$

which contradicts the assumption that $g_1, \ldots, g_\ell$ have no nonunit common divisor modulo $g_{\ell+1}$. Thus, $\mathcal{A}$ is not free.

□

Theorem 3.2.6. *Assume that $g_{\ell+1} \in S_1 \setminus \{0\}$, and that $g_1, \ldots, g_\ell \in S_{>0}$ have no nonunit common divisor modulo $g_{\ell+1}$.*

If $\mathrm{pd}_S D(\mathcal{A}) \leq 1$, then $\theta_1, \ldots, \theta_{\ell+1}$ form a minimal generating set for $D(\mathcal{A})$, and $\mathcal{A}$ is SPOG with a relation

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0.$$

Proof. By Lemma 3.2.5, we have $\mathrm{pd}_S D(\mathcal{A}) = 1$.

Extend $\{\theta_1, \ldots, \theta_\ell\}$ to a generating set $G = \{\theta_1, \ldots, \theta_\ell, \eta_{\ell+1}, \ldots, \eta_p\}$ of $D(\mathcal{A})$ such that $\eta_j \notin S(G \setminus \{\theta_j\})$ for all $j = \ell + 1, \ldots, p$.

We will prove $\theta_i \notin S(G \setminus \{\theta_i\})$ for $i = 1, \ldots, \ell$ so that G is a minimal generating set. Assume the contrary, that there exists $i \in \{1, \ldots, \ell\}$ such that

$$\theta_i = \sum_{j \neq i, j=1}^{\ell} p_j \theta_j + \sum_{j=\ell+1}^{p} h_j \eta_j, \quad (p_j \in S, h_j \in S_{>0}). \tag{3.2.5}$$

By Proposition 3.2.2, for any $j = \ell + 1, \ldots, p$,

$$g_{\ell+1} \eta_j = g_1^j \theta_1 + \cdots + g_\ell^j \theta_\ell, \tag{3.2.6}$$

where

$$g_i^j Q = (-1)^i \det M[\theta_1, \cdots, \hat{\theta}_i, \ldots, \theta_\ell, \eta_j].$$

If $h_j g_i^j \neq 0$, then g_i^j must be in $\mathbb{K}^*$ by degree reasons. However, this cannot happen by Theorem 1.2.5, since $\mathcal{A}$ is not free. Multiplying (3.2.5) by $g_{\ell+1} \neq 0$ and substituting with (3.2.6), we see that $\theta_1, \ldots, \theta_\ell$ are S-dependent, which contradicts $\Delta_{\ell+1} \neq 0$. Therefore, G is a minimal set of generators of $D(\mathcal{A})$.

Now, we will show $|G| = \ell + 1$. Since $\mathrm{pd}_S D(\mathcal{A}) = 1$, we have a minimal free resolution

$$0 \to \bigoplus_{i=\ell+1}^{w} S[-e_i] \xrightarrow{R} \bigoplus_{i=1}^{p} S[-d_i] \to D(\mathcal{A}) \to 0, \tag{3.2.7}$$

where d_i's are the degrees of minimal generators and e_i's are those of minimal relations. We see $w = p$ since $D(\mathcal{A})$ is of rank ℓ. Since $\{\theta_1, \ldots, \theta_\ell\}$ is S-independent, all relations should involve some η_j. Also, each η_j should appear in some relation. Hence, by reordering, we can assume $e_j > \deg \eta_j$ for $j = \ell + 1, \ldots, p$. By Theorem 1.2.11 and $\sum_{i=1}^{\ell} \deg \theta_i = \deg(g_{\ell+1} Q) = |\mathcal{A}| + 1$, we have

$$|\mathcal{A}| = \sum_{i=1}^{\ell} \deg \theta_i + \sum_{j=\ell+1}^{p} \deg \eta_j - \sum_{j=\ell+1}^{p} e_j = |\mathcal{A}| + 1 - \sum_{j=\ell+1}^{p} (e_j - \deg \eta_j).$$

Therefore, we have $p = \ell + 1$. The unique relation is given by (3.2.6) with $j = \ell + 1$.

We claim that $\{\theta_1, \ldots, \theta_\ell, \theta_{\ell+1}\}$ forms a minimal generating set for $D(\mathcal{A})$. By Lemma 3.2.1, we have a relation

$$g_1 \theta_1 + \cdots + g_{\ell+1} \theta_{\ell+1} = 0.$$

If $\theta_{\ell+1} \in S\theta_1 + \cdots + S\theta_\ell + \mathbb{K}^* \eta_{\ell+1}$, we are done. Otherwise, we may assume that

$$\theta_{\ell+1} = u_1 \theta_1 + \cdots + u_\ell \theta_\ell + u \eta_{\ell+1},$$

with $u \in S_{>0} \setminus \{0\}$. Similar to the proof in Lemma 3.2.5, we arrive at the contradiction that $g_1, \ldots, g_\ell$ have no nonunit common divisor modulo $g_{\ell+1}$. This completes the proof. □

When $\ell = 3$, we have $\mathrm{pd}_S D(\mathcal{A}) \leq \ell - 2 = 1$. By Proposition 3.2.3 and Theorem 3.2.6, it implies the following:

Theorem 3.2.7. *Let $\ell = 3$. $\mathcal{A}$ is SPOG, and the set $\{\theta_1, \ldots, \theta_4\}$ forms a minimal generating set for $D(\mathcal{A})$, satisfying a unique relation*

$$g_1 \theta_1 + \cdots + g_4 \theta_4 = 0$$

of minimal degree if and only if $g_4 \in S_1 \setminus \{0\}$, and $g_1, g_2, g_3 \in S_{>0}$ have no nonunit common divisor modulo g_4.

3.3 Future Work

We outline several directions related to this chapter that we believe are both promising and feasible:

(1) We consider two possible extensions of Theorem 3.2.6.

Dropping the condition $\mathrm{pd}_S D(\mathcal{A}) < 1$:

Conjecture 3.3.1. $\mathcal{A}$ is SPOG, and the set $\{\theta_1, \dots, \theta_{\ell+1}\}$ forms a minimal generating set for $D(\mathcal{A})$, satisfying the unique relation

$$g_1\theta_1 + \cdots + g_{\ell+1}\theta_{\ell+1} = 0,$$

of minimal degree if and only if $g_{\ell+1} \in S_1 \setminus \{0\}$, and $g_1, \dots, g_\ell \in S_{>0}$ have no nonunit common divisor modulo $g_{\ell+1}$.

Dropping the condition $g_{\ell+1} \in S_1$:

Conjecture 3.3.2. When $\mathrm{pd}_S D(\mathcal{A}) = 1$, there is a minimal free resolution of the following form:

$$0 \to \bigoplus_{i=\ell+1}^{p} S[-d_i - 1] \to \bigoplus_{i=1}^{p} S[-d_i] \to D(\mathcal{A}) \to 0.$$

(2) More generally, it is natural to ask how one can characterize the condition under which $\{\theta_i \in D(\mathcal{A}) \mid i \in I\}$ forms a set of (minimal) generators of $D(\mathcal{A})$, in terms of the ideal Δ_I generated by $\det M_J$, where M_J denotes the $(\ell \times \ell)$-matrix whose columns are $\{\theta_j \mid j \in J\}$ for subsets $J \subseteq I$ with $|J| = \ell$.

For example, at the opposite extreme from the free case, we conjecture:

Conjecture 3.3.3. Let $G = \{\theta_i \in D(\mathcal{A}) \mid i \in I\}$. The ideal Δ_I equals $S_{\geq k}Q$, where

$$k = (\ell - 1)(|\mathcal{A}| - \ell - 1)$$

if and only if $\mathcal{A}$ is generic and G forms a minimal generating set of $D(\mathcal{A})$.

(3) The projective dimension of the restriction:

Conjecture 3.3.4. Let $H \in \mathcal{A}$, and $\mathcal{A}$ be free. Then $\mathrm{pd}_S D(\mathcal{A}^H) \leq 1$.

For instance, this property was observed in all counterexamples to Orlik's conjecture that we studied in Chapter 1.

The exact sequence of Proposition 1.2.3, together with Conjectures 3.3.1 and 3.3.2, would provide powerful tools for studying $D(\mathcal{A}^H)$.

We believe that pursuing these directions will deepen the understanding of the algebraic and geometric structures studied in this thesis.

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