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An ergodic value distribution of certain meromorphic functions

Junghun Lee and Ade Irma Suriajaya

ABSTRACT. We calculate a certain mean-value of meromorphic functions by using specific ergodic transformations, which we call affine Boolean transformations. We use Birkhoff's ergodic theorem to transform the mean-value into a computable integral which allows us to completely determine the mean-value of this ergodic type. As examples, we introduce some applications to zeta functions and L -functions. We also prove an equivalence of the Lindelöf hypothesis of the Riemann zeta function in terms of its certain ergodic value distribution associated with affine Boolean transformations.

1. Introduction

In [LW09], M. Lifshitz and M. Weber investigated the value distribution of the *Riemann zeta function* $\zeta(s)$ by using the Cauchy random walk. They proved that almost surely

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \zeta\left(\frac{1}{2} + iS_n\right) = 1 + o\left(\frac{(\log N)^b}{N^{1/2}}\right)$$

holds for any $b > 2$ where $\{S_n\}_{n=1}^{\infty}$ is the Cauchy random walk. This result implies that most of the values of $\zeta(s)$ on the critical line are quite small. Analogous to [LW09], T. Srichan investigated the value distributions of Dirichlet L -functions and Hurwitz zeta functions by using the Cauchy random walk in [Sri15].

The first approach to investigate the ergodic value distribution of $\zeta(s)$ was done by J. Steuding. In [Ste12], he studied the ergodic value distribution of $\zeta(s)$ on vertical lines under the Boolean transformation.

We are interested in studying the ergodic value distribution of a larger class of meromorphic functions which includes but is not limited to the Selberg class (of ζ -functions and L -functions) and their derivatives, on vertical lines under more general Boolean transformations, which we shall call *affine Boolean transformation*

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$T_{\alpha,\beta} : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(1.1) \quad T_{\alpha,\beta}(x) := \begin{cases} \frac{\alpha}{2} \left(\frac{x+\beta}{\alpha} - \frac{\alpha}{x-\beta} \right), & x \neq \beta; \\ \beta, & x = \beta \end{cases}$$

for an $\alpha > 0$ and a $\beta \in \mathbb{R}$. Below is our main theorem. For a given $c \in \mathbb{R}$, we shall denote by \mathbb{H}_c and \mathbb{L}_c the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > c\}$ and the line $\{z \in \mathbb{C} \mid \operatorname{Re}(z) = c\}$.

THEOREM 1.1. *Let f be a meromorphic function on \mathbb{H}_c satisfying the following conditions.*

- (1) *There exists an $M > 0$ and a $c' > c$ such that for any $t \in \mathbb{R}$, we have*

$$|f(\{\sigma + it \mid \sigma > c'\})| \leq M.$$

- (2) *There exists a non-increasing continuous function $\nu : (c, \infty) \rightarrow \mathbb{R}$ such that if σ is sufficiently near c then $\nu(\sigma) \leq 1 + c - \sigma$, and that for any small $\epsilon > 0$, $f(\sigma + it) \ll_{f,\epsilon} |t|^{\nu(\sigma)+\epsilon}$ as $|t| \rightarrow \infty$.*
- (3) *f has at most one pole of order m in \mathbb{H}_c at $s = s_0 = \sigma_0 + it_0$, that is, we can write its Laurent expansion near $s = s_0$ as*

$$(1.2) \quad \frac{a_{-m}}{(s-s_0)^m} + \frac{a_{-(m-1)}}{(s-s_0)^{m-1}} + \cdots + \frac{a_{-1}}{s-s_0} + a_0 + \sum_{n=1}^{\infty} a_n (s-s_0)^n$$

for $m \geq 0$, where we set $m = 0$ if f has no pole in \mathbb{H}_c .

Then for any $s \in \mathbb{H}_c \setminus \mathbb{L}_{\sigma_0}$, we have

$$(1.3) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(s + iT_{\alpha,\beta}^n x) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$.

We denote the right-hand side of the above formula by $l_{\alpha,\beta}(s)$. If f has no pole in \mathbb{H}_c ,

$$(1.4) \quad l_{\alpha,\beta}(s) = f(s + \alpha + i\beta)$$

for all $s \in \mathbb{H}_c$. If f has a pole at $s = s_0 = \sigma_0 + it_0$,

$$(1.5) \quad l_{\alpha,\beta}(s) = \begin{cases} f(s + \alpha + i\beta) + B_m(s_0), & c < \operatorname{Re}(s) < \sigma_0, s \neq s_0 - \alpha - i\beta; \\ \sum_{n=0}^m \frac{a_{-n}}{(-2\alpha)^n}, & c < \operatorname{Re}(s) < \sigma_0, s = s_0 - \alpha - i\beta; \\ f(s + \alpha + i\beta), & \operatorname{Re}(s) > \sigma_0; \end{cases}$$

where

$$B_m(s_0) = \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta + i\alpha - i(s - s_0))^n} - \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta - i\alpha - i(s - s_0))^n}.$$

Moreover when $m = 1$, we can extend the result in (1.3) to the line \mathbb{L}_{σ_0} by setting

$$(1.6) \quad l_{\alpha,\beta}(\sigma_0 + it) = f(\sigma_0 + \alpha + i(t + \beta)) - \frac{a_{-1}\alpha}{\alpha^2 + (t_0 - t - \beta)^2}$$

for any $t \in \mathbb{R}$.

In the next section, we first give a few examples as applications of our main theorem, Theorem 1.1, to the Riemann zeta function, Dirichlet L -functions, Dedekind zeta functions, Hurwitz zeta functions, and their derivatives. We will briefly review some basics of ergodic theory and see an ergodic property of affine Boolean transformations in Section 3. In Section 4, we will complete the proof of Theorem 1.1.

2. Some applications to zeta functions and L -functions

In the following examples, we write $f^{(0)}$ to express f itself and we define

$$A_k(s) := \frac{(-1)^k k!}{i^{k+1}} \left(\frac{1}{(\beta + i\alpha - i(s-1))^{k+1}} - \frac{1}{(\beta - i\alpha - i(s-1))^{k+1}} \right)$$

for any non-negative integer k .

EXAMPLE 2.1 (The Riemann zeta function). For any $k \geq 0$ and $s \in \mathbb{H}_{-1/2} \setminus \mathbb{L}_1$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{(k)}(s + iT_{\alpha, \beta}^n x) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta^{(k)}(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$.

Denoting the right-hand side of the above formula by $l_{\alpha, \beta}^{(k)}(s)$, we have

$$l_{\alpha, \beta}^{(k)}(s) = \begin{cases} \zeta^{(k)}(s + \alpha + i\beta) + A_k(s), & -1/2 < \operatorname{Re}(s) < 1, s \neq 1 - \alpha - i\beta; \\ (-1)^k \gamma_k - \frac{k!}{(2\alpha)^{k+1}}, & -1/2 < \operatorname{Re}(s) < 1, s = 1 - \alpha - i\beta; \\ \zeta^{(k)}(s + \alpha + i\beta), & \operatorname{Re}(s) > 1; \end{cases}$$

where

$$\gamma_k := \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right).$$

If $k = 0$, we can extend the result to the line \mathbb{L}_1 by setting

$$l_{\alpha, \beta}^{(0)}(1 + it) = \zeta^{(0)}(1 + \alpha + i(t + \beta)) - \frac{\alpha}{\alpha^2 + (t + \beta)^2}.$$

Remark that Steuding showed Example 2.1 when $k = 0$, $\alpha = 1$, and $\beta = 0$ thus Example 2.1 is a generalization of [Ste12, Theorem 1.1].

PROOF OF EXAMPLE 2.1. We first note that for any $k \geq 0$, $\zeta^{(k)}(s)$ has an absolute convergent Dirichlet series expression when $\operatorname{Re}(s) > 1$. Thus condition (1) of Theorem 1.1 is satisfied for any $c' > 1$. From the Laurent expansion of $\zeta(s)$ near its pole $s = 1$ (see [Bri55, Theorem]), we can deduce the Laurent expansion of $\zeta^{(k)}(s)$ for any $k \geq 0$ near $s = 1$:

$$\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + (-1)^k \gamma_k + \sum_{n=k}^{\infty} \frac{(-1)^{n+1} \gamma_{n+1}}{(n-k+1)!} (s-1)^{n-k+1}.$$

Thus for $k \geq 0$, $\zeta^{(k)}(s)$ has a pole of order $k+1$ at $s = 1$. Moreover, we can show by using [Tit86, pp. 95–96] that

$$(2.1) \quad \zeta^{(k)}(\sigma + it) \ll_{k, \epsilon} |t|^{\mu(\sigma) + \epsilon}$$

holds with

$$\mu(\sigma) \leq \begin{cases} 0 & \text{if } \sigma > 1; \\ (1 - \sigma)/2 & \text{if } 0 \leq \sigma \leq 1; \\ 1/2 - \sigma & \text{if } \sigma < 0; \end{cases}$$

for any $k \geq 0$. Therefore we can apply Theorem 1.1 with $c = -1/2$, $s_0 = 1$, and $m = k + 1$ to $\zeta^{(k)}(s)$. \square

We can also show that this ergodic mean-value is related to the Lindelöf hypothesis. We first show that the Lindelöf hypothesis can be rewritten in terms of $\zeta^{(k)}(s)$.

THEOREM 2.2. *Let $k \in \mathbb{N}$. The Lindelöf hypothesis: For any $\epsilon > 0$,*

$$\zeta\left(\frac{1}{2} + it\right) \ll_{\epsilon} |t|^{\epsilon} \quad \text{as } |t| \rightarrow \infty$$

holds if and only if, for any $\epsilon > 0$,

$$\zeta^{(k)}\left(\frac{1}{2} + it\right) \ll_{k,\epsilon} |t|^{\epsilon} \quad \text{as } |t| \rightarrow \infty$$

holds.

The above theorem implies that we can restate the Lindelöf hypothesis as:

$$\text{For any } \epsilon > 0, \zeta^{(k)}\left(\frac{1}{2} + it\right) \ll_{k,\epsilon} |t|^{\epsilon} \quad \text{as } |t| \rightarrow \infty$$

for any non-negative integer k .

PROOF OF THEOREM 2.2. Suppose that the Lindelöf hypothesis is true. Thus by using the functional equation for $\zeta(s)$,

$$\zeta(\sigma + it) \ll_{\epsilon} |t|^{\mu(\sigma) + \epsilon/2}$$

holds for any $\epsilon > 0$ with

$$\mu(\sigma) \leq \begin{cases} 0 & \text{if } \sigma \geq 1/2; \\ 1/2 - \sigma & \text{if } \sigma < 1/2. \end{cases}$$

Then by Cauchy's integral theorem, for any $k \in \mathbb{N}$ we have

$$\zeta^{(k)}\left(\frac{1}{2} + it\right) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{\zeta(z)}{(z - 1/2 - it)^{k+1}} dz,$$

where $\gamma_r := \{z \in \mathbb{C} \mid |z - 1/2 - it| = r\}$. Taking $r = \epsilon/2$,

$$\begin{aligned} \left| \zeta^{(k)}\left(\frac{1}{2} + it\right) \right| &\ll_k \int_{\gamma_r} \frac{|\zeta(z)|}{|z - 1/2 - it|^{k+1}} |dz| \ll_{k,\epsilon} |t + \epsilon/2|^{\mu(1/2 - \epsilon/2) + \epsilon/2} \\ &\ll_{\epsilon} |t|^{\mu(1/2 - \epsilon/2) + \epsilon/2} \leq |t|^{1/2 - (1/2 - \epsilon/2) + \epsilon/2} = |t|^{\epsilon}. \end{aligned}$$

Now suppose that for some $k \in \mathbb{N}$,

$$\zeta^{(k)}\left(\frac{1}{2} + it\right) \ll_{k,\epsilon} |t|^{\epsilon}$$

holds for any $\epsilon > 0$. Then $\zeta^{(k)}(\sigma + it) \ll_{k,\epsilon} |t|^{\mu(\sigma)+\epsilon}$ for $\sigma \geq 1/2$. Note that

$$\left| \zeta^{(k-1)}\left(\frac{1}{2} + it\right) \right| \leq \left| \zeta^{(k-1)}(3 + it) \right| + \int_{1/2+it}^{3+it} \left| \zeta^{(k)}(z) \right| |dz| \ll_{k,\epsilon} |t|^\epsilon.$$

This implies

$$\left| \zeta\left(\frac{1}{2} + it\right) \right| \ll_\epsilon |t|^\epsilon.$$

□

We can then reformulate the Lindelöf hypothesis in terms of ergodic value distribution of $\zeta^{(k)}(s)$ on vertical lines under affine Boolean transformations as follows:

THEOREM 2.3. *Let k be a non-negative integer. The Lindelöf hypothesis is true if and only if, there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that for any $l \in \mathbb{N}$,*

$$(2.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \zeta^{(k)}(1/2 + iT_{\alpha,\beta}^n x) \right|^{2l}$$

exists for almost all $x \in \mathbb{R}$.

PROOF OF THEOREM 2.3. From Theorem 2.2, we can restate the Lindelöf hypothesis as

$$(2.3) \quad \zeta^{(k)}\left(\frac{1}{2} + it\right) \ll_{k,\epsilon} |t|^\epsilon \quad \text{as } |t| \rightarrow \infty$$

for any non-negative integer k . We then show that the hypothesis in the form (2.3) is equivalent to the existence of the limit in (2.2).

Replacing the function $\zeta(s)$ by $\zeta^{(k)}(s)$ in the proof of Theorem 4.1 in [Ste12], we can easily show the necessary condition for the Lindelöf hypothesis (in the form (2.3)).

To show the sufficient condition for the Lindelöf hypothesis, we note that

$$\zeta^{(k)}(s) = (-1)^{k-1} \int_1^\infty \frac{[x] - x + 1/2}{x^{s+1}} (\log x)^{k-1} (-s \log x + k) dx + \frac{(-1)^k k!}{(s-1)^{k+1}}$$

so that $|\zeta^{(k+1)}(1/2 + it)| < C_k |t|$ holds for any $|t| \geq 1$ for some $C_k > 0$ which may depend only on k . Further, for $\tau \geq 1$,

$$\frac{1}{\alpha^2 + (\tau - \beta)^2} = \frac{1}{\tau^2(1 + (\alpha/\tau)^2 + 2|\beta|/\tau + (\beta/\tau)^2)} \geq C_{\alpha,\beta} \frac{1}{\tau^2} \geq C_{\alpha,\beta} \frac{1}{1 + \tau^2}$$

for some $C_{\alpha,\beta} > 0$ that depends only on α and β . Then again we can replace the function $\zeta(s)$ by $\zeta^{(k)}(s)$ in the proof of Theorem 4.1 in [Ste12] to obtain the sufficient condition for the Lindelöf hypothesis (in the form (2.3)). This completes our proof of Theorem 2.3. □

EXAMPLE 2.4 (Dirichlet L -functions). Let $L(s, \chi)$ be the *Dirichlet L -function* associated with Dirichlet character χ .

(i) If χ is non-principal, for any $s \in \mathbb{H}_{-1/2}$, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} L^{(k)}(s + iT_{\alpha, \beta}^n x, \chi) &= \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{L^{(k)}(s + i\tau, \chi)}{\alpha^2 + (\tau - \beta)^2} d\tau \\ &= L^{(k)}(s + \alpha + i\beta, \chi) \end{aligned}$$

for almost all $x \in \mathbb{R}$.

(ii) If $\chi = \chi_0$ is principal, for any $s \in \mathbb{H}_{-1/2} \setminus \mathbb{L}_1$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} L^{(k)}(s + iT_{\alpha, \beta}^n x, \chi_0) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{L^{(k)}(s + i\tau, \chi_0)}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$. Denoting the right-hand side of the above formula by $l_{\alpha, \beta}^{(k)}(s, \chi_0)$, we have

$$l_{\alpha, \beta}^{(k)}(s, \chi_0) = \begin{cases} L^{(k)}(s + \alpha + i\beta, \chi_0) + \gamma_{-1}(\chi_0)A_k(s), & -1/2 < \operatorname{Re}(s) < 1, s \neq 1 - \alpha - i\beta; \\ \gamma_k(\chi_0) - \frac{k! \gamma_{-1}(\chi_0)}{(2\alpha)^{k+1}}, & -1/2 < \operatorname{Re}(s) < 1, s = 1 - \alpha - i\beta; \\ L^{(k)}(s + \alpha + i\beta, \chi_0), & \operatorname{Re}(s) > 1; \end{cases}$$

where $\gamma_{-1}(\chi_0)$, $\gamma_k(\chi_0)$'s are constants that depend only on χ_0 . They are coefficients of the Laurent expansion of $L^{(k)}(s, \chi_0)$ near $s = 1$. If $k = 0$, we can also show the result on \mathbb{L}_1 by setting

$$l_{\alpha, \beta}^{(0)}(1 + it, \chi_0) = L^{(0)}(1 + \alpha + i(t + \beta), \chi_0) - \frac{\alpha \gamma_{-1}(\chi_0)}{\alpha^2 + (t + \beta)^2}.$$

PROOF OF EXAMPLE 2.4. As in the proof of Example 2.1, for any non-negative integer k , $L^{(k)}(s, \chi)$ has an absolute convergent Dirichlet series expression when $\operatorname{Re}(s) > 1$. Referring to [Red82, Lemma 2], we know that $L^{(k)}(s, \chi)$ also satisfies an inequality similar to (2.1).

If χ is non-principal, $L^{(k)}(s, \chi)$ is entire for all $k \geq 0$. Thus $L^{(k)}(s, \chi)$ satisfies (1.4) of Theorem 1.1 for all $s \in \mathbb{H}_{-1/2}$.

Otherwise (that is, when $\chi = \chi_0$), $L^{(k)}(s, \chi_0)$ ($k \geq 1$) has a pole of order $k+1$ at $s = 1$. Hence we can also apply Theorem 1.1 with $c = -1/2$, $s_0 = 1$, and $m = k+1$ to $L^{(k)}(s, \chi_0)$ with the Laurent coefficients as discussed in [IK99, Theorem 2]. \square

EXAMPLE 2.5 (Dedekind ζ -functions). Let $\zeta_{\mathbb{K}}(s)$ be the *Dedekind ζ -function* of a number field \mathbb{K} over \mathbb{Q} of degree $d_{\mathbb{K}}$. Then for any $k \geq 0$ and $s \in \mathbb{H}_{1/2-1/d_{\mathbb{K}}} \setminus \mathbb{L}_1$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta_{\mathbb{K}}^{(k)}(s + iT_{\alpha, \beta}^n x) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta_{\mathbb{K}}^{(k)}(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$.

Denoting the right-hand side of the above formula by $l_{\mathbb{K}\alpha,\beta}^{(k)}(s)$, we have

$$l_{\mathbb{K}\alpha,\beta}^{(k)}(s) = \begin{cases} \zeta_{\mathbb{K}}^{(k)}(s + \alpha + i\beta) + \gamma_{-1}(\mathbb{K})A_k(s), & 1/2 - 1/d_{\mathbb{K}} < \operatorname{Re}(s) < 1, s \neq 1 - \alpha - i\beta; \\ k!\gamma_k(\mathbb{K}) - \frac{k!\gamma_{-1}(\mathbb{K})}{(2\alpha)^{k+1}}, & 1/2 - 1/d_{\mathbb{K}} < \operatorname{Re}(s) < 1, s = 1 - \alpha - i\beta; \\ \zeta_{\mathbb{K}}^{(k)}(s + \alpha + i\beta), & \operatorname{Re}(s) > 1; \end{cases}$$

where $\gamma_{-1}(\mathbb{K})$, $\gamma_k(\mathbb{K})$'s are constants that depend only on \mathbb{K} . They are coefficients of the Laurent expansion of $\zeta_{\mathbb{K}}^{(k)}(s)$ near $s = 1$. If $k = 0$, we can also show the result on \mathbb{L}_1 by setting

$$l_{\mathbb{K}\alpha,\beta}^{(0)}(1 + it) = \zeta_{\mathbb{K}}^{(0)}(1 + \alpha + i(t + \beta)) - \frac{\alpha\gamma_{-1}(\mathbb{K})}{\alpha^2 + (t + \beta)^2}.$$

PROOF OF EXAMPLE 2.5. We refer to [Ste03, Theorem 2] for the bound of the form (2.1) and to [HIKW04, pp. 496–497] for the Laurent coefficients of $\zeta_{\mathbb{K}}^{(k)}(s)$ near its pole at $s = 1$. The rest of the proof proceeds as in the proof of Example 2.1 with $c = 1/2 - 1/d_{\mathbb{K}}$, $s_0 = 1$, and $m = k + 1$. \square

REMARK 2.6. We can also show results analogous to Theorems 2.2 and 2.3 for Dirichlet L -functions associated with primitive Dirichlet characters and Dedekind zeta functions, if we formulate the *extended Lindelöf hypothesis* as:

$$\text{For any } \epsilon > 0, f\left(\frac{1}{2} + it\right) \ll_{f,\epsilon} |t|^\epsilon \quad \text{as } |t| \rightarrow \infty$$

for these functions (f is any of these ζ -functions and L -functions). We do not discuss this further but we remark that we can show these analogous results by using methods similar to the methods used in proving Theorems 2.2 and 2.3.

EXAMPLE 2.7 (Hurwitz zeta functions). For non-negative integer k , $0 < \mathbf{a} \leq 1$, and any s satisfying $\operatorname{Re}(s) > -1/2$ and $\operatorname{Re}(s) \neq 1$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{(k)}(s + iT_{\alpha,\beta}^n x, \mathbf{a}) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta^{(k)}(s + i\tau, \mathbf{a})}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all x in \mathbb{R} .

Denoting the right-hand side of the above formula by $l_{\alpha,\beta}^{(k)}(s, \mathbf{a})$, we have

$$l_{\alpha,\beta}^{(k)}(s, \mathbf{a}) = \begin{cases} \zeta^{(k)}(s + \alpha + i\beta, \mathbf{a}) + A_k(s), & -1/2 < \operatorname{Re}(s) < 1, s \neq 1 - \alpha - i\beta; \\ k!\gamma_k(\mathbf{a}) - \frac{k!}{(2\alpha)^{k+1}}, & -1/2 < \operatorname{Re}(s) < 1, s = 1 - \alpha - i\beta; \\ \zeta^{(k)}(s + \alpha + i\beta, \mathbf{a}), & \operatorname{Re}(s) > 1; \end{cases}$$

where

$$\gamma_k(\mathbf{a}) := \frac{(-1)^k}{k!} \lim_{N \rightarrow \infty} \left(\sum_{n=0}^N \frac{\log^k(n + \mathbf{a})}{n + \mathbf{a}} - \frac{\log^{k+1}(N + \mathbf{a})}{k + 1} \right)$$

is a coefficient of the Laurent expansion of $\zeta_{\mathbb{K}}^{(k)}(s)$ near $s = 1$. If $k = 0$, we can also show the result on \mathbb{L}_1 by setting

$$l_{\alpha,\beta}^{(0)}(1 + it, \mathbf{a}) = \zeta^{(0)}(1 + \alpha + i(t + \beta), \mathbf{a}) - \frac{\alpha}{\alpha^2 + (t + \beta)^2}.$$

PROOF OF EXAMPLE 2.7. The proof also follows that of Example 2.1 where we put $c = -1/2$, $s_0 = 1$, and $m = k + 1$. Here, we refer to [Red82, Lemma 2] for the bound of the form (2.1) and to [Ber72, Theorem 1] for the Laurent coefficients of $\zeta^{(k)}(s, \mathbf{a})$ near its pole at $s = 1$. \square

3. Affine Boolean transformations

In this section, we will show the ergodicity of $T_{\alpha,\beta}$ defined in (1.1) with respect to a proper measure. To state our main theorem, let us recall some basic notation. We denote by \mathcal{B} and ν the Borel σ -algebra on \mathbb{R} and the Lebesgue measure on \mathcal{B} . For a given $\alpha > 0, \beta \in \mathbb{R}$, let us define the function $\mu_{\alpha,\beta}$ by

$$\mu_{\alpha,\beta}(A) := \frac{\alpha}{\pi} \int_A \frac{d\tau}{\alpha^2 + (\tau - \beta)^2}$$

for any $A \in \mathcal{B}$. One can easily check that $\mu_{\alpha,\beta}$ is a probability on \mathcal{B} and

$$(3.1) \quad \mu_{\alpha,\beta}(A) = \frac{\alpha}{\pi} \int_A \frac{d\tau}{\alpha^2 + (\tau - \beta)^2} \leq \int_A \frac{d\tau}{\alpha\pi} = \frac{1}{\alpha\pi} \nu(A)$$

for any $A \in \mathcal{B}$. In particular, this implies that $\mu_{\alpha,\beta}(A) = 0$ if $\nu(A) = 0$.

THEOREM 3.1. *For given $\alpha > 0, \beta \in \mathbb{R}$, $T_{\alpha,\beta} : \mathbb{R} \rightarrow \mathbb{R}$ is measure preserving with respect to $\mu_{\alpha,\beta}$, that is, for any $A \in \mathcal{B}$, we have*

$$\mu_{\alpha,\beta}(T_{\alpha,\beta}^{-1}(A)) = \mu_{\alpha,\beta}(A).$$

Moreover, it is ergodic, that is, if $T_{\alpha,\beta}^{-1}(A) = A$, then either $\mu_{\alpha,\beta}(A) = 1$ or $\mu_{\alpha,\beta}(A) = 0$.

Applying Birkhoff's ergodic theorem, we have an ergodic mean-value of an integrable function. Let us denote by $T_{\alpha,\beta}^n$ the n -th iteration of $T_{\alpha,\beta}$, that is,

$$T_{\alpha,\beta}^n := \underbrace{T_{\alpha,\beta} \circ T_{\alpha,\beta} \circ \cdots \circ T_{\alpha,\beta}}_{n \text{ times}}.$$

COROLLARY 3.2. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is integrable with respect to $\mu_{\alpha,\beta}$, then*

$$(3.2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T_{\alpha,\beta}^n x = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(\tau) d\tau}{\alpha^2 + (\tau - \beta)^2}$$

for almost all $x \in \mathbb{R}$.

See [EW11, Theorem 2.30] for the proof of Birkhoff's ergodic theorem. Corollary 3.2 follows immediately from Birkhoff's ergodic theorem and Theorem 3.1.

Birkhoff's ergodic theorem describes the relation between the space average of a function and the time average along the orbit. In the next section, we will apply Corollary 3.2 to transform a mean-value of ergodic type into a computable integral.

In the rest of this section, we complete the proof of Theorem 3.1. We first recall the famous result given by R. Adler and B. Weiss.

LEMMA 3.3. *The Boolean transformation $T_{1,0}$ is measure preserving with respect to ν . Moreover, it is ergodic.*

See [AW73, Theorem and Main Theorem] for the proof of Lemma 3.3.

PROOF OF THEOREM 3.1. We first check that $T := T_{1,0}$ is measure preserving and ergodic with respect to $\mu := \mu_{1,0}$. Let us denote by χ_A the *indicator function* of $A \subset \mathbb{R}$. It follows from a simple calculation that

$$\begin{aligned}\mu(T^{-1}(A)) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_A(T(\tau)) d\tau}{1 + \tau^2} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_A(T(\tau)) dT(\tau)}{1 + T(\tau)^2} = \mu(A)\end{aligned}$$

for any $A \in \mathcal{B}$. Thus T is measure preserving with respect to μ . If $T^{-1}(A) = A$, it follows from Lemma 3.3 that either $\nu(A)$ or $\nu(X \setminus A)$ is 0. Hence, by (3.1), either $\mu(A)$ or $\mu(X \setminus A)$ must be 0.

Next, let us consider the general case. Defining the affine transformation $\phi_{\alpha,\beta} : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi_{\alpha,\beta}(x) := \alpha x + \beta$, we can easily check that

$$T_{\alpha,\beta} = \phi_{\alpha,\beta} \circ T \circ \phi_{\alpha,\beta}^{-1}$$

and

$$\mu_{\alpha,\beta}(A) = \mu(\phi_{\alpha,\beta}^{-1}(A)).$$

Since T is measure preserving with respect to μ , we have

$$\begin{aligned}\mu_{\alpha,\beta}(T_{\alpha,\beta}^{-1}(A)) &= \mu(\phi_{\alpha,\beta}^{-1}(T_{\alpha,\beta}^{-1}(A))) \\ &= \mu(\phi_{\alpha,\beta}^{-1}(\phi_{\alpha,\beta}(T^{-1}(\phi_{\alpha,\beta}^{-1}(A)))))) \\ &= \mu(T^{-1}(\phi_{\alpha,\beta}^{-1}(A))) \\ &= \mu(\phi_{\alpha,\beta}^{-1}(A)) = \mu_{\alpha,\beta}(A).\end{aligned}$$

Moreover, if $T_{\alpha,\beta}^{-1}(A) = A$, we have

$$T^{-1}(\phi_{\alpha,\beta}^{-1}(A)) = \phi_{\alpha,\beta}^{-1}(T_{\alpha,\beta}^{-1}(A)) = \phi_{\alpha,\beta}^{-1}(A).$$

Since T is ergodic with respect to μ , either $\mu_{\alpha,\beta}(A) = \mu(\phi_{\alpha,\beta}^{-1}(A))$ or $\mu_{\alpha,\beta}(X \setminus A) = \mu(X \setminus \phi_{\alpha,\beta}^{-1}(A))$ is 0. \square

4. Proof of the main theorem

PROOF OF THEOREM 1.1. It follows from Corollary 3.2 that (1.3) holds. For the case $m = 1$, we set the values of the integrand to be the principal value on the line \mathbb{L}_{σ_0} and since this is integrable, as we shall see below in Case 3 of the evaluation of $l_{\alpha,\beta}$, we can now apply Corollary 3.2 for all $s \in \mathbb{H}_c$. In the rest of this section, we evaluate $l_{\alpha,\beta}$ to complete the proof of Theorem 1.1.

Suppose that f has no pole in \mathbb{H}_c . The poles of the integrand in $l_{\alpha,\beta}$ in \mathbb{H}_c are coming only from the zeros of $\alpha^2 + (\tau - \beta)^2$. For any $s = \sigma + it \in \mathbb{H}_c$, we consider the counterclockwise oriented semicircle Γ_R of a sufficiently large radius $R > |s| + \alpha + |\beta|$ centered at the origin. Then applying Cauchy's integral theorem, we have

$$\begin{aligned}l_{\alpha,\beta}(s) &= \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau \\ &= \frac{\alpha}{\pi} \left(\lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{f(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau - 2\pi i \operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(s + i\tau)}{\alpha^2 + (\tau - \beta)^2} \right).\end{aligned}$$

Note that we can find a $\sigma' \in (c, \sigma)$ sufficiently near c . Setting $\epsilon = (\sigma' - c)/2$, we have

$$\begin{aligned}
\int_{\Gamma_R} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau &= \int_{\pi}^{2\pi} \frac{f(s+iRe^{i\theta})}{\alpha^2 + (Re^{i\theta} - \beta)^2} iRe^{i\theta} d\theta \\
&\ll_{\alpha, \beta} \frac{1}{R} \left(\int_{\pi}^{5\pi/4} + \int_{5\pi/4}^{7\pi/4} + \int_{7\pi/4}^{2\pi} \right) |f(s+iRe^{i\theta})| d\theta \\
&\ll \frac{1}{R} \left(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |f(\sigma - R \sin \theta + i(t + R \cos \theta))| \right. \\
&\quad \left. + \max_{\theta \in [5\pi/4, 7\pi/4]} |f(\sigma - R \sin \theta + i(t + R \cos \theta))| \right) \\
&\leq \frac{1}{R} \left(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |t + R \cos \theta|^{\nu(\sigma - R \sin \theta) + \epsilon} + M \right) \\
&\leq \frac{1}{R} \left(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |t + R \cos \theta|^{1+c-\sigma'+\epsilon} + M \right) \\
&\ll_{f, \epsilon} R^{c-\sigma'+\epsilon} \left(\frac{|t|}{R} + 1 \right)^{1+c-\sigma'+\epsilon} + \frac{M}{R},
\end{aligned}$$

thus the integral on Γ_R vanishes as R tends to ∞ . By simple calculations, we find that

$$\begin{aligned}
(4.1) \quad \operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} &= \lim_{\tau \rightarrow \beta-i\alpha} (\tau - \beta + i\alpha) \times \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} \\
&= \frac{f(s + \alpha + i\beta)}{-2\alpha i}.
\end{aligned}$$

Hence we obtain (1.4) for all $s \in \mathbb{H}_c$.

Suppose that f has a pole at $s = s_0 = \sigma_0 + it_0$ and $\sigma_0 > c$. Now for $s = \sigma + it \in \mathbb{H}_c$, the integrand has three simple poles: $\tau = \beta + i\alpha$, $\tau = \beta - i\alpha$, and $\tau = i(s - s_0)$. Here we divide the proof into three cases according to the condition whether the pole $\tau = i(s - s_0)$ is below ($c < \sigma < \sigma_0$), above ($\sigma > \sigma_0$), or on the real line ($\sigma = \sigma_0$) of the τ -plane.

Case 1: $\operatorname{Im}(i(s - s_0)) < 0$.

We first consider when $i(s - s_0) \neq \beta - i\alpha$ and let Γ_R be the counterclockwise oriented semicircle of a large radius $R > |s| + |s_0| + \alpha + |\beta|$ centered at the origin. Again by applying Cauchy's integral theorem, we can show that

$$\begin{aligned}
l_{\alpha, \beta}(s) &= \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} d\tau \\
&= -2\alpha i \left(\operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} + \operatorname{Res}_{\tau=i(s-s_0)} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2} \right).
\end{aligned}$$

Substituting (4.1) into the above, we obtain

$$l_{\alpha, \beta}(s) = f(s + \alpha + i\beta) - 2\alpha i \operatorname{Res}_{\tau=i(s-s_0)} \frac{f(s+i\tau)}{\alpha^2 + (\tau - \beta)^2}.$$

From the Laurent expansion (1.2) of f , we can calculate

$$\begin{aligned} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} &= \frac{i}{2\alpha} \left(\sum_{n=0}^{\infty} \frac{(\tau-i(s-s_0))^n}{(\beta+i\alpha-i(s-s_0))^{n+1}} - \sum_{n=0}^{\infty} \frac{(\tau-i(s-s_0))^n}{(\beta-i\alpha-i(s-s_0))^{n+1}} \right) \\ &\quad \times \sum_{n=-m}^{\infty} a_n i^n (\tau-i(s-s_0))^n. \end{aligned}$$

Thus

$$\begin{aligned} -2\alpha i \operatorname{Res}_{\tau=i(s-s_0)} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} \\ = \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta+i\alpha-i(s-s_0))^n} - \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta-i\alpha-i(s-s_0))^n}. \end{aligned}$$

Thus combining the above calculations and setting

$$B_m(s_0) := \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta+i\alpha-i(s-s_0))^n} - \sum_{n=1}^m \frac{a_{-n}}{i^n (\beta-i\alpha-i(s-s_0))^n},$$

we obtain

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau = f(s+\alpha+i\beta) + B_m(s_0).$$

This is the first equation of (1.5).

Now suppose that $i(s-s_0) = \beta - i\alpha$. This case only appears when $\sigma_0 - \alpha > c$. By calculations similar to the above, we have

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau = -2\alpha i \operatorname{Res}_{\substack{\tau=\beta-i\alpha \\ =i(s-s_0)}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}.$$

We consider the Laurent expansion of the integrand near $\tau = \beta - i\alpha = i(s-s_0)$:

$$\begin{aligned} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} &= \left(\sum_{n=-m}^{\infty} a_n i^n (\tau-\beta+i\alpha)^n \right) \\ &\quad \times \frac{1}{\tau-\beta+i\alpha} \times \frac{1}{-2\alpha i} \times \frac{1}{1-\frac{\tau-\beta+i\alpha}{2\alpha i}} \\ &= \frac{1}{-2\alpha i} \times \frac{1}{\tau-\beta+i\alpha} \times \frac{1}{1-\frac{\tau-\beta+i\alpha}{2\alpha i}} \sum_{n=-m}^{\infty} a_n i^n (\tau-\beta+i\alpha)^n \\ &= \frac{1}{-2\alpha i} \frac{1}{\tau-\beta+i\alpha} \sum_{n=0}^{\infty} \left(\frac{\tau-\beta+i\alpha}{2\alpha i} \right)^n \sum_{n=-m}^{\infty} a_n i^n (\tau-\beta+i\alpha)^n. \end{aligned}$$

Hence,

$$\operatorname{Res}_{\substack{\tau=\beta-i\alpha \\ =i(s-s_0)}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} = \frac{1}{-2\alpha i} \sum_{n=0}^m \frac{a_{-n}}{(-2\alpha)^n}$$

and so we obtain the second equation of (1.5).

Case 2: $\operatorname{Im}(i(s-s_0)) > 0$.

In this case, the integrand of $l_{\alpha,\beta}(s)$ has only one pole in the lower half-plane. Thus by a method similar to the case when f has no pole in \mathbb{H}_c , we can show that

$$\begin{aligned} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau &= -2\pi i \operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} \stackrel{(4.1)}{=} -2\pi i \times \frac{f(s+\alpha+i\beta)}{-2\alpha i} \\ &= \frac{\pi}{\alpha} f(s+\alpha+i\beta). \end{aligned}$$

Thus

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau = f(s+\alpha+i\beta).$$

This is the third equation of (1.5).

Case 3: $\operatorname{Im}(i(s-s_0)) = 0$ (only for the case $m = 1$).

Since $\operatorname{Im}(i(s-s_0)) = 0$ ($s_0 = \sigma_0 + it_0$), s satisfies $\operatorname{Re}(s) = \sigma_0$ in this case. For convenience, we write $s = \sigma_0 + it$. In this case, we take the principal value of the integrand as in [Ste12, p. 367] and so we obtain

$$(4.2) \quad \int_{\mathbb{R}} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau = \lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0^+}} \left(\int_{C_R} - \int_{C_\epsilon} \right) \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau - 2\pi i \operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2},$$

where C_R and C_ϵ are the counterclockwise oriented semicircles of radius R ($R > 1 + |s| + \alpha + |\beta|$) and ϵ centered at $\tau = t_0 - t$ located in the lower half of the τ -plane.

As the other cases, the integral along C_R vanishes as R tends to ∞ . On the other hand, the integral along C_ϵ is evaluated as

$$\begin{aligned} \int_{C_\epsilon} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau &= \int_{\pi}^{2\pi} \frac{f(\sigma_0 + i(t_0 + \epsilon e^{i\theta}))}{\alpha^2 + (t_0 - t + \epsilon e^{i\theta} - \beta)^2} i\epsilon e^{i\theta} d\theta \\ &= \int_{\pi}^{2\pi} \left(\frac{a_{-1}}{i\epsilon e^{i\theta}} + O(1) \right) \frac{i\epsilon e^{i\theta}}{\alpha^2 + (t_0 - t + \epsilon e^{i\theta} - \beta)^2} d\theta \end{aligned}$$

hence

$$\lim_{\epsilon \rightarrow 0^+} \int_{C_\epsilon} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau = \frac{a_{-1}\pi}{\alpha^2 + (t_0 - t - \beta)^2}.$$

Again from (4.1),

$$\operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} = \frac{f(\sigma_0 + \alpha + i(t+\beta))}{-2\alpha i}.$$

These imply that (1.6) holds. \square

Remark that the method used in Case 3 in the proof does not work if $m > 1$.

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