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The Recurrence and Transience of Random Walks on Growing Graphs

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Abstract

It is a celebrated fact that a simple random walk on $\mathbb{Z}$ and $\mathbb{Z}^2$ returns to the initial vertex v infinitely many times during infinitely many transitions, it is called *recurrent*, while it returns to v only finite times on $\mathbb{Z}^d$ for $d \geq 3$, it is called *transient*. On the other hand, it is evident that random walks on finite regions of $\mathbb{Z}^d$ are recurrent for any $d = 1, 2, 3, \ldots$. Two results lead to the following question: are random walks on growing graphs recurrent or transient? A question for growing graphs is also significant concerning utilizing changing graphs for our lives like social media and algorithms. The recurrence and transience of random walks on growing graphs is determined by growing speed. In particular, it is also known that a simple random walk on a growing region on $\mathbb{Z}^d$ can be recurrent depending on the increasing speed of any *fixed* d and the recurrence and transience of random walks on growing graphs under the assumptions of a random walk being *lazy*, growing speed being slow, and a growing graph having a *uniformly bounded* are determined by the growing speed. These results can be derived using various techniques such as *Brownian motion*, *evolving sets* and *Cheeger constant*. In this paper, we show that random walks on growing graphs can be recurrent depending on the growing speed by focusing on *coupling* techniques. Specifically, we introduce the notion of *less-homesick as graph growing* (LHaGG), which is a natural property of random walks on growing graphs. An LHaGG follows the property that the slower the growing speed, the greater the return probability to the origin. Then, we show that random walks on growing graphs satisfy LHaGG by coupling method. As a result, we give the recurrence and transience of random walks on growing graphs such as growing complete k-ary tree, growing box, and $\{0,1\}^n$ with an increasing n. Likewise, we consider a lazy random walk on $\{0,\ldots,N\}^n$ with an increasing n. It is challenging to show that a $\frac{1}{2}$- lazy random walk on $\{0,\ldots,N\}^n$ with an increasing n satisfies LHaGG. Therefore, we extend LHaGG to *weakly less-homesick as graph growing* (weakly LHaGG). A weakly LHaGG follows the property: the slower the growing speed, the greater the expected number of returning to the origin. However, it is also hard to prove that random walks on growing graphs satisfy weakly LHaGG using the same coupling as LHaGG. To address this problem, we introduce a new coupling technique, (*pausing coupling*). This approach extends the coupling technique of the previous work regarding the Bernoulli Growth Random Walk (BGRW) and provides the sufficient conditions for the recurrence and transience of a $\frac{1}{2}$- lazy random walk on $\{0,\ldots,N\}^n$, spider trees, and random trees. This paper contains various contributions as this approach is simpler than the previous works, can remove the assumptions for the existing works, and can be applied to bipartite graphs with *reversible* such trees and $\{0,1\}^n$.

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Contents

1 Introduction

The recurrence or transience is a classical and fundamental topic of random walks on *infinite* graphs, see e.g., [20, 21, 49]: let $X_0, X_1, X_2, \ldots$ be a random walk (or a Markov chain)[1] on a countable state space V, e.g., $V = \mathbb{Z}$, with $X_0 = v$ for $v \in V$. For convenience, let

$$R(t) = \Pr[X_t = v] \qquad (= \Pr[X_t = v \mid X_0 = v])$$

denote the probability that a random walk returns to the initial state at time step t ($t = 1, 2, \ldots$), and then the initial point v is *recurrent* by the random walk if

$$\sum_{t=1}^{\infty} R(t) = \infty \tag{1}$$

holds, otherwise it is *transient*. Intuitively, (1) means that the random walk is "expected" to return to the initial state infinitely many times. It is well known that a simple random walk on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$, while it is transient for $d \geq 3$, cf. [20, 21, 49]. Another celebrated fact is that a simple random walk on an infinite k-ary tree is transient [38, 40].

Analysis of random walks on *dynamic graphs* has been developed in several contexts. In probability theory, random walks in random environments are a major topic, where self-interacting random walks including reinforced random walks and excited random walks have been intensively investigated as a relatively tractable non-Markovian process, see e.g., [13, 7, 19, 50, 51, 32, 23]. The recurrence or transience of a random walk in a random environment is a major topic there, particularly random walks on growing subgraphs of $\mathbb{Z}^d$ or on infinitely growing trees are the major targets [24, 15, 16, 27, 2]. In distributed computing, analysis of algorithms, including random walk, on dynamic graph attracts increasing attention due to the fact that real networks are often dynamic [10, 33, 4, 45]. Searching or covering networks, related to hitting or cover times of random walks, are major topics there [11, 5, 18, 6, 37, 8, 31].

1.1 Existing works and contribution of the paper

The recurrence/transience of random walks on dynamic graphs has been mainly developed in the context of random walks in random environments including reinforced random walks and excited walks. Here we briefly review some existing works concerning the recurrence of a random walk on $\mathbb{Z}^d$ and infinite (or infinitely growing) trees, directly related to this paper.

Random walks on (asymptotically) $\mathbb{Z}^d$. It is a celebrated fact that the initial point, say origin $\mathbf{0}$, in the infinite integer grid $\mathbb{Z}^d$ is recurrent when $d = 1$ and 2 by a simple random walk, and it is transient for $d \geq 3$, see e.g., [20, 21, 49]. On the other hand, it is known that a random walk on the finite region in $\mathbb{Z}^d$ is recurrent for any $d = 1, 2, 3, \ldots$.

Amir et al. [2] introduced random walks in changing environments model, and investigated the recurrence and transience of random walks in the model. They focused on the condition for recurrence and transience of a random walk in a changing environment on $\mathbb{N}$ and $\mathbb{Z}^2$, and got the sufficient condition by comparing a random walk in a changing environment to a random walk on an infinite graph. Additionally, they provided a conjecture regarding the relationship between a random walk in a changing environment and a random walk on an infinite graph such that the recurrence and transience of a random walk on a growing graph depends on the recurrence and transience of a random walk on an infinite graph. In 2024, Park and Ray

[1]This paper is concerned with discrete time and space processes. We will be mainly concerned with time-*inhomogeneous* Markov chains, but here you may assume a time-homogeneous chain, i.e., the transition probability $\Pr[X_{t+1} = v \mid X_t = u]$ is independent of the time t, but depends on u, v.

investigated the sufficient condition for a recurrence and transience of a random walk in slowly changing environments [48]. They claimed that a recurrence and transience of random walks in changing environments with non-decrease conductance depends on the origin graph, and hence this result is a little different from the conjecture by Amir et al [2]. Furthermore, they gave a sufficient condition for a recurrence and transience of a simple random walk in changing environments such as depending on infinite graphs. This result is similar to the conjecture by Amir et al [2]. Their result extends prior work [2] concerning generalizing graphs instead of slowly changing conductance.

Dembo et al. [16] investigated a random walk on an infinitely growing subgraph of $\mathbb{Z}^d$, and gave a phase transition, that is roughly speaking a random walk is recurrent if and only if $\sum_{t=1}^{\infty} \pi_t(\mathbf{0}) = \infty$ holds under a certain condition, where π_t denotes the stationary distribution of the transition matrix at time t. The proofs are based on coupling, Brownian motion, and the central limit theorem, on the assumption that the growing region at time t is sandwiched between the two different growing balls for any $t \geq 0$ that have the same growing speed. In the same year, Dembo et al. [15] focused on the recurrence and transience of a simple random walk on monotone interaction growing graphs which is about the growth by visiting the graph boundary (they call open-by-touch). Additionally, they considered the recurrence and transience of a general open-by-touch model (they call expanding glassy spheres), the growth depends on the number of visiting the graph boundary. They provided a bound of the return probability of a growing graph by comparing it to the return probability on an infinite graph with uniformly bounded, and hence they obtained sufficient conditions for the recurrence and transience of random walks on monotone interaction growing graphs with uniformly bounded. This growth model depends on the walker as opposed to [16] and our models. In 2017, Dembo et al. [14] provided a tight upper bound by heat kernel for lazy random walks on graphs with changing conductance as a new approach (it is called *evolving sets*) and *Cheeger constant*. To obtain a bound for the mixing time and the heat kernel for an irreducible Markov chain on a countable state space, this new approach was utilized by Morris and Peres [46, 47]. Since their technique focused mainly on random walks on graphs with non-decrease conductance in $\mathbb{Z}^d$, they assumed the condition that $c^{(t)}(x, y) \in \left[C^{-1}, C\right]$ holds for any t, $x \in V_t$ and $(x, y) \in E$ (it is called *uniformly bounded*), where $c^{(t)}(x, y)$ be a random walk in time-varying edge conductance and C is a positive constant for t. As a result, they extended the prior work [16] and proved a part of the conjecture by Amir et al. [2]. In 2019, Dembo et al. [17] focused on the bound for the heat kernel of random walks on graphs with changing finite conductance. They obtained the bound by the heat kernel on the condition that a random walk on graphs with non-decrease conductance satisfy $C_v^{-1} \leq \frac{c^{(t)}(\mathbb{B}(x,r))}{v(r)} \leq C_v$ for any $x \in V$, $r \geq 0$ and $t \geq 0$, where $v(\cdot)$ is non-decreasing, $v(2r) \leq C_v v(r)$ and $v(0) = v(1) = 1$ (they call uniform volume growth with $v(r)$ doubling), and uniform Poincaré inequality (this inequality is the relation for the eigenvalue of the transition probability matrix). As a result, they extended the previous works [14, 27].

Huang [27] extended the argument of [14, 16] and gave a similar or essentially the same phase transition for more general graphs. The proofs are based on the Cheeger constant, Evolving sets, and Radon-Nikodyn density, on the assumption that every vertex of the growing graph has a degree at most *constant* to time (or the size of the graph), and the random walk is "lazy" such that it has at least a *constant* probability of self-loops at every vertex. In the same year, Huang [28] discussed a random walk on a growing Internal Diffusion Limited Aggregation (IDLA) in $\mathbb{Z}^d$. He provided the Cheeger constant on an IDLA by using a growing IDLA characteristic which a growing IDLA sandwiched between the two balls at a sufficiently large enough time. By his previous technique [27], he proved that a random walk on a growing IDLA is recurrent for any $d \geq 3$.

Random walks on infinitely growing trees. Lyons [38] studied sufficient conditions for a random walk to be recurrent/transient, see also [40]. Roughly speaking, the initial point, say the root r, is recurrent if and only if the random walk is enough *homesick*, meaning that a random walk probabilistically tends to

Table 1: The previous works about the recurrence and transience of a random walk on $\mathbb{Z}^d$

——Author——	——Condition / Assumption ——	——Result——
Amir et al. [2]	Random walk on $\mathbb{N}$ and $\mathbb{Z}^2$ / changing conductance	The recurrence and transience depends on infinite graph and give the Conjecture
Dembo et al. [16]	Simple random walk on an infinitely growing subgraph of $\mathbb{Z}^d$ / sandwiched between the two different growing balls and slow growth	v is recurrent if and only if $\sum \pi_t(v) = \infty$
Dembo et al. [14]	Lazy random walk on graphs with changing conductance / non-decrease conductance, uniformly bounded and slow growth	Obtain the upper bound of the heat kernel and extend the previous work [16]
Dembo et al. [14]	Random walk on graphs with changing conductance / non-decrease conductance, uniformly bounded, uniformly volume growth with $v(r)$ doubling and uniform Poincare inequality	Obtain the bound for the heat kernel and extent the previous work [14, 27]
Huang [27]	Lazy random walk on growing graph / uniform bounded degree, slow growth and Radon-Nikodyn density	v is recurrent if and only if $\sum \pi_t(v) = \infty$
Huang [28]	Random walk on growing IDLA in $\mathbb{Z}^d$ / sandwiched between the two balls	Recurrent for any $d \geq 3$

choose the direction to the root.

Amir et al. [2] introduced random walks in changing environments model, and investigated the recurrence and transience of a random walk in the model. Their technique compares a random walk in a changing environment with a random walk on an infinite graph. As a result, they gave a conjecture about the conditions for the recurrence and transience regarding a changing graph and an infinite graph and proved it for trees. Their result led to the beginning of the analysis of the recurrence and transience of a random walk in a changing environment. Huang's work [27], which we mentioned above, implies that a simple random walk starting from a vertex v on a growing k-ary tree is recurrent if and only if $\sum \pi_t(v) = \infty$, that is similar to or essentially the same as the main result of this paper under a certain condition. We remark that a k-ary tree with height n is not an (edge-induced) subgraph of $\mathbb{Z}^d$ for a *constant* d. Figueiredo et al. [22] studied the recurrence and transience of the Bernoulli Growth Random Walk (BGRW), BGRW is a random walk on a growing graph such that the start graph is a finite tree and a new leaf is added to the current location with probability p every steps. They proved that BGRW is transient for any $0 < p \leq 1$ by coupling and central limit theorem. Their approach is a special technique that compares the two walkers with different time scales. They are enabled by focusing on the tree feature of no cycles.

There is a lot of work on the recurrence or transience of a random walk on a growing tree, related to self-interacting random walks including reinforced random walks and excited random walks, e.g., [3, 9, 29, 22, 30]. They are non-Markovian processes, and in a bit different line from [16, 2, 27] and this paper.

Table 2: The previous works about the recurrence and transience of a random walk on a tree

——Author——	——Condition / Assumption ——	——Result——
Amir et al. [2]	Random walk on infinite tree / changing conductance	The recurrence and transience depends on infinite graph and gives the Conjecture
Huang [27]	Lazy random walk on growing graph / uniform bounded degree, slow growth and Radon-Nikodyn density	v is recurrent if and only if $\sum \pi_t(v) = \infty$
Figueiredo et al. [22]	BGRW / Tree	Transient for any $0 < p \leq 1$

Other Related works - Temporal graph Mertzios et al. focused on a temporal graph [41, 42]. This graph contains the nature number label for every edge and this edge of the graph can only be used at the labeling times. It is interesting to obtain the path between two different nodes (it is called a journey) since temporal graphs appear or disappear from the edge for the time change. They investigated an algorithm to compute the foremost journey, which contains the minimum arriving time, for a temporal graph.

Michail surveyed a result on the temporal graph and temporal graph problems [43] and investigated the random temporal graphs, in which the edge labels are chosen randomly in $\{1, \dots, r\}$. He proved that random temporal graphs contain a journey of length 4 if $r \geq 4$. One year later, Michail and Spirakis [44] investigated the traveling salesman problems on temporal graphs. They focused on the temporal exploration problem and claimed that it is difficult to approximate in polynomial time unless $P = NP$. Then, they gave an approximation algorithm for the traveling salesman problems on the complete graph with weight $\{1, 2\}$.

Other Related works - The cover time of a random walk on a changing graph Related to the cover time, is another major topic on random walks, Avin et al. [5] investigated the cover time of a simple random walk on dynamic undirected graphs, which edges are inserted or deleted for every step. They gave an upper and lower bound for the cover time of a dynamic undirected graph. Then, they proved that the cover time for the simple random walk on the Bernoulli evolving graph satisfies $\mathrm{O}(n^3 \log n)$, the maximum hitting time for the simple random walk on the Bernoulli evolving graph satisfies $\mathrm{O}(n^3)$ and the cover time for the simple random walk on d-regular evolving graph satisfies $\mathrm{O}(d^2 n^3 \log^2 n)$. In 2018, Avin et al. [6] investigated the cover time and the mixing time of simple random walks on dynamic undirected graphs. They gave the mixing time of a lazy random walk on dynamic undirected graphs on the assumption that there is a unique stationary distribution and extended the previous work [5] by focusing on the maximum value of the degree in the dynamic graphs. Then, they gave a bound for the cover time, hitting time, and mixing time of a special case, such as the Bernoulli evolving graph and d-regular dynamic graphs. Cooper and Frieze [11] investigated the covering rate of a random walk on the "web-graph" model, where the graph grows at a constant speed. Kijima et al. [31] introduced the RWoGG model, where the growing (inverse) speed of a graph is parameterized by $\mathfrak{d} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, and they investigated its covering rate.

1.2 Objective, contribution and result of this paper

This paper is concerned with a specific model of random walks on growing graphs (RWoGG) [31]. The network gradually grows such that the growing network keeps its shape $G(n)$ for $\mathfrak{d}(n)$ steps, then changes the shape to $G(n+1)$ by adding some vertices to $G(n)$ (see Section 2.2 for detail), and gives a phase transition by the growing speed regarding a random walk being recurrent/transient. The phase transition is very similar to or essentially the same as [16, 27], while this paper contains mainly six contributions.

In the first of this paper, we show the fact that a simple random walk on an infinitely growing complete

k-ary tree can be recurrent *depending on the growing speed* of the tree, while a simple random walk on an infinite k-ary tree is transient (see [38]). Then, we show a phase transition between the recurrence and transience of a random walk on a growing complete k-ary tree, regarding the growing speed of the graph (see Section 4,). For proof, this work develops the notion of *less-homesick as graph growing* (LHaGG), which is a quite natural property of RWoGG, and gives a simple proof by a *coupling* argument, which is an elementary technique of random walks or Markov chains based on a comparison method. This work employs a coupling argument while the existing works are based on Cheeger constant, evolving sets, Brownian motion, and a central limit theorem. Since the coupling technique is relatively simple, we can drop three assumptions in the previous works, namely a random walk being *lazy*, growing speed being slow, and a growing graph having a *uniformly bounded*, which is naturally required in the Cheeger constant argument to make the arguments simple; this is the first contribution. Furthermore, this paper is mainly concerned with reversible random walks of *period 2*, which contains simple random walks on undirected bipartite graphs; this is the second contribution. We show sufficient conditions for an LHaGG RWoGG to be recurrent (Lemma. 3.2) or transient (Lemma. 3.4). Then, we apply LHaGG lemmas to growing a complete k-ary tree and give the threshold $\sum_{k=1}^{\infty} \mathfrak{d}(k)\mathring{p}(k) = \infty$ of the phase transition (Theorem 4.2), where $\mathring{p}(k)$ denote an even stationary distribution of $o \in G(k)$. We also show some other examples of the phase transition, including as a random walk on $\{0,1\}^n$ with infinitely growing n (see Section 6). This growing graph is not uniformly bounded degree since the degree of a vertex is $\mathrm{O}(n)$. Then, we get the threshold $\sum_{k=1}^{\infty} \mathfrak{d}(k)\mathring{p}(k) = \infty$ of the phase transition (Theorem 6.1). Additionally, to claim that we apply LHaGG theory to a lazy random walk and compare my result and the prior work [16], we investigate a random walk on a growing box (see Section 5). Then, we have the threshold of the phase transition (Theorem 5.1).

However, the coupling technique is not simply applicable to a $\frac{1}{2}$ - lazy random walk on $\{0,\dots,N\}^n$ with an increasing n (and a fixed N). As a result, we introduce a *weakly less-homesick as graph growing* (weakly LHaGG), which is an extension of LHaGG and a natural property of RWoGG. It is difficult for us to prove that this growing graph satisfies weakly LHaGG using the LHaGG coupling technique. As a result, to use weakly LHaGG, we introduce the new coupling technique (write *pausing coupling*), which is based on the coupling technique [22]. Our coupling techniques compare the two growing graphs with the different growing speeds. For this reason, we can apply the coupling technique of [22] to another growing graph such as a lazy random walk on $\{0,\dots,N\}^n$ with increasing n, growing spider tree, and random tree. In particular, a lazy random walk on $\{0,\dots,N\}^n$ with increasing n has a cycle and does not use a tree property; this is the third contribution. Additionally, we also show an example of random walk on $\{0,\dots,N\}^n$ with increasing n, where the (maximum) degree of the dynamic graph, that is n, infinitely grows; this is the fourth contribution. In the middle of this paper, we show sufficient conditions for a weakly LHaGG RWoGG to be recurrent (Theorem. 7.3) or transient (Theorem. 7.8). Then, we apply weakly LHaGG theorems to $\{0,\dots,N\}^n$ with increasing n and give the threshold $\sum_{k=1}^{\infty} \mathfrak{d}(k)/(2N)^k = \infty$ of the phase transition (Theorem 8.1). Additionally, we cover the other application examples of weakly LHaGG such as random walks on growing spider trees and random trees. A growing spider tree is a distinctive example since each vertex has a different condition for recurrence and transience. In detail, the root r is always recurrent (Theorem 9.1). But the adjacent vertex to r (write v_1) has the threshold $\sum_{k=1}^{\infty} \mathfrak{d}(k)\mathring{\pi}_k(1) = \infty$ of the phase transition (Theorem 9.3), where $\mathring{\pi}_k(1)$ be an even stationary distribution of v_1 in $G(k)$. It is interesting to obtain the sufficient condition for recurrence and transience of a random walk on a growing random tree since these growing graphs are the generalization of a growing complete k - ary tree. Then, we give the sufficient condition for recurrence, if $\lim_{n\to\infty} \mathring{p}(n) \sum_{k=1}^{n} \mathfrak{d}(n) = \infty$ holds, then the root r is recurrent (Theorem 9.10). We cannot give the threshold of recurrence and transience since random trees cannot be regarded as another graph that satisfies the assumption of general theorems for mixing time.

In the last of this paper, we analyze a recurrence and transience from a slightly different point of view. We discuss the recurrence and transience of a random walk on a growing added box (see Section 10). This growing graph is a characteristic example in which the growth occurs at the internal node and edge. This implies that a coupling argument and graph growing in the internals are not compatible since the monotone logic is easy to drop. This implies that it is not necessarily that we can get the threshold of recurrence and transience by using the same approach as complete k -ary tree and $\{0,\dots,N\}^n$. However, it is possible to break through this problem by using an extension of the Dembo coupling technique [16] and LHaGG; this is the fifth contribution. Then, we obtain the threshold $\sum_{k=1}^{\infty}\mathfrak{d}(k)p(k)=\infty$ of the phase transition. Finally, we discuss the condition for a null recurrence and positive recurrence of a random walk on a growing complete graph (see Section 11). It is known that a random walk on the infinite region (such as an infinite tree and $\mathbb{Z}^3$) is transient and a finite region (such as a finite tree and boxes) is recurrent, that is the motivation for studying the recurrence and transience of random walks on growing graphs. This is true for the first return time of growing graphs. Although we can construct the assumption that depends on growing speed, it is difficult to evaluate the expectation of the first return time since the estimation of the first return probability on a growing graph is hard. For this reason, we consider a random walk on a growing complete graph since it is easy to get the first return probability and we can utilize LHaGG. Then, we obtain the sufficient condition for a null recurrence and positive recurrence by the LHaGG coupling technique such that if $\mathfrak{d}(n)=1$ for any $n\geq 1$ then the origin vertex is null recurrent if $\mathfrak{d}(n)>1$ for any $n\geq 1$ then the origin vertex is positive recurrent (Theorem 11); this is the sixth contribution.

While the coupling technique is relatively easy, it often selects the applicable target. The results by [16, 27] are widely applied to the general setting as far as it satisfies appropriate assumptions, while our result is limited to specific targets. Such an argument about conductance and coupling seems known as an implicit knowledge in the literature of mixing time analysis, cf. [36, 25, 46, 47]. However, we emphasize that the coupling technique often gives easy proof of interesting phenomena, as this paper shows.

1.3 Organization

As a preliminary, we describe the model of random walks on growing graphs (RWoGG) in Section 2.2. Section 3 introduces the notion of less homesickness as graph growing (LHaGG), and presents some general theorems for sufficient conditions of a RWoGG being recurrent/transient. Section 4 shows a phase transition between the recurrence and transience of a random walk on a growing k-ary tree. Section 5 shows the sufficient condition for the recurrence and transience of a lazy random walk on a growing box. Section 6 shows a phase transition for a random walk on $\{0,1\}^n$ with an increasing n. Section 7 introduces the notion of weakly LHaGG, and presents some general theorems for sufficient conditions that a weakly LHaGG RWoGG is recurrent/transient. Section 8 shows a phase transition between the recurrence and transience of a lazy simple random walk on $\{0,\dots,N\}^n$ with an increasing n. Section 9 gives a sufficient condition of the recurrence and transience of a random walk on a growing spider tree and a random tree by weakly LHaGG. Section 10 shows a phase transition between recurrence and transience of a random walk on growing added boxes. Section 11 shows a phase transition between null recurrence and positive recurrence of a random walk on a growing complete graph.

2 Preliminaries

2.1 Terminology

In this section, we introduce the terminology of this thesis, following [1], [26], [39], [40] and [52].

Markov chain, transition probability and stationary distribution A sequence of random variables $(X_0,X_1,\dots)$ is a *Markov chain with a countable state space V and transition probability matrix P* : $(V,V)\to[0,1]$ if for all $x,y\in V$, all $t\geq 1$, and all events $H_{t-1}=\bigcap_{s=0}^{t-1}\{X_s=x_s\}$ satisfying $\Pr(H_{t-1}\cap$

$\{X_t = x\}) > 0$, we have

$$\Pr[X_{t+1} = y | H_{t-1} \cap \{X_t = x\}] = \Pr[X_{t+1} = y | X_t = x] = P(x, y).$$

An expectation of a random variable X (denoted by $E(X)$) is defined by $E(X) := \sum_{t=0}^{\infty} \Pr[X > t]$. A probability distribution π on a finite state space V is a *stationary distribution* if it satisfies $\sum_{v \in V} \pi(v) = 1$ and $\sum_{u \in V} \pi(u)P(u, v) = \pi(v)$. A random walk is *reversible* if there exists a positive function $\mu\colon V \to \mathbb{R}_{>0}$ such that

$$\mu(u)P(u, v) = \mu(v)P(v, u) \tag{2}$$

hold for all $u, v \in V$. We call the equation (2) a *detailed balance equation*. By the detailed balance equation, we get the following proposition.

Proposition 2.1 ([39]). *Let P be the transition matrix of a Markov chain with state space V. Any distribution π satisfying* (2) *is a stationary distribution for P.*

Proof. By summing the left-hand side of (2), we get

$$\sum_{u \in V} \mu(u)P(u, v) = \sum_{u \in V} \mu(v)P(v, u) = \mu(v),$$

and hence the distribution μ satisfies the definition of the stationary distribution. □

A transition matrix P is *irreducible* if $\forall u, v \in V$, $\exists t > 0$, $(P^t)(u, v) > 0$. The period of P is given by $\mathrm{period}(P) = \min_{v \in V} \gcd\{t > 0 : (P^t)(v, v) > 0\}$. It is well known that $\gcd\{t > 0 : (P^t)(v, v) > 0\}$ is common for any $v \in V$ if P is irreducible. If $\mathrm{period}(P) = 1$ then P is said to be *aperiodic*. By using an irreducible and aperiodic, we get the following propositions.

Proposition 2.2 ([26], [39]). *Let P be a transition matrix of an irreducible and aperiodic Markov chain on a finite space V. There exists at least one stationary distribution π.*

Proof. Let X_t be a random variable with P. Let $\tilde{\pi}(y) := \sum_{t=0}^{\infty} \Pr[X_t = y,\ \tau_z^+ > t | X_0 = z]$, where $\tau_x^+ := \min\{t \geq 1\ ;\ X_t = x\}$. Then, we get

$$\begin{aligned}
&\sum_{x \in V} \tilde{\pi}(x)P(x, y) = \sum_{x \in V} \sum_{t=0}^{\infty} \Pr\left[X_t = x,\ \tau_z^+ > t \middle| X_0 = z\right] P(x, y) \\
&= \sum_{t=0}^{\infty} \Pr\left[X_{t+1} = y,\ \tau_z^+ \geq t+1 \middle| X_0 = z\right] = \sum_{t=1}^{\infty} \Pr\left[X_t = y,\ \tau_z^+ \geq t \middle| X_0 = z\right] \\
&= -\Pr\left[X_0 = y,\ \tau_z^+ \geq 0 \middle| X_0 = z\right] + \sum_{t=0}^{\infty} \Pr\left[X_t = y,\ \tau_z^+ \geq t \middle| X_0 = z\right] \\
&= -\Pr\left[X_0 = y,\ \tau_z^+ \geq 0 \middle| X_0 = z\right] + \sum_{t=0}^{\infty} \Pr\left[X_t = y,\ \tau_z^+ > t \middle| X_0 = z\right] \\
&+ \sum_{t=0}^{\infty} \Pr\left[X_t = y,\ \tau_z^+ = t \middle| X_0 = z\right] \\
&= -\Pr[X_0 = y | X_0 = z] + \tilde{\pi}(y) + \Pr\left[X_{\tau_z^+} = y \middle| X_0 = z\right]
\end{aligned} \tag{3}$$

for $y \in V$. If $y \neq z$, then (3) $= -0 + \tilde{\pi}(y) + 0 = \tilde{\pi}(y)$ holds for $y \in V \setminus \{z\}$.

If $y = z$, then (3) $= -1 + \tilde{\pi}(y) + 1 = \tilde{\pi}(y)$ holds. Therefore, we obtain $\sum_{u \in V} \tilde{\pi}(u)P(u, v) = \tilde{\pi}(v)$. Since $\sum_{u \in V} \tilde{\pi}(u) = \sum_{t=0}^{\infty} \Pr[\tau_z^+ > t | X_0 = z] = E_z[\tau_z^+]$ hold, we give $\sum_{u \in V} \pi(u)P(u, v) = \pi(v)$ and $\sum_{u \in V} \pi(u)$, where $\pi(v) = \frac{\tilde{\pi}(v)}{E_z[\tau_z^+]}$. Therefore, we obtain the claim. □

Proposition 2.3 ([26], [39])**.** *Let P be a transition matrix of an irreducible and aperiodic Markov chain on V. There exists an unique stationary distribution π satisfying $\pi(z) = \frac{1}{E_z(\tau_z^+)}$ for any z, where $\tau_z^+ := \min\{t \geq 1\ ;\ Z_t = z\}$ and Z_t is a random walk on V according to P.*

Proof. Let π and π' respectively be a stationary distribution for P. We must prove that $\pi = \pi'$. We get $||\pi' - \pi||_{TV} = ||\pi' P^t - \pi||_{TV} \leq \sup_\mu ||\mu P^t - \pi||_{TV} = d(t)$ by $\pi' P^t = \pi'$ and Proposition 2.8. Since P is an irreducible and aperiodic, we have $\lim_{t\to\infty} ||\pi' - \pi||_{TV} = ||\pi' - \pi||_{TV} \leq \lim_{t\to\infty} d(t) = 0$ by Proposition 2.11. Therefore, $\pi = \pi'$.

By Proposition 2.2, $\pi(z) = \frac{1}{E_z\left[\tau_z^+\right]}$ holds for any $z \in V$. Thus, we obtain the claim. □

Total variation distance and coupling To evaluate a return probability, we consider the total variation distance. The *total variation distance* between μ and ν is defined by

$$||\mu - \nu||_{TV} := \frac{1}{2} \sum_{x \in V} |\mu(x) - \nu(x)|.$$

By total variation distance, we get the following propositions.

Proposition 2.4 ([26], [39], [52])**.** *Let μ and ν be two probability distributions on V. Then, we get*

$$||\mu - \nu||_{TV} = \max_{A \subseteq V} |\mu(A) - \nu(A)|.$$

Proof. Let $B := \{x \in V\ ;\ \mu(x) \geq \nu(x)\}$. Then,

$$\frac{1}{2} \sum_{x \in V} |\mu(x) - \nu(x)| = \frac{1}{2} \left[\mu(B) - \nu(B) + \nu(B^c) - \mu(B^c)\right] \tag{4}$$

$$\mu(A) - \nu(A) \leq \mu(A \cap B) - \nu(A \cap B) \leq \mu(B) - \nu(B) \tag{5}$$

$$\nu(A) - \mu(A) \leq \nu(A \cap B^c) - \mu(A \cap B^c) \leq \nu(B^c) - \mu(B^c) \tag{6}$$

hold for any $A \subseteq V$. This means that $\max_{A \subseteq V} |\mu(A) - \nu(A)| = \max\{\mu(B) - \nu(B), \nu(B^c) - \mu(B^c)\}$ Since

$$\mu(B) + \mu(B^c) = 1$$
$$\nu(B) + \nu(B^c) = 1$$

hold for any two probability distribution μ and ν on V, we have

$$\mu(B) - \nu(B) = \nu(B^c) - \mu(B^c).$$

Then, we obtain (4) $= \mu(B) - \nu(B) = \nu(B^c) - \mu(B^c) = \max_{A \subseteq V} |\mu(A) - \nu(A)|$. Thus, we obtain the claim. □

Proposition 2.5 ([26], [39])**.** *Let μ, ν and η be probability distributions on V. We obtain*

$$||\mu - \nu||_{TV} \leq ||\mu - \eta||_{TV} + ||\eta - \nu||_{TV}.$$

Proof. By triangle inequality, we obtain

$$\frac{1}{2} \sum_{x \in V} |\mu(x) - \nu(x)| = \frac{1}{2} \sum_{x \in V} |\mu(x) - \eta(x) + \eta(x) - \nu(x)|$$
$$\leq \frac{1}{2} \sum_{x \in V} |\mu(x) - \eta(x)| + \frac{1}{2} \sum_{x \in V} |\eta(x) - \nu(x)| = ||\mu - \eta||_{TV} + ||\eta - \nu||_{TV}.$$

□

To discuss a total variation distance, we focus on the coupling. A *coupling* of two probability distributions μ and ν is a pair of random variables (X, Y) defined on a single probability space such that the marginal distribution of X is μ and the marginal distribution of Y is ν. That is, a coupling (X, Y) satisfies $P\{X = x\} = \mu(x)$ and $P\{Y = y\} = \nu(y)$. By using the coupling, we obtain the following.

Proposition 2.6 ([26], [39], [52]). *Let μ and ν be two probability distributions on V. Then,*

$$||\mu - \nu||_{TV} = \inf\{P\{X \neq Y\} \ ; \ (X, Y) \text{ is a coupling of } \mu \text{ and } \nu \ \}. \tag{7}$$

Proof. First, we note that for any coupling (X, Y) of μ and ν and any event $A \subset V$,

$$\begin{aligned} \mu(A) - \nu(A) &= P\{X \in A\} - P\{Y \in A\} \\ &\leq P\{X \in A, Y \notin A\} \\ &\leq P\{X \notin Y\}. \end{aligned}$$

It immediately follows that

$$||\mu - \nu||_{TV} \leq \inf\{P\{X \neq Y\} \ ; \ (X, Y) \text{ is a coupling of } \mu \text{ and } \nu \ \}.$$

It will suffice to construct a coupling for which $P\{X \neq Y\}$ is exactly equal $||\mu - \nu||_{TV}$. We will do so by forcing X and Y to be equal as often as they possibly can be. Let $B := \{x \in V \ ; \ \mu(x) \geq \nu(x)\}$ Region I has area $\mu(B) - \nu(B)$. Region II has area $\nu(B^c) - \mu(B^c)$. Region III, bounded by $\mu(x) \wedge \nu(x) = \min\{\mu(x), \nu(x)\}$, can be seen as the overlap between the two distributions. Informally, our coupling proceeds by choosing a point in the union of regions I and III, and setting X to be the x-coordinate of this point. If the point is in III, we set $Y = X$ and if it is in I, then we choose independently a point at random from region II, and set Y to be the x-coordinate of the newly selected point. In the second scenario, $X = Y$, since the two regions are disjoint. More formally, we use the following procedure to general X and Y. Let

$$p := \sum_{x \in V} \mu(x) \wedge \nu(x).$$

Write

$$\sum_{x \in V} \mu(x) \wedge \nu(x) = \sum_{x \in V, \mu(x) \leq \nu(x)} \mu(x) + \sum_{x \in V, \mu(x) > \nu(x)} \nu(x).$$

Adding and subtracting $\sum_{x;\mu(x)>\nu(x)} \mu(x)$ to the right-hand side above shows that

$$\sum_{x \in V} \mu(x) \wedge \nu(x) = 1 - \sum_{x \in V, \mu(x) > \nu(x)} [\mu(x) - \nu(x)].$$

Since $\sum_{x \in V, \mu(x) > \nu(x)} [\mu(x) - \nu(x)] = ||\mu - \nu||_{TV}$, we give

$$\sum_{x \in V} \mu(x) \wedge \nu(x) = 1 - ||\mu - \nu||_{TV} = p.$$

Flip a coin with probability of heads equal to p. (i) If the coin comes up heads, then choose a value Z according to the probability distribution

$$\gamma_{\text{III}}(x) = \frac{\mu(x) \wedge \nu(x)}{p},$$

and set $X = Y = Z$.

(ii) If the coin comes up tails, choose X according to the probability distribution

$$\gamma_{\mathrm{I}}(x) = \begin{cases} \frac{\mu(x)-\nu(x)}{||\mu-\nu||_{TV}} & \text{if } \mu(x) > \nu(x), \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and independently choose Y according to the probability distribution

$$\gamma_{\mathrm{II}}(x) = \begin{cases} \frac{\nu(x)-\mu(x)}{||\mu-\nu||_{TV}} & \text{if } \nu(x) > \mu(x), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Note that γ_{I} and γ_{II} are probability distributions.

Clearly,

$$p\gamma_{\mathrm{III}} + (1-p)\gamma_{\mathrm{I}} = \mu,$$
$$p\gamma_{\mathrm{III}} + (1-p)\gamma_{\mathrm{II}} = \nu,$$

so that the distribution of X is μ and the distribution of Y is ν. Note that in the case that the coin lands tails up, $X \neq Y$ since γ_{I} and γ_{II} are positive on disjoint subsets of V. Thus $X = Y$ if and only if the coin toss is heads. We conclude that

$$P\{X \neq Y\} = ||\mu - \nu||_{TV}.$$

□

In particular, let $d(t)$ and $\overline{d}(t)$ respectively denote $d(t) := \max_{x\in\mathcal{X}} ||P^t(x,\cdot) - \pi||_{TV}$ and $\overline{d}(t) := \max_{x,y\in\mathcal{X}} ||P^t(x,\cdot) - P^t(y,\cdot)||_{TV}$. $d(t)$ and $\overline{d}(t)$ satisfy the following propositions.

Proposition 2.7 ([26], [39], [52])**.** *Suppose P is irreducible and aperiodic. $d(t)$ and $\overline{d}(t)$ satisfy*

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

Proof. Using Proposition 2.5, we get

$$||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} \leq ||P^t(x,\cdot) - \pi(\cdot)||_{TV} + ||\pi(\cdot) - P^t(y,\cdot)||_{TV}$$
$$\overline{d}(t) \leq \max_{x,y} ||P^t(x,\cdot) - \pi(\cdot)||_{TV} + \max_{x,y} ||P^t(y,\cdot) - \pi(\cdot)||_{TV} = 2d(t).$$

Therefore, $\overline{d}(t) \leq 2d(t)$ holds.

Since $\pi(A) = \sum_{x\in V} \pi(x)P^t(x,A)$, we obtain

$$\begin{aligned} |P^t(x,A) - \pi(A)| &= \left| \sum_{x\in V} \pi(x) \left[P^t(x,A) - P^t(y,A)\right] \right| \\ &\leq \sum_{x\in V} \pi(x)||P^t(x,A) - P^t(y,A)||_{TV} \leq \overline{d}(t). \end{aligned}$$

Therefore, $d(t) \leq \overline{d}(t)$ holds, and hence we obtain the claim. □

Proposition 2.8 ([39])**.** *$d(t)$ and $\overline{d}(t)$ satisfy*

$$d(t) = \sup_{\mu} ||\mu P^t - \pi||_{TV}$$
$$\overline{d}(t) = \sup_{\mu,\nu} ||\mu P^t - \nu P^t||_{TV},$$

where μ and ν vary over probability distributions on V.

Proof. By the definition of the total variation distance, we get

$$\begin{aligned}
||\mu P^t - \pi||_{TV} &= \frac{1}{2}\sum_{v\in V}|\mu P^t(v) - \pi(v)| \\
&= \frac{1}{2}\sum_{v\in V}|\sum_{x\in V}\mu(x)P^t(x,v) - \sum_{x\in V}\mu(x)\pi(v)| \\
&\leq \frac{1}{2}\sum_{v\in V}\sum_{x\in V}\mu(x)|P^t(x,v) - \pi(v)| \\
&= \sum_{x\in V}\mu(x)\frac{1}{2}\sum_{v\in V}|P^t(x,v) - \pi(v)| \\
&= \sum_{x\in V}\mu(x)||P^t(x,\cdot) - \pi||_{TV} \\
&\leq \max_{x\in V}||P^t(x,\cdot) - \pi||_{TV}\sum_{x\in V}\mu(x) = \max_{x\in V}||P^t(x,\cdot) - \pi||_{TV}
\end{aligned}$$

for any μ. Therefore, we have

$$\sup_{\mu}||\mu P^t - \pi||_{TV} \leq \max_{x\in V}||P^t(x,\cdot) - \pi||_{TV} = d(t).$$

The opposite inequality holds, since the set of probabilities on V includes the point masses.

Similarly, if α and β are two probabilities on V, we give

$$\begin{aligned}
||\alpha P^t - \beta P^t||_{TV} &= \frac{1}{2}\sum_{z\in V}|\alpha P^t(z) - \sum_{\omega\in V}\beta(\omega)P^t(\omega,z)| \qquad (10) \\
&= \frac{1}{2}\sum_{z\in V}\left|\sum_{\omega\in V}\beta(\omega)\alpha P^t(z) - \sum_{\omega\in V}\beta(\omega)P^t(\omega,z)\right| \\
&\leq \frac{1}{2}\sum_{z\in V}\sum_{\omega\in V}\beta(\omega)|\alpha P^t(z) - P^t(\omega,z)| \\
&= \sum_{\omega\in V}\beta(\omega)\frac{1}{2}\sum_{z\in V}|\alpha P^t(z) - P^t(\omega,z)| \\
&= \sum_{\omega\in V}\beta(\omega)||\alpha P^t - P^t(\omega,\cdot)||_{TV} \\
&\leq \max_{\omega\in V}||\alpha P^t - P^t(\omega,\cdot)||_{TV}\sum_{\omega\in V}\beta(\omega) = \max_{\omega\in V}||\alpha P^t - P^t(\omega,\cdot)||_{TV}. \qquad (11)
\end{aligned}$$

Thus, applying (11) to $\alpha = \mu$ and $\beta = \nu$ gives that

$$||\mu P^t - \nu P^t||_{TV} \leq \max_{y\in V}||\mu P^t - P^t(y,\cdot)||_{TV}. \qquad (12)$$

Applying (11) to $\alpha = \delta_y$, where $\delta_y(z) = \mathbf{1}_{\{z=y\}}$, and $\beta = \mu$ shows that

$$||P^t(y,\cdot) - \mu P^t||_{TV} \leq \max_{x\in V}||P^t(y,\cdot) - P^t(x,\cdot)||_{TV}. \qquad (13)$$

Combining (12) and (13) shows that

$$\sup_{\mu,\nu}||\mu P^t - \nu P^t||_{TV} \leq \max_{x,y\in V}||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} = \bar{d}(t).$$

The opposite inequality also holds, and hence we obtain the claim. □

Proposition 2.9 ([39], [52])**.** *Let P be the transition matrix of a Markov chain with state space V and let μ and ν be any two distributions on V. We can prove that*

$$||\mu P - \nu P||_{TV} \leq ||\mu - \nu||_{TV},$$

and hence $d(t+1) \leq d(t)$ and $\overline{d}(t+1) \leq \overline{d}(t)$ hold for any $t \geq 0$.

Proof. By the definition of the total variation distance, we get

$$\begin{aligned}
||\mu P - \nu P||_{TV} &= \frac{1}{2}\sum_{x\in V} |\mu P(x) - \nu P(x)| \\
&= \frac{1}{2}\sum_{x\in V} \left|\sum_{y\in V} \mu(y)P(y,x) - \sum_{y\in V} \nu(y)P(y,x)\right| \\
&= \frac{1}{2}\sum_{x\in V} \left|\sum_{y\in V} P(y,x)[\mu(y) - \nu(y)]\right| \\
&\leq \frac{1}{2}\sum_{x\in V}\sum_{y\in V} P(y,x)|\mu(y) - \nu(y)| \\
&= \frac{1}{2}\sum_{y\in V} |\mu(y) - \nu(y)| \sum_{x\in V} P(y,x) \\
&= \frac{1}{2}\sum_{y\in V} |\mu(y) - \nu(y)| = ||\mu - \nu||_{TV}.
\end{aligned}$$

Thus, we obtain the claim. □

Proposition 2.10 ([39], [52])**.** *Suppose P is irreducible and aperiodic. $\overline{d}(t)$ satisfies*

$$\overline{d}(s+t) \leq \overline{d}(t)\overline{d}(s).$$

Proof. Let $X_0 = x$ and $Y_0 = y$. By Proposition 2.6, there exist a coupling of X_{t+s} and Y_{t+s} such that

$$\Pr\{X_t \neq Y_t\} = ||P^t(x,\cdot) - P^t(y,\cdot)||_{TV}.$$

We then construct a coupling of X_{t+s} and Y_{t+s} as follows: (i) If $X_t = Y_t$ then set $X_{t+i} = Y_{t+i}$ for $i = 0, 1, \ldots, s$. (ii) Otherwise, let $X_t = x'$ and $Y_t = y' \neq x'$. Use Proposition 2.6 to couple the distributions of X_{t+s} and Y_{t+s}, conditioned on $X_t = x'$ and $Y_t = y'$, such that

$$\Pr\left[X_{t+s} \neq Y_{t+s} \middle| X_t = x', Y_t = y'\right] = ||P^s(x,\cdot) - P^s(y,\cdot)||_{TV} \leq \overline{d}(s).$$

The last inequality holds by the definition of $\overline{d}(s)$. We now have

$$\begin{aligned}
||P^{t+s}(x,\cdot) - P^{t+s}(y,\cdot)||_{TV} &\leq \Pr\{X_{t+s} \neq Y_{t+s}\} \\
&\leq \overline{d}(s)||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} \\
&\leq \overline{d}(s)\overline{d}(t).
\end{aligned}$$

Since this holds for all x and y, we get $\overline{d}(s+t) \leq \overline{d}(s)\overline{d}(t)$. □

Proposition 2.11 ([26], [39], [52])**.** *Suppose P is irreducible and aperiodic. We have $\lim_{t\to\infty} d(t) = 0$.*

To prove Proposition 2.11, we must prove Proposition 2.12.

Proposition 2.12 ([26], [39], [52]). *Suppose P is irreducible and aperiodic. There is a finite t such that $\overline{d}(t) < 1$ holds.*

Proof. By assumption, there exists t_0 such that $P^t(x, y) > 0$ for any $x, y \in V$ and $t \geq t_0$. Meaning that $\overline{d}(t_0) < 1$, and hence the proposition follows. □

Proof of Proposition 2.11. Fix $s, s' \in \mathbb{N}$. By Proposition 2.7, 2.9, 2.10 and 2.12, we give

$$\begin{aligned} d(n+s') &\leq \overline{d}(n+s') \qquad \text{(By Proposition 2.7)} \\ &\leq \overline{d}(\lfloor \frac{n}{s} \rfloor s + s') \qquad \text{(By Proposition 2.9)} \\ &\leq \overline{d}(s')\overline{d}(\lfloor \frac{n}{s} \rfloor s) \qquad \text{(By Proposition 2.10)} \\ &\leq \overline{d}(s')\overline{d}(s)^{\lfloor \frac{n}{s} \rfloor}. \end{aligned}$$

By Proposition 2.12, there exist a finite s such that $\overline{d}(s) < 1$. Meaning that we have

$$\lim_{t \to \infty} d(t) = \lim_{n \to \infty} d(n+s') \leq \overline{d}(s') \lim_{n \to \infty} \overline{d}(s)^{\lfloor \frac{n}{s} \rfloor} = 0.$$

Thus, we obtain the claim. □

Graph and tree In this paragraph, we introduce the terminology of graphs and trees. A *graph* $G = (V, E)$ consists of a vertex set V and edge set E, where the elements of E are unordered pairs of vertices: $E \subset \{\{x, y\} \ ; \ x, y \in V, x \neq y\}$. When $\{x, y\} \in E$, we write $x \sim y$ and say that y is a neighbor of x (and also that x is a neighbor of y). The *degree* of a vertex $x \in V$ $\deg(x)$ is the number of neighbors of x. More precisely, we defined by $\deg(x) := |\{y \ ; \ x \sim y\}|$. A graph $G = (V, E)$ is *connected* when, for two vertices x and y of G, there exists a sequence of vertices $x_0, x_1, \ldots, x_k$ such that $x_0 = x$, $x_k = y$ and $\{x_i, x_{i+1}\} \in E$ for $0 \leq i \leq k-1$. A graph $T = (V, E)$ is a *tree* if T does not contain a cycle and is connected. A *leaf* is a vertex of degree 1 in the tree. We define a simple random walk on G to be the Markov chain with a state space V and transition matrix

$$P(x, y) = \begin{cases} \frac{1}{\deg(x)} & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Mixing time In this paragraph, we introduce the mixing time and coupling. The *mixing time* of P is given by

$$\tau(\epsilon) := \min \{t \ ; \ t \in \mathbb{Z}_{>0}, \ d(t) \leq \epsilon\} \tag{15}$$

for $\epsilon \in (0, 1)$.

Remark 2.13 ([39], [52]). *Let π_o be a stationary distribution on $o \in V$. If $t \geq \tau\left(\frac{\pi_o}{4}\right)$ then*

$$d(t) \leq 2d\left(\tau\left(\frac{\pi_o}{4}\right)\right) \leq \frac{\pi_o}{2} \tag{16}$$

holds.

Proof. For convenience, set $t = \tau(\frac{\pi_o}{4}) + s$ for $s \geq 0$. It is known that $d(t) \leq \overline{d}(t)$ (Proposition 2.7). By the submultiplicativity of $\overline{d}$ (Proposition 2.10 [39]), $\overline{d}(t) \leq \overline{d}\left(\tau(\frac{\pi_o}{4})\right)\overline{d}(s)$ holds. Since $\overline{d}(s) \leq 1$ clearly, $\overline{d}\left(\tau(\frac{\pi_o}{4})\right)\overline{d}(s) \leq \overline{d}\left(\tau(\frac{\pi_o}{4})\right)$ holds. Since $\overline{d}(t') \leq 2d(t')$ by Proposition 2.7, $\overline{d}\left(\tau(\frac{\pi_o}{4})\right) \leq 2d\left(\tau(\frac{\pi_o}{4})\right)$. Since $d\left(\tau(\frac{\pi_o}{4})\right) \leq \frac{\pi_o}{4}$ by definition, we obtain $d(t) \leq \frac{\pi_o}{2}$. □

$\tau(\epsilon)$ satisfies the following Lemma.

Lemma 2.14 ([39], [52])**.** *Suppose P is irreducible and aperiodic. The mixing time $\tau(\epsilon)$ satisfies*

$$\tau(\epsilon) \leq \lceil \log_2 \epsilon^{-1} \rceil \tau\left(\frac{1}{4}\right).$$

Proof. By Proposition 2.7 and 2.10, we get

$$d(l\tau(\frac{1}{4})) \leq \overline{d}(l\tau(\frac{1}{4})) \leq \overline{d}(\tau(\frac{1}{4}))^l \leq \left(2d(\tau(\frac{1}{4}))\right)^l \leq 2^{-l}.$$

By $l = \log_2 \epsilon^{-1}$, we provide $d\left(\log_2 \epsilon^{-1}\tau(\frac{1}{4})\right) \leq \epsilon$. Therefore, we obtain the claim. □

Theorem 2.15 ([39], [52])**.** *Let $\{(X_t, Y_t)\}_{t\geq 0}$ be a coupling satisfying*

$$\textit{if } X_s = Y_s \textit{ then } X_t = Y_t \textit{ for } t \geq s$$

for which $X_0 = x$ and $Y_0 = y$. Let τ_{couple} be the coalescence time of the chains:

$$\tau_{couple} := \min\{t\ ;\ X_s = Y_s \textit{ for all } s \geq t\}.$$

Suppose X_t and Y_t are irreducible and aperiodic. Then

$$\overline{d}(t) \leq P_{x,y}\{\tau_{couple} > t\}.$$

Proof. Notice that $P^t(x,z) = P_{x,y}\{X_t = z\}$ and $P^t(y,z) = \boldsymbol{P}_{x,y}\{Y_t = z\}$. Proposition 2.6 implies that

$$||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} \leq P_{x,y}\{X_t \neq Y_t\}.$$

By the definition of the coupling of X and Y, $P_{x,y}\{X_t \neq Y_t\} = P_{x,y}\{\tau_{couple} > t\}$ holds. Therefore, we obtain the claim. □

Corollary 2.16 ([39], [52])**.** *Suppose P is irreducible and aperiodic. $d(t)$ satisfies*

$$d(t) \leq \max_{x,y\in\chi} P_{x,y}\{\tau_{couple} > t\},$$

and hence we get

$$\tau(\epsilon) \leq 4\lceil \log_2 \epsilon^{-1} \rceil \max_{x,y} E_{x,y}(\tau_{couple}).$$

2.2 Model : growing graph

A growing graph is a sequence of (static) graphs $\boldsymbol{\mathcal{G}} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ where $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ for $t = 0, 1, 2, \ldots$ denotes a graph[2] with a finite vertex set $\mathcal{V}_t$ and an edge set $\mathcal{E}_t \subseteq \binom{\mathcal{V}_t}{2}$. For simplicity, this paper assumes[3] $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$. In this paper, we assume $|\mathcal{V}_\infty| = \infty$, otherwise the subject (recurrence) is trivial. A random walk on a growing graph is a Markovian series $X_t \in \mathcal{V}_t$ $(t = 0, 1, 2, \ldots)$.

In particular, this paper is concerned with a specific model, described as follows, cf. [31]. A *random walk on a growing graph* (*RWoGG*), in this paper, is formally characterized by a 3-tuple of functions $\mathcal{D} = (\mathfrak{d}, G, P)$. The function $\mathfrak{d} \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ denotes the duration. For convenience, let $T_n^{\mathfrak{d}} = \sum_{i=1}^{n} \mathfrak{d}(i)$ for $n = 1, 2, \ldots$[4] and $T_0^{\mathfrak{d}} = 0$. We call the time interval $[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}}]$ *phase* n for $n = 1, 2, \ldots$; thus $T_{n-1}^{\mathfrak{d}} = \sum_{i=1}^{n-1} \mathfrak{d}(i)$ is the beginning of the n-th phase, but we also say that $T_{n-1}^{\mathfrak{d}}$ is the end of the $(n-1)$-st phase, for convenience. The function $G \colon \mathbb{Z}_{>0} \to \mathfrak{G}$ represents the graph $G(n) = (V(n), E(n))$ for the phase n, where $\mathfrak{G}$ denotes the set of all (static) graphs, i.e., our growing graph $\boldsymbol{\mathcal{G}}$ satisfies $\mathcal{G}_t = G(n)$ for $t \in [T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})$. Similarly, the function $P \colon \mathbb{Z}_{>0} \to \mathfrak{M}$ is a function that represents the "transition probability" of a random walk on graph $G(n)$ where $\mathfrak{M}$ denotes the set of all stochastic matrices.

A RWoGG X_t $(t = 0, 1, 2, \ldots)$ characterized by $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ is temporally a time-homogeneous finite Markov chain according to $P(n)$ with the state space $V(n)$ during the time interval $[T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}}]$; precisely, a transition from X_t to X_{t+1} follows $P(n)$ for any $t \in [T_{n-1}^{\mathfrak{d}}, T_n^{\mathfrak{d}})$ (see Figure 1). We specially remark for $t = T_n^{\mathfrak{d}}$ that $X_t \in V(n) \subseteq V(n+1)$, meaning that X_t is a state of $V(n+1)$ but actually X_t must be in $V(n)$ by the definition of the transition. Suppose $X_0 = v$ for $v \in V(1)$.

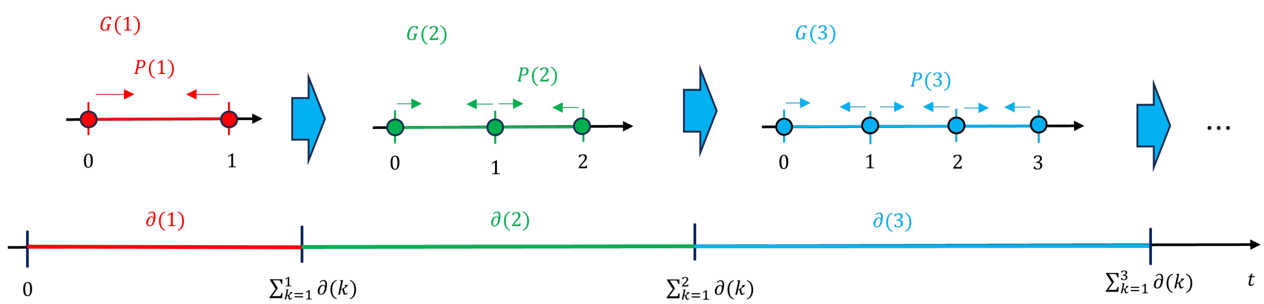

Figure 1: Random walks on growing graphs

We define the return probability at v by

$$R_{\mathfrak{d}}(t) = \Pr[X_t = v] \quad (= \Pr[X_t = v \mid X_0 = v]) \tag{17}$$

at each time $t = 0, 1, 2, \ldots$. We say v is *recurrent* by RWoGG $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ if

$$\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) = \infty \tag{18}$$

holds, otherwise, i.e., $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t)$ is finite, v is *transient* by $\mathcal{D}_{\mathfrak{d}}$. Furthermore, suppose v is *recurrent* by RWoGG $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$. We call v is *null recurrent* if

$$E_v[\tau_v^+] = \infty \tag{19}$$

holds, otherwise, i.e., $E_v[\tau_v^+]$ is finite, v is *positive recurrent* by $\mathcal{D}_{\mathfrak{d}}$.

[2]Every static graph is simple and undirected in this paper, for simplicity of the arguments.
[3]Thus, the current position does not disappear in the next step.
[4]We do not exclude $T_{n-1}^{\mathfrak{d}} = T_n^{\mathfrak{d}}$; if $\mathfrak{d}(n) = 0$ then $T_{n-1}^{\mathfrak{d}} = T_n^{\mathfrak{d}}$.

2.3 Terminology on time-homogeneous Markov chains

We here briefly introduce some terminology for random walks on static graphs, or time-homogeneous Markov chains, according to [39].

Ergodic random walks A transition matrix P is *ergodic* if it is irreducible and aperiodic. We say a random walk is (γ-)*lazy* if $P(v, v) \geq \gamma$ holds for any $v \in V$ for a constant γ ($0 < \gamma < 1$). A lazy random walk is aperiodic.

Proposition 2.17 ([39]). *If P is irreducible, reversible and $\frac{1}{2}$ - lazy then $\pi(v) \leq P^{t+1}(v, v) \leq P^t(v, v)$ for any $t = 0, 1, 2, \ldots$.*

Proof. Define $Q = 2P - I$. Enlarge the state space by adding a new state $m_{xy} = m_{yx}$ for each pair of state $x, y \in V$ with $Q(x, y) > 0$.

On the larger state space V_K define a transition matrix K by

$$K(x, m_{xy}) = Q(x, y) \qquad \text{for } x, y \in V$$
$$K(m_{xy}, x) = K(m_{xy}, y) \qquad \text{for } x \neq y$$
$$K(m_{xx}, x) = 1 \qquad \text{for all } x$$

other transitions having K - probability 0. Then K is reversible with stationary measure π_K given by $\pi_K(x) = \frac{\pi(x)}{2}$ for $x \in V$ and

$$\pi_k(m_{xy}) = \begin{cases} \pi(x)Q(x, y) & \text{if } x \neq y \\ \pi(x)\frac{Q(x,y)}{2} & \text{if } x = y. \end{cases} \tag{20}$$

Clearly $K^2(x, y) = P(x, y)$ for $x, y \in V$, so $K^{2t}(x, y) = P^t(x, y)$. Applying K to Proposition 2.21,

$$P^t(v, v) \geq P^{t+1}(v, v) \geq \ldots \geq \pi(v)$$

for any $v \in V$ and $t = 0, 1, 2, \ldots$, and hence we obtain the claim. □

2.4 Random walk with period 2

A *simple* random walk (or "busy" simple random walk) on an undirected graph $G = (V, E)$ is given by $P(u, v) = 1/\deg(u)$ for $\{u, v\} \in E$ where $\deg(u)$ denotes the degree of $u \in V$ on G. This paper is mainly concerned with bipartite graphs, such as trees, integer grids, and 0-1 hypercubes, and then the most targeted random walks are irreducible and reversible, but *not aperiodic*.

Observation 2.18. *If P is reversible then its period is at most 2.*

Suppose P is irreducible and reversible, and it has period 2. Then, the underlying graph is a connected bipartite $(U, \overline{U}; E)$, where $U = \{u \in V \ ; \ \exists t', \ P^{2t'}(v, u) \neq 0\}$ for any $v \in U$, $\overline{U} = \{u \ ; \ \forall t, \ P^{2t}(v, u) = 0\}$, i.e., $\overline{U} = V \setminus \overline{U}$, and $E = \{\{u, v\} \in V^2 \ ; \ P(u, v) > 0\}$. Notice that E does not contain any self-loop, otherwise, P is aperiodic. Let $\mathring{E} = \{\{u, v\} \in U^2 \ ; \ P^2(u, v) > 0\}$

Proposition 2.19 ([39]). *A random walk on $(U, \mathring{E})$ with the transition probability matrix $Q(u, v) = P^2(u, v)$ for any $u, v \in U$ satisfies irreducible and aperiodic.*

Proof. By the definition of U, there exists an integer t (with respected to u and v) such that $P^{2t}(u, v) = Q^t(u, v) > 0$ for any $u, v \in U$. Therefore, a random walk on $(U, \mathring{E})$ with the transition probability matrix Q is irreducible.

Since P is irreducible and it has period 2, there exists a vertex y such that $P(x,y)P(y,x) > 0$ for any $x \in V$. This implies that, for any $u \in U$, there exists a $v \in U$ such that $P^2(u,u) > P(u,v)P(v,u) > 0$. Therefore, $Q(u,u) > 0$ holds, and hence a random walk on $(U, \mathring{E})$ with the transition probability matrix Q is aperiodic. □

Here, we introduce some unfamiliar terminology for periodic Markov chains. We say $\mathring{x} \in \mathbb{R}^V_{\geq 0}$ is *even-time distribution* if it satisfies $\sum_{v \in V} \mathring{x}(v) = 1$ and $\mathring{x}(u) = 0$ for any $u \in \overline{U}$. We say $\mathring{\pi} \in \mathbb{R}^V_{\geq 0}$ is *even-time stationary distribution* if it is an even-time distribution and satisfies $\mathring{\pi} P^2 = \mathring{\pi}$.

Proposition 2.20 (limit distribution)**.** *Suppose P is irreducible and reversible, and it has period 2. Then, P has an unique even-time stationary distribution $\mathring{\pi}$, and $\lim_{t\to\infty} \mathring{x} P^{2t} = \mathring{\pi}$ for any even-time distribution $\mathring{x}$.*

Proof. Let π be a stationary distribution according to P. Consider a random walk on $(U, \mathring{E})$ which has the transition probability matrix $Q(u,v) = P^2(u,v)$ for any $u, v \in U$. By Proposition 2.3 and Proposition 2.19, there exists a unique stationary distribution $\mathring{\pi}$ such that $\mathring{\pi} = \mathring{\pi} Q$. Let π' denote

$$\pi'(u) = \begin{cases} 0 & (\text{if } u \in \overline{U}) \\ \frac{\pi(u)}{\sum_{v \in U} \pi(v)} & (\text{if } u \in U). \end{cases} \tag{21}$$

Since $\sum_{x \in U} \pi(x) P^2(x,y) = \pi(y)$ for any $y \in U$, $\sum_{x \in U} \pi'(x) P^2(x,y) = \pi'(y)$ holds for any $y \in U$. Therefore,

$$\sum_{u \in U} \mathring{\pi}(u) Q(u,v) = \mathring{\pi}(v) \text{ and } \sum_{u \in U} \mathring{\pi}(u) = 1$$

hold for any $v \in U$, where $\mathring{\pi}(u) = \frac{\pi(u)}{\sum_{v \in U} \pi(v)}$. □

By Proposition 2.19 and 2.20, we get the following corollaries.

Corollary 2.21. *Let X_t denote a random walk on $(U, \mathring{E})$ with the transition probability matrix Q. Then, we can apply X to Proposition 2.1 – Corollary 2.16*

Corollary 2.22. *Suppose P is irreducible, a simple random walk on a bipartite graph $G = (V, E)$ and period 2. Then, $\mathring{\pi}(u) = \frac{\deg(u)}{|E|}$ holds.*

Proof. By Proposition 2.20, we provide

$$\pi'(u) = \begin{cases} 0 & (\text{if } u \in \overline{U}) \\ \frac{\deg(u)}{\sum_{v \in U} \deg(v)} & (\text{if } u \in U). \end{cases} \tag{22}$$

By assumption, the all edges $\{x, y\} \in E$ satisfy $x \in U$ and $y \in \overline{U}$, or $x \in \overline{U}$ and $y \in U$. Therefore,

$$\sum_{v \in U} \deg(v) = |\left\{(x,y) \in E \,;\, x \in U, y \in \overline{U}\right\}|$$

holds. Furthermore, we get

$$|\left\{(x,y) \in E \,;\, x \in U, y \in \overline{U}\right\}| = |\left\{(y,x) \in E \,;\, y \in \overline{U}, x \in U\right\}| = |E|.$$

Thus, we obtain $\mathring{\pi}(u) = \frac{\deg(u)}{|E|}$. □

We define the *even mixing-time* of P by

$$\mathring{\tau}(\epsilon) = \min\left\{2t' \;;\; t' \in \mathbb{Z}_{>0},\ \max_{u\in U} ||P^{2t'}(u,\cdot) - \mathring{\pi}||_{TV} \le \epsilon\right\} \tag{23}$$

for $\epsilon \in (0,1)$. We remark that the even mixing-time of P is equal to the twice of the mixing time of $P^2[U]$, where $P^2[U]$ denotes the submatrix of P induced by U. Thus, we can use some standard arguments, e.g., coupling technique, about the even mixing time of P. Finally, we remark on a proposition, that plays a key role in our analysis.

Proposition 2.23 ([39])**.** *If P is reversible then $\mathring{\pi}(v) \le P^{2t+2}(v,v) \le P^{2t}(v,v)$ for any $t = 0,1,2,\ldots$.*

Proof. Since $P^{2t+2}(x,x) = \sum_{y,z\in\chi} P^t(x,y)P^2(y,z)P^t(z,x)$, we have

$$\pi(x)P^{2t+2}(x,x) = \sum_{y,z\in\chi} P^t(y,x)\pi(y)P^2(y,z)P^t(z,x) = \sum_{y,z\in\chi} \psi(y,z)\psi(z,y),$$

where $\psi(y,z) = P^t(y,x)\sqrt{\pi(y)P^2(y,z)}$. By Cauchy-Schwarz, we obtain

$$\begin{aligned}
\sum_{y,z\in\chi} \psi(y,z)\psi(z,y) &\le \sum_{y,z\in\chi} \psi(y,z)^2 = \sum_{y,z\in\chi} [P^t(y,x)]^2\pi(y) \\
&= \sum_{y,z\in\chi} [P^t(y,x)]^2\pi(y) = \sum_{y\in\chi} P^t(y,x)\pi(y)P^t(y,x) \\
&= \sum_{y\in\chi} P^t(y,x)\pi(x)P^t(x,y) = \pi(x)P^{2t}(x,x).
\end{aligned}$$

Therefore, $P^{2t+2}(x,x) \le P^{2t}(x,x)$ holds for any $t \ge 1$. This means that we have

$$P^{2t}(x,x) \ge P^{2t+2}(x,x) \ge \ldots \ge \lim_{t\to\infty} P^{2t}(x,x) = \mathring{\pi}(x).$$

□

2.5 Coupon collector

To prove the mixing time of random walk on $\{0,1\}^n$, we consider the coupon collector (This section follows [39]). "A company issues n different types of coupons. A collector desires a complete set. We suppose each coupon he acquires is equally likely to be each of the n types. How many coupons must he obtain so that his collection contains all n types? " Let X_t denote the number of collecting coupons at time t. Suppose that $X_t = k$. There exists a $n-k$ type of coupon which does not get yet. This implies

$$\Pr[X_{t+1} = k+1 | X_t = k] = \frac{n-k}{n}$$
$$\Pr[X_{t+1} = k | X_t = k] = \frac{k}{n}.$$

The time of collecting an all coupon is the same as the time t which satisfies $X_t = n$. Therefore, we consider the time t which $X_t = n$.

Proposition 2.24 ([39], [52])**.** *Let $\tau := \min\{t \;;\; X_t = n\}$. Then*

$$E[\tau] = n\sum_{k=1}^{n} \frac{1}{k} = \mathrm{O}(n\log n)$$

holds for any $n \ge 1$.

Proof. This proof follows [39]. Let τ_k be the time at which the collector gets k difference coupon. Then

$$\tau = \tau_n = \tau_1 + (\tau_2 - \tau_1) + \ldots + (\tau_n - \tau_{n-1})$$

holds. Furthermore, $\tau_k - \tau_{k-1}$ is a geometric random variable with success probability $\frac{n-k+1}{n}$: after collecting τ_{k-1} coupons, there are $n-k+1$ types missing from the collection. Each subsequent coupon drawn has the same probability $\frac{n-k+1}{n}$ of being a type not already collected, until a new type is finally drawn. Thus $E(\tau_k - \tau_{k-1}) = \frac{n}{n-k+1}$ and

$$E(\tau) = \sum_{k=1}^{n} E(\tau_k - \tau_{k-1}) = n \sum_{k=1}^{n} \frac{1}{n-k+1} = n \sum_{k=1}^{n} \frac{1}{k} = \mathrm{O}(n \log n),$$

where $\tau_0 = 0$. Thus, we obtain the claim. □

3 Analytical Framework: LHaGG

This section introduces the notion of less homesickness as graph growing (LHaGG), and presents general theorems (Lemmas 3.2 and 3.4) describing some sufficient conditions for an RWoGG to be recurrent or transient. See the following sections for specific RWoGGs, namely, a random walk on a growing k-ary tree in Section 4, a random walk on a growing box in Section 5, a random walk on $\{0,1\}^n$ hypercube skeleton with increasing n in Section 6, etc.

3.1 Less-homesick as graph growing

Let $\mathcal{D} = (f, G, P)$ and $\mathcal{D}' = (f', G', P')$ be RWoGG, and let $R(t)$ and $R'(t)$ respectively denote their return probabilities to respective initial vertices at time $t = 1, 2, \ldots$. We say $\mathcal{D}$ is *less-homesick* than $\mathcal{D}' = (f', G', P')$ at time t if $R(t) \leq R'(t)$ holds.

In particular, this paper is mainly concerned with the less-homesick relationship between $\mathcal{D}_f = (f, G, P)$ and $\mathcal{D}_g = (g, G, P)$ with the same P, G and the initial vertex v. We say $(\cdot, G, P)$ is *less-homesick as graph growing* (*LHaGG*)[5] if $\mathcal{D}_g = (g, G, P)$ is less-homesick than for any $\mathcal{D}_f = (f, G, P)$ satisfying that

$$\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i) \tag{24}$$

for any $n \in \mathbb{Z}_{>0}$. The condition (24) intuitively implies that the graph in $\mathcal{D}_g$ grows faster than $\mathcal{D}_f$. For instance, we will prove that the simple random walk on growing k-regular tree is LHaGG, in Section 4.

Lemma 3.1. *Suppose RWoGG $(\cdot, G, P)$ is LHaGG. Let X_t ($t = 0, 1, 2, \ldots$) be a RWoGG according to $\mathcal{D}_f = (f, G, P)$ with $X_0 = v \in V(1)$. Let Y_t ($t = 0, 1, 2, \ldots$) be a random walk on (a static graph) $G(n)$ according to $P(n)$ with $Y_0 = v$, where G, P and v are common with $\mathcal{D}_f$. Then, Y_t is less-homesick than X_t at any time $t \in [T_n^f, T_{n+1}^f]$, i.e., $R_f(t) \geq R_g(t)$ holds for $t \in [T_n^f, T_{n+1}^f]$, where $R_f(t) = \Pr[X_t = v]$ and $R_g(t) = \Pr[Y_t = v]$.*

Proof. Let

$$g(i) = \begin{cases} 0 & (i < n), \\ \sum_{j=1}^{n} f(j) & (i = n), \\ f(i) & (i > n). \end{cases}$$

[5] Strictly speaking, LHaGG should be a property of the sequence of transition matrices $P(1), P(2), P(3), \ldots$. For the convenience of the notation, we say $(\cdot, G, P)$ is LHaGG, in this paper.

Then, the static random walk Y_t on $G(n)$ also follows $\mathcal{D}_g = (g, G, P)$ for $t \leq T^f_{n+1}$. Clearly, $\sum_{i=1}^n f(i) \geq \sum_{i=1}^n g(i)$ for any n. Since $\mathcal{D}$ is LHaGG by the hypothesis, $R_f(t) \geq R_g(t)$. ☐

We remark that if all P_n takes period 2 then $R_f(t) = R_g(t) = 0$ for any odd t.

3.2 Recurrent - irreducible and periodic 2

We prove the following lemma, presenting a sufficient condition for a RWoGG to be recurrent.

Lemma 3.2. *Suppose that RWoGG* $(\cdot, G, P)$ *is LHaGG, and that every* $P(n) = P_n$ $(n = 1, 2, \ldots)$ *is irreducible, reversible and* $\mathrm{period}(P_n) = 2$*. Let* $\mathring{p}(n) = \mathring{\pi}_n(v)$ *where* $\mathring{\pi}_n$ *denote the even-time stationary distribution of* P_n*. If* $\mathfrak{d}$ *satisfies*

$$\sum_{n=1}^{\infty} (\mathfrak{d}(n) - 1)\mathring{p}(n) = \infty \tag{25}$$

then v *is recurrent by* $\mathcal{D}_{\mathfrak{d}}$.

Proof. Let $f(n) = 2\lfloor \frac{\mathfrak{d}(n)}{2} \rfloor$, i.e., $f(n) = \mathfrak{d}(n)$ if $\mathfrak{d}(n)$ is even, otherwise $f(n) = \mathfrak{d}(n) - 1$. For convenience, let $T^f_n = \sum_{k=1}^n f(k)$ for $n = 1, 2, \ldots$, and let $T^f_0 = 0$. Let X_t (resp. X'_t) for $t = 0, 1, 2, \ldots$ be a RWoGG according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ (resp. $\mathcal{D}_f = (f, G, P)$), and let $R_{\mathfrak{d}}(t)$ (resp. $R_f(t)$) denote the return probability of X_t (resp. X'_t). The hypothesis LHaGG implies $R_{\mathfrak{d}}(t) \geq R_f(t)$. Let Y^n_t $(t = 0, 1, \ldots, T^f_n)$ be a time-homogeneous random walk according to $P(n)$, and let $R''_n(t)$ $(t = 1, \ldots, T^f_n)$ denote the return probability of Y^n_t. The hypothesis LHaGG and Lemma 3.1 implies

$$R_f(t) \geq R''_n(t) \tag{26}$$

for $t \in (T^f_{n-1}, T^f_n]$. Then, we obtain

$$\begin{aligned}
\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) &\geq \sum_{t=1}^{\infty} R_f(t) && \text{(By LHaGG)} \\
&= \sum_{n=1}^{\infty} \sum_{t=T^f_{n-1}+1}^{T^f_n} R_f(t) \\
&\geq \sum_{n=1}^{\infty} \sum_{t=T^f_{n-1}+1}^{T^f_n} R''_n(t) && \text{(By (26))} \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{f(n)} R''_n(T^f_{n-1} + i) && \text{(Recall } T^f_n = T^f_{n-1} + f(n)\text{)} \\
&= \sum_{n=1}^{\infty} \sum_{i'=1}^{\frac{f(n)}{2}} R''_n(T^f_{n-1} + 2i') && \text{(Notice that } R''_n(T^f_{n-1} + 2i' - 1) = 0\text{)} \\
&\geq \sum_{n=1}^{\infty} \sum_{i'=1}^{\frac{f(n)}{2}} \mathring{p}(n) && \text{(By Proposition 2.23)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} f(n)\mathring{p}(n)
\end{aligned}$$

$$\geq \frac{1}{2}\sum_{n=1}^{\infty}(\mathfrak{d}(n)-1)\mathring{p}(n). \tag{27}$$

If (25) holds then (27) is ∞, meaning that v is recurrent by $\mathcal{D}_{\mathfrak{d}}$. □

3.2.1 Recurrent-irreducible and aperiodic

We get also the sufficient condition for the recurrence over irreducible and aperiodic.

Lemma 3.3. *Suppose that RWoGG $(\cdot, G, P)$ is LHaGG, and that every $P(n) = P_n$ ($n = 1, 2, \ldots$) is irreducible, reversible and* $\mathrm{period}(P_n) = 1$. *Let $p(n) = \pi_n(v)$ where π_n denote the stationary distribution of P_n. If $\mathfrak{d}$ satisfies*

$$\sum_{n=1}^{\infty}\mathfrak{d}(n)p(n) = \infty \tag{28}$$

then v is recurrent by $\mathcal{D}_{\mathfrak{d}}$.

Proof. Let X_t for $t = 0, 1, 2, \ldots$ be a RWoGG according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$, and let $R_{\mathfrak{d}}(t)$ denote the return probability of X_t. Let Y^n_t ($t = 0, 1, \ldots, T^{\mathfrak{d}}_n$) be a time-homogeneous random walk according to $P(n)$, and let $R''_n(t)$ ($t = 1, \ldots, T^{\mathfrak{d}}_n$) denote the return probability of Y^n_t. The hypothesis LHaGG and Lemma 3.1 implies

$$R_{\mathfrak{d}}(t) \geq R''_n(t) \tag{29}$$

for $t \in (T^{\mathfrak{d}}_{n-1}, T^{\mathfrak{d}}_n]$. By Proposition 2.17, $R''_n(t) \geq p(n)$ holds for $t \in (T^{\mathfrak{d}}_{n-1}, T^{\mathfrak{d}}_n]$. Therefore, we obtain the claim using the similar technique as Lemma 3.2. □

3.3 Transient - irreducible and period 2

This section establishes the following lemma, which suggests Lemma 3.2 is nearly optimal. We will provide an example of a random walk on a growing k-ary tree in Section 4, which shows a tight example of Lemma 3.2.

Lemma 3.4. *Suppose that a RWoGG $(\cdot, G, P)$ is LHaGG, and that every $P(n) = P_n$ ($n = 1, 2, \ldots$) is irreducible and reversible with* $\mathrm{period}(P_n) = 2$. *Let $\mathring{p}(n) = \mathring{\pi}_n(v)$ where $\mathring{\pi}_n$ denote the even-time stationary distribution of P_n. Let $\mathring{\tau}_n(\epsilon)$ denote the even mixing-time of $P(n)$, and let*

$$\mathring{\mathfrak{t}}(n) = \mathring{\tau}_n\left(\frac{\mathring{p}(n)}{4}\right)$$

for $n = 2, 3, \ldots$. If

$$\max\left\{\mathfrak{d}(1), \mathring{\mathfrak{t}}(1)\right\} + \sum_{n=2}^{\infty}\max\left\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\right\}\mathring{p}(n-1) < \infty \tag{30}$$

holds then v is transient by $\mathcal{D}_{\mathfrak{d}}$.

Proof. Let

$$g(n) = \max\left\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\right\}$$

for $n = 1, 2, 3, \ldots$. Let $R_{\mathfrak{d}}(t)$ and $R_g(t)$ respectively denote the return probabilities of $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ and $\mathcal{D}_g = (g, G, P)$. Clearly, $g(n) \geq \mathfrak{d}(n)$ for any n, LHaGG implies

$$R_g(t) \geq R_{\mathfrak{d}}(t) \tag{31}$$

for any $t = 0, 1, 2, \ldots$. For convenience, let

$$T_n^g = \sum_{k=1}^{n} g(k) \tag{32}$$

for $n = 1, 2, \ldots$.

We carry a tricky argument in the following: roughly speaking we compare $\mathcal{D}_g$ with P_{n-1} in the n-th round, i.e., $[T_{n-1}^g, T_n^g]$, for $n = 2, 3, \ldots$. Let

$$f_{n-1}(k) = \begin{cases} g(k) & (k \leq n-2) \\ \infty & (k = n-1) \end{cases}$$

for $n = 2, 3, \ldots$. Let $Z_t^{(n-1)}$ $(t = 0, 1, 2, \ldots)$ denote a RWoGG (f_{n-1}, G, P), where $Z_0^{(n-1)} = v$. Let $R''_{n-1}(t)$ denote the return probability of $Z_t^{(n-1)}$. Clearly, $\sum_{i=1}^{j} g(i) \leq \sum_{i=1}^{j} f_{n-1}(i)$ holds for any j, hence the LHaGG assumption implies

$$R''_{n-1}(t) \geq R_g(t) \tag{33}$$

for any $t = 0, 1, 2 \ldots$ and $n = 2, 3, \ldots$.

Notice that $Z_t^{(n-1)}$ for $t \in [T_{n-2}^g, T_n^g]$ is nothing but a time-homogeneous random walk according to P_{n-1} with the "initial state" $Z_{T_{n-2}^g} = v$ for $n = 2, 3, \ldots$. Since

$$T_{n-1}^g = T_{n-2}^g + g(n-1) \geq T_{n-2}^g + \mathring{\mathfrak{t}}(n-1) = T_{n-2}^g + \mathring{\tau}_{n-1}\left(\frac{\mathring{p}(n-1)}{4}\right) \tag{34}$$

$Z_t^{(n-1)}$ mixes well for $t > T_{n-1}^g$, meaning that, by Remark 2.13, $|\Pr[Z_t^{(n-1)} = v] - \mathring{p}(n-1)| \leq \frac{\mathring{p}(n-1)}{2}$ for any even $t \in (T_{n-1}^g, T_n^g]$. This implies

$$R''_{n-1}(t) = \Pr[Z_t^{(n-1)} = v] \leq \mathring{p}(n-1) + \frac{\mathring{p}(n-1)}{2} = \frac{3}{2}\mathring{p}(n-1) \tag{35}$$

hold [6] for $t \in (T_{n-1}^g, T_n^g]$, where we remark that $R''_{n-1}(t) = 0$ for any odd t. Then,

$$\begin{aligned} \sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) &\leq \sum_{t=1}^{\infty} R_g(t) && \text{(by (31))} \\ &= \sum_{n=1}^{\infty} \sum_{t=T_{n-1}^g+1}^{T_n^g} R_g(t) \\ &\leq g(1) + \sum_{n=2}^{\infty} \sum_{t=T_{n-1}^g+1}^{T_n^g} R_g(t) \end{aligned}$$

[6] We remark this argument requires only *point-wise additive error* bound, instead of total variation. Clearly, a point-wise additive error is upper bounded by total variation. We here use the mixing time for total variation just because it has been better analyzed than the other.

$$
\begin{aligned}
&\leq g(1) + \sum_{n=2}^{\infty} \sum_{t=T_{n-1}^g+1}^{T_n^g} R''_{n-1}(t) && \text{(by (33))} \\
&\leq g(1) + \frac{3}{2} \sum_{n=2}^{\infty} \sum_{t=T_{n-1}^g+1}^{T_n^g} \mathring{p}(n-1) && \text{(by (35))} \\
&= g(1) + \frac{3}{2} \sum_{n=2}^{\infty} g(n)\mathring{p}(n-1)
\end{aligned}
$$

holds. Now it is easy to see that (30) implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) < \infty$, meaning that v is transient by $\mathcal{D}_{\mathfrak{d}}$. □

It is simple to see that a similar proposition holds for *lazy* random walks.

3.3.1 Transient - irreducible and aperiodic

In this section, we provides the sufficient condition for the transience of random walks on growing graphs which is irreducible and aperiodic.

Lemma 3.5. *Suppose that a RWoGG $(\cdot, G, P)$ is LHaGG, irreducible and* $\mathrm{period}(P_n) = 1$. *Let $p(n) = \pi_n(v)$ where π_n denote the stationary distribution of P_n. Let $\tau_n(\epsilon)$ denote the mixing-time of $P(n)$, and let*

$$
\mathfrak{t}(n) := \tau_n\left(\frac{p(n)}{4}\right)
$$

for $n = 2, 3, \ldots$. If

$$
\max\{\mathfrak{d}(1), \mathfrak{t}(1)\} + \sum_{n=2}^{\infty} \max\{\mathfrak{d}(n), \mathfrak{t}(n)\}\, p(n-1) < \infty \tag{36}
$$

holds then v is transient by $\mathcal{D}_{\mathfrak{d}}$.

Proof. We consider $\mathring{\mathfrak{t}}(n) = \mathfrak{t}(n)$ and $\mathring{p}(n) = p(n)$ in Lemma 3.4. Obviously, (31), (33) and (34) hold. By irreducible and aperiodic, this case satisfies (35) for $t \in (T_{n-1}^g, T_n^g]$. Therefore, the lemma follows by the same proof as Lemma 3.4. □

4 Random walk on a growing complete k-ary tree

Lyons gave sufficient conditions that a random walk on an infinite tree gets recurrent or transient at the root (initial point), cf. [38, 40], as a consequence, it is a celebrated fact that a simple random walk on an infinite k-ary tree is transient. This section shows that a simple random walk on a *moderately* growing complete k-ary tree is recurrent at the root.

4.1 Result summary

Let k be an integer greater than one, and let $G_n = (V_n, E_n)$ denote a *complete k-ary tree* with height n for $n = 1, 2, \ldots$, i.e., $|V_n| = \sum_{i=0}^{n} k^i = \frac{k^{n+1}-1}{k-1}$, every internal node (including the root) has exactly k children, and every leaf places the same height n. Let $r \in V_n$ denote the root, that is the unique vertex of height 0. For convenience, let $h(v)$ denote the height of vertex $v \in V_n$, i.e., $h(r) = 0$, and $h(v) = n$ if and only if v is a leaf of G_n. Let

$$
U_n = \{v \in V_n \mid h(v) \equiv 0 \pmod 2\} \tag{37}
$$

denote the vertices of even heights, and thus $\overline{U}_n = V_n \setminus U_n$ is the vertices of odd heights. Clearly, $G_n = (U_n, \overline{U}_n; E_n)$ is a bipartite graph. See [12] for a standard terminology about a complete k-ary tree, e.g., parent, child, root, internal node, leaf, height.

Next, we define a transition probability of a random walk over G_n according to [38, 40]. Let λ be a fixed positive real[7], and we define a transition probability on the k-ary tree G_n with height n by

$$P_n(u, v) = \begin{cases} \frac{1}{k} & \text{if } u = r \text{ and } v \text{ is a child of } u, \\ \frac{1}{\lambda+k} & \text{if } u \neq r \text{ and } v \text{ is a child of } u, \\ \frac{\lambda}{\lambda+k} & \text{if } u \text{ is an internal node and } v \text{ is the parent of } u, \\ 1 & \text{if } u \text{ is a leaf and } v \text{ is the parent of } u, \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

for $u, v \in V_n$. Notice that (38) denotes a *simple random walk* over T_n when $\lambda = 1$. We also remark that λ and k are constants to n. As a consequence of [38], we know the following fact about a random walk on an infinite k-ary tree T_∞.

Proposition 4.1 ([38, 40])**.** *If $\lambda \geq k$ (resp. $\lambda < k$) then the root r is recurrent (resp. transient) by P_∞.*

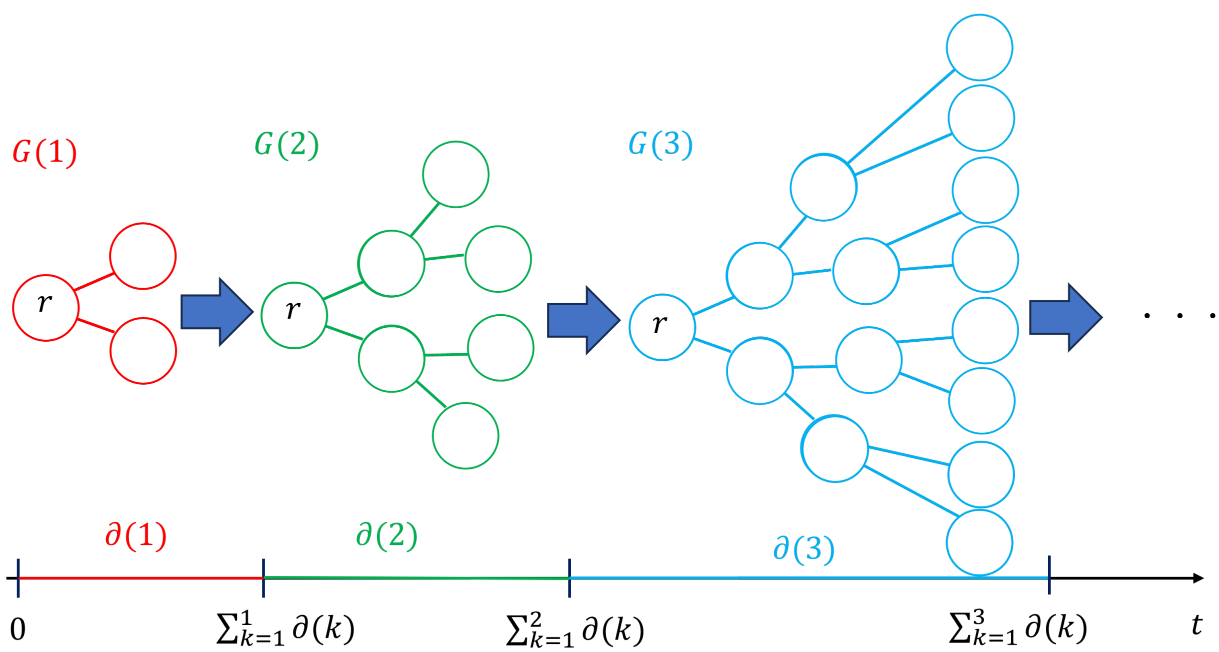

Figure 2: growing complete 2-ary tree

Then, we are concerned with a RWoGG $\mathcal{D}_{\mathfrak{d}}^{\mathrm{T}} = (\mathfrak{d}, G, P)$ (see Figure 2) starting from the root r where $G(n) = G_n$ and $P(n) = P_n$. Our goal of the section is to establish the following theorem.

Theorem 4.2. *Let $k \geq 2$ and $\lambda > 0$ be constants to n. Then, the root r is recurrent by $\mathcal{D}_{\mathfrak{d}}^{\mathrm{T}}$ if*

$$\sum_{n=1}^{\infty} \mathfrak{d}(n) \left(\frac{\lambda}{k}\right)^n = \infty \tag{39}$$

[7]For simplicity of notation, Lyons [38] and Lyons and Peres [40] assume $\lambda > 1$, but many arguments are naturally extended to $\lambda > 0$ by modifications with some bothering notations.

holds, otherwise, transient.

For instance, Theorem 4.2 implies the following corollary, about a simple random walk on an infinitely growing k-ary tree.

Corollary 4.3. *Let $\lambda = 1$, i.e., every P_n denotes a* simple *random walk on the complete k-ary tree T_n. If $\mathfrak{d}(n) = \Omega(k^n/(n \log n))$ then r is recurrent by $\mathcal{D}_{\mathfrak{d}}^{\mathrm{T}}$. If $\mathfrak{d}(1) < \infty$ and $\mathfrak{d}(n) = \mathrm{O}(k^n/(n(\log n)^{1+\epsilon}))$ for $n \geq 2$ with a constant $\epsilon > 0$ then r is transient by $\mathcal{D}_{\mathfrak{d}}^{\mathrm{T}}$.*

Proof. Suppose $\mathfrak{d}(n) \geq ck^n/(n \log n)$ for some constant $c > 0$. Then, $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{1}{k})^n \geq \sum_{n=1}^{\infty} c\frac{k^n}{n \log n}(\frac{1}{k})^n = c\sum_{n=1}^{\infty} \frac{1}{n \log n} \geq c\int_2^{\infty} \frac{1}{n \log n} = c[\log \log n]_2^{\infty} = \infty$, and Theorem 4.2 implies that r is recurrent.

Suppose $\mathfrak{d}(n) \leq c'k^n/(n(\log n)^{1+\epsilon})$for some constant $c' > 0$. Then, $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{1}{k})^n \leq \mathfrak{d}(1) + \sum_{n=2}^{\infty} c'\frac{k^n}{n(\log n)^{1+\epsilon}}(\frac{1}{k})^n \leq \mathfrak{d}(1) + c'\frac{1}{2(\log 2)^{1+\epsilon}} + c'\int_2^{\infty} \frac{1}{x(\log x)^{1+\epsilon}}\mathrm{d}x = \mathfrak{d}(1) + c'\frac{1}{2(\log 2)^{1+\epsilon}} + c'k\left[-\frac{1}{\epsilon(\log x)^{\epsilon}}\right]_2^{\infty} < \infty$, and Theorem 4.2 implies that r is transient. □

4.2 Proof of Theorem 4.2

We prove Theorem 4.2. As a preliminary step, we remark on the following two facts.

Lemma 4.4. *(i) Every P_n ($n = 1, 2, \ldots$) is reversible: precisely, let*

$$\phi(v) = \begin{cases} \frac{k}{\lambda+k} & \text{if } h(v) = 0 \text{ (i.e., } v = r\text{)}, \\ \lambda^{-h(v)} & \text{if } 0 < h(v) < n, \\ \frac{\lambda}{\lambda+k}\lambda^{-n} & \text{if } h(v) = n \text{ (i.e., } v \text{ is a leaf)}. \end{cases} \tag{40}$$

Then, the detailed balance equation

$$\phi(u)P_n(u, v) = \phi(v)P_n(v, u)$$

holds for any $u, v \in V_n$. (ii) Every P_n is irreducible and $\mathrm{period}(P_n) = 2$. *Thus the even-time stationary distribution of P_n is*

$$\mathring{\pi}_n(v) = \frac{\phi(v)}{\sum_{u \in U_n} \phi(u)} \tag{41}$$

for any $v \in U_n$.

Let $\mathring{p}(n) = \mathring{\pi}_n(r)$, then

$$\mathring{p}(n) = \begin{cases} \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} & \text{if } n \text{ is odd}, \\ \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} & \text{if } n \text{ is even} \end{cases} \tag{42}$$

by (40) and (41) considering the fact $|\{v \in V_n \mid h(v) = i\}| = k^i$ for $i = 0, 1, \ldots, n$.

Lemma 4.5. *If $\lambda < k$, then*

$$\tfrac{k-\lambda}{k}\left(\frac{\lambda}{k}\right)^{n+1} \leq \mathring{p}(n) \leq \left(\frac{\lambda}{k}\right)^{n-1}. \tag{43}$$

Proof. Firstly, we prove the upper bound of (43). When n is odd,

$$\mathring{p}(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{k}{\lambda}\right)^{2i}} \leq \frac{1}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{k}{\lambda}\right)^{2i}} \leq \frac{1}{\left(\frac{k}{\lambda}\right)^{2\lfloor\frac{n}{2}\rfloor}} = \left(\frac{\lambda}{k}\right)^{2\lfloor\frac{n}{2}\rfloor} = \left(\frac{\lambda}{k}\right)^{n-1}$$

and we obtain the upper bound in the case. When n is even, similarly,

$$\begin{aligned}\mathring{p}(n) &= \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1}\left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} = \frac{1}{1 + \frac{\lambda+k}{k}\sum_{i=1}^{\frac{n}{2}-1}\left(\frac{k}{\lambda}\right)^{2i} + \left(\frac{k}{\lambda}\right)^{n}} \\ &\leq \frac{1}{\left(\frac{k}{\lambda}\right)^{n}} \leq \left(\frac{\lambda}{k}\right)^{n-1}\end{aligned}$$

and we obtain the upper bound. Then, we prove the lower bound of (43). When n is odd,

$$\mathring{p}(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{k}{\lambda}\right)^{2i}} \geq \frac{\frac{k}{\lambda+k}}{1 + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{k}{\lambda}\right)^{2i}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\left(\frac{k}{\lambda}\right)^{2}\right)^{\lfloor\frac{n}{2}\rfloor+1}-1}{\left(\frac{k}{\lambda}\right)^{2}-1}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+1}-1}{\left(\frac{k}{\lambda}\right)^{2}-1}}$$

and we obtain the lower bound in the case. When n is even,

$$\mathring{p}(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1}\left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} \geq \frac{\frac{k}{\lambda+k}}{1 + \sum_{i=1}^{\frac{n}{2}-1}\left(\frac{k}{\lambda}\right)^{2i} + \left(\frac{k}{\lambda}\right)^{n}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+2}-1}{\left(\frac{k}{\lambda}\right)^{2}-1}}$$

holds. In both cases,

$$\begin{aligned}\mathring{p}(n) &\geq \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+2}-1}{\left(\frac{k}{\lambda}\right)^{2}-1}} = \frac{k}{\lambda+k}\left(\left(\frac{k}{\lambda}\right)^{2}-1\right)\frac{1}{\left(\frac{k}{\lambda}\right)^{n+2}-1} \\ &\geq \frac{k}{\lambda+k}\left(\left(\frac{k}{\lambda}\right)^{2}-1\right)\frac{1}{\left(\frac{k}{\lambda}\right)^{n+2}} = \frac{k-\lambda}{k}\left(\frac{\lambda}{k}\right)^{n+1}\end{aligned}$$

holds and we obtain the lower bound. □

The following lemma is a key of the proof of Theorem, 4.2.

Lemma 4.6. *If $\lambda < k$ then $(\cdot, G, P)$ is LHaGG.*

Proof. Let f and g satisfy $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$ for any $n = 1, 2, \ldots$, and let X_t and Y_t ($t = 0, 1, 2, \ldots$) respectively follow (f, G, P) and (g, G, P), i.e., the tree of (f, G, P) grows faster than (g, G, P). Let $X_0 = Y_0 = r$, and we prove $\Pr[X_t = r] \geq \Pr[Y_t = r]$ for any $t = 1, 2, \ldots$ (recall Section 3.1 for LHaGG).

We construct a coupling of $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$ such that $h(X_t) \leq h(Y_t)$ holds for any $t = 1, 2, \ldots$. The proof is an induction concerning t. Clearly, $h(X_0) = h(Y_0) = 0$. Inductively assuming $h(X_t) \leq h(Y_t)$, we prove $h(X_{t+1}) \leq h(Y_{t+1})$. If $h(X_t) < h(Y_t)$ then $h(X_t) \leq h(Y_t) - 2$ since every P_n is $\mathrm{period}(P_n) = 2$ for $n = 1, 2, \ldots$. It is easy to see that $h(X_{t+1}) \leq h(X_t) + 1 \leq h(Y_t) - 1 \leq h(Y_{t+1})$, and we obtain $h(X_{t+1}) \leq h(Y_{t+1})$ in the case.

Suppose $h(X_t) = h(Y_t)$. We consider four cases: (i) $X_t = Y_t = r$, (ii) both X_t and Y_t are internal nodes, (iii) both X_t and Y_t are leaves, i.e., both trees of (f, G, P) and(g, G, P) take the same height at time

t, (iv) X_t is not a leaf but Y_t is a leaf, i.e., the tree of (g, G, P) is higher than that of (f, G, P) at time t. In the case (i),

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = \Pr[h(Y_{t+1}) = h(Y_t) + 1] = 1$$

hold, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (ii), since both X_t and Y_t are internal nodes,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \Pr[h(Y_{t+1}) = h(Y_t) - 1] = \frac{\lambda}{k + \lambda}$$

and

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = \Pr[h(Y_{t+1}) = h(Y_t) + 1] = \frac{k}{k + \lambda}$$

hold, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (iii), since both X_t and Y_t are leaves,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \Pr[h(Y_{t+1}) = h(Y_t) - 1] = 1$$

holds, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (iv), since X_t is not a leaf but Y_t is a leaf,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = 1 \geq \Pr[h(Y_{t+1}) = h(Y_t) - 1] = \frac{\lambda}{k + \lambda}$$

holds, and hence we can couple them to satisfy $h(X_{t+1}) \leq h(Y_{t+1})$.

Now we obtain a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $h(X_t) \leq h(Y_t)$ hold for any $t = 1, 2, \ldots$, which implies that $h(Y_t) = 0$ as long as $h(X_t) = 0$. This means that $\Pr[X_t = r] \geq \Pr[Y_t = r]$ for any $t = 1, 2, \ldots$. We obtain the claim. □

By Lemma 3.2 with Lemma 4.6, we get a sufficient condition for recurrence in Theorem 4.2. On the other hand, we cannot directly apply Lemma 3.4 to the sufficient condition for transient in Theorem 4.2, because the "mixing time" of P_n is proportional to k^n, see e.g., [39]. Then, we estimate $R(t)$ by another random walk.

Let $Z_t = h(X_t)$, where X_t is a random walk on a growing k-ary tree $\mathcal{D}^{\mathrm{T}}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$. Then Z_t is a RWoGG $\mathcal{D}^{\mathrm{L}}_{\mathfrak{d}} = (\mathfrak{d}, L, Q)$ where $L(n) = (\{0, 1, \ldots, n\}, \{\{i, i+1\} \mid i = 0, 1, 2, \ldots, n-1\})$ is a path graph of length n, and the transition probability matrix $Q(n) = Q_n$ is given by

$$\begin{cases} Q_n(0,1) = 1, \\ Q_n(i, i+1) = \frac{k}{\lambda + k} & \text{for } i = 1, 2, \ldots, n-1, \\ Q_n(i, i-1) = \frac{\lambda}{\lambda + k} & \text{for } i = 1, 2, \ldots, n-1, \\ Q_n(n, n-1) = 1. \end{cases}$$

The following Lemmas 4.7 and 4.8 are easy to observe.

Lemma 4.7. *Let X_t (resp. Z_t) follow $\mathcal{D}^{\mathrm{T}}_f = (f, G, P)$ (resp. $\mathcal{D}^{\mathrm{L}}_f = (f, L, Q)$). Let $R(t) = \Pr[X_t = r]$ (resp. $R'(t) = \Pr[Z_t = r]$), and let $\mathring{\pi}_n$ (resp. $\mathring{\pi}'_n$) denote the even-time stationary distribution of P_n (resp. Q_n). Then, $R(t) = R'(t)$ for any $t = 1, 2, \ldots$, as well as $\mathring{\pi}_n(r) = \mathring{\pi}'_n(r)$.*

Lemma 4.8. *If $\lambda < k$ then $(\cdot, L, Q)$ is LHaGG.*

The following lemma about the mixing time of Q_n is easily obtained by a standard coupling argument for the mixing time, and we here omit the proof.

Lemma 4.9. *Let $\mathring{\tau}'_n(\epsilon)$ denote the even mixing-time of Q_n then $\mathring{\tau}'_n(\epsilon) \le 4n^2 \left(\log_2 \epsilon^{-1} + 1\right)$.*

Proof. By a standard coupling argument for the mixing time and Corollary 2.16, it is easy to see that $\mathring{\tau}'_n(\epsilon) \le 4n^2 \left(\log_2 \epsilon^{-1} + 1\right)$ where n^2 is the hitting time of a unbiased random walk on a path of length n. □

Then, we can prove the condition for $\mathcal{D}_{\mathrm{L}}$ being transient from Lemma 3.4.

Lemma 4.10. *If $\lambda < k$ and $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$ then 0 is transient by $\mathcal{D}^{\mathrm{L}}_{\mathfrak{d}}$.*

Proof. Let $\mathring{p}(n) = \mathring{\pi}_n(r)$ and $\mathring{p}'(n) = \mathring{\pi}'_n(r)$, then $\mathring{p}(n) = \mathring{p}'(n)$ by Lemma 4.7. By Lemma 4.9, $\mathring{\mathfrak{t}}'(n) = \mathring{\tau}'_n(\mathring{p}(n-1)) \le 4n^2 \left(\log(\mathring{p}(n-1)) + 1\right) \le 4n^2 \left(\log\left((\frac{\lambda}{k})^n\right) + 1\right) \le c'n^3$, and hence $\sum_{n=1}^{\infty} \mathring{\mathfrak{t}}'(n)\mathring{p}(n-1) \le \sum_{n=1}^{\infty} n^3 c'(\frac{\lambda}{k})^{n-1} < \infty$. If $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$, then $\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathring{\mathfrak{t}}'(n)\}\mathring{p}(n-1) \le \sum_{n=1}^{\infty}(\mathfrak{d}(n) + \mathring{\mathfrak{t}}'(n))(\frac{\lambda}{k})^{n-1} < \infty$, which implies $\sum_{t=1}^{\infty} R'(t) < \infty$ by Lemma 3.4 with Lemma 4.8. □

Now, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. First, we consider the (interesting) case $\lambda < k$.

(Recurrent) Assuming $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n = \infty$, we prove $\sum_{n=1}^{\infty}(\mathfrak{d}(n) - 1)\mathring{p}(n) = \infty$. Notice that $\sum_{n=1}^{\infty} \mathring{p}(n) \le c\sum_{n=1}^{\infty}(\frac{\lambda}{k})^n = \frac{1}{1-\frac{\lambda}{k}} < \infty$. Let $C = \sum_{n=1}^{\infty} \mathring{p}(n)$, then $\sum_{n=1}^{\infty}(\mathfrak{d}(n)-1)\mathring{p}(n) = \sum_{n=1}^{\infty} \mathfrak{d}(n)\mathring{p}(n) - C \ge \sum_{n=1}^{\infty} \mathfrak{d}(n)c(\frac{\lambda}{k})^n - C$, which is ∞ from the assumption. Thus, r is recurrent by Lemma 3.2.

(Transient) By Lemma 4.7, $\sum_{t=1}^{\infty} R(t) = \sum_{t=1}^{\infty} R'(t)$. If $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$ then $\sum_{t=1}^{\infty} R'(t) < \infty$ by Lemma 4.10, meaning that r is transient.

In the case of $\lambda \ge k$, it is always recurrent. The proof follows that of Lemma 3.2, but here we omit the proof. □

5 Random walk on a growing box

Dembo et al. [16] investigated sufficient conditions for the recurrence and transience of random walks on infinitely growing subgraphs of $\mathbb{Z}^d$ under the assumption of being sandwiched between the two growing balls with different sizes. This section examines a random walk on a growing box in terms of the coupling. As a result, we give the same sufficient conditions without the assumption.

5.1 Definition and main result

Let $G_n = (V_n, E_n)$ be a graph given by

$$V_n := \left\{ v = (v^1, \dots, v^{n_0}) \in \mathbb{Z}^{n_0} \; ; \; v \in \bigcap_{i=1}^{n_0} V_{i,n} \right\}$$
$$E_n := \{\{x, y\} \; ; \; x, y \in V_n, \|x - y\|_1 = 1\}$$

where n is a positive integer, the function $a_i(n)$ satisfies $a_i(1) = N$ ($N \ge 2$) and $a_i(n)$ is non-decrease to n for any $i \in \{1, \dots, n_0\}$, and the set $V_{i,n}$ given by

$$V_{i,n} := \left\{ v = (v^1, \dots, v^{n_0}) \in \mathbb{Z}^{n_0} \; ; \; v_i \in \{0, \dots, a_i(n)\} \right\}$$

for $i \in \{1, \dots, n_0\}$. Let o denote the origin. Let P_n for $n \geq 1$ denote a transition probability of a random walk on a static graph G_n, where

$$P_n(x,y) = \begin{cases} \frac{1}{2} & (\text{if } x = y) \\ \frac{1}{4n_0} & (\text{if } \|x-y\|_1 = 1,\ x^i \neq y^i \text{ and } x^i \notin \{0, a_i(n)\}) \\ \frac{1}{2n_0} & (\text{if } \|x-y\|_1 = 1,\ x^i \neq y^i \text{ and } x^i \in \{0, a_i(n)\}) \\ 0 & (\text{otherwise}) \end{cases}$$

for $x, y \in V_n$ and $i \in \{1, \dots, n_0\}$. Let $a(n) := \max\{a_1(n), \dots, a_{n_0}(n)\}$. Then, we are concerned with a RWoGG $X_t = (X_t^1, \dots, X_t^{n_0})$ according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$, where $G(n) = G_n$ (see Figure 3).

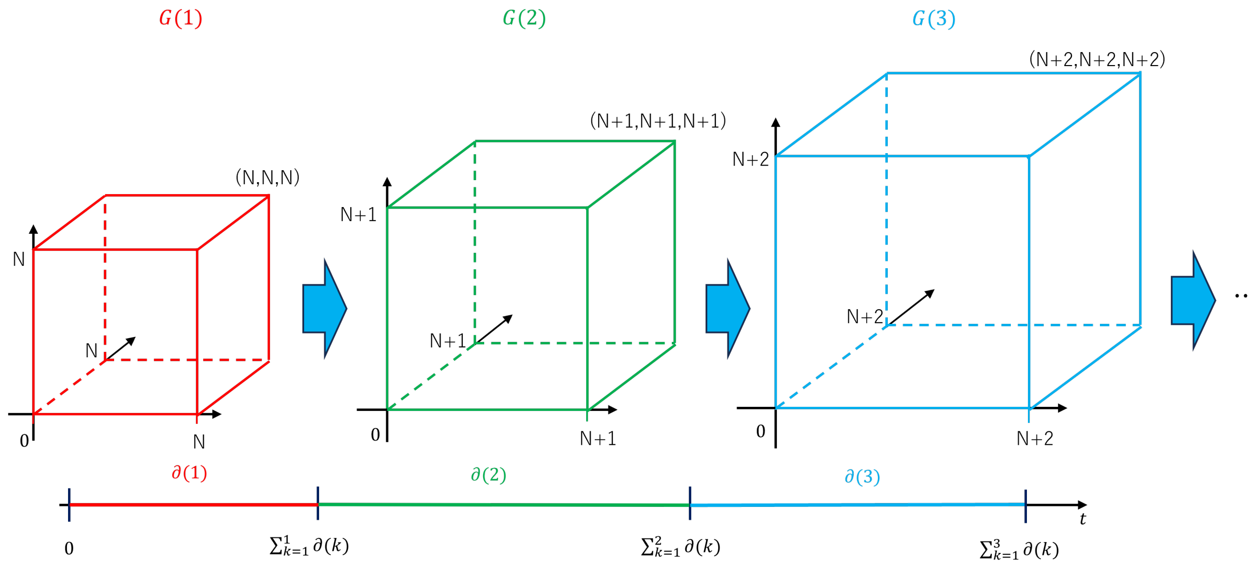

Figure 3: growing box

Theorem 5.1. *If $\mathcal{D}_{\mathfrak{d}}$ satisfies*

$$\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{2^{n_0} \prod_{i=1}^{n_0} a_i(k)} = \infty,$$

o is recurrent. If $\mathcal{D}_{\mathfrak{d}}$ satisfies

$$\sum_{k=2}^{\infty} \frac{\max\left\{\mathfrak{d}(k), 8(n_0)^3 a^2(k) \log_2(2a(k))\right\}}{2^{n_0} \prod_{i=1}^{n_0} a_i(k-1)} < \infty,$$

o is transient.

5.2 Proof of Theorem 5.1

To prove Theorem 5.1, we must prove the following lemmas.

Lemma 5.2. *A random walk on a growing box satisfies LHaGG.*

Proof. Let f and g satisfy $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$ for $n = 1, 2, \ldots$, and let X_t and Y_t ($t = 0, 1, 2, \ldots$) respectively follow (f, G, P) and (g, G, P). For convenience, let $n_0^f := 0$ and $n_t^f := n - 1$ for $t \in (T_{n-1}^f, T_n^f]$. Clearly, we get $n_t^f \leq n_t^g$ for any $t \geq 0$ by hypothesis of f and g. Let $X_0 = Y_0 = o$, and we prove $\Pr[X_t = 0] \geq \Pr[Y_t = 0]$ for any $t = 1, 2, \ldots$. Let $I : \mathbb{N} \to \{1, \ldots, n_0\}$ and $J : \mathbb{N} \to \{1, \ldots, n_0\}$ respectively denote the random variables which are selected by changing from X_{t-1} to X_t and from Y_{t-1} to Y_t. We can construction a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $X_t^i \leq Y_t^i$ for any $t \geq 0$ and $i \in \{1, \ldots, n_0\}$. The proof is an induction concerning t. Clearly, $X_0^i \leq Y_0^i$ holds for any $i \in \{1, \ldots, n_0\}$.

Inductively assuming $X_t^i \leq Y_t^i$ holds for any $i \in \{1, \ldots, n_0\}$, we prove $X_{t+1}^i \leq Y_{t+1}^i$. We consider four cases: (i) $X_{t+1}^i < Y_{t+1}^i$, (ii) $X_{t+1}^i = Y_{t+1}^i = 0$, (iii) $X_t^i = Y_t^i = a_i(n_t^g)$ and (iv) $X_t^i = Y_t^i$ and $X_t^i, Y_t^i \neq 0, a_i(n_t^g)$. In case (i), since

$$\Pr\left[I(t) = i\right] = \Pr\left[J(t) = i\right] = \frac{1}{n_0}$$
$$\Pr\left[|X_{t+1}^i - X_t^i| = 0 \mid I(t) = i\right] = \Pr\left[|Y_{t+1}^i - Y_t^i| = 1 \mid J(t) = i\right] = \frac{1}{2}$$
$$\Pr\left[|X_{t+1}^i - X_t^i| = 1 \mid I(t) = i\right] = \Pr\left[|Y_{t+1}^i - Y_t^i| = 0 \mid J(t) = i\right] = \frac{1}{2}$$

hold for any $i \in \{1, \ldots, n_0\}$, it follows that there exist two cases: (i-a)$|X_{t+1}^i - X_t^i| = 1$ as long as $Y_{t+1}^i = Y_t^i$, and (i-b):$X_{t+1}^i = X_t^i$ as long as $|Y_{t+1}^i - Y_t^i| = 1$. Recall that $X_t^i < Y_t^i$ implies $X_t^i + 1 \leq Y_t^i$ and $X_t^i \leq Y_t^i - 1$. This means that if (i-a) then $X_{t+1}^i \leq X_t^i + 1 \leq Y_t^i = Y_{t+1}^i$ and if (i-b) then $X_{t+1}^i = X_t^i \leq Y_t^i - 1 \leq Y_{t+1}^i$, and hence we obtain the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1}^i \leq Y_{t+1}^i$ in (i).

In case (ii), since

$$\Pr\left[I(t) = i\right] = \Pr\left[J(t) = i\right] = \frac{1}{n_0}$$
$$\Pr\left[X_{t+1}^i - X_t^i = 0 \mid I(t) = i, X_t^i = 0\right] = \Pr\left[Y_{t+1}^i - Y_t^i = 0 \mid J(t) = i, Y_t^i = 0\right] = \frac{1}{2}$$
$$\Pr\left[X_{t+1}^i - X_t^i = 1 \mid I(t) = i, X_t^i = 0\right] = \Pr\left[Y_{t+1}^i - Y_t^i = 1 \mid J(t) = i, Y_t^i = 0\right] = \frac{1}{2}$$

hold for any $i \in \{1, \ldots, n_0\}$, we construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1}^i \leq Y_{t+1}^i$.

In case (iii), it follows that there exist two cases: (iii-a) $n_t^f = n_t^g$ and (iii-b) $n_t^f < n_t^g$. For (iii-a), since $X_t^i \in \left\{0, \ldots, n_t^f\right\}$, $Y_t^i \in \{0, \ldots, n_t^g\}$ and $n_t^f = n_t^g$, we provide

$$\Pr\left[I(t) = i\right] = \Pr\left[J(t) = i\right] = \frac{1}{n_0}$$
$$\Pr\left[X_{t+1}^i - X_t^i = 0 \mid I(t) = i, X_t^i = a_i(n_t^f)\right] = \Pr\left[Y_{t+1}^i - Y_t^i = 0 \mid J(t) = i, Y_t^i = a_i(n_t^g)\right] = \frac{1}{2}$$
$$\Pr\left[X_{t+1}^i - X_t^i = -1 \mid I(t) = i, X_t^i = a_i(n_t^f)\right] = \Pr\left[Y_{t+1}^i - Y_t^i = -1 \mid J(t) = i, Y_t^i = a_i(n_t^g)\right] = \frac{1}{2}$$

for any $i \in \{1, \ldots, n_0\}$. Then, we provide the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1}^i = Y_{t+1}^i$ in (iii-a).

For (iii-b), since $X_t^i \in \left\{0, \ldots, n_t^f\right\}$, $Y_t^i \in \{0, \ldots, n_t^g\}$ and $n_t^f < n_t^g$, we have

$$\Pr\left[I(t) = i\right] = \Pr\left[J(t) = i\right] = \frac{1}{n_0}$$
$$\Pr\left[X_{t+1}^i - X_t^i = 0 \mid I(t) = i, X_t^i = a_i(n_t^g)\right] = \Pr\left[Y_{t+1}^i - Y_t^i = 0 \mid J(t) = i, Y_t^i = a_i(n_t^g)\right] = \frac{1}{2}$$

$$\Pr\left[Y^i_{t+1} - Y^i_t = -1 \mid I(t) = i, Y^i_t = a_i(n^g_t)\right] = \frac{1}{4}$$
$$\Pr\left[Y^i_{t+1} - Y^i_t = 1 \mid I(t) = i, Y^i_t = a_i(n^g_t)\right] = \frac{1}{4}$$
$$\Pr\left[X^i_{t+1} - X^i_t = -1 \mid J(t) = i, X^i_t = a_i(n^g_t)\right] = \frac{1}{2}$$

for any $i \in \{1, \dots, n_0\}$. Then, we give the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X^i_{t+1} \le Y^i_{t+1}$ in (iii-b). By (iii-a) and (iii-b), we can construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X^i_{t+1} \le Y^i_{t+1}$ in (iii).

In case (iv), since

$$\Pr\left[I(t) = i\right] = \Pr\left[J(t) = i\right] = \frac{1}{n_0}$$
$$\Pr\left[X^i_{t+1} - X^i_t = 0 \mid I(t) = i\right] = \Pr\left[Y^i_{t+1} - Y^i_t = 0 \mid J(t) = i\right] = \frac{1}{2}$$
$$\Pr\left[X^i_{t+1} - X^i_t = 1 \mid I(t) = i\right] = \Pr\left[Y^i_{t+1} - Y^i_t = 1 \mid J(t) = i\right] = \frac{1}{4}$$
$$\Pr\left[X^i_{t+1} - X^i_t = -1 \mid I(t) = i\right] = \Pr\left[Y^i_{t+1} - Y^i_t = -1 \mid J(t) = i\right] = \frac{1}{4}$$

hold for any $i \in \{1, \dots, n_0\}$, we can construct the coupling of X and Y such that $X^i_{t+1} \le Y^i_{t+1}$ in (iv). By (i), (ii), (iii), and (iv), we obtain $X^i_t \le Y^i_t$ for any $t \ge 0$ and $i \in \{1, \dots, n_0\}$.

Now we obtain a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X^i_{t+1} \le Y^i_{t+1}$. This means that $Y_t = o$ as long as $X_t = o$ for any $t \ge 0$, and hence we get $\Pr[X_t = 0] \ge \Pr[Y_t = 0]$ for any $t = 1, 2, \dots$. Thus, we obtain the claim. □

Lemma 5.3. *G_n satisfies*

$$p(n) = \frac{1}{2^{n_0} \prod_{i=1}^{n_0} a_i(n)}.$$

Proof. Let $\pi_n(x)$ ($x \in V_n$) denote

$$\pi_n(x) = \frac{1}{2^{W_x} \prod_{i=1}^{n_0} a_i(n)},$$

where $W_x := |\left\{i \in \{1, \dots, n_0\} \ ; \ x^i \in \{0, a_i(n)\}\right\}|$. π_n satisfies

$$\sum_{x \in V_n} \pi_n(x) P(x, y) = \pi_n(y),$$
$$\sum_{x \in V_n} \pi_n(x) = 1.$$

This means that π_n is a stationary distribution on $G_n = (V_n, E_n)$. □

Lemma 5.4 ([39]). *G_n satisfies* $\mathfrak{t}(n) \le 8(n_0)^3 a^2(n) \log_2(2a(n))$.

As a preliminary of proving Lemma 5.4, the graph $L_N = (V'_N, E'_N)$ is given by

$$V'_N := \{x \ ; \ x \in \mathbb{Z}, \ 0 \le x \le N\} \quad \text{and} \quad E'_N := \left\{\{x, y\} \ ; \ x, y \in V'_N, \ |x - y| = 1\right\},$$

where $N \in \mathbb{N}$. The transition probability matrix of a random walk on L_N, denoted by P'_N, is defined as

$$P'_N(i,j) := \begin{cases} p_i & \text{if } j = i+1, \\ q_i & \text{if } j = i-1, \\ r_i & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \tag{44}$$

where we have set

$$p_i := \begin{cases} \frac{1}{2} & \text{if } i = 0, \\ \frac{1}{4} & \text{if } i \notin \{0, N\}, \\ 0 & \text{otherwise,} \end{cases}, \; q_i := \begin{cases} \frac{1}{4} & \text{if } i \notin \{0, N\}, \\ \frac{1}{2} & \text{if } i = N, \\ 0 & \text{otherwise,} \end{cases}, \; r_i := \begin{cases} \frac{1}{2} & \text{if } i \in \{0, N\}, \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Then, we give the Lemma 5.5.

Lemma 5.5 ([39]). *Let X_t and Y_t denote random walks on L_N given by (44), where $X_0 = x_0$ and $Y_0 = y_0$. Set $\tau^{L_N}_{couple} := \min\{t \geq 0\ ;\ X_t = Y_t\}$. Then, we obtain*

$$E_{x,y}\left[\tau^{L_N}_{couple}\right] \leq 2N^2. \tag{46}$$

To show Lemma 5.5, we prove the following Lemma 5.6.

Lemma 5.6 ([39]). *Let Z_t be a random walk on L_N with the transition probability matrix P'_N. Set $\tau_x := \min\{t\ ;\ Z_t = x\}$. Then, we have*

$$E_{l-1}[\tau_l] = 2(2l-1) \tag{47}$$

for $l \in \{1, \ldots, N\}$. Therefore,

$$E_0[\tau_N] = 2N^2 \tag{48}$$

holds for any $N \geq 1$.

Proof. The proof follows [39]. We first show that (47). Let Z'_t be a random walk on L_l with the transition probability matrix P'_l. Set

$$\tau'_x := \min\left\{t\ ;\ Z'_t = x\right\} \quad \text{and} \quad \tau'^{+}_x := \min\left\{t \geq 1\ ;\ Z'_t = x\right\}.$$

Consider the expectation for return time at the vertex $l \in V'_l$. We give

$$\begin{aligned} E_l\left[\tau'^{+}_l\right] &= 1 \cdot \Pr\left[Z'_{t+1} = l \mid Z'_t = l\right] + (1 + E_{l-1}[\tau_l]) \Pr\left[Z'_{t+1} = l-1 \mid Z'_t = l\right] \\ &= \frac{1}{2} + \frac{1}{2}\left(1 + E_{l-1}\left[\tau'_l\right]\right) = 1 + \frac{1}{2}E_{l-1}\left[\tau'_l\right] \end{aligned} \tag{49}$$

since $\Pr\left[Z'_{t+1} = l \mid Z'_t = l\right] = \frac{1}{2}$ and $\Pr\left[Z'_{t+1} = l-1 \mid Z'_t = l\right] = \frac{1}{2}$ hold. By Lemma 5.3,

$$\pi_l(x) = \begin{cases} \frac{1}{l} & \text{if } 0 < x < l \\ \frac{1}{2l} & \text{if } x = 0, l \end{cases}$$

hold. Notice that $\pi_l(l) = \frac{1}{E_l(\tau'^{+}_l)}$ (see Proposition 2.3), it follows that

$$E_{l-1}\left[\tau'_l\right] = 2(2l-1) \tag{50}$$

by (49). Recall that $P'_l(v, v-1) = P'_N(v, v-1)$ for $v \in \{1, l-1\}$, $P'_l(v, v) = P'_N(v, v)$ and $P'_l(v, v+1) = P'_N(v, v+1)$ hold for $v \in \{0, l-1\}$, this implies

$$E_{l-1}[\tau_l] = E_{l-1}[\tau'_l] = 2(2l-1)$$

by (50), and hence (47) is proved.

We next show that (48). Since $E_0[\tau_N] = \sum_{l=1}^{N} E_{l-1}[\tau_l]$, it follows that

$$E_0[\tau_N] = \sum_{l=1}^{N} E_{l-1}[\tau_l] = \sum_{l=1}^{N} 2(2l-1) = 2N^2$$

by (47), and hence (48) is proved. □

Proof of Lemma 5.5. Let X_t and Y_t denote random walks on L_N with transition probability matrix P'_N. Without loss of generality, we assume $x_0 \le y_0$. We first show that there is a coupling of $\boldsymbol{X} = \{X_t\}_{t \ge 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \ge 0}$ such that $X_t \le Y_t$ for any $t \ge 0$. The proof is by induction on t. Clearly we get $X_0 \le Y_0$. Inductively assuming $X_t \le Y_t$, we prove $X_{t+1} \le Y_{t+1}$. We consider two transitions X_t to X_{t+1} and Y_t to Y_{t+1}. There are four cases: (i) $X_t < Y_t$, (ii) $X_t = Y_t = 0$, (iii) $X_t = Y_t = N$, (iv) $X_t = Y_t$ and $X_t, Y_t \notin \{0, N\}$. In case (i), since

$$\Pr[|X_{t+1} - X_t| = 1] = \Pr[Y_{t+1} - Y_t = 0] = \frac{1}{2}$$
$$\Pr[X_{t+1} - X_t = 0] = \Pr[|Y_{t+1} - Y_t| = 1] = \frac{1}{2}$$

hold, it follows that there exist two cases: (a) $|X_{t+1} - X_t| = 1$ as long as $Y_{t+1} = Y_t$. (b) $X_{t+1} = X_t$ as long as $|Y_{t+1} - Y_t| = 1$. Recall that $X_t < Y_t$ implies $X_t + 1 \le Y_t$ and $X_t \le Y_t - 1$. This means that if (a) then $X_{t+1} \le X_t + 1 \le Y_t = Y_{t+1}$ and if (b) then $X_{t+1} = X_t \le Y_t - 1 \le Y_{t+1}$, and hence we get the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1} \le Y_{t+1}$.

In case (ii), since $X_t = Y_t = 0$,

$$\Pr[X_{t+1} = X_t + 1] = \Pr[Y_{t+1} = Y_t + 1] = \frac{1}{2}$$
$$\Pr[X_{t+1} - X_t = 0] = \Pr[Y_{t+1} - Y_t = 0] = \frac{1}{2}$$

hold, and hence we give the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1} = Y_{t+1}$.

In case (iii), since $X_t = Y_t = N$,

$$\Pr[X_{t+1} = X_t - 1] = \Pr[Y_{t+1} = Y_t - 1] = \frac{1}{2}$$
$$\Pr[X_{t+1} - X_t = 0] = \Pr[Y_{t+1} - Y_t = 0] = \frac{1}{2}$$

hold, and hence we provide the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1} = Y_{t+1}$.

In case (iv), since $X_t = Y_t$ and $X_t, Y_t \neq \{0, N\}$,

$$\Pr[X_{t+1} = X_t + 1] = \Pr[Y_{t+1} = Y_t + 1] = \frac{1}{4}$$
$$\Pr[X_{t+1} = X_t - 1] = \Pr[Y_{t+1} = Y_t - 1] = \frac{1}{4}$$
$$\Pr[X_{t+1} - X_t = 0] = \Pr[Y_{t+1} - Y_t = 0] = \frac{1}{2}$$

hold, and hence we construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_{t+1} = Y_{t+1}$.

Now we get a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $X_t \leq Y_t$ for any $t \geq 0$, which implies that $Y_t = N$ as long as $X_t = N$. Therefore, we obtain

$$E_{x,y}\left(\tau^{L_N}_{couple}\right) \leq E_{X_0}\left(\tau_N\right) \leq E_0\left(\tau_N\right) = 2N^2 \tag{51}$$

by Lemma 5.6, and the lemma follows. □

Proof of Lemma 5.4. The proof follows [39]. Let X_t and Y_t denote random walks on G_n. Let

$$\tau^{G_n}_{couple} := \min\left\{t \geq 0\,;\ X_t = Y_t\right\}, \tag{52}$$

and we estimate the upper bound of $E_{x,y}[\tau^{G_n}_{couple}]$. Let $\tau_i := \min\left\{t \geq 0\,;\ X^i_t = Y^i_t\right\}$ for $i \in \{1, \ldots, n_0\}$. Then, it is not difficult to see that $E_{x,y}(\tau_i) \leq n_0 E(\tau^{L_{a_i(n)}}_{couple})$. Using Lemma 5.5, we give

$$E_{x,y}(\tau_i) \leq n_0 E(\tau^{L_{a_i(n)}}_{couple}) \leq 2n_0 a_i^2(n). \tag{53}$$

Notice that $\tau^{G(n)}_{couple} = \max\{\tau_1, \ldots, \tau_{n_0}\}$, this means that $\tau^{G(n)}_{couple} \leq \sum_{i=1}^{n_0} \tau_i$. As a result, it follows that

$$\begin{aligned} E_{x,y}[\tau^{G(n)}_{couple}] &\leq \sum_{i=1}^{n_0} E_{x,y}(\tau_i) \leq \sum_{i=1}^{n_0} \max_i E(\tau^{L_{a_i(n)}}_{couple}) \\ &= \sum_{i=1}^{n_0} E(\tau^{L_{a(n)}}_{couple}) \leq 2(n_0)^2 a^2(n) \end{aligned}$$

by (53). Recall that $\mathfrak{t}(n) = \tau(\frac{p(n)}{4})$, then we obtain

$$\begin{aligned} \mathfrak{t}(n) &\leq 4E_{x,y}[\tau^{G(n)}_{couple}]\left\lceil \log_2\left(\frac{2^{n_0}\prod_{i=1}^{n_0} a_i(n)}{4}\right)\right\rceil \\ &= 4E_{x,y}[\tau^{G(n)}_{couple}]\left\lceil \log_2\left(2^{n_0}\prod_{i=1}^{n_0} a_i(n)\right) - 2\right\rceil \\ &\leq 4E_{x,y}[\tau^{G(n)}_{couple}] \log_2\left(2^{n_0}\prod_{i=1}^{n_0} a_i(n)\right) \\ &\leq 4E_{x,y}[\tau^{G(n)}_{couple}] \log_2\left(2^{n_0} a^{n_0}(n)\right) \\ &= 4n_0 E_{x,y}[\tau^{G(n)}_{couple}] \log_2\left(2a(n)\right) \\ &\leq 8(n_0)^3 a^2(n) \log_2\left(2a(n)\right) \end{aligned}$$

for any $n \geq 1$, and the lemma follows. □

Proof of Theorem 5.1. If $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{2^{n_0}\prod_{i=1}^{n_0} a_i(k)} = \infty,$$

o is recurrent by Lemma 3.3. If $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{k=2}^{\infty} \frac{\max\left\{\mathfrak{d}(k), 8(n_0)^3 a^2(k)\log_2\left(2a(k)\right)\right\}}{2^{n_0}\prod_{i=1}^{n_0} a_i(k-1)} < \infty,$$

o is transient by Lemma 3.4 and Lemma 5.4. □

6 Random walk on $\{0,1\}^n$ with increasing n

6.1 Definition and main result

This section shows an interesting example. Let $C_n = (V_n, E_n)$ where

$$V_n = \{0,1\}^n \tag{54}$$

$$E_n = \left\{ \{\boldsymbol{u},\boldsymbol{v}\} \in \binom{V_n}{2} \mid \|\boldsymbol{u}-\boldsymbol{v}\|_1 = 1 \right\} \tag{55}$$

for $n = 1, 2, \ldots$. Let $\mathbf{0} \in V_n$ denote the (common) origin vertex $(0, \ldots, 0)$ for each n. Let

$$P_n(\boldsymbol{u},\boldsymbol{v}) = \begin{cases} \frac{1}{n} & \text{if } \|\boldsymbol{u}-\boldsymbol{v}\|_1 = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{56}$$

for $\boldsymbol{u},\boldsymbol{v} \in V_n$. Then, we are concerned with $\mathcal{D}_{\mathfrak{d}}^{\mathrm{C}} = (\mathfrak{d}, G, P)$ starting from $\mathbf{0}$ where $G(n) = C_n$ and $P(n) = P_n$ (see Figure 4).

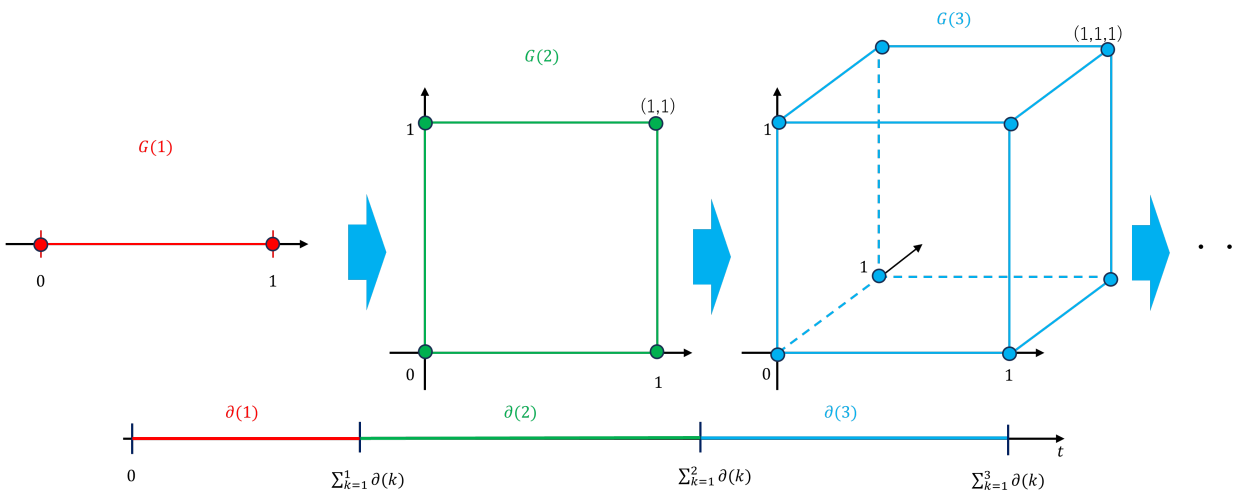

Figure 4: growing dimension hypercube

Theorem 6.1. *If $\mathcal{D}_{\mathfrak{d}}^{\mathrm{C}}$ satisfies*

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty \tag{57}$$

then $\mathbf{0}$ *is recurrent, otherwise* $\mathbf{0}$ *is transient.*

The following lemma is not very difficult, but nontrivial.

Lemma 6.2. *$(\cdot, G, P)$ is LHaGG.*

Proof. Let f and g satisfy $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$ for any $n = 1, 2, \ldots$, and let X_t and Y_t $(t = 0, 1, 2, \ldots)$ respectively follow (f, G, P) and (g, G, P), i.e., the box of (g, G, P) grows faster than (f, G, P). For convenience, let $n_0^f := 0$ and $n_t^f := n - 1$ for $t \in (T_{n-1}^f, T_n^f]$. and then notice that $n_t^f \leq n_t^g$ hold for any $t = 0, 1, \dots$ by the assumption that (g, G, P) grows faster. Let $X_0 = Y_0 = \mathbf{0}$, and we prove $\Pr[X_t = \mathbf{0}] \geq \Pr[Y_t = \mathbf{0}]$ for any $t = 1, 2, \dots$.

Let $h(\boldsymbol{u}) = |\{i \in \{1, \dots, n\} \ ; \ u_i = 1\}|$ for $\boldsymbol{u} = (u^1, \dots, u^n) \in V_n$. We construct a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $h(X_t) \leq h(Y_t)$ holds for any $t = 1, 2, \dots$. The proof is an induction concerning t. Clearly, $h(X_0) = h(Y_0) = 0$. Inductively assuming $h(X_t) \leq h(Y_t)$, we prove $h(X_{t+1}) \leq h(Y_{t+1})$. If $h(X_t) < h(Y_t)$ then $h(X_t) - 2 \leq h(Y_t)$ since every P_n is $\mathrm{period}(P_n) = 2$ for $n = 1, 2, \dots$. It is easy to see that $h(X_{t+1}) \leq h(X_t) + 1 \leq h(Y_t) - 1 \leq h(Y_{t+1})$, and we obtain $h(X_{t+1}) \leq h(Y_{t+1})$ in the case. Suppose $h(X_t) = h(Y_t)$. Then,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \frac{h(X_t)}{n_t^f} \geq \frac{h(Y_t)}{n_t^g} = \Pr[h(Y_{t+1}) = h(Y_t) - 1], \qquad \text{and}$$

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = 1 - \frac{h(X_t)}{n_t^f} \leq 1 - \frac{h(Y_t)}{n_t^g} = \Pr[h(Y_{t+1}) = h(Y_t) + 1]$$

hold, which implies that a coupling exists such that $h(X_{t+1}) \leq h(Y_{t+1})$.

Now we obtain a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ satisfying $h(X_t) \leq h(Y_t)$ for any $t = 1, 2, \dots$, which implies that $h(Y_t) = 0$ as long as $h(X_t) = 0$. This means that $\Pr[X_t = \mathbf{0}] \geq \Pr[Y_t = \mathbf{0}]$ for any $t = 1, 2, \dots$. We obtain the claim. □

The following two lemmas are well known.

Lemma 6.3. *Then,* $\mathring{\tau}_n(\epsilon) = \mathrm{O}(n \log(n) \log_2(\epsilon^{-1}))$ *holds for any* $n \geq 1$.

Proof. Let $X_t = (X_t^1, \dots, X_t^n) \in \{0,1\}^n$ and $Y_t = (Y_t^1, \dots, Y_t^n) \in \{0,1\}^n$ respectively denote a random walk on G_n, where $X_0 = x_0$, $Y_0 = y_0$ and $h(x_0) \equiv h(y_0) \equiv 0 \pmod 2$. Let $C(t) := \left\{i \ ; \ ||X_t^i - Y_t^i||_1 = 1\right\} = \left\{c_0, c_1, \dots, c_{|C(t)|-1}\right\}$ Suppose that $h(X_0) \leq h(Y_0)$. We consider the following coupling of X_t and Y_t : (i) If $X_t^i \neq X_{t+1}^i$ and $i \notin C(t)$ then $Y_t^i \neq Y_{t+1}^i$. (ii) If $X_t^i \neq X_{t+1}^i$, $i \in C(t)$ and $c_j = i$ then $Y_t^{c_{j+1}} \neq Y_{t+1}^{c_{j+1}}$, where $c_{|C(t)|} := c_0$.

In case (i), since

$$\Pr\left[i \notin C(t)\right] = \Pr\left[i \notin C(t)\right] = \frac{n - |C(t)|}{n} \tag{58}$$
$$\Pr\left[X_t^i \neq X_{t+1}^i \middle| i \notin C(t)\right] = \Pr\left[Y_t^i \neq Y_{t+1}^i \middle| i \notin C(t)\right] = \frac{1}{n - |C(t)|}$$

hold for any $i \notin C(t)$, we obtain the coupling of X_t and Y_t such that (i) holds.

In case (ii), since

$$\Pr\left[i \in C(t)\right] = \Pr\left[i \in C(t)\right] = \frac{|C(t)|}{n} \tag{59}$$
$$\Pr\left[X_t^i \neq X_{t+1}^i \middle| i = c_j \in C(t)\right] = \Pr\left[Y_t^{c_{j+1}} \neq Y_{t+1}^{c_{j+1}} \middle| i = c_j \in C(t)\right] = \frac{1}{|C(t)|}$$

holds for any $i = c_j \in C(t)$, we get the coupling of X and Y such that (ii) holds.

We repeat (i) and (ii) until $|C(t)| = 0$ since $|C(t)| = 0$ implies $X_t = Y_t$ but X_t does not always satisfy $h(X_t) \equiv 0 \pmod 2$. Therefore, we provide

$$E_{x,y}[\mathring{\tau}_{couple}^{C_n}] \leq E_{x,y}[\min\{t \ ; \ |C(t)| = 0\}] + 1.$$

In (59), $|C(t)|$ decreases 2 for once coupling, and $|C(t)| \leq n$, and $|C(t)|$ is an even number for any t. This implies that τ in Proposition 2.24 is greater than or equal to $\{t\ ;\ |C(t)| = 0\}$. Hence,

$$E_{x,y}[\min\{t\ ;\ |C(t)| = 0\}] \leq \mathrm{O}(n\log(n))$$

holds for any $x, y \in \{0,1\}^n$. Therefore, we obtain

$$\mathring{\tau}_n(\epsilon) \leq \mathrm{O}(n\log(n)\log_2(\epsilon^{-1}))$$

by Corollary 2.16 and 2.21. □

Lemma 6.4. $\mathring{p}(n) = \frac{1}{\frac{2^n}{2}} = 2^{-n+1}$.

Proof. By Corollary 2.22, we get $\mathring{p}(n) = \frac{\deg(o)}{|E_n|} = \frac{n}{n2^{n-1}} = 2^{-n+1}$. □

Proof of Theorem 6.1. (Recurrence) By Lemma 6.2, $\mathcal{D}^{\mathrm{C}}_{\mathfrak{d}}$ is LHaGG. Since $\mathring{p}(n) = 2^{-n+1}$ by Lemma 6.4, Lemma 3.2 implies that if $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then **0** is recurrent.

(Transience) By Lemma 6.3, $\mathring{\mathfrak{t}}(n) = \mathring{\tau}_n(\mathring{p}(n-1)) \leq n\log n \log_2 \mathring{p}(n-1) \leq n\log(n)\log_2(2^{n-2}) \leq c'n^2\log n$, and hence $\sum_{n=1}^{\infty} \mathring{\mathfrak{t}}(n)\mathring{p}(n-1) \leq \sum_{n=1}^{\infty} c'n^2\log n \frac{1}{2^{n-2}} < \infty$. If $\sum_{n=1}^{\infty}\frac{\mathfrak{d}(n)}{2^n} < \infty$, then $\sum_{n=1}^{\infty}\max\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\}\mathring{p}(n-1) \leq \sum_{n=1}^{\infty}(\mathfrak{d}(n) + \mathring{\mathfrak{t}}(n))\frac{1}{2^n} < \infty$, which implies $\sum_{t=1}^{\infty} R(t) < \infty$ by Lemma 3.4 with Lemma 6.2. □

6.2 An Interesting fact: every finite point becomes recurrent

We can easily observe the following fact from Theorem 6.1.

Corollary 6.5. *If $\mathfrak{d}(n) = \Omega(2^n/n)$ then* **0** *is recurrent. If $\mathfrak{d}(n) = \mathrm{O}(2^n/n^{1+\epsilon})$ then* **0** *is transient.*

Notice that the maximum degree of $G(n)$ is unbounded asymptotic to n. Nevertheless, we can see the following interesting facts.

Proposition 6.6. *If $\mathfrak{d}(n) = \Omega(n2^n)$ then $\mathcal{D}^{\mathrm{C}}_{\mathfrak{d}}$ starting from* **0** *visits $\boldsymbol{v} \in V_m$ infinitely many times for any $m < \infty$.*

Proof. Notice that $\mathring{\tau}_n(2^{-n-1}) = \mathrm{O}(n\log(n2^{n+1})) = \mathrm{O}(n^2\log n)$ by Lemma 6.3. Thus, in the n-th phase, i.e, $[T^{\mathfrak{d}}_{n-1}, T^{\mathfrak{d}}_n]$, the random walk X visits $\boldsymbol{v} \in V_n$ with probability at least 2^{-n-1} in every $\mathrm{O}(n^2\log n)$ steps (even if $\boldsymbol{v} \in \overline{U}_n$, here we omit the proof). Thus the probability that X never visit $\boldsymbol{v}$ during the n-th phase is at most $(1 - 2^{-n-1})^{n2^{n+1}/n^2\log n} \leq \exp(-\frac{1}{n\log n})$. This implies that the probability that X never visits $\boldsymbol{v} \in V_m$ forever is at most $\prod_m^{\infty}\exp(-\frac{1}{n\log n}) = \exp(-\sum_{n=m}^{\infty}\frac{1}{n\log n}) \leq \exp(-\int_m^{\infty}\frac{1}{x\log x}\mathrm{d}x) = \exp(-[\log\log x]_m^{\infty}) = \exp(-\infty) = 0$. This means that the RWoGG X visits $\boldsymbol{v} \in V_m$ at least once in finite steps with probability 1.

Once we know that X visits $\boldsymbol{v}$ in a finite steps, the claim is trivial thanks to the vertex transitivity of the hypercube skeleton. □

We think that the hypothesis of Proposition 6.6 can be relaxed from $\Omega(n2^n)$ to $\Omega(2^n/n)$, but we are not sure.

7 Recurrence and Transience - weakly LHaGG

It is very difficult to prove that growing graphs such as a random walk on $\{0, \ldots, N\}^n$ with an increasing n and random tree satisfy LHaGG. Therefore, we introduce the new notion of weakly LHaGG and give the sufficient condition for the recurrence and transience.

7.1 Recurrence - irreducible and period 2

In this section, we give sufficient conditions for the recurrence of random walks on growing graphs with irreducible and periodic 2.

Theorem 7.1. *Suppose that RWoGG* $\mathcal{D} = (\cdot, G, P)$ *is*

$$\sum_{k=1}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k) < \infty, \tag{60}$$

and that every $P(n) = P_n$ $(n = 1, 2, \ldots)$ *is irreducible and* $\mathrm{period}(P_n) = 2$*. Let* $\mathring{p}(n) = \mathring{\pi}_n(o)$ *where* $\mathring{\pi}_n$ *denote the even-time stationary distribution of* P_n*. If* $\mathfrak{d}$ *satisfies*

$$\sum_{k=1}^{\infty} (\mathfrak{d}(k) - 3)\mathring{p}(k) = \infty \tag{61}$$

then the origin vertex o *is recurrent by* $\mathcal{D}_{\mathfrak{d}}$*.*

To prove Theorem 7.1, we must provide Lemma 7.2.

Lemma 7.2. $\mathcal{D}_{\mathfrak{d}}$ satisfies

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq \frac{1}{4}\sum_{k=1}^{n} \left(f(k) - 2\right)\mathring{p}(k) - \frac{1}{4}\sum_{k=1}^{n} \mathring{\mathfrak{t}}(k)\mathring{p}(k) \tag{62}$$

for $n = 1, 2, \ldots$, where $f(n) = 2\lfloor \frac{\mathfrak{d}(n)}{2} \rfloor$.

Proof. We prove the claim by induction for n. For $n = 1$, we must prove

$$\sum_{t=1}^{T_1^{\mathfrak{d}}} R_{\mathfrak{d}}(t) = \sum_{t=1}^{f(1)} R_{\mathfrak{d}}(t) \geq \frac{1}{4}\left(f(1) - 2 - \mathring{\mathfrak{t}}(1)\right)\mathring{p}(1). \tag{63}$$

We consider two cases whether $f(1) \leq \mathring{\mathfrak{t}}(1)$ or not. If $f(1) \leq \mathring{\mathfrak{t}}(1)$ then the right-hand side of (62) ≤ 0. The left-hand side of (62) ≥ 0, and we obtain (62). If $f(1) > \mathring{\mathfrak{t}}(1)$ then

$$\begin{aligned}
&\sum_{t=1}^{\mathfrak{d}(1)} R_{\mathfrak{d}}(t) = \sum_{t=1}^{f(1)} R_{\mathfrak{d}}(t) \geq \sum_{t=\mathring{\mathfrak{t}}(1)}^{f(1)} R_{\mathfrak{d}}(t) \\
&= \sum_{t=\frac{\mathring{\mathfrak{t}}(1)}{2}}^{\frac{f(1)}{2}} R_{\mathfrak{d}}(2t) \geq \frac{1}{2}\sum_{t=\frac{\mathring{\mathfrak{t}}(1)}{2}}^{\frac{f(1)}{2}} \mathring{p}(1) = \frac{1}{2}\left(\frac{f(1)}{2} - \frac{\mathring{\mathfrak{t}}(1)}{2}\right)\mathring{p}(1) = \frac{1}{4}\left(f(1) - \mathring{\mathfrak{t}}(1)\right)\mathring{p}(1) \\
&\geq \frac{1}{4}\left(f(1) - 2 - \mathring{\mathfrak{t}}(1)\right)\mathring{p}(1)
\end{aligned}$$

hold by Remark 2.13. Then, we obtain (63), and hence (62) holds in case for $n = 1$.

Inductively assuming (62) holds for n, we prove it for $n+1$. Noting that $T_{n+1}^{\mathfrak{d}} = T_n^{\mathfrak{d}} + \mathfrak{d}(n+1)$,

$$\sum_{t=1}^{T_{n+1}^{\mathfrak{d}}} R_{\mathfrak{d}}(t) = \sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) + \sum_{t=1}^{\mathfrak{d}(n+1)} R_{\mathfrak{d}}(T_n^{\mathfrak{d}} + t)$$

$$\geq \frac{1}{4}\sum_{k=1}^{n}\left(f(k)-2\right)\mathring{p}(k)-\frac{1}{4}\sum_{k=1}^{n}\mathring{\mathfrak{t}}(k)\mathring{p}(k)+\sum_{t=1}^{\mathfrak{d}(n+1)}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+t) \tag{64}$$

hold since $\sum_{t=1}^{T_n^{\mathfrak{d}}}R_{\mathfrak{d}}(t)\geq \frac{1}{2}\sum_{k=1}^{n}f(k)\mathring{p}(k)-\frac{1}{2}\sum_{k=1}^{n}\mathring{\mathfrak{t}}(k)\mathring{p}(k)$ holds by the inductive assumption. We consider two cases: (i) $T_n^{\mathfrak{d}}$ is even and (ii) $T_n^{\mathfrak{d}}$ is odd. In case (i), concerning the third term of (64),

$$\sum_{t=1}^{\mathfrak{d}(n+1)}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+t)\geq \frac{1}{4}\left(f(n+1)-2-\mathring{\mathfrak{t}}(n+1)\right)\mathring{p}(n+1)$$

holds similar to (63). Then, we get

$$\begin{aligned}(64)&\geq \frac{1}{4}\sum_{k=1}^{n}\left(f(k)-2\right)\mathring{p}(k)-\frac{1}{4}\sum_{k=1}^{n}\mathring{\mathfrak{t}}(k)\mathring{p}(k)+\frac{1}{4}\left(f(n+1)-2-\mathring{\mathfrak{t}}(n+1)\right)\mathring{p}(n+1)\\ &=\frac{1}{4}\sum_{k=1}^{n+1}\left(f(k)-2\right)\mathring{p}(k)-\frac{1}{4}\sum_{k=1}^{n+1}\mathring{\mathfrak{t}}(k)\mathring{p}(k).\end{aligned}$$

Thus, we obtain the claim in case (i).

In case (ii), we consider the third term of (64). Since $T_n^{\mathfrak{d}}$ is an odd number,

$$\begin{aligned}\sum_{t=1}^{\mathfrak{d}(n+1)}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+t)&=\sum_{t=0}^{\mathfrak{d}(n+1)-1}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+1+i)=\sum_{t=0}^{\lfloor\frac{\mathfrak{d}(n+1)-1}{2}\rfloor}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+1+2i)\\ &\geq \sum_{t=\frac{\mathring{\mathfrak{t}}(n+1)}{2}}^{\lfloor\frac{\mathfrak{d}(n+1)-1}{2}\rfloor}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+1+2i)\geq \sum_{t=\frac{\mathring{\mathfrak{t}}(n+1)}{2}}^{\frac{f(n+1)}{2}-1}R_{\mathfrak{d}}(T_n^{\mathfrak{d}}+1+2i)\end{aligned} \tag{65}$$

holds since $\lfloor\frac{\mathfrak{d}(n+1)-1}{2}\rfloor\geq\lfloor\frac{\mathfrak{d}(n+1)}{2}\rfloor-1=\frac{f(n+1)}{2}-1$. We consider two cases whether $f(n+1)-2\leq\mathring{\mathfrak{t}}(n+1)$ or not. If $f(n+1)-2\leq\mathring{\mathfrak{t}}(n+1)$ then $\left(f(n+1)-2-\mathring{\mathfrak{t}}(n+1)\right)\frac{\mathring{p}(n+1)}{4}\leq 0$. The left-hand side of (65) ≥ 0, and we obtain (62). If $f(n+1)-2>\mathring{\mathfrak{t}}(n+1)$ then we have

$$(65)\geq\sum_{t=\frac{\mathring{\mathfrak{t}}(1)}{2}}^{\frac{f(n+1)}{2}-1}\frac{\mathring{p}(n+1)}{2}=\left(\frac{f(n+1)}{2}-1-\frac{\mathring{\mathfrak{t}}(1)}{2}\right)\frac{\mathring{p}(n+1)}{2}=\left(f(n+1)-2-\mathring{\mathfrak{t}}(n+1)\right)\frac{\mathring{p}(n+1)}{4},$$

and hence (62) holds for case (ii). Thus, we obtain the claim. □

Proof of Theorem 7.1. Recall (60), meaning that

$$\begin{aligned}\sum_{t=1}^{T_n^{\mathfrak{d}}}R_{\mathfrak{d}}(t)&\geq\frac{1}{4}\sum_{k=1}^{n}\left(f(k)-2\right)\mathring{p}(k)-\frac{1}{4}\sum_{k=1}^{n}\mathring{\mathfrak{t}}(k)\mathring{p}(k)\qquad\text{(By Lemma 7.2)}\\ &\geq\frac{1}{4}\sum_{k=1}^{n}\left(f(k)-2\right)\mathring{p}(k)-\frac{1}{4}\sum_{k=1}^{\infty}\mathring{\mathfrak{t}}(k)\mathring{p}(k)\qquad(\mathring{\mathfrak{t}}(k)\mathring{p}(k)\geq 0\text{ for any }k)\\ &=\frac{1}{4}\sum_{k=1}^{n}\left(f(k)-2\right)\mathring{p}(k)-C_1\qquad\text{(By assumption)}\end{aligned}$$

$$\geq \frac{1}{4} \sum_{k=1}^{n} (\mathfrak{d}(k) - 3)\, \mathring{p}(k) - C_1 \qquad (f(k) \geq \mathfrak{d}(k) - 1 \text{ for any } k\ ) \tag{66}$$

hold, where C_1 is a positive constant. Note that we have $\sum_{k=1}^{\infty} (\mathfrak{d}(k) - 3)\, \mathring{p}(k) = \infty$ by the hypothesis, then (66) with $n \to \infty$ implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) = \infty$ holds, and hence we obtain the claim. □

7.2 Recurrence-irreducible and aperiodic

By utilizing the same technique as irreducible and periodic 2, we give sufficient conditions for o being recurrent by $\mathcal{D}_{\mathfrak{d}}$ with irreducible and aperiodic.

Theorem 7.3. *Suppose that RWoGG $\mathcal{D} = (\cdot, G, P)$ is*

$$\sum_{k=1}^{\infty} \mathfrak{t}(k) p(k) < \infty, \tag{67}$$

and that every $P(n) = P_n$ ($n = 1, 2, \dots$) is irreducible and $\mathrm{period}(P_n) = 1$. *Let $p(n) = \pi_n(o)$ where π_n denote the stationary distribution of P_n. If $\mathfrak{d}$ satisfies*

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) = \infty \tag{68}$$

then the origin vertex o is recurrent by $\mathcal{D}_{\mathfrak{d}}$.

Proof of Theorem 7.3. Using the same logic as Lemma 7.2, we can prove that $\mathcal{D}_{\mathfrak{d}}$ satisfies

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq \frac{1}{2} \sum_{k=1}^{n} \mathfrak{d}(k) p(k) - \frac{1}{2} \sum_{k=1}^{n} \mathfrak{t}(k) p(k) \tag{69}$$

for $n = 1, 2, \dots$, and hence we obtain the claim performing the same transformation as (66). □

7.3 Weakly less-homesick as graph growing

Before giving sufficient conditions for the transience, we introduce the notion of weakly less-homesickness as graph growing, which is a relation between RWoGGs playing an important role in our analysis. Let $\mathcal{D}_f = (f, G, P)$ and $\mathcal{D}_{f'} = (f', G', P')$ be RWoGG, and let X_t and Y_t respectively denote a random walk on growing graph according to $\mathcal{D}_f$ and $\mathcal{D}_{f'}$, where $X_0 = x_0$ and $Y_0 = y_0$ We say $\mathcal{D}_{f'} = (f', G', P')$ is *weakly less-homesick* than $\mathcal{D}_f = (f, G, P)$ at time T if

$$\sum_{t=1}^{T} \Pr[X_t = x_0] \geq \sum_{t=1}^{T} \Pr[Y_t = y_0] \tag{70}$$

holds.

In particular, this paper is mainly concerned with the weakly less-homesick relation between $\mathcal{D}_f = (f, G, P)$ and $\mathcal{D}_g = (g, G, P)$ with the same P (and G). we say $(\cdot, G, P)$ is *weakly less-homesick as graph growing* (*weakly LHaGG*)[8] if $\mathcal{D}_g = (g, G, P)$ is weakly less-homesick than $\mathcal{D}_f = (f, G, P)$ whenever

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{71}$$

[8]Strictly speaking, weakly LHaGG should be a property of the sequence of transition matrices $P(1), P(2), P(3), \dots$. For the convenience of the notation, we say $\mathcal{D} = (f, G, P)$ is weakly LHaGG, in this paper.

holds for any $n \in \mathbb{Z}_{>0}$, where we remark that G and P are common in $\mathcal{D}_f$ and $\mathcal{D}_g$. The condition (71) implies the graph in $\mathcal{D}_g$ grows faster than $\mathcal{D}_f$, intuitively.

7.4 Transience - irreducible and periodic 2

Using weakly LHaGG, we provide sufficient conditions for the transience of random walks on growing graphs with irreducible and periodic 2.

Theorem 7.4. *Suppose RWoGG $(\cdot, G, P)$ is weakly LHaGG, irreducible and* $\mathrm{period}(P_n) = 2$. *And it satisfies* $\sum_{k=2}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) < \infty$. *If $\mathfrak{d}$ satisfies*

$$\sum_{k=2}^{\infty} \mathfrak{d}(k)\mathring{p}(k-1) < \infty \tag{72}$$

then the origin vertex o is transient by $\mathcal{D}_{\mathfrak{d}}$.

Remark 7.5. *We can regard the assumption of* $\sum_{k=2}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) < \infty$ *as* $\sum_{k=1}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k) < \infty$ *in Theorem 7.4 if* $C_1\mathring{p}(k) \leq \mathring{p}(k-1) \leq C_2\mathring{p}(k)$ *hold for $k \geq 1$, where C_1 and C_2 are positive constants for k.*

To prove Theorem 7.4, we prove Lemma 7.6.

Lemma 7.6. *Suppose RWoGG $(\cdot, G, P)$ is weakly LHaGG, irreducible and* $\mathrm{period}(P_n) = 2$. *Let*

$$g(k) := \max\left\{\mathfrak{d}(k), \mathring{\mathfrak{t}}(k)\right\} \tag{73}$$

for $k = 1, 2, \ldots$. $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T_n^g} R_g(t) = \sum_{k=1}^{n} \sum_{t=T_{k-1}^g+1}^{T_k^g} R_g(t) \leq g(1) + \frac{3}{2}\sum_{k=2}^{n} g(k)\mathring{p}(k-1) \tag{74}$$

for $n = 2, 3, \ldots$.

To prove Lemma 7.6, we must prove Lemma 7.7.

Lemma 7.7. *Suppose RWoGG $(\cdot, G, P)$ is weakly LHaGG, irreducible and* $\mathrm{period}(P_n) = 2$. *$\mathcal{D}_g$ satisfies*

$$\sum_{t=1}^{T_n^g} R_g(t) \leq \sum_{t=1}^{T_{n-1}^g} R_g(t) + \frac{3}{2} g(n)\mathring{p}(n-1) \tag{75}$$

for any $n = 2, 3, \ldots$.

Proof of Lemma 7.7. Let

$$f_{n-1}(k) := \begin{cases} g(k) & (k \leq n-2) \\ \infty & (k = n-1). \end{cases} \tag{76}$$

Let $Z_t^{(n-1)}$ $(t = 0, 1, 2, \ldots)$ denote a RWoGG $\mathcal{D}_{f_{n-1}} = (f_{n-1}, G, P)$, where $Z_0^{(n-1)} = v$. Let $R''_{n-1}(t)$ denote the return probability of $Z_t^{(n-1)}$. By the definition of f_{n-1}, we obtain

$$R''_{n-1}(t) = R_g(t) \tag{77}$$

for $t \in [0, T^g_{n-1}]$ and $n = 2, 3, \ldots$. Clearly, $\sum_{i=1}^{j} g(i) \leq \sum_{i=1}^{j} f_{n-1}(i)$ holds for $j = 1, 2, \ldots$, and hence the weakly LHaGG assumption implies

$$\sum_{t=1}^{T} R''_{n-1}(t) \geq \sum_{t=1}^{T} R_g(t) \tag{78}$$

for $T = 1, 2, \ldots$ and $n = 2, 3, \ldots$. By the proof of Lemma 3.4,

$$R''_{n-1}(t) = \Pr[Z_t^{(n-1)} = o] \leq \mathring{p}(n-1) + \frac{\mathring{p}(n-1)}{2} = \frac{3}{2}\mathring{p}(n-1) \tag{79}$$

hold for $t \in (T^g_{n-1}, T^g_n]$, where we remark that $R''_{n-1}(t) = 0$ for any odd t. This implies that

$$\begin{aligned}
\sum_{t=1}^{T^g_n} R_g(t) &\leq \sum_{t=1}^{T^g_n} R''_{n-1}(t) \qquad \text{(By (78))}\\
&= \sum_{t=1}^{T^g_{n-1}} R''_{n-1}(t) + \sum_{t=T^g_{n-1}+1}^{T^g_n} R''_{n-1}(t)\\
&= \sum_{t=1}^{T^g_{n-1}} R_g(t) + \sum_{t=T^g_{n-1}+1}^{T^g_n} R''_{n-1}(t) \qquad \text{(By (77))}\\
&\leq \sum_{t=1}^{T^g_{n-1}} R_g(t) + \sum_{t=T^g_{n-1}+1}^{T^g_n} \frac{3}{2}\mathring{p}(n-1) \qquad \text{(By (79))}\\
&= \sum_{t=1}^{T^g_{n-1}} R_g(t) + \frac{3}{2}g(n)\mathring{p}(n-1)
\end{aligned}$$

hold for any $n = 2, 3, \ldots$. Therefore, Lemma 7.7 holds. □

Proof of Lemma 7.6. Clearly, $T^g_s \geq T^{\mathfrak{d}}_s$ holds for $s = 1, 2, \ldots$, and hence the weakly LHaGG assumption implies

$$\sum_{t=1}^{T} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T} R_g(t) \tag{80}$$

for $T = 1, 2, \ldots$. By Lemma 7.7 for $n = 2, 3, \ldots$, we obtain

$$\begin{aligned}
&\sum_{t=1}^{T^g_n} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T^g_n} R_g(t) \qquad \text{(By (80))}\\
&\leq \sum_{t=1}^{T^g_{n-1}} R_g(t) + \frac{3}{2}g(n)\mathring{p}(n-1) \qquad \text{(By Lemma 7.7 for } n \text{)}\\
&\leq \sum_{t=1}^{T^g_{n-2}} R_g(t) + \frac{3}{2}g(n-1)\mathring{p}(n-2) + \frac{3}{2}g(n)\mathring{p}(n-1) \qquad \text{(By Lemma 7.7 for } n-1 \text{)}\\
&\vdots
\end{aligned}$$

$$\leq \sum_{t=1}^{T_2^g} R_g(t) + \frac{3}{2}\sum_{k=3}^{n} g(k)\mathring{p}(k-1) \qquad \text{(By Lemma 7.7 for 3)}$$
$$\leq \sum_{t=1}^{T_1^g} R_g(t) + \frac{3}{2}\sum_{k=2}^{n} g(k)\mathring{p}(k-1) \qquad \text{(By Lemma 7.7 for 2)}$$
$$\leq g(1) + \frac{3}{2}\sum_{k=2}^{n} g(k)\mathring{p}(k-1)$$

for $n = 2, 3, \ldots$. Therefore, Lemma 7.6 holds. □

Proof of Theorem 7.4. Since $\sum_{k=2}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) < \infty$ and $\mathring{\mathfrak{t}}(k) \geq 2$ for any $k = 1, 2, \ldots$, we provide $\sum_{k=1}^{\infty} \mathring{p}(k) < \infty$. We calculate $\sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t)$ using Lemma 7.6:

$$\begin{aligned}
\sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t) = \sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T_n^g} R_g(t) \qquad &\text{(By the definition of weakly LHaGG)} \\
\leq g(1) + \frac{3}{2}\sum_{k=2}^{n} g(k)\mathring{p}(k-1) \qquad &\text{(By Lemma 7.6)} \\
= g(1) + \frac{3}{2}\sum_{k=2}^{n} g(k)\mathring{p}(k-1) & \qquad (81) \\
\leq \mathfrak{d}(1) + 1 + \mathring{\mathfrak{t}}(1) + \frac{3}{2}\sum_{k=2}^{n} \mathfrak{d}(k)\mathring{p}(k-1) + \frac{3}{2}\sum_{k=2}^{n} \mathring{p}(k-1) + \frac{3}{2}\sum_{k=2}^{n} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) \\
\leq \mathfrak{d}(1) + 1 + \mathring{\mathfrak{t}}(1) + \frac{3}{2}\sum_{k=2}^{n} \mathfrak{d}(k)\mathring{p}(k-1) + \frac{3}{2}\sum_{k=2}^{\infty} \mathring{p}(k-1) + \frac{3}{2}\sum_{k=2}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) \\
\leq \frac{3}{2}\sum_{k=2}^{n} \mathfrak{d}(k)\mathring{p}(k-1) + C \qquad &\text{(By } \textstyle\sum_{k=2}^{\infty} \mathring{\mathfrak{t}}(k)\mathring{p}(k-1) < \infty \text{ and } \sum_{k=1}^{\infty} \mathring{p}(k) < \infty)
\end{aligned}$$

hold. Now it is easy to see that (72) implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) < \infty$ by $n \to \infty$, meaning that $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ is transient. □

7.5 Transience - irreducible and aperiodic

By using a similar approach as irreducible and periodic 2, we provide sufficient conditions for the transience of random walks on growing graphs with irreducible and aperiodic.

Theorem 7.8. *Suppose RWoGG $(\cdot, G, P)$ is weakly LHaGG, irreducible and* $\mathrm{period}(P_n) = 1$*. And it satisfies $\sum_{k=2}^{\infty} \mathfrak{t}(k)p(k-1) < \infty$. If $\mathfrak{d}$ satisfies*

$$\sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) < \infty \qquad (82)$$

then the initial vertex o is transient by $\mathcal{D}_{\mathfrak{d}}$.

Remark 7.9. *This case satisfies the same statement as Remark 7.5.*

Proof of Theorem 7.8. By the same theory as Lemma 7.6 and 7.7, we can prove that $\mathcal{D}_{\mathfrak{d}}$ satisfies

$$\sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t) \leq g(1) + \frac{3}{2}\sum_{k=2}^{n} g(k)p(k-1)$$

for $n = 2, 3, \ldots$, where $g(k) := \max\{\mathfrak{d}(k), \mathfrak{t}(k)\}$. Performing the similar technique as Theorem 7.4, we provide

$$\sum_{t=1}^{T_n^g} R_{\mathfrak{d}}(t) \leq \frac{3}{2}\sum_{k=2}^{n} \mathfrak{d}(k)p(k-1) + C$$

for $n = 2, 3, \ldots$. Now it is easy to see that (82) implies $\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) < \infty$ by $n \to \infty$, meaning that $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ is transient. □

8 Random walk on $\{0, \ldots, N\}^n$ with an increasing n

8.1 Definition and main result

This section is concerned with a $\frac{1}{2}$ - lazy random walk on $\{0, \ldots, N\}^n$ with an increasing n. Let $G_n = (V_n, E_n)$ be a graph given by

$$V_n := \{0, \ldots, N\}^{n_0+n-1}$$
$$E_n := \{\{x, y\} \ ; \ x, y \in V_n, \|x - y\|_1 = 1\}$$

where n and N are (fixed) positive integers. Let $o = (0, \ldots, 0)$ denote the origin. Let P_n for $n = 1, 2, \ldots$ denote the transition probability of a random walk on a static graph G_n, where

$$P_n(x, y) = \begin{cases} \frac{1}{2} & (\text{if } x = y) \\ \frac{1}{4(n_0+n-1)} & (\text{if } \|x - y\|_1 = 1,\ x_k \neq y_k \text{ and } x_k \notin \{0, N\}) \\ \frac{1}{2(n_0+n-1)} & (\text{if } \|x - y\|_1 = 1,\ x_k \neq y_k \text{ and } x_k \in \{0, N\}) \\ 0 & (\text{otherwise}) \end{cases}$$

for $x, y \in V_n$. Then, we are concerned with a RWoGG X_t according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ (see Figure 5), where $G(n) = G_n$. If the graph grows at time t, we assume $X_t = (x_1, \ldots, x_{n_0+n-1}) = (x_1, \ldots, x_{n_0+n-1}, 0)$.

Theorem 8.1. If $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} = \infty,$$

o is recurrent, otherwise o is transient.

8.2 Proof of Theorem 8.1

We will prove Theorem 8.1 based on Theorem 7.3 and 7.8. For this purpose, we prove that a $\frac{1}{2}$ - lazy random walk on $\{0, \ldots, N\}^n$ with an increasing n satisfies Lemma 8.2 and 8.3.

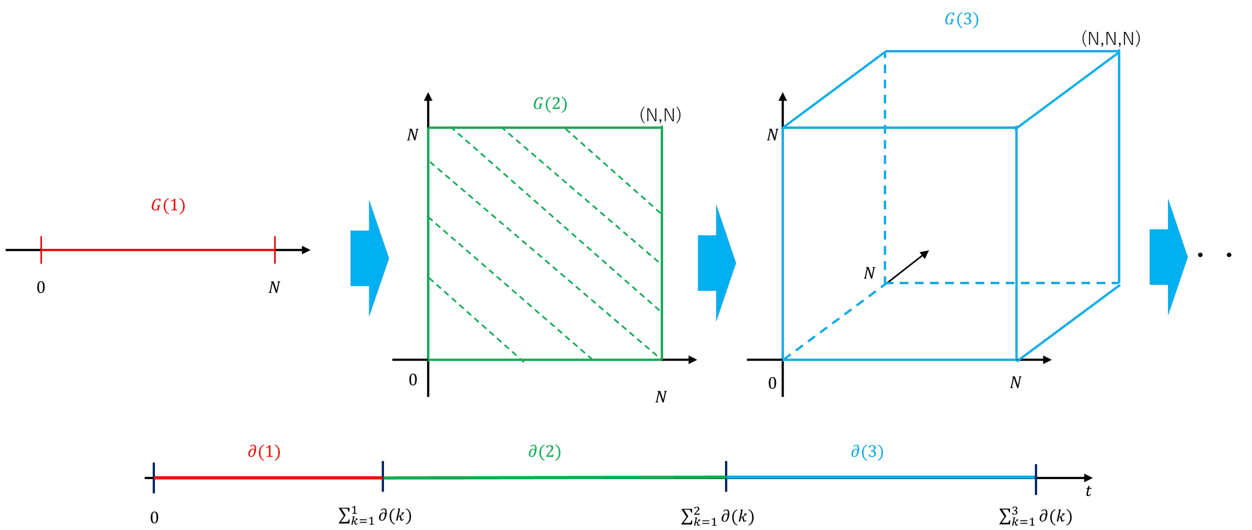

Figure 5: growing dimension box

Lemma 8.2 ([39]). *$G(n)$ satisfies*

$$p(n) = \frac{1}{(2N)^{n_0+n-1}} \tag{83}$$

$$\mathfrak{t}(n) \leq 8N^2 \log_2(2N)(n_0+n-1)^3 \tag{84}$$

for any $n \geq 1$. Therefore, $\mathcal{D} = (\cdot, G, P)$ satisfies (67).

We obtain Lemma 8.2 using Lemma 5.3 and 5.4.

Lemma 8.3. *A $\frac{1}{2}$ - lazy random walk on $\{0,\ldots,N\}^n$ with an increasing n is weakly LHaGG.*

Here we define an essential symbol to prove Lemma 8.3 Let X_t $(t = 0,1,2,\ldots)$ be a random walk on $\{0,\ldots,N\}^n$ with an increasing n according to $\mathcal{D}_f = (f, G, P)$, and let $R_f(t)$ $(t = 1,2,\ldots)$ denote the return probability of X_t. Let $\boldsymbol{X} = X_0, X_1, \ldots$, for convenience. Similarly, let Y_t $(t = 0,1,2,\ldots)$ be a random walk on $\{0,\ldots,N\}^n$ with an increasing n according to $\mathcal{D}_g = (g, G, P)$, and let $R_g(t)$ $(t = 1,2,\ldots)$ denote the return probability of Y_t. Let $\boldsymbol{Y} = Y_0, Y_1, \ldots$. For convenience, let $n_0^f := 0$ and $n_t^f := n-1$ for $t \in (T_{n-1}^f, T_n^f]$. Note that $X_0 = (X_0^1, \ldots, X_0^{n_0+n_0^f}) = (0,\ldots,0) = o$ and $X_t = (X_t^1, \ldots, X_t^{n_0+n_t^f})$ for $t \geq 1$. Suppose that

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{85}$$

for any $n \geq 1$. Let $\Delta : \mathbb{N} \to \{-1, 0_{-1}, 0_1, 1\}$ and $\Delta' : \mathbb{N} \to \{-1, 0_{-1}, 0_1, 1\}$. Let $S_t^f := \{1, \ldots, n_0 + n_t^f\}$ and $S_t^g := \{1, \ldots, n_0 + n_t^g\}$. Let $I : \mathbb{N} \to S_t^f$ and $J : \mathbb{N} \to S_t^g$ respectively denote the random variables which are selected by changing from X_{t-1} to X_t and from Y_{t-1} to Y_t. The transition of X and Y can be respectively represented by the random variables sequence $(\Delta_1, I_1), (\Delta_2, I_2), \ldots \in \prod_{t=1}^{\infty} \Delta_t \times I_t$ and $(\Delta_1', J_1), (\Delta_2', J_2), \ldots \in \prod_{t=1}^{\infty} \Delta_t' \times J_t$. We use the following Lemma 8.4 to prove Lemma 8.3.

Lemma 8.4. *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy*

$$X_t = o,\ Y_{t'} = o$$

for $0 \leq t \leq t'$ and $\min\{\phi\} = \infty$. There is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that

$$\tau^X \leq \tau^Y, \tag{86}$$

where $\tau^X := \min\{r\ ;\ r > 0,\ X_{t+r} = o\}$ and $\tau^Y := \min\{r\ ;\ r > 0,\ Y_{t'+r} = o\}$, i.e., X returns to the origin vertex o in a fewer steps than Y.

To show Lemma 8.4, we must prove the following lemmas.

Lemma 8.5. *Suppose that (85) holds. We can construct a coupling of I and J such that $J_{t_2+s} \notin S^f_{t_1+1}$ for $s = 1, \ldots, T(1) - 1$ and $J_{t_2+T(1)} = I_{t_1+1}$, where $t_1 \leq t_2$ and $T(1)$ is uniformly chosen on $\mathbb{N} \cup \{\infty\}$.*

To prove Lemma 8.5, we must prove Lemma 8.6.

Lemma 8.6. *Suppose that $\alpha_0 = 0$ and $0 \leq \alpha_i \leq 1$ for any $i \in \mathbb{N}$. Then, we have*

$$\sum_{t=1}^{\infty} \alpha_t \prod_{j=0}^{t-1} (1 - \alpha_j) = 1 - \prod_{j=0}^{\infty} (1 - \alpha_j).$$

Proof. Let $\beta_t := \prod_{j=0}^{t}(1 - \alpha_j)$. Then,

$$\sum_{t=1}^{M} \alpha_t \prod_{j=0}^{t-1} (1 - \alpha_j) = \sum_{t=1}^{M} (\beta_{t-1} - \beta_t) = \beta_0 - \beta_M = 1 - \prod_{j=0}^{M} (1 - \alpha_j)$$

holds, and the lemma follows by letting $M \to \infty$. □

Proof of Lemma 8.5. Clearly, $I_{t_1+1} \in S^f_{t_1+1}$. Then, we have

$$\Pr\left[I_{t_1+1} = a\right] = \frac{1}{|S^f_{t_1+1}|}.$$

A set $\Psi_j(S^f_{t_1+1})$ is given by

$$\Psi_j(S^f_{t_1+1}) := \begin{cases} \left\{\omega \in \prod_{i=1}^{\infty} S^g_{t_2+i}\ ;\ \omega_{j'} \notin S^f_{t_1+1} \text{ for } j' < j \text{ and } \omega_j \in S^f_{t_1+1}\right\} & (\text{if } j \in \mathbb{N}) \\ \left\{\omega \in \prod_{i=1}^{\infty} S^g_{t_2+i}\ ;\ \omega_{j'} \notin S^f_{t_1+1} \text{ for } j' \geq 1 \right\} & (\text{if } j = \infty). \end{cases}$$

Then, we get

$$\Pr\left[\{J_{t_2+i}\}_{i=1,\ldots} \in \Psi_j(S^f_{t_1+1})\right] = \begin{cases} \frac{|S^f_{t_1+1}|}{|S^g_{t_2+1}|} & (\text{if } j = 1) \\ \frac{|S^f_{t_1+1}|}{|S^g_{t_2+j}|} \prod_{i=1}^{j-1} \left(1 - \frac{|S^f_{t_1+1}|}{|S^g_{t_2+i}|}\right) & (\text{if } j \neq 1 \text{ and } j \in \mathbb{N}\,) \\ \prod_{i=1}^{\infty} \left(1 - \frac{|S^f_{t_1+1}|}{|S^g_{t_2+i}|}\right) & (\text{if } j = \infty\,.) \end{cases} \tag{87}$$

By Lemma 8.6, we have

$$\sum_{j \in \mathbb{N} \cup \{\infty\}} \Pr\left[\{J_{t_2+i}\}_{i=1,\ldots} \in \Psi_j(S^f_{t_1+1})\right]$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{N}} \alpha_j \prod_{i=1}^{j-1}(1-\alpha_i) + \prod_{i=1}^{\infty}(1-\alpha_i) \\
&= 1 - \prod_{j=0}^{\infty}(1-\alpha_j) + \prod_{i=1}^{\infty}(1-\alpha_i) \qquad \text{(by Lemma 8.6)} \\
&= 1,
\end{aligned}
$$

where $\alpha_0 = 0$ and $\alpha_i = \frac{|S^f_{t_1+1}|}{|S^g_{t_2+i}|}$. Then, we obtain

$$
\begin{aligned}
&\sum_{T(1)\in\mathbb{N}\cup\{\infty\}} \Pr\left[J_{t_2+1} \notin S^f_{t_1+1}, \ldots, J_{t_2+T(1)-1} \notin S^f_{t_1+1}, J_{t_2+T(1)} = a\right] \qquad (88)\\
&= \sum_{j\in\mathbb{N}\cup\{\infty\}} \Pr\left(\{J_{t_2+i}\}_{i=1,\ldots} \in \Psi_j(S^f_{t_1+1})\right) \Pr\left[J_{t_2+j} = a \middle| \{J_{t_2+i}\}_{i=1,\ldots} \in \Psi_j(S^f_{t_1+1})\right] \\
&= \frac{1}{|S^f_{t_1+1}|} \sum_{j\in\mathbb{N}\cup\{\infty\}} \Pr\left(\{J_{t_2+i}\}_{i=1,\ldots} \in \Psi_j(S^f_{t_1+1})\right) = \frac{1}{|S^f_{t_1+1}|},
\end{aligned}
$$

where $\Pr\left[J_{t_2+\infty} = a \middle| \{J_{t_2+i}\}_{i=1,\ldots,\infty} \in \Psi_\infty(S^f_{t_1+1})\right] = \frac{1}{|S^f_{t_1+1}|}$. Hence

$$
\Pr\left[I_{t_1+1} = a\right] = \sum_{T(1)\in\mathbb{N}\cup\{\infty\}} \Pr\left[J_{t_2+1} \notin S^f_{t_1+1}, \ldots, J_{t_2+T(1)-1} \notin S^f_{t_1+1}, J_{t_2+T(1)} = I_{t_1+1}\right] = \frac{1}{|S^f_{t_1+1}|}
$$

for any $a \in S^f_{t_1+1}$. This means that $J_{t_2+s} \notin S^f_{t_1+1}$ for $s < T(1)$ and $J_{t_2+T(1)} = I_{t+1}$ as long as $I_{t_1+1} = a$, and the lemma follows. □

Lemma 8.7. *Suppose that $X^i_{t_1} \le Y^i_{t_2}$ holds for $i \in S^f_{t_1}$, where $t_1 \le t_2$. There is a coupling of X and Y such that*

$$
X^i_{t_1} \le Y^i_{t_2+s} \quad \text{and} \quad X^i_{t_1+1} \le Y^i_{t_2+T(1)}
$$

hold for $s < T(1)$, where $T(1)$ is given by Lemma 8.5.

Proof. Lemma 8.5 implies

$$
Y^i_{t_2} = \ldots = Y^i_{t_2+T(1)-1} \qquad (89)
$$

for $i \in S^f_{t_1}$. Then, by $X^i_{t_1} \le Y^i_{t_2}$, we have $X^i_{t_1} \le Y^i_{t_2+s}$ for $s < T(1)$. We consider two cases whether $X^i_{t_1} = Y^i_{t_2}$ or not.

(i) For $X^i_{t_1} = Y^i_{t_2}$. We prove that

$$
\begin{aligned}
X^i_{t_1} = Y^i_{t_2} = \ldots = Y^i_{t_2+T(1)-1}, \qquad (90)\\
X^i_{t_1+1} = Y^i_{t_2+T(1)}. \qquad (91)
\end{aligned}
$$

Clearly, we get (90) by (89) and $X^i_{t_1} = Y^i_{t_2}$. Then, we give

$$
\Pr\left[\Delta_{t_1+1} = z \mid X_{t_1},\ I_{t_1+1} = i\right] = \Pr\left[\Delta'_{t_2+T(1)} = z \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right]
$$

for $z \in \{-1, 0_{-1}, 0_1, 1\}$. Recall that $X^i_{t_1} = Y^i_{t_2} = \ldots = Y^i_{t_2+T(1)-1}$. The coupling implies $X^i_{t_1+1} = X^i_{t_1} + z = Y^i_{t_2+T(1)-1} + z = Y^i_{t_2+T(1)}$ hold, where $X^i_{t_1} + 0_1 := X^i_{t_1} + 0$ and $X^i_{t_1} + 0_{-1} := X^i_{t_1} + 0$. Thus, we obtain (91).

(ii)For $X^i_{t_1} < Y^i_{t_2}$. We prove that

$$X^i_{t_1} < Y^i_{t_2} = \ldots = Y^i_{t_2+T(1)-1}, \tag{92}$$
$$X^i_{t_1+1} \le Y^i_{t_2+T(1)}. \tag{93}$$

We get (92) by (89) and $X^i_{t_1} < Y^i_{t_2}$. Recall that

$$\begin{aligned}
&\Pr\left[\Delta_{t_1+1} = 0_1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta_{t_1+1} = 0_{-1} \mid X_{t_1},\ I_{t_1+1} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta_{t_1+1} = 1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta_{t_1+1} = -1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta'_{t_2+T(1)} = 0_1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta'_{t_2+T(1)} = 0_{-1} \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta'_{t_2+T(1)} = 1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right] = \frac{1}{4},\\
&\Pr\left[\Delta'_{t_2+T(1)} = -1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right] = \frac{1}{4}
\end{aligned}$$

hold. Then, we can couple the transitions $X^i_{t_1} \mapsto X^i_{t_1+1}$ and $Y^i_{t_2+T(1)-1} \mapsto Y^i_{t_2+T(1)}$ such that

$$\begin{aligned}
&\Pr\left[\Delta_{t_1+1} = 0_1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \Pr\left[\Delta'_{t_2+T(1)} = 1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right]\\
&\Pr\left[\Delta_{t_1+1} = 0_{-1} \mid X_{t_1},\ I_{t_1+1} = i\right] = \Pr\left[\Delta'_{t_2+T(1)} = -1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right]\\
&\Pr\left[\Delta_{t_1+1} = 1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \Pr\left[\Delta'_{t_2+T(1)} = 0_1 \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right]\\
&\Pr\left[\Delta_{t_1+1} = -1 \mid X_{t_1},\ I_{t_1+1} = i\right] = \Pr\left[\Delta'_{t_2+T(1)} = 0_{-1} \,\middle|\, Y_{t_2+T(1)-1},\ J_{t_2+T(1)} = i\right]
\end{aligned}$$

hold. Recall that $X^i_{t_1} < Y^i_{t_2} = \ldots = Y^i_{t_1+T(1)-1}$ implies $X^i_{t_1+1} = X^i_{t_1} + \Delta_{t_1+1} \le Y^i_{t_2+T(1)-1} + \Delta'_{t_2+T(1)} = Y^i_{t_2+T(1)}$. The coupling implies $X^i_{t_1+1} \le Y^i_{t_2+T(1)}$. Thus, we obtain (93). □

Proof of Lemma 8.4. We consider the two cases: (i) $\tau^X = 1$ and (ii) $\tau^X > 1$.

Part (i): Clearly, we get

$$\Pr\left[\Delta_{t+1} = 0_{-1}, 0_1 | X_t = o\right] = \frac{1}{2}$$
$$\Pr\left[\Delta_{t+1} = 1 | X_t = o\right] = \frac{1}{2}$$

for any $t \ge 1$. On the other hands, we obtain

$$\Pr\left[\Delta'_{t'+1} = 0_{-1}, 0_1 \middle| Y_{t'} = o\right] = \frac{1}{2}$$

$$\Pr\left[\Delta'_{t'+1}=1\middle|Y_{t'}=o\right]=\frac{1}{2}$$

for any $t'\geq 1$. Hence

$$\begin{aligned}&\Pr\left[\Delta_{t+1}=0_{-1},0_1|X_t=o\right]=\Pr\left[\Delta'_{t'+1}=0_{-1},0_1\middle|Y_{t'}=o\right]\\&\Pr\left[\tau^X=1\middle|X_t=o\right]=\Pr\left[\tau^Y=1\middle|Y_{t'}=o\right]\end{aligned}$$

for any $0\leq t\leq t'$. This means that we can construct a coupling of X and Y such that $\tau^X=\tau^Y=1$.

Part(ii): Since

$$\Pr\left[\Delta_{t+1}=1|X_t=o\right]=\Pr\left[\Delta'_{t'+1}=1\middle|Y_{t'}=o\right]=\frac{1}{2},$$

we obtain

$$\Pr\left[|X_{t+1}|=1|X_t=o\right]=\Pr\left[|Y_{t'+1}|=1|Y_{t'}=o\right]=\frac{1}{2}$$

for any $0\leq t\leq t'$. Therefore, without loss of generality we can assume that [9] $X_{t''}=(X^1_{t''},X^2_{t''},\dots,X^{n_0+n^f_{t''}}_{t''})=(1,0,\dots,0)$ and $Y_{t'''}=(Y^1_{t'''},Y^2_{t'''},\dots,Y^{n_0+n^g_{t'''}}_{t'''})=(1,0,\dots,0)$, where $t'':=t+1$ and $t''':=t'+1$. Let ξ_j denote

$$\xi_j:=\min\left\{r\ ;\ r>0,\ J_{t'''+T(j-1)+r}\in S^f_{t''+j}\right\},$$

where $T(0):=0$ and $T(j):=\sum_{i=1}^j\xi_i$.

Let

$$\tilde{\Delta}_s:=\begin{cases}\Delta_s & (\text{if } X^{I_s}_{s-1}=Y^{J_{T(s)}}_{T(s)-1}\ )\\ 0_{-1} & (\text{if } X^{I_s}_{s-1}<Y^{J_{T(s)}}_{T(s)-1} \text{ and } \Delta_s=-1\ )\\ 0_1 & (\text{if } X^{I_s}_{s-1}<Y^{J_{T(s)}}_{T(s)-1} \text{ and } \Delta_s=1\ )\\ -1 & (\text{if } X^{I_s}_{s-1}<Y^{J_{T(s)}}_{T(s)-1} \text{ and } \Delta_s=0_{-1}\ )\\ 1 & (\text{if } X^{I_s}_{s-1}<Y^{J_{T(s)}}_{T(s)-1} \text{ and } \Delta_s=0_1\ ).\end{cases}\tag{94}$$

Let

$$A_{t'',\tau^X}:=\left\{\omega=\{\omega_i\}_{i\in\mathbb{N}}\in\prod_{i=1}^{\infty}(I_i\times\Delta_i)\ ;\ \omega_{t''+1}=(\theta_{t''+1},\Delta_{t''+1}),\dots,\omega_{t''+\tau^X}=(\theta_{t''+\tau^X},\Delta_{t''+\tau^X})\right\},$$

$$B_{t''',\xi_1,\dots,\xi_{\tau^X}}$$
$$:=\left\{\omega=\{\omega_i\}_{i\in\mathbb{N}}\in\prod_{i=1}^{\infty}\left(J_i\times\Delta'_i\right)\ ;\ \omega_{t'''+T(1)}=(\theta_{t''+1},\tilde{\Delta}_{t''+1}),\dots,\omega_{t'''+T(\tau^X)}=(\theta_{t''+\tau^X},\tilde{\Delta}_{t''+\tau^X})\right\},$$

$$C_{s,0}:=\left\{\omega=\{\omega_i\}_{i\in\mathbb{N}}\in\prod_{i=1}^{\infty}\left(J_i\times\Delta'_i\right)\ ;\ Y_s=o\right\},$$

$$C_{s,j}:=\left\{\omega=\{\omega_i\}_{i\in\mathbb{N}}\in\prod_{i=1}^{\infty}\left(J_i\times\Delta'_i\right)\ ;\ Y_s\neq o,\dots,Y_{s+j-1}\neq o,Y_{s+j}=o\right\}.$$

[9] Suppose that $|Y_t|=|Y'_t|=1$. Let $Y^i_t=Y'^j_t=1$. There is the coupling of $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ such that $Y_t=o$ if and only if $Y'_t=o$.

By Lemma 8.5, we obtain

$$\begin{aligned}
&\Pr\left[(I_{t''+j}, \Delta_{t''+j}) = (\theta_{t''+j}, \Delta_{t''+j}) \middle| I_{t''+j-1}, \Delta_{t''+j-1}\right] \\
&= \Pr\left[I_{t''+j} = \theta_{t''+j} \middle| I_{t''+j-1}, \Delta_{t''+j-1}\right] \Pr\left[\Delta_{t''+j} = \Delta_{t''+j} \middle| I_{t''+j} = \theta_{t''+j}\right] \\
&= \sum_{\xi_j \in \mathbb{N}\cup\{\infty\}} \Pr\left[J_{t'''+T(j-1)+\xi_j} = \theta_{t''+j} \middle| J_{t'''+T(j-1)}, \Delta'_{t'''+T(j-1)}\right] \\
&\cdot \Pr\left[\Delta'_{t'''+T(j-1)+\xi_j} = \tilde{\Delta}_{t''+j} \middle| J_{t'''+T(j-1)+\xi_j} = \theta_{t''+j}\right] \\
&= \sum_{\xi_j \in \mathbb{N}\cup\{\infty\}} \Pr\left[(J_{t'''+T(j-1)+\xi_j}, \Delta'_{t'''+T(j-1)+\xi_j}) = (\theta_{t''+j}, \tilde{\Delta}_{t''+j}) \middle| J_{t'''+T(j-1)}, \Delta'_{t'''+T(j-1)}\right]
\end{aligned}$$

for $j = 1, \ldots, \tau^X$. Repeated application of Lemma 8.5 shows that

$$\Pr\left[A_{t'',\tau^X} \middle| X_t = o, \{\Delta_{t+1} = 1\}\right] = \sum_{\xi_1,\ldots,\xi_{\tau^X} \in \mathbb{N}\cup\{\infty\}} \Pr\left[B_{t''',\xi_1,\ldots,\xi_{\tau^X}} \middle| Y_{t'} = o, \left\{\Delta'_{t'+1} = 1\right\}\right]$$

for any $\tau^X \geq 1$. Recall that $X^i_{t''} \leq Y^i_{t'''}$, Lemma 8.7 implies

$$\begin{aligned}
X^i_{t''+j-1} &\leq Y^i_{t'''+T(j-1)} = \ldots = Y^i_{t'''+T(j)-1} \\
X^i_{t''+j} &\leq Y^i_{t'''+T(j)}
\end{aligned}$$

for $j = 1, \ldots, \tau^X$. This means that $Y_{t'''+s-1} \neq o$ for $s = 1, \ldots, T(\tau^X)$. Since

$$\begin{aligned}
&\sum_{j \in \{0\}\cup\mathbb{N}\cup\{\infty\}} \Pr\left[C_{t'''+T(\tau^X),j} \middle| Y_{t'} = o, \left\{\Delta'_{t'+1} = 1\right\}, B_{t''',\xi_1,\ldots,\xi_{\tau^X}}\right] \\
&= \sum_{j \in \{0\}\cup\mathbb{N}\cup\{\infty\}} \Pr\left[Y_{t'''+T(\tau^X)} \neq o, \ldots, Y_{t'''+T(\tau^X)+j} = o \middle| Y_{t'''+T(\tau^X)}\right] = 1,
\end{aligned}$$

we provide

$$\begin{aligned}
&\sum_{j \in \{0\}\cup\mathbb{N}\cup\{\infty\}} \sum_{\xi_1,\ldots,\xi_{\tau^X} \in \mathbb{N}\cup\{\infty\}} \Pr\left[B_{t''',\xi_1,\ldots,\xi_{\tau^X}} \cap C_{t'''+T(\tau^X),j} \middle| Y_{t'} = o, \left\{\Delta'_{t'+1} = 1\right\}\right] \\
&= \sum_{j \in \{0\}\cup\mathbb{N}\cup\{\infty\}} \sum_{\xi_1,\ldots,\xi_{\tau^X} \in \mathbb{N}\cup\{\infty\}} \Pr\left[B_{t''',\xi_1,\ldots,\xi_{\tau^X}} \middle| Y_{t'} = o, \left\{\Delta'_{t'+1} = 1\right\}\right] \Pr\left[C_{t'''+T(\tau^X),j} \middle| B_{\xi_1,\ldots,\xi_{\tau^X}}\right] \\
&= \sum_{\xi_1,\ldots,\xi_{\tau^X} \in \mathbb{N}\cup\{\infty\}} \Pr\left[B_{t''',\xi_1,\ldots,\xi_{\tau^X}} \middle| Y_{t'} = o, \left\{\Delta'_{t'+1} = 1\right\}\right] = \Pr\left[A_{t'',\tau^X} \middle| X_t = o, \{\Delta_{t+1} = 1\}\right].
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\Pr\left[\{\Delta_{t+1} = 1\} \cap A_{t'',\tau^X} \middle| X_t = o\right] \\
&= \sum_{j \in \{0\}\cup\mathbb{N}\cup\{\infty\}} \sum_{\xi_1,\ldots,\xi_{\tau^X} \in \mathbb{N}\cup\{\infty\}} \Pr\left[\left\{\Delta'_{t'+1} = 1\right\} \cap B_{t''',\xi_1,\ldots,\xi_{\tau^X}} \cap C_{t'''+T(\tau^X),j} \middle| Y_{t'} = o\right]
\end{aligned}$$

for any $0 \leq t \leq t'$ and $\tau^X > 1$. Furthermore, since $t \leq t'$ and $\tau^X \leq T(\tau^X)$, we get

$$t'' + \tau^X \leq t''' + T(\tau^X) \leq t''' + T(\tau^X) + j = t''' + \tau^Y.$$

This means that we can construct a coupling of X and Y such that $\tau^X \leq \tau^Y$. □

Lemma 8.8. *Let* $Z_t(\boldsymbol{X}) := \sum_{s=0}^{t} 1_{\{X_s=o\}}$ *and* $Z_t(\boldsymbol{Y}) := \sum_{s=0}^{t} 1_{\{Y_s=o\}}$*. Let* $\tau_o^X(n) := \min\{t\,;\, Z_t(X) = n\}$ *and* $\tau_o^Y(n) := \min\{t\,;\, Z_t(Y) = n\}$*. There is a coupling of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *such that* $\tau_o^X(n) \leq \tau_o^Y(n)$ *for* $n = 0, 1, 2, \ldots$*.*

Proof. The proof is by induction on n.

(1) For $n = 0$. It is clear that $\tau_o^X(0) \leq \tau_o^Y(0)$ since $\tau_o^X(0) = 0$ and $\tau_o^Y(0) = 0$. Therefore, the lemma follows for $n = 0$.

(2) Assuming Lemma 8.8 to hold for n, we will prove it for $n+1$. By assumption, $\tau_o^X(n) \leq \tau_o^Y(n)$.

(a)For $\tau_o^X(n+1) = \tau_o^X(n) + 1$. By Lemma 8.4, we obtain

$$\begin{aligned}\Pr\left[\tau_o^X(n+1) = \tau_o^X(n) + 1\right] &= \Pr\left[\Delta_{\tau_o^X(n)+1} = 0_{-1}, 0_1\right]\\ &= \Pr\left[\Delta'_{\tau_o^Y(n)+1} = 0_{-1}, 0_1\right] = \Pr\left[\tau_o^Y(n+1) = \tau_o^Y(n) + 1\right].\end{aligned}$$

Recall that $\tau_o^X(n) \leq \tau_o^Y(n)$ implies $\tau_o^X(n+1) = \tau_o^X(n) + 1 \leq \tau_o^Y(n) + 1 = \tau_o^Y(n+1)$, and hence the lemma follows for (a).

(b)For $\tau_o^X(n+1) > \tau_o^X(n) + 1$. If $\tau_o^Y(n+1) = \infty$ we have $\tau_o^X(n+1) \leq \tau_o^Y(n+1)$ clearly, and we obtain the claim in this case. Then, we consider $\tau_o^Y(n+1) < \infty$. By Lemma 8.4, we get

$$\begin{aligned}&\Pr\left[\left\{\Delta_{\tau_o^X(n)+1} = 1\right\} \cap A_{\tau^X} \middle| \tau_o^X(n)\right]\\ &= \sum_{j\in\{0\}\cup\mathbb{N}\cup\{\infty\}} \sum_{\xi_1,\ldots,\xi_{\tau^X}\in\mathbb{N}\cup\{\infty\}} \Pr\left[\left\{\Delta'_{\tau_o^Y(n)+1} = 1\right\} \cap B_{\tau_o^Y(n),\xi_1,\ldots,\xi_{\tau^X}} \cap C_{\tau_o^Y(n)+T(\tau^X),j} \middle| \tau_o^Y(n)\right].\end{aligned}$$

This implies that $\tau_o^Y(n+1) = \tau_o^Y(n) + T(\tau^X) + j \geq \tau_o^X(n) + \tau^X = \tau_o^X(n+1)$ by $\tau_o^X(n) \leq \tau_o^Y(n)$. Hence the lemma follows for (b). Therefore, we obtain the claim. □

Proof of Lemma 8.3. Lemma 8.8 implies there is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $Z_T(\mathbf{X}) \geq n$ as long as $Z_T(\mathbf{Y}) \geq n$ for any $0 \leq n \leq T$ and $T \geq 1$. This means that

$$\Pr\left[Z_T(\mathbf{X}) \geq n\right] \geq \Pr\left[Z_T(\mathbf{Y}) \geq n\right]$$

for $n = 0, \ldots, T$ and $T = 1, 2, \ldots$, and hence it follows that $E\left[Z_T(\mathbf{X})\right] \geq E\left[Z_T(\mathbf{Y})\right]$. Therefore, we have

$$\begin{aligned}E\left[\sum_{t=1}^{T} 1_{\{X_t=o\}}\right] &\geq E\left[\sum_{t=1}^{T} 1_{\{Y_t=o\}}\right]\\ \sum_{t=1}^{T} E\left[1_{\{X_t=o\}}\right] &\geq \sum_{t=1}^{T} E\left[1_{\{Y_t=o\}}\right]\\ \sum_{t=1}^{T} R_f(t) &\geq \sum_{t=1}^{T} R_g(t)\end{aligned}$$

for any $T \geq 1$ since $Z_T(\mathbf{X}) = \sum_{t=1}^{T} 1_{\{X_t=o\}}$, and the proof is complete. □

Proof of Theorem 8.1. By Lemma 8.2,

$$C_1 p(n) \leq p(n-1) \leq C_2 p(n) \tag{95}$$

hold for any $n = 2, 3, \ldots$, where C_1 and C_2 are positive constants to n. Since $\mathcal{D}_\mathfrak{d}$ satisfies Lemma 8.2 and Lemma 8.3, if $\mathcal{D}_\mathfrak{d}$ satisfies

$$\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) \geq C_1 \sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} = \infty,$$

o is recurrent by Theorem 7.3, and if $\mathcal{D}_\mathfrak{d}$ satisfies

$$\begin{aligned} \sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) &\leq C_2 \sum_{k=2}^{\infty} \mathfrak{d}(k)p(k) \qquad \text{(by (95))} \\ &\leq C_2 \sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2N)^k} < \infty, \end{aligned} \tag{96}$$

o is transient by Theorem 7.8. We obtain the claim. □

9 Other example of weakly LHaGG

9.1 Random walk on a growing spider tree

This section is concerned with a random walk on a growing spider tree. This example of application is different recurrence and transience for each vertex. Let $C_n = (V_n, E_n)$ be a tree which has the height n, where $V_1 = \{r, v_1\}$ and $E_1 = \{\{r, v_1\}\}$. For convenience, let $h(v)$ denote the height of vertex $v \in V_n$, i.e., $h(r) = 0$, and $h(v_1) = 1$. Let $H_n := \{v \in C_n \ ; \ h(v) = n\}$. And let $L_t^f := \left\{v \in C_{n_t^f} \ ; \ \deg_t^f(v) = 1, v \neq r\right\}$. By using V_n, E_n and w, we inductively construct $C_{n+1} = (V_{n+1}, E_{n+1})$ such that

$$\begin{aligned} V_{n+1} &:= V_n \cup \tilde{V}_n \\ E_{n+1} &:= E_n \cup \tilde{E}_n \end{aligned}$$

for $n \geq 1$, where

$$\begin{aligned} \tilde{V}_n &:= \bigcup_{i=1}^{k^n} \bigcup_{j=1}^{n+1} \{v_j^i\} \\ \tilde{E}_n &:= \bigcup_{i=1}^{k^n} \{\{r, v_1^i\}\} \bigcup_{j=1}^{n} \{\{v_j^i, v_{j+1}^i\}\} \end{aligned}$$

for $n \geq 1$, where $k \geq 2$. We consider the recurrence of a simple random walk on a growing spider tree $\mathcal{D}_\mathfrak{d} = (\mathfrak{d}, C, P)$ (see Figure 6).

Theorem 9.1. *r is always recurrent.*

To prove Theorem 9.1, we must prove Lemma 9.2.

Lemma 9.2. *Let X_t be a simple random walk on a growing spider tree according to $\mathcal{D}_\mathfrak{d} = (\mathfrak{d}, C, P)$ Let Z_t be a random walk on $\mathbb{Z}$. Then, we obtain*

$$\Pr[X_t = r | X_0 = r] \geq \Pr[Z_t = 0 | Z_0 = 0]$$

for any $t \geq 0$.

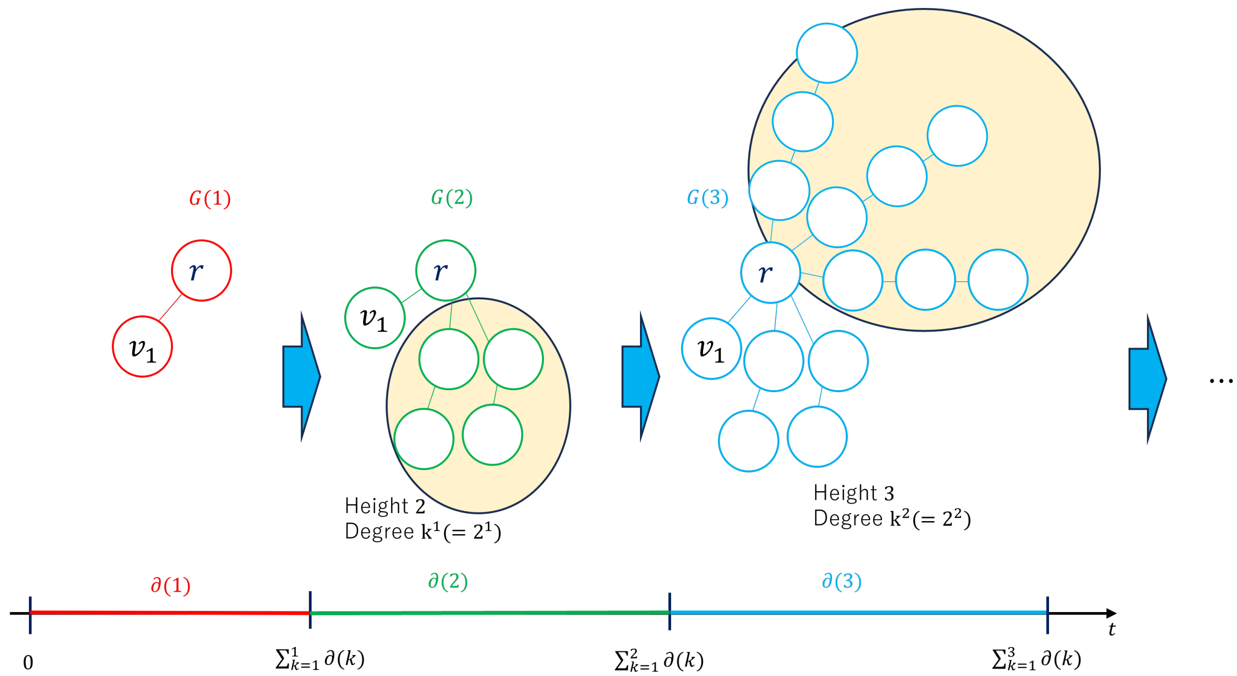

Figure 6: growing spider tree (k=2)

Proof. To prove the lemma, we prove that there is a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Z} = \{Z_t\}_{t \geq 0}$ such that $h(X_t) \leq |Z_t|$ for any $t \geq 0$. The proof is by induction on t. Clearly $h(X_0) = |Z_0| = 0$. Inductively assuming $h(X_t) \leq |Z_t|$, we prove $h(X_{t+1}) \leq |Z_{t+1}|$. If $h(X_t) > |Z_t|$ then $h(X_t) + 2 \leq |Z_t|$ since a random walk on $\mathcal{D}_{\mathfrak{d}}$ and $\mathbb{Z}$ are periodic 2. It is easy to see that $h(X_{t+1}) \leq h(X_t) + 1 \leq |Z_t| - 1 \leq |Z_{t+1}|$, and hence we obtain $h(X_t) \leq |Z_t|$.

Suppose that $h(X_t) = |Z_t|$. We consider three cases: (i) $h(X_t) = |Z_t| = 0$, (ii) X_t is internal nodes and (iii) X_t is a leaf. In case (i), since

$$\Pr\left[h(X_{t+1}) = h(X_t) + 1\right] = \Pr\left[|Z_{t+1}| = |Z_t| + 1\right] = 1$$

hold, we get a coupling of $\boldsymbol{X}$ and $\boldsymbol{Z}$ such that $h(X_{t+1}) = |Z_{t+1}|$. In case (ii), since

$$\Pr\left[h(X_{t+1}) = h(X_t) + 1\right] = \Pr\left[|Z_{t+1}| = |Z_t| + 1\right] = \frac{1}{2}$$
$$\Pr\left[h(X_{t+1}) = h(X_t) - 1\right] = \Pr\left[|Z_{t+1}| = |Z_t| - 1\right] = \frac{1}{2}$$

hold, we construct a coupling of $\boldsymbol{X}$ and $\boldsymbol{Z}$ such that $h(X_{t+1}) = |Z_{t+1}|$. In case (iii), since

$$\Pr\left[h(X_{t+1}) = h(X_t) - 1\right] = 1$$
$$\Pr\left[|Z_{t+1}| = |Z_t| + 1\right] = \frac{1}{2}$$
$$\Pr\left[|Z_{t+1}| = |Z_t| - 1\right] = \frac{1}{2}$$

hold, we give a coupling of $\boldsymbol{X}$ and $\boldsymbol{Z}$ such that $h(X_{t+1}) \leq |Z_{t+1}|$.

Now we obtain a coupling of $\boldsymbol{X}$ and $\boldsymbol{Z}$ satisfying $h(X_t) \leq |Z_t|$ for any $t \geq 0$. This means that $\Pr[X_t = r] \geq \Pr[Z_t = 0]$ for any $t \geq 0$. Thus, we obtain the claim. $\square$

Proof of Theorem 9.1 . Since a random walk on $\mathbb{Z}$ is recurrent, we have

$$\sum_{t=0}^{\infty} \Pr[X_t = r] \geq \sum_{t=0}^{\infty} \Pr[Z_t = 0] = \infty$$

by Lemma 9.2. Thus, we obtain the claim. □

Next, we consider the recurrence and transience at v_1.

Theorem 9.3. *Let $\mathring{\pi}_n(1)$ be an even stationary distribution at v_1 in C_n. If $\mathfrak{d}$ satisfies*

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) \mathring{\pi}_k(1) = \infty$$

then v_1 is recurrent, otherwise v_1 is transient.

To prove Theorem 9.3, we must prove the following lemmas.

Lemma 9.4. *A simple random walk on a growing spider tree according to $(\cdot, C, P)$ satisfies $\mathring{\pi}_n(1) = \frac{(k-1)^2}{k^n\{n(k-1)-1\}+1}$. This implies that $C_1 \mathring{\pi}_n(1) \leq \mathring{\pi}_{n-1}(1) \leq C_2 \mathring{\pi}_n(1)$ for $n = 1, 2, \ldots$, where C_1 and C_2 are positive constants to n.*

Proof. By Corollary 2.22, we get

$$\mathring{\pi}_n(1) = \frac{\deg(v_1)}{|E_n|} = \frac{1}{\sum_{i=1}^{n} i \cdot k^{i-1}} = \frac{(k-1)^2}{k^n \{n(k-1)-1\} + 1}$$

for $n = 1, 2, \ldots$ and $k = 2, 3, \ldots$. □

Lemma 9.5. *A simple random walk on a growing spider tree according to $(\cdot, C, P)$ satisfies $\sum_{n=1}^{\infty} \mathring{\pi}_n(1) \mathring{\mathfrak{t}}(n) < \infty$.*

To prove Lemma 9.5, we must prove Lemma 9.6.

Lemma 9.6. *Let $\mathring{\tau}_{couple}^{G_n}$ be an even coupling time of a simple random walk on G_n. Then,*

$$E_{x,y}[\mathring{\tau}_{couple}^{G_n}] \leq n^2 + 1$$

holds for any $n \geq 1$.

Proof. Let X_t and Y_t respectively denote random walks on G_n, where $X_0 = x_0$, $Y_0 = y_0$ and $h(x_0) \equiv h(y_0) \pmod 2$. Suppose that $h(x_0) \leq h(y_0)$. By the definition of $\mathring{\tau}_{couple}^{G_n}$, we can regard as $\mathring{\tau}_{couple}^{G_n} = \min\{2t \,;\, X_{2t} = Y_{2t}\}$. We consider the following coupling of $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$: (i) If X and Y are the different heights, two particles perform respectively the independent transitions until the two particles are the same heights. (ii) If X and Y are the same height vertex, X and Y perform the same directions. We repeat (i) and (ii). Then, we can construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that

$$h(X_t) \leq h(Y_t),\ X_{\tau_r} = Y_{\tau_r} = r$$

hold for any $t \leq \tau_r$, where $\tau_r := \min\{t \,;\, Y_t = r\}$. Suppose τ_r is an even number. Then, we get $\mathring{\tau}_{couple}^{G_n} \leq \tau_r$. Suppose τ_r is an even number. Then, we give $\mathring{\tau}_{couple}^{G_n} \leq \tau_r + 1$. Therefore, we obtain

$$\begin{aligned} E_{x,y}[\mathring{\tau}_{couple}^{G_n}] &\leq E_{y_0}[\tau_r] + 1 \\ &\leq E_{v_n^1}[\tau_r] + 1 = n^2 + 1 \end{aligned}$$

for $n = 1, 2, \ldots$. Thus, we obtain the claim. □

Proof of Lemma 9.5. By Corollary 2.16, Corollary 2.21 and Lemma 9.4, we obtain

$$\begin{aligned}\mathring{\mathfrak{t}}(n) &\leq \mathrm{O}(n^2 \log_2(\mathring{\pi}_n(1)^{-1})) \\ &= \mathrm{O}(n^2 \log_2(k^n)) = \mathrm{O}(n^3 \log_2(k)).\end{aligned}$$

Therefore, we provide

$$\begin{aligned}\sum_{n=1}^{\infty} \mathring{\pi}_n(1)\mathring{\mathfrak{t}}(n) &\leq \sum_{n=1}^{\infty} C_1 \frac{n^3}{nk^n} \\ &\leq C_1 \sum_{n=1}^{\infty} \frac{n^2}{k^n} < \infty,\end{aligned}$$

where C_1 is a positive constant to n. Thus, we obtain the claim. □

Lemma 9.7. *A simple random walk on a growing spider tree is weakly LHaGG.*

To prove Lemma 9.7, we must prove the following lemmas.

Lemma 9.8. *Let X_t ($t = 0, 1, 2, \dots$) be a simple random walk on a growing spider tree according to $\mathcal{D}_f = (f, C, P)$, where $X_0 = v_1$, and let $R_f(t)$ ($t = 1, 2, \dots$) denote the return probability of X_t. Similarly, let Y_t ($t = 0, 1, 2, \dots$) be a simple random walk on a growing spider tree according to $\mathcal{D}_g = (g, C, P)$, where $Y_0 = v_1$, and let $R_g(t)$ ($t = 1, 2, \dots$) denote the return probability of Y_t. Let $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$, for convenience. Suppose that*

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{97}$$

for $n = 1, 2, \dots$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy

$$X_t = v_1, Y_{t'} = v_1$$

for $t \leq t'$ and $\min\{\phi\} = \infty$. There is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that

$$\min\{s\ ;\ s > 0,\ X_{t+s} = v_1\} \leq \min\{s\ ;\ s > 0,\ Y_{t'+s} = v_1\}, \tag{98}$$

i.e., X returns to the origin vertex r in a fewer steps than Y.

Proof. For convenience, let $n_0^f := 0$ and $n_t^f := n - 1$ for $t \in (T_{n-1}^f, T_n^f]$. Let $\deg_t^f(v)$ be a degree of the vertex v at time t in $\mathcal{D}_f$. Equation (97) means that the leaf in $\mathcal{D}_g$ is higher than $\mathcal{D}_f$ for any time. We write the coupling as follows: (i) If Y locates the same height vertex at X except of root r and leaf, X and Y perform the same directions. (ii) If X transits from root r to the adjacent vertex, Y transits from root r to the same or another adjacent vertex which is deeper than a leaf of X. (iii) If X and Y are the same heights and X only locates leaf, X transits from leaf node to parent node and pause until Y returns to the same height as X. In case (i), since

$$\begin{aligned}\Pr\left[X_{t_1+1} = v' \middle| X_{t_1} = v\right] &= \frac{1}{\deg_t^f(v)}, \\ \Pr\left[Y_{t_2+1} = v''' \middle| Y_{t_2} = v''\right] &= \frac{1}{\deg_t^g(v'')} = \frac{1}{\deg_t^f(v)} \qquad (v \text{ and } v'' \text{ are internal nodes and not root } r)\end{aligned}$$

hold for any $t_1 \le t_2$, $h(v) = h(v'')$ and $h(v') = h(v''')$, we have

$$\Pr\left[X_{t_1+1} = v' \middle| X_{t_1} = v\right] = \Pr\left[Y_{t_2+1} = v''' \middle| Y_{t_2} = v''\right]$$

for any $t_1 \le t_2$. This implies that we get the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $h(X_{t_1+1}) = h(Y_{t_2+1})$.

In case (ii), X satisfies

$$\Pr\left[X_{t_1+1} = v | X_{t_1} = r\right] = \frac{1}{\deg^f_{t_1}(r)}$$

for any $t_1 \ge 0$ and $v \in H_1$. On the other hands, Y satisfies

$$\Pr\left[Y_{t_2+1} = v | Y_{t_2} = r\right] + \frac{1}{\deg^f_{t_1}(r)} \Pr\left[Y_{t_2+1} = v' \in V_{n^g_{t_2}} \setminus V_{n^f_{t_1}} \middle| Y_{t_2} = r\right] = \frac{1}{\deg^f_{t_1}(r)}$$

for any $t_1 \le t_2$ and $v \in H_1$. This implies that we provide the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that (ii) holds.

In case (iii), X satisfies

$$\Pr[X_{t_1+1} = v' \mid X_{t_1} = v] = 1$$

for any $v \in L^f_{t_1}$ and $\{v, v'\} \in E_{n^f_t}$. We consider the transition probability of Y from v to v'. Let $\tau^Y_{v,v'} := \min\{t' \; ; \; Y_{t_2+t'} = v'\}$. By the definition of $\tau^Y_{v,v'}$, we provide $\sum_{s \in \mathbb{N} \cup \{\infty\}} \Pr\left[\tau^Y_{v,v'} = s \middle| Y_{t_2} = v\right] = 1$ for any $v \in L^f_{t_1}$ and $\{v, v'\} \in E_{n^f_t}$. This implies that we construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that (iii) holds.

We repeat this coupling until Y returns to the vertex v_1. Furthermore, Y cannot go to the root v_1 without going to v' via v. Meaning that Y does not go to the root v_1 among case (iii) at all. Two particles of the cost of time in case (i) and (ii) are the same but in case (iii), the time of Y is longer than or equal to X. Thus, we obtain the claim. □

Lemma 9.9. *Let* $Z_t(\boldsymbol{X}) := \sum_{s=0}^{t} 1_{\{X_s = v_1\}}$ *and* $Z_t(\boldsymbol{Y}) := \sum_{s=0}^{t} 1_{\{Y_s = v_1\}}$. *Let* $\tau^X_{v_1}(n) := \min\{t \; ; \; Z_t(\boldsymbol{X}) = n\}$ *and* $\tau^Y_{v_1}(n) := \min\{t \; ; \; Z_t(\boldsymbol{Y}) = n\}$. *There is a coupling of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *such that* $\tau^X_{v_1}(n) \le \tau^Y_{v_1}(n)$ *for* $n = 0, 1, \ldots$.

Proof. The proof is by induction on n. (1) For $n = 0$. It is clear that $\tau^X_{v_1}(0) \le \tau^Y_{v_1}(0)$ since $\tau^X_{v_1}(0) = 0$ and $\tau^Y_{v_1}(0) = 0$, and hence Lemma 9.9 follows for $n = 0$.

(2) Assuming Lemma 9.9 to hold for n, we will prove it for $n + 1$. If $\tau^Y_{v_1}(n+1) = \infty$ we have $\tau^X_{v_1}(n+1) \le \tau^Y_{v_1}(n+1)$ clearly, and we get the claim in this case. Then, we consider $\tau^Y_{v_1}(n+1) < \infty$. By assumption, $\tau^X_{v_1}(n) \le \tau^Y_{v_1}(n)$ holds, and hence we give

$$\min\left\{s \; ; \; s > 0, X_{\tau^X_{v_1}(n)+s}\right\} \le \min\left\{s \; ; \; s > 0, Y_{\tau^Y_{v_1}(n)+s}\right\}$$

by Lemma 9.8. This implies that $\tau^X_{v_1}(n+1) \le \tau^Y_{v_1}(n+1)$. Hence the lemma follows for $n+1$. Therefore, we obtain the claim. □

Proof of Lemma 9.7. Using the same proof as Lemma 8.3, we obtain

$$\sum_{t=1}^{T} R_f(t) \ge \sum_{t=1}^{T} R_g(t)$$

for $T = 1, 2, \ldots$, and the proof is complete. □

Proof of Theorem 9.3. Since Lemma 9.5 holds, by Theorem 7.1, we can prove that v_1 is recurrent by $\mathcal{D}_{\mathfrak{d}}$ if

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) \mathring{\pi}_k(1) = \infty$$

holds. Using Theorem 7.4, Lemma 9.4 and Lemma 9.7, we can prove that v_1 is transient in $\mathcal{D}_{\mathfrak{d}}$ if

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) \mathring{\pi}_k(1) < \infty$$

holds. Thus, we obtain the claim. □

9.2 Random walk on a growing random tree

This section is concerned with a random walk on a growing random tree. This tree is a generalization of a complete k- ary tree. Let $G_n = (V_n, E_n)$ be a tree which has the height n, where $V_1 = \{r, v_1, \ldots, v_N\}$ and $E_1 = \bigcup_{i=1}^{N} \{\{r, v_i\}\}$. For convenience, let $h(v)$ denote the height at $v \in V_n$, i.e., $h(r) = 0$, and $h(v) = n$ if and only if v is a leaf of G_n. Let $H_n := \{v \in G_n \ ; \ h(v) = n\}$. And let $w : V \to \{1, \ldots, N'\}$ be an uniformly chosen on $\{1, \ldots, N'\}$. Using V_n, E_n and w, we inductively construct $G_{n+1} = (V_{n+1}, E_{n+1})$ such that

$$V_{n+1} := V_n \cup \tilde{V}_n$$
$$E_{n+1} := E_n \cup \tilde{E}_n$$

for $n \geq 1$, where

$$\tilde{V}_n := \bigcup_{i \in H_n} \bigcup_{j=1}^{w(i)} \{i_j\}$$
$$\tilde{E}_n := \bigcup_{i \in H_n} \bigcup_{j=1}^{w(i)} \{\{i, i_j\}\}$$

for $n \geq 1$. To consider the recurrence of a simple random walk on a growing random tree $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ (see Figure 7), we give Theorem 9.10.

Theorem 9.10. *If $\mathcal{D}_{\mathfrak{d}}$ satisfies*

$$\lim_{n \to \infty} \mathring{p}(n) \sum_{k=1}^{n} \mathfrak{d}(k) = \infty$$

then the origin vertex o is recurrent.

To prove Theorem 9.10, we must prove the following lemmas.

Lemma 9.11. *Suppose RWoGG $(\cdot, G, P)$ is weakly LHaGG with irreducible, reversible and period 2. Then, $\mathcal{D}_{\mathfrak{d}}$ satisfies*

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq C_1 \mathring{p}(n) \sum_{k=1}^{n} \mathfrak{d}(k)$$

for any $n = 1, 2, \ldots$, where C_1 is a positive constant with respect to n.

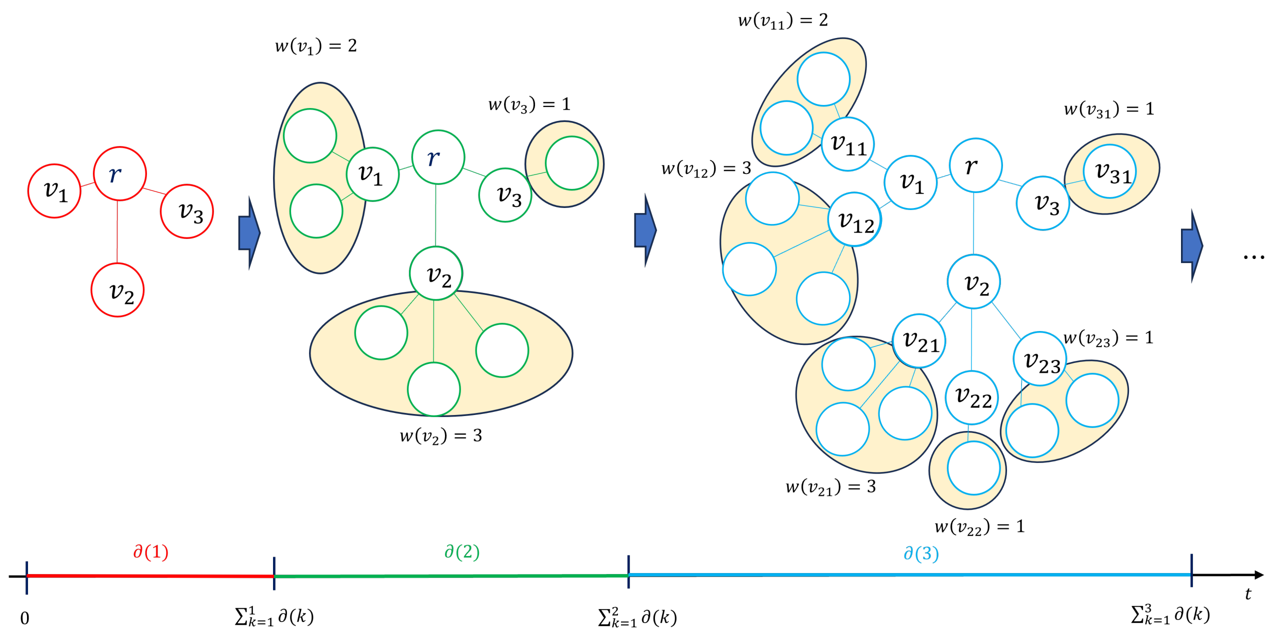

Figure 7: growing random tree (N=3, N'=3)

Proof. Let

$$\mathfrak{d}_n(i) := \begin{cases} 0 & (i < n), \\ \sum_{j=1}^{n} \mathfrak{d}(j) & (i = n), \\ \mathfrak{d}(i) & (i > n). \end{cases}$$

Obviously, $T_{n'}^{\mathfrak{d}} \geq T_{n'}^{\mathfrak{d}_n}$ holds for $n = 1, 2, \ldots$ and $n' = 1, 2, \ldots$, and hence the weakly LHaGG assumption implies

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq \sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}_n}(t)$$

for any $n \geq 1$. Since $\mathcal{D}_{\mathfrak{d}_n}$ regards as a random walk on a static graph $G(n)$ at the duration from 1 to $T_n^{\mathfrak{d}_n}$, we provide $R_{\mathfrak{d}_n}(t) \geq \mathring{p}(n)$ by Lemma 2.4. Meaning that

$$\sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}_n}(t) = \sum_{t=1}^{\lfloor \frac{T_n^{\mathfrak{d}}}{2} \rfloor} R_{\mathfrak{d}_n}(2t) \geq \sum_{t=1}^{\lfloor \frac{T_n^{\mathfrak{d}}}{2} \rfloor} \mathring{p}(n) = \mathring{p}(n) \sum_{t=1}^{\lfloor \frac{T_n^{\mathfrak{d}}}{2} \rfloor} 1 = \mathring{p}(n) \lfloor \frac{T_n^{\mathfrak{d}}}{2} \rfloor \geq C_1 \mathring{p}(n) \sum_{i=1}^{n} \mathfrak{d}(i)$$

hold for $n = 1, 2, \ldots$, where C_1 is a positive constant with respected to n. Thus, we obtain the claim. □

Lemma 9.12. *A random walk on a growing random tree $(\cdot, G, P)$ is weakly LHaGG.*

To prove Lemma 9.12, we must prove the following lemmas.

Lemma 9.13. *Let X_t $(t = 0, 1, 2, \ldots)$ be a random walk on a growing random tree according to $\mathcal{D}_f = (f, G, P)$, and let $R_f(t)$ $(t = 1, 2, \ldots)$ denote the return probability of X_t. Similarly, let Y_t $(t = 0, 1, 2, \ldots)$*

be a random walk on a growing random tree according to $\mathcal{D}_g = (g, G, P)$*, and let* $R_g(t)$ $(t = 1, 2, \ldots)$ *denote the return probability of* Y_t*. Let* $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ *and* $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$*. Suppose that*

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k) \tag{99}$$

for $n = 1, 2, \ldots$*,* $\min\{\phi\} = \infty$ *and*

$$X_t = r\; Y_{t'} = r,$$

where $t \leq t'$*. There is a coupling of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *such that*

$$\min\{s\; ;\; s > 0,\; X_{t+s} = r\} \leq \min\{s\; ;\; s > 0,\; Y_{t'+s} = r\}, \tag{100}$$

i.e., X *returns to the origin vertex* r *in a fewer steps than* Y*.*

Proof. For convenience, let $n_0^f := 0$ and $n_t^f := n - 1$ for $t \in (T_{n-1}^f, T_n^f]$. Let $\deg_t^f(v)$ be a degree of the vertex v at time t on $G_{n_t^f}$. Equation (99) means that the leaf in $\mathcal{D}_g$ is deeper than $\mathcal{D}_f$ for any time. We write the coupling as follows: (i) If Y locates no leaf vertex, X and Y perform the same transitions. (ii) If X and Y are same heights and X only locates leaf, X transits from leaf node to parent node and pause until Y returns to the same vertex as X.

In case (i), since

$$\Pr[X_{t_1+1} = v_2 \mid X_{t_1} = v_1] = \frac{1}{\deg_{t_1}^f(v_1)} = \frac{1}{w(v_1)+1}$$

$$\Pr[Y_{t_2+1} = v_2 \mid Y_{t_2} = v_1] = \frac{1}{\deg_{t_2}^g(v_1)} = \frac{1}{w(v_1)+1} \qquad (v_1 \text{ is a internal node.})$$

hold for any $v_1 \in V_{n_{t_1}^f}$ and $\{v_1, v_2\} \in E_{n_{t_1}^f}$, where $t_1 \leq t_2$, we have

$$\Pr[X_{t_1+1} = v_2 \mid X_{t_1} = v_1] = \Pr[Y_{t_2+1} = v_2 \mid Y_{t_2} = v_1]$$

for any $t_1 \leq t_2$. This means that there is a coupling of X and Y such that $X_{t_1+1} = Y_{t_2+1} = v_2$.

In case (ii), X satisfies

$$\Pr[X_{t_1+1} = v' \in H_{n_{t_1}^f - 1} \mid X_{t_1} = v \in H_{n_{t_1}^f}] = 1$$

for any $v \in H_{n_{t_1}^f}$. Consider the transition probability of Y from v to v'. Let $\tau_{v,v'}^Y := \min\{t'\; ;\; Y_{t_2+t'} = v'\}$. By the definition of $\tau_{v,v'}^Y$, we get $\sum_{s\in\mathbb{N}\cup\{\infty\}} \Pr\left[\tau_{v,v'}^Y = s \middle| Y_{t_2} = v\right] = 1$.

We repeat this coupling by visiting the root r. Furthermore, Y cannot go to the root r without going to v' via v. Meaning that Y does not go to the root r among case (ii) at all. Two particles of the cost of time in case (i) are the same but in case (ii), the time of Y is longer than or equal to X. Thus, we obtain the claim. □

Proof of Lemma 9.12. Performing the similar proof as Lemma 9.9, we can construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $\tau_r^X(n) \leq \tau_r^Y(n)$ for $n = 0, 1, 2, \ldots$, where $\tau_r^X(n) := \min\left\{t\; ;\; \sum_{s=0}^{t} 1_{\{X_s=r\}} = n\right\}$ and $\tau_r^Y(n) := \min\left\{t\; ;\; \sum_{s=0}^{t} 1_{\{Y_s=r\}} = n\right\}$. Using the same proof as Lemma 9.7, we obtain

$$\sum_{t=1}^{T} R_f(t) \geq \sum_{t=1}^{T} R_g(t)$$

for $T = 1, 2, \ldots$, and the proof is complete. □

Proof of Theorem 9.10. By Lemma 9.11 and 9.12, we provide

$$\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) = \lim_{n\to\infty} \sum_{t=1}^{T_n^{\mathfrak{d}}} R_{\mathfrak{d}}(t) \geq C_1 \lim_{n\to\infty} \mathring{p}(n) \sum_{i=1}^{n} \mathfrak{d}(i).$$

Therefore, we obtain the claim. □

We cannot apply the weakly LHaGG theorems (Theorem 7.1 and 7.4) to a growing random tree since a random walk on a static random tree cannot be regarded as other graphs satisfying (60) and (72). For this reason, we obtain only the sufficient condition for the recurrence.

10 Random walk on a growing added box

10.1 Definition and main result

This section is concerned with a random walk on a growing added box. Let $G_n = (V_n, E_n)$ be a graph given by

$$V_n = V_{n-1} \bigcup_{x \in V_{n-1} \setminus \{N\}} \left\{ x + \frac{1}{2^{n-1}} \right\}$$

$$E_n = \left\{ \{x, y\} \ ; \ x, y \in V_n, \ ||x - y||_1 = \frac{1}{2^{n-1}} \right\}$$

where n and N are (fixed) positive integers and $V_0 = \{0\}$. Let $G'_n = (V'_n, E'_n)$ denote

$$G'_n = G_n \times G_n \ldots \times G_n. \qquad (n_0 \text{ times})$$

Let $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ be a random walk on a growing added box, where $G(n) = G'_n$. Let o denote the origin. Let P_n for $n \geq 1$ denote a transition probability of a random walk on a static graph G'_n, where

$$P_n(x, y) = \begin{cases} \frac{1}{2} & (\text{if } x = y) \\ \frac{1}{4n_0} & (\text{if } \{x, y\} \in E'_n, \ x_k \neq y_k \text{ and } x_k \notin \{0, N\}) \\ \frac{1}{2n_0} & (\text{if } \{x, y\} \in E'_n, \ x_k \neq y_k \text{ and } x_k \in \{0, N\}) \\ 0 & (\text{otherwise}) \end{cases}$$

for $x, y \in V_n$. Then, we are concerned with a RWoGG X_t according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ (see Figure 8).

Theorem 10.1. *Suppose that $n_0 \geq 3$. If $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies*

$$\sum_{k=1}^{\infty} \frac{\mathfrak{d}(k)}{(2^k N)^{n_0}} = \infty,$$

o is recurrent, otherwise o is transient.

10.2 Proof of Theorem 10.1

We can immediately prove the sufficient condition for the recurrent by using Theorem 7.3. Therefore, we need to give a new technique for getting the transience. To prove the transience, we must prove the following lemmas.

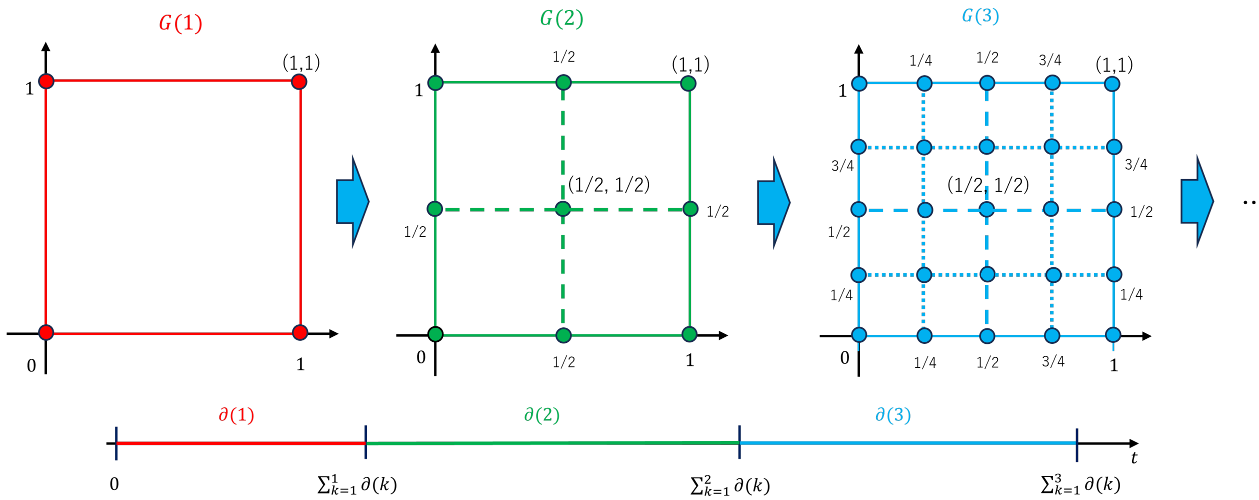

Figure 8: growing added box (N=1)

Lemma 10.2. *G'_n satisfies*

$$p(n) = \frac{1}{(2^n N)^{n_0}}$$
$$\mathfrak{t}(n) \leq 8N^2 n_0^3 (2^{n-1})^2 \{n + \log_2 N\}.$$

Therefore, $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ satisfies (67) *and* (95).

Proof. By Lemma 5.3 and 5.4, we obtain the claim. □

Lemma 10.3. *Let Y_t be a random walk on a growing box $\mathcal{D}''_{\mathfrak{d}} = (\mathfrak{d}, G'', P'')$, where $G''(n) = \left\{0, \ldots, N2^{n-1}\right\}^{n_0}$. Let $R''_{\mathfrak{d}}(t)$ denote the return probability of Y_t. Then, we obtain $R_{\mathfrak{d}}(t) \leq R''_{\mathfrak{d}}(t)$ for any $t \geq 1$.*

Proof. For convenience, let $n_0^f := 0$ and $n_t^f := n-1$ for $t \in (T_{n-1}^f, T_n^f]$. Let $D(X_t^i) := \min\left\{s \,;\, X_{t+s}^i = 0\right\}$ and $D(Y_t^i) := \min\left\{s \,;\, Y_{t+s}^i = 0\right\}$. We construct a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $D(X_t^i) \geq D(Y_t^i)$ holds for any $1 \leq i \leq n_0$ and $t \geq 1$. The proof is an induction concerning t. Let $I : \mathbb{N} \to \{1, \ldots, n_0\}$ and $J : \mathbb{N} \to \{1, \ldots, n_0\}$ respectively denote the random variables which are selected by changing from X_{t-1} to X_t and from Y_{t-1} to Y_t.

Clearly, $D(X_0^i) = D(Y_0^i) = 0$ for any $1 \leq i \leq n_0$. Inductively assuming $D(X_t^i) \geq D(Y_t^i)$, we prove $D(X_{t+1}^i) \geq D(Y_{t+1}^i)$. We consider four cases: (i) $D(X_t^i) > D(Y_t^i)$, (ii) $D(X_t^i) = D(Y_t^i) = 0$, (iii) $D(X_t^i) = D(Y_t^i) = N2^{n_t^g}$ and (iv) $D(X_t^i) = D(Y_t^i)$ and $D(X_t^i), D(Y_t^i) \notin \left\{0, N2^{n_t^g}\right\}$. In case (i), since

$$\Pr\left[|D(X_{t+1}^i) - D(X_t^i)| = 1 \mid I_{t+1} = i\right] = \Pr\left[|D(Y_{t+1}^i) - D(Y_t^i)| = 0 \mid J_{t+1} = i\right] = \frac{1}{2}$$
$$\Pr\left[|D(X_{t+1}^i) - D(X_t^i)| = 0 \mid I_{t+1} = i\right] = \Pr\left[|D(Y_{t+1}^i) - D(Y_t^i)| = 1 \mid J_{t+1} = i\right] = \frac{1}{2}$$

hold, it follows that there exists a two cases coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that : (a) $|D(X_{t+1}^i) - D(X_t^i)| = 1$ as long as $D(Y_{t+1}^i) = D(Y_t^i)$. (b) $D(X_{t+1}^i) = D(X_t^i)$ as long as $|D(Y_{t+1}^i) - D(Y_t^i)| = 1$. Recall that $D(X_t^i) > D(Y_t^i)$ implies $D(X_t^i) - 1 \geq D(Y_t^i)$ and $D(X_t^i) \geq D(Y_t^i) + 1$. This means that if (a) then $D(X_{t+1}^i) \geq D(X_t^i) - 1 \geq D(Y_t^i) = D(Y_{t+1}^i)$ and if (b) then $D(X_{t+1}^i) = D(X_t^i) \geq D(Y_t^i) + 1 \geq D(Y_{t+1}^i)$, and hence we construct the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_{t+1}^i) \geq D(Y_{t+1}^i)$.

In case (ii), since $D(X_t^i) = D(Y_t^i) = 0$,

$$\Pr\left[D(X_{t+1}^i) = D(X_t^i) + 1 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) = D(Y_t^i) + 1 \mid J_{t+1} = i\right] = \frac{1}{2}$$
$$\Pr\left[D(X_{t+1}^i) - D(X_t^i) = 0 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) - D(Y_t^i) = 0 \mid J_{t+1} = i\right] = \frac{1}{2}$$

hold, and hence we provide the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_{t+1}^i) = D(Y_{t+1}^i)$.

In case (iii), since $D(X_t^i) = D(Y_t^i) = N2^{n_t^g}$,

$$\Pr\left[D(X_{t+1}^i) = D(X_t^i) - 1 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) = D(Y_t^i) - 1 \mid J_{t+1} = i\right] = \frac{1}{2}$$
$$\Pr\left[D(X_{t+1}^i) - D(X_t^i) = 0 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) - D(Y_t^i) = 0 \mid J_{t+1} = i\right] = \frac{1}{2}$$

hold, and hence we get the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_{t+1}^i) = D(Y_{t+1}^i)$.

In case (iv), since $D(X_t^i) = D(Y_t^i)$ and $D(X_t^i), D(Y_t^i) \notin \left\{0, N2^{n_t^g}\right\}$,

$$\Pr\left[D(X_{t+1}^i) = D(X_t^i) + 1 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) = D(Y_t^i) + 1 \mid J_{t+1} = i\right] = \frac{1}{4}$$
$$\Pr\left[D(X_{t+1}^i) = D(X_t^i) - 1 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) = D(Y_t^i) - 1 \mid J_{t+1} = i\right] = \frac{1}{4}$$
$$\Pr\left[D(X_{t+1}^i) - D(X_t^i) = 0 \mid I_{t+1} = i\right] = \Pr\left[D(Y_{t+1}^i) - D(Y_t^i) = 0 \mid J_{t+1} = i\right] = \frac{1}{2}$$

hold, and hence we give the coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_{t+1}^i) = D(Y_{t+1}^i)$.

Now we obtain a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_t^i) \geq D(Y_t^i)$ for any $t \geq 0$ and $1 \leq i \leq n_0$, which implies that $D(Y_t^i) = 0$ as long as $D(X_t^i) = 0$. Meaning that $R_{\mathfrak{d}}(t) \leq R''_{\mathfrak{d}}(t)$ for any $t \geq 1$, and hence the lemma follows □

Proof of Theorem 10.1 . By Theorem 7.3 and Lemma 10.2, o is recurrent by $\mathcal{D}_{\mathfrak{d}}$ if $\mathfrak{d}$ satisfies

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) = \infty.$$

Utilizing Theorem 7.8, Lemma 10.2 and Lemma 5.2, if $\mathfrak{d}$ satisfies

$$\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) < \infty \tag{101}$$

then $\sum_{t=1}^{\infty} R''_{\mathfrak{d}}(t) < \infty$. This means that if (101) holds, we obtain

$$\sum_{t=1}^{\infty} R_{\mathfrak{d}}(t) < \sum_{t=1}^{\infty} R''_{\mathfrak{d}}(t) < \infty$$

by Lemma 10.3. Thus, we obtain the claim. □

11 The threshold of the null recurrence and positive recurrence of a random walk on a growing complete graph

This section focuses on the null recurrence and positive recurrence of a random walk on a growing complete graph. Let $G_n = (V_n, E_n)$ be a graph given by

$$\begin{aligned} V_n &:= \{1, \ldots, N+n\} \\ E_n &:= \{\{x, y\} \ ; \ x, y \in V_n\}, \end{aligned}$$

where $N \geq 2$. We define the transition probability on G_n by

$$P_n(x, y) = \begin{cases} \frac{1}{N+n} & \text{if } x, y \in V_n \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in V_n$. We consider a random walk on a growing complete graph $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$ (see Figure 9), where $o = 1 \in V_n$.

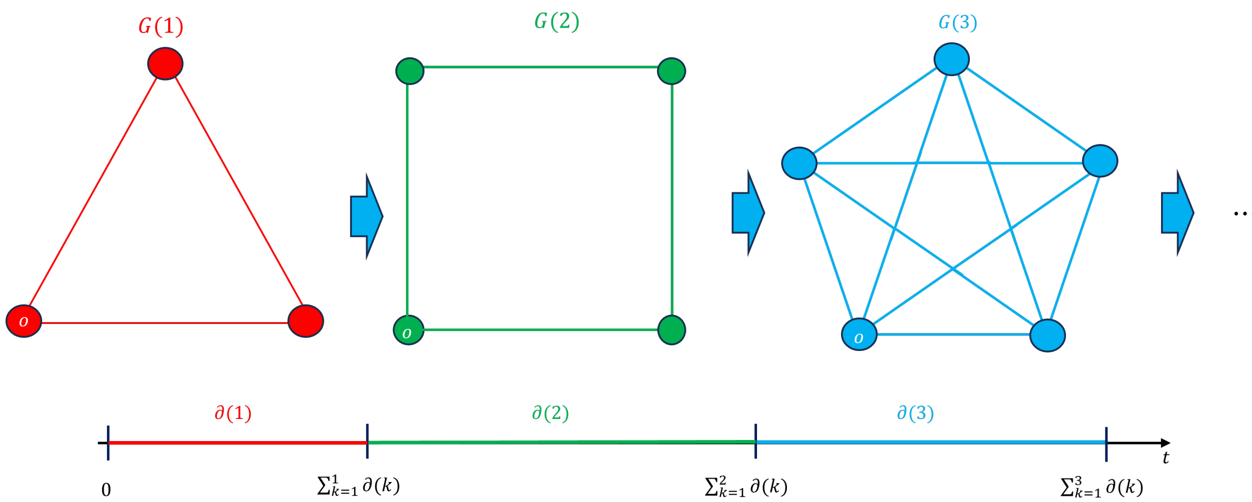

Figure 9: growing complete graph (N=2)

Theorem 11.1. *If $\mathfrak{d}(n) = 1$ for $n = 1, 2, \ldots$ then o is null recurrent by $\mathcal{D}_{\mathfrak{d}}$. If $\mathfrak{d}(n) > 1$ for $n = 1, 2, \ldots$ then o is positive recurrent by $\mathcal{D}_{\mathfrak{d}}$.*

To prove the theorem, we must prove the following lemmas.

Lemma 11.2. *Let X_t be a random walk on a growing complete graph according to $\mathcal{D}_f = (f, G, P)$. Let Y_t be a random walk on a growing complete graph according to $\mathcal{D}_g = (g, G, P)$. Let $D(X_t^i)$ and $D(Y_t^i)$ respectively denote the distances from o to X_t and o to Y_t. Suppose that*

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k)$$

for $n = 1, 2, \ldots$. We can construct the coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $D(X_t^i) \leq D(Y_t^i)$ for any $t \geq 1$.

Proof. For convenience, let $n_0^f := 0$ and $n_t^f := n-1$ for $t \in (T_{n-1}^f, T_n^f]$. The proof is an induction concerning t. Clearly $D(X_0^i) = D(Y_0^i) = 0$. Inductively assuming $D(X_t^i) \le D(Y_t^i)$, we prove $D(X_{t+1}^i) \le D(Y_{t+1}^i)$. Since $\sum_{k=1}^n f(k) \ge \sum_{k=1}^n g(k)$, we obtain $n_t^f \le n_t^g$ for any $t \ge 0$. Suppose that $D(X_t^i) < D(Y_t^i)$ then $D(X_t^i) = 0$ and $D(Y_t^i) = 1$. Since

$$\begin{aligned}
\Pr\left[D(X_{t+1}^i) = 0 \mid D(X_t^i) = 0\right] &= \frac{1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 0 \mid D(Y_t^i) = 1\right] &= \frac{1}{N + n_t^g} \\
\Pr\left[D(X_{t+1}^i) = 1 \mid D(X_t^i) = 0\right] &= \frac{N + n_t^f - 1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 1 \mid D(Y_t^i) = 1\right] &= \frac{N + n_t^g - 1}{N + n_t^g},
\end{aligned}$$

we get

$$\begin{aligned}
\Pr\left[D(X_{t+1}^i) = 0 \mid D(X_t^i) = 0\right] &\ge \Pr\left[D(Y_{t+1}^i) = 0 \mid D(Y_t^i) = 1\right] \\
\Pr\left[D(X_{t+1}^i) = 1 \mid D(X_t^i) = 0\right] &\le \Pr\left[D(Y_{t+1}^i) = 1 \mid D(Y_t^i) = 1\right].
\end{aligned}$$

This mean that there is a coupling of $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $D(X_{t+1}^i) \le D(Y_{t+1}^i)$.

Suppose that $D(X_t^i) = D(Y_t^i)$. We consider two cases: (i) $D(X_t^i) = D(Y_t^i) = 0$ and (ii) $D(X_t^i) = D(Y_t^i) = 1$. In case (i), since

$$\begin{aligned}
\Pr\left[D(X_{t+1}^i) = 0 \mid D(X_t^i) = 0\right] &= \frac{1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 0 \mid D(Y_t^i) = 0\right] &= \frac{1}{N + n_t^g} \\
\Pr\left[D(X_{t+1}^i) = 1 \mid D(X_t^i) = 0\right] &= \frac{N + n_t^f - 1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 1 \mid D(Y_t^i) = 0\right] &= \frac{N + n_t^g - 1}{N + n_t^g}
\end{aligned}$$

hold, we give

$$\begin{aligned}
\Pr\left[D(X_{t+1}^i) = 0 \mid D(X_t^i) = 0\right] &\ge \Pr\left[D(Y_{t+1}^i) = 0 \mid D(Y_t^i) = 0\right] \\
\Pr\left[D(X_{t+1}^i) = 1 \mid D(X_t^i) = 0\right] &\le \Pr\left[D(Y_{t+1}^i) = 1 \mid D(Y_t^i) = 0\right].
\end{aligned}$$

Therefore, we obtain $D(X_{t+1}^i) \le D(Y_{t+1}^i)$.

In case (ii), since

$$\begin{aligned}
\Pr\left[D(X_{t+1}^i) = 0 \mid D(X_t^i) = 1\right] &= \frac{1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 0 \mid D(Y_t^i) = 1\right] &= \frac{1}{N + n_t^g} \\
\Pr\left[D(X_{t+1}^i) = 1 \mid D(X_t^i) = 1\right] &= \frac{N + n_t^f - 1}{N + n_t^f} \\
\Pr\left[D(Y_{t+1}^i) = 1 \mid D(Y_t^i) = 1\right] &= \frac{N + n_t^g - 1}{N + n_t^g}
\end{aligned}$$

hold, we have

$$\Pr\left[D(X_{t+1}^i)=0 \mid D(X_t^i)=1\right] \geq \Pr\left[D(Y_{t+1}^i)=0 \mid D(Y_t^i)=1\right]$$
$$\Pr\left[D(X_{t+1}^i)=1 \mid D(X_t^i)=1\right] \leq \Pr\left[D(Y_{t+1}^i)=1 \mid D(Y_t^i)=1\right].$$

Therefore, we obtain $D(X_{t+1}^i) \leq D(Y_{t+1}^i)$, and hence the lemma follows. □

By Lemma 11.2, we get the following corollaries.

Corollary 11.3. *Suppose that*

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} g(k)$$

for $n=1,2,\ldots$. *There is a coupling of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *such that*

$$\min\{t>0\,;\ X_t=o\} \leq \min\{t>0\,;\ Y_t=o\}.$$

Corollary 11.4. *A random walk on a growing graph* $(\cdot, G, P)$ *is LHaGG.*

Lemma 11.5. *Let* $F_{\mathfrak{d}}(t)$ *denote the first return probability at time* t *on* $\mathcal{D}_{\mathfrak{d}}$. *Suppose that* $\mathfrak{d}(n)=1$ *for* $n=1,2,\ldots$. *Then,*

$$F_{\mathfrak{d}}(t)=\frac{N-1}{(N+t-2)(N+t-1)}$$

holds for any $t \geq 1$.

Proof. Let X_t be a random walk on a growing graph according to $\mathcal{D}_{\mathfrak{d}}=(\mathfrak{d}, G, P)$, where $\mathfrak{d}(n)=1$ for $n=1,2,\ldots$. By the definition of $F_{\mathfrak{d}}(t)$, we get

$$\begin{aligned}F_{\mathfrak{d}}(t)&=\Pr\left[X_1\neq o,\ldots,X_{t-1}\neq o, X_t=o | X_0=o\right]\\&=\frac{N-1}{N}\cdot\frac{N}{N+1}\cdots\frac{N+t-3}{N+t-2}\cdot\frac{1}{N+t-1}\\&=\frac{N-1}{(N+t-2)(N+t-1)}\end{aligned}$$

for any $t \geq 1$. □

Lemma 11.6. *Suppose that* $\mathfrak{d}(n)=2$ *for* $n=1,2,\ldots$. *Then,*

$$F_{\mathfrak{d}}(t)=\begin{cases}\frac{(N-1)^2}{(N+k-2)(N+k-1)^2} & \textit{(if } t=2k\textit{)}\\ \frac{(N-1)^2}{(N+k-1)^2(N+k)} & \textit{(if } t=2k+1.\textit{)}\end{cases} \tag{102}$$

hold for any $t \geq 1$.

Proof. Let X_t be a random walk on a growing complete graph according to $\mathcal{D}_{\mathfrak{d}}=(\mathfrak{d}, G, P)$, where $\mathfrak{d}(n)=2$ for $n=1,2,\ldots$. By the definition of $F_{\mathfrak{d}}(t)$, we get

$$F_{\mathfrak{d}}(t)=\begin{cases}\left(\prod_{i=1}^{k-1}\left(\frac{N+i-2}{N+i-1}\right)^2\right)\frac{N+k-2}{N+k-1}\frac{1}{N+k-1} & \text{(if } t=2k\text{)}\\ \left(\prod_{i=1}^{k}\left(\frac{N+i-2}{N+i-1}\right)^2\right)\frac{1}{N+k} & \text{(if } t=2k+1\text{)}\end{cases} \tag{103}$$

for any $t \geq 1$. This implies that we obtain

$$F_{\mathfrak{d}}(t) = \begin{cases} \frac{(N-1)^2}{(N+k-2)(N+k-1)^2} & (\text{if } t = 2k) \\ \frac{(N-1)^2}{(N+k-1)^2(N+k)} & (\text{if } t = 2k+1.) \end{cases} \tag{104}$$

□

Lemma 11.7. *Let Z_t denote a random walk on a growing complete graph according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$. Suppose that $\mathfrak{d}(n) = 1$ for $n = 1, 2, \ldots$. Then,*

$$\Pr[Z_t = o] = \frac{1}{N+t-1}$$

holds for any $t \geq 1$. Therefore, the origin vertex o is recurrent by $\mathcal{D}_{\mathfrak{d}}$.

Proof. Since $\Pr[Z_t = o | Z_{t-1} = x] = \frac{1}{N+t-1}$ for any $t \geq 1$ and $x \in G_{n^{\mathfrak{d}}_{t-1}}$, we get

$$\begin{aligned}\Pr[Z_t = o] &= \sum_{x \in V_{n^{\mathfrak{d}}_{t-1}}} \Pr[Z_{t-1} = x | Z_0 = o] \Pr[Z_t = o | Z_{t-1} = x] \\ &= \frac{1}{N+t-1} \sum_{x \in V_{n^{\mathfrak{d}}_{t-1}}} \Pr[Z_{t-1} = x | Z_0 = o] = \frac{1}{N+t-1}\end{aligned}$$

for any $t \geq 1$. Thus, we obtain the claim. □

Proof of Theorem 11.1. Let Z_t denote a random walk on a growing complete graph according to $\mathcal{D}_{\mathfrak{d}} = (\mathfrak{d}, G, P)$, where $\mathfrak{d}(n) \geq 1$ for $n = 1, 2, \ldots$. Let Z'_t denote a random walk on a growing complete graph according to $\mathcal{D}_{\mathfrak{d}'} = (\mathfrak{d}', G, P)$, where $\mathfrak{d}'(n) = 2$ for $n = 1, 2, \ldots$. By Lemma 11.4, we obtain

$$\Pr[Z'_t = o] \geq \Pr[Z_t = o]$$

for any $t \geq 1$. Therefore, we provide

$$\sum_{t=1}^{\infty} \Pr[Z'_t = o] \geq \sum_{t=1}^{\infty} \Pr[Z_t = o] = \infty$$

by Lemma 11.7. This means that if $\mathfrak{d}(n) \geq 1$ then o is recurrent by $\mathcal{D}_{\mathfrak{d}}$. Let Z''_t denote a random walk on a growing complete graph according to $\mathcal{D}_{\mathfrak{d}''} = (\mathfrak{d}'', G, P)$, where $\mathfrak{d}''(n) \geq 2$ for $n = 1, 2, \ldots$. Using Lemma 11.5 and 11.6, we get

$$\begin{aligned}E_o[\min\{t > 0\,;\ Z_t = o\}] &= \sum_{t=1}^{\infty} \frac{N-1}{(N+t-2)(N+t-1)} \cdot t \geq C_1 \sum_{t=1}^{\infty} \frac{1}{t+1} = \infty \\ E_o[\min\{t > 0\,;\ Z'_t = o\}] &= \sum_{t=1}^{\infty} t F_{\mathfrak{d}''}(t) \\ &= \sum_{k=1}^{\infty} \left\{ \frac{2k(N-1)^2}{(N+k-2)(N+k-1)^2} + \frac{(2k+1)(N-1)^2}{(N+k-1)^2(N+k)} \right\} \\ &\leq C_2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} < \infty,\end{aligned}$$

where C_1 and C_2 are a positive constants with respect to t and k. By Corollary 11.3, we obtain

$$E_o[\min\{t > 0\,;\ Z''_t = o\}] \leq E_o[\min\{t > 0\,;\ Z'_t = o\}] < \infty.$$

Therefore, if $\mathfrak{d}(n) \geq 2$ for $n = 1, 2, \ldots$, o is positive recurrent by $\mathcal{D}_{\mathfrak{d}}$. If $\mathfrak{d}(n) = 1$ for $n = 1, 2, \ldots$, o is null recurrent by $\mathcal{D}_{\mathfrak{d}}$. □

12 Conclusion and Future work

In this paper, we have introduced a coupling method to prove the recurrence and transience of random walks on growing graphs by defining the notion of LHaGG and weakly LHaGG. Then, we showed the phase transition between the recurrence and transience of random walks on growing graphs such as k-ary tree (Theorem 4.2) and on $\{0, \dots, N\}^n$ with an increasing n (Theorem 8.1). We also have other examples of growing graphs, such as box (Theorem 5.1), $\{0, 1\}^n$ with an increasing n (Theorem 6.1), spider tree (Theorem 9.1 and 9.3), growing random tree (Theorem 9.10), added box (Theorem 10.1) and complete graph (Theorem 11.1). It is a future work to develop an extended technique to prove the phase transitions for more general growing graphs.

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