

Exploring Quantum Nature of Gravity for Macroscopic Quantum Systems under Quantum Control

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PhD Thesis

**Exploring Quantum Nature of Gravity for Macroscopic
Quantum Systems under Quantum Control**

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for the Degree of Doctor of Science

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Abstract

The theory of quantum gravity has been extensively explored to understand phenomena such as the early universe and black holes. Despite significant theoretical advancements, no experimental evidence for quantum gravity has yet been obtained, and it remains theoretically possible that the gravitational field may not require quantization. Verifying the quantum nature of gravity has long been regarded as a challenging problem due to the need for high-energy experiments. However, recent advancements in quantum technologies have opened new avenues for testing quantum gravity. One promising approach is investigating quantum entanglement induced by gravitational interactions in the non-relativistic regime proposed by Bose et al. and Marletto and Vedral. Since classical gravity alone cannot generate entanglement without introducing measurements, the observation of gravity-induced entanglement would serve as critical evidence of the quantum nature of gravity. Furthermore, generating entanglement necessitates the presence of quantized dynamical degrees of freedom in the gravitational field. Therefore, the experimental verification of entanglement is a crucial step toward a deeper understanding of quantum gravity.

Since gravitational effects are extremely weak, verifying quantum gravity requires preparing quantum states in macroscopic objects. Optomechanical systems consisting of macroscopic mirrors and cavities are promising models for achieving macroscopic quantum systems. Applying quantum control techniques makes it possible to suppress mechanical noise and reduce fluctuations to the quantum level. This thesis explores the quantum nature of gravity in optomechanical systems under quantum control based on the measurements. We examine the entanglement generation between macroscopic objects in an interferometric setup. Two macroscopic mirrors at the ends of the interferometer arms are controlled by the measurements of output fields. A power-recycled mirror at the common mode makes the asymmetry between common and differential modes, leading to quantum entanglement in the individual cavity light fields. Consequently, entanglement is also generated between the macroscopic mirrors coupled to the cavity light.

Subsequently, we investigate entanglement generation in optomechanical systems mediated by gravitational interactions. The gravity affects the differential mode, which induces an asymmetry between the common and differential modes and generates entanglement between the center-of-mass modes of the mirrors. We derive the conditions required for gravity-induced entanglement in real and Fourier spaces. Furthermore, we analyze the time evolution of the optomechanical system in real space, providing estimates for the time scales

necessary to generate entanglement and for the signal-to-noise ratio to exceed unity. The results presented in this thesis provide essential theoretical foundations for testing the quantum nature of gravity using macroscopic optomechanical systems.

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Publication List

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- [2] **Daisuke Miki**, Akira Matsumura, and Kazuhiro Yamamoto,
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The following six publications [4–10] are not directly related to this thesis.

- [4] **Daisuke Miki**, Akira Matsumura, and Kazuhiro Yamamoto,
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- [5] Masaki Tani, Kosei Hatakeyama, **Daisuke Miki**, Yuki Yamasaki,
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“Violation of the two-time Leggett-Garg inequalities for a coarse-grained quantum field”,
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- [6] Kosei Hatakeyama, **Daisuke Miki**, Masaki Tani, Yuuki Yamasaki, Satoshi Iso,
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“Non-Gaussian entanglement in gravitating masses: the role of cumulants”,
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- [10] **Daisuke Miki**, Akira Matsumura, and Kazuhiro Yamamoto,
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1 Introduction

Unifying quantum mechanics with general relativity remains one of the most fundamental and challenging problems in theoretical physics [11]. Quantum gravity theory is expected to help us understand the beginning of the universe and the quantum aspects of black holes. Among the various approaches, superstring theory is a promising candidate for quantum gravity (e.g., [12]). However, we have found no empirical evidence of quantum gravity and still need to complete a quantum gravity theory. One of the reasons for uncompleting quantum gravity is that there are almost no experimental studies on quantum gravity due to the extremely high energy required, corresponding to the Plank mass (10^{19} GeV). Consequently, most research on quantum gravity has been restricted to theoretical frameworks. Only a few experiments have been conducted to test quantum mechanics in a classical gravitational field. The Collela-Overhauser-Werner (COW) experiment was the first to reveal that the Earth's gravitational field induces the quantum phase shifts in the wave function of a neutron [13]. An experiment of a bouncing neutron in a uniform gravitational field has also helped investigate quantum mechanics in a gravitational field (Refs. [14–16]). These experiments showed that classical gravity affects the quantum systems, but thus far, whether gravity follows quantum mechanics has not been answered.

In recent years, an intriguing possibility has emerged: testing the quantum nature of gravity in low-energy, non-relativistic regimes. The key question, first discussed by Feynman at the Chapel Hill conference, is how the gravitational field appears for a spatially superposed particle. If gravity follows quantum mechanics, the gravitational field is generated as a superposition. This insight led to the proposal of quantum gravity-induced entanglement of masses (QGEM) protocol [17–19]. In this framework, two particles, each in a spatial superposition state, interact gravitationally via the Newtonian gravitational potential

$$\hat{H}_g = -\frac{Gm^2}{|\hat{Q}_A - \hat{Q}_B|}, \quad (1.1)$$

where G is the gravitational constant, m is the particle mass, and $\hat{Q}_{A/B}$ is position operator. This Newtonian potential operator is a natural result of the non-relativistic limit of a quantum field theory of linearized gravity. The authors showed that two particles evolve into an entangled state through the Newtonian potential operator, although classical gravity cannot generate the entanglement. Hence, detecting gravity-induced entanglement may provide

evidence that gravity follows quantum mechanics. Gravity-induced entanglement has stimulated several researches [10, 20–27]. For entanglement to emerge, the particles must satisfy strict experimental conditions: a mass of approximately 10^{-14}kg , a sizeable spatial superposition of order $100\mu\text{m}$, and coherence times on the order of one-second [28–30]. Achieving such conditions remains experimentally challenging due to decoherence effects, electromagnetic interactions, and thermal noise. Various strategies have been proposed to mitigate these issues, including enhanced isolation techniques [31, 32], suppression of decoherence [33], and electromagnetic shielding [28, 34–36].

Parallel to these efforts, theoretical discussions regarding the quantum nature of gravity have explored the role of gravitons and quantum field theory [37–44]. Refs. [37–39] analyze gravity-induced entanglement within the frameworks of complementarity in quantum mechanics and causality in general relativity. They argue that inconsistencies arising from gravitational superposition can be resolved through the vacuum fluctuation and the radiation of the gravitational fields. Refs. [41, 42] clarified this statement based on the quantum electrodynamics. The authors showed that the contradiction between complementarity and causality due to the superposition of the electromagnetic field is resolved by the vacuum fluctuation in the photon field and photon emission. Refs. [43, 44] extended the discussion of the quantum electrodynamics to the gravitational field. Ref. [40] indicated that gravity-induced entanglement could verify the quantized dynamical degree of freedom in the gravitational field under unitarity and Lorentz invariance assumptions. Therefore, verifying gravity-induced entanglement is the first step toward understanding quantum gravity and one of the most facilitative approaches to experimental verification of quantum gravity.

Optomechanics.—A significant challenge in detecting gravity-induced entanglement lies in the extremely weak nature of gravitational interactions. Cavity optomechanics is a promising model for overcoming this challenge by enabling precise quantum control of mechanical systems [45–48]. The interaction between a mechanical oscillator and an optical mode allows us to prepare the quantum states of the oscillators. Ground-state cooling of mechanical oscillators has been demonstrated using cavity detuning and feedback control techniques [49]. Coulomb-induced entanglement between levitated nanospheres has also been proposed for exploring quantum entanglement in mechanical systems [50]. Significant experimental milestones have been achieved in optomechanical systems, including the demonstration of entanglement between mechanical oscillators with masses as small as 70 picograms [51–

53] and the observation of quantum back-action noise in nanogram-scale oscillators [54]. Quantum squeezing of macroscopic mechanical mirrors near the quantum limit has been reported for systems with masses of 7 milligrams [55–57]. Quantum correlations between light and kilogram-scale mirrors in the Laser Interferometer Gravitational-Wave Observatory (LIGO) have been measured, underscoring the feasibility of optomechanical systems for probing quantum effects in macroscopic objects [58]. With advancements in quantum control techniques, optomechanical systems are being explored as candidates for verifying gravity-induced entanglement. Refs. [9, 59, 60] propose using single-photon interactions to generate entanglement between mechanical mirrors and optical fields. Additionally, inverted harmonic potentials have been shown to enhance optomechanical entanglement [61, 62].

The present doctoral thesis investigates the quantum nature of gravity in optomechanical systems under quantum control. The primary contributions of this work are the following two topics: (i) Entanglement between macroscopic mirrors: In Chapter 4, we explore the feasibility of generating entanglement between macroscopic mirrors via quantum measurement and feedback control [3]. While entanglement in picogram-scale systems has been demonstrated, extending these results to heavier, milligram-scale objects remains an open challenge. The measurement of quantum phenomena on the milligram scale is also important for verifying the quantum effects of gravity since the gravitational constant has experimentally been measured using the 90-milligram objects [63]. Therefore, we discuss whether quantum entanglement can be generated between two macroscopic mirrors by quantum control based on the measurement—the authors of Refs. [64–66] analyzed the entanglement in a comparable situation where the power-recycled mirror (PRM) asymmetrically makes the mechanical common and differential modes. However, their analysis was limited to high-frequency regions, where the mirrors were regarded as free mass. Namely, they only demonstrated entanglement generation between Fourier modes of the mechanical motions in high-frequency regions. Quantum control of macroscopic mirrors around resonant frequencies is important for entanglement generation (e.g., Refs. [67, 68]). Then, our analysis is wider than high-frequency regions.

(ii) Gravity-induced entanglement in optomechanical systems: In Chapter 5, we investigate gravity-induced entanglement between mechanical mirrors in optomechanical systems [1, 2]. In Section 5.2, we consider the time evolution of gravity-induced entanglement [1]. We show that gravity-induced entanglement can be generated by reducing thermal noise and realizing a quantum state through the quantum Kalman filtering process. The time evolution of gravity-induced entanglement, which is important for evaluating the feasibility

of the experiment, has yet to be clarified. We also determine the optimized setup of an experiment by investigating the signal-to-noise ratio. In Section 5.3, we focus on gravity-induced entanglement in the Fourier domain [2]. Ref. [67] investigated the gravity-induced entanglement between two output optical modes in the Fourier space, and the authors of Ref. [68] investigated the gravity-induced squeezing of output lights in the Fourier space. However, they do not discuss the quantum control of mechanical mirrors, and the entanglement and the squeezing of the mechanical modes of mirrors in the Fourier space have yet to be investigated. Therefore, we evaluate the gravity-induced entanglement between mirrors and the squeezing of mirrors in the Fourier space.

Entanglement.—Quantum entanglement is one of the fundamental properties of quantum mechanics. When two systems are in a quantum entangled state, a measurement on one subsystem immediately affects the other. Unlike classical correlations, the state of each system is not determined before the measurement, and quantum entanglement does not disappear when the measurement basis is changed. Quantum entanglement is one of the central researches in quantum information, which has been applied to quantum teleportation [69–72].

We first introduce the general density matrix of a single system $\hat{\rho} = \sum_i q_i |\psi_i\rangle \langle \psi_i|$, where $|\psi_i\rangle$ is the pure quantum state and q_i is the non-negative value satisfying $\sum_i q_i = 1$. The state is pure if and only if the one of q_i is 1 and all others are 0, $\hat{\rho} = |\psi\rangle \langle \psi|$. The density matrix is positive semi-definite, $\langle \phi | \hat{\rho} | \phi \rangle \geq 0$ for an arbitrary state $|\phi\rangle$, and $\text{tr}[\hat{\rho}] = 1$. Next, we consider a system composed of subsystem A and subsystem B. If the system is separable (not entangled), then the density matrix of the system is

$$\hat{\rho} = \sum_i p_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i, \quad (1.2)$$

where $\hat{\rho}_{A/B}^i$ is the density matrix of the subsystem A/B and p_i is the non-negative value that satisfies $\sum_i p_i = 1$. Taking the partially transposed state with subsystem A, we have

$$\hat{\rho}^{\text{T}_A} = \sum_i p_i (\hat{\rho}_A^i)^{\text{T}} \otimes \hat{\rho}_B^i. \quad (1.3)$$

Therefore, the partially transposed density matrix is positive semi-definite for any separable state because the density matrix $\hat{\rho}_{A/B}^i$ is positive semi-definite. However, the entangled state

is defined as one that cannot be expressed in Eq. (1.2). Considering the counterpart, if the partially transposed density matrix is negative, the state is non-separable, i.e., entangled. This is well-known as the positive partial transpose (PPT) criterion [73–75]. According to this criterion, we investigate the entanglement by entanglement negativity defined as

$$\mathcal{N} = \sum_{u_i < 0} |u_i|, \quad (1.4)$$

where u_i is the eigenvalue of the partial transposition of the density matrix. The system is entangled if the negativity is a positive value. Positive entanglement negativity is generally a sufficient condition for a state to be quantum entangled. However, it is a necessary and sufficient condition, for instance, considering two qubits and two-mode Gaussian states [76].

It is well known that Local Operations and Classical Communication (LOCC) cannot generate and increase the entanglement [73–75]. For instance, the local operation on the subsystem A is expressed as $\hat{\rho}_{LA} = \sum_i (\hat{A}_i \otimes \hat{1}) \hat{\rho} (\hat{A}_i \otimes \hat{1})^\dagger$, where $\hat{A}_i$ is the Kraus operator satisfying $\sum_i \hat{A}_i^\dagger \hat{A}_i = \hat{1}$. Then, we consider the case that we get the measurement result i_0 for the subsystem A and classically communicate with the subsystem B. Considering the operation on the subsystem B based on the measurement result i_0 , the density matrix is expressed as $\hat{\rho}_{LA}^{i_0} = \sum_i (\hat{A}_{i_0} \otimes \hat{B}_i^{i_0}) \hat{\rho} (\hat{A}_{i_0} \otimes \hat{B}_i^{i_0})^\dagger$, where $\hat{B}_i^{i_0}$ is the operation depended on the measurement results i_0 and satisfies $\sum_i \hat{B}_i^{i_0 \dagger} \hat{B}_i^{i_0} = \hat{1}$. Since the whole system remains in the product state throughout these operations, quantum entanglement is not caused by LOCC. Additionally, we introduce separable operations

$$\hat{\rho}' = \sum_i (\hat{M}_i \otimes \hat{N}_i) \hat{\rho} (\hat{M}_i \otimes \hat{N}_i)^\dagger, \quad (1.5)$$

where $\sum_i \hat{M}_i^\dagger \hat{M}_i \otimes \hat{N}_i^\dagger \hat{N}_i = \hat{1} \otimes \hat{1}$. The separable operations are the broad class of the LOCC; every LOCC is the separable operation, but not every separable operation is LOCC. The entanglement cannot occur in either operation, and we refer to both operations in the present thesis simply as LOCC. Therefore, generating and increasing quantum entanglement requires the non-LOCC operations. Using this property, detecting gravity-induced entanglement means that gravitational interaction is a non-LOCC operation. Assuming gravity is based on a local theory, generating QGEM provides evidence that gravity is a quantum entity explained by quantum communication.

Gaussian state.—Here, we introduce the Gaussian state based on Ref. [76], whose wave function is Gaussian and characterized by an expectation value and a covariance matrix. Gaussian states are widely used in quantum optics and quantum information science, and quantum entanglement in Gaussian states has been well studied [76–79].

We consider a second-order Hamiltonian, expressed as a second-order polynomial of position and momentum. We can generally write the second-order Hamiltonian as

$$\hat{H} = \frac{1}{2} \hat{\mathbf{s}}^T H \hat{\mathbf{s}} + \mathbf{s} \hat{\mathbf{s}}^T, \quad (1.6)$$

where $\hat{\mathbf{s}} = (\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)^T$ is the dimensionless operators satisfying $[\hat{q}_i, \hat{p}_i] = i$. $\mathbf{s}$ is the 4-dimensional real vector, and we can assume that the matrix H is the arbitrary symmetric matrix because the anti-symmetric parts can be ignored. We henceforth restrict the Hamiltonian matrix to be positive definite $H > 0$, which corresponds to a stable system. Using this second-order Hamiltonian, the general Gaussian state is defined as

$$\hat{\rho}_G = \frac{e^{-\beta \hat{H}}}{\text{tr}[e^{-\beta \hat{H}}]}, \quad (1.7)$$

with a positive real value β , which corresponds to the inverse temperature $\beta = 1/k_B T$ with a Boltzmann constant k_B . Redefining the real vector as $\bar{\mathbf{s}} = -H^{-1} \mathbf{s}$, the Hamiltonian is rewritten as $\hat{H} = \frac{1}{2} (\hat{\mathbf{s}} - \bar{\mathbf{s}})^T H (\hat{\mathbf{s}} - \bar{\mathbf{s}})$, where we ignore the constant. Then, we can derive the expected value and the covariance matrix of the Gaussian state as

$$\langle \hat{\mathbf{s}} \rangle = \text{tr}[\hat{\rho}_G \hat{\mathbf{s}}] = \bar{\mathbf{s}} \quad (1.8)$$

$$\boldsymbol{\sigma} = \text{tr}[\hat{\rho}_G \{(\hat{\mathbf{s}} - \bar{\mathbf{s}}), (\hat{\mathbf{s}} - \bar{\mathbf{s}})^T\}] = S \left(\oplus_{j=1}^2 \nu_j \mathbf{1}_2 \right) S^T = \begin{pmatrix} \boldsymbol{\sigma}_A & \boldsymbol{\sigma}_{AB} \\ \boldsymbol{\sigma}_{AB}^T & \boldsymbol{\sigma}_B \end{pmatrix}, \quad (1.9)$$

where S is the real symplectic group, which is defined $SJS^T = J$ where J is the anti-symmetric matrix defined as $[\hat{\mathbf{s}}, \hat{\mathbf{s}}^T] = iJ$. $\nu_j \geq 1$ is the symplectic eigenvalues.

Next, we discuss the properties of the covariance matrix. The covariance matrix is the symmetric matrix and positive definite $\boldsymbol{\sigma} > 0$. Moreover, from the uncertainty relation, we derive the following inequality:

$$\boldsymbol{\sigma} + iJ \geq 0, \quad (1.10)$$

which is equivalent to $\nu_j \geq 1$. We can also derive $\sigma > 0$ from this inequality since $\mathbf{v}^T J \mathbf{v} = 0$ for all real vector $\mathbf{v}$. Therefore, since these conditions are based on the uncertainty principle, the matrix σ can be regarded as a covariance matrix of the physical state if and only if these conditions are satisfied. The properties of Gaussian states are characterized only by the expectation value and covariance matrix. Here, we introduce the purity, the Wigner function, and quantum entanglement.

The purity is the indicator of the quantumness of the systems, defined by $P = \text{tr}[\hat{\rho}^2]$. We have $P = 1$ for the pure quantum state and $P < 1$ for the mixed state. For a Gaussian state, the purity is

$$P = \text{tr}[\hat{\rho}_G^2] = \frac{1}{\sqrt{\det \sigma}}, \quad (1.11)$$

and the Gaussian state is pure if and only if $\det \sigma = 1$ ($\nu_1 = \nu_2 = 1$).

Next, we consider the Wigner function, which describes a distribution function in the phase space. An interesting property of the Wigner function is that it can be negative in the quantum state, whereas the classical phase distribution function is only positive. Therefore, the Wigner function is commonly used in quantum optics as a quantum signature [80–82]. The definition of the Wigner function is

$$W(q, p) = \frac{1}{\pi^2} \int_{\mathbf{R}} dq' e^{ipq'} \langle q + \frac{q'}{2} | \hat{\rho} | q - \frac{q'}{2} \rangle, \quad (1.12)$$

and for the Gaussian state, we have

$$W_G(\mathbf{s}) = \frac{4}{\pi^2 \sqrt{\det \sigma}} e^{-(\mathbf{s} - \bar{\mathbf{s}})^T \sigma^{-1} (\mathbf{s} - \bar{\mathbf{s}})}. \quad (1.13)$$

The center of the function of the distribution is the location of the expected value $\bar{\mathbf{s}}$, and the spread is characterized by the covariance matrix σ . We plot the Wigner ellipses as a contour satisfying $W = e^{-1} W_{\max}$ where $W_{\max}$ is the maximum value of the Wigner function, i.e., $(\mathbf{s} - \bar{\mathbf{s}})^T \sigma^{-1} (\mathbf{s} - \bar{\mathbf{s}}) = 1$.

We then consider the entanglement criterion of two-mode Gaussian states. As we have already described, the quantum entanglement can be evaluated from the eigenvalues of the state after partial transposition. The operation of the partial transposition on the state corresponds to $\sigma \rightarrow T \sigma T$ in the covariance matrix, where $T = \mathbb{1}_2 \oplus \sigma_z$ with the 2×2

identity matrix $\mathbb{1}_2$ and the Pauli matrix σ_z . Then, we can derive the entanglement criterion: the two-mode Gaussian state is entangled if and only if

$$T\boldsymbol{\sigma}T + iJ < 0. \quad (1.14)$$

To investigate the entanglement, we define the logarithmic negativity as

$$E_N = \log_2 \|\hat{\rho}^{\text{T}_A}\|_1 = \max\{0, -\log_2 \tilde{\nu}\}, \quad (1.15)$$

where $\|\hat{\rho}\|_1$ is the trace norm of $\hat{\rho}$. $\tilde{\nu}$ is the minimum symplectic eigenvalue of the partially transposed covariance matrix $T\boldsymbol{\sigma}T$, given by

$$\tilde{\nu}^2 = \frac{\Sigma - \sqrt{\Sigma^2 - 4\det\boldsymbol{\sigma}}}{2}, \quad (1.16)$$

where $\Sigma = \det\boldsymbol{\sigma}_A + \det\boldsymbol{\sigma}_B - 2\det\boldsymbol{\sigma}_{AB}$. The inequality (1.14) is equivalent to $\tilde{\nu} < 1$, which leads to $E_N > 0$.

Remainder of the present paper.—This thesis is organized as follows. In Chapter 2, we review the models of gravity-induced entanglement using two superposed particles and two harmonic oscillators. We explain that Newtonian gravitational potential generates the entanglement of two masses as gravity obeys quantum mechanics. In Chapter 3, we introduce the optomechanical systems consisting of the mechanical oscillating mirror and cavity light. We also describe the quantum control of the mechanical mirror based on measurements. The main parts of the present thesis are Chapters 4-5. In Chapter 4, we explore the macroscopic quantum mechanics in optomechanical systems. We consider the entanglement between two macroscopic masses in the Fabry-Pérot-Michelson interferometer. In Chapter 5, we discuss gravity-induced entanglement in optomechanical systems under quantum control. We derive the condition of generating the entanglement, and investigate the time scale of the gravitational entanglement. In Appendix A, we compute the quantum Wiener filtering based on the measurement of the optical output field. In Appendix B, we discuss the input-output relation for the interferometer setup with a power-recycled mirror. In Appendix C, we present an exact formula of the entanglement negativity in the power-recycled interferometer. In Appendix D, we review the entanglement and squeezing of optical modes induced by gravitational interaction.

2 Testing quantum nature of gravity

The development of a complete theory of quantum gravity remains an ongoing challenge, mainly due to the need for experimental validation. This difficulty stems from the challenges of conducting experiments at extremely high energy scales where quantum gravitational effects are expected to appear. However, recent advancements in quantum technology have introduced new possibilities for testing quantum gravity. One promising approach involves examining quantum gravity-induced entanglement of masses (QGEM). In this Chapter, we briefly review gravity-induced entanglement between two superposed particles, proposed by Bose et al. [17] and Marletto and Vedral [18].

2.1 COW experiment

Before starting the discussion of QGEM, we introduce the Collela-Overhauser-Werner (COW) experiment [13], which is an experiment that showed the effect of a classical gravitational field on a quantum system. The experiment setup is shown in Figure 1: a neutron with mass m and wavelength λ is split into two paths, ACD and ABD, by the beam splitter at A. The heights of the two paths differ by $h \sin \alpha$, and neutrons passing through each path are affected by different Earth's gravitational fields. Finally, the neutron interferes at D, and we measure the intensity difference δI . Since the energy of neutrons passing through each path is conserved, we obtain

$$\frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2 k^2}{2m} + mgh \sin \alpha, \quad (2.1)$$

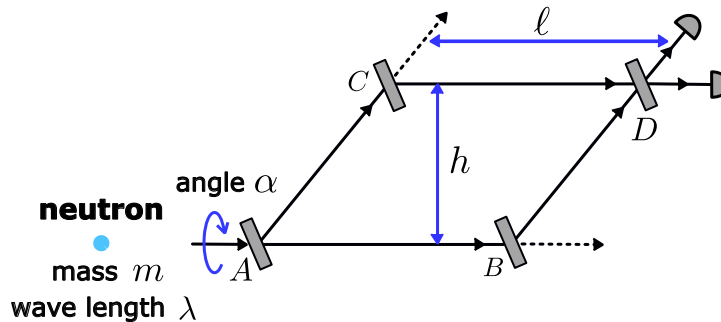


FIG 1: A schematic plot of the COW experiment. The interferometer is rotated by an angle α , which causes the effect of the Earth's gravitational field to be different in the two paths.

where g is the gravitational acceleration and k is the wavenumber. This indicates that the kinetic energy is reduced on ACD by converting the gravitational potential energy. Therefore, the difference of the two paths is

$$\beta = k\ell - k_0\ell \sim -\frac{\lambda m^2 g}{2\pi\hbar^2} h\ell \sin \alpha. \quad (2.2)$$

Since the intensity difference ΔI is proportional to $\cos \beta$ in the Mach-Zehnder interferometer, the Earth's classical gravitational field affects the phase of the neutron wave function. However, this experiment did not answer whether gravity follows quantum mechanics.

2.2 Gravity-induced entanglement

2.2.1 Two superposed particles

Here, we consider the two superposed particles interacting with the Newtonian potential operator. As shown in Fig. 2, we can consider the linear setup in the original paper [17, 18] and the parallel setup [10, 25]. Although the expressions of both cases are almost identical, Ref. [28] pointed out that the parallel setup makes it easier to reach the time scale requirements than the linear case. Therefore, we focus on the parallel setup in this section.

Two particles of the same mass m are separated by a distance of d . Each particle is initially prepared as the superposition of two spatially localized states separated by distance L along the same direction. We use notations $|\uparrow_{A/B}\rangle$ and $|\downarrow_{A/B}\rangle$ to represent the states of the particle at the left and right paths, respectively. We consider the case where the initial

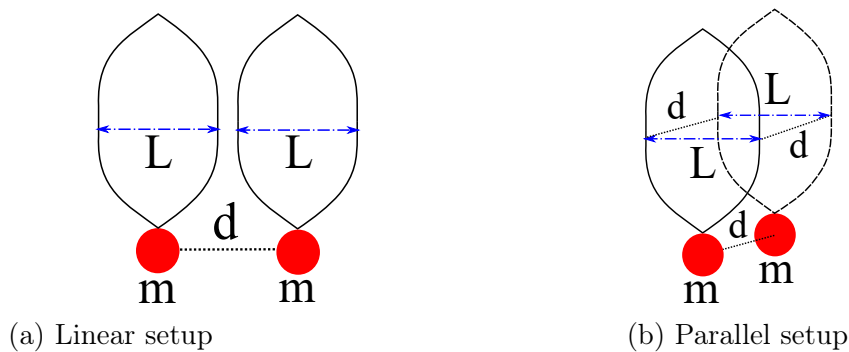


FIG 2: Sketch of the two superposed particles: (a) in the linear setup, and (b) the parallel setup. In both configurations, the particles are separated by d , and L separates the spatial superpositions.

state of the total system is separable state as

$$|\Psi(0)\rangle = |\psi_A(0)\rangle \otimes |\psi_B(0)\rangle \quad (2.3)$$

where $|\psi_{A/B}(0)\rangle$ is the initial state

$$|\psi_{A/B}(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow_{A/B}\rangle + |\downarrow_{A/B}\rangle). \quad (2.4)$$

The initial state evolves under the gravitational interaction described by

$$\hat{H} = -\frac{\Delta_{AB}}{2}\hat{\sigma}_z^A \otimes \hat{\sigma}_z^B \quad \text{with} \quad \Delta_{AB} = \frac{Gm^2}{d} \left(1 - \frac{1}{\sqrt{1 + L^2/d^2}}\right), \quad (2.5)$$

where we ignore the constant term proportional to the identity operator. Because we assume that the wave packet of each particle does not spread, the kinetic term is neglected. The state of the total system at time t is $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi(0)\rangle$. The density matrix of a given pure state is

$$\begin{aligned} \hat{\rho}(t) &= e^{-i\hat{H}t/\hbar} |\Psi(0)\rangle \langle \Psi(0)| e^{i\hat{H}t/\hbar} \\ &= \frac{1}{4} \begin{pmatrix} 1 & e^{i\frac{\Delta_{AB}}{\hbar}t} & e^{i\frac{\Delta_{AB}}{\hbar}t} & 1 \\ e^{-i\frac{\Delta_{AB}}{\hbar}t} & 1 & 1 & e^{-i\frac{\Delta_{AB}}{\hbar}t} \\ e^{-i\frac{\Delta_{AB}}{\hbar}t} & 1 & 1 & e^{-i\frac{\Delta_{AB}}{\hbar}t} \\ 1 & e^{i\frac{\Delta_{AB}}{\hbar}t} & e^{i\frac{\Delta_{AB}}{\hbar}t} & 1 \end{pmatrix}. \end{aligned} \quad (2.6)$$

We analyze the entanglement using the positive partial transpose (PPT) criterion introduced in Chapter 1. The eigenvalues of the partial transposed matrix are

$$u_{\pm} = \pm \frac{1}{2} \sin\left[\frac{\Delta_{AB}}{\hbar}t\right], \quad v_{\pm} = \frac{1}{2} \left(1 \pm \cos\left[\frac{\Delta_{AB}}{\hbar}t\right]\right). \quad (2.7)$$

$u'_{\pm}$ s are always positive, but either u_+ or u_- is always negative or zero. Therefore, the negativity is

$$\mathcal{N} = \frac{1}{2} \left| \sin\left[\frac{\Delta_{AB}}{\hbar}t\right] \right|. \quad (2.8)$$

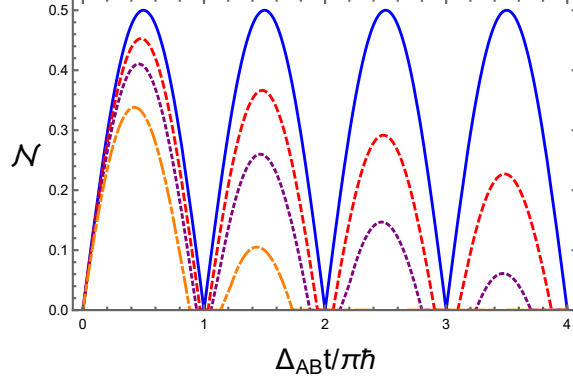


FIG 3: Behavior of the entanglement negativity $\mathcal{N}$ as a function of the dimensionless time $\Delta_{AB}t/\pi\hbar$ for $L/d = 1$. The state of two particles is entangled except when $\mathcal{N} = 0$. The blue curve is the case without the dissipation. The red dashed, purple dotted, and orange dashed-dotted curves correspond to the case for $\pi\Gamma\hbar/\Delta_{AB} = 0.1, 0.3, 0.6$, respectively.

The blue curve in Fig. 3 shows $\mathcal{N}$ as a function of the dimensionless time $\Delta_{AB}t/\pi\hbar$. The state of the two particles periodically oscillates between the maximally entangled and non-entangled states because the system is a pure quantum state.

In the previous analysis, We assumed a closed quantum system and did not consider the influence of the environment. However, in experimental verification, the effect of decoherence due to the environment has a significant impact. Refs. [29, 33] estimated the decoherence effects of two particles due to the environment. Here, we review these studies. Assuming the initial state is pure, given by Eq. (2.3), the density matrix is given by

$$\hat{\rho}'(t) = \frac{1}{4} \begin{pmatrix} 1 & e^{i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & e^{i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & e^{-2\Gamma t} \\ e^{-i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & 1 & e^{-2\Gamma t} & e^{-i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} \\ e^{-i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & e^{-2\Gamma t} & 1 & e^{-i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} \\ e^{-2\Gamma t} & e^{i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & e^{i\frac{\Delta_{AB}}{\hbar}t - \Gamma t} & 1 \end{pmatrix}, \quad (2.9)$$

where Γ is the dissipation rate. Then, the eigenvalues of the partially transposed density matrix are

$$u'_{\pm} = \frac{1}{4} - \frac{1}{4}e^{-\Gamma t} \left(e^{-\Gamma t} \pm 2 \sin\left[\frac{\Delta_{AB}}{\hbar}t\right] \right) \quad (2.10)$$

$$v'_{\pm} = \frac{1}{4} + \frac{1}{4}e^{-\Gamma t} \left(e^{-\Gamma t} \pm 2 \cos\left[\frac{\Delta_{AB}}{\hbar}t\right] \right). \quad (2.11)$$

The red dashed, purple, and orange dashed-dotted curves in Fig. 3 show the entanglement

negativity with environmental dissipation. For quantum entanglement to occur, the timescale of the gravitational interaction is faster than the timescale of the dissipation.

2.2.2 Harmonic oscillators

In this section, we briefly explain the model of generating gravity-induced entanglement between two harmonic oscillators [23, 24]. Unlike the previous case of using two superposed particles, gravity-induced entanglement occurs even in Gaussian states. The evaluation of quantum entanglement in Gaussian states is well established, as introduced in Chapter 1, which simplifies the analysis. However, this model is very much affected by thermal fluctuations. One useful way to reduce thermal noise is quantum control, which will be introduced in Chapter 3. Gravity-induced entanglement under quantum control will be discussed in Chapter 5. From another perspective, the realization of non-Gaussian states in continuous variable systems may play an important role in the generation of quantum entanglement [9, 23, 55]. Non-Gaussian state realizations have been discussed in Ref. [83]. Here, we review gravity-induced entanglement between harmonic oscillators and thermal noise, which is the basis of the analysis in Chapter 5.

The Hamiltonian of the system is

$$\hat{H} = \frac{1}{2m}\hat{Q}_A^2 + \frac{1}{2}m\Omega^2\hat{P}_A^2 + \frac{1}{2m}\hat{Q}_B^2 + \frac{1}{2}m\Omega^2\hat{P}_B^2 - \frac{Gm^2}{d + \hat{Q}_A - \hat{Q}_B}, \quad (2.12)$$

where we assume the symmetric oscillators with the mass m and the frequency Ω , and d is a distance of two oscillators. $\hat{Q}_{A/B}$ and $\hat{P}_{A/B}$ are the dimensional canonical operators satisfying $[\hat{Q}_{A/B}, \hat{P}_{A/B}] = i\hbar$. Assuming the mechanical displacement is much smaller than the distance of two particles d , we can expand the gravitational potential as

$$\hat{H}_G = -\frac{Gm^2}{d + \hat{Q}_A - \hat{Q}_B} \simeq -\frac{Gm^2}{d} + \frac{Gm^2}{d^2}(\hat{Q}_A - \hat{Q}_B) - \frac{Gm^2}{d^3}(\hat{Q}_A - \hat{Q}_B)^2, \quad (2.13)$$

where the first term is constant, and the second term is the local part that cannot increase the entanglement between two oscillators. Hence, the third term generates the entanglement. We ignored higher order terms after the third order because the position displacement is small, which allows for analysis based on the Gaussian states.

We can derive the Heisenberg equation of the two oscillators as

$$\dot{\hat{Q}}_A = \frac{1}{m}\hat{P}_A, \quad \dot{\hat{Q}}_B = \frac{1}{m}\hat{P}_B, \quad (2.14)$$

$$\dot{\hat{P}}_A = -m\Omega^2(1 - \frac{\epsilon}{2})\hat{Q}_A - m\Omega^2\frac{\epsilon}{2}\hat{Q}_B - \nu \quad (2.15)$$

$$\dot{\hat{P}}_B = -m\Omega^2(1 - \frac{\epsilon}{2})\hat{Q}_B - \Omega\frac{\epsilon}{2}\hat{Q}_A + \nu, \quad (2.16)$$

where $\nu = Gm^2/d^2$ is the constant and ϵ is the gravitational coupling given by

$$\epsilon = \frac{4Gm}{d^3\Omega^2}. \quad (2.17)$$

These equations are simple cases without thermal dissipation, but the macroscopic objects are affected by environmental noises. Then, the system's dynamics is given by the Langevin equation (e.g., [84]). Considering the macroscopic quantum open systems, we have

$$\dot{\hat{Q}}_A = \frac{1}{m}\hat{P}_A, \quad \dot{\hat{Q}}_B = \frac{1}{m}\hat{P}_B, \quad (2.18)$$

$$\dot{\hat{P}}_A = -m\Omega^2(1 - \frac{\epsilon}{2})\hat{Q}_A - m\Omega^2\frac{\epsilon}{2}\hat{Q}_B - \nu - \Gamma\hat{P}_A + \sqrt{2\Gamma}\hat{P}_A^{\text{th}} \quad (2.19)$$

$$\dot{\hat{P}}_B = -m\Omega^2(1 - \frac{\epsilon}{2})\hat{Q}_B - \Omega\frac{\epsilon}{2}\hat{Q}_A + \nu - \Gamma\hat{P}_B + \sqrt{2\Gamma}\hat{P}_B^{\text{th}}, \quad (2.20)$$

where Γ is the dissipation rate, and $\hat{P}_{A/B}^{\text{th}}$ is the thermal fluctuation with zero mean value $\langle \hat{P}_{A/B}^{\text{th}} \rangle = 0$ and its correlation function is

$$\langle \hat{P}_{A/B}^{\text{th}}(t)\hat{P}_{A/B}^{\text{th}}(t') \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{m\hbar\omega}{2} e^{-i\omega(t-t')} \left[\coth\left(\frac{\hbar\omega}{2k_B T}\right) - 1 \right], \quad (2.21)$$

where k_B is the Boltzmann constant and T is the environmental temperature. Since we have $k_B T/\hbar \sim 10^{11} \text{ s}^{-1}$ even for low temperature $T = 1 \text{ K}$, we can use the approximation $\langle \{\hat{P}_{A/B}^{\text{th}}(t), \hat{P}_{A/B}^{\text{th}}(t')\} \rangle \simeq m\hbar\Omega(2n_{\text{th}} + 1)\delta(t - t')$, where $n_{\text{th}} = (e^{\hbar\Omega/k_B T} - 1)^{-1} \sim k_B T/\hbar\Omega$ is the thermal occupation number. We here introduce the dimensionless operators as

$$\hat{q}_{A/B} = \sqrt{\frac{2m\Omega}{\hbar}}\hat{Q}_{A/B}, \quad \hat{p}_{A/B} = \sqrt{\frac{2}{m\hbar\Omega}}\hat{P}_{A/B}, \quad (2.22)$$

that satisfy $[\hat{q}_{A/B}, \hat{p}_{A/B}] = 2i$. Then, the Langevin equations are

$$\dot{\hat{q}}_A = \Omega \hat{p}_A, \quad \dot{\hat{q}}_B = \Omega \hat{p}_B \quad (2.23)$$

$$\dot{\hat{p}}_A = -\Omega(1 - \frac{\epsilon}{2})\hat{q}_A - \Omega\frac{\epsilon}{2}\hat{q}_B - \nu' - \Gamma\hat{p}_A + \sqrt{2\Gamma}\hat{p}_A^{\text{th}} \quad (2.24)$$

$$\dot{\hat{p}}_B = -\Omega(1 - \frac{\epsilon}{2})\hat{q}_B - \Omega\frac{\epsilon}{2}\hat{q}_A - \nu' - \Gamma\hat{p}_B + \sqrt{2\Gamma}\hat{p}_B^{\text{th}}, \quad (2.25)$$

where $\nu' = \nu\sqrt{2/m\hbar\Omega}$ and $\hat{p}_{A/B}^{\text{th}} = \hat{P}_{A/B}^{\text{th}}\sqrt{2/m\hbar\Omega}$.

Introducing the vector of dimensionless operators $\hat{\mathbf{r}} = (\hat{q}_A, \hat{p}_A, \hat{q}_B, \hat{p}_B)^T$, we rewrite the Langevin equations as $\dot{\hat{\mathbf{r}}} = \mathbf{A}\hat{\mathbf{r}} + \mathbf{v}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ -\Omega(1 - \epsilon/2) & -\Gamma & -\Omega\epsilon/2 & 0 \\ 0 & 0 & 0 & \Omega \\ -\Omega\epsilon/2 & 0 & -\Omega(1 - \epsilon/2) & -\Gamma \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ -\nu' + \sqrt{2\Gamma}\hat{p}_A^{\text{th}} \\ 0 \\ \nu' + \sqrt{2\Gamma}\hat{p}_B^{\text{th}} \end{pmatrix}. \quad (2.26)$$

Then, the covariance matrix $\mathbf{V} = \langle \{\hat{\mathbf{r}} - \langle \hat{\mathbf{r}} \rangle, (\hat{\mathbf{r}} - \langle \hat{\mathbf{r}} \rangle)^T\} \rangle / 2$ follows the Lyapunov equation as

$$\dot{\mathbf{V}} = \mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}^T + \mathbf{N}, \quad (2.27)$$

where $\mathbf{N}$ describes the thermal noise defined as

$$\mathbf{N} = \text{diag}\{0, 2\Gamma(2n_{\text{th}} + 1), 0, 2\Gamma(2n_{\text{th}} + 1)\}. \quad (2.28)$$

Solving the Lyapunov equation, we can derive the covariance matrix as

$$\mathbf{V}(t) = e^{\mathbf{A}t}\mathbf{V}(0)e^{\mathbf{A}^T t} + \int_0^t dt' e^{\mathbf{A}(t-t')}\mathbf{N}(t')e^{-\mathbf{A}(t-t')}. \quad (2.29)$$

The logarithmic negativity for the Gaussian state in Eq. (1.15) can evaluate the entanglement.

The left panel of Figure 4 shows the behavior of gravity-induced entanglement as a function of time for initial vacuum state $\mathbf{V}(0) = \mathbf{1}_4$. Here the mechanical frequency $\Omega/2\pi = 10^{-3}$ Hz, the mass density $m/d^3 = 2 \times 20$ g/cm³, and the dissipation rate $\Gamma/2\pi = 10^{-6}$ Hz

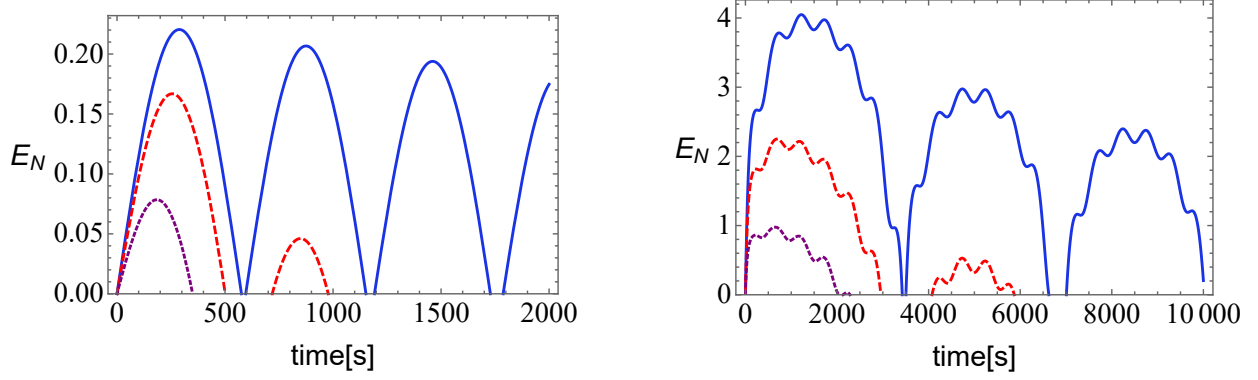


FIG 4: Behavior of logarithmic negativity for initial vacuum state (left panel) and initial squeezed state (right panel). We fix the mechanical frequency $\Omega/2\pi = 10^{-3}$, the mass density $m/d^3 = 2 \times 20 \text{ g/cm}^3$, and the dissipation $\Gamma/2\pi = 10^{-6} \text{ Hz}$. Each curve correspond to $n_{\text{th}} = 1$ (blue solid curve), $n_{\text{th}} = 10$ (red dashed curve), and $n_{\text{th}} = 30$ (purple dotted curve).

are fixed. Each curve changes the effect of thermal noise, with the blue curve corresponding to $n_{\text{th}} = 1$, the red dashed curve to $n_{\text{th}} = 10$, and the purple dotted curve to $n_{\text{th}} = 30$. The time scale of generating maximum entanglement is estimated by $t \sim \pi/2\Omega(1 - \epsilon/2)$, where we assume the time scale of gravitational interaction is much faster than that of the thermal decoherence. The right panel of Figure 4 shows the behavior of the entanglement for initial squeezed state $\mathbf{V}(0) = \text{diag}\{e^{-2r_{\text{sq}}}, e^{2r_{\text{sq}}}, e^{-2r_{\text{sq}}}, e^{2r_{\text{sq}}}\}$ with a squeezing parameter $r_{\text{sq}} = 1.73$. Then, the time scale of maximum entanglement is $t \sim \pi/\Omega\epsilon$ under the same condition as the vacuum case. It can be seen that the entanglement for the initial squeezed state is more resistant to thermal noise; the entanglement for the initial squeezed state is larger and survives longer than that in initial vacuum state. However, the entanglement is strongly lost if we consider the initial thermal state or initial thermal squeezed state. The entanglement with initial thermal state is derived by $E_N \sim (\epsilon/2)/\ln 2 - \log_2(2n_{\text{th}} + 1)$, while the entanglement for initial thermal squeezed state is $E_N \sim 2|r_{\text{sq}}|/\ln 2 - \log_2(2n_{\text{th}} + 1)$. For example, the entanglement does not occur in both cases if the initial thermal noise is set to $n_{\text{th}} \gtrsim 10$. Therefore, we need extremely low temperatures and low dissipation rates.

2.2.3 Schrödinger-Newton gravity

Since quantum effects of gravity have not been observed to date, the possibility that gravity is classical remains [85–87]. We here consider the mutual gravity in Schrödinger-Newton (SN) theory for two objects [67, 85, 88, 89]. SN gravity is described by the expectation value

of the Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle \hat{T}_{\mu\nu} \rangle, \quad (2.30)$$

where $G_{\mu\nu}$ is the Einstein tensor of classical spacetime, and $\hat{T}_{\mu\nu}$ is the operator of the energy-stress tensor. This means that the system is quantum, but gravity is classical. Considering the non-relativistic regime, the Hamiltonian of SN gravity is

$$\hat{H}_{\text{SN}} = -\frac{Gm}{d^3}(\hat{q}_A - q_{cB})^2 - \frac{Gm}{d^3}(\hat{q}_B - q_{cA})^2, \quad (2.31)$$

where $q_{cA/cB}$ is the expectation value of the mechanical position. The first term represents the effect of classical gravity with system A as the source of quantum system B, and the second term is vice versa. It has been assumed that in many gravity-induced entanglement models, the expectation value can always be prepared to zero $q_{cA/cB} = 0$ and that no SN-induced correlations appear between two objects because SN gravity is expressed by LOCC. This is the correct description when we do not consider the measurement process. However, when the measurement process is considered, the expected value changes due to the wavefunction collapse. Ref. [88] showed that the difference in how the expectation value is determined leads to differences in the behavior of classical gravity. In particular, when considering SN gravity based on the causal estimation, the measurement process to system A produces a change in the expectation value q_{cA} , which in turn affects the motion of system B. This has very similar effects to quantum gravity. We are currently working on a theoretical model to distinguish the signature of SN and quantum gravity using time-delayed measurement [90].

2.3 Conclusions

We briefly reviewed gravity-induced entanglement to test the quantum nature of gravity. First, we described the COW experiment as an experimental validation of gravitational effects on quantum systems. In this experiment, we explained that Earth's gravity causes a phase shift of the neutron's wavefunction. This is one of the gravitational effects on quantum systems, but gravity itself is classical. Next, we explained gravity-induced entanglement models of spatially superposed particles and harmonic oscillators. We demonstrated the periodic quantum entanglement based on the Schrödinger equation for pure quantum systems. How-

ever, in both models, we need to consider the effect of environmental decoherence towards the experimental verification. We simply showed the behavior of quantum entanglement with environmental decoherence and the entanglement is very sensitive to thermal noise, which requires a low-dissipation and low-temperature environment. Finally, we discussed the semiclassical gravity in Schrödinger-Newton theory. The influence of classical gravity has been neglected so far, but it may be necessary to consider its influence when considering the measurement process. We have been working on the effects of semiclassical gravity now. In the next Chapter, we discuss the realization of macroscopic quantum systems by quantum control in optomechanical systems in order to reduce the influence of noises.

3 Macroscopic Quantum mechanics

This section concisely reviews cavity optomechanical systems with a single movable mirror under quantum control. This topic has gained significant attention for advancing macroscopic quantum phenomena [45–48]. These systems have been extensively investigated to explore and realize quantum behaviors in macroscopic mechanical oscillators, offering insights into the quantum-classical boundary. A theoretical framework for cooling mechanical oscillators to their ground state through cavity detuning and feedback control was presented in Ref. [49]. Additionally, References [64–66, 91] discussed a framework for quantum estimation, employing a quantum filtering process to prepare and stabilize quantum states in mechanical mirrors. These frameworks have laid the foundation for experimental advancements in controlling mechanical systems at the quantum level. Recent experimental progress has demonstrated the feasibility of bringing milligram-scale mirrors close to quantum states by detuning, quantum feedback, and quantum filtering methods [55–57]. These advancements pave the way for realizing quantum control over significantly larger masses, marking substantial steps toward macroscopic quantum systems. These milestones underscore the potential of optomechanical systems in entangling massive objects, essential for probing fundamental questions in quantum mechanics. Realizing macroscopic quantum states in optomechanical systems is pivotal for probing the boundaries between quantum and classical physics and offers a promising platform to test the quantum nature of gravity. The following sections delve into the dynamics of optomechanical systems and subsequently review quantum feedback control and quantum filtering, which play critical roles in achieving and sustaining quantum states in optomechanical systems.

3.1 Dynamics of optomechanics

3.1.1 Optomechanical interaction

We consider an optomechanical system consisting of an optical cavity with a single mechanically movable end mirror, as depicted in Figure 5. In this system, the interaction between the optical field inside the cavity and the mechanical mirror arises from radiation pressure exerted by the intracavity light, which induces a displacement in the mirror’s position. When the optical field is resonant with the cavity length ℓ , the resonant frequency of the cavity mode ω_c is given by $\omega_c = n\pi c/\ell$, where c represents the speed of light and n is the mode number of the cavity mode. As the radiation pressure from the intracavity photons displaces

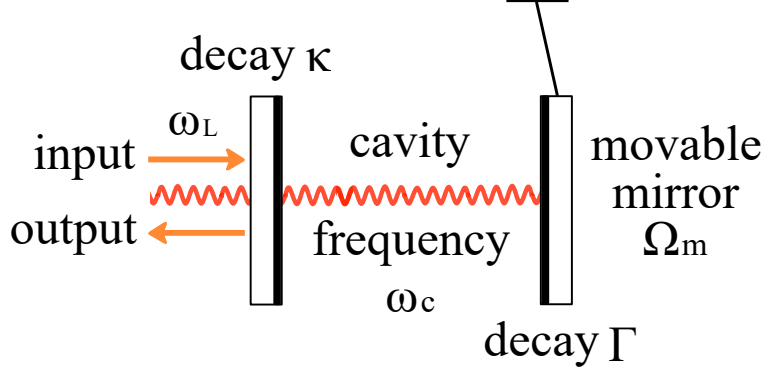


FIG 5: A schematic plot of the optomechanical system with a cavity and a movable end mirror.

the mirror by $\hat{Q}$, the effective cavity length changes to $\ell + \hat{Q}$. Consequently, this displacement shifts the cavity's frequency to

$$\omega'_c = \frac{n\pi c}{\ell - \hat{Q}} = \frac{\omega_c}{1 - \hat{Q}/\ell}. \quad (3.1)$$

This frequency shift induces a coupling between the optical field within the cavity and the mirror's mechanical motion.

The Hamiltonian describing this optomechanical system can be expressed as

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\Omega^2\hat{Q}^2 + \frac{\hbar\omega_c}{1 - \hat{Q}/\ell}\hat{a}^\dagger\hat{a} + i\hbar E(\hat{a}^\dagger e^{-i\omega_L t} - \hat{a}e^{i\omega_L t}). \quad (3.2)$$

The first and second terms describe the mechanical oscillator with mass m and resonant frequency Ω . $\hat{Q}$ and $\hat{P}$ denote the mechanical position and momentum operators, satisfying the commutation relation $[\hat{Q}, \hat{P}] = i\hbar$. The operators $\hat{a}$ and $\hat{a}^\dagger$ are the cavity photon annihilation and creation operators, with $[\hat{a}, \hat{a}^\dagger] = 1$. The third term is the Hamiltonian of the cavity mode and the optomechanical coupling. The final term represents the input laser field with frequency ω_L and amplitude $E = \sqrt{P_{\text{in}}\kappa/\hbar\omega_L}$ where P_{in} is the input laser power, and κ is the cavity decay rate. When we consider a coherent laser with a large mean photon number in the cavity, we can linearize the system around its stationary value. Decomposing the classical stationary value and quantum fluctuation of the cavity field as $\hat{a} = \bar{a} + \delta\hat{a}$, we can also linearize the mechanical position $\hat{Q} = \bar{Q} + \delta\hat{Q}$ and momentum $\hat{P} = \bar{P} + \delta\hat{P}$. The

third term in the Hamiltonian can be expanded to yield

$$\begin{aligned}
\frac{\hbar\omega_c}{1 - \hat{Q}/\ell}\hat{a}^\dagger\hat{a} &\simeq \hbar\frac{\omega_c}{\xi}\left(1 + \frac{\delta\hat{Q}}{\xi\ell} + \frac{\delta\hat{Q}^2}{\xi^2\ell^2}\right)(|\bar{a}|^2 + \bar{a}\delta\hat{a}^\dagger + \bar{a}^*\delta\hat{a} + \delta\hat{a}^\dagger\delta\hat{a}) \\
&\simeq \hbar\frac{\omega_c}{\xi}(\bar{a}\delta\hat{a}^\dagger + \bar{a}^*\delta\hat{a}) + \hbar\frac{\omega_c}{\xi^2\ell}|\bar{a}|^2\delta\hat{Q} \\
&\quad + \hbar\frac{\omega_c}{\xi}\delta\hat{a}^\dagger\delta\hat{a} + \hbar\frac{\omega_c}{\xi^2\ell}\delta\hat{Q}(\bar{a}\delta\hat{a}^\dagger + \bar{a}^*\delta\hat{a}) + \hbar\frac{\omega_c}{\xi^3\ell^2}|\bar{a}|^2\delta\hat{Q}^2,
\end{aligned} \tag{3.3}$$

where $\xi = 1 - \bar{Q}/\ell$. As a result, the total Hamiltonian is expressed as a quadratic polynomial of the quantum fluctuation operators. Since a Gaussian initial state will remain Gaussian under a quadratic Hamiltonian, the quantum state of the system is Gaussian, fully characterized by the mean values and the covariance matrix of the quantum fluctuation operators. This property greatly simplifies the analysis of the system's evolution, allowing for comprehensive insights into its quantum dynamics.

3.1.2 Equation of motion of mechanical mirror and optical light

In a closed quantum system, the system dynamics can be fully derived from the Hamiltonian in Eq. (3.2). However, it is essential to consider the effects of dissipation and fluctuation due to interactions with the external environment for macroscopic systems. As we introduced in Chapter 2, the dynamics of the mechanical mirror is explained by the Langevin equations as

$$\delta\dot{\hat{Q}} = \frac{\hat{P}}{m} \tag{3.4}$$

$$\delta\dot{\hat{P}} = -m\Omega_m^2\delta\hat{Q} - \Gamma\delta\hat{P} - \hbar\alpha(e^{i\theta_a}\delta\hat{a}^\dagger + e^{-i\theta_a}\delta\hat{a}) + \sqrt{2\Gamma}\hat{P}^{\text{th}}, \tag{3.5}$$

where $\alpha = \omega_c|\bar{a}|/\xi^2\ell$ represents the optomechanical coupling strength, and $\bar{a} = |\bar{a}|e^{i\theta_a}$ denotes the steady-state intracavity field amplitude. The term $\Omega_m^2 = \Omega^2 + 2\hbar\omega_c|\bar{a}|^2/m\xi^3\ell^2$ represents the mechanical frequency, shifted due to the radiation pressure of quadratic position term in Eq. (3.3). When the mechanical displacement is much smaller than the cavity length $\bar{Q} \ll \ell$, we have $\xi = 1$ and $\Omega_m = \Omega$. This approximation is commonly used for conventional optomechanical systems, but may be better not to use it for gravitationally interacting optomechanical systems because the gravity effect is comparably small. Although we have not discussed this effect in the present thesis, it may be necessary for precise prediction to observe the weak effects of gravity.

Next, we consider the equation of the intracavity light; the photon leakage from the cavity results in dissipation of the intracavity field. Assuming perfect reflectivity at the movable end mirror and all photon leakage at the fixed mirror, the equation of motion for the intracavity field is

$$\dot{\hat{a}} = i\Delta\hat{a} - i\alpha e^{i\theta_a}\delta\hat{Q} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\hat{a}^{\text{in}}, \quad (3.6)$$

where we have transformed $\hat{a} \rightarrow e^{i\omega_L t}\hat{a}$ to the rotating frame. Here, $\Delta = \omega_L - \omega_c/\xi$ represents the cavity detuning, which induces an optical spring effect, discussed in a later section. The input noise operator $\hat{a}^{\text{in}}$ satisfies the following correlation functions:

$$\langle \hat{a}^{\text{in}}(t)\hat{a}^{\text{in}\dagger}(t') \rangle = (N_{\text{th}} + 1)\delta(t - t'), \quad \langle \hat{a}^{\text{in}\dagger}(t)\hat{a}^{\text{in}}(t') \rangle = N_{\text{th}}\delta(t - t'), \quad (3.7)$$

where $N_{\text{th}} = (e^{\hbar\omega_c/k_B T} - 1)^{-1}$ denotes the equilibrium thermal photon number, which can be assumed to be zero $N_{\text{th}} \simeq 0$ because the optical frequency is high $\hbar\omega_c/k_B T \gg 1$. To further analyze the dynamics, we introduce the amplitude quadrature $\delta\hat{x} = e^{-i\theta_a}\delta\hat{a} + e^{i\theta_a}\delta\hat{a}^\dagger$ and the phase quadrature $\delta\hat{y} = (e^{-i\theta_a}\delta\hat{a} - e^{i\theta_a}\delta\hat{a}^\dagger)/i$, yielding the equations of motion:

$$\dot{\delta\hat{x}} = -\Delta\delta\hat{y} - \frac{\kappa}{2}\delta\hat{x} + \sqrt{\kappa}\hat{x}^{\text{in}} \quad (3.8)$$

$$\dot{\delta\hat{y}} = -2\alpha\delta\hat{Q} + \Delta\delta\hat{x} - \frac{\kappa}{2}\delta\hat{y} + \sqrt{\kappa}\hat{y}^{\text{in}}, \quad (3.9)$$

where the input noise operators $\hat{x}^{\text{in}}$ and $\hat{y}^{\text{in}}$ satisfy

$$\langle \hat{x}^{\text{in}}(t)\hat{x}^{\text{in}}(t') \rangle = \langle \hat{y}^{\text{in}}(t)\hat{y}^{\text{in}}(t') \rangle = (2N_{\text{th}} + 1)\delta(t - t'). \quad (3.10)$$

The quadratures $\delta\hat{x}$ and $\delta\hat{y}$ are Hermitian operators satisfying the commutation relation $[\delta\hat{x}, \delta\hat{y}] = 2i$. In the resonant case ($\Delta = 0$), the amplitude quadrature $\delta\hat{x}$ becomes decoupled from the phase quadrature $\delta\hat{y}$, and the amplitude quadrature affects the mechanical position displacement as radiation pressure. Then, the mechanical displacement changes the cavity length, affecting the phase quadrature equation. In the detuning case, the mechanical displacement affects both optical quadratures since the amplitude quadrature is coupled with the phase quadrature. Hence, detuning is necessary to monitor mechanical motion by measuring the amplitude quadrature, while the tuned cavity is good for monitoring mechanical motion from the measurement of the phase quadrature. We note that homodyne detection is

necessary to measure the phase quadrature, while we can measure the amplitude quadrature by photon counting without the homodyne measurement in a linear treatment.

The classical stationary values can be derived as follows:

$$\bar{Q} = -\frac{\hbar\omega_c}{m\Omega_m^2\xi^2\ell}|\bar{a}|^2 \quad (3.11)$$

$$\bar{P} = 0 \quad (3.12)$$

$$\bar{a} = \frac{2E}{\kappa - 2i\Delta}. \quad (3.13)$$

These equations are inherently nonlinear since ξ and Δ depend on the stationary position $\bar{Q}$. However, if $\bar{Q}$ remains small, Eq. (3.13) yields a stationary amplitude $\bar{a}$ that is effectively independent of $\bar{Q}$.

3.1.3 Optical spring effects

When the mechanical mirror is displaced from the resonant position of the optical cavity, it experiences the optical spring effect [92]. This phenomenon arises because the intracavity power varies with the displacement of the mirror, creating a position-dependent restoring force. Specifically, when the cavity length extends beyond the resonance point, the intracavity power decreases, inducing a restoring force that pulls the mirror back toward its original position. Conversely, when the cavity length shortens, the increased intracavity power exerts a force that pushes the mirror back. This optical spring effect effectively increases the mechanical resonant frequency and enhances its quality factor. In this section, we analyze how detuning affects the system dynamics based on the theoretical framework presented in Ref. [49].

To analyze the optical spring effects, we use the Fourier transformation $F(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \tilde{F}(t)$, which yields the mechanical position in the Fourier domain as:

$$\delta\hat{Q}(\omega) = \chi_{\text{sp}}(\omega) \left(\sqrt{2\Gamma} \hat{P}^{\text{th}}(\omega) - \frac{2\hbar\alpha\sqrt{\kappa}}{(\kappa - 2i\omega)^2 + 4\Delta^2} \left((\kappa - 2i\omega)\hat{x}^{\text{in}}(\omega) - 2\Delta\hat{y}^{\text{in}}(\omega) \right) \right), \quad (3.14)$$

where $\chi_{\text{sp}} = (m(\Omega_{\text{sp}}^2 - i\Gamma_{\text{sp}}\omega - \omega^2))^{-1}$ is the effective susceptibility of the mirror. Here, $\Omega_{\text{sp}}(\omega)$ and $\Gamma_{\text{sp}}(\omega)$ are the effective frequency and dissipation rate shifted by the detuning,

given by

$$\Omega_{\text{sp}}(\omega) = \sqrt{\Omega_m^2 + \frac{8\hbar\alpha^2\Delta(\kappa^2 + 4\Delta^2 - 4\omega^2)}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}} \quad (3.15)$$

$$\Gamma_{\text{sp}}(\omega) = \Gamma - \frac{32\hbar\alpha^2\kappa\Delta}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}. \quad (3.16)$$

Assuming the tuned cavity $\Delta = 0$, we have $\Omega_{\text{sp}} = \Omega_m$ and $\Gamma_{\text{sp}} = \Gamma$.

For the blue-detuned case ($\Delta > 0$), the optical spring effect enhances the mechanical frequency Ω_{sp} . This frequency shift, induced by radiation pressure, can substantially enhance the quality factor for low-frequency mechanical mirrors. The quality factor, defined by $Q_{\text{sq}} = \Omega_{\text{sp}}/\Gamma_{\text{sp}}$, is crucial as it determines how long the mechanical oscillator can sustain coherent motion. However, the effective dissipation increases the thermal and radiation pressure noises, scaling as $1/\Gamma_{\text{sp}}$. This is because the mechanical mirror is heated as it receives energy from the cavity light. Thus, techniques such as quantum feedback cooling and quantum filtering are necessary to mitigate these noises, as will be discussed in subsequent sections.

For the red-detuned case ($\Delta < 0$), the optical spring effect primarily increases the effective dissipation rate Γ_{sp} while slightly reducing the effective frequency Ω_{sp} . Assuming the systems with high bare mechanical frequency, the optical spring effects hardly affect the mechanical frequency, i.e., $\Omega_{\text{sp}} \simeq \Omega$. However, the increased dissipation rate Γ_{sp} significantly reduces thermal and radiation pressure noises by $1/\Gamma_{\text{sp}}$, facilitating detuning-induced cooling. Ground-state cooling can thus be achieved if the optomechanical coupling is sufficiently strong to enhance the effective dissipation rate. It is important to note that stability considerations constrain the detuning values in this regime [49].

3.1.4 Continuous measurement

In this subsection, we consider the adiabatic limit ($\kappa \gg \omega$), where the cavity field dissipates faster than the mechanical oscillation timescale. In this regime, information about mechanical motion can be continuously monitored by measuring cavity output. This is called the continuous measurement. Under this approximation, we can ignore the time derivative of

the amplitude and the phase quadratures $\dot{\hat{x}} = \dot{\hat{y}} = 0$, yielding

$$\delta\hat{x} = \frac{8\alpha\Delta}{\kappa^2 + 4\Delta^2}\delta\hat{Q} + \frac{2\kappa\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{x}^{\text{in}} - \frac{4\Delta\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{y}^{\text{in}} \quad (3.17)$$

$$\delta\hat{y} = -\frac{4\alpha\kappa}{\kappa^2 + 4\Delta^2}\delta\hat{Q} + \frac{4\Delta\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{x}^{\text{in}} + \frac{2\kappa\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{y}^{\text{in}}. \quad (3.18)$$

Then, we derive the effective mechanical frequency and the mechanical dissipation rate due to detuning as $\Omega_{\text{sp}} \simeq \omega_m$ and $\Gamma_{\text{sp}} \simeq \Gamma$ with

$$\omega_m = \sqrt{\Omega_m^2 + \Omega_m \frac{16g^2\Delta}{\kappa^2 + 4\Delta^2}}, \quad (3.19)$$

where we redefine the optomechanical coupling

$$g = \alpha \sqrt{\frac{\hbar}{2m\Omega_m}}. \quad (3.20)$$

In this adiabatic limit, the optical spring effect primarily shifts in the mechanical frequency while the dissipation rate remains close to its bare value Γ .

3.1.5 Mechanical dissipation

Mechanical dissipation in optomechanical systems is commonly modeled using viscous and structural damping. These dissipation mechanisms impact the system's quality factor and response.

We first introduce the viscous damping, primarily from collisions with air molecules and the scattering, absorption, and emission of thermal photons [33]. These dissipative processes are influenced by the environmental temperature T and the material properties of the system but are independent of frequency. Assuming a sphere of radius R_0 , each dissipation rates are defined as

$$\Gamma_{\text{air}} = \frac{8\sqrt{2\pi}}{3} \frac{P_0 R_0^2 \sqrt{m_{\text{air}}}}{m \sqrt{k_B T}}, \quad (3.21)$$

$$\Gamma_{\text{sc}} = 8! \frac{8}{9} \left(\frac{k_B T}{\hbar c} \right)^8 \frac{\hbar R^6 \zeta(9)}{m \pi} \text{Re} \left[\frac{\epsilon - 1}{\epsilon + 2} \right]^2, \quad (3.22)$$

$$\Gamma_{\text{ab}} = \Gamma_{\text{em}} = \frac{16}{189} \left(\frac{k_B T}{\hbar c} \right)^5 \frac{\hbar R^3 \pi^5}{m} \text{Im} \left[\frac{\epsilon - 1}{\epsilon + 2} \right], \quad (3.23)$$

where P_0 is the pressure, m_{air} is the mass of the molecules of the residual air, $\zeta(n)$ is the Riemann zeta function, and ϵ is the dielectric constant. Therefore, the total mechanical damping is $\Gamma = \Gamma_{\text{air}} + \Gamma_{\text{sc}} + \Gamma_{\text{ab}} + \Gamma_{\text{em}}$.

Next, we consider structural damping, a frequency-dependent dissipation due to internal friction [7, 8, 93, 94]. The mechanical dissipation rate for structural damping can be expressed as

$$\omega\Gamma(\omega) = \Omega_m\Gamma(\Omega_m), \quad (3.24)$$

where $\Gamma(\Omega_m)$ is the damping rate at the mechanical resonant frequency Ω_m . This frequency dependence is relevant in high-quality factor materials, where internal friction mechanisms contribute to energy dissipation at different rates depending on the oscillation frequency. In Chapter 4, we consider the theoretical model to test the entanglement between suspended macroscopic mirrors with structural damping.

3.2 Quantum feedback control

In the previous section, we describe the increase or decrease in mechanical frequency and mechanical dissipation rate due to the optical spring effect. In this section, we introduce quantum feedback control based on linear measurement of the mechanical position to reduce the noises [49, 57, 94, 95]. Since this control scheme primarily employs external optical fields with a negligible thermal photon number, it effectively increases the mechanical dissipation rate without introducing additional thermal noises. The feedback mechanism operates by applying a force to the mirror, effectively reducing its velocity and thus damping its motion.

Here, we consider feedback control based on the measurement of either the phase quadrature or the amplitude quadrature of the optical field. The setup of the optical field measurement is shown in Fig. 6. The output optical field, which serves as the measurement signal, can be obtained through input-output relations [46, 47, 96], given by

$$\hat{x}^{\text{out}} = \hat{x}^{\text{in}} - \sqrt{\kappa}\delta\hat{x} \quad (3.25)$$

$$\hat{y}^{\text{out}} = \hat{y}^{\text{in}} - \sqrt{\kappa}\delta\hat{y} \quad (3.26)$$

where $\hat{x}^{\text{out}}$ and $\hat{y}^{\text{out}}$ represent the output amplitude and phase quadratures of the light field, respectively. To account for imperfections in measurement, we introduce additional noise

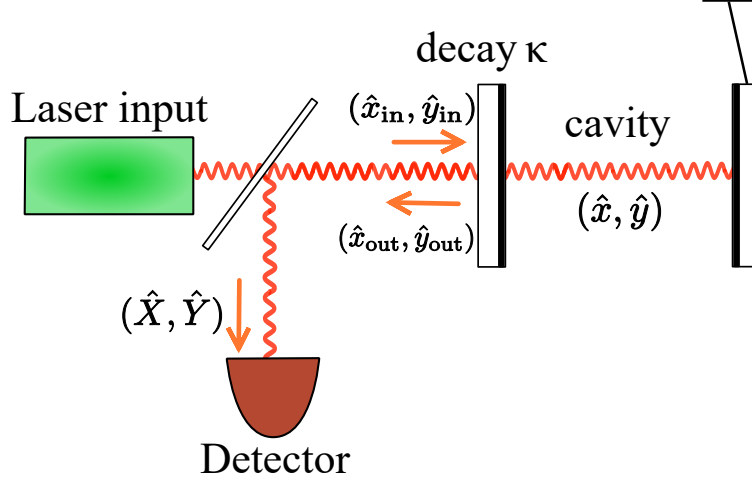


FIG 6: Configuration of the measurement of the output optical field. The moving mirror is assumed to be perfectly reflective, and the light leaking from the fixed mirror of the cavity with the dissipation rate κ is measured by a detector.

terms $(\hat{x}^{\text{add}}, \hat{y}^{\text{add}})$, which represent vacuum noise from the detection process. The detected output field quadratures, denoted by $\hat{X}$ and $\hat{Y}$, are then given by

$$\hat{X} = \sqrt{\eta_x} \hat{x}^{\text{out}} + \sqrt{1 - \eta_x} \hat{x}^{\text{add}} \quad (3.27)$$

$$\hat{Y} = \sqrt{\eta_y} \hat{y}^{\text{out}} + \sqrt{1 - \eta_y} \hat{y}^{\text{add}}, \quad (3.28)$$

where η_x (η_y) $\in [0, 1]$ denote the detection efficiencies for the amplitude and phase quadratures, respectively. When $\eta_x(\eta_y) = 1$, a perfect measurement is achieved, and we have $\hat{X} = \hat{x}^{\text{out}}$ ($\hat{Y} = \hat{y}^{\text{out}}$).

The feedback force $\hat{F}_{\text{fb}}$ is described by convolution of the measured signal with a causal feedback function $f_{\text{fb}}(t)$:

$$\hat{F}_{\text{fb}} = - \int_{-\infty}^t dt' f_{\text{fb}}(t - t') \hat{I}(t'), \quad (3.29)$$

where $\hat{I}$ represents the measured quadrature ($\hat{X}$ or $\hat{Y}$) and $f_{\text{fb}}(t)$ is the causal filter function that indicates the feedback response. For the ideal derivative feedback control model, $f_{\text{fb}}(t)$ is proportional to the derivative of a Dirac delta function, i.e., $f_{\text{fb}}(t) = g_{\text{fb}} \delta'(t)$, where g_{fb} is the feedback gain. This form of feedback effectively applies a force proportional to the velocity of the mirror, thereby enhancing its damping rate. The feedback force term $\hat{F}_{\text{fb}}$ is incorporated into the equation of motion for the mirror's momentum, resulting in an

increased effective dissipation rate. This term increases the mechanical effective dissipation, which reduces the thermal fluctuation. Therefore, the feedback control provides a pathway toward achieving ground-state cooling in the macroscopic mechanical mirror. The following sections will elaborate on how quantum feedback affects noise reduction.

3.2.1 Feedback based on output phase field

We first discuss feedback control based on measuring the phase quadrature of the optical field. In the Fourier domain, the output phase quadrature $\hat{Y}$ can be expressed as:

$$\begin{aligned} \hat{Y} = & \frac{4\alpha(\kappa - 2i\omega)\sqrt{\kappa\eta_y}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\delta\hat{Q} - \frac{4\kappa\Delta\sqrt{\eta_y}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\hat{x}^{\text{in}} - \frac{(\kappa^2 - 4\Delta^2 + 4\omega^2)\sqrt{\eta_y}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\hat{y}^{\text{in}} \\ & + \sqrt{1 - \eta_y}\hat{y}^{\text{add}}. \end{aligned} \quad (3.30)$$

The coefficient of $\delta\hat{Q}$, known as the sensitivity coefficient, is optimized for a tuned cavity ($\Delta = 0$). The mechanical position can be estimated by calibrating the measurement results. A simple calibration method divides the measurement result by the sensitivity coefficient, while optimal estimation techniques are discussed in the later section. The spectral density of the output phase quadrature $\hat{Y}$ is given by:

$$S_{YY}(\omega) = \frac{16\alpha^2(\kappa^2 + 4\omega^2)\kappa\eta_y}{|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}S_{\delta Q\delta Q}(\omega) + 2\eta_y N_{\text{th}} + 1, \quad (3.31)$$

where we define the spectral density for operators $\hat{A}$ and $\hat{B}$ as

$$2\pi\delta(\omega - \omega')S_{AB} = \langle \hat{A}(\omega)\hat{B}^\dagger(\omega') + \hat{B}^\dagger(\omega')\hat{A}(\omega) \rangle / 2. \quad (3.32)$$

Once the mechanical position is estimated from the measurement results, a feedback force $\hat{F}_{\text{fb}}$ is applied to control the mirror, increasing the dissipation rate and thereby effectively lowering thermal noise by reducing the kinetic energy of the mirror. The equation of motion

for the mirror's position operator $\delta\hat{Q}$ is obtained by

$$\begin{aligned} \delta\hat{Q} = \chi_{\text{fby}} \left(\sqrt{2\Gamma} \hat{P}^{\text{th}} - \frac{2\hbar\alpha\sqrt{\kappa}}{(\kappa - 2i\omega)^2 + 4\Delta^2} \left((\kappa - 2i\omega)\hat{x}^{\text{in}} - 2\Delta\hat{y}^{\text{in}} \right) \right. \\ \left. + i\omega g_{\text{fby}} \left(\frac{-4\kappa\Delta\sqrt{\eta_y}}{(\kappa - 2i\omega)^2 + 4\Delta^2} \hat{x}^{\text{in}} + \frac{-(\kappa^2 - 4\Delta^2 + 4\omega^2)\sqrt{\eta_y}}{(\kappa - 2i\omega)^2 + 4\Delta^2} \hat{y}^{\text{in}} + \sqrt{1 - \eta_y} \hat{y}^{\text{add}} \right) \right), \end{aligned} \quad (3.33)$$

where we assume ideal derivative feedback $f_{\text{fby}}(\omega) = -i\omega g_{\text{fby}}$, with a positive feedback gain $g_{\text{fby}} > 0$. The effective susceptibility of the mechanical mirror under feedback control is denoted by $\chi_{\text{fby}} = (m(\Omega_{\text{fby}}^2 - i\Gamma_{\text{fby}}\omega - \omega^2))^{-1}$, with the effective frequency and dissipation rate

$$\Omega_{\text{fby}}^2(\omega) = \Omega_{\text{sp}}^2 + \frac{8\alpha\omega^2(\kappa^2 - 4\Delta^2 + 4\omega^2)\sqrt{\kappa\eta_y}}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2} g_{\text{fby}} \quad (3.34)$$

$$\Gamma_{\text{fby}}(\omega) = \Gamma_{\text{sp}} + \frac{4\alpha\kappa(\kappa^2 + 4\Delta^2 + 4\omega^2)\sqrt{\kappa\eta_y}}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2} g_{\text{fby}}. \quad (3.35)$$

Feedback can either increase or decrease the effective frequency Ω_{fby} depending on the detuning. In contrast, the feedback control enhances the effective dissipation rate Γ_{fby} irrespective of the detuning.

In the adiabatic limit ($\kappa \gg \omega$), which allows continuous position measurements, we have $\Omega_{\text{fby}}(\omega) \simeq \Omega_m$ and $\Gamma_{\text{fby}}(\omega) \simeq \Gamma + \gamma_{my}$, with

$$\gamma_{my} = \frac{4\alpha\kappa\sqrt{\kappa\eta_y}}{m(\kappa^2 + 4\Delta^2)} g_{\text{fby}} > 0. \quad (3.36)$$

In this regime, the feedback control increases the dissipation rate without significantly affecting the mechanical frequency. The spectral density of the mechanical mirror's position $S_{\delta Q \delta Q}$ can be expressed as:

$$S_{\delta Q \delta Q} = |\chi_{\text{fby}}|^2 \left(m\hbar\omega\Gamma \coth\left(\frac{\hbar\omega}{2k_B T}\right) + \frac{4\hbar^2\alpha^2\kappa}{\kappa^2 + 4\Delta^2} (2N_{\text{th}} + 1) + g_{\text{fby}}^2\omega^2(2\eta_y N_{\text{th}} + 1) \right). \quad (3.37)$$

The first term represents thermal noise, the second represents radiation pressure, and the third represents feedback-induced noise. For high temperatures, the thermal noise can be

approximated by:

$$m\hbar\omega\Gamma \coth\left(\frac{\hbar\omega}{2k_BT}\right) \simeq m\hbar\Omega_m\Gamma(2n_{\text{th}} + 1), \quad (3.38)$$

where $n_{\text{th}} = (e^{\hbar\Omega_m/k_BT} - 1)^{-1} \simeq k_BT/\hbar\Omega_m$ is the thermal phonon number. The variance $\mathcal{V}_{\delta Q\delta Q} = \langle \delta\hat{Q}^2 \rangle / 2 = \int_{-\infty}^{\infty} d\omega S_{\delta Q\delta Q} / (2\pi)$ is given by:

$$\mathcal{V}_{\delta Q\delta Q} = \frac{\hbar}{2m\Omega_m} \left(\frac{\Gamma}{\Gamma_{\text{fby}}} (2n_{\text{th}} + 1) + \frac{8g^2\kappa}{\Gamma_{\text{fby}}(\kappa^2 + 4\Delta^2)} (2N_{\text{th}} + 1) + \frac{g_{\text{fby}}^2\Omega_m}{m\hbar\Gamma_{\text{fb}}} (2\eta_y N_{\text{th}} + 1) \right). \quad (3.39)$$

Since the effective dissipation rate Γ_{fby} is greater than the bare dissipation rate Γ , feedback control reduces thermal noise by a factor of $\Gamma/\Gamma_{\text{fby}}$. The second term represents the radiation pressure noise, while the third term is inversely proportional to the second term. The minimum variance is constrained by the zero-point uncertainty. Using the arithmetic-geometric mean inequality and assuming perfect detection ($\eta = 1$), we obtain

$$\mathcal{V}_{\delta Q\delta Q} \geq \frac{\hbar}{2m\Omega_m} \left(2n_{\text{th}} \frac{\Gamma}{\Gamma_{\text{fby}}} + 1 \right), \quad (3.40)$$

where the zero-point fluctuation is obtained when the thermal noise is zero $n_{\text{th}} = 0$. The variance of the momentum operator can be derived similarly, ensuring the uncertainty principle is preserved under quantum feedback control. If feedback influences dissipation without altering the frequency, the feedback force can be approximated as a resistive force, $\hat{F}_{\text{fb}} \simeq -\gamma_{\text{my}}\delta\hat{P}$. The Langevin equations then become

$$\dot{\delta\hat{Q}} = \frac{\delta\hat{P}}{m} \quad (3.41)$$

$$\dot{\delta\hat{P}} = -m\Omega_m^2\delta\hat{Q} - \Gamma_{\text{fby}}\delta\hat{P} - \hbar\alpha\delta\hat{x} + \sqrt{2\Gamma_{\text{fby}}}\hat{P}^{\text{eff}}, \quad (3.42)$$

where the thermal noise satisfies

$$\langle \{P^{\text{eff}}(t), P^{\text{eff}}(t')\} \rangle \simeq m\hbar\Omega_m(2n_{\text{thfb}} + 1)\delta(t - t'), \quad \text{with} \quad n_{\text{thfb}} = n_{\text{th}} \frac{\Gamma}{\Gamma_{\text{fby}}}. \quad (3.43)$$

3.2.2 Feedback based on output amplitude field

Next, we consider the feedback cooling based on the output amplitude quadrature $\hat{X}$ measurements. The amplitude quadrature in the Fourier domain can be expressed as

$$\begin{aligned}\hat{X} = & -\frac{8\alpha\Delta\sqrt{\kappa\eta_x}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\delta\hat{Q} - \frac{(\kappa^2 - 4\Delta^2 + 4\omega^2)\sqrt{\eta_x}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\hat{x}^{\text{in}} + \frac{4\kappa\Delta\sqrt{\eta_x}}{(\kappa - 2i\omega)^2 + 4\Delta^2}\hat{y}^{\text{in}} \\ & + \sqrt{1 - \eta_x}\hat{x}^{\text{add}},\end{aligned}\quad (3.44)$$

and its spectral density is given by

$$S_{XX}(\omega) = \frac{64\alpha^2\kappa\Delta^2\eta_x}{|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}S_{\delta Q\delta Q}(\omega) + 2\eta_x N_{\text{th}} + 1. \quad (3.45)$$

The coefficient of $\delta\hat{Q}$ in the amplitude quadrature $\hat{X}$ is negative, requiring the feedback gain g_{fbx} also to be negative to enhance mechanical dissipation. Unlike the phase quadrature measurement, the sensitivity coefficient of the amplitude quadrature measurement becomes zero at $\Delta = 0$, meaning that position information is inaccessible in a perfectly tuned cavity. Thus, detuning must be adjusted to optimize measurement sensitivity. The effective mechanical susceptibility with feedback control, $\chi_{\text{fbx}} = (m(\Omega_{\text{fbx}}^2 - i\Gamma_{\text{fbx}}\omega - \omega^2))^{-1}$, is characterized by the effective frequency Ω_{fbx} and dissipation rate Γ_{fbx} , given by

$$\Omega_{\text{fbx}}^2 = \Omega_{\text{sp}}^2 - \frac{32\alpha\omega^2\kappa\Delta\sqrt{\kappa\eta_x}}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}g_{\text{fbx}} \quad (3.46)$$

$$\Gamma_{\text{fbx}} = \Gamma_{\text{sp}} - \frac{8\alpha\Delta(\kappa^2 + 4\Delta^2 - 4\omega^2)\sqrt{\kappa\eta_x}}{m|(\kappa - 2i\omega)^2 + 4\Delta^2|^2}g_{\text{fbx}} \quad (3.47)$$

In the adiabatic limit ($\kappa \gg \Omega$), the effective mechanical frequency and dissipation simplify to $\Omega_{\text{fbx}} \simeq \omega_m$ and $\Gamma_{\text{fbx}} \simeq \Gamma + \gamma_{mx}$, with

$$\gamma_{mx} = -\frac{8\alpha\Delta\sqrt{\kappa\eta_x}}{m(\kappa^2 + 4\Delta^2)}g_{\text{fbx}}. \quad (3.48)$$

Under this approximation, the feedback control hardly affects the mechanical frequency, but only by the shift due to the optical spring effects. The effective dissipation rate, however, depends on both the detuning and feedback. For $\Delta > 0$, feedback increases effective dissipation, thereby achieving cooling, whereas it reduces dissipation for $\Delta < 0$. Thus, feedback-induced

cooling occurs under positive detuning. Henceforth, we denote the dissipation by feedback control as γ_m , regardless of the measurement.

The spectral density of the mechanical position in the adiabatic limit is

$$S_{\delta Q \delta Q} = |\chi_{\text{fbx}}|^2 \left(m \hbar \omega \Gamma \coth \left(\frac{2k_B T}{\hbar \omega} \right) + \frac{4 \hbar^2 \alpha^2 \kappa}{\kappa^2 + 4 \Delta^2} (2N_{\text{th}} + 1) + g_{\text{fbx}}^2 \omega^2 (2\eta_x N_{\text{th}} + 1) \right), \quad (3.49)$$

and the position variance $\mathcal{V}_{\delta Q \delta Q} = \langle \delta \hat{Q}^2 \rangle / 2$ is given by

$$\mathcal{V}_{\delta Q \delta Q} = \frac{\hbar}{2m\omega_m} \left(\frac{\Gamma}{\Gamma_{\text{fbx}}} (2n_{\text{th}} + 1) + \frac{8g_m^2 \kappa}{\Gamma_{\text{fbx}} (\kappa^2 + 4\Delta^2)} (2N_{\text{th}} + 1) + \frac{g_{\text{fbx}}^2 \Omega_m}{m \hbar \Gamma_{\text{fbx}}} (2\eta_x N_{\text{th}} + 1) \right), \quad (3.50)$$

where we define the rescaled optomechanical coupling $g_m = g \sqrt{\Omega_m / \omega_m}$. As with phase quadrature measurements, the variance is bounded by quantum fluctuations, and the feedback force effectively acts as a dissipative term.

3.3 Quantum filtering

To read out the mechanical mirror information and estimate the position of the mirror, we need to calibrate the measurement of the output optical field. The simplest estimation is to divide the measured quantity by the sensitivity coefficient. However, we need a better estimation method because this method retains shot noise. We introduce two optimal estimation methods conditioned on the measurement results: Wiener filtering [64–66, 91], based on the Fourier domain analysis, and Kalman filtering [57, 97], based on the time domain analysis. Using these filterings, the quantum state of the macroscopic mirror can be achieved by reducing the thermal noise and the radiation pressure noise. In this section, we consider the general output optical field $\hat{b}$ defined as

$$\hat{b} = \sqrt{\eta} \hat{b}^{\text{out}} + \sqrt{1 - \eta} \hat{b}^{\text{add}} \quad (3.51)$$

$$\hat{b}^{\text{out}} = \hat{x}^{\text{out}} \cos \theta + \hat{y}^{\text{out}} \sin \theta, \quad (3.52)$$

where $\eta \in [0, 1]$ is the detection efficiency, $\langle (\hat{b}^{\text{add}})^2 \rangle = 1$ is the vacuum noise input due to imperfect detection, and θ is the homodyne angle. The case for $\theta = 0$ ($\theta = \pi/2$) corresponds to

the amplitude (phase) field quadrature measurement. In this thesis, we focus our discussion on the case of phase and amplitude measurements. The output field quadrature $\hat{b}$ satisfies the commutation relation $[\hat{b}(t), \hat{b}(t')] = 0$, which implies that we can regard the measurement as the classical stochastic process. We note that the output $\hat{b}$ satisfies the quantum nondemolition condition, $[\delta\hat{Q}(t), \hat{b}(t')] = 0$ for $t > t'$, which means the measurement output at t' does not affect the future system observable.

3.3.1 Quantum Wiener filtering

First, we discuss Wiener filtering, the estimation method for stationary states. The estimated values of the mechanical position and momentum are given by the convolution $\delta Q_{\text{est}} = H_q(t) \otimes \hat{b}(t)$ and $\delta P_{\text{est}} = H_p(t) \otimes \hat{b}(t)$, where H_q and H_p are the causal filter function. We determine these filter functions to optimize the conditional variance and covariance $\mathcal{V}_{\delta A \delta B}^{\text{wf}} = \langle \{ \delta \hat{A} - \delta A_{\text{est}}, \delta \hat{B} - \delta B_{\text{est}} \} \rangle / 2$ for $A, B = Q, P$. In the Fourier domain, we have $\delta Q_{\text{est}}(\omega) = H_q(\omega) \hat{b}(\omega)$ and $\delta P_{\text{est}}(\omega) = H_p(\omega) \hat{b}(\omega)$, where the filter functions are derived by using the Wiener-Hopf method [66, 98],

$$H_q(\omega) = \frac{1}{\phi_+(\omega)} \left[\frac{S_{\delta Q b}(\omega)}{\phi_-(\omega)} \right]_+ \quad (3.53)$$

$$H_p(\omega) = \frac{1}{\phi_+(\omega)} \left[\frac{S_{\delta P b}(\omega)}{\phi_-(\omega)} \right]_+. \quad (3.54)$$

$[f]_+$ denotes the causal parts of f , which has poles only in the lower-half complex plane, while $[f]_-$ denotes the non-causal parts of f , which has poles only in the upper-half complex plane. This indicates that the filter functions are independent of the future measurement results. ϕ_+ and ϕ_- are factors of the spectral density of the output light $S_{bb}(\omega)$ with

$$S_{bb}(\omega) = \phi_+(\omega) \phi_-(\omega), \quad (3.55)$$

and $\phi_+(\omega)$ has poles on the lower-half complex plane, while $\phi_-(\omega)$ has poles on the upper-half complex plane. The conditional variance and covariance of the mechanical mirror are

derived by integration of the spectral density

$$\mathcal{V}_{\delta Q \delta Q}^{\text{wf}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (S_{\delta Q \delta Q} - |G_q(\omega)|^2) \quad (3.56)$$

$$\mathcal{V}_{\delta P \delta P}^{\text{wf}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (S_{\delta P \delta P} - |G_p(\omega)|^2) \quad (3.57)$$

$$\mathcal{V}_{\delta Q \delta P}^{\text{wf}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Re}(S_{\delta Q \delta P} - G_q(\omega)G_p(\omega)^*), \quad (3.58)$$

with

$$G_q(\omega) = \left[\frac{S_{\delta Q b}(\omega)}{\phi_-(\omega)} \right]_+ = \phi_+(\omega) H_q(\omega). \quad (3.59)$$

$$G_p(\omega) = \left[\frac{S_{\delta P b}(\omega)}{\phi_-(\omega)} \right]_+ = \phi_+(\omega) H_p(\omega). \quad (3.60)$$

In the following, let us consider the adiabatic limit $\kappa \gg \omega$ for simplicity. In this case, the mechanical frequency is shifted to ω_m defined in Eqs. (3.19) due to the optical spring effects, and the mechanical dissipation rate is replaced by $\Gamma_{\text{fb}} = \Gamma + \gamma_m$ due to feedback control. We can then obtain the mechanical operators as

$$\delta \hat{Q}(\omega) = \chi_{\text{eff}}(\omega) \left(\sqrt{2\Gamma_{\text{fb}}} \hat{P}^{\text{eff}}(\omega) - \frac{2\hbar\alpha\kappa\sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{x}^{\text{in}}(\omega) + \frac{4\hbar\alpha\Delta\sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}^{\text{in}}(\omega) \right) \quad (3.61)$$

$$\delta \hat{P}(\omega) = -im\omega \delta \hat{Q}(\omega), \quad (3.62)$$

where $\chi_{\text{eff}}(\omega) = (m(\omega_m^2 - i\Gamma_{\text{fb}}\omega - \omega^2))^{-1}$. The output field of the amplitude and phase quadratures are

$$\hat{x}^{\text{out}} = -\frac{8\alpha\Delta\sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \delta \hat{Q} - \frac{\kappa^2 - 4\Delta^2}{\kappa^2 + 4\Delta^2} \hat{x}^{\text{in}} + \frac{4\kappa\Delta}{\kappa^2 + 4\Delta^2} \hat{y}^{\text{in}} \quad (3.63)$$

$$\hat{y}^{\text{out}} = \frac{4\alpha\kappa\sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \delta \hat{Q} - \frac{4\kappa\Delta}{\kappa^2 + 4\Delta^2} \hat{x}^{\text{in}} - \frac{\kappa^2 - 4\Delta^2}{\kappa^2 + 4\Delta^2} \hat{y}^{\text{in}}. \quad (3.64)$$

We can obtain the filter function from these expressions by computing the spectral density and its causal and non-causal parts. We show the details of the computation of the Wiener filtering in Appendix A, and we can obtain the conditional variance and covariance of the mechanical mode for the stationary state. Let us consider the measurement of either phase field quadrature $\theta = \pi/2$ or amplitude field quadrature $\theta = 0$. Then, we can derive the

simple form of the conditional variance and covariance as

$$\mathcal{V}_{\delta Q \delta Q}^{\text{wf}} = \frac{\hbar}{2m\omega_m} \frac{\gamma_I - \Gamma_{\text{fb}}}{\lambda_I} \quad (3.65)$$

$$\mathcal{V}_{\delta P \delta P}^{\text{wf}} = \frac{m\hbar\omega_m}{2} \frac{(\gamma_I - \Gamma_{\text{fb}})(2\omega_m(\omega_m + \Lambda_I) + \gamma_I^2 - \Gamma_{\text{fb}}\gamma_I)}{2\lambda_I\omega_m^2} \quad (3.66)$$

$$\mathcal{V}_{\delta Q \delta P}^{\text{wf}} = \frac{\hbar}{2} \frac{(\gamma_I - \Gamma_{\text{fb}})^2}{2\lambda_I\omega_m}, \quad (3.67)$$

where

$$\lambda_X = \frac{64g_m^2\kappa\Delta^2\eta}{(2\eta N_{\text{th}} + 1)(\kappa^2 + 4\Delta^2)^2}, \quad \lambda_Y = \frac{16g_m^2\kappa^3\eta}{(2\eta N_{\text{th}} + 1)(\kappa^2 + 4\Delta^2)^2}, \quad (3.68)$$

$$\Lambda_X = -\Lambda_Y = -\frac{32g_m^2\kappa^2\Delta(2N_{\text{th}} + 1)\eta}{(\kappa^2 + 4\Delta^2)^2(2\eta N_{\text{th}} + 1)}, \quad (3.69)$$

$$\gamma_I = \sqrt{\Gamma_{\text{fb}}^2 - 2\omega_m(\omega_m + \Lambda_I) + 2\omega_m\sqrt{\omega_m(\omega_m + 2\Lambda_I) + \bar{n}\lambda_I}}, \quad (3.70)$$

$$\bar{n} = 2\Gamma_{\text{fb}}(2n_{\text{thfb}} + 1) + \frac{16g_m^2\kappa}{\kappa^2 + 4\Delta^2}(2N_{\text{th}} + 1), \quad (3.71)$$

and $g_m = g\sqrt{\Omega_m/\omega_m} = \alpha\sqrt{\hbar/2m\omega_m}$ is the optomechanical coupling. Here, we have $I = Y$ for the phase field measurement ($\theta = \pi/2$), while $I = X$ corresponds to the amplitude field measurement ($\theta = 0$).

Using quantum filtering, the optimal estimated values are obtained based on the measurement of the mechanical position, which causes conditional mechanical squeezing [57, 91]. We here define the squeezing level in dB as

$$\mathcal{S} = -10 \log_{10} \frac{V_{\min}}{V_{\text{vac}}}, \quad (3.72)$$

where $V_{\min}$ is the minimum eigenvalue of the conditional covariance matrix

$$V_{\min} = \frac{V_{\delta q \delta q}^{\text{wf}} + V_{\delta p \delta p}^{\text{wf}}}{2} - \sqrt{\frac{(V_{\delta q \delta q}^{\text{wf}} - V_{\delta p \delta p}^{\text{wf}})^2}{4} + (V_{\delta q \delta p}^{\text{wf}})^2}. \quad (3.73)$$

We here use the dimensionless mechanical operators

$$\hat{q} = \sqrt{\frac{2m\omega_m}{\hbar}} \hat{Q}, \quad \hat{p} = \sqrt{\frac{2}{m\hbar\omega_m}} \hat{P}, \quad (3.74)$$

and its variances and covariance

$$V_{\delta q \delta q}^{\text{wf}} = \frac{2m\omega_m}{\hbar} \mathcal{V}_{\delta Q \delta Q}^{\text{wf}}, \quad V_{\delta p \delta p}^{\text{wf}} = \frac{2}{m\hbar\omega_m} \mathcal{V}_{\delta P \delta P}^{\text{wf}}, \quad \mathcal{V}_{\delta q \delta p} = \frac{2}{\hbar} V_{\delta Q \delta P}. \quad (3.75)$$

$V_{\text{vac}} = 1$ is the eigenvalue of the vacuum state. The quantum squeezing beyond the vacuum fluctuation appears for $\mathcal{S} > 0$.

Here we assume the structural damping $\Gamma(\omega_m) = \Gamma(\Omega)\Omega/\omega_m$, introduced in Eq. (3.24). Then, the thermal phonon number is approximately given by

$$n_{\text{thfb}} = \frac{k_B T \Gamma(\Omega) \Omega}{\hbar \gamma_m \omega_m^2}. \quad (3.76)$$

Figure 7 shows the behavior of the purity and the squeezing level using the parameters in Table 1, some have already been achieved [55–57], whereas others are conservative parameters expected in the mid-term future. The red curve shows the results using the Wiener filtering based on phase field measurement, while the blue curve corresponds to the amplitude field measurement with the detuned cavity. We also show a case without the Wiener filtering with the purple dotted line. Without the quantum filter, the mechanical mirror is in a thermal state of minimal purity, indicating no squeezing. Using the quantum filter, a quantum state close to the pure state can be achieved by lowering the noise. Figure 8 shows the Wigner ellipse of the mechanical state conditioned on the measurement of the phase field quadrature (red curve) and the amplitude field quadrature (blue curve), compared to that of the vacuum state. We here use the dissipation rate $\Gamma/2\pi = 10^{-6}$ Hz and room temperature $T = 300$ K. It can be seen that the quantum squeezing is realized in both cases.

3.3.2 Quantum Kalman filtering

Next, we consider Kalman filtering, the optimal estimation method in the time domain. With the quantum Kalman filter, we can track the behavior of $(\delta\hat{Q}, \delta\hat{P})$ conditioned on the measurement result, and its fluctuation is represented by the conditional covariance matrix following the Riccati equation [97]. Notably, the covariance matrix without the filter is always larger than the covariance matrix conditioned on the measurement. On the other hand, without the quantum filter, we only have the average behavior of $(\delta\hat{Q}, \delta\hat{P})$. Then, the quantum squeezing does not appear, as shown in Figure 7. This is essential for the entanglement between mechanical mirrors in the next Chapter.

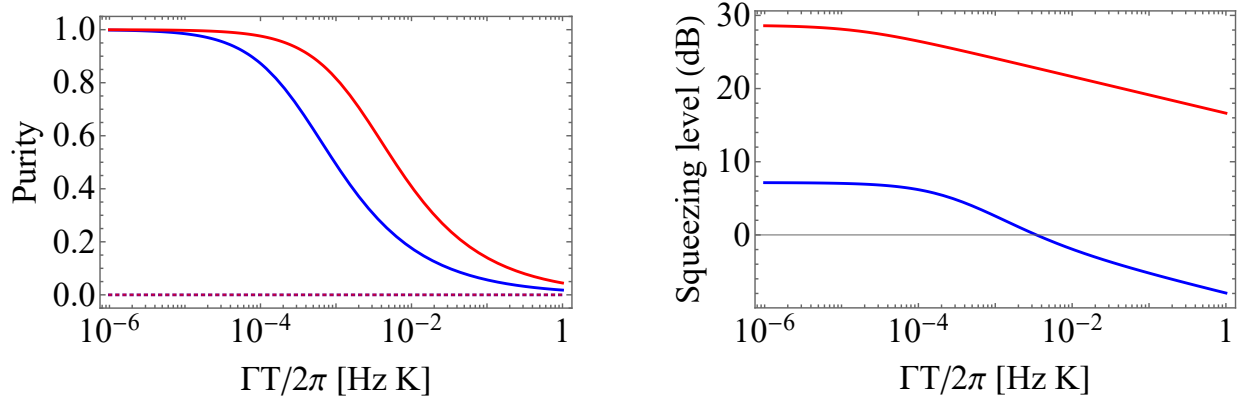


FIG 7: The behavior of the purity (left panel) and the squeezing level (right panel) of the conditional mechanical state using the parameters in Table 1 except for T and Γ . We use the Wiener filtering to measure the phase field (red curves) and the amplitude field (blue curves). The dotted curve close to 0 represents the case without the Wiener filter.

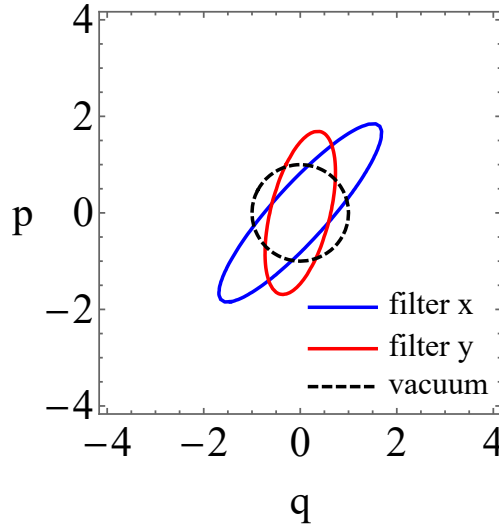


FIG 8: The Wigner ellipses of the mechanical conditional state are based on the measurement of the phase quadrature (red curve) and the amplitude quadrature (blue curve) using the parameters in Table 1. The black dashed curve describes the phase distribution of the vacuum state. The conditional state is quantum squeezed for both cases.

Name	Symbol	Value	Reference
Mechanical frequency	$\Omega/2\pi$	2.2 Hz	[55–57]
Mechanical decay rate	$\Gamma(\Omega)/2\pi$	10^{-6} Hz	[55–57]
Bath temperature	T	300 K	[55–57]
Mirror mass	m	7.71 mg	[55–57]
Cavity length	ℓ	10 cm	[55–57]
Laser wavelength	$\omega_L (\simeq \omega_c)/2\pi$	3×10^{14} Hz	[55–57]
Optical decay rate	$\kappa/2\pi$	1.64×10^6 Hz	[55–57]
Input laser power	P_{in}	30 mW	[55–57]
Detection efficiency	η	0.92	[55–57]
Effective mechanical decay rate	$\gamma_m/2\pi$	6.9×10^{-3} Hz	
Thermal photon number	N_{th}	0	
Detuning	Δ	0.2κ	

Table 1: Parameters employed in Ref. [55–57] and expected from the state-of-art technique.

We here consider the adiabatic limit, and the optical quadratures are given by Eqs. (3.17) and (3.18). Then, the Langevin equation of the mechanical motion is

$$\delta\dot{\hat{q}} = \omega_m \delta\hat{p} \quad (3.77)$$

$$\delta\dot{\hat{p}} = -\omega_m \delta\hat{q} - \Gamma_{fb} \hat{p} + \sqrt{2\Gamma_{fb}} \hat{p}^{\text{eff}} - \frac{4g_m \kappa \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{x}^{\text{in}} + \frac{8g_m \Delta \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}^{\text{in}}, \quad (3.78)$$

where we use the dimensionless operators defined in Eq. (3.74). We rewrite the Langevin equation in matrix form as

$$\dot{\hat{\mathbf{r}}} = \mathbf{A} \hat{\mathbf{r}} + \begin{pmatrix} 0 \\ \hat{w} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \omega_m \\ \omega_m & -\Gamma_{fb} \end{pmatrix} \quad (3.79)$$

$$\hat{w} = \sqrt{2\Gamma_{fb}} \hat{p}^{\text{eff}} - \frac{4g_m \kappa \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{x}^{\text{in}} + \frac{8g_m \Delta \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}^{\text{in}}, \quad (3.80)$$

where we define the vector $\hat{\mathbf{r}} = (\delta\hat{q}, \delta\hat{p})^T$. The general output field is also rewritten as

$$\hat{b} = \mathbf{C} \hat{\mathbf{r}} + \hat{v}, \quad \mathbf{C} = \begin{pmatrix} \frac{8g_m \sqrt{\kappa\eta}}{\kappa^2 + 4\Delta^2} (-2\Delta \cos \theta + \kappa \sin \theta) & 0 \end{pmatrix}, \quad (3.81)$$

$$\hat{v} = \frac{-(\kappa^2 - 4\Delta^2) \cos \theta - 4\kappa \Delta \sin \theta}{\kappa^2 + 4\Delta^2} \sqrt{\eta} \hat{x}^{\text{in}} + \frac{4\kappa \Delta \cos \theta - (\kappa^2 - 4\Delta^2) \sin \theta}{\kappa^2 + 4\Delta^2} \sqrt{\eta} \hat{y}^{\text{in}} + \sqrt{1 - \eta} \hat{b}^{\text{add}} \quad (3.82)$$

For the Kalman filter [97, 99], we obtained the time evolution of the optimized covariance

matrix $\mathbf{V}^{\text{kf}} = \langle \{\hat{\mathbf{r}} - \mathbf{r}_{\text{est}}, (\hat{\mathbf{r}} - \mathbf{r}_{\text{est}})^{\text{T}}\} \rangle / 2$ as the following Riccati equation:

$$\dot{\mathbf{V}}^{\text{kf}} = \mathbf{A}\mathbf{V}^{\text{kf}} + \mathbf{V}^{\text{kf}}\mathbf{A}^{\text{T}} + \mathbf{N} - (\mathbf{V}^{\text{kf}}\mathbf{C}^{\text{T}} + \mathbf{L})M^{-1}(\mathbf{V}^{\text{kf}}\mathbf{C}^{\text{T}} + \mathbf{L})^{\text{T}} \quad (3.83)$$

where $M = \langle \hat{v}^2 \rangle = 2\eta N_{\text{th}} + 1$ and each matrix is

$$\mathbf{V}^{\text{kf}} = \begin{pmatrix} V_{\delta q \delta p}^{\text{kf}} & V_{\delta q \delta q}^{\text{kf}} \\ V_{\delta p \delta p}^{\text{kf}} & V_{\delta p \delta q}^{\text{kf}} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & \langle \hat{w}^2 \rangle \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 \\ \langle \hat{w} \hat{v} \rangle \end{pmatrix}, \quad (3.84)$$

with

$$\bar{n} \equiv \langle \hat{w}^2 \rangle = 2\Gamma_{\text{fb}}(2n_{\text{thfb}} + 1) + \frac{16g_m^2\kappa}{\kappa^2 + 4\Delta^2}(2N_{\text{th}} + 1), \quad (3.85)$$

$$\bar{\ell} \equiv \langle \hat{w} \hat{v} \rangle = \frac{4g_m(\kappa \cos \theta + 2\Delta \sin \theta)\sqrt{\kappa\eta}(2N_{\text{th}} + 1)}{\kappa^2 + 4\Delta^2}. \quad (3.86)$$

Here, we used the estimated value $\mathbf{r}_{\text{est}} = (\delta q_{\text{est}}, \delta p_{\text{est}})^{\text{T}}$ to optimized the covariance matrix conditioned on the measurement results, which follows

$$\dot{\mathbf{r}}_{\text{est}} = \mathbf{A}\mathbf{r}_{\text{est}} + (\mathbf{V}\mathbf{C}^{\text{T}} + \mathbf{L})M^{-1}(\hat{\mathbf{b}} - \mathbf{C}\mathbf{r}_{\text{est}}). \quad (3.87)$$

In the absence of the quantum filter, the covariance matrix follows the Lyapunov equation, $\dot{\mathbf{V}} = \mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}^{\text{T}} + \mathbf{N}$, whose components are always larger than those with the filter. As discussed in Chapter 2, the covariance matrix without the quantum filter is derived in Eq. (2.29). For $t \rightarrow \infty$, the mechanical state evolves into the thermal state with $V_{\delta q \delta q} = \bar{n}/2\Gamma_{\text{fb}}$, $V_{\delta p \delta p} = \bar{n}/2\Gamma_{\text{fb}}$, and $V_{\delta q \delta p} = 0$. Hence, we need the quantum Kalman filter to suppress the thermal fluctuation.

Here, we consider the conditional covariance matrix based on the measurement of either phase ($\theta = \pi/2$) or amplitude ($\theta = 0$). Considering the steady state $\dot{\mathbf{V}}^{\text{kf}} = 0$, the components of the conditional covariance matrix satisfy the following equations:

$$2\omega_m V_{\delta q \delta p}^{\text{kf}} - \lambda_I (V_{\delta q \delta p}^{\text{kf}})^2 = 0 \quad (3.88)$$

$$(\Gamma_{\text{fb}} + \lambda_I V_{\delta q \delta q}^{\text{kf}}) V_{\delta q \delta p}^{\text{kf}} + (V_{\delta q \delta q}^{\text{kf}} - V_{\delta p \delta p}^{\text{kf}}) \omega_m + \Lambda_I V_{\delta q \delta q}^{\text{kf}} = 0 \quad (3.89)$$

$$2\Gamma_{\text{fb}} V_{\delta p \delta p}^{\text{kf}} + 2\omega_m V_{\delta q \delta p}^{\text{kf}} + (\sqrt{\lambda_I} V_{\delta q \delta p}^{\text{kf}} + \Lambda_I / \sqrt{\lambda_I})^2 - \bar{n} = 0, \quad (3.90)$$

where λ_I , Λ_I , and $\bar{n}$ are defined in previous subsection. Solving these equations, we can derive the same steady-state solution as the case with the Wiener filtering, $\mathbf{V}^{\text{wf}} = \mathbf{V}^{\text{kf}}$. In the following chapters, we denote the conditional variance with the quantum Kalman filter, which can treat both time evolution and stationary solutions, denote simply as $\mathbf{V}$.

We demonstrate the time evolution of all components of the covariance matrix conditioned on the amplitude field measurement ($\theta = 0$), $V_{\delta q \delta q}^{\text{kf}}$ (black solid curve), $V_{\delta q \delta p}^{\text{kf}}$ (orange dashed curve), and $V_{\delta p \delta p}^{\text{kf}}$ (purple dashed-dotted curve) in Fig. 9. We use the parameters in Table 1. We assume that the initial state is the steady state without the filtering, which is the solution of the Lyapunov equation $\mathbf{V}^{\text{kf}}(0) = \mathbf{V}$. Using a quantum Kalman filter, the conditional variance of the mechanical position and momentum can be minimal, which means that fluctuations in phase space are small and close to the quantum regime. The time scale to reach the steady state is approximately obtained by $4/\gamma_X \sim 7 \times 10^{-4}$ s.

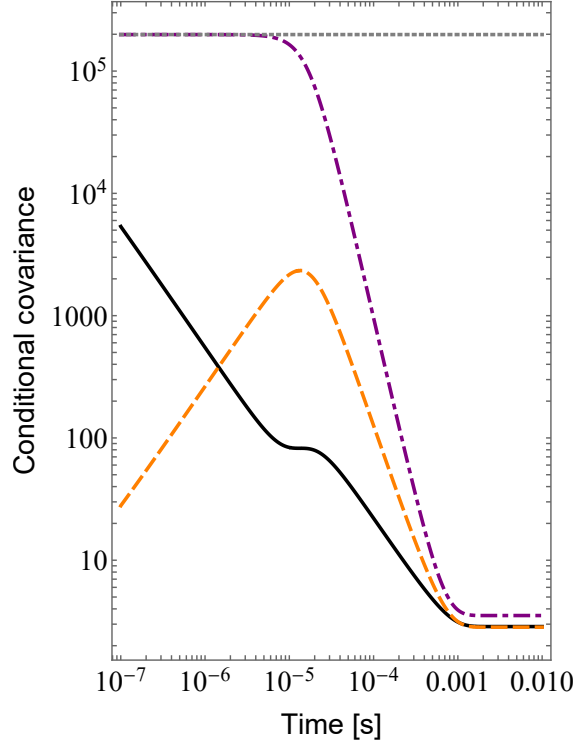


FIG 9: Time evolution of the dimensionless conditional variance and covariance, $V_{\delta q \delta q}^{\text{kf}}$ (black solid curve), $V_{\delta q \delta p}^{\text{kf}}$ (orange dashed curve), and $V_{\delta p \delta p}^{\text{kf}}$ (purple dashed-dotted curve). The black dotted curve describes the initial thermal state, $V_{\delta q \delta q} = V_{\delta p \delta p} = \bar{n}/2\Gamma_{\text{fb}}$ and $V_{\delta q \delta p} = 0$. We use the parameters in Table 1.

3.3.3 Retrodiction

We cannot experimentally access unconditional states. On the other hand, a technique called retrodiction has been proposed as a way to experimentally validate the conditional covariance matrix obtained by quantum filtering [57, 100, 101]. In the quantum filtering described in the previous sections, the state at time t_0 has been predicted from measurements up to time $t < t_0$, but retrodiction is an estimation in the opposite direction, estimating the state at t_0 from measurements at $t' > t_0$.

We here review the references [57, 100, 101]. The estimated value for the retrodiction is obtained by

$$\dot{\overleftarrow{\mathbf{r}}}_{\text{est}} = -\mathbf{A}\overleftarrow{\mathbf{r}}_{\text{est}} + (\mathbf{V}_E\mathbf{C}^T - \mathbf{L})M^{-1}(\hat{b} - \mathbf{L}\overleftarrow{\mathbf{r}}), \quad (3.91)$$

which is a time derivative given in Eq. (3.87) in the reverse direction. We here used the retrodictive covariance matrix $\mathbf{V}_E = \langle \{\hat{\mathbf{r}} - \overleftarrow{\mathbf{r}}_{\text{est}}, (\hat{\mathbf{r}} - \overleftarrow{\mathbf{r}}_{\text{est}})^T\} \rangle$. Then, the retrodictive Riccati equation is expressed by

$$\dot{\mathbf{V}}_E = -\mathbf{A}\mathbf{V}_E - \mathbf{V}_E\mathbf{A}^T + \mathbf{N} - (\mathbf{V}_E\mathbf{C}^T - \mathbf{L})M^{-1}(\mathbf{V}_E\mathbf{C}^T - \mathbf{L})^T. \quad (3.92)$$

Here, we consider the measurement of either phase quadrature or amplitude quadrature. Solving this equation for the steady-state, we have

$$V_{E\delta q\delta q} = \frac{\gamma_I + \Gamma_{\text{fb}}}{\lambda_I} \quad (3.93)$$

$$V_{E\delta p\delta p} = \frac{V_{E\delta q\delta q}}{2\omega_m^2} (2\omega_m(\omega_m + \Lambda_I) + \gamma_I V_{E\delta q\delta q} \lambda_I) \quad (3.94)$$

$$V_{E\delta q\delta p} = -\frac{V_{E\delta q\delta q}^2 \lambda_I}{2\omega_m}, \quad (3.95)$$

where $I = X$ or Y . Under the assumption $\lambda_I \gg \Gamma_{\text{fb}}$, we can derive $V_{E\delta q\delta q} \simeq V_{\delta q\delta q}$,

$V_{E\delta q\delta p} \simeq -V_{\delta q\delta p}$, and $V_{E\delta p\delta p} \simeq V_{\delta p\delta p}$. This expression is also written in the Fourier space as

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (|H_q(\omega)|^2 - |\overleftarrow{H}_q(\omega)|^2) S_{II} \simeq 2V_{\delta q\delta q} \quad (3.96)$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (|H_p(\omega)|^2 - |\overleftarrow{H}_p(\omega)|^2) S_{II} \simeq 2V_{\delta p\delta p} \quad (3.97)$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (H_q(\omega)H_p^*(\omega) - \overleftarrow{H}_q(\omega)\overleftarrow{H}_p^*(\omega)) S_{II} \simeq 0, \quad (3.98)$$

where $\overleftarrow{H}_q(\omega)$ and $\overleftarrow{H}_p(\omega)$ are the retrodictive filter function with $\overleftarrow{q}_{\text{est}} = \overleftarrow{H}_q(\omega)\hat{I}^\dagger(\omega)$ and $\overleftarrow{p}_{\text{est}} = \overleftarrow{H}_p(\omega)\hat{I}^\dagger(\omega)$. Hence, we can derive the conditional covariance matrix from the filter functions for $\lambda_I \gg \Gamma_{\text{fb}}$.

3.4 Conclusions

We described the dynamics and quantum control of optomechanical systems. The optomechanical interaction arises from the displacement of the mirror due to the radiation pressure of the cavity light. We also explained that the frequency and dissipation of the mechanical mirror can be shifted by using the cavity detuning. This is an effect that helps to increase the effective q factor. Furthermore, the displacement of the mirror can be estimated by measuring the light leaking out of the cavity. Then, we reviewed the quantum feedback control and the quantum filtering to reduce the noises based on the measurements. Combining these results, we showed that the milligram-scale macroscopic quantum state of the mechanical mirror is feasible in the midterm future. In the next Chapter, we discuss how these quantum controls can be used to create quantum entanglement between milligram-scale mirrors. This could be a fundamental theory of quantum control toward the quantum nature of gravity.

4 Macroscopic entanglement

In this section, we investigate the feasibility of generating Gaussian entanglement between macroscopic mirrors in optomechanical systems under feedback control and quantum filtering. This study builds upon the results of Ref. [2], consisting of a power-recycled Fabry-Pérot-Michelson interferometer. In this configuration, the common mode of the output light is partially reflected by the power-recycling mirror (PRM) and subsequently reinjected into the interferometer. This reinjection introduces an asymmetry between the common and differential modes of the output optical fields, leading to entanglement between the outputs from cavities A and B. Such entanglement is analogous to the well-known entanglement between two squeezed beams passing through a half-beam splitter in quantum optics (e.g., Ref. [69]). The mechanical asymmetry of mirrors A and B is also generated due to the linear coupling between the mechanical mirror positions and the optical field quadratures. This asymmetry facilitates entanglement generation between the two mechanical mirrors under quantum control.

In earlier work, the authors of Ref. [64] analyzed entanglement conditioned on measuring the optical phase quadrature. However, their study treated the mechanical mirrors as free-mass objects, neglecting the effects of mechanical resonances. The mechanical resonances are critical for the generation of entanglement [24, 67]. In the present work, we extend the analysis beyond the free-mass approximation to include the effects of mechanical resonance frequencies explicitly. We consider a more general measurement scheme that includes the phase quadrature and the amplitude quadrature. While phase quadrature measurements have been the primary focus in previous studies, amplitude field measurements do not require high power, which might be helpful for experimental verification. Our analysis shows that entanglement can still be achieved even when employing amplitude quadrature measurements.

4.1 Dynamics of the interferometer

We consider the power-recycled Fabry-Pérot-Michelson interferometer, as shown in Fig. 10. The light incident from the laser is split by a beam splitter and injected into two cavities. Each cavity light interacts optomechanically with the mechanical end mirror, and the output light interferes again with the beamsplitters. We assume, as in the previous Chapter, that the movable end mirror is perfectly reflective and that the dissipation of the cavity light

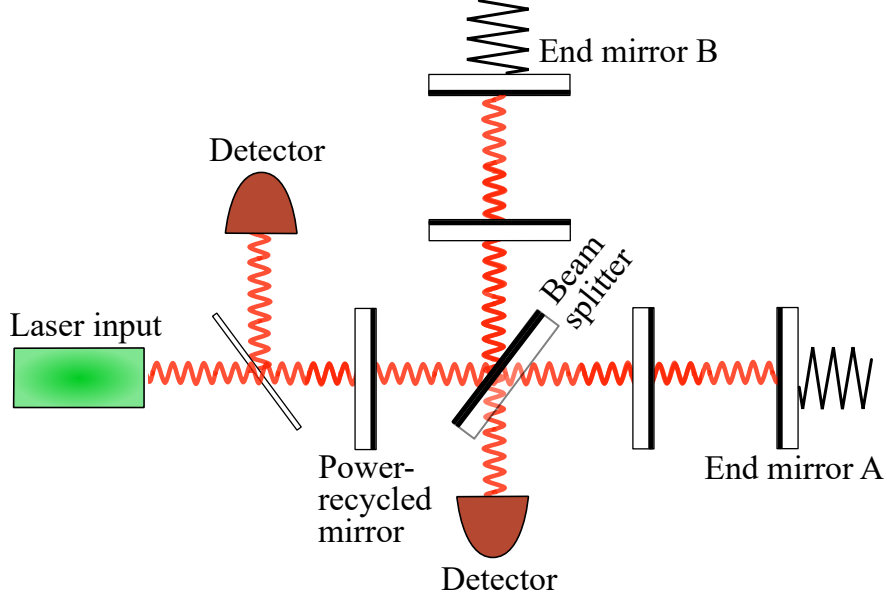


FIG 10: Schematic plot of power-recycled Fabry-Pérot-Michelson interferometer. The input laser beam is split by a beam splitter into two cavities and interacts with mechanical mirrors.

occurs at the other cavity mirror. The output light of the differential mode is detected straight through, but the power-recycled mirror partially reflects the light of the common mode. This asymmetry between the optical common and differential modes is a factor in the generation of the quantum entanglement.

We first consider the Fabry-Pérot-Michelson interferometer without the power-recycled mirror. The Hamiltonian is

$$\begin{aligned} \hat{H} = & \frac{1}{2m}(\hat{P}_A^2 + \hat{P}_B^2) + \frac{1}{2}m\Omega^2(\hat{Q}_A^2 + \hat{Q}_B^2) + \frac{\hbar\omega_c}{1 - \hat{Q}_A/\ell}\hat{a}_A^\dagger\hat{a}_A + \frac{\hbar\omega_c}{1 - \hat{Q}_B/\ell}\hat{a}_B^\dagger\hat{a}_B \\ & + i\hbar E\left(\frac{\hat{a}_A^\dagger + \hat{a}_B^\dagger}{2}e^{-i\omega_L t} - \frac{\hat{a}_A + \hat{a}_B}{2}e^{i\omega_L t}\right), \end{aligned} \quad (4.1)$$

and we henceforth assume the symmetrical setup and that $\bar{Q}_{A/B}/\ell$ is small, yielding $\xi = 1$ and $\Omega_m = \Omega$ for simplicity. Using linearization, we can derive the equations of motion of mechanical mirrors and cavity lights like the previous section. Introducing the common and differential modes $\delta\hat{Q}_\pm = (\delta\hat{Q}_A \pm \delta\hat{Q}_B)/\sqrt{2}$, $\delta\hat{P}_\pm = (\delta\hat{P}_A \pm \delta\hat{P}_B)/\sqrt{2}$, and $\delta\hat{a}_\pm =$

$(\delta\hat{a}_A \pm \delta\hat{a}_B)/\sqrt{2}$, the equations of the mechanical and optical modes are

$$\delta\dot{\hat{Q}}_{\pm} = \frac{\hat{P}_{\pm}}{m} \quad (4.2)$$

$$\delta\dot{\hat{P}}_{\pm} = -m\Omega^2\delta\hat{Q}_{\pm} - \Gamma_{\text{fb}}\delta\hat{P}_{\pm} - \hbar\alpha\delta\hat{x}_{\pm} + \sqrt{2\Gamma_{\text{fb}}}\hat{P}_{\pm}^{\text{th}} - \int_{-\infty}^t dt' f_{\text{fb}}(t-t')\hat{I}_{\pm}(t') \quad (4.3)$$

$$\delta\dot{\hat{x}}_{\pm} = -\Delta\delta\hat{y}_{\pm} - \frac{\kappa}{2}\delta\hat{x}_{\pm} + \sqrt{\kappa}\hat{x}_{\pm}^{\text{in}} \quad (4.4)$$

$$\delta\dot{\hat{y}}_{\pm} = -2\alpha\delta\hat{Q}_{\pm} + \Delta\delta\hat{x}_{\pm} - \frac{\kappa}{2}\delta\hat{y}_{\pm} + \sqrt{\kappa}\hat{y}_{\pm}^{\text{in}}, \quad (4.5)$$

where $\alpha = \omega_c|\bar{a}_A|/\ell$. The classical stationary values are

$$\bar{Q}_A = -\frac{\hbar\omega_c}{m\Omega^2\ell}|\bar{a}_A|^2 \quad (4.6)$$

$$\bar{P}_A = 0 \quad (4.7)$$

$$\bar{a}_A = \frac{\sqrt{2}E}{\kappa - 2i\Delta}, \quad (4.8)$$

and $(\bar{Q}_B, \bar{P}_B, \bar{a}_B) = (\bar{Q}_A, \bar{P}_A, \bar{a}_A)$, where $E = \sqrt{P_{\text{in}}\kappa/\hbar\omega_L}$ and $\Delta = \omega_L - \omega_c(1 + \bar{Q}_A/\ell)$. Without the power-recycled mirror, the common mode and differential mode follow the same dynamics. Hence, no asymmetry of the optical common and differential modes appear, and cavity A's light is decoupled with cavity B's light.

4.1.1 Asymmetry due to power-recycled mirror

Next, we consider the effects of the power-recycled mirror. The power-recycled mirror creates an asymmetry between the common and differential modes of the lights. The power-recycled mirror partially reflects the optical common mode. Therefore, the optical decay rate of the common mode κ_+ is smaller than that of the differential mode $\kappa_- = \kappa$. Considering the input-output relation of the optical common mode and differential mode, we introduce the parameter $\zeta \geq 1$ to describe the asymmetry using

$$\kappa_+ = \frac{1}{\zeta}\kappa_-. \quad (4.9)$$

Therefore, the optical decay rate ratio ζ describes the asymmetry between optical common and differential modes, which causes entanglement owing to the half-beam splitter. This entanglement is due to the beamsplitter, which is well-known in quantum optics.

Considering the effect of a power-recycled mirror, the equations of motion (4.4) and (4.5) for the common mode and differential mode are rewritten as follows:

$$\dot{\delta\hat{x}}_{\pm} = -\Delta\delta\hat{y}_{\pm} - \frac{\kappa_{\pm}}{2}\delta\hat{x}_{\pm} + \sqrt{\kappa_{\pm}}\hat{x}_{\pm}^{\text{in}} \quad (4.10)$$

$$\dot{\delta\hat{y}}_{\pm} = -2\alpha\delta\hat{Q}_{\pm} + \Delta\delta\hat{x}_{\pm} - \frac{\kappa_{\pm}}{2}\delta\hat{y}_{\pm} + \sqrt{\kappa_{\pm}}\hat{y}_{\pm}^{\text{in}}, \quad (4.11)$$

where we replace the optical decay rate of common mode as $\kappa \rightarrow \kappa_+$. The classical stationary values are also rewritten as

$$\bar{Q}_A = -\frac{\hbar\omega_c}{m\Omega^2\ell}|\bar{a}_A|^2 \quad (4.12)$$

$$\bar{P}_A = 0 \quad (4.13)$$

$$\bar{a}_A = \frac{\sqrt{2}E}{\kappa_+ - 2i\Delta}, \quad (4.14)$$

and $(\bar{Q}_B, \bar{P}_B, \bar{a}_B) = (\bar{Q}_A, \bar{P}_A, \bar{a}_A)$, where $E = \sqrt{P_{\text{in}}\kappa_+/\hbar\omega_L}$ and $\Delta = \omega_L - \omega_c(1 + \bar{Q}_A/\ell)$.

4.1.2 Quantum filtering

In this Chapter, we consider the adiabatic limit $\kappa_{\pm} \gg \Omega$ for simplicity. Then, the optical amplitude and phase field quadratures are

$$\delta\hat{x}_{\pm} = \frac{8\alpha\Delta}{\kappa_{\pm}^2 + 4\Delta^2}\delta\hat{Q}_{\pm} + \frac{2\kappa_{\pm}\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{x}_{\pm}^{\text{in}} - \frac{4\Delta\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{y}_{\pm}^{\text{in}} \quad (4.15)$$

$$\delta\hat{y}_{\pm} = -\frac{4\alpha\kappa_{\pm}}{\kappa_{\pm}^2 + 4\Delta^2}\delta\hat{Q}_{\pm} + \frac{4\Delta\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{x}_{\pm}^{\text{in}} + \frac{2\kappa_{\pm}\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{y}_{\pm}^{\text{in}}. \quad (4.16)$$

Introducing the dimensionless operators as

$$\hat{q}_{\pm} = \sqrt{\frac{2m\omega_m^{\pm}}{\hbar}}\hat{Q}_{\pm}, \quad \hat{p}_{\pm} = \sqrt{\frac{2}{m\hbar\omega_m^{\pm}}}\hat{P}_{\pm}, \quad (4.17)$$

we have the Langevin equation as

$$\dot{\delta\hat{q}}_{\pm} = \omega_m^{\pm}\delta\hat{p}_{\pm} \quad (4.18)$$

$$\dot{\delta\hat{p}}_{\pm} = -\omega_m^{\pm}\delta\hat{q}_{\pm} - \Gamma_{\text{fb}}\hat{p}_{\pm} + \sqrt{2\Gamma_{\text{fb}}}\hat{p}_{\pm}^{\text{eff}} - \frac{4g_m^{\pm}\kappa_{\pm}\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{x}_{\pm}^{\text{in}} + \frac{8g_m^{\pm}\Delta\sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2}\hat{y}_{\pm}^{\text{in}}, \quad (4.19)$$

where $\omega_m^\pm$ is the shifted frequency by optical spring effects and $g_m^\pm$ is the optomechaical coupling defined as

$$\omega_m^\pm = \sqrt{\Omega^2 + \Omega \frac{16g^2\Delta}{\kappa_\pm^2 + 4\Delta^2}}, \quad g_m^\pm = g\sqrt{\frac{\Omega}{\omega_m^\pm}} \quad (4.20)$$

where $g = \alpha\sqrt{\hbar/2m\Omega}$.

The general output field quadratures of the common and differential modes are

$$\hat{b}_\pm = \sqrt{\eta}\hat{b}_\pm^{\text{out}} + \sqrt{1-\eta}\hat{b}_\pm^{\text{add}} \quad (4.21)$$

$$\hat{b}_\pm^{\text{out}} = \hat{x}_\pm^{\text{out}} \cos \theta_\pm + \hat{y}_\pm^{\text{out}} \sin \theta_\pm, \quad (4.22)$$

where $\eta \in [0, 1]$ and $\langle (\hat{b}_\pm^{\text{add}})^2 \rangle = 1$. Here, we employ the quantum Kalman filter to reduce noises and increase entanglement; the results of the Wiener filter are consistent with the stationary solution of the Kalman filter, as described in the previous section. Since we assume the symmetrical setup, we can independently derive the Riccati equation for common and differential modes as

$$\dot{\mathbf{V}}_\pm = \mathbf{A}_\pm \mathbf{V}_\pm + \mathbf{V}_\pm \mathbf{A}_\pm^T + \mathbf{N}_\pm - (\mathbf{V}_\pm \mathbf{C}_\pm^T + \mathbf{L}_\pm) M^{-1} (\mathbf{V}_\pm \mathbf{C}_\pm^T + \mathbf{L}_\pm)^T, \quad (4.23)$$

and each matrix is

$$\begin{aligned} \mathbf{V}_\pm &= \begin{pmatrix} V_{\delta q \delta q}^\pm & V_{\delta q \delta p}^\pm \\ V_{\delta q \delta p}^\pm & V_{\delta p \delta p}^\pm \end{pmatrix}, \quad \mathbf{A}_\pm = \begin{pmatrix} 0 & \omega_m^\pm \\ \omega_m^\pm & -\Gamma_{\text{fb}} \end{pmatrix}, \quad \mathbf{L}_\pm = \begin{pmatrix} 0 \\ \bar{\ell}_\pm \end{pmatrix} \\ \mathbf{C} &= \begin{pmatrix} \frac{4g_m^\pm \sqrt{\kappa_\pm \eta}}{\kappa_\pm^2 + 4\Delta^2} (-2\Delta \cos \theta + \kappa_\pm \sin \theta) & 0 \end{pmatrix}, \quad \mathbf{N}_\pm = \begin{pmatrix} 0 & 0 \\ 0 & \bar{n}_\pm \end{pmatrix}, \end{aligned} \quad (4.24)$$

where $M = 2\eta N_{\text{th}} + 1$ and

$$\bar{n}_\pm = 2\Gamma_{\text{fb}}(2n_{\text{thfb}}^\pm + 1) + \frac{16(g_m^\pm)^2 \kappa_\pm}{\kappa_\pm^2 + 4\Delta^2} (2N_{\text{th}} + 1), \quad (4.25)$$

$$\bar{\ell}_\pm = \frac{4g_m^\pm (\kappa_\pm \cos \theta + 2\Delta \sin \theta) \sqrt{\kappa_\pm \eta} (2N_{\text{th}} + 1)}{\kappa_\pm^2 + 4\Delta^2}. \quad (4.26)$$

$n_{\text{thfb}}^\pm = k_B T \Gamma / \hbar \omega_m^\pm \Gamma_{\text{fb}}$ is the thermal phonon number. The distinction between common

and differential modes depends only on the difference in the optical dissipation rate $\kappa_{\pm}$ due to the power-recycled mirror. This difference causes the change of the effective mechanical frequency ω_m and the optomechanical coupling $g_m^{\pm}$.

To investigate the entanglement, we analyze the covariance matrix of the individual end mirrors A and B. Each mirror's canonical operator is obtained by

$$\begin{pmatrix} \delta Q_A \\ \delta P_A \\ \delta Q_B \\ \delta P_B \end{pmatrix} = \mathbf{S} \begin{pmatrix} \delta Q_+ \\ \delta P_+ \\ \delta Q_- \\ \delta P_- \end{pmatrix}, \quad \mathbf{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad (4.27)$$

where the matrix $\mathbf{S}$ describes the operation of the half-beam splitter. Then, we derive the dimensional covariance matrix of the individual end mirrors as

$$\mathbf{v} = \mathbf{S} \begin{pmatrix} \mathbf{v}_+ & 0 \\ 0 & \mathbf{v}_- \end{pmatrix} \mathbf{S} \equiv \begin{pmatrix} \mathbf{v}_A & \mathbf{v}_{AB} \\ \mathbf{v}_{AB} & \mathbf{v}_B \end{pmatrix}, \quad (4.28)$$

where $\mathbf{v}_{\pm}$ is the covariance matrix with the basis $(\delta\hat{Q}_{\pm}, \delta\hat{P}_{\pm})$ defined as:

$$\mathbf{v}_{\pm} = \begin{pmatrix} \frac{\hbar}{2m\omega_m^{\pm}} V_{\delta q \delta q}^{\pm} & \frac{\hbar}{2} V_{\delta q \delta p}^{\pm} \\ \frac{\hbar}{2} V_{\delta q \delta p}^{\pm} & \frac{m\hbar\omega_m^{\pm}}{2} V_{\delta p \delta p}^{\pm} \end{pmatrix}, \quad (4.29)$$

and $\mathbf{v}_A$, $\mathbf{v}_{AB}$, and $\mathbf{v}_B$ are 2×2 -component matrices. The logarithmic negativity for the dimensional covariance matrix with the basis $(\delta\hat{Q}_A, \delta\hat{P}_A, \delta\hat{Q}_B, \delta\hat{P}_B)$ is given by

$$E_N = \max \left\{ 0, -\log_2 \left(\frac{2}{\hbar} \sqrt{\frac{\Sigma - \sqrt{\Sigma^2 - 4\det\mathbf{v}}}{2}} \right) \right\}, \quad (4.30)$$

where $\Sigma = \det\mathbf{v}_A + \det\mathbf{v}_B - 2\det\mathbf{v}_{AB}$. For a two-mode Gaussian state, the system is only entangled if the logarithmic negativity is positive $E_N > 0$, as discussed in Chapter 1. We here define the critical value ϵ_{cr} as the second part of the brace in Eq. (4.30),

$$\epsilon_{\text{cr}} = -\log_2 \left(\frac{2}{\hbar} \sqrt{\frac{\Sigma - \sqrt{\Sigma^2 - 4\det\mathbf{v}}}{2}} \right) \quad (4.31)$$

and $\epsilon_{\text{cr}} > 0$ shows that the state is entangled.

4.2 Entanglement

4.2.1 Time evolution

We first consider the time evolution of the entanglement between two individual mechanical mirrors. Figure 11 shows the time evolution of the conditional variances and covariance based on the amplitude field measurement, $V_{\delta q \delta q}^+$ (black solid curve), $V_{\delta q \delta p}^+$ (orange dashed curve), and $V_{\delta p \delta p}^+$ (purple dashed-dotted curve). We use the future feasible parameters in Table 2 and assume the initial thermal state derived by the Lyapunov equation. The state reaches the steady state on a time scale of $t \simeq 4/\gamma_X^+ \sim 5 \times 10^{-4}$ s, where

$$\lambda_X^\pm = \frac{64(g_m^\pm)^2 \kappa_\pm \Delta^2 \eta}{(2\eta N_{\text{th}} + 1)(\kappa_\pm^2 + 4\Delta^2)^2}, \quad \lambda_Y^\pm = \frac{16(g_m^\pm)^2 \kappa_\pm^3 \eta}{(2\eta N_{\text{th}} + 1)(\kappa_\pm^2 + 4\Delta^2)^2}, \quad (4.32)$$

$$\Lambda_X^\pm = -\Lambda_Y^\pm = -\frac{32(g_m^\pm)^2 \kappa_\pm^2 \Delta (2N_{\text{th}} + 1) \eta}{(\kappa_\pm^2 + 4\Delta^2)^2 (2\eta N_{\text{th}} + 1)}, \quad (4.33)$$

$$\gamma_I^\pm = \sqrt{\Gamma_{\text{fb}}^2 - 2\omega_m^\pm(\omega_m^\pm + \Lambda_I^\pm) + 2\omega_m^\pm \sqrt{\omega_m^\pm(\omega_m^\pm + 2\Lambda_I^\pm) + \bar{n}^\pm \lambda_I^\pm}}, \quad (4.34)$$

$$\bar{n}_\pm = 2\Gamma_{\text{fb}}(2n_{\text{thfb}}^\pm + 1) + \frac{16(g_m^\pm)^2 \kappa_\pm}{\kappa_\pm^2 + 4\Delta^2} (2N_{\text{th}} + 1), \quad (4.35)$$

The entanglement between two mirrors also appears on the same time scale $t \simeq 4/\gamma_X^+$, as shown in Fig. 12. This time scale of reaching the steady state and generating the entanglement based on the measurement of the phase quadrature is also the same, $t \simeq 4/\gamma_Y^+$.

4.2.2 Steady-state solution

Next, we analyze the stationary entanglement in detail. Solving the Riccati equation for steady-state $\dot{\mathbf{V}} = 0$, we derive

$$V_{\delta q \delta q}^\pm = \frac{\gamma_I^\pm - \Gamma_{\text{fb}}}{\lambda_I^\pm} \quad (4.36)$$

$$V_{\delta p \delta p}^\pm = \frac{(\gamma_I^\pm - \Gamma_{\text{fb}})(2\omega_m^\pm(\omega_m^\pm + \Lambda_I^\pm) + (\gamma_I^\pm)^2 - \Gamma_{\text{fb}}\gamma_I^\pm)}{2\lambda_I^\pm(\omega_m^\pm)^2} \quad (4.37)$$

$$V_{\delta q \delta p}^\pm = \frac{(\gamma_I^\pm - \Gamma_{\text{fb}})^2}{2\lambda_I^\pm \omega_m^\pm}, \quad (4.38)$$

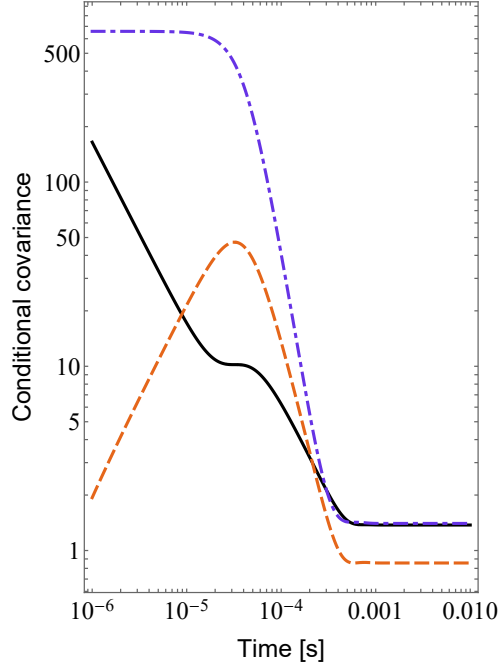


FIG 11: Behavior of the time evolution of the dimensionless conditional variances and covariance of the mechanical common mode, $V_{\delta q \delta q}^+$ (black solid curve), $V_{\delta q \delta p}^+$ (orange dashed curve), and $V_{\delta p \delta p}^+$. We use the parameters expected in the short term in Table 2.

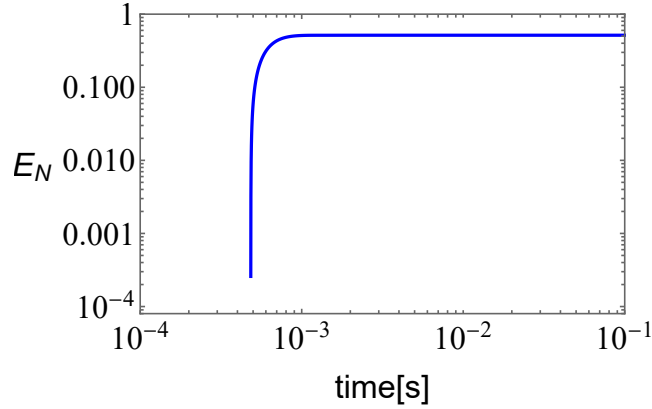


FIG 12: Behavior of the logarithmic negativity conditioned on the amplitude field measurement. We find that the time scale required for the entanglement to occur is the same as the time scale required for the conditional variance and covariance to reach steady-state solutions.

Name	Symbol	Value	Reference
Mechanical frequency	$\Omega/2\pi$	2.2 Hz	[55–57]
Mechanical decay rate	$\Gamma(\Omega)/2\pi$	10^{-6} Hz	[55–57]
Bath temperature	T	300 K	[55–57]
Mirror mass	m	7.71 mg	[55–57]
Cavity length	ℓ	10 cm	[55–57]
Laser wave length	$\omega_L(\simeq \omega_c)/2\pi$	3×10^{14} Hz	[55–57]
Optical decay rate	$\kappa_-/2\pi$	1.64×10^6 Hz	[55–57]
Input laser power	P_{in}	30 mW	[55–57]
Detection efficiency	η	0.92	[55–57]
Effective mechanical decay rate	$\gamma_m/2\pi$	6.9×10^{-3} Hz	
Thermal photon number	N_{th}	0	
Detuning	Δ	$0.2\kappa_-$	
Ratio of optical decay rate	$\zeta = \kappa/\kappa_+$	3	
Finesse	$F = 2\pi c/\ell\kappa_-$	1.8×10^3	[56, 57]
Expectation value of cavity photon field	$ \bar{a}_A $	1.27×10^5	
Optomechanical coupling	$g/2\pi$	2.68×10^5 Hz	
Quality factor	$Q_- = \omega_m^-/\Gamma_{fb}$	7.5×10^4	
	$Q_+ = \omega_m^+/\Gamma_{fb}$	1.6×10^5	
Cooperativity	$C_- = 4(g_m^-)^2/\gamma_m\kappa_-$	1.1×10^5	
	$C_+ = 4(g_m^+)^2/\gamma_m\kappa_+$	1.6×10^5	
Thermal phonon number	n_{th}^-	7.5×10^3	
	n_{th}^+	1.8×10^3	

Table 2: Parameters employed in Ref. [55–57] and expected from the state-of-art technique.

Using these variances and covariances, the entanglement can be characterized by the quality factor $Q_{\pm}$, cooperativity $C_{\pm}$, thermal phonon number n_{thfb} , photon number N_{th} , detuning $\Delta/\kappa_{\pm}$, and detection efficiency η . Here, we introduce the quality factor and cooperativity as

$$Q_{\pm} = \frac{\omega_m^{\pm}}{\gamma_m}, \quad C_{\pm} = \frac{4(g_m^{\pm})^2}{\gamma_m\kappa_{\pm}}. \quad (4.39)$$

The exact formula of the logarithmic negativity is shown in Appendix C. Whereas the mechanical frequency is written as $\Omega/\gamma_m = \sqrt{Q(Q - 4C\delta + 4Q\delta^2)/(1 + 4\delta^2)}$, the stability condition $Q_{\pm}(1 + 4\delta_{\pm}^2) > 4C_{\pm}\delta_{\pm}$ is always satisfied for the normalized detuning $\delta_{\pm} = \Delta/\kappa_{\pm} > 0$. As a result, we note that the stability condition is satisfied for any $(C_{\pm}, Q_{\pm}, \delta_{\pm})$ as long as $\delta_{\pm} > 0$. Here, we adopt the structural damping $\Gamma(\omega_m^{\pm}) = \Gamma(\Omega)\Omega/\omega_m^{\pm}$, which leads to

$$n_{thfb}^{\pm} = \frac{k_B T \Gamma(\Omega) \Omega}{\hbar \gamma_m (\omega_m^{\pm})^2}. \quad (4.40)$$

The upper left panel of Figure 13 plots the critical value ϵ_{cr} in Eq. (4.31) for the measurement of both the optical amplitude quadratures $X_{\pm}$ as a function of the quantum cooperativity C_-/n_{thf}^- and the normalized detuning $\delta_- = \Delta/\kappa_-$. The upper right panel of Fig. 13 shows the same plot as a function of the mechanical dissipation rate Γ and the detuning δ_- . We adopt the parameters in Table 2; the variable parameters are the bare mechanical dissipation rate Γ , the detuning Δ , and the others are fixed. In these parameter regions, the measurement of the optical phase quadrature is experimentally challenging. Thus, we primarily consider the measurement of the optical amplitude quadratures. Entanglement appears for $\epsilon_{\text{cr}} > 0$, which is achieved for the dark brown regions in the upper left and right panels of Fig. 13. The required value of the quantum cooperativity to generate the entanglement is $C_-/n_{\text{th}}^- \simeq 3$. $\delta_- \simeq 0.1 \sim 0.2$ is advantageous in generating entanglement for the amplitude field measurement because this detuning optimizes the sensitivity coefficient of the amplitude quadrature. In this region, the effective frequency $\omega_m^{\pm}$ is also optimized, as shown in the lower panels of Fig. 13.

The quantum-squeezed state does not always mean the entanglement between the mirrors. Namely, there is no one-to-one connection between the quantum-squeezed state and the entanglement, as shown in Figures 14 and 15. Figure 14 shows the natural logarithm of the minimum eigenvalue for the mechanical covariance matrix. The state is quantum-squeezed when the minimum eigenvalue is less than 1 because the squeezed uncertainty is less than the vacuum fluctuations. Therefore, the region, $C_-/n_{\text{th}}^- \gtrsim 1$ and $0.1 \lesssim \delta_- \lesssim 0.8$ roughly satisfies the condition of a quantum-squeezed state. In particular, Fig. 15 shows the curves as the contour for the Wigner function satisfying $W = e^{-1}W_{\text{max}}$. The red dashed, blue solid, and black dotted curves correspond to the Wigner ellipses of the mechanical common mode, differential mode, and ground state. Panel (a) in Fig. 15, which corresponds to the green circle, shows the case when neither mode is quantum-squeezed and the entanglement does not occur. Panels (b) and (c), corresponding to the blue and red circles, illustrate the case in which only mechanical differential modes and both the mechanical common and differential modes are quantum-squeezed, respectively. However, no entanglement occurs in both cases (b) and (c). Panel (d), corresponding to the black circle, represents an experimentally feasible parameter using Γ and Δ in Table 2. In this case, both the modes are quantum-squeezed and the entanglement is generated. From these results, we find that quantum squeezing does not necessarily imply the generation of entanglement.

We infer that purity and quantum squeezing play an important role in entanglement

generation. Figure 16 plots the purity as a function of C_-/n_{th}^- and δ_- for the mechanical differential mode (upper panel) and common mode (lower panel). We find that high purity and quantum squeezing are necessary for generating entanglement. To be more specific, we roughly need $P_- \gtrsim 0.5$ and $P_+ \gtrsim 0.8$ for generating the entanglement between the two mirrors at $\delta_- = 0.2$. The purity is also optimized at $\delta \sim 0.2$, where the sensitivity coefficient is maximized.

We next discuss the entanglement behavior and phase distribution for the phase field measurement in Figs. 17 and 18, similar to the amplitude field measurement. Unlike the X measurement, Figure 17 shows that entanglement is larger in the region of small detuning. This is because the sensitivity coefficient is optimized at $\Delta = 0$ in the phase field measurements. The ellipses in Fig. 18 show the significance of squeezing for the Y measurement, where each panel corresponds to the parameters specified by the colored circles in Fig. 17. From an experimental point of view, we notice that performing the homodyne measurement to detect the phase field quadrature assumed in Table 2 is challenging due to the problem of detection of such a high-power laser. Therefore, the entanglement generation based on the amplitude quadratures is advantageous under the parameters in Table 2. Finesse can be enhanced to avoid this difficulty, but it reduces the optical cavity's linear range, making cavity length control difficult.

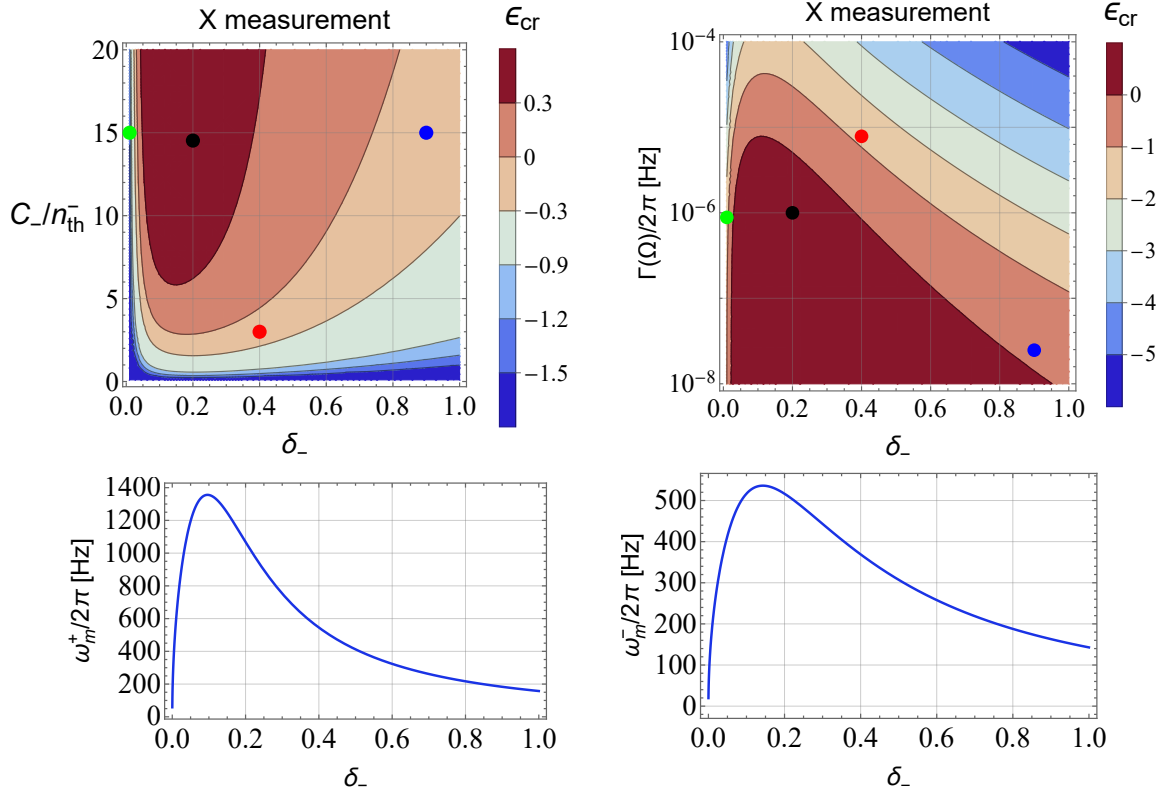


FIG 13: Upper panels: The critical value ϵ_{cr} as a function of C_-/n_{th}^- and δ_- is shown in the upper left panel, while the same is shown as a function of $\Gamma(\Omega)/2\pi$ and δ_- in the upper right panel. We consider the X measurement of both the optical amplitude quadratures of the common and differential modes. The entanglement occurs for the region in dark brown $\epsilon_{\text{cr}} > 0$ in both panels. Lower panels: The lower panels show $\omega_m^\pm$ as functions of δ_- .

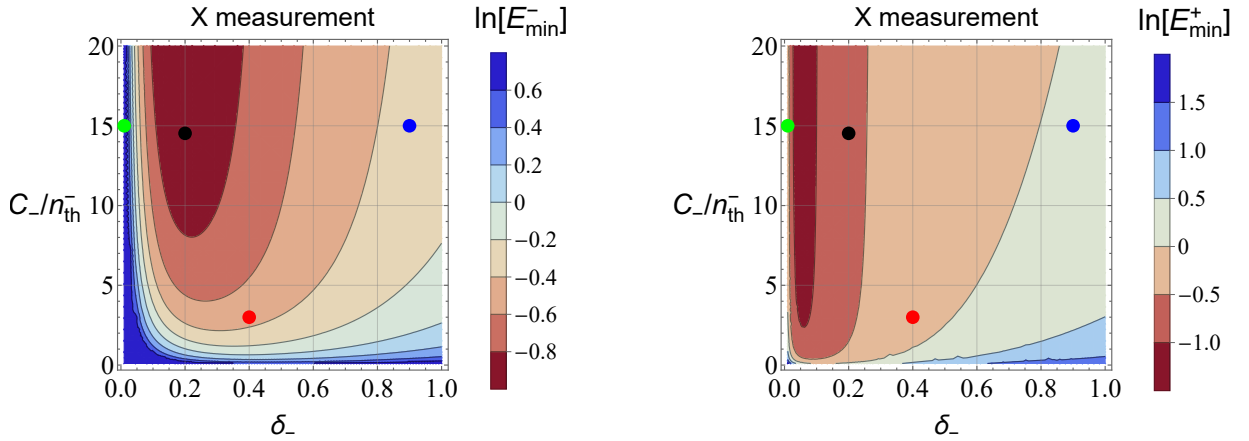


FIG 14: The behavior of the natural logarithm of the minimum eigenvalue for the covariance matrix of the mechanical common mode and differential mode $E_{\text{min}}^\pm$. Here, the covariance matrix is normalized at the frequency $\omega_m^\pm$. When $E_{\text{min}}^\pm < 1$, the squeezed uncertainty is less than the vacuum fluctuation.

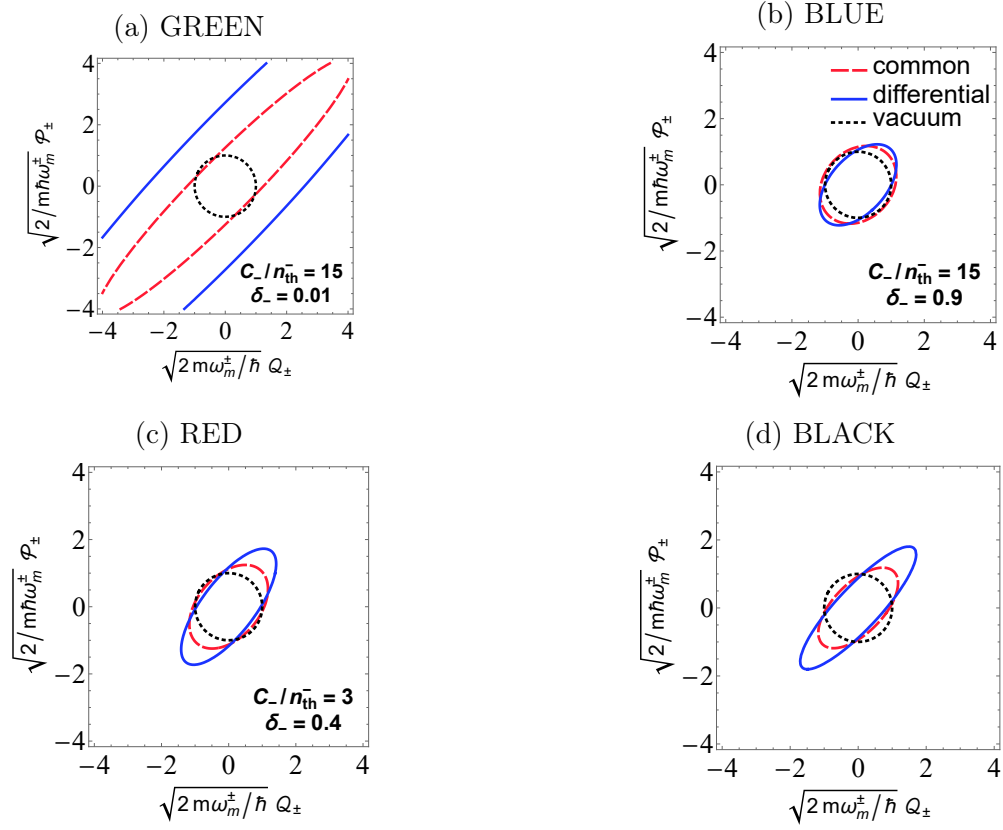


FIG 15: The Wigner ellipses of the mechanical common mode (red dashed line), mechanical differential mode (blue solid line), and ground state (black dotted line). The mechanical covariance matrices are normalized with the frequency $\omega_m^\pm$. Each panel corresponds to the parameters specified by the colored circle in Fig. 13, (a) is the green circle, (b) is the blue circle, (c) is the red circle, and (d) is the black circle.

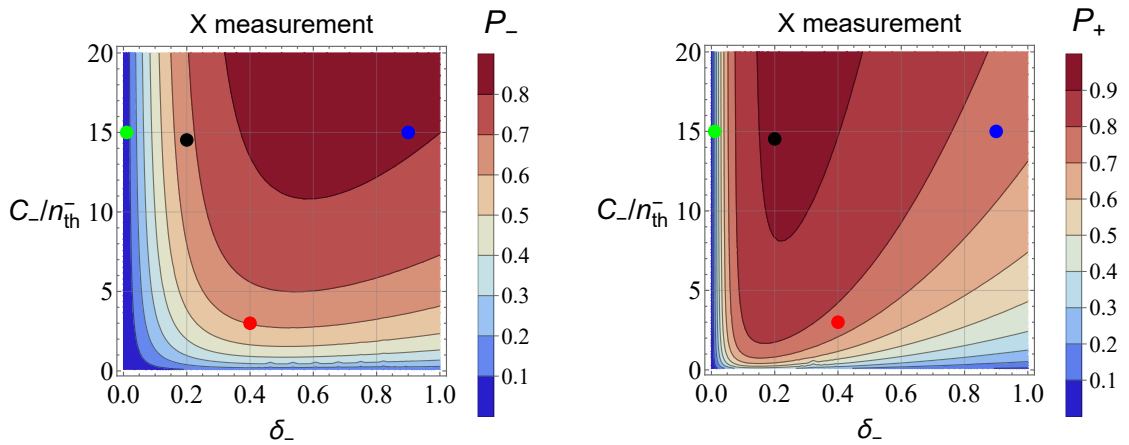


FIG 16: The behavior of the purity of differential (upper panel) and common (lower panel) modes. By combining this figure and the upper panels of Fig. 13, one can roughly read that $P_- \gtrsim 0.5$ and $P_+ \gtrsim 0.8$ are necessary at $\delta_- = 0.2$ for generating the entanglement.

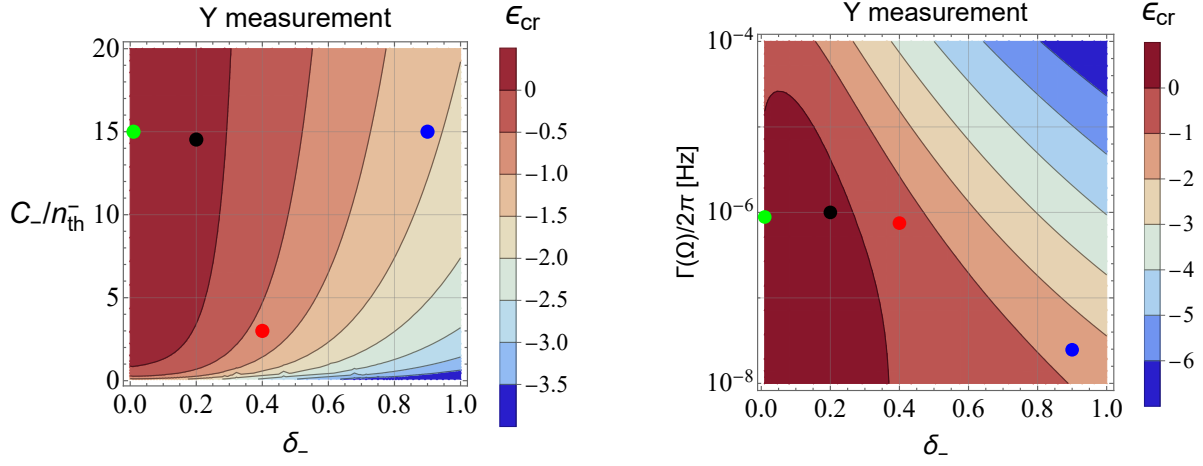


FIG 17: Same as the upper panels of Fig. 13 but for the Y measurement. We consider the Y measurement of the optical phase quadratures $Y_{\pm}$.

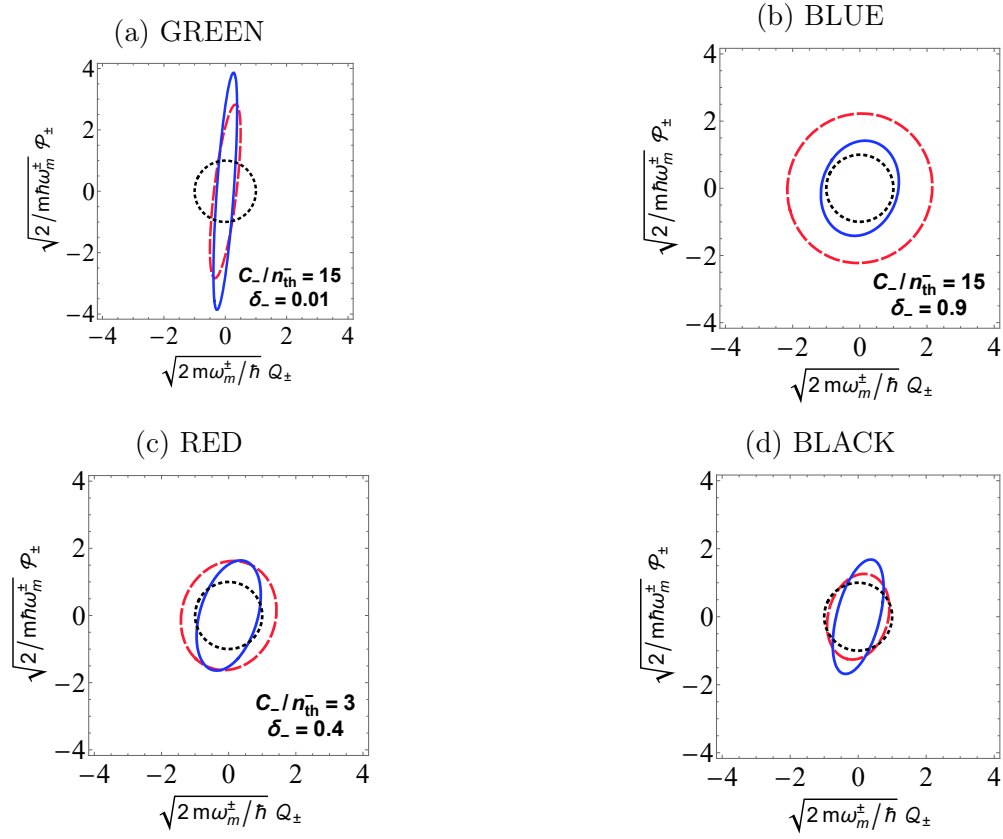


FIG 18: Same as Fig. 15 but for the Y measurement.

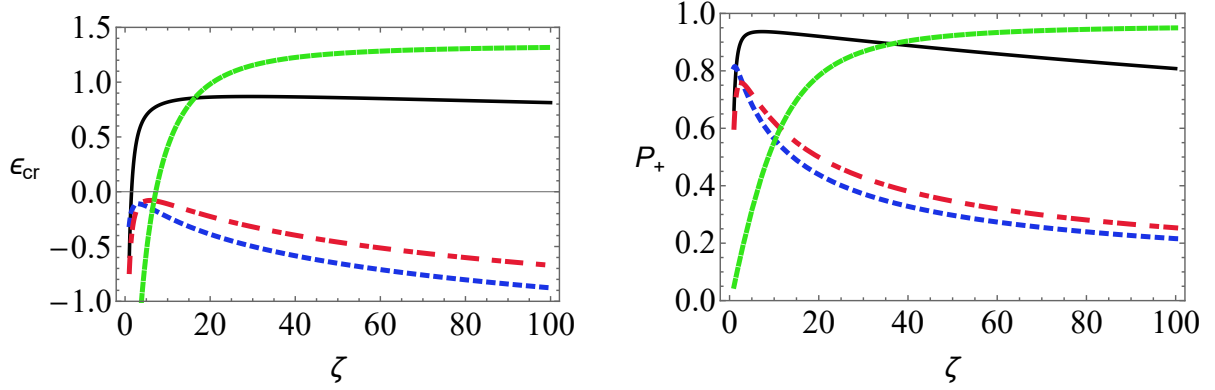


FIG 19: The behavior of the logarithmic negativity (left panel) and purity of the mechanical common mode (right panel) for the X measurement. Each curve in the four colors assumes the same parameters as those of the circle in the same corresponding color in Fig. 13. In the right panel, P_- is fixed for each curve as $P_- = 0.67$ (black solid curve), 0.05 (green solid curve), 0.81 (blue dashed curve), 0.60 (red dash-dotted curve).

In the above analysis, we fixed the parameter $\zeta = 3$, which characterizes the asymmetry of the mechanical common and differential modes. Here, we discuss how the entanglement behavior depends on the parameter ζ . Figure 19 shows the logarithmic negativity for the X measurement (left panel) and the frequency ratio ω_m^+/ω_m^- (right panel) as a function of ζ . The entanglement with the parameters in Table 2, the solid black curve, saturates for $\zeta \gtrsim 10$. The green curve in the left panel of Fig. 19 increases significantly as ζ increases. We infer that the joint effect of the mechanical common mode and the differential mode on the entanglement is important. The right panel of Fig. 19 plots the purity of the common mode P_+ as functions of ζ , where the purity of the differential model P_- is fixed for each curve as $P_- = 0.67$ (black solid curve), 0.05 (green solid curve), 0.81 (blue dashed curve), 0.60 (red dash-dotted curve). The green curves in the right panel of Fig. 19 and the left panel of Fig. 19 demonstrate that the entanglement appears by increasing the purity of the common mode when ζ increases, even when the purity of the differential mode is small. We note that this statement relies on the fact that the squeezing level of the differential mode is fixed. Furthermore, the output amplitude quadrature's sensitivity coefficient plays an important role in squeezing through the detuning parameter when ζ changes. Thus, ζ is important for entanglement to control the purity and the asymmetry of the squeezing between the mechanical common and the differential modes, which is caused by the asymmetric measurement efficiency.

4.3 Conclusions

We investigated the feasibility of generating macroscopic Gaussian entanglement between mechanical mirrors coupled to optical cavity modes under feedback control and quantum filtering. The mechanical mirrors are optically trapped via the optical spring effect, which arises from cavity detuning, enabling control of their dynamics in the quantum regime. Mechanical squeezing is induced through quantum control based on continuous measurements of the output light field. Specifically, we analyze a Fabry–Pérot–Michelson interferometer equipped with a power-recycled mirror designed to introduce asymmetry between the common and differential modes of the mechanical oscillators. This configuration facilitates entanglement between two mechanical oscillators via the optical fields interacting at the interferometer’s half-beam splitter. The entanglement mechanism is analogous to the well-known generation of optical entanglement between two squeezed beams passing through a half-beam splitter in quantum optics.

In our optomechanical setup, the squeezed states of the mechanical modes are generated and optimized using a Kalman filter, which minimizes the uncertainty of the conditional states by continuously updating their state based on the measurement outcomes. The steady-state covariance matrix of the mechanical mirrors is obtained by solving the Riccati equation, considering the effects of optical detuning, feedback control, and mechanical noise. We extended our analysis beyond the free-mass approximation and analytically derived the logarithmic negativity to measure entanglement for X and Y quadrature measurements.

We further examined the entanglement behavior by analyzing the logarithmic negativity and the Wigner distribution of the mechanical modes under experimentally feasible conditions expected for tabletop-scale optomechanical systems [55–57]. Two key parameters govern the degree of entanglement: the quantum cooperativity $C_{\pm}/n_{\text{th}}^{\pm}$, which quantifies the strength of the optomechanical interaction relative to thermal noise, and the normalized detuning parameter $\delta_{\pm} = \Delta/\kappa_{-}$. While quantum squeezing of the common and differential modes is achievable for $C_{\pm}/n_{\text{th}}^{\pm} \gtrsim 1$, we found that squeezing alone is insufficient to generate entanglement. In particular, entanglement does not emerge in regions of low purity, even when both mechanical modes exhibit quantum squeezing. This highlights the necessity of achieving quantum-squeezed states with high purity to realize entanglement between the mechanical oscillators.

The conditions required for entanglement generation depend on the cavity detuning and the choice of measurement quadrature. For X quadrature measurements, we demonstrate

that a quantum cooperativity condition of $C_-/n_{\text{th}}^- \gtrsim 3$ is required, assuming realistic parameters such as a measurement efficiency $\zeta = 3$ and $\delta_- = 0.2$. These conditions are within reach of mid-term future optomechanical technologies and thus represent a promising pathway for the near-term experimental demonstration of quantum entanglement between milligram-scale mirrors. In contrast, the tuned cavity is advantageous in generating the entanglement for Y quadrature measurements because the sensitivity coefficient is optimized. However, homodyne detection in the Y quadrature with a high-power laser remains experimentally challenging due to its technical complexity. Our results represent a significant advance toward the experimental realization of macroscopic entanglement in optomechanical systems. Achieving such entanglement would not only validate the predictions of quantum mechanics in the macroscopic regime but also pave the way for testing the quantum nature of gravity, a critical milestone in the quest to reconcile quantum mechanics with general relativity.

Despite these promising results, several practical challenges are addressed to facilitate experimental verification. In the present analysis, we have neglected the contribution of coating thermal noise, which is typically negligible within the operational bandwidths around 1 kHz [56]. However, its impact must be carefully assessed for broader frequency ranges and larger-scale systems. Furthermore, in optomechanical systems with suspended mirrors, higher-order mechanical modes, such as rotational and violin modes, can introduce additional noise and coupling effects that complicate the system dynamics [7, 43, 93].

5 Gravity-induced entanglement between mechanical mirrors

This Chapter investigates gravity-induced entanglement between two macroscopic mirrors in optomechanical systems under quantum control. This research is based on Refs. [1, 2]. As explained in Chapter 2, the verification of gravity-induced entanglement as a means to establish the quantum nature of gravity has opened up new possibilities for exploring quantum gravity. However, significant experimental challenges are posed since the gravity-induced entanglement signal is extremely weak. Therefore, this Chapter analyzes the conditions under which gravity-induced entanglement can appear between macroscopic quantum systems through quantum control in optomechanical systems.

Reference [67] investigated gravity-induced entanglement between two output optical modes in the Fourier space. Reference [68] demonstrated the conditions for generating gravity-induced squeezing of optical output fields at the mechanical resonance frequency. However, these studies did not account for the quantum control of the mechanical mirrors. In the present research, we evaluate the mechanical entanglement and squeezing induced by gravity under quantum control and compare these findings to the optical entanglement and squeezing conditions. In Ref. [1], we evaluate the time scales associated with gravity-induced entanglement under quantum control in optomechanical systems. Additionally, we estimate the total experimental duration required for the signal-to-noise ratio of the quantum entanglement signal to exceed unity. In Ref. [2], as in previous studies, the evaluation primarily focuses on the mechanical resonance frequencies in Fourier space. We demonstrate that thermal noise effects and optomechanical interactions introduce the conditions for producing the entanglement between mirrors. Furthermore, we point out that while gravity-induced quantum squeezing of mirrors occurs, such squeezing alone may not constitute definitive evidence for the quantum nature of gravity. These results provide critical insights into the feasibility and design of future experiments aiming to probe the quantum aspects of gravity using macroscopic quantum systems.

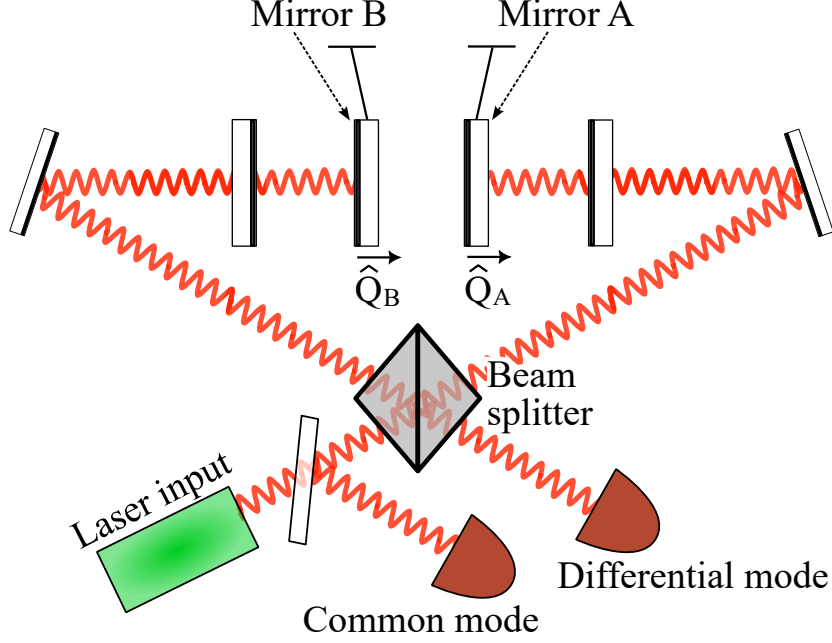


FIG 20: Configuration of the optomechanical systems to generate gravity-induced entanglement between mirror A and mirror B. We use the terms “differential mode” and “common mode” of the optical lights to refer to those coupled to the differential mode and the common mode of the mechanical mode, respectively.

5.1 Dynamics of gravitating optomechanical systems

We consider the optomechanical systems to consist of two cavities and two movable mechanical mirrors, as shown in Fig. 20. We assume mirrors A and B are perfectly reflective, and light dissipates out of the cavity only from the other mirrors of the cavities. We assume that the system is symmetric and that the mirror masses, mechanical bare frequencies, and mechanical bare dissipation rates are the same. We do not use power-recycled mirrors; the asymmetry arises only from gravity.

5.1.1 Gravity-induced asymmetry

The Hamiltonian of the gravitating optomechanical systems is

$$\begin{aligned} \hat{H} = & \frac{1}{2m}(\hat{P}_A^2 + \hat{P}_B^2) + \frac{1}{2}m\Omega^2(\hat{Q}_A^2 + \hat{Q}_B^2) + \frac{\hbar\omega_c}{1 - \hat{Q}_A/\ell}\hat{a}_A^\dagger\hat{a}_A + \frac{\hbar\omega_c}{1 + \hat{Q}_B/\ell}\hat{a}_B^\dagger\hat{a}_B \\ & + i\hbar E\left(\frac{\hat{a}_A^\dagger + \hat{a}_B^\dagger}{2}e^{-i\omega_L t} - \frac{\hat{a}_A + \hat{a}_B}{2}e^{i\omega_L t}\right) + \hat{H}_G, \end{aligned} \quad (5.1)$$

which is the same Hamiltonian as the previous section except for the last term. Considering the linearization $\hat{Q} \rightarrow \bar{Q} + \delta\hat{Q}$, $\hat{P} \rightarrow \bar{P} + \delta\hat{P}$, and $\hat{a} \rightarrow \bar{a} + \delta\hat{a}$, the optomechanical terms can be expand as in Eq. (3.3). The last term describes the quantum gravity in the Newtonian limit given by

$$\hat{H}_G = -\frac{Gm^2}{L} \frac{1}{1 + \hat{Q}_A - \hat{Q}_B} \simeq \frac{Gm^2}{d\Xi} \left(1 - \frac{\delta\hat{Q}_A - \delta\hat{Q}_B}{d\Xi} + \frac{(\delta\hat{Q}_A - \delta\hat{Q}_B)^2}{d^2\Xi^2} \right), \quad (5.2)$$

where we use the linearization $\hat{Q}_{A/B} = \bar{Q}_{A/B} + \delta\hat{Q}_{A/B}$, and $\Xi = 1 + \bar{Q}_A - \bar{Q}_B$. Considering the small displacements $\bar{Q}_{A/B} \ll d, \ell$, we henceforth assume $\Xi \simeq 1$ and $\xi = 1 \pm \bar{Q}_{A/B}/\ell \simeq 1$. Here, we introduce the common mode and differential mode as

$$\hat{Q}_{\pm} = \frac{\hat{Q}_A \pm \hat{Q}_B}{\sqrt{2}}, \quad \hat{P}_{\pm} = \frac{\hat{P}_A \pm \hat{P}_B}{\sqrt{2}}, \quad \hat{a}_{\pm} = \frac{\hat{a}_A \mp \hat{a}_B}{\sqrt{2}}, \quad (5.3)$$

where the optical common (differential) mode is defined as the mode of light coupled with the mechanical common (differential) mode. Then, the gravitational interaction is

$$\hat{H}_G \simeq \frac{1}{2} m \Omega_-^2 \hat{Q}_-^2, \quad (5.4)$$

where we ignore the constant terms. The gravitational interaction shifts the mechanical frequency of the differential mode as

$$\Omega_+ = \Omega, \quad \Omega_- = \Omega \sqrt{1 - \epsilon}, \quad (5.5)$$

where ϵ is the gravitational coupling defined by

$$\epsilon = \frac{4Gm}{d^3\Omega^2} \quad (5.6)$$

The important point is that the gravitational interaction affects only the mechanical differential mode. This means that gravity produces asymmetry between common and differential modes, replacing the power-recycled mirror of the previous Chapter. Hence, the entanglement between individual mirror A and mirror B occurs due to gravity. The equations of the

motion for the mechanical modes and the cavity lights are derived by

$$\delta\dot{\hat{Q}}_{\pm} = \frac{1}{m}\delta\hat{P}_{\pm} \quad (5.7)$$

$$\delta\dot{\hat{P}}_{\pm} = -m\Omega_{\pm}^2\delta\hat{Q}_{\pm} - \Gamma_{\text{fb}}\delta\hat{P}_{\pm} - \hbar\alpha\delta\hat{x}_{\pm} + \sqrt{2\Gamma_{\text{fb}}}\hat{P}_{\pm}^{\text{eff}} \quad (5.8)$$

$$\delta\dot{\hat{x}}_{\pm} = -\Delta\delta\hat{y}_{\pm} - \frac{\kappa}{2}\delta\hat{x}_{\pm} + \sqrt{\kappa}\hat{x}_{\pm}^{\text{in}} \quad (5.9)$$

$$\delta\dot{\hat{y}}_{\pm} = -2\alpha\delta\hat{Q}_{\pm} + \Delta\delta\hat{x}_{\pm} - \frac{\kappa}{2}\delta\hat{y}_{\pm} + \sqrt{\kappa}\hat{y}_{\pm}^{\text{in}}, \quad (5.10)$$

where we replaced $\hat{a}$ by $\hat{a}e^{i\omega_L t}$ and introduced the detuning $\Delta = \omega_L - \omega_c(1 + \bar{Q}_-/ \ell)$. We assumed that the effective mechanical dissipation rate under feedback control $\Gamma_{\text{fb}} = \Gamma + \gamma_m$ is the same for the mechanical common and differential modes. The classical mean values are

$$\dot{\bar{Q}}_- = \frac{\bar{P}_-}{m}, \quad (5.11)$$

$$\dot{\bar{P}}_- = -m\Omega_-^2\bar{Q}_- - \frac{\sqrt{2}Gm^2}{L^2} - \frac{\hbar\omega_c}{\sqrt{2}\ell}|\bar{a}_-|^2 - \Gamma\bar{P}_-, \quad (5.12)$$

$$\dot{\bar{a}}_- = i\Delta\bar{a}_- + E - \frac{\kappa}{2}\bar{a}_-, \quad (5.13)$$

and $(\bar{Q}_2, \bar{P}_2, \bar{a}_2) = (-\bar{Q}_1, -\bar{P}_1, \bar{a}_1)$ and $\bar{Q}_+ = \bar{P}_+ = \bar{a}_+ = 0$ for the symmetrical setup.

5.1.2 Electromagnetic forces and other noises

Even if we prepare the object neutrally, electromagnetic interactions due to intermolecular forces will still occur. One advantage of optomechanical systems to test the quantum nature of gravity is that the effects of non-gravitational interactions can be relatively small since the mass can be large. Here, we estimate the ratio of the magnitudes of some noise and gravity interactions.

We first consider the Casimir-Polder potential described by $V(r) = 23\hbar c R^6 / 4\pi r^7 (\epsilon - 1/\epsilon + 2)^2$, where R is the radius of the mass, r is the separation of the two masses, ϵ is the dielectric constant of the mass, and c is the light velocity [28, 34, 35, 102]. Assuming $R \sim r$, we can neglect the Casimir-Polder potential for $m \gg 10^{-7}$ kg by comparison with the gravitational potential. Next, we consider the dipole-dipole interaction given by $V(\mathbf{r}) = (\mathbf{d}_A \cdot \mathbf{d}_B - 3(\mathbf{r} \cdot \mathbf{d}_A/|\mathbf{r}|)(\mathbf{r} \cdot \mathbf{d}_B/|\mathbf{r}|))/4\pi\epsilon_0|\mathbf{r}|^3$, where $\mathbf{d}_{A,B}$ is the electric dipole and ϵ_0 is the permeability of space [28, 35, 36, 102]. Assuming $\mathbf{d}_A = \mathbf{d}_B = eNR\mathbf{r}/|\mathbf{r}|$, where e is

the electron charge and N is the number of the electron, the dipole-dipole interaction can be negligible for $m \gg 10^{-9}N$ kg. In the present paper, we consider macroscopic mirrors with large masses greater than the milligram scale and ignore these interactions. This is an advantage of optomechanical systems in verifying gravity-induced entanglement, which means that gravity can dominate the interactions between masses.

Acceleration noise and gravity gradient noise [26, 102, 103] lose the gravity-induced entanglement and the thermal noise. However, we can neglect the acceleration noise when gravitational interaction is dominant because electromagnetic interactions induce acceleration noise. The gravity gradient noise can be ignored when considering the ground-based experiments (e.g., Advanced LIGO [104]) because the thermal noise is larger than the gravity gradient noise.

5.2 Time domain analysis

In this section, we consider the time evolution of gravity-induced entanglement under quantum Kalman filtering introduced in Chapter 3. This research is based on Ref. [1]. Here, we assume the adiabatic limit $\kappa \gg \Omega$, which allows continuous measurement. Under the this limit, Eqs. (5.7)-(5.10) are rewritten as

$$\delta\dot{\hat{q}}_{\pm} = \omega_m^{\pm} \delta\hat{p}_{\pm} \quad (5.14)$$

$$\delta\dot{\hat{p}}_{\pm} = -\omega_m^{\pm} \delta\hat{q}_{\pm} - \Gamma_{\text{fb}} \hat{p}_{\pm} + \sqrt{2\Gamma_{\text{fb}}} \hat{p}_{\pm}^{\text{eff}} - \frac{4g_m^{\pm} \kappa \sqrt{\kappa_{\pm}}}{\kappa_{\pm}^2 + 4\Delta^2} \hat{x}_{\pm}^{\text{in}} + \frac{8g_m^{\pm} \Delta \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}_{\pm}^{\text{in}}, \quad (5.15)$$

$$\delta\hat{x}_{\pm} = \frac{8g_m^{\pm} \Delta}{\kappa^2 + 4\Delta^2} \delta\hat{q}_{\pm} + \frac{2\kappa \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{x}_{\pm}^{\text{in}} - \frac{4\Delta \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}_{\pm}^{\text{in}} \quad (5.16)$$

$$\delta\hat{y}_{\pm} = -\frac{4g_m^{\pm} \kappa}{\kappa^2 + 4\Delta^2} \delta\hat{q}_{\pm} + \frac{4\Delta \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{x}_{\pm}^{\text{in}} + \frac{2\kappa \sqrt{\kappa}}{\kappa^2 + 4\Delta^2} \hat{y}_{\pm}^{\text{in}}. \quad (5.17)$$

where we introduce the dimensionless operators as

$$\hat{q}_{\pm} = \sqrt{\frac{2m\omega_m^{\pm}}{\hbar}} \hat{Q}_{\pm}, \quad \hat{p}_{\pm} = \sqrt{\frac{2}{m\hbar\omega_m^{\pm}}} \hat{P}_{\pm}, \quad (5.18)$$

where $[\hat{q}_\pm, \hat{p}_\pm] = 2i$. The mechanical frequency and the optomechanical coupling are shifted by gravitational interaction as

$$\omega_m = \sqrt{\Omega_\pm^2 + \Omega \frac{16g^2\Delta}{\kappa^2 + 4\Delta^2}}, \quad g_m^\pm = g\sqrt{\frac{\Omega}{\omega_m^\pm}}. \quad (5.19)$$

In this section, we consider the general output field $\hat{b}_\pm$ defined as

$$\hat{b}_\pm = \hat{x}_\pm^{\text{out}} \cos \theta_\pm + \hat{y}_\pm^{\text{out}} \sin \theta_\pm, \quad (5.20)$$

where we assume the perfect detection $\eta = 1$ and optical vacuum noise input $N_{\text{th}} = 0$. The Langevin equation of the mechanical motion is rewritten in the matrix form as

$$\dot{\hat{\mathbf{r}}}_\pm = \mathbf{A}_\pm \hat{\mathbf{r}}_\pm + \begin{pmatrix} 0 \\ \hat{w}_\pm \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \omega_m^\pm \\ \omega_m^\pm & -\Gamma_{\text{fb}} \end{pmatrix} \quad (5.21)$$

$$\hat{w}_\pm = \sqrt{2\Gamma_{\text{fb}}}\hat{p}_\pm^{\text{eff}} - \frac{4g_m^\pm\kappa\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{x}_\pm^{\text{in}} + \frac{8g_m^\pm\Delta\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}\hat{y}_\pm^{\text{in}}, \quad (5.22)$$

where $\hat{\mathbf{r}}_\pm = (\delta\hat{q}_\pm, \delta\hat{p}_\pm)^T$. We also derive the general output quadrature in the matrix form as

$$\hat{b}_\pm = \mathbf{C}_\pm \hat{\mathbf{r}}_\pm + \hat{v}_\pm, \quad \mathbf{C}_\pm = \begin{pmatrix} \frac{8g_m^\pm\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}(-2\Delta \cos \theta_\pm + \kappa \sin \theta_\pm) & 0 \end{pmatrix}, \quad (5.23)$$

$$\hat{v}_\pm = \frac{-(\kappa^2 - 4\Delta^2) \cos \theta_\pm - 4\kappa\Delta \sin \theta_\pm}{\kappa^2 + 4\Delta^2}\hat{x}_\pm^{\text{in}} + \frac{4\kappa\Delta \cos \theta_\pm - (\kappa^2 - 4\Delta^2) \sin \theta_\pm}{\kappa^2 + 4\Delta^2}\hat{y}_\pm^{\text{in}}. \quad (5.24)$$

For the Kalman filtering method used in Chapter 3, the conditional covariance matrix follows the Riccati equation as

$$\dot{\mathbf{V}}_\pm = \mathbf{A}_\pm \mathbf{V}_\pm + \mathbf{V}_\pm \mathbf{A}_\pm^T + \mathbf{N}_\pm - (\mathbf{V}_\pm \mathbf{C}_\pm^T + \mathbf{L}_\pm)M^{-1}(\mathbf{V}_\pm \mathbf{C}_\pm^T + \mathbf{L}_\pm)^T, \quad (5.25)$$

where $M = \langle \hat{v}_\pm^2 \rangle = 1$, and

$$\mathbf{V}_\pm = \begin{pmatrix} V_{\delta q \delta p}^\pm & V_{\delta q \delta p}^\pm \\ V_{\delta q \delta p}^\pm & V_{\delta p \delta p}^\pm \end{pmatrix}, \quad \mathbf{N}_\pm = \begin{pmatrix} 0 & 0 \\ 0 & \langle \hat{w}_\pm^2 \rangle \end{pmatrix}, \quad \mathbf{L}_\pm = \begin{pmatrix} 0 \\ \langle \hat{w}_\pm \hat{v}_\pm \rangle \end{pmatrix}, \quad (5.26)$$

with

$$\bar{n}_{\pm} \equiv \langle \hat{w}_{\pm}^2 \rangle = 2\Gamma_{\text{fb}}(2n_{\text{thfb}}^{\pm} + 1) + \frac{16(g_m^{\pm})^2\kappa}{\kappa^2 + 4\Delta^2}, \quad (5.27)$$

$$\bar{\ell}_{\pm} \equiv \langle \hat{w}_{\pm}\hat{v}_{\pm} \rangle = \frac{4g_m^{\pm}(\kappa \cos \theta_{\pm} + 2\Delta \sin \theta_{\pm})\sqrt{\kappa}}{\kappa^2 + 4\Delta^2}. \quad (5.28)$$

5.2.1 Time evolution of the entanglement

In this section, we consider the time evolution of gravity-induced entanglement in optomechanical systems. We solved the Riccati equation using a numerical method, assuming that the two mirrors were initially in a separable thermal state: $\mathbf{V}(0) = (\bar{n}_+/2\gamma_m)\mathbf{1}_4$, where $\mathbf{V}$ is the covariance matrix of the individual mirrors with the basis $(\delta\hat{Q}_A, \delta\hat{P}_A, \delta\hat{Q}_B, \delta\hat{P}_B)$ defined as Eq. (4.28). Figure 21 shows the behavior of the components of the covariance matrix of the mechanical common mode conditioned on the phase quadrature measurement (left panel) and the amplitude quadrature measurement (right panel). The steady-state solution is derived by solving the Riccati equation for $\dot{\mathbf{V}}_{\pm} = 0$ as

$$V_{\delta q \delta q}^{\pm} = \frac{\gamma_I^{\pm} - \Gamma_{\text{fb}}}{\lambda_I^{\pm}} \quad (5.29)$$

$$V_{\delta p \delta p}^{\pm} = \frac{(\gamma_I^{\pm} - \Gamma_{\text{fb}})(2\omega_m^{\pm}(\omega_m^{\pm} + \Lambda_I^{\pm}) + \gamma_I^{\pm} - \gamma_I^{\pm}\Gamma_{\text{fb}})}{2\lambda_I^{\pm}(\omega_m^{\pm})^2} \quad (5.30)$$

$$V_{\delta q \delta p}^{\pm} = \frac{(\gamma_I^{\pm} - \Gamma_{\text{fb}})^2}{2\lambda_I^{\pm}\omega_m^{\pm}} \quad (5.31)$$

where

$$\lambda_X^{\pm} = \frac{64(g_m^{\pm})^2\kappa\Delta^2}{(\kappa^2 + 4\Delta^2)^2}, \quad \lambda_Y^{\pm} = \frac{16(g_m^{\pm})^2\kappa^3}{(\kappa^2 + 4\Delta^2)^2}, \quad (5.32)$$

$$\Lambda_X^{\pm} = -\Lambda_Y^{\pm} = -\frac{32(g_m^{\pm})^2\kappa^2\Delta}{(\kappa^2 + 4\Delta^2)^2}, \quad (5.33)$$

$$\gamma_I^{\pm} = \sqrt{\Gamma_{\text{fb}}^2 - 2\omega_m^{\pm}(\omega_m^{\pm} + \Lambda_I^{\pm}) + 2\omega_m^{\pm}\sqrt{\omega_m^{\pm}(\omega_m^{\pm} + 2\Lambda_I^{\pm}) + \bar{n}^{\pm}\lambda_I^{\pm}}}, \quad (5.34)$$

$$\bar{n}_{\pm} = 2\Gamma_{\text{fb}}(2n_{\text{thfb}}^{\pm} + 1) + \frac{16(g_m^{\pm})^2\kappa}{\kappa^2 + 4\Delta^2}, \quad n_{\text{thfb}}^{\pm} = \frac{k_B T \Gamma}{\hbar \omega_m^{\pm} \Gamma_{\text{fb}}}. \quad (5.35)$$

These variables are the almost same as Eqs. (4.32)-(4.35), but the asymmetry appears in $\omega_m^{\pm}$ and $g_m^{\pm}$. Considering the phase quadrature measurement ($I = Y$), the compo-

nents of the conditional covariance for $t < 10$ s is approximately derived by $V_{qq}^\pm(t) = \kappa / (16t(g_m^\pm)^2)$, $V_{qp}^\pm(t) = \Omega_\pm (n_{\text{thfb}}^\pm + 1/2) t$, $V_{pp}^\pm(t) = 2n_{\text{thfb}}^\pm + 1$, where $V_{pp}^\pm(t)$ and $V_{qp}^\pm(t)$ are determined using the initial values of $V_{pp}^\pm(0)$. However, $V_{qq}^\pm(t)$ is an attractor solution that does not depend on the initial conditions, $V_{qq}^\pm(0)$. Subsequently, the solution for $30 \text{ s} < t < 300 \text{ s}$ can be approximated as $V_{qq}^\pm(t) = \kappa / (4t(g_m^\pm)^2)$, $V_{qp}^\pm(t) = 3\kappa / (8t^2(g_m^\pm)^2\Omega_\pm)$, $V_{pp}^\pm(t) = 3\kappa / (8t^3(g_m^\pm)^2\Omega_\pm^2)$, where all components are attractor solutions, which did not depend on initial conditions, considering the steady-state solution, the time scale of achieving the steady-state is derived by $t \simeq 4/\gamma_{X/Y}^\pm$, which is the same as in the Chapter 3 and 4. We here assumed that the initial state is the thermal state, and the components $V_{qq}^+(t)$ and $V_{pp}^+(t)$ are significantly reduced from the initial values determined using the thermal phonon number. Hence, the mechanical modes become quantum-squeezed states through the Kalman filter process. In amplitude field measurement, the structure of the early time attractor solution changes. There are also regions where momentum squeezing occurs due to detuning and optomechanical coupling. We have ongoing work on these amplitude field measurements.

Figure 22 shows the behavior of the time evolution of the logarithmic negativity. From the stationary solutions of the covariance matrix (5.31), we estimate the logarithmic negativity as

$$E_N \simeq -\frac{1}{2} \log_2 \left[1 - \frac{\kappa \Gamma_{\text{fb}}}{16g^2} \left(\frac{\Omega}{\sqrt{2}\Gamma_{\text{fb}}} \epsilon - 4n_{\text{thfb}}^+ \right) \right], \quad (5.36)$$

where the approximations $\bar{n}_+ \lambda_+ \gg \Omega_+^2 \gg \Gamma_{\text{fb}}^2$ and $4g^2/\kappa \Gamma_{\text{fb}} \gg n_{\text{thfb}}^+ \geq 1$ were used. We also assumed the phase quadrature measurement with tuned cavities. The entanglement criterion is as follows:

$$\Omega \epsilon > 4\sqrt{2}\Gamma_{\text{fb}} n_{\text{thfb}}^+, \quad (5.37)$$

which is nearly the same as the entanglement criterion between the output lights [67, 68]. This means that the time scale of generating gravity-induced entanglement is faster than the time scale of thermal decoherence. Figure 23 shows the stationary entanglement between two mirrors under the measurement of the phase quadrature. The condition of Eq. (5.37) determines the boundary of $\Gamma T/2\pi$. Additionally, the requirement for quantum cooperativity

Name	Symbol	Value
Mechanical frequency	$\Omega/2\pi$	10^{-3} Hz
Mechanical decay rate	$\Gamma(\Omega)/2\pi$	10^{-18} Hz
Bath temperature	T	1 K
Mirror mass	m	100 g
Cavity length	ℓ	10 cm
Laser wavelength	$\omega_L(\simeq \omega_c)/2\pi$	3×10^{14} Hz
Optical decay rate	$\kappa/2\pi$	10^8 Hz
Input laser power	P_{in}	10^{-5} or 10^{-6} W
Effective mechanical decay rate	$\gamma_m/2\pi$	10^{-6} Hz
Detuning	Δ	0.05κ or 0
Mass density	m/d^3	40 g/cm ³

Table 3: Parameters of the numerical solution in Fig. 21. Ref. [105] discussed the experimental situation to achieve low mechanical dissipation. We adopt $P_{in} = 10^{-6}$ W and $\Delta = 0$ for the phase measurement, while $P_{in} = 10^{-5}$ W and $\Delta = 0.05\kappa$ for the amplitude measurement.

is roughly given by

$$\frac{\gamma_Y^+}{4} > 4\Gamma_{fb}n_{thfb}^+, \quad (5.38)$$

which indicates the time scale of reaching the steady state must be faster than the time scale of thermal decoherence. This condition was absent in the quantum entanglement between output lights and comes from the quantum control of mirrors. The same condition appears in the amplitude quadrature measurement.

As discussed in Chapter 2 (also see Ref. [24]), the entanglement between two harmonic oscillators does not appear for the initial thermal state with the thermal phonon number $n_{th} \gtrsim 10$. However, the entanglement occurs even in the initial thermal state using quantum filtering and feedback control. For the initial pure quantum state, the time scale of generating entanglement between two harmonic oscillators in the squeezed state is estimated by

$$t_{en} \simeq \frac{\pi}{\Omega\epsilon} = 1.8 \times 10^3 \text{ s} \left(\frac{\Omega/2\pi}{10^{-3} \text{ Hz}} \right) \left(\frac{m/d^3}{40 \text{ g/cm}^3} \right)^{-1}. \quad (5.39)$$

Our study's entanglement generation time is $t \simeq 4/\gamma_Y^+ \simeq 600$ s for the initial thermal state. Therefore, the use of the quantum Kalman filter is advantageous in generating entanglement and preserving the quantum coherence of the system. This entanglement generation time is almost the same for the case based on the amplitude field measurements.

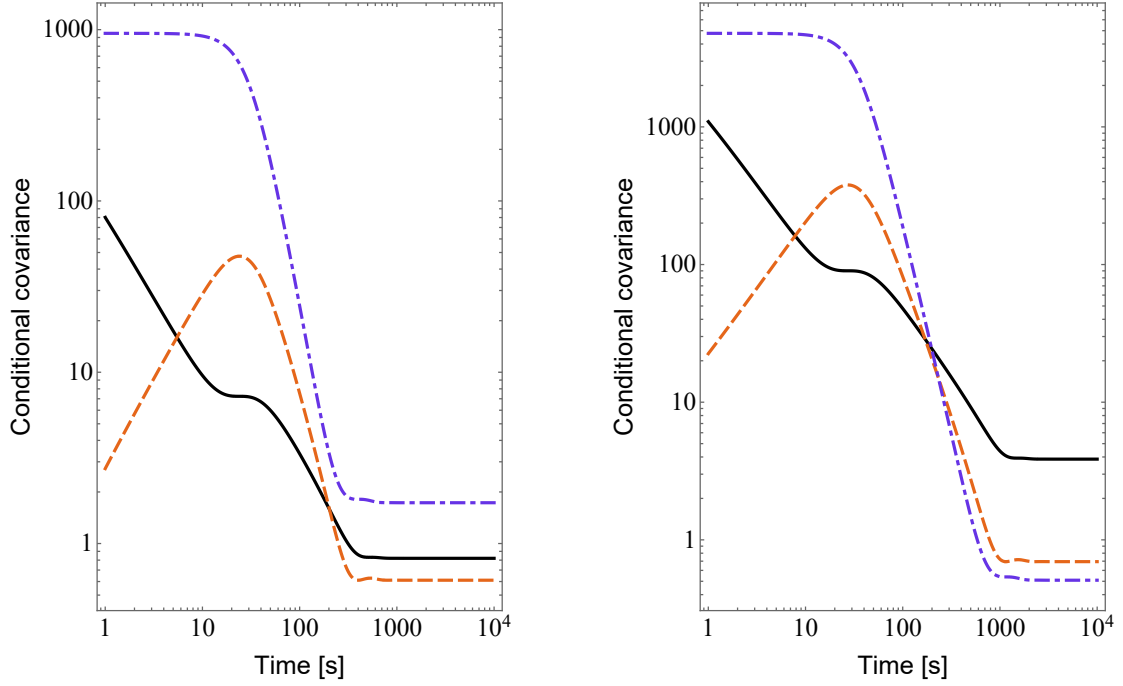


FIG 21: Time evolution of the components of the conditional covariance matrix elements, $V_{qq}^+(t)$ (black solid curve), $V_{qp}^+(t)$ (orange dashed curve), $V_{pp}^+(t)$ (purple dashed-dotted curve). The left and right panels show the case based on the phase quadrature and amplitude quadrature measurements, respectively. The parameters are listed in Table 3, and $\Delta = 0$ for the phase quadrature measurement and $\Delta = 0.05\kappa$ for the amplitude measurement. The behavior of the differential mode is nearly the same.

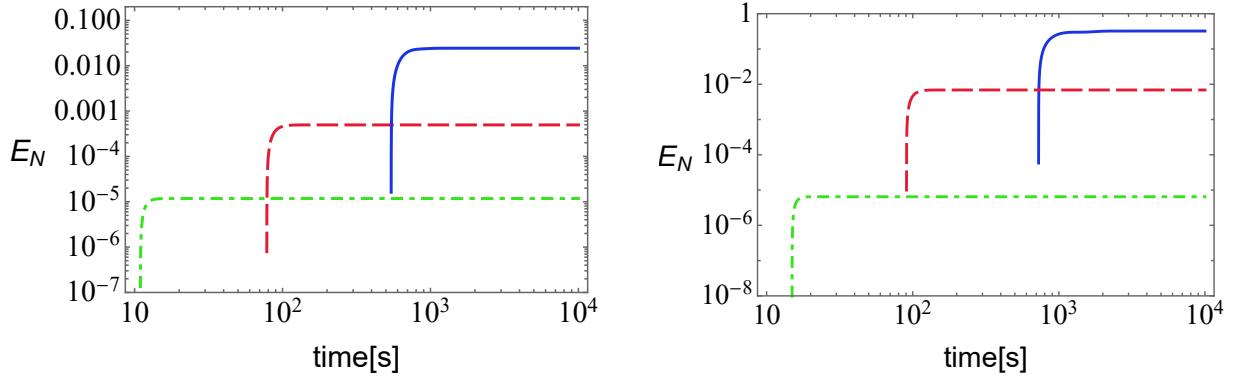


FIG 22: Behavior of the logarithmic negativity between mechanical mirrors under the Kalman filtering. The left and right panels correspond to the results based on the phase and amplitude quadratures measurement. Each curve assumes $\kappa/2\pi = 10^8\text{Hz}$ (blue solid curve), 10^7Hz (red dashed curve), 10^6Hz (green dashed-dotted curve), respectively. The others are the same as those in Table 3.

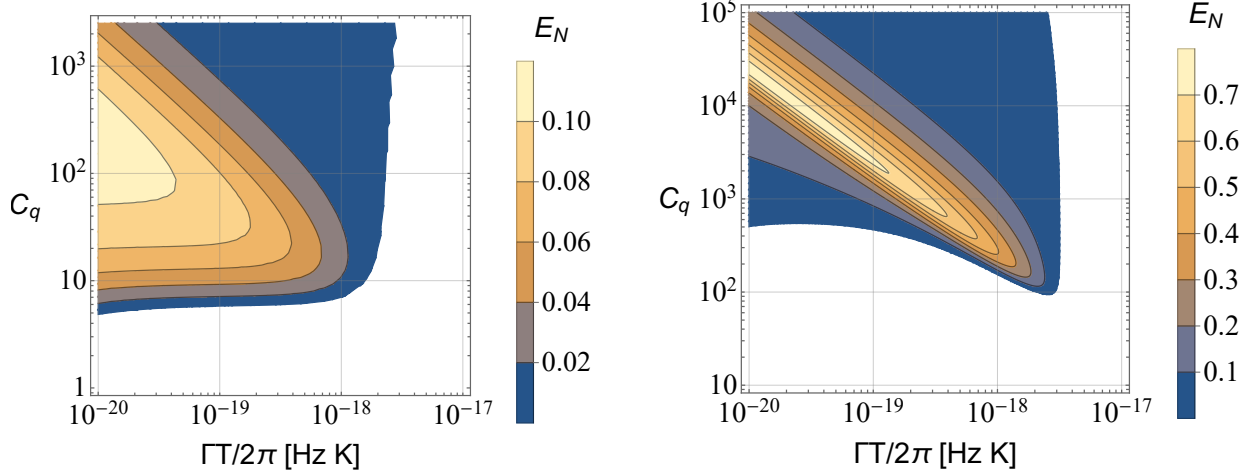


FIG 23: Behavior of the logarithmic negativity for the stationary mechanical states as a function of quantum cooperativity $C_q = C_+/n_{\text{thfb}}^+$ and $\Gamma T/2\pi$. We adopt the measurement of the phase quadrature in the left panel and the amplitude quadrature in the right panel. The required values of the entanglement are $C_q \gtrsim 5$ and $\Gamma T/2\pi \lesssim 10^{-18}$ Hz K.

5.2.2 Purity and squeezing

In this section, we demonstrate the time evolution of the conditional state under the Kalman filtering. The left panel of Fig. 24 shows the time evolution of the purity for the measurement of phase quadrature (red curves) and the amplitude quadrature (blue curves). The solid and dashed curves correspond to the mechanical common and differential modes, respectively. Slight asymmetry appears in the purity, and the steady state approaches a sufficiently pure quantum state. The right panel of Fig. 24 shows the squeezing level; if it is greater than unity, quantum squeezing beyond vacuum fluctuations occurs. We note that the quantum filtering process produces mechanical squeezing independent of gravity, as shown in Chapters 3 and 4. Therefore, gravity-induced squeezing appears differently in the common and differential modes.

5.2.3 Signal-to-noise ratio

Finally, we discuss the error in the measurement of E_N , which can be estimated by

$$\begin{aligned} \Delta E_N(V) &= \sum_{j=\pm} \left(\frac{\partial E_N}{\partial V_{qq}^j} \Delta V_{qq}^j + \frac{\partial E_N}{\partial V_{qp}^j} \Delta V_{qp}^j + \frac{\partial E_N}{\partial V_{pp}^j} \Delta V_{pp}^j \right). \end{aligned} \quad (5.40)$$

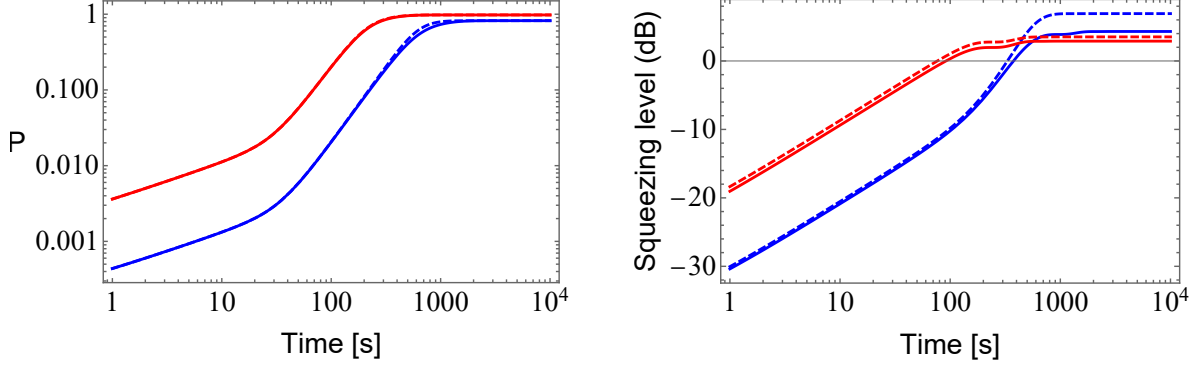


FIG 24: Left: Behavior of the purity as a function of a time under quantum Kalman filtering based on the measurement of the phase quadrature (red curves) and amplitude quadrature (blue curve). The solid and dashed curves correspond to the mechanical common and differential modes. Right: Behavior of the squeezing level.

Assuming that the errors $q_{\pm}$ and $p_{\pm}$ follow a Gaussian distribution around the estimated values, we may write $\langle \Delta V_{qq}^2 \rangle = 2V_{qq}^2$, $\langle \Delta V_{qp}^2 \rangle = V_{qq}V_{pp} + \bar{V}_{qp}^2$, $\langle \Delta V_{pp}^2 \rangle = 2V_{pp}^2$, $\langle \Delta V_{qq}\Delta V_{qp} \rangle = 2V_{qq}V_{qp}$, $\langle \Delta V_{qq}\Delta V_{pp} \rangle = 2V_{qp}^2$. Then, the variance in the error of $\Delta E_N(V)$ is given by

$$\begin{aligned}
\langle \Delta E_N^2(V) \rangle = & 2\left(\frac{\partial E_N}{\partial V_{qq}^+}\right)^2 V_{qq}^{+2} + \left(\frac{\partial E_N}{\partial V_{qp}^+}\right)^2 (V_{qp}^{+2} + V_{qq}^+ V_{pp}^+) \\
& + 2\left(\frac{\partial E_N}{\partial V_{pp}^+}\right)^2 V_{pp}^{+2} + 4\left(\frac{\partial E_N}{\partial V_{qq}^+}\right)\left(\frac{\partial E_N}{\partial V_{pp}^+}\right) V_{qp}^{+2} \\
& + 4\left(\frac{\partial E_N}{\partial V_{qq}^+}\right)\left(\frac{\partial E_N}{\partial V_{qp}^+}\right) V_{qq}^+ V_{qp}^+ + 4\left(\frac{\partial E_N}{\partial V_{pp}^+}\right)\left(\frac{\partial E_N}{\partial V_{qp}^+}\right) V_{pp}^+ V_{qp}^+ \\
& + 2\left(\frac{\partial E_N}{\partial V_{qq}^-}\right)^2 V_{qq}^{-2} + \left(\frac{\partial E_N}{\partial V_{qp}^-}\right)^2 (V_{qp}^{-2} + V_{qq}^- V_{pp}^-) \\
& + 2\left(\frac{\partial E_N}{\partial V_{pp}^-}\right)^2 V_{pp}^{-2} + 4\left(\frac{\partial E_N}{\partial V_{qq}^-}\right)\left(\frac{\partial E_N}{\partial V_{pp}^-}\right) \bar{V}_{qp}^{-2} \\
& + 4\left(\frac{\partial E_N}{\partial V_{qq}^-}\right)\left(\frac{\partial E_N}{\partial V_{qp}^-}\right) V_{qq}^- V_{qp}^- + 4\left(\frac{\partial E_N}{\partial V_{pp}^-}\right)\left(\frac{\partial E_N}{\partial V_{qp}^-}\right) V_{pp}^- V_{qp}^-.
\end{aligned} \tag{5.41}$$

Assuming that we perform $\mathcal{N}$ measurements of gravity-induced entanglement, we estimated the signal-to-noise ratio as

$$S/N = \sqrt{\mathcal{N}} \frac{E_N}{\Delta E_N}. \tag{5.42}$$

To achieve $S/N = 1$, the required number of measurements is $\mathcal{N} = \langle \Delta E_N^2(V) \rangle / E_N^2$. Provided that the condition $\Omega\epsilon \gg 4\sqrt{2}\gamma_m n_{\text{th}}^+$ is satisfied, E_N is approximately estimated as $E_N \simeq \kappa G\rho / (8\sqrt{2}g_+^2\Omega)$. Since the generation time of gravity-induced entanglement is estimated as $4/\gamma_Y^+$, the total time τ required to achieve $S/N = 1$ is proportional to $4/\gamma_Y^+(E_N)^2$, where we assume that ΔE_N is a constant of $\mathcal{O}(1)$. For the parameters in Table I, we have $\tau \simeq 2 \times 10^6$ s. On the other hand, as shown in Fig. 22, the logarithmic negativity is larger for the amplitude measurement than for the phase measurement. This may be due to the fact that the position squeezing is rotated, resulting in momentum squeezing. Detailed analysis is in progress, but by using the value of quantum entanglement and the time scale $4/\gamma_X^+$, the total time is obtained as $\tau \simeq 2 \times 10^5$ s. This indicates that amplitude field measurements may be a better way to verify quantum entanglement due to gravity.

5.2.4 Conclusions

We investigated the feasibility of detecting gravity-induced entanglement using optomechanical systems under quantum Kalman filtering. For the first time, we determined a feasible set of experimental parameters for achieving a signal-to-noise ratio $S/N=1$. This is achieved by solving the Riccati equation to determine the time evolution of the conditional state of macroscopic mechanical mirrors in optomechanical systems under quantum control using the Kalman filtering process. When the two mirrors are coupled via gravitational interaction, the quantum-squeezed states for the common and differential modes are slightly different, which gives rise to gravity-induced entanglement. We showed that the time scale for generating entanglement, associated with the quantum state preparation using the Kalman filtering process, is faster than the well-known timescale t_{en} . We also demonstrated that gravity-induced entanglement occurs when the two mirrors are in the initial thermal states. The rapid development of gravity-induced entanglement is an advantage of optomechanical systems. The parameters in Table 3 can be an optimal set. The feasibility of experiments for the measurement of the amplitude quadrature to detect gravity-induced entanglement should be investigated in more detail in the future.

5.3 Fourier domain analysis

In the previous section, we investigated the time evolution of gravity-induced entanglement. Here, we focus on the entanglement in the Fourier space at the mechanical resonance frequency, which is based on Ref. [2]. This research is motivated by Refs. [67, 68]; they analyzed the entanglement between two output lights and the squeezing of optical modes in this Fourier domain. Thus, we compare the entanglement between two mechanical mirrors and the squeezing of the mechanical modes to their results.

We consider the same system shown in the previous section in Figure 20, the Hamiltonian is Eq. 5.1 and the equation of motion is Eqs. (5.7)-(5.10). We use the Wiener filtering introduced in Chapter 3 to analyze the entanglement in the Fourier domain. Additionally, we focus on the measurement of the phase field quadrature with tuned cavities for simplicity. As we will show later, quantum Wiener filtering has little effect on gravity-induced entanglement in the Fourier space based on either phase quadrature or amplitude quadrature measurements. Therefore, the results of amplitude quadrature measurement are similar to the case of the phase quadrature measurement.

5.3.1 Wiener filter for optical phase measurement

We define the variables in the Fourier domain as $F(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} F(t)$. From the equation of the motion, we derive the equation of the motion for the mechanical modes and the optical modes in the Fourier domain as

$$\delta\hat{q}_{\pm}(\omega) = \frac{\Omega_{\pm}}{\Omega_{\pm}^2 - i\Gamma_{\text{fb}}\omega - \omega^2} \left(\sqrt{2\Gamma_{\text{fb}}} \hat{p}_{\pm}^{\text{eff}}(\omega) - \frac{2ig_{\pm}\sqrt{\kappa}}{\omega + i\kappa/2} \hat{x}_{\pm}^{\text{in}}(\omega) \right), \quad (5.43)$$

$$\delta\hat{p}_{\pm}(\omega) = -\frac{i\omega}{\Omega_{\pm}} \delta\hat{q}_{\pm}(\omega), \quad (5.44)$$

$$\delta\hat{x}_{\pm}(\omega) = \frac{i\sqrt{\kappa}}{\omega + i\kappa/2} \hat{x}_{\pm}^{\text{in}}(\omega), \quad (5.45)$$

$$\delta\hat{y}_{\pm}(\omega) = \frac{i}{\omega + i\kappa/2} \left(-2g_{\pm} \delta\hat{q}_{\pm}(\omega) + \sqrt{\kappa} \hat{y}_{\pm}^{\text{in}}(\omega) \right), \quad (5.46)$$

where we assume the tuned cavity $\Delta = 0$, which can be optimized for the sensitivity coefficient for the phase field quadrature. We obtain the mechanical spectral density as

$$S_{qq}^{\pm}(\omega) = \frac{\Omega_{\pm}^2}{(\Omega_{\pm}^2 - \omega^2)^2 + \Gamma_{\text{fb}}^2 \omega^2} \left(2\Gamma_{\text{fb}}(2n_{\text{thfb}}^{\pm} + 1) + \frac{4g_{\pm}^2 \kappa}{\omega^2 + \kappa^2/4} (2N_{\text{th}} + 1) \right), \quad (5.47)$$

$$S_{qp}^{\pm}(\omega) = \frac{i\omega}{\Omega_{\pm}} S_{qq}^{\pm}(\omega), \quad S_{pp}^{\pm}(\omega) = \frac{\omega^2}{\Omega_{\pm}^2} S_{qq}^{\pm}(\omega), \quad (5.48)$$

where we used the correlation functions of the fluctuation given by

$$\langle \{\hat{p}_{\pm}^{\text{eff}}(\omega), \hat{p}_{\pm}^{\text{eff}}(\omega')\} \rangle \simeq 2(2n_{\text{thfb}}^{\pm} + 1) \times 2\pi\delta(\omega + \omega'), \quad (5.49)$$

$$\langle \{\hat{x}_{\pm}^{\text{in}}(\omega), \hat{x}_{\pm}^{\text{in}}(\omega')\} \rangle = \langle \{\hat{y}_{\pm}^{\text{in}}(\omega), \hat{y}_{\pm}^{\text{in}}(\omega')\} \rangle = 2(2N_{\text{th}} + 1) \times 2\pi\delta(\omega + \omega'). \quad (5.50)$$

Here, we focus on the measurement of the optical phase quadrature, which is linearly coupled with the mechanical position operator. Considering the input-output relation

$$\hat{y}_{\pm}^{\text{out}} = \hat{y}_{\pm}^{\text{in}} - \sqrt{\kappa}\delta\hat{y}_{\pm}, \quad (5.51)$$

and the output optical phase quadrature is obtained as

$$\hat{Y}_{\pm}(\omega) = \sqrt{\eta}\hat{y}_{\pm}^{\text{out}} + \sqrt{1-\eta}\hat{y}_{\pm}^{\text{add}} \quad (5.52)$$

$$= \frac{2ig_{\pm}\sqrt{\kappa\eta}}{\omega + i\kappa/2}\delta\hat{q}_{\pm} + \frac{\omega - i\kappa/2}{\omega + i\kappa/2}\sqrt{\eta}\hat{y}_{\pm}^{\text{in}} + \sqrt{1-\eta}\hat{y}_{\pm}^{\text{add}} \quad (5.53)$$

$$= \frac{1}{F_{\pm}(\omega)} \left(2ig_{\pm}\Omega_{\pm}\sqrt{2\Gamma_{\text{fb}}\kappa\eta}\left(\omega + \frac{i\kappa}{2}\right)\hat{p}_{\pm}^{\text{eff}}(\omega) + 4g_{\pm}^2\Omega_{\pm}\kappa\sqrt{\eta}\hat{x}_{\pm}^{\text{in}}(\omega) \right. \\ \left. + F_{\pm}(\omega)\frac{\omega - i\kappa/2}{\omega + i\kappa/2}\sqrt{\eta}\hat{y}_{\pm}^{\text{in}}(\omega) + F_{\pm}(\omega)\sqrt{1-\eta}\hat{y}_{\pm}^{\text{add}} \right), \quad (5.54)$$

where η is the detection efficiency. $F_{\pm}(\omega)$ is given by

$$F_{\pm}(\omega) = (\Omega_{\pm}^2 - i\Gamma_{\text{fb}}\omega - \omega^2) \left(\omega + \frac{i\kappa}{2} \right)^2. \quad (5.55)$$

We note that $F_{\pm}(\omega)$ is a causal function where the imaginary part of the solution for $F_{\pm}(\omega) = 0$ is negative. The spectral densities S_{YY} , S_{qY} , and S_{pY} are computed as

$$S_{YY}^{\pm}(\omega) = \frac{2\eta N_{\text{th}} + 1}{|F_{\pm}(\omega)|^2} \left(|F_{\pm}(\omega)|^2 + 8g_{\pm}^2 \Omega_{\pm}^2 \Gamma_{\text{fb}} \kappa \eta (\omega^2 + \frac{\kappa^2}{4}) \frac{2n_{\text{thfb}}^{\pm} + 1}{2\eta N_{\text{th}} + 1} + 16g_{\pm}^4 \Omega_{\pm}^2 \kappa^2 \eta \frac{2N_{\text{th}} + 1}{2\eta N_{\text{th}} + 1} \right), \quad (5.56)$$

$$S_{qY}^{\pm}(\omega) = \frac{(2N_{\text{th}} + 1)}{|F_{\pm}(\omega)|^2} \left(-4ig_{\pm} \Omega_{\pm}^2 \Gamma_{\text{fb}} \sqrt{\kappa \eta} (\omega^2 + \frac{\kappa^2}{4}) (\omega + \frac{i\kappa}{2}) \frac{2n_{\text{thfb}}^{\pm} + 1}{2N_{\text{th}} + 1} - 8ig_{\pm}^3 \Omega_{\pm}^2 \kappa \sqrt{\kappa \eta} (\omega + \frac{i\kappa}{2}) \right), \quad (5.57)$$

$$S_{pY}^{\pm}(\omega) = \frac{-i\omega}{\Omega_{\pm}} S_{qY}^{\pm}(\omega). \quad (5.58)$$

Introducing the quantum Wiener filters discussed in Chapter 3 as

$$H_q^{\pm}(\omega) = \frac{1}{[S_{YY}^{\pm}(\omega)]_+} \left[\frac{S_{qY}^{\pm}(\omega)}{[S_{YY}^{\pm}(\omega)]_-} \right]_+, \quad (5.59)$$

$$H_p^{\pm}(\omega) = \frac{1}{[S_{YY}^{\pm}(\omega)]_+} \left[\frac{S_{pY}^{\pm}(\omega)}{[S_{YY}^{\pm}(\omega)]_-} \right]_+, \quad (5.60)$$

where $[f(\omega)]_+$ and $[f(\omega)]_-$ describe the causal and non-causal part of $f(\omega)$, respectively. Then, we derive the conditional mechanical quadratures as

$$\tilde{q}_{\pm}(\omega) = \delta \hat{q}_{\pm}(\omega) - H_q^{\pm}(\omega) \hat{Y}_{\pm}(\omega), \quad (5.61)$$

$$\tilde{p}_{\pm}(\omega) = \delta \hat{p}_{\pm}(\omega) - H_p^{\pm}(\omega) \hat{Y}_{\pm}(\omega). \quad (5.62)$$

The causal part and the non-causal part of the spectral density S_{YY} are

$$[S_{YY}^{\pm}(\omega)]_+ = \frac{F'_{\pm}(\omega)}{F_{\pm}(\omega)}, \quad [S_{YY}^{\pm}(\omega)]_- = \frac{F'_{\pm}^*(\omega)}{F_{\pm}^*(\omega)}, \quad (5.63)$$

where $F'_{\pm}(\omega)$ is the causal function given as the solution of the following equation:

$$|F'_{\pm}(\omega)|^2 = |F_{\pm}(\omega)|^2 + 8g_{\pm}^2 \Omega_{\pm}^2 \Gamma_{\text{fb}} \kappa \eta (\omega^2 + \frac{\kappa^2}{4}) \frac{2n_{\text{thfb}}^{\pm} + 1}{2\eta N_{\text{th}} + 1} + 16g_{\pm}^4 \Omega_{\pm}^2 \kappa^2 \eta \frac{2N_{\text{th}} + 1}{2\eta N_{\text{th}} + 1}. \quad (5.64)$$

We decide the filter functions $H_q^\pm(\omega)$ and $H_p^\pm(\omega)$ to optimize the mechanical covariance matrix in the same way as Appendix A.

5.3.2 Entanglement in Fourier domain

In the Fourier space, it is hard to define entanglement negativity because the commutation relation between $\delta\hat{q}$ and $\delta\hat{p}$ changes from that in a real space. Here, we use the degree of entanglement introduced in Ref. [106]. They showed the entanglement criterion in the Fourier domain as

$$E(\omega) = \frac{\langle \hat{R}_{\tilde{q}_+}(\omega)^2 \rangle \langle \hat{R}_{\tilde{p}_-}(\omega)^2 \rangle \Omega_- / \Omega}{|\langle [\hat{R}_{\tilde{q}_A}(\omega), \hat{R}_{\tilde{p}_A}(\omega)] \rangle|^2} < 1, \quad (5.65)$$

where $\hat{R}_Z(\omega)$ is defined by $\hat{R}_Z(\omega) = (\hat{Z}(\omega) + \hat{Z}(-\omega))/2$ with $Z = \tilde{q}_+, \tilde{p}_-, \tilde{q}_A$, and $\tilde{p}_A$, and $\hat{R}_Z(\omega)$ is the Hermitian operator in Fourier domain. $\tilde{q}_A(\omega) = (\tilde{q}_+(\omega) + \tilde{q}_-(\omega)\sqrt{\Omega/\Omega_-})/2$ and $\tilde{p}_A(\omega) = (\tilde{p}_+(\omega) + \tilde{p}_-(\omega)\sqrt{\Omega_-/\Omega})/2$ are the conditional canonical operators of the mechanical mirror A. According to the reference [84], the commutation relations of the fluctuations are

$$[\hat{p}_\pm^{\text{eff}}(\omega), \hat{p}_\pm^{\text{eff}}(\omega')] = \frac{2\omega}{\Omega_\pm} \times 2\pi\delta(\omega + \omega'), \quad (5.66)$$

$$[\hat{x}_\pm^{\text{in}}(\omega), \hat{y}_\pm^{\text{in}}(\omega')] = 2i \times 2\pi\delta(\omega + \omega'). \quad (5.67)$$

Here we note that the commutation relation of $\hat{p}_\pm^{\text{eff}}(\omega)$ was found in Ref. [84]. We also note that the inequality (5.65) is the only sufficient condition for the entanglement, which means that this criterion may not capture the full entanglement.

We here define quality factor $Q_\pm$ and cooperativity $C_\pm$ for the optomechanical systems as follows:

$$Q_\pm = \frac{\Omega_\pm}{\gamma_m}, \quad C_\pm = \frac{4g_\pm^2}{\Gamma_{\text{fb}}\kappa}, \quad (5.68)$$

where the asymmetry between common and differential modes originates from the gravity-shifted frequency $\Omega_\pm$. Hereafter, we fix the Fourier frequency to the resonant frequency of the mechanical mirror $\omega = \Omega$, which is the same as the previous researches [67, 68]. For simplicity, we also assume the perfect detection $\eta = 1$ and vacuum noise input $N_{\text{th}} = 0$. Then, the entanglement between the mechanical mirrors or between the output lights is determined only by the five dimensionless parameters $(C_+, Q_+, n_{\text{thfb}}^+, \kappa/\Gamma_{\text{fb}}, \epsilon)$. We first consider the case

without quantum filtering. The analytic solution of the degree of the entanglement without the Wiener filter is obtained by

$$E(\Omega) = \frac{4Q_+Q_-}{(Q_+^4 + Q_-^4 - 2Q_+^2(Q_-^2 - 1))^2(1 + 4\Omega^2/\kappa^2)}(Q_+^4 + Q_-^4 - Q_+^2(2Q_-^2 - 1)) \\ \times (1 + 2C_+ + 2n_{\text{thfb}}^+ + 4(2n_{\text{thfb}}^+ + 1)\Omega^2/\kappa^2)(1 + 2C_- + 2n_{\text{thfb}}^- + 4(2n_{\text{thfb}}^- + 1)\Omega^2/\kappa^2). \quad (5.69)$$

Here, we consider the entanglement criterion under the two assumptions. One is that the quantum cooperativity is larger than unity $C_q^+ \equiv C_+/n_{\text{thfb}}^+ \gg 1$, which means that the radiation pressure noise is dominant compared to the thermal noise. Because the radiation pressure noise is adjusted by controlling lights, this condition guarantees that we can control the motion of the mirrors through the interaction with lights. Another assumption is the condition of an adiabatic limit $\kappa \gg \Omega$, which allows us to monitor the mechanical mirror's position continuously. These assumptions are necessary for quantum control and continuous measurements. Then we found that the entanglement criterion yields

$$Q_{+\epsilon} > 4C_+, \quad (5.70)$$

which is equivalent to

$$\frac{Gm}{L^3\Omega} > 4\frac{g_+^2}{\kappa}. \quad (5.71)$$

This inequality represents that the gravitational interaction between the mechanical mirrors is dominant against the measurement process using the cavity lights. This guarantees the generation of the gravity-induced entanglement between the mirrors.

On the other hand, Ref. [67, 68] showed that the condition for generating the gravity-induced entanglement between output lights is

$$Q_{+\epsilon} > 4n_{\text{thfb}}^+, \quad (5.72)$$

under the same assumptions. This is the necessary condition for the inequality (5.70) as long as $C_q^+ \gg 1$. It is, therefore, more severe to verify the gravity-induced entanglement between the mirrors than between the output lights, as demonstrated in Figure 25.

Figure 25 illustrates the degree of entanglement $E(\Omega)$ as a function of the quality factor Q_+ and the quantum cooperativity $C_q^+ (= C_+/n_{\text{thfb}}^+)$ in the absence of the Wiener filter. The thermal phonon number is set to $n_{\text{thfb}}^+ = 1$ for the left panel and $n_{\text{thfb}}^+ = 10$ for the right panel. The red dashed line in both panels corresponds to the boundary defined by the relation (5.70). This boundary agrees well with the numerical results for gravity-induced entanglement generation in the regime $C_q^+ \gg 1$. The black vertical dashed line represents the condition $Q_+ \epsilon = 4n_{\text{thfb}}^+$, which serves as the threshold for generating entanglement between the output light fields in Fourier space, as characterized by the entanglement negativity [67, 68]. A brief review of the definition of the entanglement negativity for the output light fields is provided in Appendix D. The gravity-induced entanglement between the output light fields occurs in the region to the right of the black dashed line, where the quantum cooperativity dominates over the thermal noise contributions.

Figure 26 presents the same analysis as Fig. 25, but now incorporating the Wiener filter. Implementing the Wiener filter enhances the entanglement generation, particularly in the regime of large quantum cooperativity and high mechanical quality factor. In this region, the degree of quantum entanglement $E(\Omega)$ increases by approximately 3 compared to the case without the Wiener filter. In contrast, for regions characterized by small quantum cooperativity and low mechanical quality factor, the entangled region shows only minor changes when applying the Wiener filter. Our results demonstrate that gravity-induced entanglement between the mechanical mirrors inevitably leads to entanglement between the output light fields, regardless of whether the Wiener filter is used. However, since we cannot access the unconditional variable experimentally, the result with the Wiener filtering is valid for experimental verification. As discussed in Chapter 3, we obtain the conditional state by retrodiction for $\lambda_I^\pm > \Gamma_{\text{fb}}$, which restricts the lower bound of quantum cooperativity C_q .

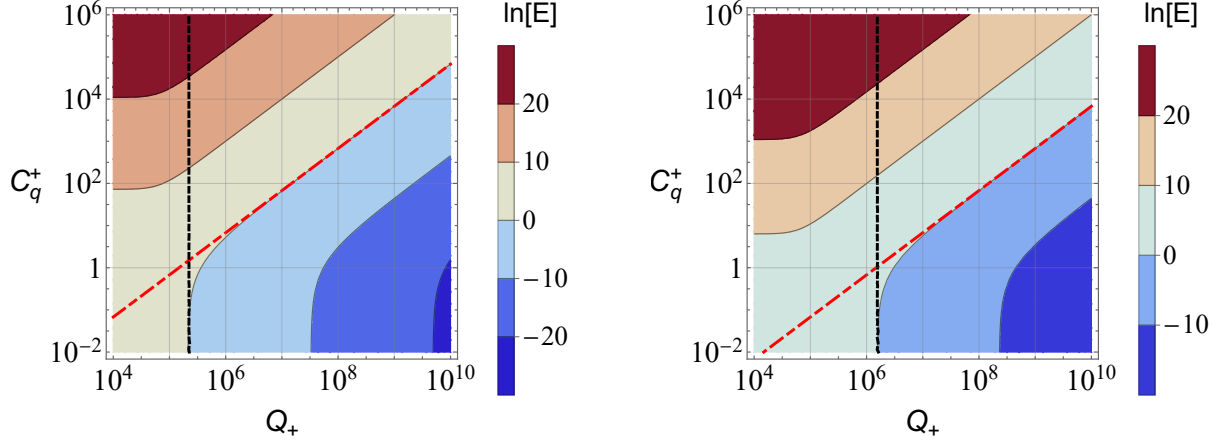


FIG 25: Entanglement behavior of $\ln[E(\Omega)]$ as a function of quality factor Q_+ and quantum cooperativity $C_q^+ = C_+/n_{\text{thfb}}^+$ without the quantum Wiener filter. If $\ln[E(\Omega)] < 0$, the two mechanical mirrors are entangled. Parameters are $\kappa/\Gamma_{\text{fb}} = 10^{13}$, $\epsilon = 2.7 \times 10^{-5}$, $n_{\text{thfb}}^+ = 1$ (left panel), and $n_{\text{thfb}}^+ = 10$ (right panel). The red dashed line represents the boundary of the approximate entanglement condition Eq. (5.70). Gravity-induced entanglement between output lights occurs in the region to the right of the black vertical dashed line.

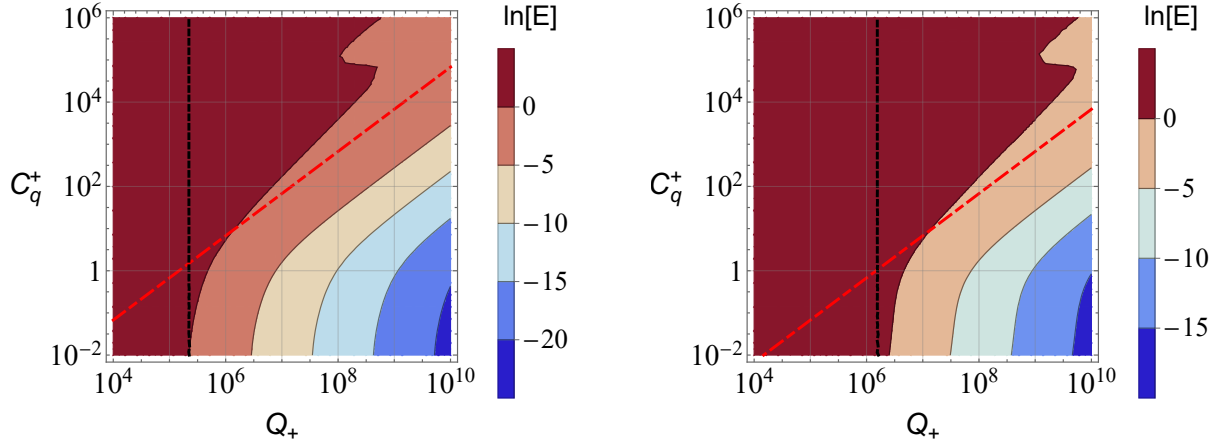


FIG 26: Same as Fig. 25 but with the quantum Wiener filter.

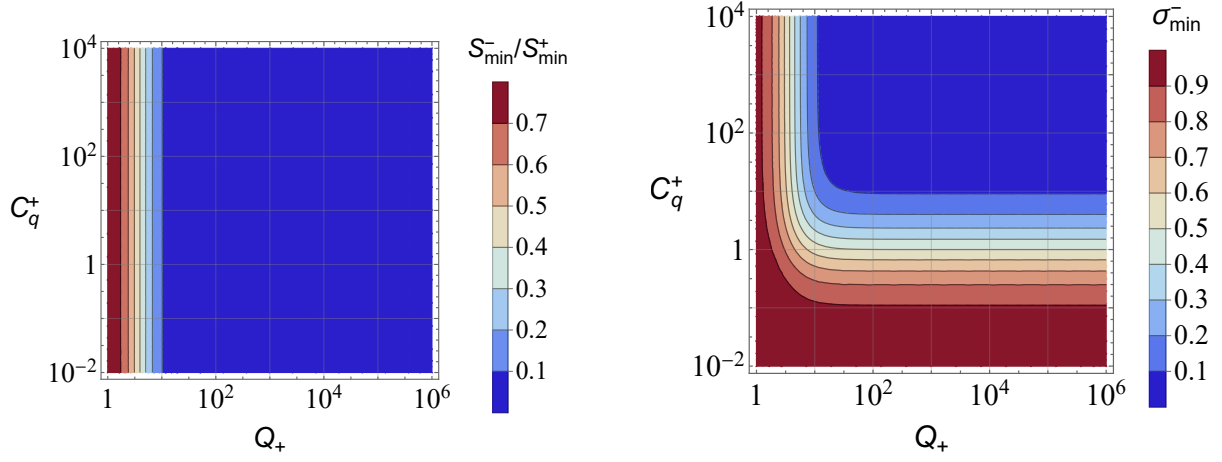


FIG 27: The left panel shows the ratio of the minimum variances of the mechanical differential mode to the common mode on the plane of the quantum cooperativity $C_q^+ = C_+/n_{\text{thfb}}^+$ and the quality factor Q_+ without the Wiener filter. The right panel shows the minimum variance of the optical differential mode. Parameters are $\kappa/\Gamma_{\text{fb}} = 10^{10}$, $\epsilon = 0.27$, and $n_{\text{thfb}}^+ = 2.1 \times 10^{11}$.

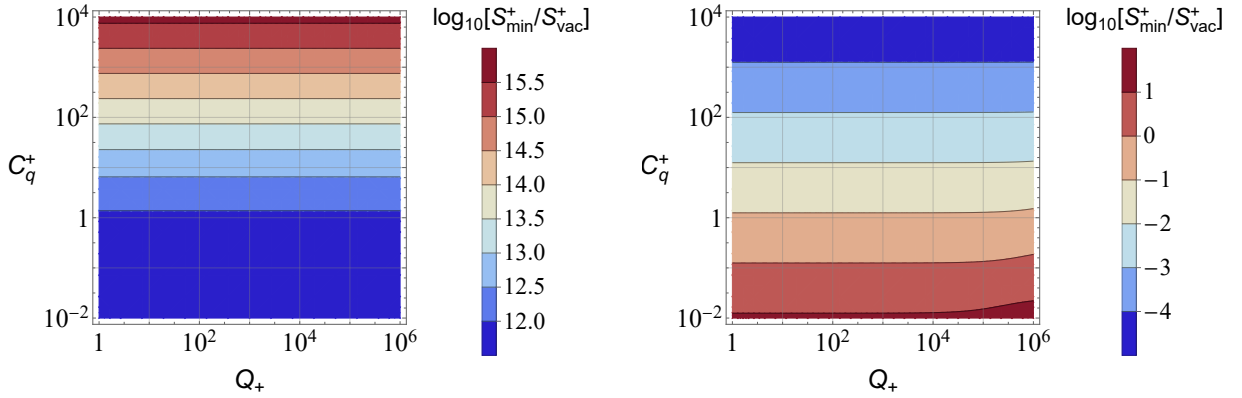


FIG 28: The behaviors of $S_{\min}^+/S_{\text{vac}}^+$ without the filter (left panel) and with the filter (right panel), where the parameters are the same as those of Fig. 27.

5.3.3 Gravity-induced squeezing

This section focuses on the gravity-induced squeezing of mirrors, which might be a quantum signature of gravity. Our method in this section is an application of the investigation for the gravity-induced squeezing of output lights in Ref. [68] to mechanical mirrors. We introduce the mechanical quadrature

$$\tilde{q}_\theta^\pm = \tilde{q}_\pm \cos \theta + \tilde{p}_\pm \sin \theta, \quad (5.73)$$

where θ is a rotation angle in the phase space. The spectral density is

$$S_\pm(\omega, \theta) = S_{\tilde{q}\tilde{q}}^\pm(\omega) \cos^2 \theta + S_{\tilde{p}\tilde{p}}^\pm(\omega) \sin^2 \theta + (S_{\tilde{q}\tilde{p}}^\pm(\omega) + S_{\tilde{p}\tilde{q}}^\pm(\omega)) \cos \theta \sin \theta, \quad (5.74)$$

and its minimum value is

$$S_\pm^{\min}(\omega) = \frac{S_{\tilde{q}\tilde{q}}(\omega) + S_{\tilde{p}\tilde{p}}(\omega)}{2} - \frac{\sqrt{(S_{\tilde{q}\tilde{q}}(\omega) - S_{\tilde{p}\tilde{p}}(\omega))^2 + (S_{\tilde{q}\tilde{p}}(\omega) + S_{\tilde{p}\tilde{q}}(\omega))^2}}{2}. \quad (5.75)$$

We first analyze the case without the quantum filter, focusing on the mechanical resonant frequency $\omega = \Omega$. Under this condition, we derived the ratio of the spectral densities of the differential mode to the common mode as

$$\frac{S_-^{\min}(\Omega)}{S_+^{\min}(\Omega)} = \frac{\sqrt{1 - \epsilon}}{1 + Q_+^2 \epsilon^2}, \quad (5.76)$$

where $n_{\text{thfb}}^\pm = k_B T \Gamma / \hbar \Omega_\pm \Gamma_{\text{fb}} \gg 1$ is assumed. This expression is independent of both the quantum cooperativity and the thermal noise of the mirrors, remaining valid even in the low quantum cooperativity regime. The left panel of Figure 27 visualizes the contour of Eq. (5.76) in the plane of the quality factor Q_+ and quantum cooperativity C_q^+ . The right panel of Figure 27 shows the gravity-induced squeezing in the output light of the differential mode, $\sigma_{\min}^-$. The gravity-induced squeezing for the output lights is briefly reviewed in Appendix D, following the analysis in Refs. [67, 68], the gravity-induced squeezing for the common mode output light is always unity, $\sigma_{\min}^+ = 1$. Thus, the right panel of Fig. 27 plots the ratio $\sigma_{\min}^- / \sigma_{\min}^+$, comparing the differential mode to the common mode. The calculations adopt the parameters $n_{\text{th}}^+ = 2.1 \times 10^{11}$ and $\epsilon = 0.27$, derived from $T = 1\text{K}$, $\Gamma/2\pi = 10^{-6}\text{Hz}$, $\gamma_m/2\pi = 10^{-4}\text{Hz}$, $\Omega/2\pi = 10^{-3}\text{Hz}$, $m/L^3 = \rho\Lambda = 2 \times 20\text{g/cm}^3$.

The asymmetry between the differential and common modes arises solely from the gravitational interaction characterized by ϵ . Consequently, the deviations from unity in the ratios $S_{\min}^-/S_{\min}^+$ and $\sigma_{\min}^-/\sigma_{\min}^+$ in Fig. 27 reflect gravitational interaction effects. However, the interpretation of gravity-induced squeezing requires caution. Figure 28 presents the spectral density of the mechanical common mode $S_{\min}^+$, normalized by the vacuum state spectral density S_{vac}^+ , as a function of the quantum cooperativity and quality factor. The vacuum state spectral density is given by

$$S_{\text{vac}}^+ = \frac{1}{2} (S_{qq,\text{vac}}^+(\Omega) + S_{pp,\text{vac}}^+(\Omega)) = \frac{2}{\Gamma_{\text{fb}}}, \quad (5.77)$$

which is derived from Eq. (5.75) by neglecting the thermal phonon number and optical noise inputs. The left panel of Fig. 28 demonstrates that applying a quantum Wiener filter eliminates thermal noise, enabling conditional quantum-squeezed states for $C_q^+ \gtrsim 0.1$. We note that the panels of Fig. 27 are obtained with no use of the quantum Wiener filter, and then the state of the mirrors is dominated by the thermal noises. These facts suggest that we need to be careful to interpret the ratio $S_{\min}^-/S_{\min}^+$ in the left panel of Fig. 27 as a quantum nature of gravity. There is no support that the gravity-induced squeezing in the optomechanical mode comes from the quantum nature of gravity.

5.3.4 Conclusions

We analyzed the quantum signature of gravity in optomechanical systems under quantum control and found the condition for generating the gravity-induced entanglement between the mechanical mirrors with/without the quantum filter in Fourier space. We summarized the conditions for generating gravity-induced entanglement and gravity-induced squeezing in Table 4. The condition of generating gravity-induced entanglement between mirrors is expressed by $Q_{+\epsilon} > 4C_+$, as long as the condition $C_q^+ \gg 1$ is satisfied. The entanglement between output lights occurs for $Q_{+\epsilon} > 4n_{\text{th}}^+$ in Ref. [67, 68]. As shown in Fig. 25, the condition for generating the entanglement between the mirrors found in the present paper is more severe than the condition for the entanglement between the output lights. This is true even with the quantum filter, as shown in Fig. 26. Hence, the gravity-induced entanglement between the output lights may be easier to detect than between the mirrors.

We also investigated the conditions for gravity-induced mechanical squeezing to compare with the gravity-induced squeezing of optical modes [68]. We found the ratio of the gravity-

induced squeezing of mechanical modes, $S_-^{\min}/S_+^{\min} = \sqrt{1-\epsilon}/(1+Q_+^2\epsilon^2)$, which is similar to that of optical modes Eq. (D.14) in the region $C_q^+ \gg 1$. However, gravity-induced squeezing appears when the thermal (classical) noises dominate, and the ratio of the gravity-induced squeezing of the differential mode to the common mode does not depend on the quantum cooperativity. Such a situation implies caution when interpreting gravity-induced squeezing, which appears in the dominant regime of thermal noise, as a quantum nature of gravity.

Previous research	Oscillators	Optomechanical systems	
		Output lights	
Entanglement	Ref. [23, 24, 68]	Ref. [67, 68] and Eq. (5.72)	
Squeezing	Ref. [68]	Ref. [68] and Eq. (D.11)	
Our research	Optomechanical systems		
	Mechanical mirrors		
	Time domain		Fourier domain
	With filter	With filter	Without filter
Entanglement	Eqs. (5.37)-(5.38), and Fig. 22	Fig. 26	Eq. (5.70) and Fig. 25
Squeezing	Fig. 24	Fig. 28	Eq. (5.76) and Fig. 27

Table 4: Summary of the conditions for gravity-induced entanglement and gravity-induced squeezing.

6 Conclusions and Future Work

Quantum gravity theory is a fundamental question in physics that has remained unresolved for many years. Verifying quantum gravity effects requires high-energy experiments corresponding to the Planck mass scale, which has long been considered experimentally unfeasible. However, recent advancements in quantum technology have opened up new possibilities for testing quantum gravity, specifically by verifying gravity-induced entanglement. Gravity-induced entanglement is a consequence of the quantum field theory of linearized gravity in the non-relativistic limit, and the relation of gravity-induced entanglement in aspects of the gravitons has been discussed. However, the signal of gravity-induced entanglement is small because the gravitational interaction is very weak. This thesis has covered two main topics: Realizing macroscopic quantum systems in optomechanical systems under quantum control and their application to verifying gravity-induced entanglement.

In Chapter 2, we reviewed the proposed model for verifying gravity-induced entanglement between two spatially superposed particles. This model demonstrates that Newtonian gravity generates the entanglement between two superposed particles if gravity follows quantum mechanics. We showed the behavior of the entanglement, which indicated the periodical modulation of the entanglement. Achieving gravity-induced entanglement requires a coherence time $\mathcal{O}(1)$ s for an object with a mass of 10^{-14} kg in a spatial superposition separated by $100\text{ }\mu\text{m}$. This is a challenging problem due to the effects of environmental decoherence. Considering the thermal noise, the entanglement generation time scale must be faster than the time scale of the thermal decoherence, or quantum entanglement will be lost. Experimental verification of gravity-induced entanglement requires a reduction in the effects of thermal noise.

One approach to realizing the quantum macroscopic state is optomechanics. In Chapter 3, we outlined the fundamentals of optomechanical system dynamics and the principles of quantum control required to realize macroscopic quantum systems where gravitational effects are measurable. In optomechanical systems, macroscopic mirrors interact with the cavity light through the radiation pressure of light, and we can read out the displacements in the mirror's position from the output light. We explained that the influence of thermal noise can be mitigated by employing quantum feedback control based on the measurements. Moreover, quantum filtering techniques conditioned on the measurement results can reduce mirror fluctuations to the quantum fluctuation level. Using these quantum controls, we have demonstrated that high purity and quantum squeezing occur using the expected parameters

in the midterm. These insights represent a significant step toward experimentally verifying the quantum nature of gravity using macroscopic optomechanical systems under quantum control.

In Chapter 4, we investigated the feasibility of generating the entanglement between two milligram-scale mirrors in a power-recycled Fabry-Pérot-Michelson interferometer. The mechanical mirrors are entangled by the optical beam passing the half-beam splitter, which is an analogy to the fact that when two squeezing states are injected into a half-beam splitter, the post-injection states are entangled. This is also an analogy for generating the EPR state. We analyzed the entanglement behavior as a function of the cavity detuning Δ and the quantum cooperativity C_-/n_{thfb} . The required value of the quantum squeezing of the mechanical modes is $C_-/n_{\text{thfb}} \gtrsim 1$, but this condition is not enough to generate the entanglement. We demonstrated that quantum squeezing and high purity are necessary to generate the entanglement. For the measurement of the amplitude quadrature, the required value is $C_-/n_{\text{thfb}} \gtrsim 3$ for $\zeta = 3$ and $\Delta = 0.2\kappa_-$. Therefore, we can achieve the entanglement between two milligram-scale mass objects in the short term.

In Chapter 5, we evaluated gravity-induced entanglement between two gram-scale mirrors in optomechanical systems. we first investigated the time evolution of gravity-induced entanglement between mechanical mirrors. Solving the Riccati equation, we estimate the time scale of generating the entanglement as $t \simeq 4/\gamma_Y^+ \simeq 600$ s, the time scale of reaching the steady state. We derive the conditions for generating gravity-induced entanglement as $\Omega\epsilon > 4\sqrt{2}\Gamma_{\text{fb}}n_{\text{thfb}}^+$ and $\gamma_Y^+ > 16\Gamma_{\text{fb}}n_{\text{thfb}}$. Additionally, we determined the feasible parameters using the signal-to-noise ratio, which is $\tau \simeq 2 \times 10^6$ s for the phase field measurement. These predictions of the quantum nature of gravity may assist in testing and understanding quantum gravity theory.

Moreover, we first discussed the entanglement in the Fourier domain at the mechanical resonance frequency. Without quantum filtering, the condition for generating gravity-induced entanglement is expressed as $Q_{+\epsilon} > 4C_+$ under the condition $C_+/n_{\text{thfb}} \gg 1$. When we use the quantum filter, the entanglement condition is similar to the case without the quantum filtering. However, since we do not have direct access to the variances and covariance of the mechanical mirror, the results with the quantum filter for $\lambda_I^\pm > \Gamma_{\text{fb}}$ are valid for experimental verification. We also showed that the entanglement condition found in the present research is more severe than the condition for the entanglement between output lights, discussed in Refs. [67, 68]. Hence, the gravity-induced entanglement between the

output lights may be easier to detect than between the mirrors in the Fourier space. We also evaluated gravity-induced squeezing of the mechanical modes, while Ref. [68] discussed gravity-induced squeezing of output lights. We evaluated the gravity-induced squeezing, similar to the squeezing of output light under $C_+/n_{\text{thfb}} \gg 1$. However, gravity-induced squeezing also appears even for large thermal noises. Ref. [88, 89] has reported that semiclassical gravity produces similar squeezing. Therefore, we should pay attention to the relationship between gravity-induced squeezing and the quantum nature of gravity.

There are still many problems that need to be solved. In Chapter 4, we discussed the entanglement generation between macroscopic mirrors using the power-recycled mirror. We showed that asymmetry between the common mode and the differential mode is the essence of generating quantum entanglement. It is also possible to introduce the signal-recycled mirror on the differential mode side [45, 66], which may increase the asymmetry of the common and differential modes. In addition, the use of incident squeezing light to increase sensitivity [45, 66] is interesting approach to test the entanglement. From the perspective of macroscopic quantum phenomena, the application of the Leggett-Garg inequality [5, 6, 107] to optomechanical systems to verify macrorealism, and the verification of negative values of the Wigner function using non-Gaussian states [108, 109] are also interesting. In Chapter 5, we discussed the generation of entanglement by gravity, but the analysis based on the amplitude field measurements is insufficient. It is necessary to clarify how the detuning and measurement process affect the mechanical squeezing and asymmetry of the common and differential modes. Furthermore, it is interesting approach to distinguish the effects of semiclassical gravity and quantum gravity. Although semiclassical gravity can produce squeezing and correlations through the measurement process [88, 89], we have been working on a model that can distinguish between semiclassical and quantum gravity signals [90].

A Calculation of the causal part for Wiener filter

Using Eqs. (3.63) and (3.64), the spectral density of the general output field is

$$S_{bb} = \phi_+ \phi_- = (2\eta N_{\text{th}} + 1) \frac{|\chi'_{\text{eff}}|^2}{|\chi_{\text{eff}}|^2}, \quad (\text{A.1})$$

and the causal part and non-causal parts are

$$\phi_+ = \sqrt{2\eta N_{\text{th}} + 1} \frac{\chi_{\text{eff}}}{\chi'_{\text{eff}}}, \quad \phi_- = \sqrt{2\eta N_{\text{th}} + 1} \frac{\chi_{\text{eff}}^*}{\chi_{\text{eff}}'^*} \quad (\text{A.2})$$

$$\chi_{\text{eff}} = \frac{1}{m(\omega_m^2 - i\Gamma_{\text{fb}}\omega - \omega^2)}, \quad \chi'_{\text{eff}} = \frac{1}{m(\Omega'^2 - i\Gamma'\omega - \omega^2)}, \quad (\text{A.3})$$

where $\Omega' = (\beta')^{1/4}$ and $\Gamma' = \sqrt{\alpha' + 2\sqrt{\beta'}}$ with

$$\alpha' = \Gamma_{\text{fb}}^2 - 2\omega_m^2 + \frac{16g_m^2\eta\kappa\omega_m(4\kappa\Delta\cos 2\theta - (\kappa^2 - 4\Delta^2)\sin 2\theta)(2N_{\text{th}} + 1)\eta}{(\kappa^2 + 4\Delta^2)^2} \frac{1}{2\eta N_{\text{th}} + 1} \quad (\text{A.4})$$

$$\begin{aligned} \beta' = & \omega_m^4 + \frac{16g_m^2\omega_m^2\Gamma_{\text{fb}}\kappa(2n_{\text{th}} + 1)\eta}{\kappa^2 + 4\Delta^2} \frac{1}{2\eta N_{\text{th}} + 1} + \frac{128g_m^4\omega_m^2\kappa^2}{(\kappa^2 + 4\Delta^2)^2} - \frac{16g_m^2\omega_m^2\kappa}{(\kappa^2 + 4\Delta^2)^2} \\ & \times \left[\left(\Gamma_{\text{fb}}(\kappa^2 - 4\Delta^2) \frac{(2n_{\text{th}} + 1)\eta}{2\eta N_{\text{th}} + 1} + 4\kappa(\omega_m\Delta + \frac{2g_m^2(\kappa^2 - 4\Delta^2)}{\kappa^2 + 4\Delta^2}) \frac{(2N_{\text{th}} + 1)\eta}{2\eta N_{\text{th}} + 1} \right) \cos 2\theta \right. \\ & \left. + \left(4\Gamma_{\text{fb}}\kappa\Delta \frac{(2n_{\text{th}} + 1)\eta}{2\eta N_{\text{th}} + 1} + ((\kappa^2 - 4\Delta^2)\omega_m + \frac{32g_m^2\kappa^2\Delta}{\kappa^2 + 4\Delta^2}) \frac{(2N_{\text{th}} + 1)\eta}{2\eta N_{\text{th}} + 1} \right) \sin 2\theta \right]. \quad (\text{A.5}) \end{aligned}$$

g_m is defined in Eq. 3.50, which is the optomechanical coupling rescaled at ω_m . The correlation spectral density of $\delta\hat{Q}$ and $\hat{b}$ is

$$\begin{aligned} S_{\delta Q b} = & \sqrt{\frac{\hbar}{2m\omega_m}} \frac{4g_m\omega_m\sqrt{\kappa\eta}(\kappa\cos\theta + 2\Delta\sin\theta)(2N_{\text{th}} + 1)}{\kappa^2 + 4\Delta^2} |m\chi_{\text{eff}}|^2 (\tilde{\Omega}^2 + i\Gamma_{\text{eff}}\omega - \omega^2) \\ \tilde{\Omega}^2 = & \omega_m^2 \left(\left(1 - \frac{32g_m^2\Delta}{\omega_m(\kappa^2 + 4\Delta^2)} - \frac{4\Gamma_{\text{fb}}\Delta}{\kappa\omega_m} \frac{2n_{\text{th}} + 1}{2N_{\text{th}} + 1} \right) \frac{\kappa\cos\theta}{\kappa\cos\theta + 2\Delta\sin\theta} \right. \\ & \left. + \left(1 + \frac{8g_m^2\kappa^2}{\omega_m\Delta(\kappa^2 + 4\Delta^2)} + \frac{\Gamma_{\text{fb}}\kappa}{\omega_m\Delta} \frac{2n_{\text{th}} + 1}{2N_{\text{th}} + 1} \right) \frac{2\Delta\sin\theta}{\kappa\cos\theta + 2\Delta\sin\theta} \right), \quad (\text{A.6}) \end{aligned}$$

and $S_{\delta Pb} = -im\omega S_{\delta Qb}$. Using Eq. (3.59), we have

$$G_q = \sqrt{\frac{\hbar}{2m\omega_m}} \frac{4g_m\omega_m(\kappa \cos \theta + 2\Delta \sin \theta)(2N_{\text{th}} + 1)\sqrt{\kappa\eta}}{(\kappa^2 + 4\Delta^2)\sqrt{2\eta N_{\text{th}} + 1}} m\chi_{\text{eff}}(\gamma_1\omega + \delta_1) \quad (\text{A.7})$$

$$G_p = \sqrt{\frac{m\hbar\omega_m}{2}} \frac{4g_m(\kappa \cos \theta + 2\Delta \sin \theta)(2N_{\text{th}} + 1)\sqrt{\kappa\eta}}{(\kappa^2 + 4\Delta^2)\sqrt{2\eta N_{\text{th}} + 1}} m\chi_{\text{eff}}(\gamma_2\omega + \delta_2), \quad (\text{A.8})$$

where

$$\gamma_1 = -i \frac{\Gamma'(\tilde{\Omega}^2 - \omega_m^2) + \Gamma_{\text{fb}}(\omega_m^2 - 2\Omega'^2 + \tilde{\Omega}^2)}{(\omega_m^2 - \Omega'^2)^2 + (\Gamma_{\text{fb}} + \Gamma')(\omega_m^2\Gamma' + \Omega'^2\Gamma_{\text{fb}})} \quad (\text{A.9})$$

$$\delta_1 = \frac{\Gamma_{\text{fb}}(\Gamma_{\text{fb}} + \Gamma')(\omega_m^2 + \tilde{\Omega}^2) + (\omega_m^2 - \Omega'^2)(\omega_m^2 - \tilde{\Omega}^2)}{(\omega_m^2 - \Omega'^2)^2 + (\Gamma_{\text{fb}} + \Gamma')(\omega_m^2\Gamma' + \Omega'^2\Gamma_{\text{fb}})} \quad (\text{A.10})$$

$$\gamma_2 = -i \frac{2\Gamma_{\text{fb}}^2\Omega'^2 + 2\Gamma_{\text{fb}}\Gamma'\omega_m^2 + (\omega_m^2 - \Omega'^2)(\omega_m^2 - \tilde{\Omega}^2)}{(\omega_m^2 - \Omega'^2)^2 + (\Gamma_{\text{fb}} + \Gamma')(\omega_m^2\Gamma' + \Omega'^2\Gamma_{\text{fb}})} \quad (\text{A.11})$$

$$\delta_2 = -\frac{\omega_m^2(\Gamma'(\tilde{\Omega}^2 - \omega_m^2) + \Gamma_{\text{fb}}(\omega_m^2 - 2\Omega'^2 + \tilde{\Omega}^2))}{(\omega_m^2 - \Omega'^2)^2 + (\Gamma_{\text{fb}} + \Gamma')(\omega_m^2\Gamma' + \Omega'^2\Gamma_{\text{fb}})}. \quad (\text{A.12})$$

We can then derive the conditional variance and covariance as

$$\mathcal{V}_{\delta Q \delta Q}^{\text{wf}} = \frac{\hbar}{2m\omega_m} \left(2n_{\text{th}} + 1 + \frac{8g_m^2\kappa(2N_{\text{th}} + 1)}{\Gamma_{\text{fb}}(\kappa^2 + 4\Delta^2)} - \frac{8g_m^2\kappa(\kappa \cos \theta + 2\Delta \sin \theta)^2(2N_{\text{th}} + 1)^2\eta(|\gamma_1|^2\omega_m^2 + |\delta_1|^2)}{\Gamma_{\text{fb}}(\kappa^2 + 4\Delta^2)^2(2\eta N_{\text{th}} + 1)} \right) \quad (\text{A.13})$$

$$\mathcal{V}_{\delta P \delta P}^{\text{wf}} = \frac{m\hbar\omega_m}{2} \left(2n_{\text{th}} + 1 + \frac{8g_m^2\kappa(2N_{\text{th}} + 1)}{\Gamma_{\text{fb}}(\kappa^2 + 4\Delta^2)} - \frac{8g_m^2\kappa(\kappa \cos \theta + 2\Delta \sin \theta)^2(2N_{\text{th}} + 1)^2\eta(|\gamma_2|^2\omega_m^2 + |\delta_2|^2)}{\omega_m^2\Gamma_{\text{fb}}(\kappa^2 + 4\Delta^2)^2(2\eta N_{\text{th}} + 1)} \right) \quad (\text{A.14})$$

$$\mathcal{V}_{\delta Q \delta P}^{\text{wf}} = -\frac{\hbar}{2} \frac{8g_m^2\kappa(\kappa \cos \theta + 2\Delta \sin \theta)^2(2N_{\text{th}} + 1)^2\eta(-\gamma_1\gamma_2\omega_m^2 + \delta_1\delta_2)}{\omega_m\Gamma_{\text{fb}}(\kappa^2 + 4\Delta^2)^2(2\eta N_{\text{th}} + 1)}. \quad (\text{A.15})$$

These variances and covariance are identical to those derived from the Kalman filtering for the stationary solution.

B Input-Output Relation for the Interferometer

In this Appendix, we derive the input-output relation for the power-recycled Fabry-P'erot-Michelson interferometer depicted in Fig. 10. First, we consider the input-output relation for an individual Fabry-P'erot cavity. Assuming the same optical decay rate for both cavities, the input-output relations can be written as

$$\hat{x}_A^{\text{out}} = \sqrt{\kappa}\hat{x}_A - \hat{x}_A^{\text{in}}, \quad (\text{B.1})$$

$$\hat{x}_B^{\text{out}} = \sqrt{\kappa}\hat{x}_B - \hat{x}_B^{\text{in}}, \quad (\text{B.2})$$

Since no mirror is present on the differential mode side, the output optical quadrature for the differential mode is given by

$$\begin{aligned} \hat{x}_-^{\text{out}} &= \frac{\hat{x}_A^{\text{out}} - \hat{x}_B^{\text{out}}}{\sqrt{2}} = \frac{\hat{x}_A^{\text{in}} - \hat{x}_B^{\text{in}} - \sqrt{\kappa}(\delta\hat{x}_A - \delta\hat{x}_B)}{\sqrt{2}} \\ &= \sqrt{\kappa}\delta\hat{x}_- - \hat{x}_-^{\text{in}}. \end{aligned} \quad (\text{B.3})$$

Next, we consider the input-output relation for the common mode, considering the power-recycled mirror. Let T and $R = 1 - T$ denote the transmissivity and reflectivity of the power-recycled mirror, respectively. The input-output relations for the common mode can be expressed as

$$\hat{x}_+^{\text{out}} = \sqrt{R}\hat{x}_+^{\text{in}} + \sqrt{T}\hat{\rho}_+^{\text{out}}, \quad (\text{B.4})$$

$$\hat{\rho}_+^{\text{in}} = -\sqrt{R}\hat{\rho}_+^{\text{out}} + \sqrt{T}\hat{x}_+^{\text{in}}, \quad (\text{B.5})$$

where $\hat{\rho}_+^{\text{out}}$ and $\hat{\rho}_+^{\text{in}}$ are

$$\hat{\rho}_+^{\text{out}} = \frac{\hat{x}_A^{\text{out}} + \hat{x}_B^{\text{out}}}{\sqrt{2}}, \quad (\text{B.6})$$

$$\hat{\rho}_+^{\text{in}} = \frac{\hat{x}_A^{\text{in}} + \hat{x}_B^{\text{in}}}{\sqrt{2}}. \quad (\text{B.7})$$

Substituting these relations, we obtain the output optical quadrature for the common mode as

$$\begin{aligned}\hat{x}_+^{\text{out}} &= \frac{\hat{x}_A^{\text{in}} + \hat{x}_B^{\text{in}} - \sqrt{\frac{1-\sqrt{R}}{1+\sqrt{R}}}\kappa(\delta\hat{x}_A + \delta\hat{x}_B)}{\sqrt{2}} \\ &= \sqrt{\frac{1-\sqrt{R}}{1+\sqrt{R}}}\kappa\delta\hat{x}_+ - \hat{x}_+^{\text{in}}\end{aligned}\tag{B.8}$$

From this result, we derive the relationship between the optical decay rates of the common mode and the differential mode

$$\kappa_+ = \frac{1-\sqrt{R}}{1+\sqrt{R}}\kappa_- \equiv \frac{1}{\tau}\kappa_-.\tag{B.9}$$

where the power-recycled mirror introduces the asymmetry factor τ . Thus, we have derived the input-output relations for the common and differential modes of the power-recycled Fabry-P'erot-Michelson interferometer.

C Exact formula of stationary entanglement in the interferometer

In this Appendix, we derive the exact formula of the entanglement negativity in power-recycled Fabry-Pérot-Michelson interferometer. The conditional covariance matrix for the stationary state is derived by Eqs. (4.36)-(4.38). Using these components of the covariance matrix, we have

$$\Sigma = \frac{\hbar^2(\gamma_I^+ - \gamma_m)(\gamma_I^- - \gamma_m)}{8\lambda_I^+\lambda_I^-} \left(\frac{(\gamma_I^+)^2 + (\gamma_I^-)^2 - \gamma_I^+\gamma_I^- - \gamma_m^2}{\omega_m^+\omega_m^-} + 2\frac{\Lambda_I^+ + \omega_m^+}{\omega_m^-} + 2\frac{\Lambda_I^- + \omega_m^-}{\omega_m^+} \right), \quad (\text{C.1})$$

$$\det \mathbf{V} = \left(\frac{\hbar^2(\gamma_I^+ - \gamma_m)(\gamma_I^- - \gamma_m)}{16\lambda_I^+\lambda_I^-} \right)^2 \left(\frac{(\gamma_I^+)^2 - \gamma_m^2}{(\omega_m^+)^2} + 4\frac{\Lambda_I^+}{\omega_m^+} + 4 \right) \left(\frac{(\gamma_I^-)^2 - \gamma_m^2}{(\omega_m^-)^2} + 4\frac{\Lambda_I^-}{\omega_m^-} + 4 \right). \quad (\text{C.2})$$

Then, we can exactly derive the logarithmic negativity as a function of $Q_\pm$, $C_\pm$, $n_{\text{thfb}}^\pm$, N_{th} , $\Delta/\kappa_\pm$, and η . The critical value ϵ_{cr} in Eq. (4.31) is obtained by

$$\begin{aligned} \epsilon_{\text{cr}} = & -\frac{1}{2} \log_2 \left[\frac{(\gamma_I^{+'} - 1)(\gamma_I^{-'} - 1)}{4\lambda_I^{+'}\lambda_I^{-'}} \left\{ \left(\frac{(\gamma_I^{+'})^2 + (\gamma_I^{-'})^2 - \gamma_I^{+'}\gamma_I^{-'} - 1}{Q_+Q_-} + 2\frac{\Lambda_I^{+'} + Q_+}{Q_-} + 2\frac{\Lambda_I^{-'} + Q_-}{Q_+} \right) \right. \right. \\ & - \left(\frac{((\gamma_I^{+'})^2 + (\gamma_I^{-'})^2 - 1)(\gamma_I^{+'} - \gamma_I^{-'})^2}{Q_+^2Q_-^2} + 4 \left(\frac{\Lambda_I^{+'} + Q_+}{Q_-} - \frac{\Lambda_I^{-'} + Q_-}{Q_+} \right)^2 \right. \\ & \left. \left. + 4 \frac{\gamma_I^{+'}(\gamma_I^{+'} - \gamma_I^{-'})(\Lambda_I^{+'} + Q_+)}{Q_+Q_-^2} - 4 \frac{\gamma_I^{-'}(\gamma_I^{-'} - \gamma_I^{+'})(\Lambda_I^{-'} + Q_-)}{Q_+^2Q_-} \right)^{1/2} \right\} \right], \quad (\text{C.3}) \end{aligned}$$

where

$$\frac{\lambda_X^\pm}{\gamma_m} = \frac{16C_\pm \delta_\pm^2 \eta}{(2\eta N_{\text{th}} + 1)(1 + 4\delta_\pm^2)^2} \equiv \lambda_X^{\pm'}, \quad \frac{\lambda_Y^\pm}{\gamma_m} = \frac{4C_\pm \eta}{(2\eta N_{\text{th}} + 1)(1 + 4\delta_\pm^2)^2} \equiv \lambda_Y^{\pm'}, \quad (\text{C.4})$$

$$\frac{\Lambda_X^\pm}{\gamma_m} = -\frac{\Lambda_Y^\pm}{\gamma_m} = -\frac{8C_\pm \delta_\pm \eta}{(1 + 4\delta_\pm^2)^2} \frac{2N_{\text{th}} + 1}{2\eta N_{\text{th}} + 1} \equiv \Lambda_X^{\pm'}, \quad (\text{C.5})$$

$$\frac{\bar{n}_\pm}{\gamma_m} = 4n_{\text{thfb}}^\pm + 2 + \frac{4C_\pm}{(1 + 4\delta_\pm^2)}(2N_{\text{th}} + 1) \equiv \bar{n}_\pm', \quad (\text{C.6})$$

$$\frac{\gamma_I^\pm}{\gamma_m} = \sqrt{1 - 2Q_\pm^2 \left(1 + \frac{\Lambda_I^\pm}{\gamma_m Q_\pm} - \sqrt{1 + 2\frac{\Lambda_I^\pm}{\gamma_m Q_\pm} + \frac{\bar{n}_\pm \lambda_I^\pm}{\gamma_m^2 Q_\pm^2}} \right)} \equiv \gamma_I^{\pm'}, \quad (\text{C.7})$$

and $\delta_\pm = \Delta/\kappa_\pm$.

D Gravity-induced entanglement between two output lights and squeezing of output lights

In this Appendix, we review the entanglement and squeezing induced by gravitational interaction in the output optical fields in the optomechanical systems. The output quadratures in the Fourier domain are given as

$$\hat{x}_{\pm}^{\text{out}}(\omega) = \frac{\omega - i\kappa/2}{\omega + i\kappa/2} \hat{x}_{\pm}^{\text{in}}, \quad (\text{D.1})$$

$$\hat{y}_{\pm}^{\text{out}}(\omega) = \frac{2ig_{\pm}\sqrt{\kappa}}{\omega + i\kappa/2} \delta\hat{q}_{\pm} + \frac{\omega - i\kappa/2}{\omega + i\kappa/2} \hat{y}_{\pm}^{\text{in}}, \quad (\text{D.2})$$

where we assume the perfect detection $\eta = 1$. To investigate gravity-induced entanglement and squeezing, we introduce the optical quadratures at the mechanical resonance frequency as defined in Refs. [67, 68]

$$\mathcal{X}_{\pm} = \sqrt{\frac{\Delta\omega}{2\pi}} \hat{x}_{\pm}^{\text{out}}(\Omega), \quad (\text{D.3})$$

$$\mathcal{Y}_{\pm} = \sqrt{\frac{\Delta\omega}{2\pi}} \hat{y}_{\pm}^{\text{out}}(\Omega), \quad (\text{D.4})$$

where we approximated the delta function as $1/\Delta\omega$, where $\Delta\omega$ denotes the bandwidth, and the commutation relation $[\mathcal{X}_{\pm}, \mathcal{Y}_{\pm}^{\dagger}] = 2i$ is satisfied. The covariance matrix of the optical quadratures for the common and differential modes is defined as

$$\boldsymbol{\sigma}_{\pm} = \begin{bmatrix} \sigma_{\mathcal{X}\mathcal{X}}^{\pm} & \sigma_{\mathcal{X}\mathcal{Y}}^{\pm} \\ \sigma_{\mathcal{Y}\mathcal{X}}^{\pm} & \sigma_{\mathcal{Y}\mathcal{Y}}^{\pm} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \langle\{\mathcal{X}_{\pm}, \mathcal{X}_{\pm}^{\dagger}\}\rangle & \langle\{\mathcal{X}_{\pm}, \mathcal{Y}_{\pm}^{\dagger}\}\rangle \\ \langle\{\mathcal{Y}_{\pm}, \mathcal{X}_{\pm}^{\dagger}\}\rangle & \langle\{\mathcal{Y}_{\pm}, \mathcal{Y}_{\pm}^{\dagger}\}\rangle \end{bmatrix}. \quad (\text{D.5})$$

The covariance matrix for the individual optical modes A and B can be expressed in terms of the covariance matrices of the common and differential modes as

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}_A & \boldsymbol{\sigma}_{AB} \\ \boldsymbol{\sigma}_{AB}^{\text{T}} & \boldsymbol{\sigma}_B \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{+} & 0 \\ 0 & \boldsymbol{\sigma}_{-} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{bmatrix}. \quad (\text{D.6})$$

The entanglement between the optical modes A and B can be quantified using the logarithmic negativity, defined as

$$E_N = -\frac{1}{2} \log_2 \left[\frac{\Sigma - \sqrt{\Sigma^2 - 4\det\boldsymbol{\sigma}}}{2} \right], \quad (\text{D.7})$$

where $\Sigma = \det\boldsymbol{\sigma}_A + \det\boldsymbol{\sigma}_B - 2\det\boldsymbol{\sigma}_{AB}$. The output optical modes are entangled if and only if $E_N > 0$. Under the assumptions $C_+/n_{\text{th}}^+ \gg 1$ and $\kappa \gg \Omega$, we can derive the entanglement criterion as (5.72).

Next, we review the gravity-induced squeezing effects discussed in Refs. [67, 68]. Assuming $\eta = 1$ and $N_{\text{th}} = 0$, the variance of the quadrature $X_\theta = x^{\text{out}} \cos \theta + y^{\text{out}} \sin \theta$ is given by

$$\sigma_{\mathcal{X}_\theta \mathcal{X}_\theta}^\pm = \sigma_{\mathcal{X}\mathcal{X}}^\pm \cos^2 \theta + (\sigma_{\mathcal{X}\mathcal{Y}}^\pm + \sigma_{\mathcal{Y}\mathcal{X}}^\pm) \cos \theta \sin \theta + \sigma_{\mathcal{Y}\mathcal{Y}}^\pm \sin^2 \theta \quad (\text{D.8})$$

where θ is the homodyne detection angle. The minimum variance is determined as

$$\sigma_{\mathcal{X}_\theta \mathcal{X}_\theta}^\pm \geq \sigma_{\min}^\pm = \frac{\sigma_{\mathcal{X}\mathcal{X}}^\pm + \sigma_{\mathcal{Y}\mathcal{Y}}^\pm}{2} - \frac{\sqrt{(\sigma_{\mathcal{X}\mathcal{Y}}^\pm + \sigma_{\mathcal{Y}\mathcal{X}}^\pm)^2 + (\sigma_{\mathcal{X}\mathcal{X}}^\pm - \sigma_{\mathcal{Y}\mathcal{Y}}^\pm)^2}}{2}. \quad (\text{D.9})$$

For the common mode, the variance satisfies

$$\sigma_{\min}^+ = 1, \quad (\text{D.10})$$

implying that the fluctuation remains above the vacuum fluctuation level, and no squeezing occurs. In contrast, for the differential mode, the minimum variance is given by

$$\sigma_{\min}^- = 1 - \frac{4C_+(1 + 2n_{\text{th}}^+ + 2C_+) \left(\sqrt{1 + Q_+^2 \epsilon^2 / (1 + 2n_{\text{th}}^+ + 2C_+)^2} - 1 \right)}{1 + Q_+^2 \epsilon^2}, \quad (\text{D.11})$$

Under the assumption $C_+/n_{\text{th}}^+ \gg 1$, $C_+ \gg 1$, and $Q_+ \epsilon \ll C_+$, this simplifies to

$$\sigma_{\min}^- = \frac{1}{1 + Q_+^2 \epsilon^2}. \quad (\text{D.12})$$

For the alternative case $C_+/n_{\text{th}}^+ \ll 1$, $n_{\text{th}}^+ \gg 1$, and $Q_+\epsilon \ll n_{\text{th}}^+$, the minimum value is

$$\sigma_{\text{min}}^- = 1 - \frac{C_+ Q_+^2 \epsilon^2}{1 + n_{\text{th}}^+ Q_+^2 \epsilon^2}, \quad (\text{D.13})$$

which reduces to

$$\sigma_{\text{min}}^- = 1 - \frac{C_+}{n_{\text{th}}^+}, \quad (\text{D.14})$$

under the additional assumption $n_{\text{th}}^+ Q_+^2 \epsilon^2 \gg 1$.

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