Geometry of hypersurfaces with zero or pinched anisotropic mean curvature in Euclidean space

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Geometry of hypersurfaces with zero or pinched anisotropic mean curvature in Euclidean space

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Contents

1 Introduction

Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be an isometrically immersed hypersurface in Euclidean space. One of the most fundamental geometric quantity of Σ is the mean curvature which is given by the trace of the second fundamental form. In physical theoretic view point, the mean curvature is also important since it arises from the first variation of the volume of Σ. For example, surfaces with constant mean curvature describe the shapes of interfaces between two non mixing media such as soap films or bubbles. It is well known that a constraint on the mean curvature restricts the geometric invariants of a hypersurface such as the volume, the intrinsic or extrinsic diameter, or the first eigenvalue of the Laplacian (see [2, 3, 5, 14, 20, 27] for instance).

On the other hand, over the past years, many authors considered the geometric problems involving the anisotropic mean curvature (see [8, 15, 17, 18, 19, 24, 28] for instance). The setting is given as follows. Let $\gamma : \mathbb{S}^n \to \mathbb{R}_{>0}$ be a smooth, positive function satisfying the convexity condition

$$A_\gamma = (\mathrm{Hess}_\gamma^{\mathbb{S}^n} + \gamma I)_\nu > 0,$$

for any $\nu \in \mathbb{S}^n$. Here, I denotes the identity operator on $T_\nu \mathbb{S}^n$ and > 0 means the positivity of self-adjoint operators. The anisotropic mean curvature of Σ is defined as the trace of the composition of tensors $A_\gamma \circ S$, where S denotes the shape operator of Σ. As in the isotropic setting, the anisotropic mean curvature can be derived from the variation of the anisotropic surface energy $\mathcal{F}_\gamma(\Sigma) = \int_\Sigma \gamma(N) d\Sigma$, where N is the unit normal vector field on Σ. The anisotropic surface energy is considered as the free energy of surfaces of materials which have anisotropy such as crystals or nematic liquid crystals. For a function γ satisfying $A_\gamma > 0$, we can consider a unique convex set W_γ, called the Wulff shape of γ, the supporting function of which is γ.

In the recent years, lots of classical results for hypersurfaces with mean curvature constraints have been generalized to the anisotropic case. For instance, Chodosh and Li [8] considered the Bernstein problem for anisotropic minimal hypersurfaces in $\mathbb{R}^4$ and they showed that complete stable minimal hypersurface must be flat when γ is close to 1 in C^2 sense. In the case of closed hypersurfaces, Scheuer and Zhang [28] studied the quantitative stability of Wulff shape for the anisotropic Heintze–Karcher inequality and the anisotropic Alexandrov theorem.

The aim of this article is to obtain some anisotropic generalizations to the classical results on hypersurfaces in Euclidean space.

In Section 2, we consider the Morse index of anisotropic minimal surfaces. The Morse index is defined through the second variation of given functional and measures how unstable a solution to the considering variational problem is. For minimal surfaces, the Gauss map is used to relate the Morse index to the genus (see [3, 21, 30] for instance).

In this section, we study the behavior of the Gauss map of anisotropic minimal surfaces with convex integrand γ and apply it to obtain a Tysk [30] type upper bound and a

Nayatani [21] type lower bound of the Morse index. As an application of the latter lower estimate of the Morse index, we also prove the instability of anisotropic minimal surfaces with low genus.

In Section 3, we prove the Hasanis–Koutroufiotis [13] type inequality for the anisotropic extrinsic radius of hypersurfaces in Euclidean space involving the anisotropic mean curvatures. We also study the equality case and proved that an almost extremal hypersurface must be close to the Wulff shape in the sense of the Hausdorff distance.

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2 On the Gauss map of anisotropic minimal surfaces and applications to the Morse index estimates

2.1 Introduction

Let $\gamma : \mathbb{S}^2 \to \mathbb{R}_{>0}$ be a smooth positive function on the unit sphere. For an isometric immersion $\Sigma \to \mathbb{R}^3$ from an oriented surface, the *anisotropic surface energy* $\mathcal{F}_\gamma(\Sigma)$ is defined by

$$\mathcal{F}_\gamma(\Sigma) = \int_\Sigma \gamma(N) d\Sigma, \tag{2.1}$$

where N is the unit normal vector field along Σ and $d\Sigma$ is the area element of Σ.

A surface Σ is called a γ-*anisotropic minimal surface* if it satisfies $\frac{d}{dt}\big|_{t=0} \mathcal{F}_\gamma(\Sigma_t) = 0$ for all compactly supported variations. We note that if we let $\gamma = 1$ in (2.1), then $\mathcal{F}_\gamma$ coincides with the area functional. In this sense, we may consider anisotropic minimal surfaces as a generalization of minimal surfaces.

The stability and the instability of solutions to the variational problem of given functional is an important problem in differential geometry. For minimal surfaces, the Gauss map is a fundamental tool in the study of the stability. Barbosa and Do Carmo [3] gave a sufficient condition for stability of subdomains of minimal surfaces by studying the image of Gauss map. Tysk [30] studied the spectrum of the Laplacian of covering manifolds and applied it to the Gauss map of a minimal surface to get an upper bound of the Morse index in terms of the total curvature. Nayatani [21] gave a lower bound for the Morse index by considering the pre-image of a great circle in the unit sphere via the Gauss map.

In this section, we study the behavior of the Gauss map of anisotropic minimal surfaces with convex integrand γ and apply it to obtain an estimate of the Morse index. A positive function γ on the unit sphere is called *convex* if the second derivative $A_\gamma = (\nabla^{\mathbb{S}^2})^2\gamma + \gamma I$ of γ is positively definite. When γ is convex, the Euler–Lagrange equation of the variation of $\mathcal{F}_\gamma$ becomes an elliptic equation. Moreover, the Jacobi operator, which is defined through the second variation of $\mathcal{F}_\gamma$, becomes a second order linear elliptic operator and its Morse index can be well-defined. For the behavior of the Gauss map, we obtain the following result.

Theorem 2.1. *Let $\Sigma \subset \mathbb{R}^3$ be a γ-anisotropic minimal surface. Assume γ is convex. Then the critical set of the Gauss map of Σ has no accumulation points.*

This result is a generalization of the well known fact for minimal surfaces. The discreteness of the critical set of the Gauss map allows us to consider the Gauss map as a branched covering map onto the unit sphere. In the anisotropic case, we define the *anisotropic Gauss map* N_γ by the composition of the Gauss map $N : \Sigma \to \mathbb{S}^2$ and $\xi : \mathbb{S}^2 \to \mathbb{R}^3$ defined by $\xi(\nu) = \nabla^{\mathbb{S}^2}\gamma(\nu) + \gamma(\nu)\nu$ for $\nu \in \mathbb{S}^2$. Applying an argument in [30] to the anisotropic Gauss map, we obtain the following index upper bound.

Theorem 2.2. *Let $\Sigma \subset \mathbb{R}^3$ be a γ-anisotropic minimal surface with convex integrand γ. Then there exists a positive constant $C(\gamma)$ which depends only on γ such that*

$$\mathrm{Ind}(\Sigma) \leq C(\gamma)\deg(N_\gamma),$$

where $\deg(N_\gamma)$ is the mapping degree of the anisotropic Gauss map N_γ on Σ.

Another application of Theorem2.1 is the following generalization of Nayatani's Morse index lower bound in [21].

Theorem 2.3. *Let $\Sigma \subset \mathbb{R}^3$ be a γ-anisotropic minimal surface with convex integrand γ. Let S be a great circle in the unit sphere $\mathbb{S}^2$. Then, the Morse index $\mathrm{Ind}(\Sigma)$ of Σ satisfies*

$$\mathrm{Ind}(\Sigma) \geq b(N, N^{-1}(S)) + 1 - 2g,$$

where $b(N, N^{-1}(S))$ is the branching order of the Gauss map N with respect to $N^{-1}(S)$ and g is the genus of Σ.

2.2 Geometry of anisotropic minimal surfaces

In this section, we prepare some basic properties of γ-anisotropic minimal surfaces. We refer to [17] and [18]. Let $\Sigma \to \mathbb{R}^3$ be an immersion of an oriented surface. Let $\gamma : \mathbb{S}^2 \to \mathbb{R}_{>0}$ be a smooth positive function on the unit sphere. For any variation Σ_t of Σ, the first variation of the anisotropic surface energy $\mathcal{F}_\gamma$ defined by (2.1) is given by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_\gamma(\Sigma_t) = -\int_\Sigma H_\gamma u d\Sigma, \tag{2.2}$$

where u is the normal component of given variation and H_γ is the *anisotropic mean curvature.* As a consequence of (2.2), we see that Σ is a critical point of the functional $\mathcal{F}_\gamma$ if and only if its anisotropic mean curvature vanishes identically.

The integrand γ is called *convex* or *elliptic* if the hessian $A_\gamma = (\nabla^{\mathbb{S}^2})^2\gamma + \gamma I$ of γ (extended 1-homogeneously) satisfies $A_\gamma > 0$. Geometrically speaking, the convexity of γ implies that γ is the support function of the convex body

$$\bigcap_{\nu \in \mathbb{S}^2} \{x \in \mathbb{R}^3 | \langle x, \nu\rangle \leq \gamma(\nu)\}. \tag{2.3}$$

The boundary W_γ of the convex set (2.3) is called the *Wulff shape* of γ. We note that if γ is convex, the Wulff shape W_γ has a parametrization

$$W_\gamma = \xi(\mathbb{S}^2),$$

where ξ is the map, which is called the *Cahn–Hoffman map for* γ (see [17] for details), defined by $\xi(\nu) = \nabla^{\mathbb{S}^2}\gamma(\nu) + \gamma(\nu)\nu$ from $\mathbb{S}^2$ to W_γ. The following fact is a fundamental property for the Cahn–Hoffman map (see [29]).

Lemma 2.1. *Let $\gamma : \mathbb{S}^2 \to \mathbb{R}_{>0}$ be a smooth positive function on $\mathbb{S}^2$. Then the Cahn–Hoffman map ξ for γ satisfies $\langle d\xi_\nu(X), \nu\rangle = 0$ for any $\nu \in \mathbb{S}^2$ and $X \in T_\nu\mathbb{S}^2$.*

By Lemma 2.1, the tangent planes $T_\nu\mathbb{S}^2$ at $\nu \in \mathbb{S}^2$ and $T_{\xi(\nu)}W_\gamma$ at $\xi(\nu) \in W_\gamma$ are parallel. In particular, we may consider the differential $d\xi = A_\gamma$ at $\nu \in \mathbb{S}^2$ of the Cahn–Hoffman map ξ as an endomorphism on $T_\nu\mathbb{S}^2$.

For a surface Σ, the map $N_\gamma = \xi \circ \nu$ from Σ to W_γ is called the *anisotropic Gauss map*. Since the planes $T_p\Sigma$ at $p \in \Sigma$ and $T_{N_\gamma(p)}W_\gamma$ at $N_\gamma(p) \in W_\gamma$ are parallel, we may consider the differential $dN_\gamma = -A_\gamma S$ as an endomorphism on $T_p\Sigma$. Here, S denotes the usual shape operator of Σ. Hence, the anisotropic mean curvature can be written as

$$H_\gamma = \mathrm{tr}(-dN_\gamma) = \mathrm{tr}(A_\gamma S). \tag{2.4}$$

Let us assume that Σ is a γ-anisotropic minimal surface with convex integrand γ. We give a local expression of (2.4). Given $p \in \Sigma$, choose a frame $\{e_i\}_i$ diagonalizing the second fundamental form A and let $\{\kappa_i\}_i$ be the principal curvatures. We set $a_i = \langle A_\gamma e_i, e_i\rangle$. Note that a_1 and a_2 are positive by the convexity of γ. Under these notations, we have

$$H_\gamma = a_1\kappa_1 + a_2\kappa_2. \tag{2.5}$$

Using (2.5), the Gaussian curvature K^Σ of Σ is given by

$$K^\Sigma = \kappa_1\kappa_2 = -\frac{a_1}{a_2}{\kappa_1}^2 \le 0. \tag{2.6}$$

Here, we used the convexity of γ. Therefore, a γ-anisotropic minimal surface has nonpositive Gaussian curvature.

Before showing Theorem 2.1, we consider a graphical surface. Let $u = u(x, y)$ be a smooth function on a domain $\Omega \subset \mathbb{R}^2$. Since the unit normal N of Graph_u is given by

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}},$$

the anisotropic energy $\mathcal{F}_\gamma$ of Graph_u becomes

$$\mathcal{F}_\gamma(\mathrm{Graph}_u) = \int_\Omega \gamma(N)\sqrt{1 + |\nabla u|^2}\, dxdy = \int_\Omega \bar{\gamma}(-u_x, -u_y, 1)\, dxdy. \tag{2.7}$$

Taking a variation of (2.7), we obtain the γ-anisotropic minimal surface equation:

$$\sum_{i,j=1}^{2} \bar{\gamma}_{ij}(-\nabla u, 1)u_{ij} = 0. \tag{2.8}$$

We next show a curvature estimate for graphical surfaces. Using (x, y) as local coordinates on Graph_u, the induced metric g and its inverse g^{-1} can be written as

$$g = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix}, \quad g^{-1} = \frac{1}{1+|\nabla u|^2} \begin{pmatrix} 1+u_y^2 & -u_x u_y \\ -u_x u_y & 1+u_x^2 \end{pmatrix}.$$

Since the eigenvalues of g are 1 and $1+|\nabla u|^2$, g^{-1} satisfies the following inequality as a symmetric matrix

$$\frac{1}{1+|\nabla u|^2} I \leq g^{-1} \leq I. \tag{2.9}$$

Similarly, the second fundamental form A in the coordinate (x, y) is expressed as

$$A = \frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}} \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}.$$

The norm squared of A is given by $|A|^2 = A_{ij} A_{kl}\, g^{ik} g^{jl}$. Hence the matrix inequality (2.9) yields the following curvature estimate:

$$\frac{|\text{Hess}_u|^2}{(1+|\nabla u|^2)^3} \leq |A|^2 \leq \frac{|\text{Hess}_u|^2}{1+|\nabla u|^2}. \tag{2.10}$$

Finally, we recall the following fact from [9].

Lemma 2.2. *Let $\Sigma \subset \mathbb{R}^3$ be an immersed surface with*

$$16 s^2 \sup_{\Sigma} |A| \leq 1$$

for some $s > 0$. If $x_0 \in \Sigma$ and $\text{dist}^{\Sigma}(x_0, \partial\Sigma) \geq 2s$, then $B^{\Sigma}_{2s}(x_0)$ can be written as a graph of a functtion u over a domain of $T_{x_0}\Sigma$ with $|\nabla u| \leq 1$ and $\sqrt{2}|\text{Hess}_u| \leq 1$.

Proof of Therorem2.1. Let $p \in \Sigma$ be a point in Σ such that $K^{\Sigma}(p) = 0$. Let κ_i be the i-th principal curvature of Σ at p. It follows from (2.6) that

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = \kappa_1^2 + \frac{a_1^2}{a_2^2}\kappa_1^2 = \left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) \frac{a_1}{a_2}\kappa_1^2 = \left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right)(-K^{\Sigma}),$$

so $|A|^2(p) = 0$.

Up to translation and rotation, we may assume that $p = 0$, $T_p\Sigma = \{x_3 = 0\}$, and $N(p) = (0, 0, 1)$.

Since $|A|^2(0) = 0$, there exists a $\delta > 0$ such that $|A|^2 < 1$ on $B^{\Sigma}_{\delta}(0)$. Rescaling the canonical metric on $\mathbb{R}^3$ by factor $\lambda > 0$, we have

$$|\tilde{A}|^2 < \frac{1}{\lambda^2} \quad \text{on } B^{\Sigma}_{\lambda\delta}(0),$$

where $\tilde{A}$ denotes the second fundamental form of Σ given by the rescaled metric. Choosing $\lambda > 0$ large, we may assume that

$$|A|^2 \le \frac{1}{16} \quad \text{on } B_4^\Sigma(0).$$

By Lemma 2.2 the geodesic disk $B_2^\Sigma(0)$ can be written as a graph of a function u over a domain in $\mathbb{R}^2 = T_p\Sigma$ with $|\nabla u| \le 1$ and $|\mathrm{Hess}_u| \le \frac{1}{\sqrt{2}}$.

Since Σ is γ-minimal, u satisfies a quasilinear elliptic equation (2.8) with bounded coefficients. Up to linear transform of local coordinate, we may assume that $\bar{\gamma}_{ij}(-\nabla u(0), 1) = \delta_{ij}$. In this coordinate, u satisfies $\Delta u = 0$ at p.

By the theorem of Bers [4], there exists a homogeneous polynomial P of degree n and a constant $\varepsilon \in (0,1)$ such that

$$\begin{aligned} u(x) &= P(x) + \mathcal{O}(|x|^{n+\varepsilon}), \\ u_i(x) &= P_i(x) + \mathcal{O}(|x|^{n-1+\varepsilon}), \quad i = 1,2, \\ u_{ij}(x) &= P_{ij}(x) + \mathcal{O}(|x|^{n-2+\varepsilon}), \quad 1 \le i,j \le 2, \end{aligned}$$

and

$$\Delta P = 0 \tag{2.11}$$

hold around 0. Since $u(0) = 0$ and $\nabla u(0) = 0$, it follows that $n \ge 2$.

Moreover, by (2.10), we have

$$|\mathrm{Hess}_u|^2 \le (1 + |\nabla u|^2)^3 |A|^2 \le 8|A|^2,$$

which implies $u_{ij}(0) = 0$ and hence $n \ge 3$.

We now introduce the complex coordinate z on $\mathbb{R}^2 = T_p\Sigma$ by $z = x_1 + \sqrt{-1}x_2$. The harmonicity (2.11) of P yields $P_{z\bar{z}} = 0$. Hence, the function P_z is holomorphic.

Since P is a polynomial of degree $n \ge 3$ and P_z is holomorphic, there exists a holomorphic function G around 0 such that

$$P_{zz}(z) = z^{n-2}\, G(z) \quad \text{and} \quad G(0) \ne 0.$$

Assume that there exists a sequence $\{z_k\}_k$ such that $z_k \to 0$ and $|A|(z_k) = 0$. Then $u_{ij}(z_k) = 0$ for each k, and hence $u_{zz}(z_k) = 0$.

On the other hand, we have

$$\begin{aligned} u_{zz} &= P_{zz} + \mathcal{O}(|z|^{n-2+\varepsilon}) \\ &= z^{n-2}\,(G(z) + \mathcal{O}(|z|^\varepsilon)). \end{aligned}$$

For each z_k, we get

$$0 = u_{zz}(z_k) = z_k^{n-2}\,(G(z_k) + \mathcal{O}(|z_k|^\varepsilon)).$$

Since $n \ge 3$, this implies $G(z_k) = \mathcal{O}(|z_k|^\varepsilon)$ which contradicts the fact that $G(0) \ne 0$.

Therefore, the set $\{p \in \Sigma | K^\Sigma(p) = 0\}$ has no accumulation points, which proves the theorem. □

2.3 Index upper bound for anisotropic minimal surfaces

Let Σ be a complete, oriented γ-anisotropic minimal surface. Recall that the second variation formula for the anisotropic surface energy $\mathcal{F}_\gamma$ is given by

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{F}_\gamma = -\int_\Sigma uLud\Sigma,$$

where L is the *Jacobi operator* given by $L = \operatorname{div}^\Sigma(A_\gamma\nabla^\Sigma) + \operatorname{tr}(A_\gamma S^2)$. For relatively compact domain $\Omega \subset \Sigma$, we define $\operatorname{Ind}(\Omega)$ as the number of negative eigenvalues (counted with multiplicity) of the Dirichlet eigenvalue problem

$$Lu + \lambda u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Note that the number $\operatorname{Ind}(\Omega)$ is always finite by the theory of elliptic operators. The *Morse index* $\operatorname{Ind}(\Sigma)$ of Σ is defined as the supremum of the numbers $\operatorname{Ind}(\Omega)$ over all relatively compact domains in Σ. The associated quadratic form to L is given by

$$Q(u) = \int_\Sigma (\langle A_\gamma\nabla^\Sigma u, \nabla^\Sigma u\rangle - \operatorname{tr}(A_\gamma S^2)u^2)d\Sigma.$$

We set

$$\lambda_\gamma = \min_{\nu\in\mathbb{S}^2}\min_{v\in T_\nu\mathbb{S}^2,|v|=1}\langle A_\gamma v, v\rangle, \quad \Lambda_\gamma = \max_{\nu\in\mathbb{S}^2}\max_{v\in T_\nu\mathbb{S}^2,|v|=1}\langle A_\gamma v, v\rangle.$$

By the convexity of γ, we have

$$0 < \lambda_\gamma I \leq A_\gamma \leq \Lambda_\gamma I, \tag{2.12}$$

where I denotes the identity matrix.

To estimate the quadratic form, we define the *anisotropic Gaussian curvature* K_γ by

$$K_\gamma = \det(-dN_\gamma) = \det(A_\gamma S) = \det(A_\gamma)K^\Sigma. \tag{2.13}$$

Lemma 2.3. *Let Σ be a γ-anisotropic minimal surface with convex integrand γ. We have*

$$-\frac{2}{\Lambda_\gamma}K_\gamma \leq \operatorname{tr}(A_\gamma S^2) \leq -\frac{2}{\lambda_\gamma}K_\gamma.$$

Proof. Let κ_i be the principal curvature. Using the identity $a_1\kappa_1 + a_2\kappa_2 = 0$, we have

$$\operatorname{tr}(A_\gamma S^2) = a_1{\kappa_1}^2 + a_2{\kappa_2}^2 = -a_2\kappa_1\kappa_2 - a_1\kappa_1\kappa_2 = -(a_1+a_2)K^\Sigma = -\operatorname{tr}(A_\gamma)K^\Sigma = -\frac{\operatorname{tr}(A_\gamma)}{\det A_\gamma}K_\gamma.$$

If we let μ_1 and μ_2 be the eigenvalues of A_γ, we have

$$\frac{\operatorname{tr}(A_\gamma)}{\det A_\gamma} = \frac{\mu_1 + \mu_2}{\mu_1\mu_2} = \frac{1}{\mu_1} + \frac{1}{\mu_2}.$$

Moreover, since (2.12) gives $\lambda_\gamma \leq \mu_1, \mu_2 \leq \Lambda_\gamma$, we obtain

$$\frac{2}{\Lambda_\gamma} \leq \frac{\mathrm{tr}(A_\gamma)}{\det A_\gamma} \leq \frac{2}{\lambda_\gamma}.$$

□

By Lemma 2.3, the quadratic form Q can be estimated below by

$$\begin{aligned} Q(u) &= \int_\Sigma (\langle A_\gamma \nabla^\Sigma u, \nabla^\Sigma u\rangle - \mathrm{tr}(A_\gamma S^2)u^2) d\Sigma \\ &\geq \int_\Sigma \left(\lambda_\gamma |\nabla^\Sigma u|^2 + \frac{2}{\lambda_\gamma} K_\gamma u^2 \right) d\Sigma \\ &= \lambda_\gamma Q_\gamma(u), \end{aligned} \tag{2.14}$$

where we defined $Q_\gamma(u) = \int_\Sigma \left(|\nabla^\Sigma u|^2 + \frac{2}{{\lambda_\gamma}^2} K_\gamma u^2 \right) d\Sigma$, which is the quadratic form corresponding to the elliptic operator $L_\gamma = \Delta^\Sigma - \frac{2}{{\lambda_\gamma}^2} K_\gamma$. By (2.14), if u satisfies $Q(u) < 0$, it also follows that $Q_\gamma(u) < 0$. This implies that the Morse index can be bounded from above by the number of negative eigenvalues $\mathrm{Neg}(L_\gamma)$ of L_γ.

We now assume that Σ has finite total curvature. Since the Gaussian curvature of γ-anisotropic minimal surface is nonpositive by (2.6), the theorem of Osserman [23, Theorem 9.1] asserts that there exists a compact surface $\tilde{\Sigma}$ such that Σ is conformally diffeomorphic to $\tilde{\Sigma} \setminus \{p_1, \cdots p_k\}$. By the theorem of White [31], the Gauss map N can be extended continuously to $\tilde{\Sigma}$. Thus, it follows from Theorem2.1 that the set $C = \{p \in \tilde{\Sigma} | K^\Sigma(p) = 0\}$ is a finite set. Since C is the critical set of N, N is locally diffeomorphic outside of C. Letting $B = N(C) \subset \mathbb{S}^2$, we obtain the following:

Lemma 2.4. *The restricted Gauss map $N : \tilde{\Sigma} \setminus N^{-1}(B) \to \mathbb{S}^2 \setminus B$ is a covering map. Moreover, $N : \tilde{\Sigma} \to \mathbb{S}^2$ is surjective.*

Since γ is convex, (2.13) asserts that the critical set of N_γ coincides with C. Hence, the anisotropic Gauss map $N_\gamma : \tilde{\Sigma} \setminus N^{-1}(B) \to W_\gamma \setminus \xi(B)$ is also a covering map onto the Wulff shape W_γ.

Now, let h be the metric on W_γ induced from the immersion $W_\gamma \to \mathbb{R}^3$. In the case of minimal surfaces, the pullback metric from $\mathbb{S}^2$ via the Gauss map is conformal to the original one. Since the index form for the Jacobi operator on a minimal surface is invariant under conformal changes, we may relate the Morse index with the eigenvalue problem of the Laplacian on $\mathbb{S}^2$. Motivated by these facts, we consider the energy density of given function with respect to the pullback metric $N_\gamma^* h$ on Σ.

Lemma 2.5 (cf. Lemma 3.10 in [3])**.** *Let Σ be a γ-anisotropic minimal surface with convex integrand γ. Assume that p is a point at which N_γ is regular. Then there exists a constant $c(\gamma) > 0$ which depends only on γ such that it follows for any smooth function u on Σ that*

$$-c(\gamma)^{-1} \frac{1}{K_\gamma} |du|^2(p) \leq |du|^2_{N_\gamma^* h}(p) \leq -c(\gamma) \frac{1}{K_\gamma} |du|^2(p).$$

Proof. Let $\{e_i\}_i$ be an orthonormal frame around p. Since the tangent planes $T_p\Sigma$ and $T_{N_\gamma(p)}W_\gamma$ are parallel, we may consider $\{e_i(p)\}_i$ as an orthonormal basis of $T_{N_\gamma(p)}W_\gamma$. In this basis, the metric $N_\gamma^* h$ is expressed as

$$(N_\gamma^* h)_{ij} = \langle dN_\gamma(e_i), dN_\gamma(e_j)\rangle = \langle A_\gamma S e_i, A_\gamma S e_j\rangle = (SA_\gamma^2 S)_{ij}.$$

To obtain the last equality, we used the symmetry of A_γ and S.

Note that the matrix $SA_\gamma^2 S$ is symmetric. Choose $\{e_i(p)\}_i$ so that it diagonalizes $SA_\gamma^2 S$ and let λ_1 and λ_2 be the corresponding eigenvalues. Since N_γ is regular at p, the matrix $SA_\gamma^2 S$ is invertible. Hence, we have

$$|du|^2_{N_\gamma^* h}(p) = \langle (SA_\gamma^2 S)^{-1} du, du\rangle = {\lambda_1}^{-1} u_1^2 + {\lambda_2}^{-1} u_2^2.$$

This equality yields

$$\frac{1}{\mathrm{tr}(SA_\gamma^2 S)}|du|^2 = \frac{1}{\lambda_1 + \lambda_2}|du|^2 \le |du|^2_{N_\gamma^* h}(p) \le \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}|du|^2 = \frac{\mathrm{tr}(SA_\gamma^2 S)}{\det(SA_\gamma^2 S)}|du|^2. \quad (2.15)$$

Since A_γ^2 satisfies $\lambda_\gamma^2 I \le A_\gamma^2 \le \Lambda_\gamma^2 I$, we have

$$\langle SA_\gamma^2 S v, v\rangle = \langle A_\gamma^2 S v, S v\rangle \le \Lambda_\gamma^2 \langle S^2 v, v\rangle,$$

for any vector $v \in T_p\Sigma$. Hence,

$$\begin{aligned} \mathrm{tr}(SA_\gamma^2 S) &\le \Lambda_\gamma^2 \mathrm{tr}(S^2) = \Lambda_\gamma |A|^2 = -\Lambda_\gamma^2 \left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) K^\Sigma \\ &\le -\Lambda_\gamma^2 \left(\frac{\Lambda_\gamma}{\lambda_\gamma} + \frac{\lambda_\gamma}{\Lambda_\gamma}\right) K^\Sigma \\ &\le -\left(\frac{\Lambda_\gamma}{\lambda_\gamma}\right)^2 \left(\frac{\Lambda_\gamma}{\lambda_\gamma} + \frac{\lambda_\gamma}{\Lambda_\gamma}\right) K_\gamma \\ &= -c(\gamma) K_\gamma, \end{aligned}$$

by (2.13). Combining this with (2.15), we obtain

$$-c(\gamma)^{-1} \frac{1}{K_\gamma}|du|^2 \le |du|^2_{N_\gamma^* h} \le -c(\gamma)\frac{1}{K_\gamma}|du|^2.$$

□

Before the proof of the index upper bound, we prepare the following notation. For any second order linear differential operator A and any subdomain Ω of Σ, let $N_\lambda(A, \Omega)$ be the number of the eigenvalues of A less than λ on Ω.

Proof of Theorem 2.2. If the degree of N_γ is infinity, there are nothing to prove. Assume that the degree of N_γ is finite.

Since
$$-\lambda_\gamma^2 \int_\Sigma K^\Sigma d\Sigma \le \deg(N_\gamma) \le -\Lambda_\gamma^2 \int_\Sigma K^\Sigma d\Sigma,$$
by (2.12), the finiteness of $\deg(N_\gamma)$ implies the finiteness of total curvature. Thus, it follows from Lemma 2.4 that N_γ is a branched covering map from Σ to W_γ.

For each $\varepsilon > 0$, let Ω_ε be the subset of Σ subtracting the ε-neighborhood of $N^{-1}(B)$. By Lemma 2.5, we have
$$\begin{aligned} Q_\gamma(u) &= \int_\Sigma \left(|du|^2 + \frac{2}{{\lambda_\gamma}^2} K_\gamma u^2 \right) d\Sigma \\ &\ge \int_\Sigma \left(-c(\gamma)^{-1} K_\gamma |du|^2_{N_\gamma^* h} + \frac{2}{{\lambda_\gamma}^2} K_\gamma u^2 \right) d\Sigma \\ &= c(\gamma)^{-1} \int_\Sigma \left(|du|^2_{N_\gamma^* h} - \frac{2c(\gamma)}{{\lambda_\gamma}^2} u^2 \right) (-K_\gamma) d\Sigma \\ &= c(\gamma)^{-1} \int_\Sigma \left(|du|^2_{N_\gamma^* h} - \frac{2c(\gamma)}{{\lambda_\gamma}^2} u^2 \right) d(N_\gamma^* h) \end{aligned}$$
for any function u with compact support in Ω_ε. This implies that
$$N_0(L_\gamma, \Omega_\varepsilon) \le N_{c'(\gamma)}(\Delta^{N_\gamma^* h}, \Omega_\varepsilon),$$
where $c'(\gamma) = 2c(\gamma)\lambda_\gamma^{-2}$ and $\Delta^{N_\gamma^* h}$ is the Laplacian with respect to $N_\gamma^* h$. Letting $\varepsilon \to 0$, we obtain
$$\mathrm{Ind}(\Sigma) \le N_0(L_\gamma, \Sigma) \le N_{c'(\gamma)}(\Delta^{N_\gamma^* h}.\Sigma).$$

We now estimate the number of eigenvalues of $\Delta^{N_\gamma^* h}$. Let $\{\lambda_i\}_i$ and $\{\mu_i\}_i$ be the eigenvalues of the Laplacians $\Delta^{N_\gamma^* h}$ and Δ^{W_γ}. Since N_γ is a branched covering map from Σ to W_γ, we may apply Tysk's heat trace estimate [30] to N_γ and obtain for any $t > 0$ that
$$\sum_i e^{-\lambda_i t} \le \deg(N_\gamma) \sum_i e^{-\mu_i t}.$$
Combining this with the heat kernel estimate by Cheng and Li [8], we have
$$\begin{aligned} N_{c'(\gamma)}(\Delta^{N_\gamma^* h}, \Sigma) &= \sum_{\lambda_i < c'(\gamma)} 1 \\ &\le e^{c'(\gamma)t} \sum_{\lambda_i < c'(\gamma)} e^{-\lambda_i t} \\ &\le e^{c'(\gamma)t} \deg(N_\gamma) \sum_i e^{-\mu_i t} \\ &\le \deg(N_\gamma) C(W_\gamma) \frac{e^{c'(\gamma)t}}{t}, \end{aligned} \tag{2.16}$$

where $C(W_\gamma)$ is the Sobolev constant of W_γ. Taking a minimum of the right hand side of (2.16), we finally obtain that

$$\mathrm{Ind}(\Sigma) \leq c'(\gamma)^{-1} C(W_\gamma) \deg(N_\gamma) = C(\gamma) \deg(N_\gamma).$$

□

2.4 Index lower bound for anisotropic minimal surfaces

In this section, we assume that $\Sigma \to \mathbb{R}^3$ is an embedded, complete, non-planner γ-minimal surface.

Proposition 2.1. *Assume* $\mathrm{Ind}(\Sigma)$ *is finite. Let u is a nontrivial solution to the equation $Lu = 0$ and let N_u be the number of connected components of $\Sigma \setminus u^{-1}(0)$. Then it follows that*

$$\mathrm{Ind}(\Sigma) \geq N_u - 1.$$

Proof. Let $I = \mathrm{Ind}(\Sigma)$. Then u is an $(I+1)$-th eigenfunction of L. By the Courant nodal domain theorem, the number of nodal domains of u is not greater than $I + 1$. Therefore, $I \geq N_u - 1$. □

The following lemma asserts that nontrivial solutions of $Lu = 0$ can be constructed by the Gauss map N.

Lemma 2.6. *Let Σ be a γ-minimal surface in $\mathbb{R}^3$ and let $a \in \mathbb{R}^3$ be a non-zero fixed vector. Then the function $\phi_a = \langle N, a \rangle$ satisfies the equation $Lu = 0$.*

Proof. Under the variation $X_t = X + ta$ of a γ-minimal immersion $X : \Sigma \to \mathbb{R}^3$, it is clear that the anisotropic mean curvature H_γ does not change. Differentiating H_γ in t gives $L\phi_a = 0$. □

Let us consider the nodal set $\phi_a^{-1}(0)$ of ϕ_a. If $x \in \phi_a^{-1}(0)$, we have $\langle N(x), a \rangle = \phi_a(x) = 0$. This implies that $N(x)$ is contained in the great circle in $\mathbb{S}^2$ determined as the intersection of $\mathbb{S}^2$ and the plane which is normal to a. Therefore, the nodal set of ϕ_a is given by the inverse image by N of a great circle.

We observe the behavior of N around critical points. Pick any point p in the critical set C of N. By Theorem2.1, we may choose a local coordinate neighborhood U around p which has no other points of C. Let $D \subset \mathbb{S}^2$ be a coordinate disk centered at $N(p)$. Up to a linear transformation on U, we may assume that $N(p) = 0$. Since N satisfies the equation $LN = 0$, we may also assume that $\Delta N + \mathrm{tr}(A_\gamma S^2) N = 0$ at p.

By the theorem of Aronszajn [1], the vanishing order of N at p is finite. Hence, by the theorem of Bers [4], there exists a homogeneous polynomial P of order $b(p) + 1$ and a constant $\varepsilon \in (0, 1)$ such that

$$N(x, y) = P(x, y) + \mathcal{O}(r^{b(p)+1+\varepsilon}),$$

where (x, y) denotes the local coordinate around p. By Lemma2.4 of [6], there exists a C^1-diffeomorphism Ψ around p such that $N = P \circ \Psi$. Hence, for any regular value q' of N in D, we have $\# N^{-1}(q') = b(p) + 1$.

Thus, we may define *the branching order of* $p \in C$ by the number $b(p)$ defined as above. Considering a triangulation of $\mathbb{S}^2$ and its pullback on Σ via N, we obtain a Riemann–Hurwitz type formula.

Proposition 2.2. *For any γ-anisotropic minimal surface Σ which has finite total curvature with $A_\gamma > 0$, it follows that*

$$\chi(\tilde{\Sigma}) = 2\deg(N) - \sum_{p \in \tilde{\Sigma}} b(p).$$

Let S be a great circle in $\mathbb{S}^2$. The following lemma asserts that the nodal set $N^{-1}(S)$ forms a pseudograph on Σ.

Lemma 2.7. *$N^{-1}(S)$ forms a pseudograph on Σ.*

Proof. Let $q_1, \cdots, q_s$ be singular values of N on S. We may assume that $q_1, \cdots, q_s$ lie on S in this order. For each q_i, we set $\{p^i_1, \cdots, p^i_{t_i}\} = N^{-1}(q_i)$. Set $t = \sum_i t_i$ and $b = \sum_{i,j} b^i_j$, where $b^i_j = b(p^i_j)$.

Consider a local disk around p^i_j. By the above argument, the arc $q_i q_{i+1}$ is lifted to $b^i_j + 1$ curves starting from p^i_j and the terminal points are among $p^{i+1}_1, \cdots, p^{i+1}_{t_{i+1}}$ (here we interpreted as $q_{s+1} = q_1$, etc.). Since N is local homeomorphism away from its critical sets, each edge has no self-intersections and any two edges do not intersect at their interiors. Thus, $N^{-1}(S)$ forms an embedded pseudograph on Σ consisting of t-vertices and $b + t$-edges. □

For a subset A of Σ, we define the branching order $b(N, A)$ of N with respect to A by

$$b(N, A) = \sum_{p \in A} b(p).$$

To show an index lower estimate, we need the following graph theoretic lemma (see [21]).

Lemma 2.8. *Let Γ be an embedded pseudograph on the compact surface S of genus g. Suppose Γ has v-vertices and e-edges, and $S \setminus \Gamma$ has N-components. Then*

$$v - e + N \geq 2 - 2g.$$

proof of Theorem 2.3. Let $\tilde{\Sigma}$ be the compactification of Σ. By Lemma 2.7, $N^{-1}(S)$ forms a pseudograph on $\tilde{\Sigma}$ consisting of t-vertices and $b + t$-edges where b and t are the numbers defined in the proof of the lemma. Notice that $N^{-1}(S)$ coincides with the nodal set of the Jacobi function ϕ_a where a is a unit vector in $\mathbb{R}^3$ which is orthogonal to S. Hence, by Lemma 2.8, we obtain

$$N_{\phi_a} \geq b + 2 - 2g.$$

Since b is the total branching order $b(N, N^{-1}(S))$, it follows from Proposition 2.1 that

$$\mathrm{Ind}(\Sigma) \geq N_{\phi_a} - 1 \geq b(N, N^{-1}(S)) + 1 - 2g.$$

□

Corollary 2.1. *Let Σ be a γ-anisotropic minimal surface which has finite total curvature with $A_\gamma > 0$. If the image of critical points $N(C)$ is contained in some great circle in $\mathbb{S}^2$, it follows that*

$$\mathrm{Ind}(\Sigma) \geq -\frac{1}{2\pi}\int_\Sigma K^\Sigma d\Sigma - 1.$$

Proof. Let S be a great circle which contains the image of critical set $N(C)$. By Proposition 2.2, we obtain

$$b(N, N^{-1}(S)) = \sum_{p\in\tilde{\Sigma}} b(p) = 2\deg(N) + 2g - 2.$$

Since the absolute total curvature of Σ coincides with $4\pi\deg(N)$, it follows from Theorem 2.3 that

$$\mathrm{Ind}(\Sigma) \geq 2\deg(N) + 2g - 2 + 1 - 2g = -\frac{1}{2\pi}\int_\Sigma K^\Sigma d\Sigma - 1.$$

□

Using Theorem 2.3, we may prove the instability of anisotropic minimal surface with low genus.

Corollary 2.2. *Let Σ be a γ-anisotropic minimal surface which has finite total curvature with $A_\gamma > 0$. If the genus g of Σ is 0 or 1, then Σ is unstable.*

Proof. If $g = 0$, Theorem 2.3 gives $\mathrm{Ind}(\Sigma) \geq 1$.

Assume that $g = 1$. Since Σ is not a plain, the mapping degree of the Gauss map $\deg(N)$ is greater than 0. It follows from Proposition 2.2 that

$$\sum_{p\in\tilde{\Sigma}} b(p) = 2\deg(N) + 2g - 2 \geq 2g = 2.$$

This implies the existence of a great circle S which contains at least two branch points (counted with multiplicity). Hence, we have $b(N, N^{-1}(S)) \geq 2$. Thus, Theorem 2.3 gives $\mathrm{Ind}(\Sigma) \geq 2 + 1 - 2 = 1$. □

3 Anisotropic extrinsic radius pinching for hypersurfaces and the stability of the Wulff shape

3.1 Introduction

Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a closed, isometrically immersed hypersurface. The *extrinsic radius* of Σ is defined as the smallest radius of balls containing Σ. It is well-known that the extrinsic radius of Σ can be bounded from below in terms of the mean curvature. Namely, T. Hasanis and D. Koutroufiotis [13] showed that

$$R_{\mathrm{ext}}\|H\|_\infty \geq 1, \tag{3.1}$$

where R_{ext} and H denote the extrinsic radius and the mean curvature of Σ respectively. For closed hypersurfaces, this inequality can be proved from an L^2-estimate of the radius as follows (see [2]):

$$\left(\int_\Sigma |H|^2\right)^{\frac{1}{2}} \left(\int_\Sigma |X - X_0|^2\right)^{\frac{1}{2}} \geq \mathrm{Vol}(\Sigma), \tag{3.2}$$

where X_0 is the center of mass of Σ which is defined by $X_0 = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma X$ and $\mathrm{Vol}(\Sigma)$ is the volume of Σ. Moreover, the equality holds in (3.2) (hence in (3.1)) if and only if Σ is the n-dimensional sphere of radius $1/\|H\|_\infty$ centered at $X_0 \in \mathbb{R}^{n+1}$.

The aim of this section is to obtain an anisotropic generalization of the extrinsic radius estimate (3.2) and quantitative and qualitative stability results proved in [2, 5, 26] for the isotropic setting.

To state our main results, we need some notations. Let $\gamma : \mathbb{S}^n \to \mathbb{R}_{>0}$ be a smooth, positive function satisfying the convexity condition

$$A_\gamma = (\mathrm{Hess}^{\mathbb{S}^n}_\gamma + \gamma I)_\nu > 0, \tag{3.3}$$

for any $\nu \in \mathbb{S}^n$. Here, I denotes the identity operator on $T_\nu\mathbb{S}^n$ and > 0 means the positivity of self-adjoint operators. We consider the map given by

$$\begin{array}{rccl} \xi : & \mathbb{S}^n & \longrightarrow & \mathbb{R}^{n+1} \\ & \nu & \longmapsto & \gamma(\nu)\nu + \nabla^{\mathbb{S}^n}\gamma(\nu). \end{array}$$

The image $W_\gamma = \xi(\mathbb{S}^n)$ is called the *Wulff shape* with respect to γ. Note that W_γ is a convex hypersurface in $\mathbb{R}^{n+1}$ by the convexity condition (3.3).

The Wulff shape W_γ can be seen as the "round sphere" for an anisotropic norm on $\mathbb{R}^{n+1}$. Namely, if we introduce the *Minkowski norm* $\gamma^* : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ by

$$\gamma^*(x) = \sup_{|z|\gamma(\frac{z}{|z|}) \leq 1} \langle x, z \rangle, \tag{3.4}$$

then W_γ can be represented as $W_\gamma = \{\gamma^* = 1\}$ (see [29] for details). For a positive $s > 0$, we call the set $sW_\gamma = \{\gamma^* = s\}$ the *Wulff shape of radius s*. We now define the *anisotropic extrinsic radius* R^γ_{ext} for a closed, isometrically immersed hypersurface $X : \Sigma^n \to \mathbb{R}^{n+1}$ by

$$R^\gamma_{\text{ext}} = \inf_{x_0 \in \mathbb{R}^{n+1}} \max_{p \in \Sigma} \gamma^*(X(p) - x_0). \tag{3.5}$$

We note that R^γ_{ext} is a natural anisotropic generalization of R_{ext}. Indeed, we have that

$$R^\gamma_{\text{ext}} = \inf\{s > 0 | \Sigma \subset \text{Int}(sW_\gamma) + x_0 \quad \text{for some } x_0 \in \mathbb{R}^{n+1}\}$$

Throughout of this paper, we let $X_0 \in \mathbb{R}^{n+1}$ denote a point at which minimizes the right hand side of (3.5).

The *anisotropic shape operator* S_γ of Σ is defined be $S_\gamma = A_\gamma \circ S$, where S denotes the usual shape operator of Σ. We define the *anisotropic mean curvature* H_γ by $H_\gamma = (-1/n)\text{tr}S_\gamma$. It is known that the Wulff shape W_γ is a stable constant anisotropic mean curvature hypersurface with $H_\gamma = 1$, like a round sphere in the isotropic setting (see for example [17, 24]).

Finally, we define the L^p norm of a function f on Σ by

$$\|f\|_p = \left(\frac{1}{\mathcal{F}_\gamma(\Sigma)} \int_\Sigma |f|^p \gamma(N) \right)^{\frac{1}{p}},$$

where $\mathcal{F}_\gamma(\Sigma) = \int_\Sigma \gamma(N)$ is the *anisotropic surface energy* of Σ and N is the unit normal vector field along Σ.

Our first result is the following anisotropic version of the extrinsic radius estimate.

Theorem 3.1. *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a closed, isometrically immersed hypersurface. Let $\gamma : \mathbb{S}^n \to \mathbb{R}_{>0}$ be a smooth positive function satisfying the convexity condition* (3.3)*. Then, it follows that*

$$\|H_\gamma\|_2 \|\gamma^*(X - X_0)\|_2 \geq 1. \tag{3.6}$$

In particular, the anisotropic extrinsic radius of Σ satisfies

$$\|H_\gamma\|_\infty R^\gamma_{\text{ext}} \geq 1. \tag{3.7}$$

Moreover, equality occurs in (3.6) *or* (3.7) *if and only if Σ is the Wulff shape with respect to γ of radius R^γ_{ext} up to translations.*

A natural question related to the equality case in (3.6) is the following: If the equality almost holds in (3.6) (or (3.7)), is Σ close to a rescaled Wulff shape $\|H_\gamma\|_2^{-1} W_\gamma$ in a certain sense? More precisely, we consider the following pinching condition for $p > 2$ and $\varepsilon > 0$:

$$\|H_\gamma\|_p \|\gamma^*(X - X_0)\|_2 \leq 1 + \varepsilon. \tag{$P_{p,\varepsilon}$}$$

In recent years, many authors study generalizations of classical pinching results for geometric invariants of hypersurfaces to the anisotropic case. In [11], De Rosa and Gioffrè studied the anisotropic almost totally umbilical hypersurfaces and proved the stability of the Wulff shapes. More precisely, they proved that if the L^p norm of the trace-free part of the anisotropic second fundamental form of a hypersurface Σ is sufficiently small, then Σ must be close to the Wulff shape in the Sobolev $W^{2,p}$ sense. Roth [27] used their results to prove that a convex hypersurface with almost constant anisotropic mean curvatures of the first and the second order must be close to the Wulff shape. Recently, Scheuer and Zhang [28] studied the quantitative stability of Wulff shape for the anisotropic Heintze–Karcher inequality and the anisotropic Alexandrov theorem.

Our next result shows that, when $\|S_\gamma\|_q$ is bounded for some $q > n$, the pinching condition $(P_{p,\varepsilon})$ implies that Σ is close to W_γ with respect to the Hausdorff distance.

Theorem 3.2. *Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a closed, isometrically immersed hypersurface. Let $\gamma : \mathbb{S}^n \to \mathbb{R}_{>0}$ be a smooth positive function satisfying the convexity condition. Let $q > n$, $p > 2$, and $A > 0$ be some real constants. Assume that the anisotropic shape operator satisfies $\mathcal{F}_\gamma(\Sigma)^{1/n}\|S_\gamma\|_q \leq A$. Then there exists some positive constants $C = C(n,p,q,A,\gamma)$ and $\alpha = \alpha(n,q)$ such that if Σ satisfies $(P_{p,\varepsilon})$, then we have*

$$\left\|\gamma^*(X - X_0) - \frac{1}{\|H_\gamma\|_2}\right\|_\infty \leq C\varepsilon^\alpha \frac{1}{\|H_\gamma\|_2}, \tag{3.8}$$

and for any $r \in [1,p)$ there exists some positive $D = D(n,p,q,r,A,\gamma)$ such that

$$\|H_\gamma - \|H_\gamma\|_2\|_r \leq D\varepsilon^{\frac{\alpha(p-r)}{r(p-1)}}\|H_\gamma\|_2. \tag{3.9}$$

Moreover, given $\varepsilon_0 > 0$, there exist a positive $\varepsilon = \varepsilon(n,p,q,A,\gamma,\|H_\gamma\|_\infty,\varepsilon_0)$ such that the pinching condition $(P_{p,\varepsilon})$ implies $d_{\mathcal{H}}(\Sigma, \|H_\gamma\|_2^{-1}W_\gamma) < \varepsilon_0$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

3.2 Preliminaries

Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a closed, isometrically immersed hypersurface and let N be the unit normal vector field along Σ. Let $\langle\cdot,\cdot\rangle$ and ∇ denote the canonical Riemannian metric and the connection on $\mathbb{R}^{n+1}$ respectively. Let ∇^Σ be the connection on Σ with respect the induced Riemannian metric from $\langle\cdot,\cdot\rangle$. The shape operator S of Σ is a $(1,1)$-tensor on Σ defined by $Sv = \nabla_v N$.

Let $\gamma : \mathbb{S}^n \to \mathbb{R}_{>0}$ be a smooth positive function on $\mathbb{S}^n$ satisfying the convexity condition (3.3). We define the *anisotropic shape operator* S_γ by $S_\gamma = A_\gamma \circ S$. The *anisotropic mean curvature* H_γ is given by $H_\gamma = -(1/n)\mathrm{tr}S_\gamma$.

In [15], He and Li proved that the anisotropic mean curvature satisfies the Hsiung–Minkowski type formula [16] given by

$$\int_{\Sigma}(\gamma(N)+H_\gamma\langle X,N\rangle)=0. \tag{3.10}$$

Such an integral formula plays an important role in the rigidity results involving anisotropic mean curvatures (see [15, 17, 24] for example).

Let us now consider the Wulff shape W_γ for γ. Let γ^* be the dual of γ defined by (3.4). We extend γ 1-homogeneously to $\mathbb{R}^{n+1}$ by letting

$$\gamma(x)=|x|\gamma\left(\frac{x}{|x|}\right)$$

As an immediate consequence of the definition of γ^*, we have the Fenchel inequality given by

$$\langle x,y\rangle\le\gamma^*(x)\gamma(y) \tag{3.11}$$

for $x,y\in\mathbb{R}^{n+1}$. Moreover, since γ is the supporting function of W_γ, the equality holds in (3.11) if and only if x is perpendicular to the tangent plane of W_γ at $\frac{y}{\gamma^*(y)}\in W_\gamma$. We now fix a point x and let ν be the unit normal vector to W_γ at $\frac{x}{\gamma^*(x)}$. Differentiating the function $G(x)=\gamma(\nu)\gamma^*(x)-\langle\nu,x\rangle$ as in [22], we can obtain the gradient of γ^* as

$$\nabla\gamma^*(x)=\frac{\nu}{\gamma(x)}. \tag{3.12}$$

Proof of Theorem 3.1. By the Hsiung–Minkowski formula (3.10) and the Fenchel inequality (3.11), we have

$$1=\left|\frac{1}{\mathcal{F}_\gamma(\Sigma)}\int_\Sigma H_\gamma\langle X-X_0,N\rangle\right|\le\frac{1}{\mathcal{F}_\gamma(\Sigma)}\int_\Sigma|H_\gamma|\gamma^*(X-X_0)\gamma(N)\le\|H_\gamma\|_2\|\gamma^*(X-X_0)\|_2,$$

which concludes the inequality.

Assume the equality holds. Set $X_{\gamma^*}=\frac{X-X_0}{\gamma^*(X-X_0)}$. We have the equality of the Fenchel inequality (3.11), which implies that the unit normal N is perpendicular to the tangent space of W_γ at X_{γ^*}. Moreover, by (3.12), we have $\nabla^\Sigma\gamma^*(X-X_0)=0$, which implies that $\gamma^*(X-X_0)$ is constant. Therefore, we have $\Sigma=H_\gamma^{-1}W_\gamma+X_0$. □

3.3 Upper bound on the anisotropic extrinsic radius

For hypersurfaces, we have the following Michael–Simon Sobolev inequality [20]:

$$\left(\int_\Sigma |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le C(n)\left(\int_\Sigma |\nabla^\Sigma f| + \int_\Sigma |Hf|\right). \tag{3.13}$$

Let $\{e_i\}$ be an orthonormal frame along Σ which diagonalizes A_γ and set $a_i = \langle A_\gamma e_i, e_i\rangle$. By the Cauchy–Schwartz inequality, we have

$$\begin{aligned} H &= \frac{1}{n}\sum_i \langle Se_i, e_i\rangle = \frac{1}{n}\sum_i \langle A_\gamma^{-1} S_\gamma e_i, e_i\rangle = \frac{1}{n}\sum_i a_i^{-1}\langle S_\gamma e_i, e_i\rangle \\ &\le \frac{1}{n}\left(\sum_i a_i^{-2}\right)^{\frac{1}{2}}\left(\sum_i |\langle S_\gamma e_i, e_i\rangle|^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{\lambda}|S_\gamma|, \end{aligned}$$

where λ is a positive constant given by

$$\lambda = \min_{\nu\in\mathbb{S}^n, u\in\nu^\perp, |u|=1} \langle A_\gamma(\nu)u, u\rangle.$$

Combining this with (3.13), we have

$$\|f\|_{\frac{n}{n-1}} \le C(n,\gamma)\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}(\|\nabla^\Sigma f\|_1 + \||S_\gamma|f\|_1). \tag{3.14}$$

To obtain the inequality (3.8), we prove the following estimate, which is an anisotropic version of [2, Theorem 1.6].

Proposition 3.1. *Let $q > n$ be a real. There exists a constant $C = C(n,q,\gamma) > 0$ such that for any isometrically immersed hypersurface $X : \Sigma \to \mathbb{R}^{n+1}$, we have*

$$\|\gamma^*(X - X_0) - \|\gamma^*(X - X_0)\|_2\|_\infty \le C(\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\|S_\gamma\|_2)^\beta \|\gamma^*(X - X_0)\|_2 \left(1 - \frac{\|\gamma^*(X - X_0)\|_1}{\|\gamma^*(X - X_0)\|_2}\right)^{\frac{1}{2(1+\beta)}},$$

where $\beta = \frac{nq}{2(q-n)}$.

Proof. Up to translation, we may assume that $X_0 = 0$. We set $\varphi = |\gamma^*(X) - \|\gamma^*(X)\|_2|$. For a positive $a > 0$, we have $|\nabla^\Sigma \varphi^{2a}| \le 2(\min\gamma)^{-1}\varphi^{2a-1}$ by (3.12). Letting $f = \varphi^{2a}$ in (3.14), we have

$$
\begin{aligned}
\|\varphi\|^{2a}_{\frac{2an}{n-1}} &\le C(n,\gamma)\mathcal{F}_\gamma(\Sigma)(2a\|\varphi\|^{2a-1}_{2a-1} + \||S_\gamma|\varphi^{2a}\|_1) \\
&\le C(n,\gamma)\mathcal{F}_\gamma(\Sigma)(2a\|\varphi\|^{2a-1}_{2a-1} + \|S_\gamma\|_q\|\varphi\|^{2a}_{\frac{2aq}{q-1}}) \\
&\le C(n,\gamma)\mathcal{F}_\gamma(\Sigma)(2a\|\varphi\|^{2a-1}_{\frac{(2a-1)q}{q-1}} + \|S_\gamma\|_q\|\varphi\|^{2a}_{\frac{2aq}{q-1}}) \\
&\le C(n,\gamma)\mathcal{F}_\gamma(\Sigma)(2a + \|S_\gamma\|_q\|\varphi\|_\infty)\|\varphi\|^{2a-1}_{\frac{(2a-1)q}{q-1}}.
\end{aligned}
\tag{3.15}
$$

We set $\nu = \frac{n(q-1)}{(n-1)q}$ and $a = a_k\frac{q-1}{2q} + \frac{1}{2}$ where $a_{k+1} = \nu a_k + \frac{n}{n-1}$ and $a_0 = \frac{2q}{q-1}$. Plugging them into (3.15) gives

$$
\left(\frac{\|\varphi\|_{a_{k+1}}}{\|\varphi\|_\infty}\right)^{\frac{a_{k+1}}{\nu^{k+1}}} \le \left\{C(n,\gamma)\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{a_k\frac{q-1}{q}+1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{n}{\nu^{p+1}(n-1)}}\left(\frac{\|\varphi\|_{a_k}}{\|\varphi\|_\infty}\right)^{\frac{a_k}{\nu^k}}
$$

Since $q > n$ then ν and $\frac{a_k}{\nu^k}$ converges to $a_0 + \frac{qn}{q+n}$ and we have

$$
\begin{aligned}
1 &\le \left(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\right)^2 \prod_{k=0}^{\infty}\left\{2C(n,\gamma)\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}a_k\left(\frac{1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{1}{\nu^k}} \\
&\le \left(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\right)^2 \prod_{k=0}^{\infty} a_k^{\frac{1}{\nu^k}}\left\{2C(n,\gamma)\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{\nu}{\nu-1}} \\
&= C(q,n,\gamma)\left(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\right)^2\left\{\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{n(q-1)}{q-n}} \\
&\le C(q,n,\gamma)\left(\frac{\|\varphi\|_{a_0}}{\|\varphi\|_\infty}\right)^{\frac{2(q-1)}{q}}\left\{\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{n(q-1)}{q-n}},
\end{aligned}
$$

hence we have

$$
\|\varphi\|_\infty \le C(n,q,\gamma)\left\{\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{1}{\|\varphi\|_\infty} + \|S_\gamma\|_q\right)\right\}^{\frac{nq}{2(q-n)}}\|\varphi\|_2.
$$

We set $\beta = \frac{nq}{2(q-n)}$. If $\|\varphi\|_\infty \ge \|S_\gamma\|_q^{-\frac{\beta}{1+\beta}}\|\varphi\|_2^{\frac{1}{1+\beta}}$, then

$$\begin{aligned}
\|\varphi\|_\infty &\le C(q,n,\gamma)\left\{\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\left(\frac{1}{\|\varphi\|_\infty}+\|S_\gamma\|_q\right)\right\}^\beta\|\varphi\|_2\\
&\le C(q,n,\gamma)(\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}(\|S_\gamma\|_q^{\frac{\beta}{1+\beta}}\|\varphi\|_2^{-\frac{1}{1+\beta}}+\|S_\gamma\|_q))^\beta\|\varphi\|_2\\
&\le C(q,n,\gamma)(\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\|S_\gamma\|_q)^\beta(\|S_\gamma\|_q^{-\frac{1}{1+\beta}}+\|\varphi\|_2^{\frac{1}{1+\beta}})^\beta\|\varphi\|_2^{\frac{1}{1+\beta}}\\
&\le C(q,n,\gamma)(\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\|S_\gamma\|_q)^\beta(\|\gamma^*(X)\|_2^{\frac{1}{1+\beta}}+\|\varphi\|_2^{\frac{1}{1+\beta}})^\beta\|\varphi\|_2\\
&\le C(q,n,\gamma)(\mathcal{F}_\gamma(\Sigma)^{\frac{1}{n}}\|S_\gamma\|_q)^\beta\|\gamma^*(X)\|_2^{\frac{\beta}{1+\beta}}\|\varphi\|_2,
\end{aligned}$$

where we have used $1 \le \|H_\gamma\|_2\|\gamma^*(X)\|_2 \le \|S_\gamma\|_q\|\gamma^*(X)\|_2$. Since we have

$$\|\varphi\|_2^2 = \|\gamma^*(X) - \|\gamma^*(X)|_2\|_2^2 = 2\|\gamma^*(X)\|_2\left(1-\frac{\|\gamma^*(X)\|_1}{\|\gamma^*(X)\|_2}\right),$$

the desired inequality follows.

If $\|\varphi\|_\infty \le \|S_\gamma\|_q^{-\frac{\beta}{1+\beta}}\|\varphi\|_2^{\frac{1}{1+\beta}}$, the result follows immediately from the above expression of $\|\varphi\|_2$ and the fact that $\|S_\gamma\|_q\|\gamma^*(X)\|_2 \ge 1$. □

Proof of (3.8). We may assume that $X_0 = 0$. From the Hsiung–Minkowski formula (3.10), it follows that $1 \le \|H_\gamma\|_p\|\gamma^*(X)\|_{\frac{p}{p-1}}$. By the Hölder inequality and the pinching condition $(P_{p,\varepsilon})$, we have

$$\|H_\gamma\|_p\|\gamma^*(X)\|_2 \le 1+\varepsilon \le (1+\varepsilon)\|H_\gamma\|_p\|\gamma^*(X)\|_{\frac{p}{p-1}} \le (1+\varepsilon)\|H_\gamma\|_p\|\gamma^*(X)\|_1^{1-\frac{2}{p}}\|\gamma^*\|_2^{\frac{2}{p}},$$

hence

$$1-\frac{\|\gamma^*(X)\|_1}{\|\gamma^*(X)\|_2} \le 1-\frac{1}{(1+\varepsilon)^{\frac{p}{p-2}}} \le \frac{p}{p-2}2^{\frac{2}{p-2}}\varepsilon.$$

Combining this inequality with Proposition 3.1, we obtain

$$\begin{aligned}
\|\gamma^*(X) - \|\gamma^*(X)\|_2\|_\infty &\le C(n,p,q,\gamma)(\mathcal{F}_\gamma(\Sigma)\|S_\gamma\|_q)^\beta\|\gamma^*(X)\|_2\varepsilon^{\frac{1}{2(1+\beta)}}\\
&\le C(n,p,q,\gamma)A^\beta\frac{1}{\|H_\gamma\|_2}\varepsilon^{\frac{1}{2(1+\beta)}},
\end{aligned}$$

Here, we used $\|H_\gamma\|_2\|\gamma^*(X)\|_2 \le 1+\varepsilon \le 2$ to get the second inequality.

Letting $\alpha = \frac{1}{2(1+\beta)}$, we obtain

$$\left\|\gamma^*(X) - \frac{1}{\|H_\gamma\|_2}\right\|_\infty \leq \|\gamma^*(X) - \|\gamma^*(X)\|_2\|_\infty + \left\|\|\gamma^*(X)\|_2 - \frac{1}{\|H_\gamma\|_2}\right\|_\infty$$
$$\leq CA^\beta \frac{\varepsilon^\alpha}{\|H_\gamma\|_2} + \frac{\varepsilon}{\|H_\gamma\|_2}$$
$$\leq C(n,p,q,A,\gamma)\frac{\varepsilon^\alpha}{\|H_\gamma\|_2}.$$

□

Proof of (3.9). By the Hsiung–Minkowski formula (3.10) and the Fenchel inequality (3.11), we have

$$1 \leq \frac{1}{\mathcal{F}_\gamma(\Sigma)} \int_\Sigma |H_\gamma|\gamma^*(X)\gamma(N).$$

Using this and $(P_{p,\varepsilon})$, we have

$$\left\|\frac{|H_\gamma|}{\|H_\gamma\|_2^2} - \gamma^*(X)\right\|_2^2 = \|\gamma^*(X)\|_2^2 + \frac{1}{\|H_\gamma\|_2^2} - \frac{2}{\|H_\gamma\|_2^2}\frac{1}{\mathcal{F}_\gamma(\Sigma)}\int_\Sigma |H_\gamma|\gamma^*(X)\gamma(N)$$
$$\leq \|\gamma^*(X)\|_2^2 - \frac{1}{\|H_\gamma\|_2^2}$$
$$\leq \left(1 - \frac{1}{(1+\varepsilon}^2\right)\|\gamma^*(X)\|_2^2$$
$$\leq 3\varepsilon\|\gamma^*(X)\|_2^2,$$

hence

$$\|H_\gamma^2 - \|H_\gamma\|_2^2\|_1 \leq \|H_\gamma^2 - \gamma^*(X)^2\|H_\gamma\|_2^4\|_1 + \|\gamma^*(X)^2\|H_\gamma\|_2^4 - \|H_\gamma\|_2^2\|_1$$
$$= \|H_\gamma\|_2^4\left(\left\|\frac{|H_\gamma|^2}{\|H_\gamma\|_2^2} - \gamma^*(X)^2\right\|_1 + \left\|\gamma^*(X)^2 - \frac{1}{\|H_\gamma\|_2^2}\right\|_1\right)$$
$$\leq \|H_\gamma\|_2^4\left(\left\|\frac{|H_\gamma|}{\|H_\gamma\|_2} - \gamma^*(X)\right\|_2\left\|\frac{|H_\gamma|}{\|H_\gamma\|_2} + \gamma^*(X)\right\|_2 + CA^\beta\frac{\varepsilon^\alpha}{\|H_\gamma\|_2}\left\|\gamma^*(X) + \frac{1}{\|H_\gamma\|_2}\right\|_1\right)$$
$$\leq \|H_\gamma\|_2^4\left(\sqrt{3\varepsilon}\|\gamma^*(X)\|_2 + CA^\beta\frac{\varepsilon^\alpha}{\|H_\gamma\|_2}\right)\left(\|\gamma^*(X)\|_2 + \frac{1}{\|H_\gamma\|_2}\right)$$
$$\leq C(n,p,q,A,\gamma)\varepsilon^\alpha\|H_\gamma\|_2^2.$$

Therefore, we obtain

$$\||H_\gamma| - \|H_\gamma\|_2\|_1 \leq \frac{\||H_\gamma|^2 - \|H_\gamma\|_2^2\|_1}{\|H_\gamma\|_2} \leq C\varepsilon^\alpha\|H_\gamma\|_2$$

Moreover, we have

$$\||H_\gamma| - \|H_\gamma\|_2\|_p \leq 2\|H_\gamma\|_p \leq 2\|H_\gamma\|_2\|\gamma^*(X)\|_2\|H_\gamma\|_p \leq 4\|H_\gamma\|_2$$

by $(P_{p,\varepsilon})$. Hence, for any $r \in [1, p)$, we obtain

$$\begin{aligned}\||H_\gamma| - \|H_\gamma\|_2\|_r &\leq \||H_\gamma| - \|H_\gamma\|_2\|_1^{\frac{p-r}{r(p-1)}} \||H_\gamma| - \|H_\gamma\|_2\|_p^{\frac{p(r-1)}{r(p-1)}} \\ &\leq D(n, p, q, r, A, \gamma)\varepsilon^{\frac{\alpha(p-r)}{r(p-1)}} \|H_\gamma\|_2.\end{aligned}$$

□

3.4 Proof of the Hausdorff closeness

In this section, we prove the Hausdorff closeness in Theorem 3.2. To prove this, we need an anisotropic version of the lemma due to B. Colbois and J. -F. Grosjean [5, Lemma 3.2].

Lemma 3.1. *For any $R > 0$ and $\rho \in (0, 1)$, there exists a positive $\eta = \eta(R, \rho, n, \gamma) > 0$ satisfying the following property: Let $z_0 \in RW_\gamma$ and let ν_0 be the outer unit normal vector to RW_γ at z_0. Let $X : \Sigma \to \mathbb{R}^{n+1}$ be a closed hypersurface isometrically immersed in $((R + \eta)W_\gamma \setminus (R - \eta)W_\gamma) \setminus B_\rho(z_0)$. If there exists a point $p \in \Sigma$ with $\langle X(p), \nu_0\rangle > 0$, then there exists a point $q \in \Sigma$ such that $|H_\gamma(q)| > \frac{\lambda}{2n\eta}$.*

Before proving Lemma 3.1, we prepare some notations. For $\nu \in \mathbb{S}^n$ and $t > 0$, we set $\Pi_t(\nu) = \nu^\perp + t\nu$, where $\nu^\perp$ denote the n-dimensional subspace of $\mathbb{R}^{n+1}$ which is perpendicular to ν. Let $z_0 = R\xi(\nu) \in RW_\gamma$.

We now let $t_0(\nu) = \max\{t > 0 | \Pi_t(\nu) \cap (\partial B_\rho(z_0) \cap RW_\gamma) \neq \emptyset\}$. Note that for every $t \in [0, t_0(\nu)]$, the set $W(\nu, t) = RW_\gamma \cap \Pi_t(\nu)$ is convex. Considering $W(\nu, t)$ as a convex hypersurface in $\mathbb{R}^n$, we let $S^{W(\nu,t)}$ and $\kappa_i^{W(\nu,t)}$ be the shape operator of $W(\nu, t)$ and the i-th principal curvature of $W(\nu, t)$ respectively. Set $H_\gamma^{W(\nu,t)} = \mathrm{tr}(A_\gamma S^{W(\nu,t)})$. Since the function $h^\nu : [0, t_0(\nu)] \ni t \mapsto \max_{W(\nu,t)} H_\gamma^{W(\nu,t)} \in \mathbb{R}_{>0}$ is non-decreasing we have $h^\nu(t_0(\nu)) = \max h^\nu$. We now define

$$H_0 = \max_{\nu \in \mathbb{S}^n} h^\nu(t_0(\nu)) \in (0, \infty).$$

Similarly, we set

$$\kappa_0 = \max_{\nu \in \mathbb{S}^n} \max\{\kappa_i^{W(\nu,t)}(x) | t \in [0, t_0(\nu)], x \in W(\nu, t), 1 \leq i \leq n - 1\}.$$

Proof of Lemma 3.1. For $\eta \in (0, \rho)$ and $t \in [0, t_0(\nu_0)]$, consider the family of smooth maps

$$\begin{array}{rccc}\Phi_{\eta,t} : & W(\nu_0, t) \times \mathbb{S}^1 & \longrightarrow & \mathbb{R}^{n+1} \\ & (x, \theta) & \longmapsto & x - \eta\cos\theta N(x) + \eta\sin\theta\nu_0 + t\nu_0,\end{array}$$

where $N(x)$ denotes the unit normal vector of $W(\nu_0, t) \subset \mathbb{R}^n$ at x. Let $S_{\eta,t}$ denote the image of $\Phi_{\eta,t}$.

We calculate the curvature of $S_{\eta,t}$. Let $\{e_i\}_{i=1}^{n-1}$ be the orthonormal frame which diagonalize $S^{W(\nu_0,t)}$ at x. We have

$$\Phi_i = d\Phi_{\eta,t}(e_i) = (1 - \eta\cos\theta\kappa_i^{W(\nu_0,t)})e_i,$$
$$\Phi_\theta = d\Phi_{\eta,t}(\partial_\theta) = \eta(\sin\theta N(x) + \cos\theta\nu_0).$$

Note that there exists a positive $\rho_0 \in (0, \rho)$ such that $S_{\eta,t}$ is an embedded hypersurface for $\eta \in (0, \rho_0)$. Since the outer unit normal vector field of $S_{\eta,t}$ is given by $\overline{N} = -\cos\theta N + \sin\theta\nu_0$, the anisotropic shape operator $S_\gamma^{S_{\eta,t}}$ of $S_{\eta,t}$ can be calculated by

$$S_\gamma^{S_{\eta,t}} e_i = -\cos\theta A_\gamma S^{W(\nu_0,t)} e_i, \quad S_\gamma^{S_{\eta,t}}\partial_\theta = -\eta^{-1}A_\gamma\Phi_\theta.$$

Since $\langle\Phi_i, \Phi_j\rangle = \delta_{ij}$ and $\langle\Phi_i, \Phi_\theta\rangle = 0$, the anisotropic mean curvature $H_\gamma^{S_{\eta,t}}$ is given by

$$\begin{aligned}
H_\gamma^{S_{\eta,t}} &= -\frac{1}{n}\mathrm{tr}S_\gamma^{S_{\eta,t}} \\
&= -\frac{1}{n}\frac{\langle S_\gamma^{S_{\eta,t}}\partial_\theta, \partial_\theta\rangle}{|\Phi_\theta|^2} - \frac{1}{n}\sum_{i=1}^{n-1}\frac{\langle S_\gamma^{S_{\eta,t}}e_i, e_i\rangle}{|\Phi_i|^2} \\
&= \frac{\langle A_\gamma\Phi_\theta, \Phi_\theta\rangle}{n\varepsilon|\Phi_\theta|^2} - \frac{1}{n}\sum_{i}^{n-1}\frac{\cos\theta\langle A_\gamma S^{W(\nu_0,t)}e_i, e_i\rangle}{(1 - \eta\cos\theta\kappa_i^{W(\nu_0,t)})^2} \\
&\geq \frac{\lambda}{n\eta} - \frac{(n-1)H_0}{n(1-\eta\kappa_0)^2}
\end{aligned}$$

We now let $\eta = \min\left\{\frac{1}{2\kappa_0}, \frac{\lambda}{8(n-1)H_0}, \rho_0\right\}$, then

$$H_\gamma^{S_{\eta,t}} \geq \frac{\lambda}{2n\eta}$$

Since there exists a point $p \in \Sigma$ so that $\langle X(p), \nu_0\rangle$ by assumption, we can find $t \in [0, t_0(\nu_0)]$ and a point $q \in \Sigma$ which is a contact point with $S_{\eta,t}$. Therefore $|H_\gamma(q)| \geq \frac{1}{2n\eta}$. □

Hausdorff closeness in Theorem 3.2. Let $B_\varepsilon(\|H_\gamma\|_2^{-1}W_\gamma)$ denote the ε-neighborhood of $\|H_\gamma\|_2^{-1}W_\gamma$. Assume a positive $\varepsilon_0 > 0$ is given. If Σ satisfies $(P_{p,\varepsilon})$ for a small $\eta > 0$, by (3.8), it follows for a small $\varepsilon \in (0, \varepsilon_0)$ that $\Sigma \subset B_\varepsilon(\|H_\gamma\|_2^{-1}W_\gamma) \subset B_\varepsilon(\|H_\gamma\|_2^{-1}W_\gamma)$.

Assume $\Sigma \subset B_\varepsilon(\|H_\gamma\|_2^{-1}W_\gamma) \setminus B_{\varepsilon_0}(y)$ occurs for some $y \in \|H_\gamma\|_2^{-1}W_\gamma$. Choosing ε so small that $\varepsilon < \min\{\eta, n/\|H_\gamma\|_\infty\}$, where η is a positive as in Lemma 3.1, it follows from Lemma 3.1 that there exists a point $q \in \Sigma$ such that $H_\gamma(q) > \|H_\gamma\|_\infty/2$, which is a contradiction.

Hence, we have $d_{\mathcal{H}}(\Sigma, \|H_\gamma\|_2^{-1}W_\gamma) \leq \varepsilon_0$. □

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