

# A new method for calculating the decay rate of the false vacuum at finite temperature by use of the saddle-point approximation

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Ph. D. Dissertation

**A new method for calculating the  
decay rate of the false vacuum at  
finite temperature by use of the  
saddle-point approximation**

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Less is more.



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# Abstract

Our universe has undergone several phase transitions to become what it is today. By properly utilizing these transitions, many phenomena observed today can be explained in a relatively simple and elegant manner without introducing new mechanisms. However, these phase transitions occurred in the early universe at finite temperature, making their associated transition rates an intriguing research topic if they are first-order phase transitions.

When calculating the transition rate at finite temperature, the process can be made more efficient by utilizing its relationship with the imaginary part of the free energy put forward in previous research. Furthermore, due to the extensibility of calculations using the path integral approach, the free energy is often reformulated into a path integral expression, which can then be evaluated using the saddle point approximation.

In previous research, the trace in free energy has been treated as equivalent to imposing periodic boundary conditions in the path integral formulation. Thus, calculating the path integral required identifying all classical solutions satisfying periodic boundary conditions and then applying the saddle point approximation to account for contributions around these solutions. However, inconsistencies arise with this method, particularly in the choice of boundary conditions to constrain the paths and their corresponding fluctuations, as well as in the treatment of the zero eigenvalue appearing for particular classical solutions.

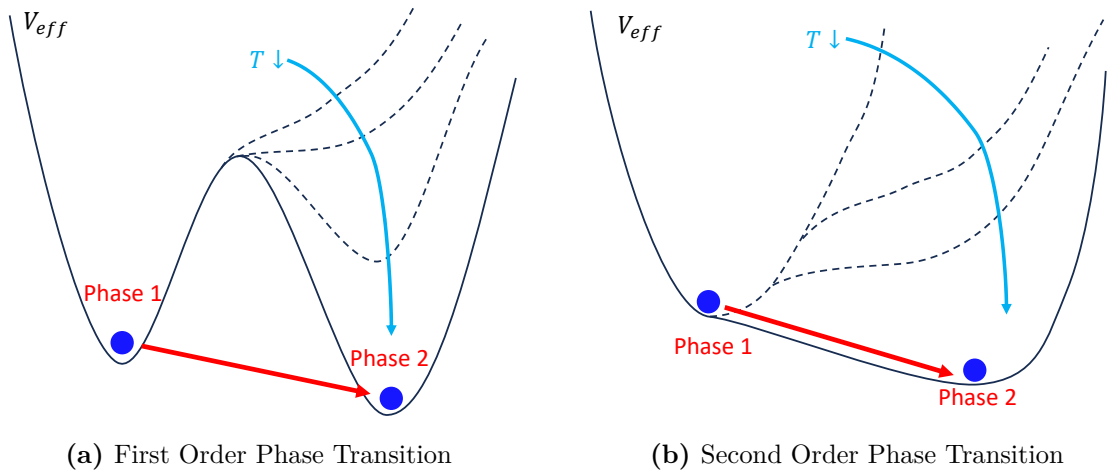
In this research, we propose a new calculation method that emphasizes a rigorous evaluation of the integral originating from the trace in the free energy. This approach introduces two key differences compared to conventional methods. First, the trace is explicitly evaluated by performing integration over the endpoints of the paths in the path integral. Second, the saddle point approximation is conducted in a functional space that satisfies Dirichlet boundary conditions, instead of the periodic boundary conditions used in previous studies.

This new approach eliminates the need for the so-called collective coordinate method to handle the zero eigenvalue in the calculation. Moreover, it provides a more intuitive way of calculating the trace in the path integral approach, avoiding the ambiguous application of periodic boundary conditions. Transition rates calculated with our method exhibit expected behavior over a broader range compared to those obtained using conventional approaches.



# 1. Introduction

## 1.1. The First Order Phase Transition at Finite Temperature



**Figure 1.1.:** Effective Potential and Phase Transition

Our universe has undergone several phase transitions to reach its current state [1–3]. One of them is called electroweak phase transition, which generated the masses of particles through the notable Higgs mechanism. By calculating the effective potential of the system at different temperatures, we can determine the order of the phase transition. To elaborate on this issue : if no potential barrier exists between the old and the new phases, then the transition between the two phases can proceed smoothly. However, if a potential barrier is present, quantum tunneling occurs, resulting in a first-order phase transition. In such cases, bubbles of the new phase form within the old phase, with a distinct bubble wall separating the two phases and propagating as the transition progresses.

Through first-order phase transitions, many interesting phenomena can be explained in a relatively simple and elegant way. We introduce some of these phenomena and their corresponding explanations through the framework of first-order phase transitions.

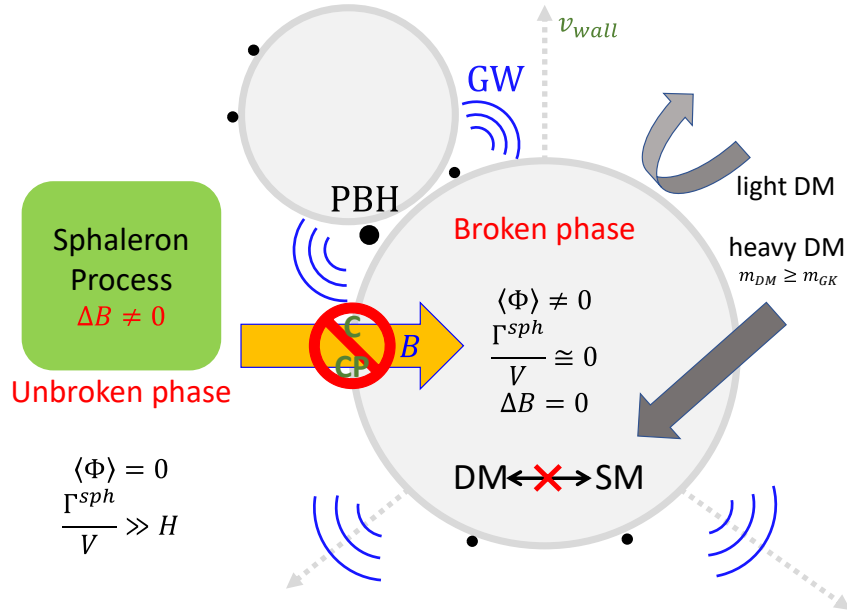


Figure 1.2.: Phenomenons

### 1.1.1. Baryogenesis

Baryogenesis refers to the physical process that produces a net positive baryon number (refer to [4–6] for more details). According to the Sakharov conditions [7], the process that realizes baryogenesis must satisfy the following three conditions.

1. The existence of the violation of baryon number conservation : This is obvious, if all reactions conserve baryon number, no net baryon can be created.
2. The existence of the violation of C and CP symmetry : The violation of C symmetry allows more baryons to be produced than anti-baryons. Additionally, CP violation ensures that the baryons produced will not cancel out with anti-baryons of opposite chirality in the same reaction.
3. The process must be out-of-equilibrium : If the process is in equilibrium, the baryons generated in one reaction will be annihilated in reverse reactions.

There are various theories which provide reasonable explanations for baryogenesis, such as

1. Baryogenesis via leptogenesis [5, 8–16]
2. Affleck–Dine baryogenesis [17–19]
3. Baryogenesis via black hole [20–24]
4. Electroweak baryogenesis [5, 8–16, 25–28]

Now, consider the situation within the Standard Model. There exists a baryon-number violating process known as the sphaleron process, which describes the tran-

sition process between gauge vacuum states with different Chern-Simons numbers. The net baryon number produced is proportional to the change in the Chern-Simons number:  $\Delta B \propto \Delta N_{CS}$ . As for the second condition, the weak interaction violates the C symmetry, and the phases in the CKM and PMNS matrix provide the violation of CP symmetry. The out-of-equilibrium condition can be satisfied by a first order phase transition. One of the most notable phase transitions in the Standard Model is the electroweak phase transition. Unfortunately, particle experiments have demonstrated that under the framework of the Standard Model, this transition is second-order or weakly first-order. However, in extended versions of the Standard Model, such as the two Higgs doublet model [28–30], the standard model endowed with supersymmetry [12, 28], it can be first-order, providing a viable mechanism.

Assuming a strong first-order electroweak phase transition occurs in an extended Standard Model, baryogenesis can be explained as follows:

During a strong first-order phase transition, bubbles of the true vacuum state, which is destined to become our universe, form within the sea of false vacuum. Inside the true vacuum bubble, where the electroweak symmetry is broken, the rate of the sphaleron process [26, 31–39] is suppressed as

$$\Gamma_{\text{sph}}^{(\text{in})} \propto e^{-\frac{E_{\text{sph}}(T)}{T}}.$$

Meanwhile, outside the true vacuum bubble, the sphaleron process rate remains high

$$\Gamma_{\text{sph}} \propto T^4,$$

potentially exceeding the Hubble parameter. Consequently, baryon number inside the true vacuum bubble is preserved while the baryon number outside is eliminated by the drastic sphaleron process.

So, where does the baryon number inside a bubble come from? A simple explanation can be done as follows : the probabilities of transmission and reflection differ for particles and anti-particles due to the interaction with the bubble wall, which leads to a net baryon number current flowing into the true vacuum bubble.

### 1.1.2. Filtered Dark Matter

As explained earlier, the bubble wall can generate a pure baryon number inside the true vacuum bubble. Moreover, it can be used to generate dark matter of WIMP<sup>1</sup> type [41].

Consider a kind of dark matter  $\chi$  as a thermal relic, whose interaction with normal matter is considered to have frozen out as the universe cooled down. The thermal relic abundance of  $\chi$  is determined by its total thermally averaged annihilation cross

---

<sup>1</sup>WIMP means weakly interacting massive particle [40].

section  $\langle\sigma_{\chi\bar{\chi}\rightarrow\text{all}}v_{\text{rel}}\rangle_{\text{fo}}$  at freeze-out.  $\langle\sigma_{\chi\bar{\chi}\rightarrow\text{all}}v_{\text{rel}}\rangle_{\text{fo}}$  can be expanded in terms of partial waves  $\sum_J\langle\sigma_Jv_{\text{rel}}\rangle$ . Due to the unitarity of the S-matrix,  $\langle\sigma_Jv_{\text{rel}}\rangle$  has an upper limit of  $\frac{4\pi}{m_\chi^2v_{\text{rel}}}(2J+1)$ .

The number density of the thermal relic  $\chi$  can be expressed as  $\Omega_\chi h^2 \approx \frac{K}{\langle\sigma_{\chi\bar{\chi}\rightarrow\text{all}}v_{\text{rel}}\rangle_{\text{fo}}}$ . Therefore, it has a lower bound. However, as just one component of dark matter, its number density must be smaller than the total dark matter observed today, which imposes a constraint on the particle mass :

$$\Omega_{\text{all}} h^2 \gtrsim \underbrace{\Omega_\chi h^2}_{\text{thermal relic}} \approx \frac{K}{\langle\sigma_{\chi\bar{\chi}\rightarrow\text{all}}v_{\text{rel}}\rangle_{\text{fo}}} = \frac{K}{\sum_J \langle\sigma_J v_{\text{rel}}\rangle} \gtrsim \frac{K}{\sum_J \frac{4\pi}{m_\chi^2 v_{\text{rel}}} (2J+1)} \sim K' m_\chi^2. \quad (1.1)$$

This mass constraint for the thermal relic type of dark matter is known as the Griest-Kamionkowski bound [42].

However, the Griest-Kamionkowski bound applies only to thermal relic dark matter, where interactions with normal matter freeze out due to the cooling of the universe. In contrast, these interactions can also be disrupted by a first-order phase transition.

According to [43], as dark matter particles enter the bubble, their interactions are abruptly driven out of equilibrium, preventing them from annihilating. As long as the thermally averaged annihilation rate inside the bubble remains smaller than the cosmological expansion rate, the dark particles that enter the bubble survive until today and constitute the relic population of dark matter. It is apparent that only particles with sufficient energy can penetrate the bubble wall and get into the bubble, where they gain mass due to the non-zero expectation value of the Higgs field. These particles could be heavy enough to exceed the Griest-Kamionkowski bound.

### 1.1.3. Primordial Black Holes

The study of black holes offers another perspective for exploring a theory of everything, as both gravitational and quantum effects are prominent near black holes. It is well established that massive black holes can form from the collapse of massive stars. Since the temperature of a black hole is inverse proportion to its mass, black holes formed from stars are relatively cold because of its large mass. Therefore, although some phenomena, such as black hole evaporation and explosions, are interesting, they are largely unobservable for stellar black holes.

However, not all black holes are formed from stellar collapse. According to [44], black holes could also form in the hot and dense early universe via other mechanisms, such as the collapse of density perturbations. These are known as primordial black holes, a term used to distinguish them from stellar black holes. The primordial black

holes can be used to explain various phenomena including the baryogenesis as well as the dark matter [23, 45–52].

The mass of a primordial black hole can vary widely. A primordial black hole with proper mass can survive to the present day. Observations of their evaporation and potential explosions could provide insight into early-universe physics and the theory of everything. Furthermore, primordial black holes can be utilized to explain many observed phenomena.

Primordial black holes can form during a first order phase transition [53–56]. Here, the filtering effect comes into play once more. Consider collisions between different bubbles. Matter trapped between two bubble walls, lacking sufficient energy to penetrate the walls and enter the bubble due to energy conservation, will accumulate be suppressed in the collision. If sufficient density perturbations occur, this accumulated matter may collapse into a primordial black hole.

### 1.1.4. Gravitational Wave

As discussed earlier, true vacuum bubbles arising from first-order phase transitions can be exploited to explain various phenomena observed in our universe. Additionally, the expansion and collision of these bubbles can directly generate observable phenomena, such as characteristic gravitational waves. The spectral properties of these waves differ from those created by other astrophysical events, such as black hole mergers [57–61].

## 1.2. The Transition Rate

There are numerous researches related to first order phase transitions, including the phenomena mentioned above as well as the motion of the bubble wall and other related dynamics (see [62–66]). Our research mainly focuses on the calculation of the transition rate at finite temperature, a topic that has garnered significant attention in past decades.

As far as we known, the earliest related studies were conducted by Langer in [67] and [68]. The thermodynamic limit of a function constructed by the partition function, which can be interpreted as the free energy, was discussed in WKB approximation from a canonical quantization perspective as well as the saddle point approximation in the functional integral framework.

Subsequently, Coleman and Callan developed a widely recognized and general method to compute the decay rate of a false vacuum state at zero temperature [69, 70]. They proposed a relation between the complex energy of the false vacuum state and the corresponding decay rate. Then they calculated the complex energy from the transition amplitude via the saddle point approximation of the equivalent path

integral formulation. While their work is conceptually similar to Langer's in essence, it is more general. However, the treatment of the zero eigenvalue encountered in the calculation is somehow different.

Regarding the transition rate at finite temperature, it was introduced in [71] by Linde in a simple manner. Then a more detailed discussion was done in [72,73]. However, Linde's analysis focused primarily on extremely high temperatures, deriving the transition rate in that regime based on Coleman and Callan's zero-temperature results. The transition rate for intermediate temperature was systematically addressed by Affleck in [74]. Affleck leveraged the connection of transition rates and free energy to analyze transition rates across different temperature ranges.

Building on the relationship between the transition rate and the free energy established by Affleck in [74], our research employs a new method to calculate the transition rate at finite temperature.

### 1.3. Structure of the thesis

This thesis is organized as follows :

In Chapter 2, we begin with an overview of previous significant research on calculations of the decay rate at zero temperature. In the final part of this chapter, some important results obtained in [75], which first introduced the concept of the shot solution, will be reviewed.

In Chapter 3, the focus shifts to the transition rate at finite temperature. The previous calculations based on WKB approximation and the saddle point approximation at finite temperature will be reviewed and some deficiencies in the previous research will be discussed. Then the zero temperature limit will also be derived. After that, we will introduce our method for calculating the transition rate at finite temperature, which contains not only the contribution from bounce solution but also the contributions from shot solutions discussed in [75]. Certainly, the zero temperature limit of our research will also be presented.

In Chapter 4, numerical calculations are shown in the first place. The dependence of the results, calculated using different methods, on different potentials and temperatures will be illustrated. Some unexpected behavior observed in the calculations is analyzed in the subsequent section, where the explanation will be given from a mathematical perspective and some inconsistencies in the previous research will be identified. These inconsistencies are explored and discussed in the chapter's concluding section.

The last chapter contains the summary of our researches, key conclusions, and comments.

Supplemental results and their proofs are provided for reference in Appendices.

## 2. Decay Rate at Zero Temperature

### 2.1. Overview

Consider the time development of a state, it can be expressed as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}E_\psi t}|\psi(0)\rangle. \quad (2.1)$$

When the energy is complex rather than real, the probability of the state becomes

$$P_\psi(t) = \langle\psi(t)|\psi(t)\rangle = e^{-\frac{i}{\hbar}(E_\psi - E_\psi^*)t} \langle\psi(0)|\psi(0)\rangle = e^{\frac{2}{\hbar}\text{Im}E_\psi t} P_\psi(0), \quad (2.2)$$

which will change with time.

For  $\text{Im}E_\psi < 0$ , the state will be metastable and finally decay to other states. The decay rate of a metastable state can then be defined as

$$\Gamma_\psi = -\frac{1}{P_\psi(t)} \frac{dP_\psi(t)}{dt} = -\frac{2}{\hbar}\text{Im}E_\psi. \quad (2.3)$$

Moreover, if the variation of the probability in a given region  $\Omega$  is considered, the decay rate in the given region can be expressed as

$$\Gamma_\Omega = -\frac{1}{P_\Omega(t)} \frac{dP_\Omega(t)}{dt}, \quad (2.4)$$

where  $P_\Omega$  is defined by the integral of the probability density function over a constrained integral region as

$$P_\Omega(t) = \int_\Omega d\mathbf{x} |\psi(t, \mathbf{x})|^2. \quad (2.5)$$

Through the discussion above, it is obvious that the decay rate of a metastable state can be derived from its complex energy. We will introduce previous studies on the derivation of the complex energy of the false vacuum state, which will finally decay to the true vacuum state.

## 2.2. WKB Approximation

One way to derive the imaginary part of the energy is to solve the Schrödinger equation and use the wave function according to [67]. The Schrödinger equation for a metastable state in the position representation in one dimension is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E_\psi\psi(x).$$

After multiplying  $\psi^*(x)$  on both sides and integral over  $x$  from  $x_i$  to  $x_f$ , the following relation can be obtained.

$$\begin{aligned} -\frac{\hbar^2}{2m} \left( [\psi^*(x)\psi'(x)]_{x_i}^{x_f} - \int_{x_i}^{x_f} dx \psi'^*(x)\psi'(x) \right) \\ + \int_{x_i}^{x_f} dx \psi^*(x)V(x)\psi(x) = E_\psi \int_{x_i}^{x_f} dx \psi^*(x)\psi(x). \end{aligned}$$

By using the complex conjugate of the equation, the following relation between the energy and the boundary values of the wave function can be acquired.

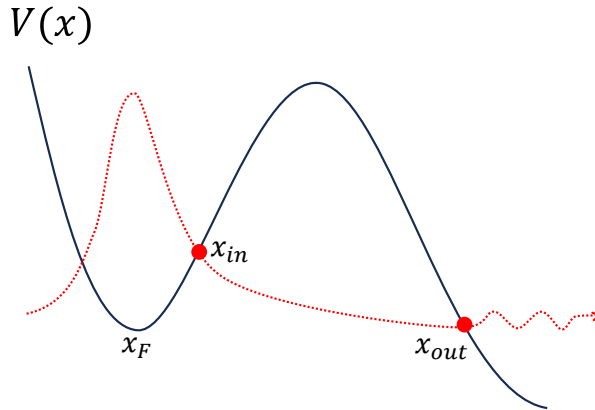
$$-\frac{\hbar^2}{2m} [\psi^*(x)\psi'(x) - \psi'^*(x)\psi(x)]_{x_i}^{x_f} = (E_\psi - E_\psi^*) \int_{x_i}^{x_f} dx \psi^*(x)\psi(x). \quad (2.6)$$

Hence, the imaginary part of the energy can be expressed as

$$\text{Im}E_\psi = -\frac{\hbar}{2} \frac{\frac{\hbar}{2mi} [\psi^*(x)\psi'(x) - \psi'^*(x)\psi(x)]_{x_i}^{x_f}}{\int_{x_i}^{x_f} dx \psi^*(x)\psi(x)} = -\frac{\hbar}{2} \frac{J(x_f) - J(x_i)}{\int_{x_i}^{x_f} dx \psi^*(x)\psi(x)}, \quad (2.7)$$

where  $J(x)$  is the probability current defined as

$$J(x) = \frac{\hbar}{2mi} [\psi^*(x)\psi'(x) - \psi'^*(x)\psi(x)].$$



**Figure 2.1.:** Potential

The most important part of the derivation of the decay rate from the imaginary part of energy lies in the explanation for the appearance of the complex energy. For a normal bound state with real energy, the current  $J(x)$  should be zero, leading to a vanished  $\text{Im}E$  expectedly. However, for a metastable state, the boundary conditions is different from the bound state, resulting in a non-vanishing current in special region. For example, when considering the outgoing wave boundary conditions, which is also known as the Gamow-Siegert outgoing boundary conditions,  $J(x)$  will be non-zero in the outgoing wave region.

Consider a potential like Figure.2.1. Choose  $x_i$  to be the point at which  $\psi'(x_i) = 0$  (generally  $x_i$  can be chosen to be the local minimum of the potential  $x_F$ ) and  $x_f \gtrsim x_{\text{out}}$ , which is in the outgoing wave region. Then  $J(x_i)$  becomes 0 while  $J(x_f) \neq 0$ . Therefore, the imaginary part of energy becomes

$$\text{Im}E_\psi = -\frac{\hbar}{2} \frac{J(x_{\text{out}})}{\int_{x_F}^{x_{\text{out}}} dx \psi^*(x) \psi(x)}, \quad (2.8)$$

where  $x_f = x_{\text{out}}$  is chosen. This formula can be understood as the probability variation in a given region by changing the integral regime and the boundary value of the current in a consistent way.

We now introduce a rough calculation using the WKB approximation. The imaginary part of  $E_\psi$  is fairly small. The wave function  $\psi(x)$  with energy  $E_\psi$  in  $x \in [x_F, x_{\text{out}}]$  can be solved  $\mathcal{O}(\hbar)$  roughly as

$$\psi(x) \propto \frac{1}{\sqrt{|p(x)|}} e^{\frac{i}{\hbar} \int_{x_{\text{in}}}^x p(y) dy},$$

where  $p(x) = \sqrt{2m(E_\psi - V(x))}$ . Consequently, the current becomes

$$J(x_{\text{out}}) \propto \frac{1}{m} e^{-\frac{2}{\hbar} \int_{x_{\text{out}}}^{x_{\text{in}}} |p(y)| dy}.$$

As a result, the imaginary part of  $E_\psi$  becomes

$$\text{Im}E_\psi = -\frac{\hbar}{2} \frac{\frac{1}{m} e^{-\frac{2}{\hbar} \int_{x_{\text{out}}}^{x_{\text{in}}} |p| dy}}{\int_{x_F}^{x_{\text{in}}} \frac{1}{|p|} dx + \int_a^b \frac{dx}{|p|} e^{-\frac{2}{\hbar} \int_{x_{\text{out}}}^{x_{\text{in}}} |p| dy}}. \quad (2.9)$$

Hence, the decay rate calculated by the WKB approximation can be expressed as

$$\Gamma_{\text{WKB}} = \frac{1}{m} \frac{e^{-\frac{2}{\hbar} \int_{x_{\text{out}}}^{x_{\text{in}}} |p| dy}}{\int_{x_F}^{x_{\text{in}}} \frac{1}{|p|} dx + \int_a^b \frac{dx}{|p|} e^{-\frac{2}{\hbar} \int_{x_{\text{out}}}^{x_{\text{in}}} |p| dy}},$$

which is derived in [75].

## 2.3. Saddle Points Approximation

### 2.3.1. Saddle Points Approximation

In this section, we will review the method put forward in [67, 68] and [69, 70] by use of the saddle points approximation (or stationary points approximation) at first and then discuss some significant problems related to the method.

The calculation can be started from the transition amplitude in Euclidean time:

$$\langle x_f | e^{-\frac{1}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle = \mathcal{N} \int \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]}, \quad (2.10)$$

where all paths in the path integral need to satisfy the boundary conditions  $x(-\frac{\mathcal{T}}{2}) = x_i$  and  $x(\frac{\mathcal{T}}{2}) = x_f$ . By use of the completeness relation of energy eigenstates  $|n\rangle$ <sup>1</sup>, the transition amplitude can be expressed as

$$\langle x_f | e^{-\frac{1}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle = \sum_n e^{-\frac{1}{\hbar} E_n \mathcal{T}} \langle x_f | n \rangle \langle n | x_i \rangle.$$

For large  $\mathcal{T}$ , the leading term reveals the energy and wave function of the lowest-lying energy eigenstate.

$$\langle x_f | e^{-\frac{1}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle \approx e^{-\frac{1}{\hbar} E_0 \mathcal{T}} \langle x_f | 0 \rangle \langle 0 | x_i \rangle.$$

Therefore, the energy can be expressed as

$$E_0 = \lim_{\mathcal{T} \rightarrow \infty} -\frac{\hbar}{\mathcal{T}} \ln \langle x_f | e^{-\frac{1}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle,$$

if  $x_i$  is chosen to be equal to  $x_f$ . It is enough to calculate the transition amplitude and then take  $\mathcal{T}$  to infinity to obtain the information of the lowest energy.

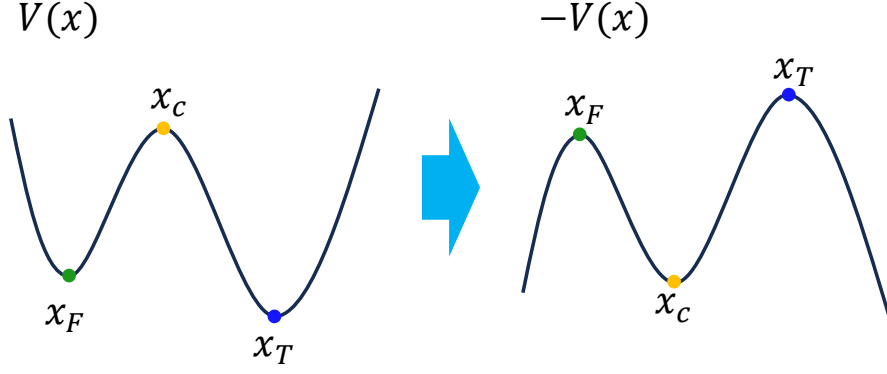
Since the canonical quantization formulation of the transition amplitude has an equivalent path integral formulation, it is sufficient and more expandable to calculate the transition amplitude from the path integral formulation rather than from the canonical quantization formulation. As for the calculation of the path integral, it can be achieved by making use of saddle points (classical solutions) of  $S_E$ , corresponding to the euclidean system in the semiclassical (small- $\hbar$ ) limit.

In order to perform the saddle points approximation, one should try to find the stationary points (classical solutions) in the euclidean system in which the corresponding potential changes to  $-V(x)$  at first. After a saddle point  $\bar{x}(\tau)$  was found, the path  $x(\tau)$  in the path integral can be expanded around  $\bar{x}(\tau)$  as

$$x(\tau) = \bar{x}(\tau) + \Delta x.$$

---

<sup>1</sup>It is necessary to pay more attention to those states.



**Figure 2.2.:** Potential and Euclidean Potential

Then the exponent  $S_E[x]$  becomes

$$\begin{aligned} S_E[x] &= S_E[\bar{x}] + \frac{1}{2} \int d\tau_1 d\tau_2 \Delta x(\tau_1) \left[ \frac{\delta^2 S_E[x]}{\delta x(\tau_1) \delta x(\tau_2)} \right]_{\bar{x}} \Delta x(\tau_2) \\ &= S_E[\bar{x}] + \frac{m}{2} \int d\tau \Delta x(\tau) \left( -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}(\tau)) \right) \Delta x(\tau). \end{aligned}$$

The fluctuation  $\Delta x(\tau)$  can be expanded by the eigenfunctions  $x_n(\tau)$  of the fluctuation operator  $\left[ -\partial^2 + \frac{1}{m} V''(\bar{x}(\tau)) \right]$  corresponding to the saddle point  $\bar{x}(\tau)$ .  $x_n(\tau)$  satisfies the eigenequation

$$\left[ -\partial^2 + \frac{1}{m} V''(\bar{x}(\tau)) \right] x_n(\tau) = \lambda_n x_n(\tau).$$

Furthermore, since  $\bar{x}(\tau)$  is a classical solution, it should satisfy the same boundary condition as  $x(\tau)$ . As a result,  $\Delta x$  and all  $x_n(\tau)$ s should satisfy the Dirichlet boundary conditions

$$\Delta x\left(+\frac{\mathcal{T}}{2}\right) = \Delta x\left(-\frac{\mathcal{T}}{2}\right) = 0$$

$$x_n\left(+\frac{\mathcal{T}}{2}\right) = x_n\left(-\frac{\mathcal{T}}{2}\right) = 0$$

in order to be in consistency. Then the expansion around  $x(\tau)$  can be expressed as

$$x(\tau) = \bar{x}(\tau) + \Delta x = \bar{x}(\tau) + \sum_n \frac{c_n}{\sqrt{m}} x_n(\tau).$$

And the contribution of  $\bar{x}$  to the original path integral can be expressed as

$$\begin{aligned}
\mathcal{N} \int \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} &\approx \mathcal{N} \int \mathcal{D}\Delta x(\tau) e^{-\frac{1}{\hbar} (S_E[\bar{x}] + \frac{m}{2} \int d\tau \Delta x(\tau) (-\partial_\tau^2 + \frac{1}{m} V''(\bar{x}(\tau))) \Delta x(\tau))} \\
&= e^{-\frac{1}{\hbar} S_E[\bar{x}]} \mathcal{N} \prod_n \int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} \frac{1}{2} \lambda_n c_n^2} \\
&= e^{-\frac{1}{\hbar} S_E[\bar{x}]} \mathcal{N} \prod_n \frac{1}{\sqrt{\lambda_n}} \\
&= \mathcal{N} \frac{e^{-\frac{1}{\hbar} S_E[\bar{x}]}}{\sqrt{\det \left[ -\partial^2 + \frac{1}{m} V''(\bar{x}(\tau)) \right]}}.
\end{aligned}$$

The fluctuation determinant here should be understood as the product of the square roots of the eigenvalues according to [76] (see also [77]).

$$\det \left[ -\partial^2 + \frac{1}{m} V''(\bar{x}(\tau)) \right] = \prod_n \lambda_n.$$

Although it is emphasized that one has to sum over all the stationary points in [70], it is not quite true according to [75], which will be explained in the next section.

In [70],  $x_i = x_f = x_F$  was chosen and two classical solutions are determined. The first one is a static solution called the false vacuum solution  $\bar{x}(\tau) = x_F$ . The action of  $\bar{x}(\tau) = x_F$  can be set to 0 by choosing  $V(x_F) = 0$ . Define  $\omega_F \equiv \sqrt{\frac{V''(x_F)}{m}}$  and then the corresponding contribution of  $\bar{x}(\tau) = x_F$  becomes

$$Z_F = \mathcal{N} \int \mathcal{D}\Delta x e^{-\frac{1}{\hbar} \frac{m}{2} \int d\tau \Delta x(\tau) (-\partial_\tau^2 + \omega_F^2) \Delta x(\tau)} = \mathcal{N} \frac{1}{\sqrt{\det [-\partial^2 + \omega_F^2]}}.$$

Here, the eigenequation can be solved with no effort and the eigenvalues can be derived as  $\lambda_n = \omega_F^2 + \left(\frac{n\pi}{\mathcal{T}}\right)^2$ . Consequently,

$$Z_F = \mathcal{N} \frac{1}{\sqrt{\prod_n \left( \omega_F^2 + \left(\frac{n\pi}{\mathcal{T}}\right)^2 \right)}} \propto \frac{1}{\sqrt{\prod \left( 1 + \left(\frac{\omega_F \mathcal{T}}{n\pi}\right)^2 \right)}} = \frac{1}{\sqrt{\frac{\sinh\left(\pi \frac{\omega_F \mathcal{T}}{\pi}\right)}{\pi \frac{\omega_F \mathcal{T}}{\pi}}}} = \sqrt{\frac{\omega_F \mathcal{T}}{\sinh(\omega_F \mathcal{T})}}.$$

Moreover, the path integral of  $\Delta x$  can be expressed as the transition amplitude of a harmonic oscillator as

$$Z_F = \langle 0 | e^{-\frac{1}{\hbar} \hat{H}_o \mathcal{T}} | 0 \rangle = \langle 0 | e^{-\frac{1}{\hbar} \left( \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_F^2 \hat{x}^2 \right) \mathcal{T}} | 0 \rangle.$$

When  $\omega_F = 0$ ,  $Z_F$  can be derived by using the completeness relation of  $|p\rangle$  as

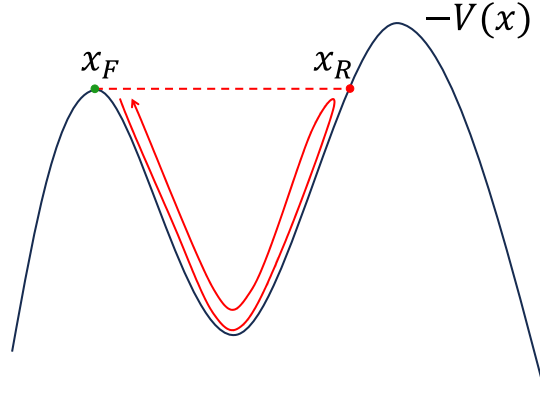
$$Z_F^{\omega_F=0} = \int dp e^{-\frac{p^2}{2m\hbar} \mathcal{T}} \langle 0 | p \rangle \langle p | 0 \rangle = \sqrt{\frac{m}{2\pi\hbar\mathcal{T}}}.$$

As a result,

$$Z_F = \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial^2 + \frac{1}{m} V''(x_F) \right]}} = \sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \mathcal{T})}}, \quad (2.11)$$

of which the dimension is the same as transition amplitude (see [78] for another derivation).

The second classical solution is a non-trivial solution called the bounce solution  $x_B(\tau)$ <sup>2</sup>. It starts from  $x_F$  with  $E = 0$  and bounces off the potential wall on  $x_R$  and returns to  $x_F$  in an infinite time interval.



**Figure 2.3.:** The Bounce Solution

Now, consider the situation for a very large  $\mathcal{T}$ . According to [70],  $n$  bounces solution could also exist and should be included<sup>3</sup>. To evaluate the functional integral, all these configurations should be included. If we denote the action of one bounce as  $S_B$ , the action for a  $n$  bounces solution is just  $nS_B$ . As for the fluctuation determinant of the  $n$  bounces solution, it can be evaluated as a product of contributions from large time intervals surrounding each bounce and even larger time intervals containing the vacant regions between different bounces following the explanation of [70]. The result can be expressed as

$$\sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \mathcal{T})}} K^n,$$

where  $K$  is defined by demanding that this expression give the right answer for one bounce. Finally, if it is assumed that all bounces in the  $n$  bounces solution are

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<sup>2</sup>A same type of solution was derived precisely in [67] since the solvable model was chosen.

<sup>3</sup>You may notice that there are several inconsistencies here. We will discuss all inconsistencies in Section.4.3.

widely separated, which is referred as thin wall approximation, then the center of every widely separated bounce  $\tau_{cn}$  can be anywhere along the time axis. And the integration over the locations of the centers should be performed:

$$\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau_{c1} \int_{-\frac{\mathcal{T}}{2}}^{\tau_{c1}} d\tau_{c2} \cdots \int_{-\frac{\mathcal{T}}{2}}^{\tau_{cn-1}} d\tau_{cn} = \frac{\mathcal{T}^n}{n!}.$$

In summary, the original path integral can be expressed as

$$\mathcal{N} \int \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} \approx \sum_n \frac{\mathcal{T}^n}{n!} \sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \mathcal{T})}} K^n e^{-\frac{1}{\hbar} n S_B} = \sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \mathcal{T})}} e^{\mathcal{T} K e^{-\frac{1}{\hbar} S_B}}. \quad (2.12)$$

For large  $\mathcal{T}$ ,

$$E_0 \approx -\frac{\hbar}{\mathcal{T}} \ln \left( \sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \mathcal{T})}} e^{\mathcal{T} K e^{-\frac{1}{\hbar} S_B}} \right) \approx \frac{\hbar\omega_F}{2} - \hbar K e^{-\frac{1}{\hbar} S_B}.$$

As for the  $K$ , it relates to the fluctuation determinant of bounce solution. Since the bounce solution is also a classical solution,  $\dot{x}_B(\tau)$  should satisfy

$$\left[ -\partial^2 + \frac{1}{m} V''(x_B(\tau)) \right] \dot{x}_B(\tau) = 0.$$

And the initial and final velocities of the bounce solution are all 0, which means that  $\dot{x}_B(\tau)$  also satisfies the boundary conditions required for every eigenfunction. As a result,  $\frac{\dot{x}_B(\tau)}{\|\dot{x}_B(\tau)\|}$  should be the eigenfunction  $x_1(\tau)$  of the fluctuation operator of the bounce solution with eigenvalue of  $c_1 = 0$ <sup>4</sup>. Moreover,  $\dot{x}_B(\tau)$  has a node, which means that there is a negative eigenvalue  $c_0$  of which the corresponding eigenfunction has no node. Therefore, the integral over  $c_0$  and  $c_1$  will both be infinite.

$$\int dc_0 e^{-\frac{1}{2} \lambda_0 c_0^2} = \infty$$

$$\int dc_1 e^{-\frac{1}{2} \lambda_1 c_1^2} = \infty.$$

In order to deal with infinity caused by the zero eigenvalue, the so-called collective coordinate method was used in [70]. According to [70], the relation between the expansion coefficient  $c_1$  of the zero eigenfunction and the time translations of the bounce solution  $\tau_c$  is given by

$$\frac{dc_1}{\sqrt{m}} \frac{\dot{x}_B(\tau)}{\|\dot{x}_B(\tau)\|} = d\tau_c$$

---

<sup>4</sup>We use  $\|f(\tau)\|$  to denote  $\|f(\tau)\|^2 = \int d\tau f^2(\tau)$ .

since the small variation over  $c_1$  can be expressed as

$$dx(\tau) = \frac{dc_1}{\sqrt{m}} x_1(\tau) = \frac{dc_1}{\sqrt{m}} \frac{\dot{x}_B(\tau)}{\|\dot{x}_B(\tau)\|} = x_B(\tau + \frac{dc_1}{\sqrt{m} \|\dot{x}_B(\tau)\|}) - x_B(\tau). \quad (2.13)$$

The relation indicates that the small variation over  $c_1$  can be transformed into the small variation over  $\tau_c$ . By use of  $x_1(\tau) \frac{dc_1}{\sqrt{m}} = \dot{x}_B(\tau) d\tau_c$  and  $\frac{1}{2} \dot{x}_B^2(\tau) - V(x_B(\tau)) = -V(x_F) = 0$ , the following relation can be obtained

$$\frac{1}{\sqrt{2\pi\hbar}} dc_1 = \sqrt{\frac{S_B}{2\pi\hbar}} d\tau_c.$$

The integral over  $c_1$  is thought to be equivalent to the integral over  $\tau_c$ . As a result, it is concluded that there is no need to include the zero eigenvalue but to include in  $K$  a factor of  $\int \sqrt{\frac{S_B}{2\pi\hbar}} d\tau_c$  since the integral over the center of the bounce has already been done.

As for the infinity caused by the negative eigenvalue  $c_0$ , it was treated with analytic continuation. The negative eigenvalue of the fluctuation determinant of the bounce solution indicates that the bounce solution is not a minimum of the action but a maximum along the special direction in the function space. By integrating over a series of paths  $\tilde{x}(\tau; z)$  parameterized by a real variable  $z$  instead of the whole function space for simplicity and the path integral can be written as

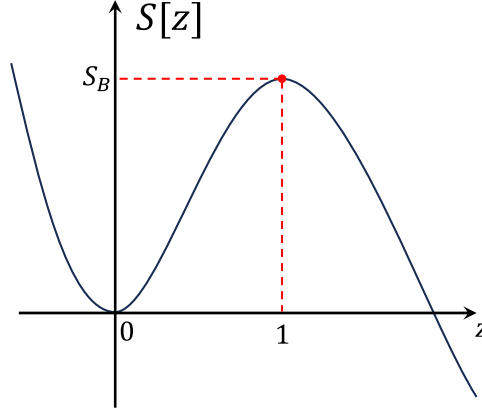
$$Z_{\{z\}} = \int \frac{dz}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} S[z]}.$$

$\tilde{x}(\tau; z)$  should satisfy  $\left. \frac{d\tilde{x}(\tau; z)}{dz} \right|_{z=1} = x_0(\tau)$  and can be set equal to the false vacuum solution at  $z = 0$  and the bounce solution at  $z = 1$  separately. Assume that all paths chosen are on the same steepest descent/ascent flow. Moreover,  $S[z]$  are assumed to reach minus infinity when  $z$  becomes large enough. The divergence of  $-\frac{1}{\hbar} S[z]$  will lead to the divergence of  $Z_{\{z\}}$ , which corresponds to the divergence of the integral over  $c_0$ . Hence, in order to obtain a reasonable result, analytic continuation is required. Actually, the analytic continuation can be conducted by changing the integral contour from the real  $z$ -axis to a new integral contour. The new integral contour can not be chosen arbitrarily but is constrained following the deformation of the potential.

When  $x = x_F$  is the absolute minimum of the potential, the static false vacuum solution is only a classical solution and the stationary point approximation should be

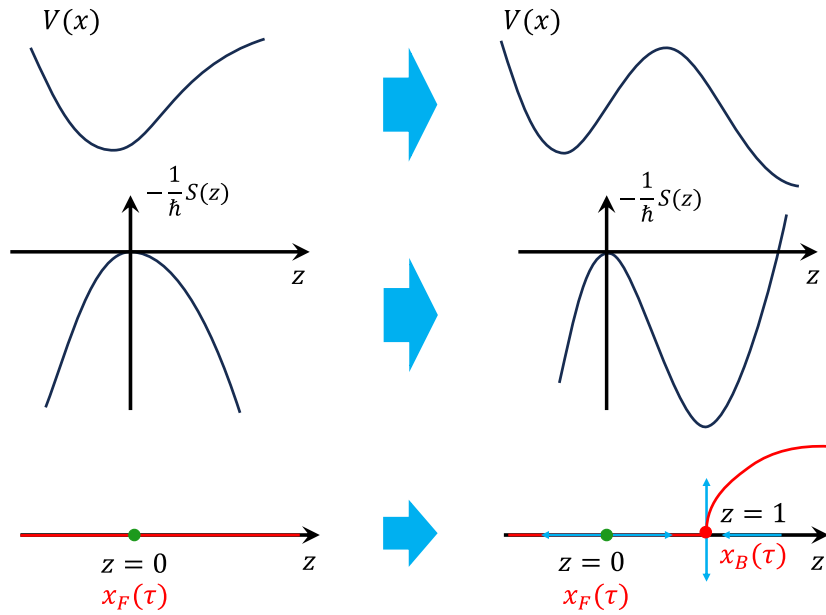
$$Z_{\{z\}} = \int \frac{dz}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} S[z]} \approx \frac{e^{-\frac{1}{\hbar} S[0]}}{\sqrt{S''[0]}}.$$

The contour of integration should be along with the whole real  $z$  axis. Then the potential can be changed analytically to the one of interest. At the same time, the



**Figure 2.4.:** Actions of the Parameterized Paths

contour of integration should be deformed according to the potential in order to keep the integral convergent. This is called potential deformation in [70] (see Figure.2.5).



**Figure 2.5.:** Potential Deformation

In the calculation of the decay rate, the contour is distorted into the upper half plane of complex  $z$ . According to the previous assumption, the real axis should correspond to the steepest ascent direction while the imaginary axis corresponds to the steepest descent direction around  $z = 1$ , which represents the bounce solution (see Appendix B.1). And along this contour, the stationary point approximation

becomes

$$Z_{\{z\}} \approx Z_{z=0} + \frac{1}{2} Z_{z=1} \approx \frac{e^{-\frac{1}{\hbar} S[0]}}{\sqrt{S''[0]}} + \frac{1}{2} i \frac{e^{-\frac{1}{\hbar} S[1]}}{\sqrt{|S''[1]|}},$$

where the factor of  $\frac{1}{2}$  arises because the integration is over only half of the Gaussian peak.

If we extend the analysis from a one-dimensional integral to the integral over a function space and only consider the false vacuum solution as well as the one-bounce solution, the stationary point approximation becomes

$$\mathcal{N} \int \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} \sim \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial^2 + \frac{1}{m} V''(x_F) \right]}} + i \frac{1}{2} \mathcal{N} \frac{\sqrt{\frac{S_B}{2\pi\hbar}} \mathcal{T} e^{-\frac{1}{\hbar} S_B}}{\sqrt{|\det' \left[ -\partial^2 + \frac{1}{m} V''(x_B) \right]|}},$$

where  $\det'$  means excluding the zero eigenvalue. Comparing this with the Eq.(2.12) (sum over  $n = 0$  and  $n = 1$ ) leads to

$$\text{Im}K = \frac{1}{2} \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{\frac{\det \left[ -\partial^2 + \frac{1}{m} V''(x_F) \right]}{|\det' \left[ -\partial^2 + \frac{1}{m} V''(x_B) \right]|}}.$$

Therefore, the final result of the imaginary part of energy can be expressed as

$$\text{Im}E_0 \approx -\frac{\hbar}{2} \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{\frac{\det \left[ -\partial^2 + \frac{1}{m} V''(x_F) \right]}{|\det' \left[ -\partial^2 + \frac{1}{m} V''(x_B) \right]|}} e^{-\frac{1}{\hbar} S_B}.$$

The decay rate related to the imaginary part of the lowest energy is  $\Gamma = -\frac{2}{\hbar} \text{Im}E_0$  and becomes

$$\Gamma_{\text{C\&C}} = \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{\frac{\det \left[ -\partial^2 + \frac{1}{m} V''(x_F) \right]}{|\det' \left[ -\partial^2 + \frac{1}{m} V''(x_B) \right]|}} e^{-\frac{1}{\hbar} S_B} \quad (2.14)$$

at  $\mathcal{O}(\hbar)$ .

### Comments

When explaining the imaginary part and the  $\frac{1}{2}$  factor, the one dimensional integral over a series of paths was used. In spite of the fact that the importance of the false vacuum solution was not emphasized in [70], it is necessary for a reasonable result as well as the occurrence of the  $\frac{1}{2}$  factor.

Furthermore, although the contributions from n-bounce solutions were included in [70], it is not necessary. The introduction of n-bounce solutions not only causes the thin-wall approximation issue but also introduces additional inconsistencies relevant to the time interval  $\mathcal{T}$  which determines the types of classical solutions that can

exist. An issue regarding the timing of when  $\mathcal{T}$  is taken to infinity will arise. In other words,  $\mathcal{T}$  was taken to infinity immediately as long as n-bounce solutions were considered since the n-bounce solutions are only reasonable for an infinite  $\mathcal{T}$ . We will elaborate on this topic in Section 4.3.

In fact, following [67], the path integral can be expressed as the sum of the contributions from the false vacuum solution and the bounce solution as

$$Z \approx Z_F + \frac{1}{2}Z_B.$$

Since  $Z_B$  has a  $e^{-\frac{1}{\hbar}S_B}$  factor, it is much smaller than  $Z_F$  in general. Consequently,

$$E_0 \approx \lim_{\mathcal{T} \rightarrow \infty} -\frac{\hbar}{\mathcal{T}} \ln \left( Z_F + \frac{1}{2}Z_B \right) \approx \lim_{\mathcal{T} \rightarrow \infty} -\frac{\hbar}{\mathcal{T}} \ln Z_F - \frac{\hbar}{\mathcal{T}} \frac{\frac{1}{2}Z_B}{Z_F},$$

which leads to the same result but with fewer problems.

Finally, it should be emphasized that taking the integral over  $\tau_c$  could cause extra problems in spite of the thin wall approximation.

### The Ratio of the Determinant

In order to obtain the final result, we must calculate the ratio of determinants in Eq.(2.14). We can examine the fluctuation operators of a portion of the full classical solutions from  $-\frac{\mathcal{P}}{2}$  to  $\frac{\mathcal{P}}{2}$  and then take  $\mathcal{P}$  to infinity. As explained in Appendix C.1.2, the ratio can be expressed as

$$\frac{\det' \left[ -\partial_\tau^2 + \frac{1}{m}V''(x_B) \right]}{\det \left[ -\partial_\tau^2 + \omega_F^2 \right]} = \frac{1}{\lambda_1} \frac{u_2^B(\frac{\mathcal{P}}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \mathcal{P})},$$

where  $u_2^B(\frac{\mathcal{P}}{2}) = -\frac{1}{\omega_F}$  and  $\lambda_1$  is Eq.(4.6). Hence, the full result of Callan and Coleman can be expressed as

$$\Gamma_{\text{C\&C}} = \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{2m\omega_F} A(\infty) e^{-\frac{1}{\hbar}S_B} (1 + \mathcal{O}(\hbar)),$$

where

$$A(\tau) \equiv \frac{\omega_F}{\sqrt{S_B}} (x_R - x_F) e^{\int_{x_B(\tau)}^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right)}.$$

### 2.3.2. Lefschetz-thimble Method

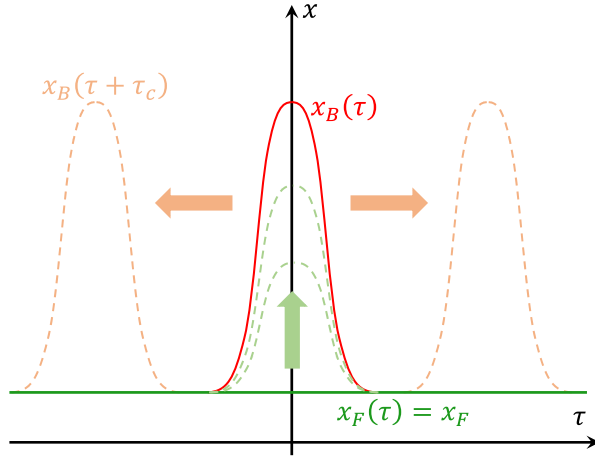
We here would introduce a new method using Lefschetz-thimble (see Appendix B.1) to calculate the path integral. It will be more intuitive when dealing with the negative eigenvalue and the zero eigenvalue.

As being elaborated on in the Appendix B.1, the Lefschetz-thimbles of a saddle point consist of all downward flows from it. The path integral can be expressed as the integral on the Lefschetz-thimbles of different saddle points. Thus, we need to find the saddle points of the action in accordance with the classical solutions at  $\mathcal{T} \rightarrow \infty$  as well. Just as the former subsection, the possible classical solutions include the false vacuum solution  $x_F(\tau)$ , the bounce solution  $x_B(\tau)$ <sup>5</sup>.

After the saddle points were found, the path integral can be expressed as the sum of the integral over the Lefschetz-thimble of each saddle point. A naive idea is that the result can be written as

$$Z = Z_F + Z_B = \mathcal{N} \int_{\mathcal{J}_F} \mathcal{D}z(\tau) e^{-\frac{1}{\hbar} S_E[z]} + \mathcal{N} \int_{\mathcal{J}_B} \mathcal{D}z(\tau) e^{-\frac{1}{\hbar} S_E[z]},$$

where  $\mathcal{J}_F$  and  $\mathcal{J}_B$  are the Lefschetz-thimble of the false vacuum solution  $x_F(\tau)$  and the bounce solution  $x_B(\tau)$ , each with  $n_B$  and  $n_F$  equal to 1 (see [79]).  $z(\tau)$  is the complexification of the real paths  $x(\tau)$ . This is true if the Lefschetz-thimble  $\mathcal{J}_F$  of the false vacuum solution  $x_F(\tau)$  and  $\mathcal{J}_B$  of the bounce solution are completely independent of each other. But in fact, there is a downward flow, referred to as Stokes rays (as discussed by [80]), which connects the two solutions in configuration space. See Figure 2.6, the green paths represent the points on the downward flow from  $x_F(\tau)$  to  $x_B(\tau)$  in the configuration space (refer to Figure 2.7). As a result, the

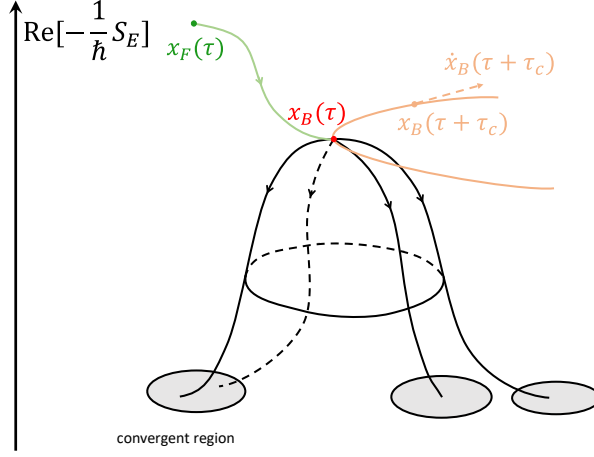


**Figure 2.6.:** Relating Solutions in Configuration Space

downward flow from  $x_F(\tau)$  must occupy one downward flow direction from  $x_B(\tau)$  to reach the convergent region, meaning  $\mathcal{J}_F$  contains part of  $\mathcal{J}_B$  (see Fig. 2.7). Therefore, to avoid double counting, only half of the integral on  $\mathcal{J}_B$  should be included:

$$Z = Z_F + \frac{1}{2} Z_B.$$

<sup>5</sup>The  $n$ -bounce solution  $x_{nB}$  can be discussed similarly.



**Figure 2.7.:** Flows in Configuration Space

Then we need to calculate  $Z_F$  and  $Z_B$ . Given that  $[-\partial_\tau^2 + \frac{1}{m}V''(x_F)]$  doesn't have non-positive eigenvalues, the eigenvalues are the flow eigenvalues and the Gaussian approximation of the integral on corresponding Lefschetz-thimble becomes

$$Z_F = \mathcal{N} \int_{\mathcal{F}_F} \mathcal{D}z e^{-\frac{1}{\hbar}S_E[z]} \approx \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m}V''(x_F) \right]}}.$$

As for the bounce solution, the corresponding fluctuation operator has a zero eigenvalue  $\lambda_1$  and a negative eigenvalue  $\lambda_0$ .

The existence of a zero eigenvalue indicates the existence of an invariant flow in the function space, which was also elaborated on in [67]. If we consider all  $x_B(\tau + \tau_0)$ s with different  $\tau_0$ s meeting the boundary conditions, they will form an invariant flow (see Figure.2.6, the orange paths represent  $x_B(\tau + \tau_0)$ s). All points along the invariant flow should contribute equivalently. The tangent direction of this invariant flow at  $x_B(\tau + \tau_0)$  is

$$\frac{\frac{\dot{x}_B(\tau + \tau_0)}{\sqrt{m}}}{\left\| \frac{\dot{x}_B(\tau + \tau_0)}{\sqrt{m}} \right\|} = \sqrt{m} \frac{\dot{x}_B(\tau + \tau_0)}{\|\dot{x}_B(\tau + \tau_0)\|},$$

and the length is

$$L = \int dL = \int \left[ \int d\tau \sqrt{m} \frac{\dot{x}_B(\tau + \tau_0)}{\|\dot{x}_B(\tau + \tau_0)\|} \frac{dx_B(\tau + \tau_0)}{d\tau_0} \right] \frac{d\tau_0}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_B}{2\pi\hbar}} \int d\tau_0 = \sqrt{\frac{S_B}{2\pi\hbar}} \mathcal{T}.$$

As for the negative eigenvalue, it corresponds to the direction of a upward flow. The flow eigenvalue  $\lambda_0$  can be changed to  $-\lambda_0$  by replacing the direction  $\chi_0^B(\tau)$  with  $+i\chi_0^B(\tau)$  or  $-i\chi_0^B(\tau)$ , which corresponds to direction of the downward flow from the

bounce solution. Hence, the Gaussian approximation around the bounce solution becomes

$$Z_B = \sqrt{\frac{S_E[x_B]}{2\pi}} \mathcal{T} \cdot \mathcal{N} \int_{\mathcal{J}_B} \mathcal{D}z e^{-\frac{1}{\hbar} S_E[z]} \approx i \mathcal{N} \frac{\sqrt{\frac{S_E[x_B]}{2\pi}} \mathcal{T} e^{-\frac{1}{\hbar} S_B}}{\sqrt{|\det'[-\partial_\tau^2 + \frac{1}{m} V''(x_B)]|}},$$

where  $i$  is the Jacobian determinant due to the transformation of the flow eigenfunction with negative eigenvalue.

Finally, we get  $\text{Im}E_0$  as

$$\begin{aligned} \text{Im}E_0 &= -\lim_{\mathcal{T} \rightarrow \infty} \frac{\hbar}{\mathcal{T}} \text{Im} \ln \left( Z_F + \frac{1}{2} Z_B \right) \\ &\approx -\lim_{\mathcal{T} \rightarrow \infty} \frac{\hbar}{\mathcal{T}} \text{Im} \frac{1}{2} \frac{Z_B}{Z_F} \\ &= -\frac{\hbar}{2} \sqrt{\frac{S_B}{2\pi\hbar}} \frac{\sqrt{\det[-\partial_\tau^2 + \frac{1}{m} V''(x_F)]}}{\sqrt{|\det'[-\partial_\tau^2 + \frac{1}{m} V''(x_B)]|}} e^{-\frac{1}{\hbar} S_B}. \end{aligned}$$

And decay rate will be

$$\Gamma_{\text{C\&C}} = -\frac{2}{\hbar} \text{Im}E_0 = \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{\frac{\det[-\partial_\tau^2 + \frac{1}{m} V''(x_F)]}{|\det'[-\partial_\tau^2 + \frac{1}{m} V''(x_B)]|}} e^{-\frac{1}{\hbar} S_B}.$$

## 2.4. How to Understand Complex Energy Eigenvalue

Although we have shown the calculation of the decay rate both from the WKB approximation and the saddle point approximation, an important problem required to be explained exists. As discussed in the previous section, the complex energy corresponds to the Gamow-Siegert boundary conditions from the perspective of canonical quantization [81, 82]. It should be understood by using the knowledge from the theory of non-unitary quantum mechanics [83]. How does one explain the appearance of the complex value in the functional integral framework since the path integral is defined by the Euclidean time? In order to demonstrate the issue, it is necessary to introduce the research of [75] at first.

For a physical potential, which is bounded from below, there is another solution called shot solution  $x_S(\tau)$  when choosing  $x_i = x_f = x_F$  at zero temperature according to [75]. If the contribution of the shot solution is included, then  $-\frac{1}{\hbar} S_E(z)$  over a series of paths  $\tilde{x}(\tau; z)$  parameterized by a real variable  $z$  has an upper bound since the shot solution is also a saddle point (see Figure.2.9a).

Therefore, the one dimensional integral

$$Z_{\{z\}} = \int \frac{dz}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} S_E[z]} \quad (2.15)$$

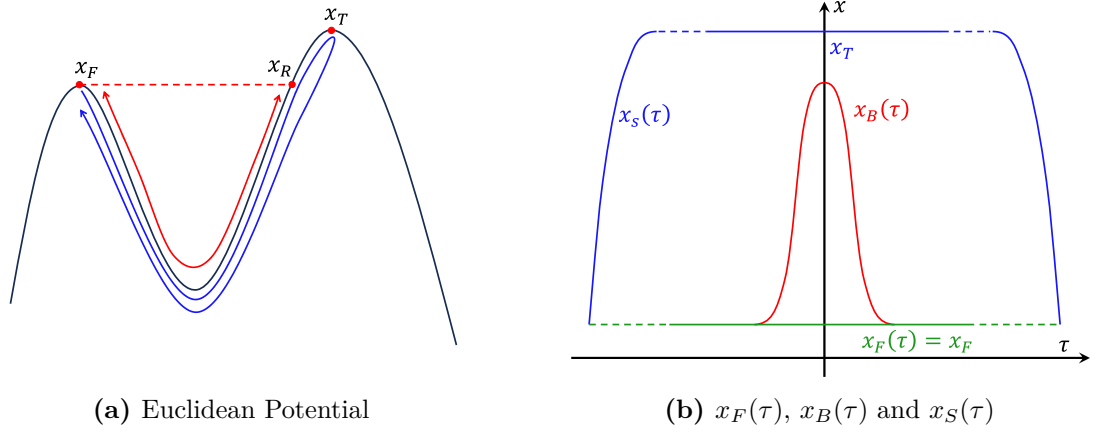


Figure 2.8.: Classical Solutions

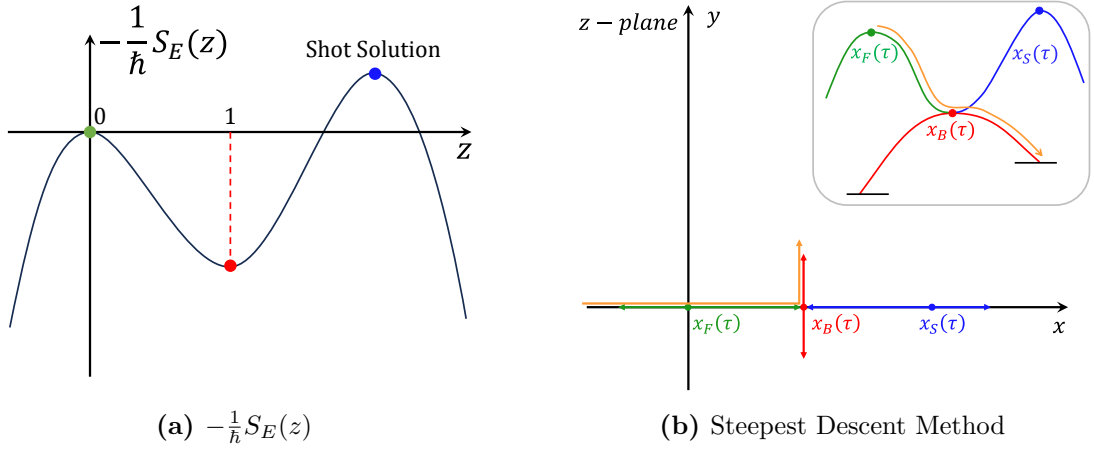


Figure 2.9.: Saddle Point Approximation

should converge and no analytic continuation is required. Then no imaginary part of the energy should appear. Furthermore, the contribution from the shot solution should be the largest.

However, the steepest-descent contour through the false vacuum solution rather than the original contour along the real  $z$ -axis was chosen following previous research. The existence of the shot solution won't affect the final results. This is the explanation for the issue that we put forward at the beginning of this section. In conclusion, the selection of the integral contour (or saddle points) should reflect the boundary conditions in the canonical quantization framework.

Nevertheless, there is still a serious problem unsolved. It is obvious that all classical solutions are real and the imaginary part of the corresponding actions are all zero. Although the imaginary part remains constant along a steepest variation flow, it is not correct that points with the imaginary parts are on the same steepest

variation flow. We make assumption here that the false vacuum solution, the bounce solution and the shot solution are on the same steepest variation flow.

In conclusion,

1. The boundary conditions of the metastable state are assumed to be corresponding to the selection of the integral contour (or saddle points).
2. The false vacuum solution, the bounce solution and the shot solution are assumed to be connected by the same steepest descent/ascent flow. This steepest descent flow from the false vacuum solution corresponds to the negative mode of the bounce solution (the steepest ascent flow from the bounce solution).



### 3. Decay Rate at Finite Temperature

Similarly to the situation at zero temperature, the decay rate can be expressed as the function of the free energy at finite temperature as

$$\Gamma^\beta = -\frac{2}{\hbar} \text{Im} F.$$

Therefore, the calculation of the decay rate at finite temperature transforms to the calculation of the imaginary part of the free energy. The fundamental concepts of quantum theory at finite temperature can be found in [84, 85].

By using the partition function  $Z^\beta = \text{Tr}[e^{-\beta\hat{H}}]$ , the free energy at finite temperature can be represented as

$$\begin{aligned} F &= -\frac{1}{\beta} \ln Z^\beta = -\frac{1}{\beta} \ln \text{Tr}[e^{-\beta\hat{H}}] \\ &= -\frac{1}{\beta} \ln \int dy \langle y | e^{-\beta\hat{H}} | y \rangle \\ &= -\frac{1}{\beta} \ln \int dy \mathcal{N} \int_{x(-\frac{\beta\hbar}{2})=x(\frac{\beta\hbar}{2})=y} \mathcal{D}x(\tau) e^{-\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau L_E[x(\tau)]}. \end{aligned}$$

What should be emphasized is that the partition function is dimensionless, unlike the transition amplitude at zero temperature, which has the dimension of  $[\text{length}]^{-1}$ . Moreover, three other significant differences arise when the temperature becomes finite.

1. The Euclidean time interval changes from infinity to a finite value  $\beta\hbar$ .
2. The endpoints of a path used in path integral should take the same value  $y$ .
3. The integral over the endpoints  $y$  must be performed due to the trace.

Before discussing our method, we will first review some results from previous researches.

## 3.1. Review of Previous Research

### 3.1.1. The Path Integral Calculated by Performing Functional Integral

In this subsection, the result from [86] will be reviewed. In [86], the stationary phase approximation is used to calculate the real time path integral

$$Z(T) \equiv \text{Tr}[e^{-i\hat{H}T}] = \int dy \mathcal{N} \int_{x(-\frac{T}{2})=y}^{x(\frac{T}{2})=y} \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x(t)]}.$$

The stationary-phase approximation involves including only the contributions from the classical periodic orbits  $\bar{x}(\tau)$ s with period  $T$ . After such classical periodic orbits  $\bar{x}(\tau)$  was found,  $Z(T)$  can be expressed as

$$Z(T) \approx e^{\frac{i}{\hbar} S[\bar{x}]} \int dy \mathcal{N} \int_{\Delta x(-\frac{T}{2})=0}^{\Delta x(\frac{T}{2})=0} \mathcal{D}\Delta x(t) e^{\frac{i}{\hbar} \tilde{S}[\Delta x]},$$

where  $\tilde{S}[\Delta x] \equiv \int_0^T dt \left( \frac{1}{2} m \dot{\Delta x}^2 - \frac{1}{2} \Delta x^2 V''(\bar{x}) \right)$ .

We should emphasize here that the periodic orbit with period  $T$  means that  $\bar{x}$  should not only satisfy  $\bar{x}(\frac{T}{2}) = \bar{x}(-\frac{T}{2})$  but also meet  $\dot{\bar{x}}(\frac{T}{2}) = \dot{\bar{x}}(-\frac{T}{2})$ . The reason for only considering the classical periodic orbits was explained in [86, 87]. A more intuitive explanation was provided in [88] as well. Only the classical periodic orbits can provide stationary phase, which means  $S[\bar{x}(t)]$  won't change with  $y$ :

$$0 = \frac{\partial S[\bar{x}(t, y)]}{\partial y} = \left[ m \dot{\bar{x}}(\tau) \partial_y \bar{x}(\tau) \right]_{-\frac{T}{2}}^{\frac{T}{2}} = m \dot{\bar{x}}\left(\frac{T}{2}\right) - m \dot{\bar{x}}\left(-\frac{T}{2}\right).$$

Consequently,  $e^{\frac{i}{\hbar} S[\bar{x}]}$  can be factored out of the integral over  $y$  after expanding  $x(t)$  around  $\bar{x}(t)$ .

The detailed derivation of the final result [86] is putted in Appendix B.2 and the final result becomes

$$Z_{\bar{x}}(T) = \sqrt{\frac{i}{2\pi\hbar}} e^{\frac{i}{\hbar} S[\bar{x}]} \frac{e^{-i\pi T}}{\sqrt{\left| \frac{dT}{dE} \right|}}.$$

The final result of the integral over  $y$  after expanding  $x(t)$  around  $\bar{x}(t)$  leads to the appearance of  $T$ .

This method can also be applied to the imaginary time path integral. Replacing  $T$  with  $-i\mathcal{T}$  yields

$$Z_{\bar{x}}(\mathcal{T}) = i \sqrt{\frac{1}{2\pi\hbar}} e^{-\frac{1}{\hbar} S_E[\bar{x}]} \frac{\mathcal{T}}{\sqrt{\left| \frac{d\mathcal{T}}{dE} \right|}},$$

where  $\bar{x}$  is now a classical periodic orbit with period of  $\mathcal{T}$  in the potential  $-V(x)$ . At finite temperature,  $\mathcal{T} = \beta\hbar$  and the classical periodic orbit corresponds to the bounce solution. Hence, the contribution from bounce solution is

$$Z_{B-\text{Da}}^\beta = ie^{-\frac{1}{\hbar}S_B^\beta} \frac{\beta\hbar}{\sqrt{2\pi\hbar \left| \frac{d\beta\hbar}{dE_B} \right|}}. \quad (3.1)$$

Here, the actions of periodic orbits with period  $\mathcal{T}$  in the potential  $-V(x)$  don't change with endpoints just as the situation in real time.

In conclusion, unlike those studies using the functional determinant, [86] chose shifting methods to derive the contribution from a periodic classical solution, which provides a relatively independent way to check other results.

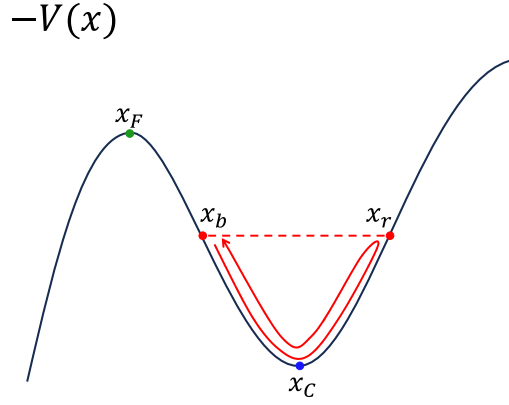
### 3.1.2. The Decay Rate at Finite Temperature Calculated with Periodic Boundary Conditions

In this section, the derivation of the decay rate from the imaginary part of the free energy in [74] will be reviewed. The basic consideration originates from the calculation of the decay rate at zero temperature. The path integral formulation of the free energy can also be calculated by the saddle point approximation. But because of the differences between the transition amplitude and the free energy, the methods should become somehow different. On the one hand, although three significant differences arise at finite temperature, some previous studies including [74] didn't carry out the integral over the endpoints of paths unlike the calculation done by [86]. On the other hand, they did contain only the contributions from the classical periodic orbits satisfying two conditions :  $x(\frac{\beta\hbar}{2}) = x(-\frac{\beta\hbar}{2})$  and  $\dot{x}(-\frac{\beta\hbar}{2}) = \dot{x}(\frac{\beta\hbar}{2})$  with a period of  $\beta\hbar$  just like what [86] did in the calculation of the real time path integral when performing the saddle point approximation. Moreover, not only are the classical paths  $\bar{x}(\tau)$  required to be periodic, but also the fluctuations  $\Delta x(\tau)$  around the saddle points are required to be periodic to be in a consistent way. As a consequence, the fluctuation determinant included in the final result is defined with the periodic boundary conditions, which is completely different from the result calculated at zero temperature. The basic consideration can be expressed as

$$\text{Tr}[e^{-\beta\hat{H}}] = \mathcal{N} \oint \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \mathcal{L}_E[x(\tau)]},$$

where  $\oint$  is used to represent the periodic boundary conditions :  $x(\frac{\beta\hbar}{2}) = x(-\frac{\beta\hbar}{2})$  and  $\dot{x}(-\frac{\beta\hbar}{2}) = \dot{x}(\frac{\beta\hbar}{2})$ .

Now, the method proposed by Affleck [74](see also [67,89]) will be reviewed. The basic consideration is



**Figure 3.1.:** Euclidean Potential

1. Find all periodic classical solutions;
2. Derive the contributions of each periodic classical solution;
3. Sum the contributions of related periodic classical solutions.

For a Euclidean potential like the one in Figure.3.1, three periodic classical orbits exist below the critical temperature<sup>1</sup>  $T_c = \frac{\hbar\omega_c}{2\pi}$ . When  $\beta\hbar \geq \frac{2\pi}{\omega_c}$ , there are two trivial saddle points  $x_F^\beta(\tau) = x_F$ ,  $x_C^\beta(\tau) = x_C$  and a bounce solution  $x_B^\beta(\tau)$  with period  $\beta\hbar^2$ .

Here, let's consider about the bounce solution in more detail. Denote the end-points of the bounce solution by  $x_b$  and the turning point to be  $x_r$ . Since the bounce solution satisfies the classical equation of motion and is even, the period of a bounce solution with energy  $E_B$  should be

$$P_B(E_B) = 2 \int_{x_b}^{x_r} dx \frac{1}{\sqrt{\frac{2}{m} (V(x) + E_B)}},$$

where  $E_B = -V(x_b) = -V(x_r)$ . When  $E_B = -V(x_F)$ , the period becomes infinite, and the bounce becomes the same as the bounce at zero temperature. As  $E_B$  approaches  $-V(x_C)$ , the period will be

$$\lim_{E \rightarrow -V(x_C)} P(E) = \frac{2\pi}{\omega_c}.$$

In addition, it is assumed that the period of bounce solution increases monotonically with energy  $E_B$ . As a result, the period of a bounce solution should satisfy  $P_B \in$

---

<sup>1</sup> $\omega_F \equiv \sqrt{\frac{V''(x_F)}{m}}$ ,  $\omega_c \equiv \sqrt{\frac{V''(x_C)}{m}}$

<sup>2</sup>The bounce solution with period  $\frac{\beta\hbar}{n}$ , where  $n$  is an integer, was ignored.

$[\frac{2\pi}{\omega_c}, \infty)$ . Since the  $P_B$  should be equal to  $\beta\hbar$ ,

$$\beta\hbar = P_B(E_B) = 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx. \quad (3.2)$$

It is apparent that the bounce solution will no longer exist when  $T > \frac{\hbar\omega_c}{2\pi}$ , or in other words,  $\beta\hbar < \frac{2\pi}{\omega_c}$ . This is the reason why  $T_c \equiv \frac{\hbar\omega_c}{2\pi}$  was called as critical temperature. Our research is only interested in the temperature below the critical temperature. That is to say, our research focuses on the situation in which the bounce solution exists.

Below the critical temperature, only  $x_F^\beta(\tau) = x_F$  and  $x_B^\beta(\tau)$  were employed to calculate the imaginary part of free energy in [74].

The action of the false vacuum solution  $S_F^\beta$  will be zero if  $V(x_F) = 0$  is chosen. Then the contribution from the false vacuum solution  $x_F^\beta(\tau) = x_F$  can be expressed as

$$Z_{F-\text{Aff}}^\beta = \mathcal{N} \int_{\text{P.B.C}} \mathcal{D}\Delta x(\tau) e^{-\frac{1}{\hbar} \frac{m}{2} \int d\tau \Delta x(\tau) (-\partial_\tau^2 + \omega_F^2) \Delta x(\tau)} = \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]}},$$

where the determinant is defined by the periodic boundary conditions. Here, if **the periodic boundary conditions are considered as the same as taking trace**, the path integral of  $x(\tau)$  can be expressed as

$$\mathcal{N} \int_{\text{P.B.C}} \mathcal{D}\Delta x(\tau) e^{-\frac{1}{\hbar} \frac{m}{2} \int d\tau \Delta x(\tau) (-\partial_\tau^2 + \omega_F^2) \Delta x(\tau)} \rightarrow \text{Tr}[e^{-\beta \hat{H}_o}],$$

where  $\hat{H}_o$  is the Hamiltonian of the harmonic oscillator. Therefore,

$$\text{Tr}[e^{-\beta \hat{H}_o}] = \sum_{n=0} e^{-\beta \omega_F (n + \frac{1}{2})} = \frac{1}{2 \sinh \frac{\omega_F \beta \hbar}{2}}.$$

The contribution from the false vacuum solution  $x_F^\beta(\tau) = x_F$  can be obtained as

$$Z_{F-\text{Aff}}^\beta = \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]}} = \frac{1}{2 \sinh \frac{\omega_F \beta \hbar}{2}}. \quad (3.3)$$

For comparison, since this is the saddle point approximation for  $\text{Tr}[e^{-\beta \hat{H}}]$ ,  $Z_F^\beta$  is dimensionless while the contribution from the false vacuum solution at zero temperature has the dimension of  $[\text{length}]^{-1}$ , which is the same as the transition amplitude.

As for the contribution from bounce solution, according to [74], just like the situation in the zero temperature case, the fluctuation operator  $\left[ -\partial^2 + \frac{1}{m} V''(x_B^\beta(\tau)) \right]$  also has a zero eigenvalue since  $\dot{x}_B^\beta(\tau)$  is **periodic** and satisfies  $\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right] \dot{x}_B^\beta(\tau) =$

0. In order to deal with the infinity caused by the zero eigenvalue, the so-called collective coordinate method was applied in [74], which led to the following result

$$\underbrace{\int \frac{dc_1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} \frac{1}{2} c_1^2 \lambda_1}}_{\text{infinite}} = \underbrace{\sqrt{\frac{m \|\dot{x}_B^\beta(\tau)\|^2}{2\pi\hbar}}}_{\text{finite}} \beta\hbar, \quad (3.4)$$

where  $c_1$  is the expansion coefficient corresponding to the eigenfunction with a zero eigenvalue

$$\varphi_1 = \frac{\dot{x}_B^\beta(\tau)}{\|\dot{x}_B^\beta(\tau)\|},$$

where  $\|\dot{x}_B^\beta(\tau)\|$  means  $\sqrt{\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau [\dot{x}_B^\beta(\tau)]^2}$ . The final result for the contribution of the bounce is

$$Z_{B-\text{Aff}}^\beta = \mathcal{N} \frac{\sqrt{\frac{m}{2\pi\hbar}} \|\dot{x}_B^\beta(\tau)\| \beta\hbar e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}}. \quad (3.5)$$

In order to obtain the final result, the fluctuation determinant must be calculated. Since both  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]$  and  $\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]$  are defined by periodic boundary conditions and  $\dot{\varphi}_1(\frac{\beta\hbar}{2}) = \frac{\dot{x}_B^\beta(\frac{\beta\hbar}{2})}{\|\dot{x}_B^\beta(\tau)\|} \neq 0$  at finite temperature, Eq.(B.30) can be applied (see Appendix B.3.4). The ratio is simply

$$\frac{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]} = \frac{m \frac{d\beta\hbar}{dE_B} \|\dot{x}_B^\beta(\tau)\|^2}{2 - 2 \cosh(\omega_F \beta\hbar)} = -\frac{m \frac{d\beta\hbar}{dE_B} \|\dot{x}_B^\beta(\tau)\|^2}{4 \sinh^2(\frac{\omega_F \beta\hbar}{2})}. \quad (3.6)$$

The ratio is apparently negative, which manifests the fluctuation operator of bounce solution has at least one negative eigenvalue. Replacing the ratio in Eq.(3.5) with Eq.(3.6) and using Eq.(3.3) gives

$$Z_{B-\text{Aff}}^\beta = i e^{-\frac{1}{\hbar} S_B^\beta} \frac{\beta\hbar}{\sqrt{2\pi\hbar \left| \frac{d\beta\hbar}{dE_B} \right|}},$$

which is exactly the same as Eq.(3.1).

The imaginary part of the free energy can be written as

$$\begin{aligned} \text{Im} F &\approx -\frac{1}{\beta} \ln \left( Z_{F-\text{Aff}}^\beta + \frac{1}{2} Z_{B-\text{Aff}}^\beta \right) \approx -\frac{1}{\beta} \frac{\frac{1}{2} Z_{B-\text{Aff}}^\beta}{Z_{F-\text{Aff}}^\beta} \\ &= -\frac{\hbar}{2} \sqrt{\frac{m}{2\pi\hbar}} \|\dot{x}_B^\beta(\tau)\| \sqrt{\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]}{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}} e^{-\frac{1}{\hbar} S_B^\beta} \\ &= -\frac{\hbar}{2} 2 \sinh \frac{\omega_F \beta\hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{2\pi\hbar \left| \frac{d\beta\hbar}{dE_B} \right|}}, \end{aligned} \quad (3.7)$$

where the  $\frac{1}{2}$  factor is introduced for the same reason as the situation at zero temperature since the false vacuum solution and the bounce solution are assumed to be on the same steepest descent/ascent flow in the periodic function space as well. Since we have proven that Eq.(3.5) is the same as Dashen's result, one can directly use the result of [86] and Eq.(3.3) to obtain the final result without relying on the ratio of the two determinants in fact.

Therefore, the decay rate is

$$\Gamma_{\text{Aff}}^\beta = -\frac{2}{\hbar} \text{Im}F = 2 \sinh \frac{\omega_F \beta \hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{2\pi \hbar \left| \frac{d\beta \hbar}{dE_B} \right|}}.$$

### The Zero Temperature Limit

Now, the zero temperature limit of  $\Gamma_{\text{Aff}}^\beta$  will be derived.

$\beta \hbar$  is the function of  $x_b$  according to Eq.(3.2). It is assumed that  $\beta \hbar$  decreases monotonically with  $x_b$  in the reasonable range. Consequently, the zero temperature limit  $\beta \hbar \rightarrow \infty$  corresponds to  $x_b \rightarrow x_F$ . The imaginary part of the free energy can be expressed as

$$\text{Im}F = -\frac{\hbar}{2} 2 \sinh \frac{\omega_F \beta \hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{2\pi \hbar \left| \frac{1}{V'(x_b)} \frac{d\beta \hbar}{dx_b} \right|}} = -\frac{\hbar}{2} 2V'(x_b) \sinh \frac{\omega_F \beta \hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{2\pi \hbar \left| V'(x_b) \frac{d\beta \hbar}{dx_b} \right|}}.$$

In Appendix C.2, the following two relations were obtained

1. The relation between  $V'(x_b)$  and  $e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}}$  :

$$\lim_{x_b \rightarrow x_F} |V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}} = 2 |V''(x_F)| (x_R - x_F) \cdot e^{\int_{x_F}^{x_R} \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} - \frac{1}{|x - x_b|}} d\bar{x}}.$$

2. The relation between  $V'(x_b)$  and  $\frac{d\beta \hbar}{dx_b}$  :

$$\lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta \hbar}{dx_b} = -2 \sqrt{\frac{m}{V''(x_b)}} V''(x_b) = -2 \sqrt{m V''(x_F)}.$$

Replacing the corresponding terms in  $\text{Im}F$ , it becomes

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \text{Im}F &= \lim_{x_b \rightarrow x_F} -\frac{\hbar}{2} 2V'(x_b) \sinh \frac{\omega_F \beta \hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{2\pi \hbar \left| V'(x_b) \frac{d\beta \hbar}{dx_b} \right|}} \\ &= -\frac{\hbar}{2} \sqrt{\frac{m \omega_F}{\pi \hbar}} \omega_F (x_R - x_F) e^{\int_{x_F}^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|x - x_F|}} d\bar{x}} e^{-\frac{1}{\hbar} S_B}. \end{aligned}$$

By denoting  $A(y)$  (similar to  $A(\tau)$ ) as

$$\begin{aligned} A(y) &\equiv \frac{\omega_F}{\sqrt{S_B}} (x_R - x_F) e^{\int_y^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right)} \\ &= \frac{\omega_F}{\sqrt{S_B}} (y - x_F) e^{\int_y^{x_R} d\bar{x} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}}}, \end{aligned}$$

the zero temperature limit of  $\text{Im}F$  can be expressed as

$$\lim_{\beta \rightarrow \infty} \text{Im}F = -\frac{\hbar}{2} \sqrt{\frac{m\omega_F}{\pi\hbar}} \sqrt{S_B} A(x_F) e^{-\frac{1}{\hbar} S_B}.$$

And the decay rate becomes

$$\Gamma_{\text{Aff}}^\infty = \sqrt{\frac{m\omega_F}{\pi\hbar}} \sqrt{S_B} A(x_F) e^{-\frac{1}{\hbar} S_B}, \quad (3.8)$$

which matches exactly with the results from [69, 70].

### 3.1.3. Discussion and Reconsideration

We here aim to give some important comments on previous methods.

The first problem we want to discuss relates to the boundary conditions of paths in the path integral. We wonder whether the periodic conditions for paths to be used is proper or not.

$$F = -\frac{1}{\beta} \ln \text{Tr}[e^{-\beta \hat{H}}] \stackrel{?}{=} -\frac{1}{\beta} \ln \mathcal{N} \oint \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \mathcal{L}_E[x(\tau)]}.$$

Through taking the trace, all of the paths with  $x(\frac{T}{2}) = x(-\frac{T}{2})$  should contribute to the functional integral, in which jagged paths are also included just as discussed in [88]. The saddle point approximation selects the classical paths, which also include those which are not periodic. When one try to calculate the real time path integral by stationary point approximation, maybe the contribution from classical paths without stationary phase cancel with each other as one could image. But the situation in Euclidean time path integral is different, paths with different actions will all have positive contributions. Therefore, we don't think it is reasonable to ignore paths which are not periodic. Furthermore, although the trace is not carried out intuitively, the fluctuation  $\Delta x(\tau) = x(\tau) - \bar{x}(\tau)$  after expanding around  $\bar{x}$  is also demanded to be periodic and the integral over  $\Delta x(\tau)$  was performed. Is it equivalent to taking trace? What is the reason for requiring  $\Delta x(\tau)$  to be periodic? In addition, double counting could appear since a  $x(\tau)$ ,  $\bar{x}(\tau)$  and  $\Delta x(\tau)$  all satisfy completely identical boundary conditions.

The second problem concerns the treatment of the zero eigenvalue of the bounce solution. There is definitely one zero eigenvalue  $\lambda_1$  corresponding to the bounce solution regardless of whether the temperature is finite or zero, and whether the boundary conditions are periodic or Dirichlet. The integral with respect to the coefficient  $c_1$  of the zero eigenfunction should be infinite. But because of the usage of the time-translation collective coordinate method, the integral was given as  $\sqrt{\frac{m}{2\pi\hbar}} \|\dot{x}_B^\beta(\tau)\| \beta\hbar$ , which is a finite value at finite temperature. The following relation is fairly suspicious.

$$\underbrace{\int \frac{dc_1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar} \frac{1}{2} c_1^2 \lambda_1}}_{\text{infinite}} = \underbrace{\sqrt{\frac{m \|\dot{x}_B^\beta(\tau)\|^2}{2\pi\hbar}}}_{\text{finite}} \beta\hbar.$$

Finally, the contributions from other periodic classical solutions with the same period are ignored. After finding a bounce solution, one can shift the center of the bounce solution to obtain a series of shifted-bounce solutions. All these shifted-bounce solutions were ignored in the previous calculation. If the point of view that the contributions from shifted-bounce solutions have already been contained when applying the collective coordinate method as will be explained in the next chapter is adopted, then the treatment for the zero eigenvalue requires additional explanation.

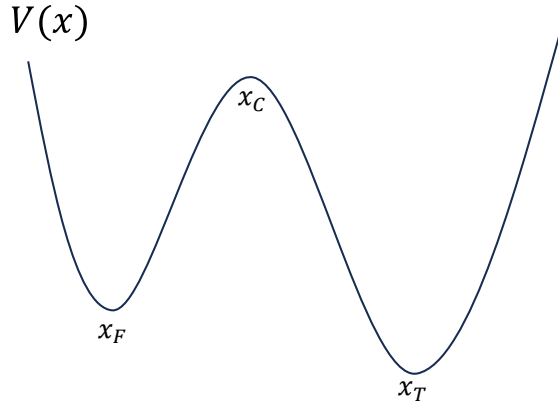
So as to avoid all three ambiguities mentioned above, we will introduce a different method to calculate the free energy by saddle point approximation. Not only the periodic paths, but all the contributions of paths with  $x(-\frac{\beta\hbar}{2}) = x(\frac{\beta\hbar}{2}) = y$  will be included in the path integral in the process of applying saddle point approximation. The periodic conditions are not required to be satisfied. The trace will be dealt with more carefully and the integral over endpoints will be preformed in an intuitive way as well. Certainly, the fluctuation determinant of a classical path will be defined with Dirichlet boundary conditions other than periodic boundary conditions.

## 3.2. Classical Solutions with Dirichlet Boundary Conditions

Since the integral over the endpoints  $y$  should be carried out, we need to find all classical solutions of which the endpoints satisfy  $x(-\frac{\beta\hbar}{2}) = x(\frac{\beta\hbar}{2}) = y$  firstly and use the saddle point approximation to calculate their contributions. After that, we will take the integral over  $y$  and derive the final result. The classical solutions can be divided into two types : static classical solutions and non-static classical solutions.

### 3.2.1. Static Classical Solutions

Consider the potential shown in Figure.3.2, three static classical solutions exist.



**Figure 3.2.:** Potential

1. False vacuum static solution  $x_F^\beta(\tau) = x_F$  with boundary condition  $x(\pm \frac{\beta\hbar}{2}) = x_F$
2. Critical point static solution  $x_C^\beta(\tau) = x_C$  with boundary condition  $x(\pm \frac{\beta\hbar}{2}) = x_C$
3. True vacuum static solution  $x_T^\beta(\tau) = x_T$  with boundary condition  $x(\pm \frac{\beta\hbar}{2}) = x_T$

Since we need to take the integral over  $y$ , those static classical solutions won't contribute to the final result because they only exist for particular isolated points, which are completely different from the treatment of those who use periodic boundary conditions.

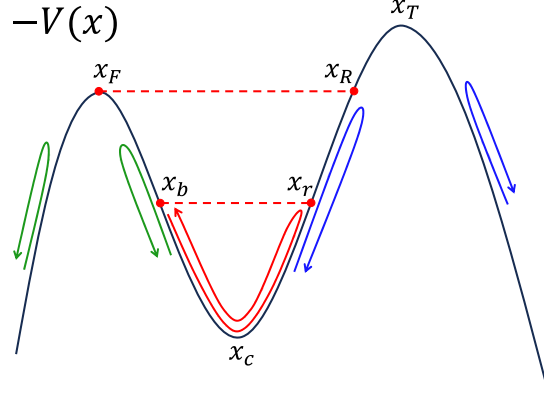
### 3.2.2. Non-static Classical Solutions

Unlike the static classical solutions, non-static classical solutions of which the endpoints are  $y$  can exist for each  $y$  in a specified range. Consequently, the contributions of those non-static classical solutions can't be ignored.

#### **Bounce Solution $x_B^\beta(\tau)$ and Shifted-bounce Solutions $x_{sB}^\beta(\tau)$ s**

The bounce solution referred to here is the same as the one used in [74]. The basic properties of the bounce solution are

1. It should be the periodic classical solution with a period of  $\beta\hbar$ .
2. Its initial velocity should vanish.
3. Its endpoints should approach the false vacuum as temperature decreases.



**Figure 3.3.:** Non-static Classical Solutions at Finite Temperature

The  $n$ -bounce solution, of which the period is  $\frac{\beta\hbar}{n}$  and  $n$  is an integer, is ignored as well in our research. Once we get a bounce solution  $x_B^\beta(\tau)$ , we can just shift the center of bounce solution by any  $\tau_c$  and get the shifted-bounce solution  $x_{sB}^\beta(\tau) = x_B^\beta(\tau - \tau_c)$ . The shifted-bounce solutions typically have non-vanishing initial velocities and its boundary conditions are different from the bounce solution.

Be careful, since the bounce solution is even, two shifted bounce solutions with the same endpoints but different initial velocities, which correspond to opposite  $\tau_c$ s, should be treated as different solutions (see Figure.3.4).

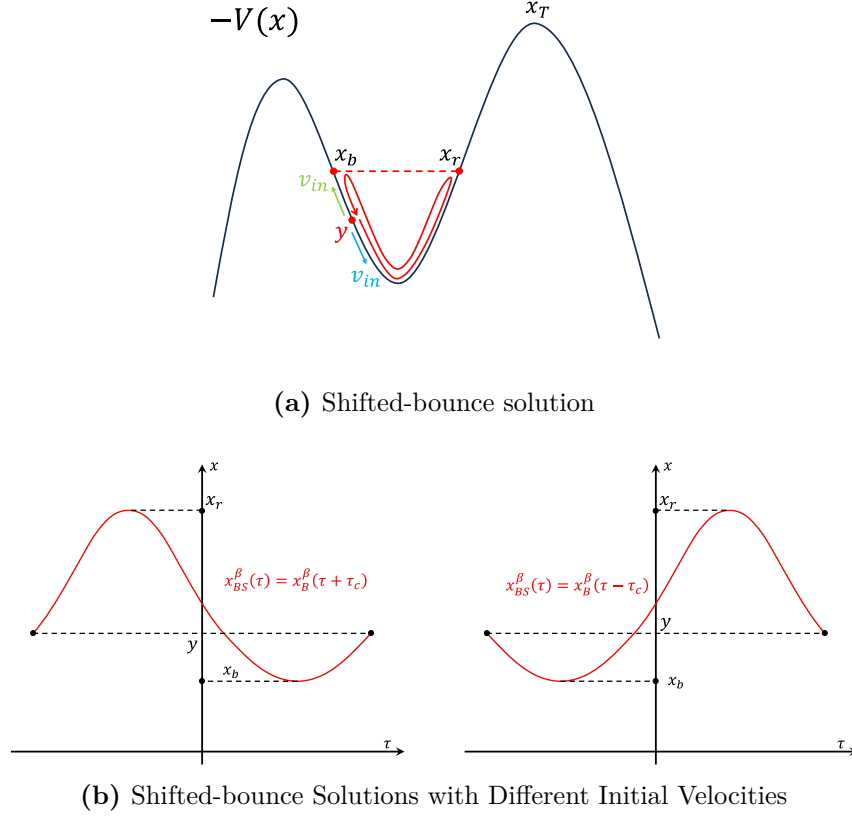
### Shot Solutions

As for the shot solutions at finite temperature, there could be at least one shot solution for every point. We now focus on the shot solutions of which the endpoints are inside the potential barrier. Then, two kinds of relevant shot solutions with different initial velocities could exist simultaneously. The one directed towards the false vacuum could be termed as shot-F solution  $x_{sF}^\beta$  and the one towards the true vacuum could be termed as shot-T solution  $x_{sT}^\beta$ . Shot solutions are also even and the turning point of shot-solution is denoted as  $x_s$  (see Figure.3.5).

Now, we will discuss some important properties of shot-F solutions. The properties of shot-T solutions can be derived similarly.

First, as we have said that a shot-F solution is also a even function and since a shot solution starting from  $y$  satisfies the boundary condition  $x(-\frac{\beta\hbar}{2}) = x(\frac{\beta\hbar}{2}) = y$ , the starting point  $y$  and the turning point  $x_s$  must meets the following condition

$$\beta\hbar = 2 \int_{x_s}^y dx \frac{1}{\sqrt{\frac{2}{m}(V(x) + E_S)}} = 2 \int_{x_s}^y dx \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_s))}}, \quad (3.9)$$



**Figure 3.4.:** Shifted-bounce Solution

where  $E_S = -V(x_s)$  is the classical energy of the shot solution. It is obvious that  $x_s$  is a function of  $y$ . Denote the integral in Eq.(3.9) as  $T_{sF}(x_s, y)$  and the following relations can be obtained:

$$\partial_y T_{sF} = \sqrt{2m} \frac{1}{\sqrt{V(y) - V(x_s)}}$$

as well as

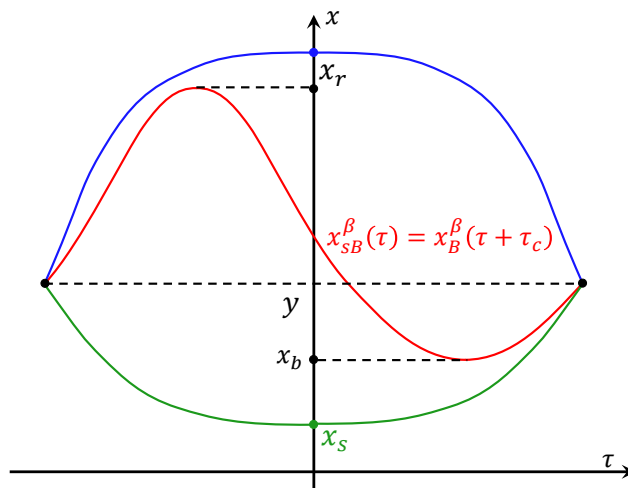
$$\partial_{x_s} T_{sF} = \sqrt{2m} \left( \frac{1}{2} \int_{x_s}^y \frac{V'(x_s)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx - \lim_{z \rightarrow x_s} \frac{1}{\sqrt{V(z) - V(x_s)}} \right) \quad (3.10)$$

$$= \sqrt{2m} \left( \frac{1}{2} \int_{x_s}^y \frac{V'(x_s) - V'(x)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx - \frac{1}{\sqrt{V(y) - V(x_s)}} \right), \quad (3.11)$$

where

$$\frac{1}{2} \int_{x_s}^y \frac{-V'(x)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx = \left[ \frac{1}{\sqrt{V(x) - V(x_s)}} \right]_{x_s}^y$$

was used. By making use of the implicit differentiation formula, the expression for

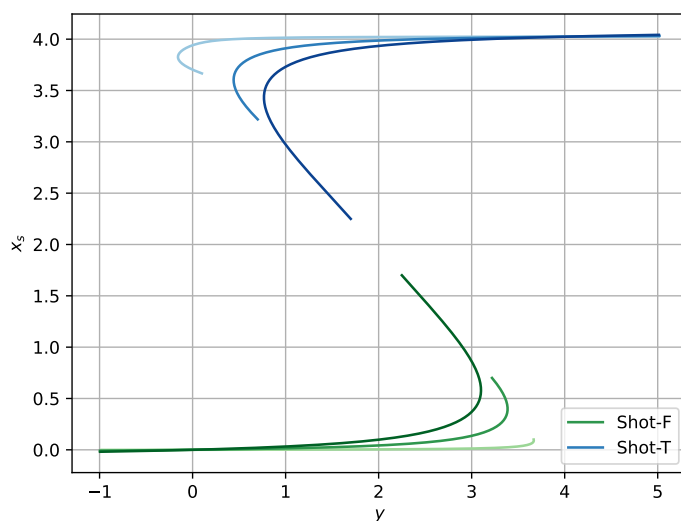


**Figure 3.5.:** The bounce solution and corresponding shot solutions

$\frac{dy}{dx_s}$  becomes

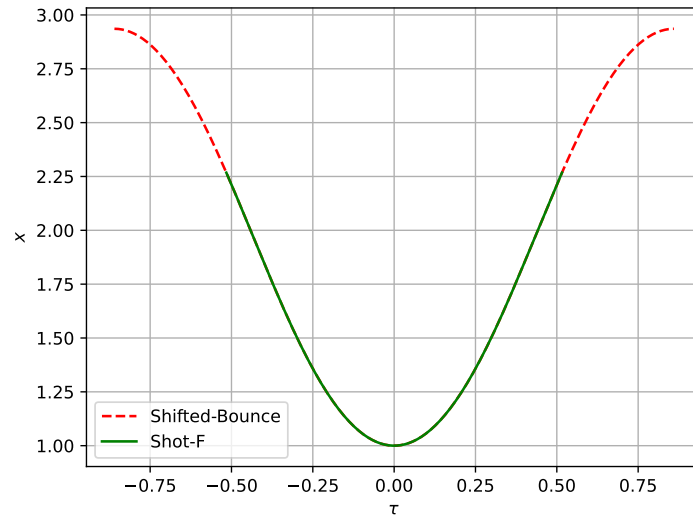
$$\frac{dy}{dx_s} = -\frac{\partial_{x_s} T_{sF}}{\partial_y T_{sF}} = \frac{1}{2} \sqrt{V(y) - V(x_s)} \int_{x_s}^y \frac{V'(x) - V'(x_s)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx + 1. \quad (3.12)$$

The numerical calculation of the relation between  $y$  and  $x_s$  is shown in Figure.3.6. Different initial conditions (shown by different color gradation in Figure.3.6), which relates to the period  $\beta\hbar$ , correspond to different branches.



**Figure 3.6.:**  $y - x_s$

Second, the turning point of a shot-F solution will be smaller than  $x_C$  while the turning point of a shot-T solution will be larger than  $x_C$ . Although the turning point of a shot-F solution with its endpoints inside the potential barrier won't go out of the range of potential barrier  $[x_F, x_R]$ , it is not true for a shot-T solution. In fact, a shot-F solution inside the potential barrier is just a part of the shifted-bounce solution with a period  $P$  larger than  $\beta\hbar$  and  $\tau_c = \frac{P}{2}$ , which indicates that  $x_s$  is just the endpoint  $x_b$  for another bounce solution with a larger period. Furthermore, with the assumption that the period of bounce solution will increase monotonically with its energy  $E = -V(x_b)$ , it is natural to realize that  $x_s < x_b$ .



**Figure 3.7.:** Shot-F Segmented from a Shifted-Bounce Solution

Third, as shown in Figure.3.6, two shot-T or shot-F solutions with the same endpoints  $y$  but different energies (corresponding to different turning points  $x_s$ ) could exist simultaneously. The two solutions with the same endpoints contribute to different parts of the free energy as we will explain in the next section.

### 3.3. Contributions from Classical Solutions

After all saddle points are identified, the path integral can be rewritten as the sum of integrals on the Lefschetz-thimble of each saddle point and then apply the Gaussian

approximation.

$$F = -\frac{1}{\beta} \ln \int dy \mathcal{N} \int_{x(-\frac{\beta\hbar}{2})=y}^{x(\frac{\beta\hbar}{2})=y} \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \mathcal{L}_E[x(\tau)]} \quad (3.13)$$

$$\approx -\frac{1}{\beta} \ln \int dy \left( \sum_{\bar{x}^\beta} Z_{\bar{x}^\beta}^\beta[y] \right) \quad (3.14)$$

$$= -\frac{1}{\beta} \ln \sum_{\bar{x}^\beta} \int dy Z_{\bar{x}^\beta}^\beta[y]. \quad (3.15)$$

Compared with the situation at zero temperature, the key difference lies in the requirements for taking the trace which is equivalent to the integral of the endpoints  $y$  when calculating the free energy at finite temperature. Apparently, because of the integral over  $y$ , static classical solutions won't contribute to the final result since they are isolated. Similarly to the situation at zero temperature, the determinant of the fluctuation operator appears in  $Z_{\bar{x}^\beta}^\beta[y]$  after applying Gaussian approximation. Moreover, the fluctuation  $\Delta x(\tau) = x(\tau) - \bar{x}(\tau)$  around a classical solution is required to satisfy the Dirichlet boundary conditions  $\Delta x(\frac{\beta\hbar}{2}) = \Delta x(-\frac{\beta\hbar}{2}) = 0$  in our research since both the path  $x(\tau)$  in the path integral as well as the classical  $\bar{x}$  satisfy the same Dirichlet boundary conditions :

$$x\left(\frac{\beta\hbar}{2}\right) = x\left(-\frac{\beta\hbar}{2}\right) = y,$$

$$\bar{x}\left(\frac{\beta\hbar}{2}\right) = \bar{x}\left(-\frac{\beta\hbar}{2}\right) = y.$$

Before delving into the calculation of the trace, it is necessary to figure out the dependence on the endpoint  $y$  of the Gaussian approximation result  $Z_{\bar{x}^\beta}^\beta[y]$ .

Now, assume that a classical solution  $\bar{x}^\beta$  of which the endpoints are  $y$  has been found. The Gaussian approximation of the integral on Lefschetz-thimble becomes

$$Z_{\bar{x}^\beta}^\beta[y] \approx \mathcal{N} \frac{e^{-S_E^\beta[\bar{x}^\beta(\tau; y)]}}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]}}, \quad (3.16)$$

where  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]$  is defined by Dirichlet boundary conditions as explained before.

In order to figure out the  $y$  dependence of  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]$ , we can make use of the Gelfand-Yaglom theorem and another known determinant  $\det \hat{O}$  defined by the same boundary conditions. As explained in Appendix B.3, the ratio of the determinant of two operators can be expressed as the ratio of  $u_2(\frac{\beta\hbar}{2})$ s related to each determinant as

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]}{\det \hat{O}} = \frac{u_{2-\bar{x}}(\frac{\beta\hbar}{2})}{u_{2-\hat{O}}(\frac{\beta\hbar}{2})},$$

where  $u_{2-\bar{x}}(\tau)$  and  $u_{2-\hat{O}}(\tau)$  are solutions to

$$\left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right] u_{2-\bar{x}}(\tau) = 0$$

$$\hat{O} u_{2-\hat{O}}(\tau) = 0$$

and satisfy the same initial conditions

$$\begin{aligned} u_{2-\bar{x}}\left(-\frac{\beta\hbar}{2}\right) &= \dot{u}_{2-\hat{O}}\left(-\frac{\beta\hbar}{2}\right) = 0 \\ u_{2-\bar{x}}\left(-\frac{\beta\hbar}{2}\right) &= \dot{u}_{2-\hat{O}}\left(-\frac{\beta\hbar}{2}\right) = 1. \end{aligned}$$

Since  $\det \hat{O}$  is already known,  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]$  can be derived immediately once the ratio  $\frac{u_{2-\bar{x}}(\frac{\beta\hbar}{2})}{u_{2-\hat{O}}(\frac{\beta\hbar}{2})}$  is determined.

$u_{2-\hat{O}}(\tau)$  depends on the selection of  $\hat{O}$  and can be decided simply. As for  $u_{2-\bar{x}}(\tau)$ , it can be expressed as

$$u_2(\tau) = \dot{\bar{x}}\left(-\frac{\beta\hbar}{2}\right) \nu_{\bar{x}}(\tau) - \dot{\bar{x}}(\tau) \nu_{\bar{x}}\left(-\frac{\beta\hbar}{2}\right)$$

by using  $\dot{\bar{x}}(\tau)$  and  $\nu_{\bar{x}}(\tau) = m \partial_E \bar{x}(\tau)$  as explained in Appendix B.3. If the  $\hat{O}$  without  $y$  dependence is chosen, then integral over  $y$  can be expressed as

$$\begin{aligned} \int dy Z_{\bar{x}^\beta}^\beta[y] &\approx \int dy \mathcal{N} \frac{e^{-S_E^\beta[\bar{x}^\beta(\tau; y)]}}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(\bar{x}^\beta(\tau; y)) \right]}} \\ &\approx \mathcal{N} \frac{1}{\sqrt{\det \hat{O}}} \int dy e^{-S_E^\beta[\bar{x}^\beta(\tau; y)]} \sqrt{\frac{u_{2-\hat{O}}(\frac{\beta\hbar}{2})}{u_{2-\bar{x}}(\frac{\beta\hbar}{2})}}. \end{aligned}$$

Now, some specific classical solutions will be discussed.

### 3.3.1. Contributions from Bounce and Shifted-bounce Solutions

For bounce solution  $x_B^\beta(\tau)$ , its time derivative  $\dot{x}_B^\beta(\tau)$  satisfies the Dirichlet boundary conditions  $\dot{x}_B^\beta(-\frac{\beta\hbar}{2}) = \dot{x}_B^\beta(\frac{\beta\hbar}{2}) = 0$ . Consequently,  $\dot{x}_B^\beta(\tau)$  is the eigenfunction of zero eigenvalue of its fluctuation operator  $\left[ -\partial_\tau^2 + \frac{1}{m} V''(\dot{x}_B^\beta) \right]$ . Since  $\dot{x}_B^\beta$  has a node,  $\left[ -\partial_\tau^2 + \frac{1}{m} V''(\dot{x}_B^\beta) \right]$  should have a negative eigenvalue, similar to the situation at zero temperature.

One might assume that a shifted-bounce solution  $x_{sB}^\beta(\tau; \tau_c) = x_B^\beta(\tau + \tau_c)$  would also have a zero eigenvalue and a negative eigenvalue, just as in the situation at

zero temperature. However, this is not correct. Although the time derivative of a shifted-bounce solution  $\dot{x}_{sB}^\beta(\tau; \tau_c)$  satisfies the eigenequation, it doesn't satisfy the Dirichlet boundary conditions at finite temperature generally, making it ineligible as an eigenfunction. So, how do we know the eigenvalue of the fluctuation operator of shifted-bounce solution without knowing the eigenfunction of zero eigenvalue? In fact, we don't need to know the situation of eigenvalues. All we need is the determinant of the fluctuation operator and we can use Gelfand-Yaglom theory to calculate it (see Appendix B.3).

First, let us consider the relation between the fluctuation determinants of the bounce solution and the shifted-bounce solutions. The ratio between the fluctuation determinant of the bounce solution  $x_B^\beta(\tau)$  and a shifted-bounce solution  $x_{sB}^\beta(\tau; \tau_c)$  can be expressed as

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta) \right]}{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]} = \frac{\left[ \ddot{x}_B^\beta(\frac{\beta\hbar}{2}) \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \right]^2}{\left\| \dot{x}_B^\beta(\tau) \right\|^2} m \frac{d\beta\hbar}{dE_B} \quad (3.17)$$

(Eq.(B.38)). As we can see,  $\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta) \right]$  is negative but not always zero since  $\dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) = \dot{x}_B^\beta(\frac{\beta\hbar}{2} + \tau_c)$  only becomes zero at special points. Here, we assume that only one negative eigenvalue exists for each fluctuation determinant of the shifted-bounce solution. We will discuss the number of negative eigenvalues in the last section of this chapter.

Now, we can take the trace and calculate the total of the contributions from both the bounce and shifted-bounce solutions. The shifted-bounce solution starting at  $y$  satisfies

$$x_{sB}^\beta(\pm \frac{\beta\hbar}{2}; \tau_c) = x_{sB}^\beta(\pm \frac{\beta\hbar}{2} + \tau_c) = y. \quad (3.18)$$

Since  $x_B^\beta(\tau) \in [x_b, x_r]$ ,  $y \in [x_b, x_r]$ . Therefore, the integral over the start point  $y$  is from  $x_b$  to  $x_r$ . Then the contributions from bounce and shifted-bounce solutions becomes :

$$\int_{x_b}^{x_r} dy Z_{sB}^\beta[y] = \int_{x_b}^{x_r} dy \mathcal{N} \frac{e^{-\frac{1}{\hbar} S_E^\beta[x_{sB}^\beta(\tau; y)]}}{\sqrt{\det[-\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta(\tau; y))]}.$$

It is apparent that the action of any shifted-bounce solution  $S_E^\beta[x_{sB}^\beta(\tau; y)]$  is the same as the action of the bounce solution  $S_B^\beta$  and has no  $y$ -dependence.  $e^{-\frac{1}{\hbar} S_E^\beta[x_{sB}^\beta(\tau; y)]} = e^{-\frac{1}{\hbar} S_B^\beta}$  can be taken out of the integral. Moreover, for every endpoint  $y \in (x_b, x_r)$ , two solutions with different initial velocities exist. The two solutions can be written as

$$x_{sB}^\beta(\tau; y) = x_B^\beta(\tau + |\tau_c|)$$

and

$$x_{sB}^\beta(\tau; y) = x_B^\beta(\tau - |\tau_c|)$$

by using  $\tau_c$ . As a result, the integral of  $y$  can be expressed as

$$\begin{aligned}
& \int_{x_b}^{x_r} dy Z_{sB}^\beta[y] \\
&= \int_{x_b}^{x_r} dy \left( Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} + |\tau_c| \right) \right] + Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} - |\tau_c| \right) \right] \right) \\
&= \int_0^{+\frac{\beta\hbar}{2}} dx_B^\beta \left( \frac{\beta\hbar}{2} + |\tau_c| \right) Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} + |\tau_c| \right) \right] + \int_0^{+\frac{\beta\hbar}{2}} dx_B^\beta \left( \frac{\beta\hbar}{2} - |\tau_c| \right) Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} - |\tau_c| \right) \right] \\
&= \int_0^{+\frac{\beta\hbar}{2}} d\tau_c \dot{x}_B^\beta \left( \frac{\beta\hbar}{2} + |\tau_c| \right) Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} + |\tau_c| \right) \right] + \int_0^{+\frac{\beta\hbar}{2}} d\tau_c \dot{x}_B^\beta \left( \frac{\beta\hbar}{2} - |\tau_c| \right) Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} - |\tau_c| \right) \right] \\
&= \int_{-\frac{\beta\hbar}{2}}^{+\frac{\beta\hbar}{2}} d\tau_c \dot{x}_B^\beta \left( \frac{\beta\hbar}{2} + \tau_c \right) Z_{sB}^\beta \left[ x_B^\beta \left( \frac{\beta\hbar}{2} + \tau_c \right) \right],
\end{aligned}$$

where Eq.(3.18) was applied.

Therefore, the integral over the start point  $y$  can be rewritten as the integral over the time shift  $\tau_c$ . and we can derive

$$Z_{sB}^\beta \equiv \int_{x_b}^{x_r} dy Z_{sB}^\beta[y] = \int_{x_b}^{x_r} dy \mathcal{N} \int_{\mathcal{J}_{sB}} \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} S_E[x]} \quad (3.19)$$

$$\approx \int_{x_b}^{x_r} dy \mathcal{N} \frac{e^{-\frac{1}{\hbar} S_E[x_{sB}^\beta(\tau; y)]}}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta(\tau; y)) \right]}} \quad (3.20)$$

$$= \mathcal{N} \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau_c \frac{|\dot{x}_B^\beta(\frac{\beta\hbar}{2} + \tau_c)| e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{\left[ \frac{\ddot{x}_B^\beta(\frac{\beta\hbar}{2}) \dot{x}_B^\beta(\frac{\beta\hbar}{2} + \tau_c)}{\|\dot{x}_B^\beta(\tau)\|^2} m \frac{d\beta\hbar}{dE_B} \det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right] \right]} \quad (3.21)$$

$$= \mathcal{N} \frac{\beta\hbar \|\dot{x}_B^\beta(\tau)\| e^{-\frac{1}{\hbar} S_E[x_B^\beta]}}{\sqrt{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}} \frac{1}{|\dot{x}_B^\beta(\frac{\beta\hbar}{2})| \sqrt{m \frac{d\beta\hbar}{dE_B}}}. \quad (3.22)$$

Although the final result we obtain looks seemingly different from Eq.(3.5), they are equal in fact. The difference results from the different boundary conditions used to define the determinant. The determinant in Eq.(3.22) is defined by Dirichlet boundary conditions while the determinant in Eq.(3.5) is defined by periodic boundary conditions.

The ratio of the fluctuation determinants of the bounce solution and the false vacuum solution is shown as

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]}{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]} = - \frac{\ddot{x}_B^\beta(\frac{\beta\hbar}{2})}{\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau u_B(\tau) \dot{x}_B^\beta(\tau)} \frac{\sinh(\omega_F \beta\hbar)}{\omega_F}$$

in Eq.(B.37). Therefore, the following relation can be obtained immediately.

$$\begin{aligned}\mathcal{N} \frac{1}{\sqrt{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}} &= i \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]}} \frac{|\ddot{x}_B^\beta(\frac{\beta\hbar}{2})|}{\|\dot{x}_B^\beta(\tau)\|} \sqrt{\frac{1}{\omega_F} \sinh(\omega_F \beta \hbar)} \\ &= i \sqrt{\frac{m}{2\pi\hbar}} \frac{|\ddot{x}_B^\beta(\frac{\beta\hbar}{2})|}{\|\dot{x}_B^\beta(\tau)\|}.\end{aligned}$$

As for  $\mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]}}$ , the value was proved in Eq.(2.11) as

$$\mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]}} = \sqrt{\frac{m\omega_F}{2\pi\hbar \sinh(\omega_F \beta \hbar)}}.$$

Consequently,  $\mathcal{N} \frac{1}{\sqrt{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}}$  becomes

$$\mathcal{N} \frac{1}{\sqrt{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}} = i \sqrt{\frac{m}{2\pi\hbar}} \frac{|\ddot{x}_B^\beta(\frac{\beta\hbar}{2})|}{\|\dot{x}_B^\beta(\tau)\|}.$$

After replacing it in  $Z_{sB}^\beta$ , the following result can be obtained

$$Z_{sB}^\beta \approx i \frac{\beta\hbar}{\sqrt{2\pi\hbar} \frac{d\beta\hbar}{dE}} e^{-\frac{1}{\hbar} S_B^\beta}, \quad (3.23)$$

which is the same as the result of Dashen and Affleck.

In fact, if Eq.(B.39)

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]} = \frac{-m \left[ \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \right]^2 \frac{d\beta\hbar}{dE}}{\frac{1}{\omega_F} \sinh(\omega_F \beta \hbar)}$$

instead of Eq.(B.37) was used in the first place, the final result would be obtained directly.

### 3.3.2. Contributions from Shot Solutions

The contributions of shot solutions can be expressed as

$$Z_S^\beta = \int dy Z_S^\beta[y] = \mathcal{N} \int dy \frac{e^{-\frac{1}{\hbar} S_E[x_S^\beta(\tau;y)]}}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_S^\beta) \right]}}. \quad (3.24)$$

It can be calculated similarly once the corresponding solution of the initial value problem is found. As explained before, the solution can be expressed as

$$u_S(\tau) = \dot{x}_S^\beta(-\frac{\beta\hbar}{2})\nu_S^\beta(\tau) - \dot{x}_S^\beta(\tau)\nu_S^\beta(-\frac{\beta\hbar}{2}).$$

And the ratio between the fluctuation determinant of a shot solution and the false vacuum solution can be expressed as

$$\frac{\det\left[-\partial_\tau^2 + \frac{1}{m}V''(x_S)\right]}{\det\left[-\partial_\tau^2 + \frac{1}{m}V''(x_F)\right]} = \frac{u_S(\frac{\beta\hbar}{2})}{\frac{1}{\omega_F}\sinh(\omega_F\beta\hbar)}.$$

What should be emphasized is that although  $x_S^\beta(\tau)$  is not a periodic solution, it is an even function with respect to time  $\tau$ . As a result,  $\dot{x}_S^\beta(\tau) = -\dot{x}_S^\beta(-\tau)$  and  $\nu_S^\beta(\tau) = \nu_S^\beta(-\tau)$  can be obtained. By using of the parity of  $\dot{x}_S^\beta(\tau)$  and  $\nu_S^\beta(\tau)$ , two useful relation can be obtained.

The first one is

$$u_S(\frac{\beta\hbar}{2}) = \dot{x}_S^\beta(-\frac{\beta\hbar}{2})\nu_S^\beta(\frac{\beta\hbar}{2}) - \dot{x}_S^\beta(\frac{\beta\hbar}{2})\nu_S^\beta(-\frac{\beta\hbar}{2}) = -\left[\dot{x}_S^\beta(\tau)\nu_S^\beta(\tau)\right]_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}}$$

and the second one is

$$u_S(\frac{\beta\hbar}{2}) = -2\dot{x}_S^\beta(\frac{\beta\hbar}{2})\nu_S^\beta(\frac{\beta\hbar}{2}) = -2\dot{x}_S^\beta(\frac{\beta\hbar}{2})m\frac{\partial x_S^\beta(\frac{\beta\hbar}{2})}{\partial E} = -2\dot{x}_S^\beta(\frac{\beta\hbar}{2})m\partial_E y. \quad (3.25)$$

Furthermore, for any classical solution  $\bar{x}(\tau)$ , there is a relation between its action and its energy :

$$\begin{aligned} \frac{\partial S_{\bar{x}}}{\partial E_{\bar{x}}} &= \frac{\partial}{\partial E_{\bar{x}}} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left( \frac{1}{2}m\dot{\bar{x}}^2 + V(\bar{x}) \right) \\ &= \left[ m\dot{\bar{x}}\partial_E \bar{x} \right]_{-\frac{T}{2}}^{\frac{T}{2}} + \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left( -m\ddot{\bar{x}}\partial_E \bar{x} + m\ddot{\bar{x}}\partial_E \bar{x} \right) \\ &= \left[ \dot{\bar{x}}\nu_{\bar{x}} \right]_{-\frac{T}{2}}^{\frac{T}{2}}, \end{aligned} \quad (3.26)$$

which indicates that  $u_S(\frac{\beta\hbar}{2}) = -\frac{\partial S_S}{\partial E_S}$ . Hence, the ratio can be expressed more briefly as

$$\frac{\det\left[-\partial_\tau^2 + \frac{1}{m}V''(x_S)\right]}{\det\left[-\partial_\tau^2 + \frac{1}{m}V''(x_F)\right]} = \frac{-\frac{\partial S_S}{\partial E_S}}{\frac{1}{\omega_F}\sinh(\omega_F\beta\hbar)} \quad \text{or} \quad \frac{-2\dot{x}_S^\beta(\frac{\beta\hbar}{2})\nu_S^\beta(\frac{\beta\hbar}{2})}{\frac{1}{\omega_F}\sinh(\omega_F\beta\hbar)}.$$

Although  $\frac{\partial S_S}{\partial E_S}$  of a shot solution is normally non-zero since different shot solutions with different energies have different actions, it can be either positive or negative for different shot solutions. If it is negative, the corresponding contribution will become imaginary after analytic continuation just as the bounce and shifted-bounce

solutions do. Like in the situation of a shifted-bounce solution, we assume that no negative eigenvalue exists for determinants with  $-\frac{\partial S_S}{\partial E_S} > 0$  and only one negative eigenvalue exists for determinants with  $-\frac{\partial S_S}{\partial E_S} < 0$ .

Then,  $Z_S^\beta$  can be expressed as

$$Z_S^\beta = \int dy Z_S^\beta[y] \quad (3.27)$$

$$= \mathcal{N} \int dy \frac{e^{-\frac{1}{\hbar} S_E[x_S^\beta(\tau; E)]}}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right] \frac{-\frac{\partial S_S}{\partial E_S}}{\frac{1}{\omega_F} \sinh(\omega_F \beta \hbar)}}}} \quad (3.28)$$

$$= \mathcal{N} \frac{1}{\sqrt{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]}} \sqrt{\frac{1}{\omega_F} \sinh(\omega_F \beta \hbar)} \int dy \frac{e^{-\frac{1}{\hbar} S_E[x_S^\beta(\tau; y)]}}{\sqrt{-\frac{\partial S_S}{\partial E_S}}}. \quad (3.29)$$

By using Eq.(2.11) to normalize, the contribution of shot solutions can be expressed as

$$Z_S^\beta = \sqrt{\frac{m}{2\pi\hbar}} \int dy \frac{e^{-\frac{1}{\hbar} S_S^\beta[y]}}{\sqrt{-\frac{\partial S_S}{\partial E_S}}}, \quad (3.30)$$

where  $S_S^\beta[y] \equiv S_E[x_S^\beta(\tau; y)]$ .

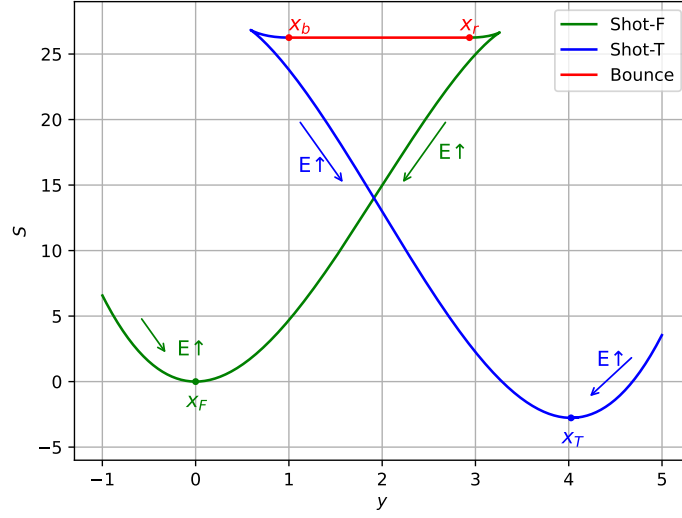
It is obvious that the formula for the contributions from the shot solution is more sophisticated than the formula of the contributions from the bounce solution and the shifted-bounce solutions. The integral range as well as the dependence of  $S_S^\beta[y]$  and  $\frac{\partial S_S}{\partial E_S}$  on  $y$  should be figured out. Although some approximation method may be used to simplify the calculation, a critical error could be introduced at the same time. In our research, the final result will be evaluated with no extra approximation method being applied.

### 3.4. The Imaginary Part of Free Energy and the Decay Rate

The contributions from different saddle points have been obtained. But in order to obtain the imaginary part of the free energy related to the decay rate, it is necessary to deal with the relation between different types of saddle point carefully.

Before we explain the treatment, we would first show the relation between different types of solutions. Since the saddle point approximation relates to the actions of the classical solutions, we can start from the dependence of the action  $S_{\bar{x}}$  on the endpoints  $y$ . The relation is shown in Figure.3.8.

The green line represents the actions of shot-F solutions, while the blue line corresponds to the actions of shot-T solutions. The energy change along each line is shown



**Figure 3.8.:** Numerical Calculation of  $S_{\bar{x}} - y$

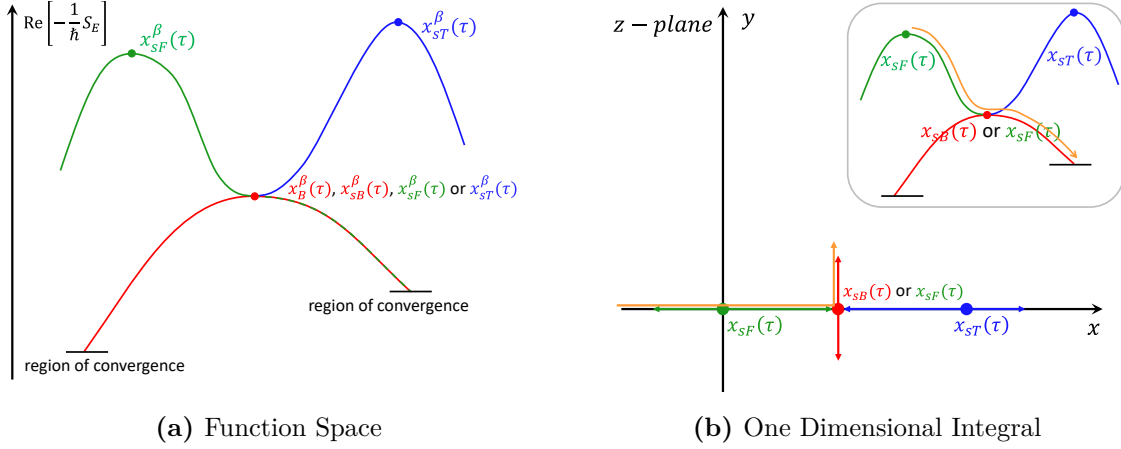
apparently. Since the actions of different shifted-bounce solutions are identical, a horizontal line from  $x_b$  to  $x_r$  can be used to represent the shifted-bounce solutions. However, for each  $y \in (x_b, x_r)$ , two distinct shifted-bounce solutions exist.

As we can see, there seems like to be three branches. The shifted-bounce line, a part of the shot-F action line connecting to the shifted-bounce line at  $y = x_r$  along with a part of the shot-T action line connecting to the shifted-bounce line at  $y = x_b$  together form the branch with larger actions. According to the energy variation indicated along the action line in the figure, it is obvious that all classical solutions at the branch with larger actions own at least one negative eigenvalue in the corresponding fluctuation determinant. The other two branches with smaller actions consist of only shot solutions, of which the fluctuation determinants are positive.

Now, focus on the range of  $y$  where three classical solutions exist. The upper limit and the lower limit of the “negative branch” are denoted as  $x_{sT-t}$  and  $x_{sF-t}$  separately. The relation of the three classical solutions with the same endpoints  $y$  in the function space can be expressed as Figure.3.9a. Two assumptions should be emphasized here.

1. Two steepest descent flows originating from  $x_{sF}^\beta(\tau)$  and  $x_{sT}^\beta(\tau)$  with larger exponent intersect at the third classical solution ( $x_{sB}^\beta(\tau)$ ,  $x_{sF}^\beta(\tau)$  or  $x_{sT}^\beta(\tau)$ ) with smaller exponent (larger action) and then go along the steepest descent flows of the Lefschetz-thimble of the third classical solution to reach the region of convergence.
2. There is only one negative eigenvalue for saddle points on the branch with a smaller exponent (larger action), whereas there are no negative eigenvalues

for saddle points on the other two branches with larger exponents (smaller action).



**Figure 3.9.:** Analytic Continuation

The first assumption is based on the conventional calculation at zero temperature.

The second assumption is based on the structure that each saddle point with a smaller exponent only connects to two other saddle points with larger exponents through steepest ascent flows, which corresponds to one negative eigenvalue of the fluctuation operator. If there are more than one negative eigenvalue, more saddle points with larger exponents could appear. As a result, the saddle point approximation results for points on the two branches with larger exponents are real, whereas the results for points on the other branch are imaginary.

Inspire by Gamow-Siegert [25,26] (GS) boundary condition, we consider that the outgoing waves boundary condition is related to the choice of saddle points necessary to be contained when one wants to calculate the decay rate from the imaginary part of free energy following the discussion of [75]. All shot-T solutions are intended to be excluded. Furthermore, we choose to include only the classical solutions with endpoints inside the range of  $[x_b, x_{sF-t}]$ . Then saddle point approximation of the imaginary part of free energy becomes

$$\begin{aligned}
 \text{Im}F &\approx -\frac{1}{\beta} \text{Im} \ln \left( Z_{sF-\text{Re}}^\beta + \frac{1}{2} (Z_{sB}^\beta + Z_{sF-\text{Im}}^\beta) \right) \approx -\frac{1}{\beta} \text{Im} \frac{\frac{1}{2} (Z_{sB}^\beta + Z_{sF-\text{Im}}^\beta)}{Z_{sF-\text{Re}}^\beta} \\
 &= -\frac{1}{\beta} \frac{\frac{1}{2} \left( \frac{\beta\hbar}{\sqrt{2\pi\hbar \frac{d\beta\hbar}{dE_B}}} e^{-\frac{1}{\hbar} S_B^\beta} + \sqrt{\frac{m}{2\pi\hbar}} \int_{x_r}^{x_{sF-t}} dy e^{-\frac{1}{\hbar} S_{sF}^\beta[y]} \right)}{\sqrt{\frac{m}{2\pi\hbar}} \int_{x_b}^{x_{sF-t}} dy e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}. \tag{3.31}
 \end{aligned}$$

From the imaginary part of free energy, we can obtain the decay rate at finite temperature.

$$\Gamma_{\text{Dir}}^\beta = -\frac{2}{\hbar} \text{Im} F = \frac{\frac{1}{\sqrt{m \frac{d\beta\hbar}{dE_B}}} e^{-\frac{1}{\hbar} S_B^\beta}}{\int_{x_b}^{x_t} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{-\frac{\partial S_{sF}^\beta}{\partial E_{sF}}}}} + \frac{1}{\beta\hbar} \frac{\int_{x_r}^{x_{sF}-t} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{-\frac{\partial S_{sF}^\beta}{\partial E_{sF}}}}}{\int_{x_b}^{x_{sF}-t} y \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{-\frac{\partial S_{sF}^\beta}{\partial E_{sF}}}}}. \quad (3.32)$$

### 3.4.1. Zero Temperature Limit

We now consider the zero temperature limit of the decay rate we obtained. As explained in the previous section, the zero temperature limit  $\beta\hbar \rightarrow \infty$  corresponds to  $x_b \rightarrow x_F$ .  $x_b \rightarrow x_F$  will be applied for convenience.

First of all, the integration range need to be specified at zero temperature. As temperature becomes lower and lower, the time interval will become larger and larger. Since we assume that the period of bounce solution will monotonically increase with energy  $E = -V(x_b)$ ,  $x_b$  will approach  $x_F$  as the temperature decreases. Therefore, it is obvious that  $x_b$  becomes  $x_F$  and  $x_r$  becomes  $x_R$  at zero temperature (see Figure.3.3). As we discussed before, the turning point  $x_s$  of shot solutions in the potential barrier satisfies  $x_s \in [x_F, x_b]$ . As a result,  $x_s$  should approach  $x_F$  faster than  $x_b$ . Moreover, the endpoints  $y$  can't excess  $x_R$ . Otherwise, the turning points  $x_s$  will no longer exist. Therefore,  $x_{sF}-t$  should become  $x_R$ , which indicates that the part of shot-F solutions which provides the imaginary contribution will disappear at zero temperature. Hence, the imaginary part of free energy becomes

$$\lim_{x_b \rightarrow x_F} \text{Im} F \approx \lim_{x_b \rightarrow x_F} -\frac{1}{\beta} \text{Im} \frac{\frac{1}{2} (Z_{sB}^\beta + Z_{sF-\text{Im}}^\beta)}{Z_{sF-\text{Re}}^\beta} = \lim_{x_b \rightarrow x_F} -\frac{1}{\beta} \text{Im} \frac{\frac{1}{2} Z_{sB}^\beta}{Z_{sF}^\beta}.$$

Now  $Z_{sB}^\beta$  and  $Z_{sF}^\beta$  will be considered separately so as to compare with the results derived by other methods in previous research.

$Z_{sB}^\beta$  can be expressed as

$$Z_{sB}^\beta \approx i \frac{\beta\hbar}{\sqrt{2\pi\hbar \frac{d\beta\hbar}{dE_B}}} e^{-\frac{1}{\hbar} S_B^\beta} = i \frac{\beta\hbar |V'(x_b)|}{\sqrt{-2\pi\hbar V'(x_b) \frac{d\beta\hbar}{dx_b}}} e^{-\frac{1}{\hbar} S_B^\beta}.$$

In Appendix C.2, the following relation (Eq.(C.9)) has already been derived

$$\lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta\hbar}{dx_b} = -2\sqrt{mV''(x_F)}.$$

After replacing it with  $Z_{sB}^\beta$ , an intermediate result of  $Z_{sB}^\beta$  can be expressed as

$$\lim_{x_b \rightarrow x_F} Z_{sB}^\beta = \lim_{x_b \rightarrow x_F} i \frac{\beta\hbar}{\sqrt{2\pi\hbar}} \frac{|V'(x_b)|}{\sqrt{2\sqrt{mV''(x_F)}}} e^{-\frac{1}{\hbar} S_B^\beta},$$

where  $\beta\hbar$  and  $V'(x_b)$  are kept for the following calculation.

As for  $Z_{sF}^\beta$ , by applying Eq.(3.25),  $\frac{\partial S_{sF}^\beta}{\partial E_{sF}}$  can be expressed as

$$-\frac{\partial S_{sF}^\beta}{\partial E_{sF}} = -2\dot{x}_{sF}^\beta\left(\frac{\beta\hbar}{2}\right)m\partial_{E_{sF}}y = 2\sqrt{2m(V(y) - V(x_s))}\frac{1}{V'(x_s)}\frac{dy}{dx_s},$$

where the final velocity of the shot-F solution  $\dot{x}_{sF}^\beta\left(\frac{\beta\hbar}{2}\right) = \sqrt{\frac{2}{m}(V(y) - V(x_s))} > 0$  and  $E_{sF} = -V(x_s)$  were used since the potential Figure.3.2 is being considered. Then  $Z_{sF}^\beta$  becomes

$$\begin{aligned}\lim_{x_b \rightarrow x_F} Z_{sF}^\beta &= \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{m}{2\pi\hbar}} \int_{x_F}^{x_R} dy \frac{1}{\sqrt{2\sqrt{2m(V(y) - V(x_s))}\frac{1}{V'(x_s)}\frac{dy}{dx_s}}} e^{-\frac{1}{\hbar}S_{sF}^\beta[y]} \\ &= \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{m}{2\pi\hbar}} \int_{x_F}^{x_R} dy \frac{|V'(x_s)|}{\sqrt{2\sqrt{2m(V(y) - V(x_s))}V'(x_s)\frac{dy}{dx_s}}} e^{-\frac{1}{\hbar}S_{sF}^\beta[y]}.\end{aligned}$$

In order to derive the final result, we need to know the expression for  $V'(x_s)$  as well as  $\frac{dy}{dx_s}$  and their zero temperature limits. Fortunately, from the time relation

$$\beta\hbar = 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx = 2 \int_{x_s}^y \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_s))}} dx. \quad (3.33)$$

which is satisfied by both bounce and shot-F solution, two two significant relations can be derived. The first one is  $\frac{dy}{dx_s}$ , which has already been obtained in the previous section (Eq.(3.12)).

$$\frac{dy}{dx_s} = -\frac{\partial_{x_s} T_S}{\partial_y T_S} = \frac{1}{2} \sqrt{V(y) - V(x_s)} \int_{x_s}^y \frac{V'(x) - V'(x_s)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx + 1. \quad (3.34)$$

The second one is a bit more non-intuitive. From Appendix C.2, it is known that the following relation Eq.(C.7) for the bounce solution can be obtained from Eq.(3.33)

$$\begin{aligned}& |V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta\hbar}{2}} \\ &= |V''(x_b)| \left| x_r - x_b + \frac{V'(x_b)}{V''(x_b)} + \sqrt{\left(x_r - x_b + \frac{V'(x_b)}{V''(x_b)}\right)^2 - \left[\frac{V'(x_b)}{V''(x_b)}\right]^2} \right| \\ &\quad \cdot e^{\sqrt{\frac{V''(x_b)}{m}} \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} - \frac{1}{\sqrt{\frac{2}{m}(V'(x_b)(x - x_b) + \frac{1}{2}V''(x_b)(x - x_b)^2)}} d\bar{x}}.\end{aligned}$$

Change  $x_b$  to  $x_s$  and  $x_r$  to  $y$ , the similar relation for the shot-F solution can be

obtained

$$\begin{aligned}
 & |V'(x_s)| e^{\sqrt{\frac{V''(x_s)}{m}} \frac{\beta \hbar}{2}} \\
 &= |V''(x_s)| \left| y - x_s + \frac{V'(x_s)}{V''(x_s)} + \sqrt{\left( y - x_s + \frac{V'(x_s)}{V''(x_s)} \right)^2 - \left[ \frac{V'(x_s)}{V''(x_s)} \right]^2} \right| \\
 &\quad \cdot e^{\sqrt{\frac{V''(x_s)}{m}} \int_{x_s}^y \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_s))}} - \frac{1}{\sqrt{\frac{2}{m}(V'(x_s)(x - x_s) + \frac{1}{2}V''(x_s)(x - x_s)^2)}} d\bar{x}}.
 \end{aligned}$$

Now, let's consider the behavior of  $\frac{dy}{dx_s}$  when  $\beta \rightarrow \infty$ . Since  $x_s \rightarrow x_F$  when  $\beta \rightarrow \infty$ ,  $\frac{dy}{dx_s}$  can be written as

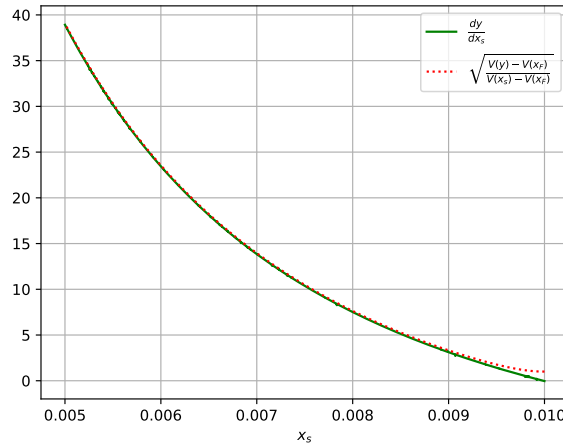
$$\lim_{\beta \rightarrow \infty} \frac{dy}{dx_s} = \lim_{x_s \leq x_b \rightarrow x_F} \frac{1}{2} \sqrt{V(y) - V(x_s)} \int_{x_s}^y \frac{V'(x) - V'(x_s)}{[V(x) - V(x_s)]^{\frac{3}{2}}} dx + 1 \quad (3.35)$$

$$= \lim_{x_s \leq x_b \rightarrow x_F} \frac{1}{2} \sqrt{V(y) - V(x_F)} \int_{x_s}^y \frac{V'(x)}{[V(x) - V(x_F)]^{\frac{3}{2}}} dx + 1 \quad (3.36)$$

$$= \sqrt{V(y) - V(x_F)} \left[ -\frac{1}{\sqrt{V(x) - V(x_F)}} \right]_{x_s}^y + 1 \quad (3.37)$$

$$= \sqrt{\frac{V(y) - V(x_F)}{V(x_s) - V(x_F)}}. \quad (3.38)$$

On the other hand, when  $x_s \rightarrow x_F$ , the following relation can be derived by using



**Figure 3.10.:** At Low Temperature

L'Hôpital's rule.

$$\lim_{x_s \leq x_b \rightarrow x_F} \frac{\sqrt{V(x_s) - V(x_F)}}{V'(x_s)} = \lim_{x_s \leq x_b \rightarrow x_F} \frac{\frac{1}{2} V'(x_s)}{V''(x_s)},$$

which indicates

$$\lim_{x_s \leq x_b \rightarrow x_F} \frac{\sqrt{V(x_s) - V(x_F)}}{V'(x_s)} = \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{1}{2V''(x_s)}} = \sqrt{\frac{1}{2V''(x_F)}}. \quad (3.39)$$

Consequently, the limit relation of  $V'(x_s) \frac{dy}{dx_s}$  becomes

$$\lim_{\beta \rightarrow \infty} V'(x_s) \frac{dy}{dx_s} = \sqrt{2V''(x_F)} \sqrt{V(y) - V(x_F)}.$$

Now, the behavior of  $V'(x_s)$  when  $\beta \rightarrow \infty$  should be considered. Since the expressions for  $|V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}}$  and  $|V'(x_s)| e^{\sqrt{\frac{V''(x_s)}{m}} \frac{\beta \hbar}{2}}$  have already been revealed, the zero temperature limit  $x_s \leq x_b \rightarrow x_F$  of their ratio can be calculated as

$$\begin{aligned} \lim_{x_s \leq x_b \rightarrow x_F} \left| \frac{V'(x_b)}{V'(x_s)} \right| &= \frac{|x_R - x_F| e^{\omega_F \int_{x_F}^{x_R} \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{\omega_F |x - x_F|}} d\bar{x}}{|y - x_F| e^{\omega_F \int_{x_F}^y \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_s))}} - \frac{1}{\omega_F |x - x_F|}} d\bar{x}} \\ &= e^{\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}. \end{aligned} \quad (3.40)$$

Substitute corresponding terms in  $Z_{sF}^\beta$  with all relations we derived and the intermediate result of  $Z_{sF}^\beta$  can be expressed as

$$\begin{aligned} &\lim_{x_b \rightarrow x_F} Z_{sF}^\beta \\ &= \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{m}{2\pi\hbar}} \int_{x_F}^{x_R} dy \frac{|V'(x_s)|}{\sqrt{2\sqrt{2m(V(y) - V(x_s))} V'(x_s) \frac{dy}{dx_s}}} e^{-\frac{1}{\hbar} S_{sF}^\beta[y]} \\ &= \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{m}{2\pi\hbar}} \int_{x_F}^{x_R} dy \frac{|V'(x_b)| e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}}{\sqrt{2\sqrt{2m(V(y) - V(x_s))} \sqrt{2V''(x_F)} \sqrt{V(y) - V(x_F)}}} e^{-\frac{1}{\hbar} S_{sF}^\beta[y]} \\ &= \lim_{x_s \leq x_b \rightarrow x_F} \sqrt{\frac{1}{2\pi\hbar}} \frac{|V'(x_b)|}{\sqrt{2\sqrt{mV''(x_F)}}} \int_{x_F}^{x_R} dy \frac{e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}. \end{aligned}$$

Combining  $Z_{sB}^\beta$  and  $Z_{sF}^\beta$  leads to the following ratio

$$\begin{aligned} \lim_{x_b \rightarrow x_F} \frac{Z_{sB}^\beta}{Z_{sF}^\beta} &= \lim_{x_b \rightarrow x_F} \frac{i \frac{\beta \hbar}{\sqrt{2\pi\hbar}} \frac{|V'(x_b)|}{\sqrt{2\sqrt{mV''(x_F)}}} e^{-\frac{1}{\hbar} S_B^\beta}}{\sqrt{\frac{1}{2\pi\hbar}} \frac{|V'(x_b)|}{\sqrt{2\sqrt{mV''(x_F)}}} \int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}} \\ &= \frac{i\beta\hbar e^{-\frac{1}{\hbar} S_B}}{\int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}}. \end{aligned}$$

The action of the shot solution at zero temperature  $S_{sF}^{\beta=\infty}[y]$  can be represented as

$$S_{sF}^{\beta=\infty}[y] = 2 \int_{x_F}^y d\bar{x} \sqrt{2m(V(\bar{x}) - V(x_F))}.$$

Then the imaginary part of free energy becomes

$$\lim_{\beta \rightarrow \infty} \text{Im} F \approx -\frac{\hbar}{2} \frac{e^{-\frac{1}{\hbar} S_B}}{\int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta=\infty}[y]}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}}$$

and the decay rate is

$$\lim_{\beta \rightarrow \infty} \Gamma_{\text{Dir}}^{\beta} = \frac{e^{-\frac{1}{\hbar} S_B}}{\int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta=\infty}[y]}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}}. \quad (3.41)$$

Now, reconsider the zero temperature limit of the result derived by Affleck, which is equivalent to the result of Callan and Coleman. The result Eq.(3.8) can be re-expressed as

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \Gamma_{\text{Aff}}^{\beta} &= \frac{e^{-\frac{1}{\hbar} S_B}}{\sqrt{\frac{\pi\hbar}{m\omega_F}} \frac{1}{\omega_F(x_R - x_F)} e^{-\int_{x_F}^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}} - \frac{1}{|x - x_F|} d\bar{x}} \\ &= \frac{e^{-\frac{1}{\hbar} S_B}}{\lim_{y \rightarrow x_F} \sqrt{\frac{\pi\hbar}{m\omega_F}} \frac{1}{\omega_F(y - x_F)} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}}} \\ &= \frac{e^{-\frac{1}{\hbar} S_B}}{\sqrt{\frac{\pi\hbar}{m\omega_F}} \left[ \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[y]}}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}} \right]_{y=x_F}}. \end{aligned}$$

It is obvious that the result derived by Affleck as well as Callan and Coleman only took the contribution from the  $y = x_F$  classical shot solution, which is the same as the false vacuum solution, into the final result. The prefactor  $\sqrt{\frac{\pi\hbar}{m\omega_F}}$ , which has a dimension of [length], might be able to be understood as the Gaussian approximation of the integral over  $y$  in the denominator of Eq.(3.41).

The  $e^{-\frac{1}{\hbar} S_{sF}^{\beta=\infty}[y]}$  can be expanded around  $y$  as

$$S_{sF}^{\beta=\infty}[y] = S_{sF}^{\beta=\infty}[x_F] + m\omega_F \Delta y^2 = m\omega_F \Delta y^2,$$

where  $\left[ \frac{dS_{sF}^{\beta=\infty}[y]}{dy} \right]_{y=x_F} = 0$  and

$$\left[ \frac{d^2 S_{sF}^{\beta=\infty}[y]}{dy^2} \right]_{y=x_F} = \lim_{y \rightarrow x_F} \frac{2V'(y)}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} = 2\sqrt{mV''(x_F)} = 2m\omega_F$$

was used (see Eq.(3.39)). Then the denominator of Eq.(3.41) becomes

$$\begin{aligned} \text{DT} &\approx \left[ \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{\frac{2}{m} (V(y) - V(x_F))}} e^{-\int_y^{x_F} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} \right]_{y=x_F} \int_{-\infty}^{+\infty} \Delta y e^{-\frac{1}{\hbar} m \omega_F \Delta y^2} \\ &= \sqrt{\frac{\pi \hbar}{m \omega_F}} \left[ \frac{e^{-\frac{1}{\hbar} S_{sF}^\beta[y]}}{\sqrt{\frac{2}{m} (V(y) - V(x_F))}} e^{-\int_y^{x_F} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} \right]_{y=x_F}. \end{aligned}$$

However, this is not a proper approximation, which could lead to drastic divergence as being explained in the next chapter.



## 4. Comparison

### 4.1. Numerical Calculation

In this section the result from the numerical calculation will be illustrated. We will first compare the decay rates under the same potential calculated through different methods at finite temperature. Then we would like to show the potential dependence of the decay rate at finite temperature. Finally, we propose to illustrate the potential dependence of the decay rates computed with different ways at zero temperature.

#### 4.1.1. Decay Rate at Finite Temperature

The results calculated by different methods will be shown through numerical calculation in this section. First of all, the non-dimensionalization is necessary to be preformed. By introducing an energy parameter  $\epsilon_0$ , Affleck's result at finite temperature can be expressed as

$$\Gamma_{\text{Aff}}^\beta \approx \frac{\epsilon_0}{\hbar} 2 \sinh \frac{\mathfrak{w}_F \mathcal{P} e^{-(\mathcal{S}_B^\beta - \mathcal{S}_F^\beta)}}{2 \sqrt{2\pi \left| \frac{d\mathcal{P}}{d\mathcal{E}_B} \right|}},$$

and our result becomes

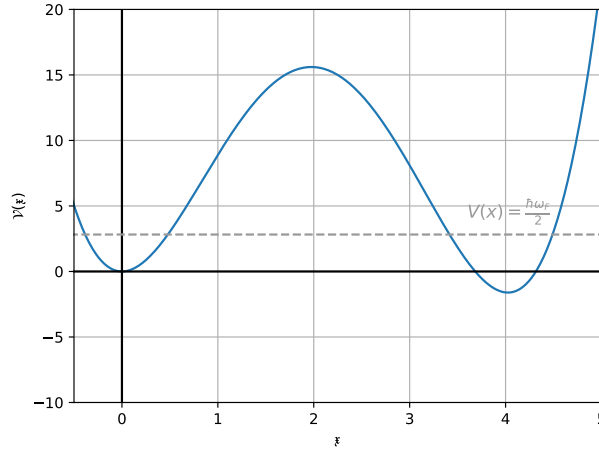
$$\Gamma_{\text{Dir}}^\beta \approx \frac{\epsilon_0}{\hbar} \frac{1}{\mathcal{P}} \frac{\frac{\mathcal{P} e^{-\mathcal{S}_B}}{\sqrt{\left| \frac{d\mathcal{P}}{d\mathcal{E}_B} \right|}} + \int_{\mathfrak{x}_r}^{\mathfrak{x}_{sF}-t} d\mathfrak{y} \frac{e^{-\mathcal{S}_{sF}}}{\sqrt{-\frac{\partial \mathcal{S}_{sF}}{\partial \mathcal{E}_{sF}}}}}{\int_{\mathfrak{x}_b}^{\mathfrak{x}_{sF}-t} d\mathfrak{y} \frac{e^{-\mathcal{S}_{sF}}}{\sqrt{-\frac{\partial \mathcal{S}_{sF}}{\partial \mathcal{E}_{sF}}}}},$$

where  $\mathfrak{w}_F = \frac{\hbar}{\epsilon_0} \omega_F$ ,  $\mathcal{P} = \frac{\epsilon_0}{\hbar} \beta \hbar$ ,  $\mathcal{S} = \frac{1}{\hbar} S_E$ ,  $\mathcal{E} = \frac{E}{\epsilon_0}$ ,  $\mathfrak{x} = \frac{\sqrt{m\epsilon_0}}{\hbar} x$  as well as  $\mathfrak{y} = \frac{\sqrt{m\epsilon_0}}{\hbar} y$  are all dimensionless quantities (see Appendix A.1).

The first dimensionless potential we choose is

$$\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 - 8\mathfrak{x}^3 + 15.9\mathfrak{x}^2.$$

The position of the false vacuum is located at  $\mathfrak{x}_F = 0$ , which results in that the action of the false vacuum solution becomes zero.



**Figure 4.1.:** Potential

After the potential is determined, the numerical calculation can be performed by Python. The process to carry out the numerical calculation will be introduced as follows.

The numerical calculation of Affleck's result is simple. Since the energy  $\mathcal{E}_B$ , the period  $\mathcal{P}$  and the action  $\mathcal{S}_B$  of bounce solutions are all relevant to the endpoints  $\mathbf{r}_b$  as

$$\begin{aligned}\mathcal{E}_B &= -\mathcal{V}(\mathbf{r}_b), \\ \mathcal{P} &= 2 \int_{\mathbf{r}_b}^{\mathbf{r}_r} \frac{1}{\sqrt{2(\mathcal{V}(\mathbf{r}) - \mathcal{V}(\mathbf{r}_b))}} d\mathbf{r}, \\ \mathcal{S}_B &= 2 \int_{\mathbf{r}_b}^{\mathbf{r}_r} d\mathbf{r} \sqrt{2(\mathcal{V}(\mathbf{r}) - \mathcal{V}(\mathbf{r}_b))} - \mathcal{E}_B \mathcal{P},\end{aligned}$$

where  $\mathcal{V}(\mathbf{r}_b) = \mathcal{V}(\mathbf{r}_r)$ ,  $\mathcal{E}_B$ ,  $\mathcal{P}$  and  $\mathcal{S}_B$  can be obtained once  $\mathbf{r}_b$  is determined. Then, all elements needed for the calculation are prepared.  $\frac{d\mathcal{P}}{d\mathcal{E}_B}$  can be calculated either by numerical differentiation or by the following relation (see Eq.(C.10))

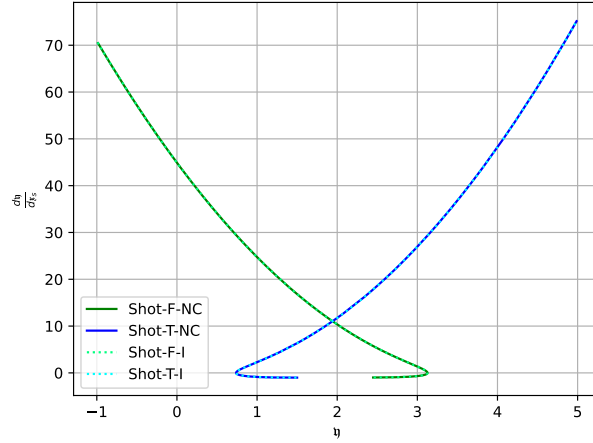
$$\frac{d\mathcal{P}}{d\mathcal{E}_B} = 2 \int_{\mathbf{r}_b}^{\mathbf{r}_r} \frac{1}{\sqrt{2(\mathcal{V}(\mathbf{r}) - \mathcal{V}(\mathbf{r}_b))}} \frac{\mathcal{V}''(\mathbf{r})}{[\mathcal{V}'(\mathbf{r})]^2} d\mathbf{r}.$$

As for our result, all quantities needed are the energy and the corresponding action of the shot-F actions with different endpoints  $\mathbf{r}$ . It is apparent that we can obtain them immediately from the following relation

$$\mathcal{E}_{sF} = -\mathcal{V}(\mathbf{r}_s)$$

and

$$\mathcal{S}_{sF} = 2 \int_{\mathbf{r}_s}^{\mathbf{r}} d\mathbf{r} \sqrt{2(\mathcal{V}(\mathbf{r}) - \mathcal{V}(\mathbf{r}_s))} - \mathcal{E}_{sF} \mathcal{P}$$



**Figure 4.2.:**  $\frac{d\eta}{dx_s}$

once we know the corresponding turning point  $x_s$  for each  $\eta$ . Although the relation between the endpoint  $\eta$  and the turning point  $x_s$  is given as

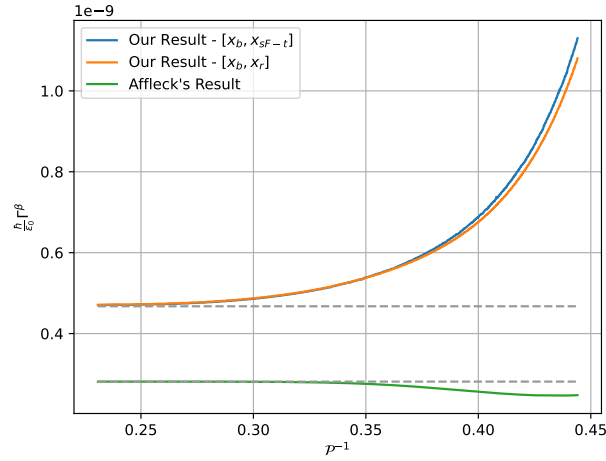
$$\frac{d\eta}{dx_s} = \frac{1}{2} \sqrt{\mathcal{V}(\eta) - \mathcal{V}(x_s)} \int_{x_s}^{\eta} \frac{\mathcal{V}'(x) - \mathcal{V}'(x_s)}{[\mathcal{V}(x) - \mathcal{V}(x_s)]^{\frac{3}{2}}} dx + 1 \quad (4.1)$$

according to Eq.(3.12), it is difficult to solve it even by the numerical calculation since Eq.(4.1) is a complicated integro-differential equation. Another possible method is required. Fortunately, since all shot-F solutions are all even classical solutions and the velocity becomes zero at the turning point, instead of solving  $\frac{d\eta}{dx_s}$ , we can solve the equation of motion with the following initial condition

$$\begin{aligned} x_{sF}(0) &= x_s \\ \dot{x}_{sF}(0) &= 0 \end{aligned}$$

and then the  $\eta$  related to a specific  $x_s$  is just the value of the solution at  $t = \frac{P}{2}$ . The range of  $x_s$  is between  $x_F$  and  $x_b$ . Therefore, we just need to solve all equations of motion for  $x_s \in (x_F, x_b)$  and find the corresponding  $\eta$ . The relation has already been shown in Figure.3.6. Moreover, the comparison of  $\frac{d\eta}{dx_s}$  calculated by Eq.(4.1) with the numerical differentiation is shown in Figure.4.2. After all  $(x_s, \eta)$  pairs are obtained, the calculation of the corresponding  $\mathcal{E}_{sF}$ ,  $\mathcal{S}_{sF}$ ,  $Z_{sF}^\beta$  as well as  $\Gamma_{\text{Dir}}^\beta$  at different temperatures can be completed smoothly.

The result of the numerical calculation is shown in Figure.4.3. The gray dashed lines represent the zero temperature limits of different results. The blue line in Figure.4.3 corresponds to our result. The orange line is also calculated by our method but the integral range of  $\eta$  is limited in  $[x_b, x_r]$ , where the contributions to the imaginary part only come from the bounce solution and the shifted-bounce solutions. They show the expected behavior as temperature increases. However,

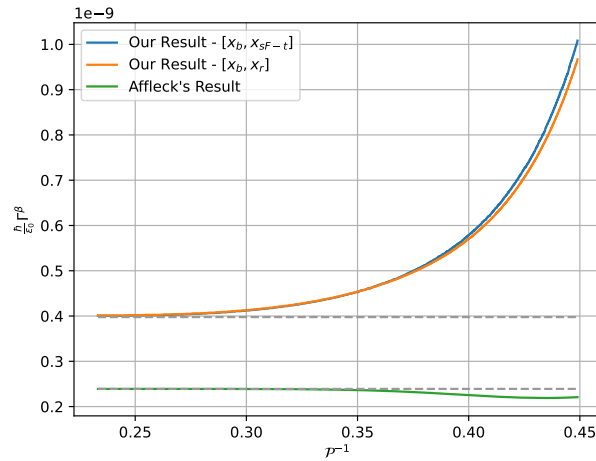


**Figure 4.3.:** Decay Rate :  $\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 - 8\mathfrak{x}^3 + 15.9\mathfrak{x}^2$

the green line representing the decay rate calculated by Affleck shows unexpected behavior at low temperature.  $\Gamma_{\text{Aff}}^\beta$  decreases as temperature increases.

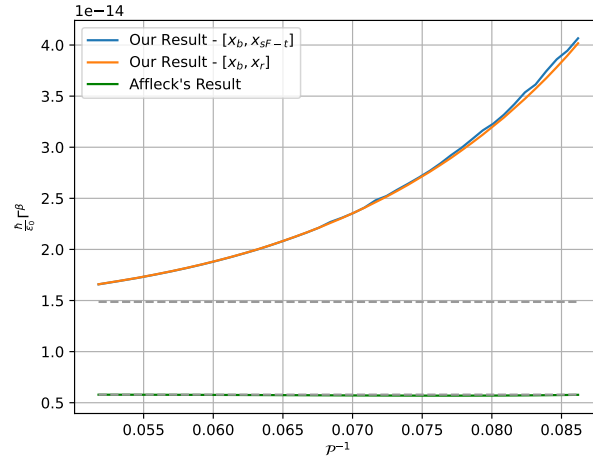
We tried other potentials and similar results were obtained.

1.  $\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 - 8.03\mathfrak{x}^3 + 16\mathfrak{x}^2$



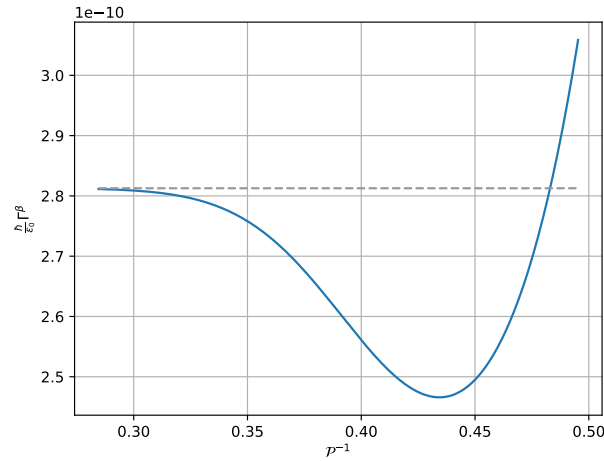
**Figure 4.4.:** Decay Rate :  $\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 - 8.03\mathfrak{x}^3 + 16\mathfrak{x}^2$

2.  $\mathcal{V}(\mathfrak{x}) = \cosh(0.2\mathfrak{x}) + \frac{10}{\cosh(\mathfrak{x}+0.7)}$



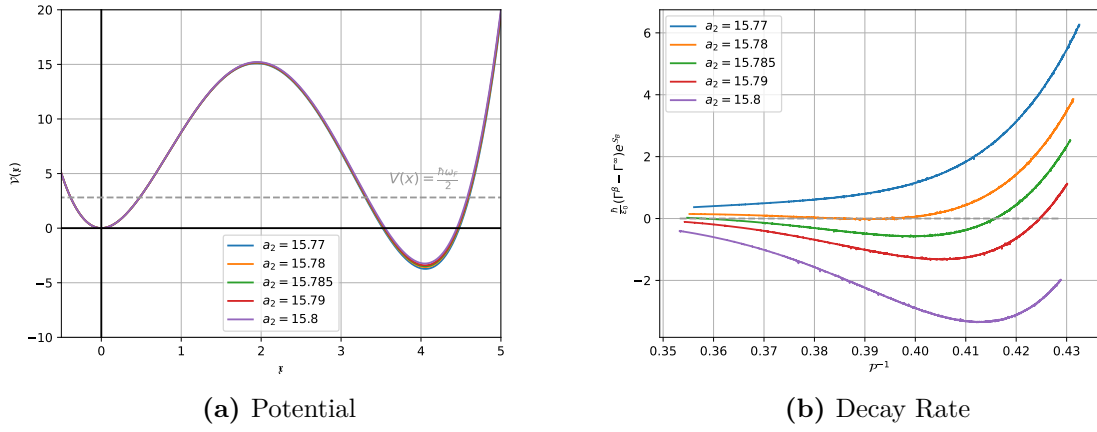
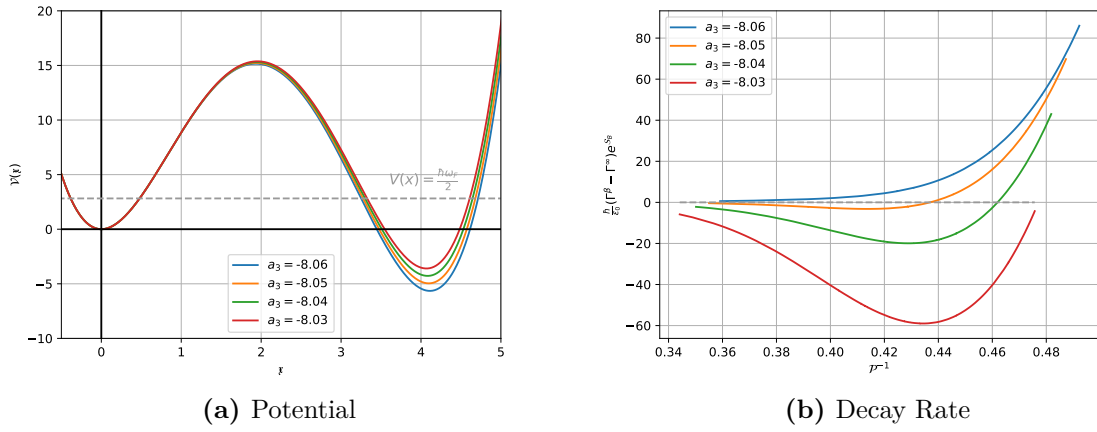
**Figure 4.5.:** Decay Rate :  $\mathcal{V}(\mathfrak{x}) = \cosh(0.2\mathfrak{x}) + \frac{10}{\cosh(\mathfrak{x}+0.7)}$

#### 4.1.2. Potential Dependence of Affleck's Result



**Figure 4.6.:** Affleck's Result

From Figure.4.3, we observed that  $\Gamma_{\text{Aff}}^\beta$  would decrease with temperature at low temperature. We checked other potentials parameterized by  $a_2$  and  $a_3$  as  $\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 - 8\mathfrak{x}^3 + a_2\mathfrak{x}^2$  and  $\mathcal{V}(\mathfrak{x}) = \mathfrak{x}^4 + a_3\mathfrak{x}^3 + 16\mathfrak{x}^2$ . Since we are only interested in the temperature dependence of  $\Gamma_{\text{Aff}}^\beta$  at low temperature,  $y$ -axis is set to be  $(\Gamma_{\text{Aff}}^\beta - \Gamma_{\text{Aff}}^{\beta=\infty}) e^{\frac{1}{\hbar} S_B}$  in order to eliminate the influence from the absolute value. The temperature dependence of the Affleck's result for different types of potentials is shown in Figure.4.7b, Figure.4.8b and Figure.4.9b.

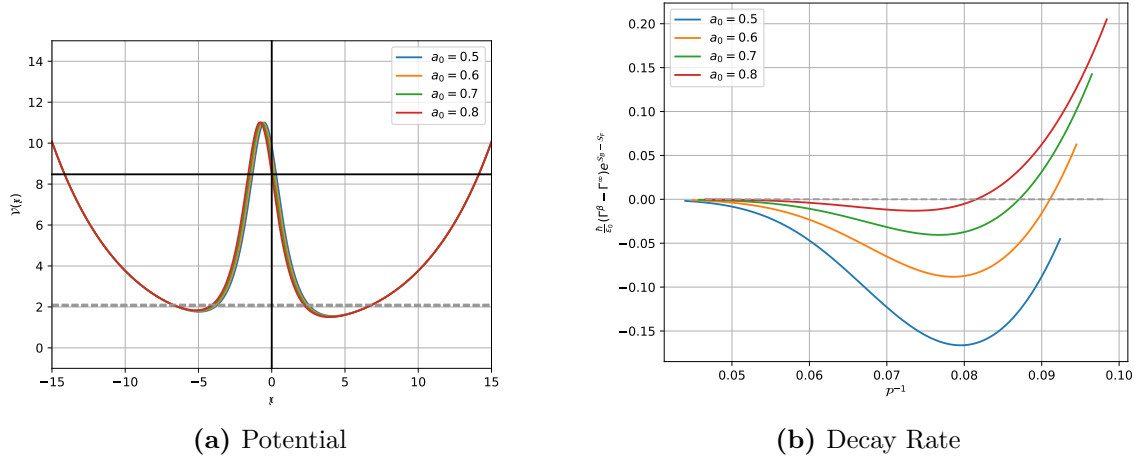
Figure 4.7.:  $\mathcal{V}(x) = x^4 - 8x^3 + a_2x^2$ Figure 4.8.:  $\mathcal{V}(x) = x^4 + a_3x^3 + 16x^2$ 

We found that it wasn't a coincidence. The closer  $\mathcal{V}'(x_R)$  is to 0, the more pronounced the decreasing behavior of  $\Gamma_{\text{Aff}}^\beta$  becomes. The questionable behavior observed will be explained in the following section.

### 4.1.3. Decay Rate at Zero Temperature

After showing the decay rate at finite temperature, we now prepare to illustrate the zero temperature limit of different results. At zero temperature limit, our result becomes

$$\Gamma_{\text{Dir}}^{\beta=\infty} = \frac{\epsilon_0}{\hbar} \frac{e^{-S_B}}{\int_{x_F}^{x_R} d\eta \frac{e^{-S_{SF}[\eta]}}{\sqrt{2(\mathcal{V}(\eta) - \mathcal{V}(x_F))}} e^{-\int_{\eta}^{x_R} \frac{\omega_F}{\sqrt{2(\mathcal{V}(\eta) - \mathcal{V}(x_F))}} d\eta}}.$$



**Figure 4.9.:**  $V(x) = \cosh(0.2x) + \frac{10}{\cosh(x+a_0)}$

The zero temperature limit of Affleck's result which is the same as Callan and Coleman's result becomes

$$\begin{aligned} \Gamma_{\text{Aff}}^{\beta=\infty} &= \Gamma_{\text{C\&C}} = \frac{\epsilon_0}{\hbar} \sqrt{\frac{w_F}{\pi}} w_F (x_R - x_F) e^{\int_{x_F}^{x_R} \left( \frac{w_F}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{|x - x_F|} \right) dx} e^{-S_B} \\ &= \frac{\epsilon_0}{\hbar} \frac{e^{-S_B}}{\sqrt{\frac{\pi}{w_F}} \left[ \frac{e^{-S_{SF}[\eta]}}{\sqrt{2(V(\eta) - V(x_F))}} e^{-\int_{\eta}^{x_R} \frac{w_F}{\sqrt{2(V(x) - V(x_F))}} dx} \right]_{\eta=x_F}}. \end{aligned}$$

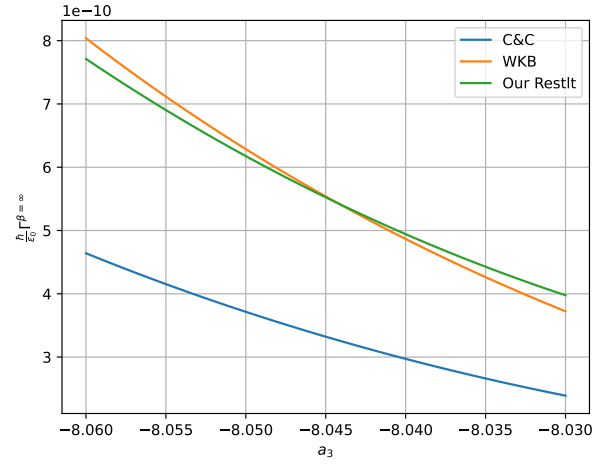
Besides the results derived through path integral by saddle point approximation, the numerical calculation of the result computed through wave function by WKB approximation

$$\Gamma_{\text{WKB}} = \frac{\epsilon_0}{\hbar} \frac{e^{-2 \int_a^b \bar{x} |p|}}{\int_{x_F}^a d\eta \frac{1}{|p|} + \int_a^b d\eta \frac{1}{|p|} e^{-2 \int_a^{\eta} \bar{x} |p|}}$$

will also be performed. Since the imaginary part of the energy is fairly small, only the real part of the energy is kept in the numerical calculation of  $\Gamma_{\text{WKB}}$ .

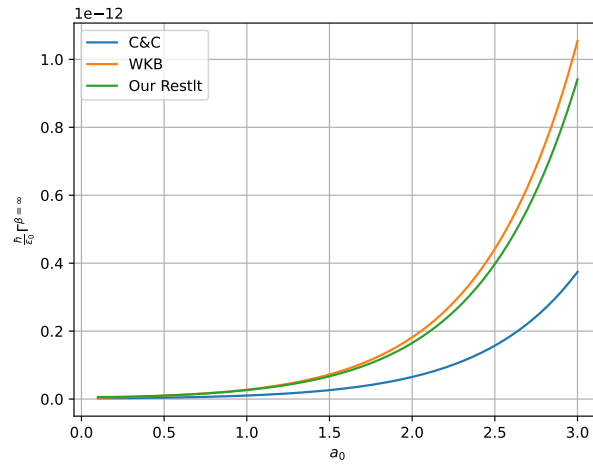
The numerical results are shown as following.

1.  $V(x) = x^4 + a_3 x^3 + 16x^2$  ( $w_F$  is kept to be the same.)



**Figure 4.10.:** Decay Rate at Zero Temperature

$$2. \mathcal{V}(\mathbf{r}) = \cosh(0.2\mathbf{r}) + \frac{10}{\cosh(\mathbf{r}+a_0)}$$



**Figure 4.11.:** Decay Rate at Zero Temperature

Our result exhibits behavior more closely aligned with the result calculated using the WKB approximation compared to the results of [70] and [74]. It is expected since [70] and [74] only included the contribution from the false vacuum solution while ignore all other classical solutions providing real contributions.

## 4.2. Potential with $\mathcal{V}'(\mathbf{r}_R) \rightarrow 0$

Through the numerical calculation, the suspicious temperature dependence of  $\Gamma_{\text{Aff}}^\beta$  at low temperature for potential with small  $V'(x_R)$  was confirmed. The key to the reason lies in the decay rate at zero temperature limit.

First, let us consider the zero temperature limit of Affleck's result :

$$\Gamma^{C\&C} = \Gamma_{\text{Affleck}}^{\beta=\infty} = \sqrt{\frac{S_B}{2\pi\hbar}} \sqrt{2m\omega_F} A(\infty) e^{-\frac{1}{\hbar}S_B},$$

where

$$A(\infty) \equiv \frac{\omega_F}{\sqrt{S_B}} (x_R - x_F) e^{\int_{x_F}^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x})-V(x_F))}} - \frac{1}{|\bar{x}-x_F|} \right)}.$$

For a potential with  $V'(x_R) \neq 0$ , the integral on the exponent should be finite since the  $\frac{1}{x}$  divergence of  $\frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x})-V(x_F))}}$  near  $\bar{x} = x_F$  is eliminated by  $\frac{1}{|\bar{x}-x_F|}$  and the  $\frac{1}{\sqrt{x}}$  divergence around  $\bar{x} = x_R$  causes no problem after integrating. However, for a potential with  $V'(x_R) = 0$ , the singularity of  $\frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x})-V(x_F))}}$  near  $\bar{x} = x_F$  was truly canceled by  $\frac{1}{|\bar{x}-x_F|}$  while the divergence around  $\bar{x} = x_R$  becomes  $\frac{1}{x}$  and leads to an infinite contribution after integrating.

As a result, for potentials with  $V'(x_R) = 0$ , such as a double-well potential,  $\Gamma^{C\&C} = \Gamma_{\text{Affleck}}^{\beta=\infty}$  will become infinite. However, at finite temperature, Affleck's result

$$\Gamma_{\text{Affleck}}^\beta = 2 \sinh \frac{\omega_F \beta \hbar}{2} \frac{e^{-\frac{1}{\hbar}S_B}}{\sqrt{2\pi\hbar} \left| \frac{d\beta\hbar}{dE_B} \right|}$$

is still definitely finite. Therefore, it must decrease with temperature at low temperature for potentials with  $V'(x_R) = 0$ .

Now, consider a series of potentials parameterized by  $\alpha$  and changes smoothly with  $\alpha$ . The curves of decay rates for different potentials are believed to change smoothly with  $\alpha$  as well. Therefore, for potentials close to the potential with  $V'(x_R) = 0$ , a temperature range in which the decay rate decreases with temperature must exist in order to avoid discontinuity.

So, how about the result derived by our methods that includes other classical solutions? The zero temperature limit can be expressed as

$$\Gamma_{\text{Dir}}^{\beta=\infty} = \frac{e^{-\frac{1}{\hbar}S_B}}{\int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar}S_F^{\beta=\infty}[y]}}{\sqrt{\frac{2}{m}(V(y)-V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x})-V(x_F))}} d\bar{x}}}.$$

Focusing on the denominator, it can be calculated by partial integration:

$$\begin{aligned}
\text{DT} &= \int_{x_F}^{x_R} dy \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta=\infty}[y]}}{\sqrt{\frac{2}{m} (V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} \\
&= \left[ \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[y]}}{\omega_F} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} \right]_{x_F}^{x_R} \\
&\quad + \frac{1}{\hbar} \int_{x_F}^{x_R} dy \frac{dS_{sF}^{\beta}[y]}{dy} \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[y]}}{\omega_F} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} \\
&= \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[x_R]}}{\omega_F} + \frac{1}{\hbar} \int_{x_F}^{x_R} dy \frac{dS_{sF}^{\beta}[y]}{dy} \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[y]}}{\omega_F} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}},
\end{aligned}$$

where  $\frac{dS_{sF}^{\beta}[y]}{dy}$  can be expressed as

$$\frac{dS_{sF}^{\beta}[y]}{dy} = \frac{d}{dy} 2 \int_y^{x_R} d\bar{x} \sqrt{2m (V(\bar{x}) - V(x_F))} = 2 \sqrt{2m (V(y) - V(x_F))},$$

which is finite for any  $y$ . For a potential with  $V'(x_R) = 0$ ,  $e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}}$  will always be 0 except for  $y = x_R$ , which results in the fact that the following integral is just 0.

$$\frac{1}{\hbar} \int_{x_F}^{x_R} dy \frac{dS_{sF}^{\beta}[y]}{dy} \frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta}[y]}}{\omega_F} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}} = 0.$$

Therefore, for a potential with  $V'(x_R) = 0$ , our result becomes

$$\lim_{\beta \rightarrow \infty} \Gamma_{\text{Dir}}^{\beta} = \frac{e^{-\frac{1}{\hbar} S_B}}{\frac{e^{-\frac{1}{\hbar} S_{sF}^{\beta=\infty}[x_R]}}{\omega_F}} = \frac{e^{-\frac{1}{\hbar} S_B}}{\frac{e^{-\frac{1}{\hbar} S_B}}{\omega_F}} = \omega_F.$$

Furthermore, if the following relation for a potential with  $V'(x_R) = 0$  is realized

$$\delta(x - x_R) = \frac{\omega_F}{\sqrt{\frac{2}{m} (V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}}, \quad (4.2)$$

the zero temperature limit of both our result and Affleck's result can be obtained more intuitively.

Denote

$$f(y) \equiv \frac{\omega_F}{\sqrt{\frac{2}{m} (V(y) - V(x_F))}} e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}}.$$

For a potential with  $V'(x_R) = 0$ ,  $e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} d\bar{x}}$  will always be 0 except for  $y = x_R$ . Therefore,  $f(y)$  can only have non-zero value at  $y = x_R$  and  $y = x_F$  since

$V(x_F) = V(x_R)$ . When  $y \rightarrow x_F$ ,

$$\lim_{y \rightarrow x_F} f(y) = \lim_{y \rightarrow x_F} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} \frac{y - x_F}{x_R - x_F} e^{-\int_y^{x_R} dx \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(x) - V(x_F))}} - \frac{1}{|x - x_F|} \right)} \quad (4.3)$$

$$= \frac{1}{x_R - x_F} e^{-\int_{x_F}^{x_R} dx \left( \frac{\omega_F}{\sqrt{2(V(x) - V(x_F))}} - \frac{1}{|x - x_F|} \right)} \quad (4.4)$$

$$= 0. \quad (4.5)$$

When  $y \rightarrow x_R$ ,

$$\lim_{y \rightarrow x_R} f(y) = \lim_{y \rightarrow x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(y) - V(x_F))}} = \infty.$$

Moreover, the integral of  $f(y)$  can be performed as

$$\int_{x_F}^{x_R} dy f(y) = \left[ e^{-\int_y^{x_R} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}} \right]_{x_F}^{x_R} = 1.$$

Therefore,  $f(y)$  can be consider as  $\delta(y - x_R)$  in  $y \in [x_F, x_R]$ .

## 4.3. Reconsideration of the Calculation at Zero Temperature

While we were calculating the decay rate at finite temperature by our methods and reconsidering previous research, we found several problems or inconsistencies in the calculation of previous research. We have already discussed some obvious problems in the calculation of Affleck at finite temperature in subsec.3.1.3. However, some similar and new problems exist in the calculation of Callan and Coleman at zero temperature as well. We will discuss all these inconsistencies in detail in this section.

### 4.3.1. Transition Amplitude and the Classical Solution

The first problem we would like to discuss relates to the timing for selecting proper endpoints and the timing for taking the time interval to infinity. The time interval relates to the selection of the classical solutions. In [70], the sequences are ambiguous. Taking the time interval  $\mathcal{T}$  to infinity seems to be the last process in the calculation, but it has already been done when finding the classical solutions.

In fact, if you select the time interval  $\mathcal{T}$  to be large but finite then you can't choose  $x_i = x_f = x_F$ . Otherwise, the classical solutions will be the false vacuum solution and the shot-T solution. On the other hand, if you choose  $x_i = x_f = x_F$

at first and attempt to include the bounce solution, then the time interval  $\mathcal{T}$  will be infinite at beginning, which leads to a calculation along with full of singularities. In other words, it is impossible to use the bounce solution starting at  $x_F$  and use a finite  $\mathcal{T}$  to derive the final result at the same time.

One might think that an finite time interval  $\mathcal{T}$  together with other endpoints deviating from  $x_F$  can be chosen in order to find a proper bounce solution with a finite period, which is corresponding to the bounce solution at finite temperature, and then  $\mathcal{T}$  can be taken to infinity to obtain the final result. However, such approach results in that the false vacuum solution shares different endpoints with the bounce solution with finite period. The same endpoints for all classical solutions are required in the saddle point approximation due to the transition amplitude  $\langle x_f | e^{-\frac{i}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle$ , which serves as the starting point for linking the decay rate to the imaginary part of the energy. If one insists on including the contribution from the false vacuum solution, the function space should be changed from the original function space in which the endpoints of all paths are  $x_F$  to a new function space in which all paths are periodic and own the same period. The boundary conditions of the periodic paths should be

$$\begin{aligned} x\left(-\frac{\mathcal{T}}{2}\right) &= x\left(\frac{\mathcal{T}}{2}\right) \\ \dot{x}\left(-\frac{\mathcal{T}}{2}\right) &= \dot{x}\left(\frac{\mathcal{T}}{2}\right). \end{aligned}$$

Then the path integral should be

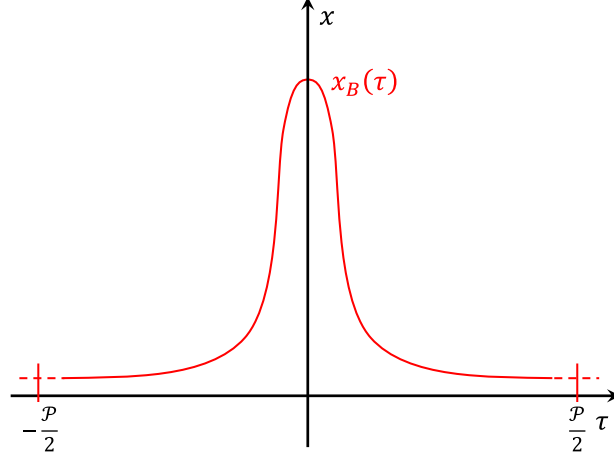
$$\mathcal{N} \int_{\text{P.B.C}} \mathcal{D}x e^{-\frac{i}{\hbar} S_E[x]}$$

instead of the one in Eq.(2.10). Moreover, the fluctuation  $\Delta x(\tau)$  around a classical solution should also satisfy the same boundary conditions. The new method will be completely identical to what Affleck did in the situation at finite temperature. Although the contribution of the false vacuum solution as well as the contribution of the bounce solution can be contained in the final result, the path integral has nothing to do with the transition amplitude, which is the basis of the calculation at zero temperature.

### 4.3.2. Zero Eigenvalue and Collective Coordinate Method

The second problem relates to the treatment for the zero eigenvalue and the use of the collective coordinate method.

Let's review what was done in previous research at first. Since the derivative of the bounce solution  $\dot{x}_B(\tau)$  satisfies the Dirichlet boundary conditions  $\dot{x}_B(\pm\infty) = 0$  and the eigenequation  $\left[-\partial_\tau^2 + \frac{1}{m} V''(x_B)\right] \dot{x}_B = 0$ , it is the eigenfunction of  $\left[-\partial_\tau^2 + \frac{1}{m} V''(x_B)\right]$  with  $\lambda_1 = 0$ . Therefore, when calculating the Gaussian integral  $\int \frac{dc_1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar} \lambda_1 c_1^2}$  after



**Figure 4.12.:** Segment of the Bounce Solution at Zero Temperature

expanding around the bounce solution, the result should be infinite. Callan and Coleman dealt with this using the so-called collective coordinate method. They considered the integral over  $c_1$  as being equivalent to the integral over the center  $\tau_c$  of the bounce solution

$$\frac{dc_1}{\sqrt{2\pi\hbar}} = \sqrt{\frac{S_B}{2\pi\hbar}} d\tau_c,$$

which finally gives the result  $\sqrt{\frac{S_B}{2\pi\hbar}} \mathcal{T}$  after carrying out the integral over  $\tau_c$ . As a result, since the integral over  $\tau_c$  has already been completed when including the contribution from all bounce solutions with different centers, they consider that there is no need to perform the integral over  $c_1$ .

However, there are two problems. The first one is the rationality of containing all the contributions from the bounce solutions with different centers. The second one is the equality of the integral over  $c_1$  and the integral over  $\tau_c$ .

Now, we start to discuss the first problem. After time translation, the original solution becomes  $x_B(\tau - \tau_c)$  and the boundary values  $x_B(\pm\frac{\mathcal{T}}{2} - \tau_c)$  differ from the original bounce solution. One might assume that the time interval is infinite or that the thin-wall approximation applies, making the change in boundary conditions negligible. But it is only reasonable for an infinite  $\mathcal{T}$  along with a finite  $\tau_c$ . However, since the integral over  $\tau_c$  is performed,  $\tau_c$  could become large enough to introduce significant effects if the fluctuation determinant is defined by Dirichlet boundary conditions. In other words, the  $K$ , which corresponds to the fluctuation determinant in Callan and Coleman's discussion, is not a constant but depends on  $\tau_c$ . There is no reasonable explanation for ignoring the dependence on  $\tau_c$  when performing the integral except for the situation that the fluctuation determinant is defined by periodic boundary conditions.

As for the second problem, the integral over  $c_1$  is not equivalent to the integral over  $\tau_c$  at least for a finite  $\mathcal{T}$ . We will explain the inequality in several dependent

ways. First, the integral over  $c_1$  should relate to  $\lambda_1$ , which is the second-smallest eigenvalue of  $[-\partial_\tau^2 + \frac{1}{m}V''(x_B)]$ . Following Coleman's explanation in [76], we can examine the fluctuation operator of a portion of the full bounce solution from  $-\frac{\mathcal{P}}{2}$  to  $\frac{\mathcal{P}}{2}$  and study how the eigenvalues depend on  $\mathcal{P}$ . Denote the bounce in  $[-\frac{\mathcal{P}}{2}, \frac{\mathcal{P}}{2}]$  as  $x_B^{\mathcal{P}/\infty}$ . When  $\mathcal{P}$  is large, the corresponding  $\lambda_1$  of  $[-\partial_\tau^2 + \frac{1}{m}V''(x_B^{\mathcal{P}/\infty})]$  will be small but non-zero. The integral over  $c_1$  can thus be expressed as

$$\int \frac{dc_1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}\lambda_1 c_1^2} = \frac{1}{\sqrt{\lambda_1}}.$$

One might expect this result to be consistent with the one obtained via the collective coordinate method, leading to the relation

$$\frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{S_B}{2\pi\hbar}} \mathcal{P},$$

as discussed in [77]. However, the small non-zero eigenvalue  $\lambda_1$  behaves like

$$\lambda_1 = 4m\omega_F A^2(\infty) e^{-\omega_F \mathcal{P}}, \quad (4.6)$$

where  $A(\infty) = \frac{\omega_F}{\sqrt{S_B}} (x_R - x_F) e^{\int_{x_F}^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right)}$  is a constant (see Appendix C.1.2). Therefore,

$$\frac{1}{\sqrt{\lambda_1}} = \frac{1}{2\sqrt{m\omega_F} A(\infty)} e^{\frac{\omega_F \mathcal{P}}{2}},$$

which is notably different from  $\sqrt{\frac{S_B}{2\pi\hbar}} \mathcal{P}$ . But be careful, the  $\mathcal{P}$  introduced here is independent of  $\mathcal{T}$ .

The second way is put forward in [75]. They considered the problem from the point of view of coordinate transformation. They noticed that the  $c_1$  can't be replaced by  $\tau_c$  in the expansion around the bounce solution:

$$x(\tau) = x_B(\tau) + \sum_n \frac{c_n}{\sqrt{m}} x_n(\tau) \neq x_B(\tau + \tau_c) + \sum_{n \neq 1} \frac{c_n}{\sqrt{m}} x_n(\tau),$$

since

$$\frac{\int d\tau (x_B(\tau + \tau_c) - x_B(\tau)) x_1(\tau)}{\int d\tau x_1^2(\tau)} = \frac{1}{\|\dot{x}_B(\tau)\|} \int d\tau x_B(\tau + \tau_c) \dot{x}_B(\tau)$$

is bounded and couldn't be equal to  $c_1$  which is unbounded. Therefore, the coordinate transformation from  $\{c_n\}$  to  $\{\tau_c, \{c_n\}_{n \neq 1}\}$  is not an equivalent transformation.

Although they considered the expansion around the bounce solution should be parameterized as

$$x(\tau) = x_B(\tau + \tau_c) + \sum_{n \neq 1} \frac{\zeta_n}{\sqrt{m}} x_n(\tau + \tau_c),$$

it is not a reasonable transformation under the Dirichlet boundary conditions.  $\tau_c$  and  $\{\zeta_n\}$  are required to satisfy extra equations

$$x(\pm \frac{\mathcal{T}}{2}) = x_B(\pm \frac{\mathcal{T}}{2} + \tau_c) + \sum_{n \neq 1} \frac{\zeta_n}{\sqrt{m}} x_n(\pm \frac{\mathcal{T}}{2} + \tau_c) = x_F \quad (4.7)$$

in order to satisfy the Dirichlet boundary conditions required for  $x(\tau)$ . The new function space denoted by  $\{\tau_c, \{\zeta_n\}_{n \neq 1}\}$  is completely different from the one in [70] at least for finite  $\mathcal{T}$ . However, if the function space consists of paths satisfying periodic boundary conditions in the first place, no extra equation required appears.

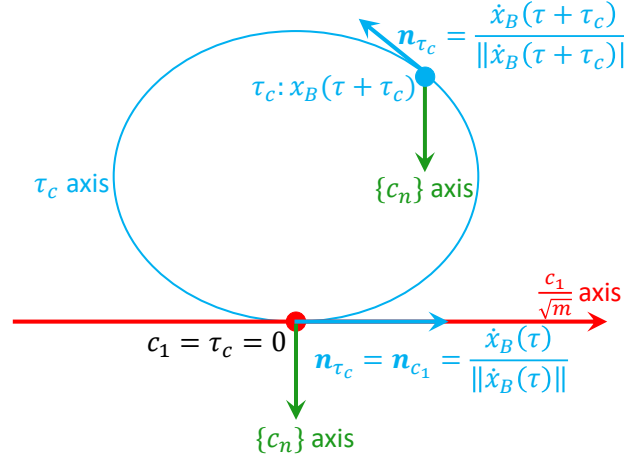
The last way comes from the situation at finite temperature. At finite temperature, there is also a zero eigenvalue when calculating the contribution from the bounce solution. However, the corresponding time interval is finite. The consideration that the zero eigenvalue results from the infinite  $\mathcal{T}$ , or the time translation invariance is somehow controversial. Furthermore, the zero eigenvalue of the fluctuation determinant of the bounce is indicated to be related to the vanished initial velocity rather than an infinite  $\mathcal{T}$  when Dirichlet boundary conditions are adopted from Eq.(B.39) :

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta(\tau + \tau_c)) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]} = - \frac{m \left[ \dot{x}_B^\beta(\frac{\beta \hbar}{2} + \tau_c) \right]^2 \frac{d\beta \hbar}{dE_B}}{\frac{1}{\omega_F} \sinh(\omega_F \beta \hbar)}.$$

The time-translation symmetry should be broken after a classical solution with corresponding boundary conditions is selected.

Then, a new question arises: What is actually being computed by the collective coordinate method? In fact, the problem was already addressed in [67, 68, 89, 90] to some extent. Through the saddle point approximation, a local coordinate system is established with the origin at  $x_B(\tau)$  in the function space. The eigenfunctions of the fluctuation operator correspond to the directions of the axes in this local coordinate system, and the expansion coefficients represent the coordinates along these axes. The integral over  $c_1$ , which is the coordinate along the  $x_1(\tau) = \frac{\dot{x}_B(\tau)}{\|\dot{x}_B\|}$  axis, corresponds to an integral over  $x_1(\tau)$  axis. On the other hand, the integral over  $\tau_c$  corresponds to the integral over  $\tau_c$  axis, which is parallel to the  $x_1(\tau)$  axis only at the origin  $\tau_c = 0$  and changes direction as  $\tau_c$  changes. The  $\tau_c$  axis consists of all configurations  $x_B(\tau + \tau_c)$ s in the function space (ignoring the change of the boundary conditions for a while) . The direction of the  $\tau_c$  axis is given by  $\frac{\dot{x}_B(\tau + \tau_c)}{\|\dot{x}_B(\tau + \tau_c)\|}$ , and this deviates from the  $\varphi_1(\tau)$  axis more and more significantly as  $\tau_c$  increases. The length increment  $dL$  at  $\tau_c$  can be expressed by the direction vector  $\frac{\dot{x}_B(\tau + \tau_c)}{\|\dot{x}_B(\tau + \tau_c)\|}$  along  $\tau_c$  axis as

$$\begin{aligned} dL &= \sqrt{m} \mathbf{n} \cdot d\mathbf{r} = \sqrt{m} \int d\tau \frac{\dot{x}_B(\tau + \tau_c)}{\|\dot{x}_B(\tau + \tau_c)\|} \dot{x}_B(\tau + \tau_c) d\tau_c \\ &= \sqrt{m} \|\dot{x}_B(\tau + \tau_c)\| d\tau_c = \sqrt{m} \|\dot{x}_B(\tau)\| d\tau_c. \end{aligned}$$



**Figure 4.13.:** The Explanation for Collective Coordinate Method

And total length becomes

$$L = \int dL = \sqrt{m} \|\dot{x}_B(\tau)\| \int d\tau_c.$$

Therefore, the collective coordinate method calculates the length of the  $\tau_c$  axis (the change of boundary conditions is ignored), which is proportional to  $\int d\tau_c$  or the period of the bounce solution, rather than the contribution along the  $c_1$  axis.

We have no idea whether the collective coordinate method is a valid method to deal with the zero eigenvalue when  $\mathcal{T}$  is infinite. For example, when  $\mathcal{T}$  is infinite, Eq.(4.7) is satisfied as long as  $\tau_c$  is finite. Moreover, the  $\tau_c$  axis might be parallel to the  $c_1$  axis when  $\mathcal{T}$  is infinite. Even if those assumptions are valid when  $\mathcal{T}$  is infinite, the calculation will be full of singularities.

In conclusion, by using the collective coordinate method, the length of the  $\tau_c$  axis is calculated rather than the contributions from the zero mode. The zero mode was just ignored rather than properly treated in research applying the collective coordinate method. Furthermore, when performing the integral over  $\tau_c$ , the change of the boundary values as well as the dependence of the fluctuation determinants on  $\tau_c$  are all ignored. In other words, all points along the  $\tau_c$  axis are included and assumed to contribute equally to the original bounce solution. This is exactly what happens in the periodic function space.

Through the discussion above, we found an impossible trinity in the saddle point approximation of the following path integral

$$\mathcal{N} \int_{\text{B.C}} \mathcal{D}x(\tau) e^{-\frac{1}{\hbar} S_E[x]}$$

with undecided boundary conditions.

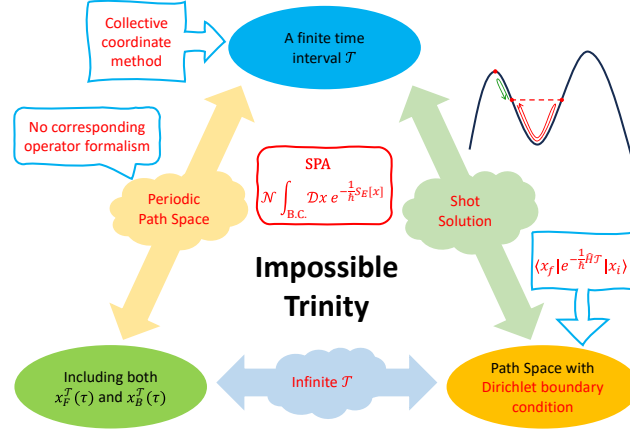


Figure 4.14.: Impossible Trinity

- ① Including both the false vacuum solution and the bounce solution
- ② Using finite  $\mathcal{T}$  (physical quantity instead of a sign)
- ③ Using the consistent Dirichlet boundary conditions

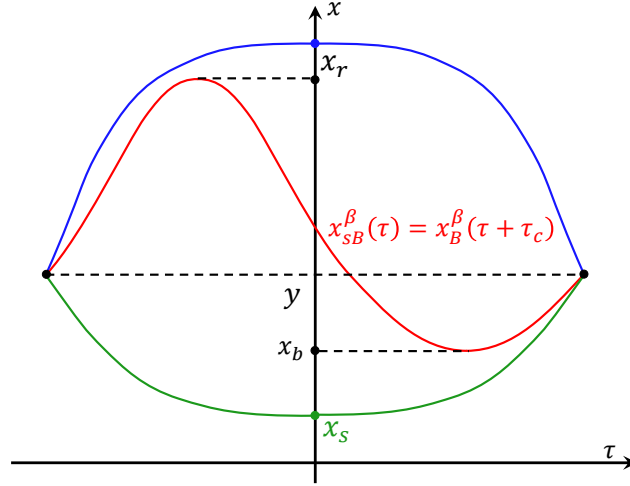
Choosing ① and ② is what Affleck did in [74], which is equivalent to what Callan and Coleman actually did in [70] since they have tempted to use a finite  $\mathcal{T}$  to perform the collective coordinate method while ignoring the change in boundary conditions. This explains why the zero temperature limit of Affleck's result is the same as Callan and Coleman's because they calculated the same thing. However, the path integral calculated by saddle point approximation can't be expressed as a canonical quantization formulation.

If one prepares to choose ① and ③, then the time interval can only be chosen to be infinite. Otherwise, the boundary conditions of the bounce solution are different from the boundary conditions of the false vacuum solution.

It is always possible to choose ② and ③ to calculate the following path integral from a transition amplitude by saddle point approximation

$$\langle x_f | e^{-\frac{1}{\hbar} \hat{H} \mathcal{T}} | x_i \rangle = \mathcal{N} \int_{x(-\frac{\mathcal{T}}{2})=x_i}^{x(\frac{\mathcal{T}}{2})=x_f} \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]}.$$

However, the classical solutions will be different from what Affleck, Callan and Coleman used in their researches. The shot solution should be contained in order to obtain a consistent result. We will explain how to calculate it in the following subsection.



**Figure 4.15.:** The Bounce Solution and Corresponding Shot Solutions

### 4.3.3. Arbitrary Endpoints

It has already been known that there is a zero eigenvalue relevant to the bounce solution with a finite period  $\mathcal{T}$  since the initial velocity is zero. However, we can choose a shift-bounce solution with non-zero velocity to avoid the zero eigenvalue. Consider a shift-bounce solution with endpoints of  $y$ . And a shot-F solution with the same endpoints can be found as well. We have already proved that there is a negative eigenvalue related to the shifted-bounce solution. As a result, the path integral can be expressed as

$$\mathcal{N} \int_{x(-\frac{\mathcal{T}}{2})=y}^{x(\frac{\mathcal{T}}{2})=y} \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} = Z_{sF}^{\mathcal{T}} + \frac{1}{2} Z_{sB}^{\mathcal{T}},$$

where the analytic continuation was done and the contributions from the shot-T solution were excluded in order to obtain an imaginary part. Then the imaginary part of the energy can be expressed as

$$\begin{aligned} \lim_{\mathcal{T} \rightarrow \infty} \text{Im} \frac{\hbar}{\mathcal{T}} \ln \mathcal{N} \int_{x(-\frac{\mathcal{T}}{2})=y}^{x(\frac{\mathcal{T}}{2})=y} \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} &\approx \lim_{\mathcal{T} \rightarrow \infty} \frac{\hbar}{\mathcal{T}} \text{Im} \frac{\frac{1}{2} Z_{sB}^{\mathcal{T}}}{Z_{sF}^{\mathcal{T}}} \\ &= \lim_{\mathcal{T} \rightarrow \infty} \frac{\hbar}{2 \mathcal{T}} \sqrt{\frac{\det \left[ -\partial_{\tau}^2 + \frac{1}{m} V''(x_{sF}^{\mathcal{T}}(\tau)) \right]}{\det \left[ -\partial_{\tau}^2 + \frac{1}{m} V''(x_{sB}^{\mathcal{T}}(\tau; y)) \right]}} e^{-\frac{1}{\hbar} [S_{sB} - S_{sF}]}. \end{aligned}$$

By using Eq.(B.39) as well as Eq.(B.40), the ratio can be expressed as

$$\begin{aligned} \frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sF}^\mathcal{T}(\tau)) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\mathcal{T}(\tau; y)) \right]} &= - \frac{-\frac{\partial S_{sF}}{\partial E_{sF}}}{m \left[ \dot{x}_{sB}^\mathcal{T}(\frac{\mathcal{T}}{2}; y) \right]^2 \frac{d\mathcal{T}}{dE_B}} \\ &= \frac{2\sqrt{2m(V(y) - V(x_s))} \frac{1}{V'(x_s)} \frac{dy}{dx_s}}{2(V(y) - V(x_b)) \frac{1}{V'(x_b)} \frac{d\mathcal{T}}{dx_b}}. \end{aligned}$$

When  $\mathcal{T} \rightarrow \infty$ ,  $x_s \leq x_b \rightarrow x_F$ , the ratio becomes

$$\begin{aligned} \lim_{\mathcal{T} \rightarrow \infty} \frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sF}^\mathcal{T}(\tau)) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\mathcal{T}(\tau; y)) \right]} &= \lim_{x_s \leq x_b \rightarrow x_F} \frac{2\sqrt{2m(V(y) - V(x_s))} \frac{1}{V'(x_s)} \frac{dy}{dx_s}}{2(V(y) - V(x_b)) \frac{1}{V'(x_b)} \frac{d\mathcal{T}}{dx_b}} \\ &= \lim_{x_s \leq x_b \rightarrow x_F} 2 \frac{\left[ \frac{V'(x_b)}{V'(x_s)} \right]^2}{\sqrt{\frac{2}{m}(V(y) - V(x_b))}} \frac{V'(x_s)}{V'(x_b)} \frac{dy}{dx_s} \\ &= -e^{2 \int_y^{x_F} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}} \end{aligned}$$

by using Eq.(3.40), Eq.(3.12), Eq.(3.39) and Eq.(C.9). Therefore,

$$\lim_{\mathcal{T} \rightarrow \infty} \text{Im} \frac{\hbar}{\mathcal{T}} \ln \mathcal{N} \int_{x(-\frac{\mathcal{T}}{2})=y}^{x(\frac{\mathcal{T}}{2})=y} \mathcal{D}x e^{-\frac{1}{\hbar} S_E[x]} \approx \lim_{\mathcal{T} \rightarrow \infty} \frac{\hbar}{2} \frac{1}{\mathcal{T}} e^{\int_y^{x_F} \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} d\bar{x}} e^{-\frac{1}{\hbar} S_B},$$

where we have set  $V(x_F) = 0$  to make  $S_{sF}^{\mathcal{T}=\infty} = 0$ .

When  $\mathcal{T} \rightarrow \infty$ ,  $y$  can be chosen to be  $x_F$  in order to obtain a finite result. However,  $y$  is selected artificially. The final result depends on the approaching behavior of  $y$  and might not be single.



# 5. Conclusion

## 5.1. Summary

In this thesis, we introduced a new method to calculate the transition rate at finite temperatures, incorporating contributions from shot-F solutions and derived the corresponding zero-temperature limit. Unlike previous approaches, our method avoids reliance on the so-called collective coordinate method, yielding more consistent results. However, our calculations, as well as those from prior research, exhibit unexpected behavior at zero temperature when the potential approaches certain shapes, which is the main reason for the unanticipated temperature dependence at intermediate temperature. This issue appears to stem from fundamental properties of metastable states and its connection with path integral, which remain unresolved in this work.

## 5.2. Discussion and Outlook

We would like to delve deeper into some of the issues discussed and discuss our future works in this section. In our thesis, the calculation of decay rates across different temperatures by use of various methods has been analyzed. Most prior research focuses solely on contributions from the false vacuum solution and the bounce solution, as dictated by their respective boundary conditions. However, as demonstrated in this thesis, such an approach is insufficiently persuasive when examining the transformation from the canonical quantization formulation to the equivalent path integral formulation carefully. Specifically, the time interval used to extract the decay rate imposes constraints on the possible classical solutions, which causes inconsistencies in the choice of classical solutions. The inconsistencies could provide a reasonable explanation for the unexpected behavior observed in numerical calculations.

Unlike previous studies, our research emphasizes the transformation from canonical quantization formulation to equivalent path integral formulation. Consequently, the contributions from the shot solutions are incorporated into our results. Although our numerical calculations still show irregularities for certain potentials, they exhibit better convergence compared to the earlier approaches. Furthermore, these unexpected behaviors of our results could potentially be resolved by considering additional shot-F solutions.

However, a more fundamental issue emerges when rigorously considering which classical solutions should be included. This problem is inherently tied to the path integral formulation associated with metastable states. While [75, 91] addressed parts of this issue, it remains unresolved in our view.

Let's elaborate on this issue at zero temperature. When calculating the complex energy with its imaginary part corresponding to the decay rate of a metastable state, the result can be obtained from

$$\langle x_f | e^{-\frac{1}{\hbar} H \mathcal{T}} | x_i \rangle = \sum_{\alpha} e^{-\frac{1}{\hbar} E_{\alpha} \mathcal{T}} \langle x_f | \psi_{\alpha} \rangle \langle \psi_{\alpha} | x_i \rangle,$$

where the completeness relation inserted pertains to metastable states, such as resonance states and scattering states. As  $\mathcal{T} \rightarrow \infty$ , the calculation isolates the metastable state with the smallest real energy component. The utilization of the specialized completeness relation should affect the selection of the classical solutions to be included, or in other words, the choice of the integral contour. Moreover, the complex energy of a metastable state can be obtained via utilizing trace without taking the infinite limit as:

$$E_{\alpha} = -\frac{1}{\mathcal{T}} \ln \langle \psi_{\alpha} | e^{-\hat{H} \mathcal{T}} | \psi_{\alpha} \rangle = -\frac{1}{\mathcal{T}} \ln \text{Tr} [e^{-\hat{H} \mathcal{T}} | \psi_{\alpha} \rangle \langle \psi_{\alpha} |],$$

which is somehow similar to the direct method used in [75, 91]. The existence of  $|\psi_{\alpha}\rangle\langle\psi_{\alpha}|$  in the trace should indicate that there are some constraints for the paths to be contained in the path integral.

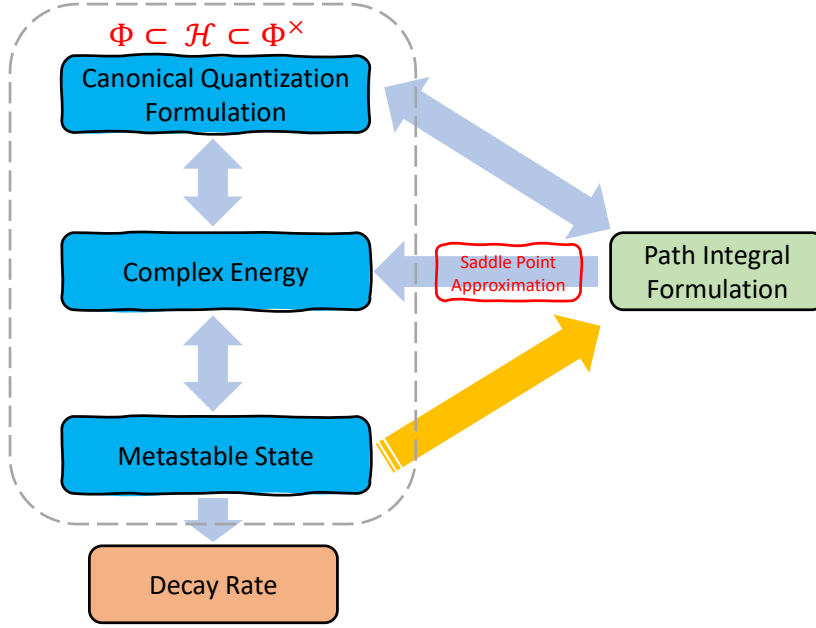
In summary, the first problem lies in the transformation from canonical quantization (relating to metastable states) to the path integral formulation, which should introduce constraints on the paths and necessitate additional considerations in the saddle-point approximation, such as the choice of saddle points or integration contours. Additionally, as illuminated in [92], the differences between different solutions of the Schrödinger equation stem ultimately from different boundary conditions. Therefore, the solution could be determined once the relation between the boundary conditions used in canonical quantization and the path integral is figured out.

However, in order to accomplish this, knowledge of conventional quantum mechanics based on the Hilbert space is definitely not sufficient. Not only the metastable states  $|\psi_{\alpha}\rangle$ s under consideration but also the position eigenvectors  $|x\rangle$ s used to derive the corresponding path integral formulation are generalized eigenvectors, which should be understood within the framework of the rigged Hilbert space, or, in other words, the Gelfand triple

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}.$$

The introduction of the rigged Hilbert space should endow the corresponding path integral formulation with additional structural properties.

Now, we want to discuss the second problem. The second issue concerns the assumption that only one negative eigenvalue exists for the relevant saddle points



**Figure 5.1.:** A Theoretical Framework

as we explained in the previous chapter. The number of the negative eigenvalues can be confirmed if there exists a zero eigenvalue and the determinant is defined by Dirichlet boundary conditions. However, determining the number of negative eigenvalues becomes challenging when no zero eigenvalue is present and periodic boundary conditions are applied. Further investigation is required to determine the number of negative eigenvalues and their impact.

Future efforts will focus on addressing the problems outlined above to provide a more complete theoretical framework. Additionally, our current study is limited to quantum mechanical systems. We believe that this approach can be extended to quantum field theory. When considering the transition rate in quantum field theory at intermediate temperature, classical solutions with  $O(3) \times U(1)$  symmetry can be obtained [71]. Therefore, shot solutions can then be acquired by segmenting bounce solutions with larger period. All necessary components for applying our method to quantum field theory should be realizable once the mathematical challenges related to high-dimensional determinants are resolved. By building on this foundation, we aim to refine our understanding of metastable states and expand the applicability of our approach to broader physical systems.



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# A. Notation and Conventions

## A.1. Nondimensionalization

In order to compare different results by numerical calculation, the nondimensionalization needs to be implemented first. We here introduce an energy parameter  $\epsilon_0$  and then all physical quantities can be expressed by the corresponding dimensionless variables as outlined in Table A.1.

| Physical quantities | Dimensionless variables                                                                                                                           | Relation                                           |
|---------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------|
| position $x$        | $\mathfrak{x}$                                                                                                                                    | $x = \frac{\hbar}{\sqrt{m\epsilon_0}}\mathfrak{x}$ |
| time $\tau$         | $\mathfrak{t}$                                                                                                                                    | $\tau = \frac{\hbar}{\epsilon_0}\mathfrak{t}$      |
| potential $V(x)$    | $\mathcal{V}(\mathfrak{x})$                                                                                                                       | $V(x) = \epsilon_0\mathcal{V}(\mathfrak{x})$       |
| energy $E$          | $\mathcal{E}$                                                                                                                                     | $E = \epsilon_0\mathcal{E}$                        |
| period $\beta\hbar$ | $\mathcal{P} = 2 \int_{\mathfrak{x}_b}^{\mathfrak{x}_r} \frac{1}{\sqrt{2(\mathcal{V}(\mathfrak{x})+\mathcal{E})}} d\mathfrak{x}$                  | $\beta\hbar = \frac{\hbar}{\epsilon_0}\mathcal{P}$ |
| action $S$          | $\mathcal{S} = 2 \int_{\mathfrak{x}_i}^{\mathfrak{x}_f} d\mathfrak{x} \sqrt{2(\mathcal{V}(\mathfrak{x}) + \mathcal{E})} - \mathcal{E}\mathcal{P}$ | $S = \hbar\mathcal{S}$                             |

**Table A.1.:** Nondimensionalization

The partition function can be reformulated as

$$Z = \mathcal{N} \int_{\mathfrak{x}_i}^{\mathfrak{x}_f} \mathcal{D}\mathfrak{x}(\mathfrak{t}) e^{-\int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} dt \left( \frac{1}{2}\dot{\mathfrak{x}}^2 + \mathcal{V}(\mathfrak{x}) \right)}, \quad (\text{A.1})$$

and the results from the saddle point approximation are expressed as

| SPA            | Results                                                                                                                     | Dimensionless                                                                                                         |
|----------------|-----------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
| $Z_F^\beta$    | $\sqrt{\frac{m\omega_F}{2\pi\hbar\sinh(\omega_F\beta\hbar)}}$                                                               | $\sqrt{\frac{\mathfrak{w}_F}{2\pi\sinh(\mathfrak{w}_F\mathcal{P})}} \frac{\mathfrak{x}}{X}$                           |
|                | $\frac{1}{2\sinh\frac{\omega_F\beta\hbar}{2}}$                                                                              | $\frac{1}{2\sinh\frac{\mathfrak{w}_F\mathcal{P}}{2}}$                                                                 |
| $Z_{sB}^\beta$ | $i\sqrt{\frac{1}{2\pi\hbar}} \frac{\beta\hbar e^{-\frac{1}{\hbar}S_B^\beta}}{\sqrt{\left \frac{d\beta\hbar}{dE_B}\right }}$ | $i\sqrt{\frac{1}{2\pi}} \frac{\mathcal{P}e^{-S_B^\beta}}{\sqrt{\left \frac{d\mathcal{P}}{d\mathcal{E}_B}\right }}$    |
| $Z_s^\beta$    | $\sqrt{\frac{m}{2\pi\hbar}} \int dy \frac{e^{-\frac{1}{\hbar}S_S^\beta}}{\sqrt{-\frac{\partial S_S}{\partial E_S}}}$        | $\sqrt{\frac{1}{2\pi}} \int d\mathfrak{y} \frac{e^{-S_S^\beta}}{\sqrt{-\frac{\partial S_S}{\partial \mathcal{E}_S}}}$ |

**Table A.2.:** Saddle point approximation

Next, by introducing dimensionless variable  $\mathfrak{w}_F \equiv \sqrt{\mathcal{V}''(\mathfrak{x}_F)} = \frac{\hbar\omega_F}{\epsilon_0}$ ,  $|\mathfrak{p}(\mathfrak{x})| \equiv \sqrt{2(\mathcal{V}(\mathfrak{x}) - \mathcal{V}(\mathfrak{x}_F))}$  and

$$\mathcal{A}(\mathfrak{y}) = \frac{A(y)}{\sqrt{\frac{\epsilon_0}{\hbar m}}} = \frac{\mathfrak{w}_F}{\sqrt{\mathcal{S}_B}} (\mathfrak{x}_R - \mathfrak{x}_F) e^{\int_{\mathfrak{y}}^{\mathfrak{x}_R} d\mathfrak{x} \left( \frac{\mathfrak{w}_F}{\sqrt{2(\mathcal{V}(\mathfrak{x}) - \mathcal{V}(\mathfrak{x}_F))}} - \frac{1}{\mathfrak{x} - \mathfrak{x}_F} \right)},$$

the zero-temperature limit of the decay rate can be written as

$$\Gamma^{C\&C} = \Gamma_{\text{Aff}}^{\beta=\infty} \approx \omega_F \sqrt{\frac{m}{\omega_F\pi\hbar}} \sqrt{\mathcal{S}_B} A(x_F) e^{-\frac{1}{\hbar}S_B} = \frac{\epsilon_0}{\hbar} \sqrt{\frac{\mathfrak{w}_F}{\pi}} \sqrt{\mathcal{S}_B} \mathcal{A}(\mathfrak{x}_F) e^{-S_B},$$

$$\Gamma_{\text{Dir}}^{\beta=\infty} \approx \frac{\epsilon_0}{\hbar} \frac{\sqrt{\mathcal{S}_B} e^{-S_B}}{\int_{\mathfrak{x}_F}^{\mathfrak{x}_R} d\mathfrak{y} \frac{\mathfrak{w}_F(\mathfrak{y} - \mathfrak{x}_F)}{\sqrt{2(\mathcal{V}(\mathfrak{y}) - \mathcal{V}(\mathfrak{x}_F))}} \frac{e^{-S_{sF}[\mathfrak{y}]}}{\mathcal{A}(\mathfrak{y})}} = \frac{\epsilon_0}{\hbar} \frac{e^{-S_B}}{\int_{\mathfrak{x}_F}^{\mathfrak{x}_R} d\mathfrak{y} \frac{e^{-S_{sF}[\mathfrak{y}]}}{|\mathfrak{p}|} e^{-\int_{\mathfrak{y}}^{\mathfrak{x}_R} d\mathfrak{x} \frac{\mathfrak{w}_F}{|\mathfrak{p}|}}}.$$

Furthermore, the decay rate calculated by WKB approximation in [75] takes the form

$$\Gamma = \frac{\epsilon_0}{\hbar} \frac{e^{-S_B}}{\int_{\mathfrak{x}_F}^{\mathfrak{x}_R} d\mathfrak{y} \frac{e^{-S_{sF}[\mathfrak{y}]}}{|\mathfrak{p}|} e^{-\int_{\mathfrak{y}}^{\mathfrak{x}_R} d\mathfrak{x} \frac{\mathfrak{w}_F}{|\mathfrak{p}|}}}.$$

In summary, the decay rates at zero temperature calculated by different methods can be rewritten in terms of dimensionless variables as follows.

| Results           | Decay rate                                                                                                                                                                                                                                                                                                                         |
|-------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| [69, 70] and [74] | $\frac{\epsilon_0}{\hbar} \sqrt{\frac{\mathfrak{w}_F}{\pi}} \sqrt{\mathcal{S}_B} \mathcal{A}(\mathfrak{x}_F) e^{-\mathcal{S}_B}$                                                                                                                                                                                                   |
| WKB [75]          | $\frac{\epsilon_0}{\hbar} \frac{e^{-2 \int_{\mathfrak{a}}^{\mathfrak{b}} \bar{\mathfrak{r}}  \mathfrak{p} }}{\int_{\mathfrak{x}_F}^{\mathfrak{a}} d\eta \frac{1}{ \mathfrak{p} } + \int_{\mathfrak{a}}^{\mathfrak{b}} d\eta \frac{1}{ \mathfrak{p} }} e^{-2 \int_{\mathfrak{a}}^{\mathfrak{b}} \bar{\mathfrak{r}}  \mathfrak{p} }$ |
| Our result        | $\frac{\epsilon_0}{\hbar} \frac{e^{-\mathcal{S}_B}}{\int_{\mathfrak{x}_F}^{\mathfrak{x}_R} d\eta \frac{e^{-\mathcal{S}_{SF}[\eta]}}{ \mathfrak{p} }} e^{-\int_{\mathfrak{y}}^{\mathfrak{x}_R} d\mathfrak{x} \frac{\mathfrak{w}_F}{ \mathfrak{p} }}$                                                                                |

**Table A.3.:** Decay rate at zero temperature

The structure of the the dimensionless result is identical to the result with  $m = 1$  and  $\hbar = 1$  before nondimensionalization.

Since the original potential should meet  $\frac{\hbar \omega_F}{2} \ll V_m$ , the potential used for numerical calculation should satisfy the following relation

$$\frac{\mathfrak{w}_F}{2} = \frac{\sqrt{\mathcal{V}''(\mathfrak{x}_F)}}{2} \ll \mathcal{V}_m.$$



## B. Review of Some Important Results

### B.1. The Lefschetz-thimble Method

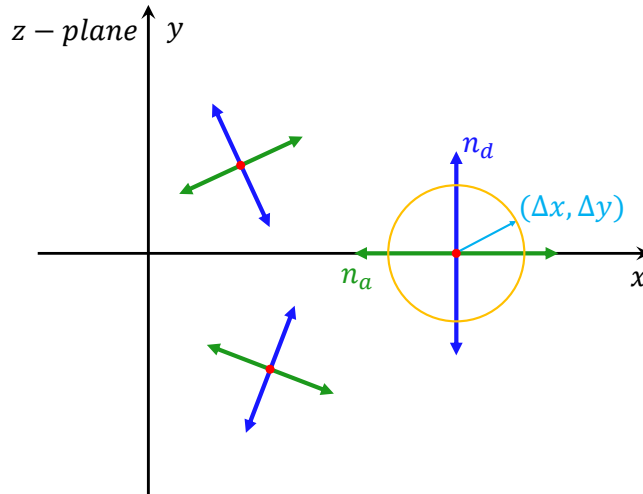
#### B.1.1. Method of Steepest Descent in One Dimensional

Consider the following complex integral

$$\int dx g(x) e^{\lambda f(x)} \rightarrow \int dz g(z) e^{\lambda f(z)}.$$

Assume  $z_0$  is a saddle point of  $f(z)$ . Then two properties around  $z_0$  can be proved.

1. The steepest descent direction  $\mathbf{n}_d$  of  $\text{Re}[f(z)]$  is orthogonal to the steepest ascent direction  $\mathbf{n}_a$ .
2. The steepest descent/ascent direction of  $\text{Re}[f(z)]$  around a real saddle point is along the real axis.



**Figure B.1.:** The Steepest Descent/Ascent Direction

$f(z)$  can be expressed as

$$f(z) = u(x, y) + iv(x, y).$$

From the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we can obtain  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ . The increment around a saddle point  $z_0 = x_0 + iy_0$  can be expressed as

$$\begin{aligned} \Delta u &= \frac{1}{2}u_{xx}(x_0, y_0)\Delta x^2 + \frac{1}{2}u_{yy}(x_0, y_0)\Delta y^2 + u_{xy}(x_0, y_0)\Delta x\Delta y \\ &= \frac{1}{2}u_{xx}(x_0, y_0) \left( \Delta x^2 - \Delta y^2 + 2G_u\Delta x\Delta y \right), \end{aligned}$$

$$\begin{aligned} \Delta v &= \frac{1}{2}v_{xx}(x_0, y_0)\Delta x^2 + \frac{1}{2}v_{yy}(x_0, y_0)\Delta y^2 + v_{xy}(x_0, y_0)\Delta x\Delta y \\ &= \frac{1}{2}v_{xx}(x_0, y_0) \left( \Delta x^2 - \Delta y^2 + 2G_v\Delta x\Delta y \right), \end{aligned}$$

where  $G_u \equiv \frac{u_{xy}(x_0, y_0)}{u_{xx}(x_0, y_0)}$  and  $G_v \equiv \frac{v_{xy}(x_0, y_0)}{v_{xx}(x_0, y_0)} = -\frac{u_{xx}(x_0, y_0)}{u_{xy}(x_0, y_0)} = -\frac{1}{G_u}$ .

The steepest descent/ascent direction  $\mathbf{r} = (\Delta x, \Delta y)$  corresponds to the direction making  $\Delta u$  or  $\Delta x^2 - \Delta y^2 + 2G_u\Delta x\Delta y$  smallest or largest under the condition  $\Delta x^2 + \Delta y^2 = d^2$ . Therefore, the corresponding  $\mathbf{r}$  can be found by using the method of Lagrange multipliers. By introducing

$$F(\Delta x, \Delta y, \lambda) = \Delta x^2 - \Delta y^2 + 2G_u\Delta x\Delta y - \lambda (\Delta x^2 + \Delta y^2 - d^2),$$

the extremum conditions become

$$\begin{aligned} \partial_{\Delta x} F &= 2(1 - \lambda)\Delta x + 2G_u\Delta y = 0 = 2(\Delta x, \Delta y) \begin{pmatrix} 1 - \lambda \\ G_u \end{pmatrix}, \\ \partial_{\Delta y} F &= 2(-1 - \lambda)\Delta y + 2G_u\Delta x = 0 = 2(\Delta x, \Delta y) \begin{pmatrix} G_u \\ -1 - \lambda \end{pmatrix}, \\ \partial_{\lambda} F &= \Delta x^2 + \Delta y^2 - d^2 = 0, \end{aligned}$$

which results in the following equation.

$$\lambda^2 = 1 + G_u^2.$$

The solutions of  $\lambda$  and the corresponding  $\mathbf{r} = (\Delta x, \Delta y)$  are

$$1. \quad \lambda = \sqrt{1 + G_u^2}$$

$$(\Delta x, \Delta y) = \frac{d}{\sqrt{G_u^2 + (1 + \sqrt{1 + G_u^2})^2}} (1 + \sqrt{1 + G_u^2}, G_u) \equiv \mathbf{r}_1 \quad (\text{B.1})$$

$$\Delta u = \frac{1}{2}u_{xx}(x_0, y_0)d^2\sqrt{1 + G_u^2} \quad (\text{B.2})$$

$$\Delta v = 0 \quad (\text{B.3})$$

$$2. \lambda = -\sqrt{1 + G^2}$$

$$(\Delta x, \Delta y) = \frac{d}{\sqrt{G_u^2 + (1 + \sqrt{1 + G_u^2})^2}} (G_u, -1 - \sqrt{1 + G_u^2}) \equiv \mathbf{r}_2 \quad (\text{B.4})$$

$$\Delta u = -\frac{1}{2} u_{xx}(x_0, y_0) d^2 \sqrt{1 + G_u^2} \quad (\text{B.5})$$

$$\Delta v = 0 \quad (\text{B.6})$$

$\mathbf{r}_1$  and  $\mathbf{r}_2$  corresponds to the steepest descent or ascent direction, depending on the sign of  $u_{xx}(x_0, y_0)$ . It is obvious that  $\mathbf{r}_1 \perp \mathbf{r}_2$ . Moreover, since  $\Delta v = 0$ , the imaginary part of  $f(z)$  remains unchanged along the steepest descent or ascent direction.

$u_{xy}$  can be expressed as

$$\frac{\partial^2 u(x, y)}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial(f(z) + f^*(z))}{\partial y} = \frac{\partial}{\partial x} \frac{\partial(f(z) + f(z^*))}{\partial y} \quad (\text{B.7})$$

$$= \frac{\partial}{\partial x} (i f'(z) - i f'(z^*)) = i(f''(z) - f''(z^*)) \quad (\text{B.8})$$

$$= -2\text{Im} f''(z). \quad (\text{B.9})$$

Therefore, for saddle points on the real axis,  $u_{xy}(x_0, y_0) = 0$ . Then,  $G = \frac{u_{xy}(x_0, y_0)}{u_{xx}(x_0, y_0)} = 0$ . Consequently,  $\mathbf{r}_1$  and  $\mathbf{r}_2$  become

$$\mathbf{r}_1 = d(1, 0),$$

$$\mathbf{r}_2 = d(0, -1).$$

$\mathbf{r}_1$  is along the real axis while  $\mathbf{r}_2$  is along the imaginary axis.

The extension of the steepest descent method in functional integral is called Lefschetz-thimble method.

### B.1.2. The Lefschetz-thimble Method

Consider a functional integral like

$$Z = \int \mathcal{D}x(\tau) e^{\mathcal{I}[x(\tau)]}$$

with boundary conditions  $x(\tau_i) = x_i$  and  $x(\tau_f) = x_f$ . The configuration space consists of all paths that adhere to the boundary conditions. As outlined in [93] and [79], the calculation of this functional integral can be approached as follows (refer to [80, 94–102] for further explanations).

1. Begin by complexifying the path, transforming  $x(\tau)$  into  $z(\tau)$ , while maintaining identical boundary conditions. Find all solutions  $\bar{z}_\sigma(\tau)$ s of the classical equation of motion  $\delta\mathcal{I}(z) = 0$  satisfying the boundary conditions in the complexified path space. All real solutions are included with no doubt.

2. Solve the downward flow equation

$$\frac{\partial z(\tau; u)}{\partial u} = -\frac{\overline{\delta \mathcal{I}[z]}}{\delta z(\tau; u)} \quad (\text{B.10})$$

$$\frac{\overline{\partial z(\tau; u)}}{\partial u} = -\frac{\delta \mathcal{I}[z]}{\delta z(\tau; u)} \quad (\text{B.11})$$

with  $z(\tau; -\infty) = z_\sigma(\tau)$ .  $u$  serves as the parameter denoting the position on a given downward flow. And the Lefschetz-thimble  $\mathcal{J}_\sigma$  associated to  $z_\sigma$  encompasses all downward flows originating from the point  $z_\sigma(\tau)$  in configuration space. Along these downward flows,

a) The real part of  $\mathcal{I}(z)$  decreases monotonically

$$\frac{\partial \text{Re}[\mathcal{I}(z)]}{\partial u} = \frac{\partial}{\partial u} \frac{\mathcal{I}(z) + \overline{\mathcal{I}(z)}}{2} = -\left| \frac{\partial z}{\partial u} \right|^2 \leq 0. \quad (\text{B.12})$$

$\text{Re}[\mathcal{I}(z)]$  is called Morse function.

b) The imaginary part of  $\mathcal{I}(z)$  remains constant

$$\frac{\partial \text{Im}[\mathcal{I}(z)]}{\partial u} = \frac{\partial}{\partial u} \frac{\mathcal{I}(z) - \overline{\mathcal{I}(z)}}{2i} = 0. \quad (\text{B.13})$$

As a result, the path integral on the Lefschetz-thimble exhibits favorable convergence properties. If all downward flows originating from  $z_\sigma$  eventually lead to convergent regions, the integral will be well-defined.

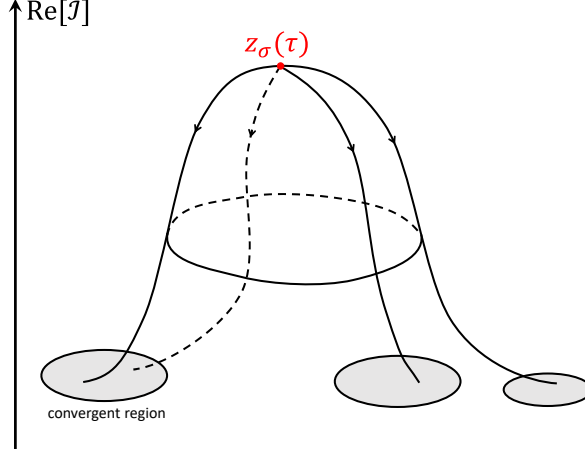
$$Z_\sigma = \int_{\mathcal{J}_\sigma} \mathcal{D}z e^{\mathcal{I}[z]} = e^{i \text{Im}[\mathcal{I}(z_\sigma)]} \int_{\mathcal{J}_\sigma} \mathcal{D}z e^{\text{Re}[\mathcal{I}[z]]}.$$

3. Let  $n_\sigma$  denote the number of intersecting points between the original integration cycle (the space of real paths) and upward flows  $\mathcal{K}_\sigma$  from  $z_\sigma$  in the homological sense. Then the functional integral can be calculated as

$$Z = \sum_\sigma n_\sigma Z_\sigma = \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}z(\tau) e^{\mathcal{I}[z(\tau)]}. \quad (\text{B.14})$$

Now, let's calculate the  $Z_\sigma$  for a real solution  $z_\sigma$ . The imaginary part  $\text{Im}[\mathcal{I}(z_\sigma)]$  is constant. To evaluate the path integral on  $\mathcal{J}_\sigma$ , we perform an expansion around  $z_\sigma$  since the real part  $\text{Re}[\mathcal{I}(z)]$  on Lefschetz-thimble converges rapidly.

$$Z_\sigma = \int_{\mathcal{J}_\sigma} \mathcal{D}z e^{\mathcal{I}[z]} \approx \int_{\mathcal{C}_\sigma} \mathcal{D}\Delta z_\sigma e^{\mathcal{I}[z_\sigma] + \frac{1}{2} \int d\tau_1 d\tau_2 \Delta z_\sigma(\tau_1) \left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau_1) \delta z(\tau_2)} \right]_{z_\sigma} \Delta z_\sigma(\tau_2)}. \quad (\text{B.15})$$



**Figure B.2.:** Downward Flows and Lefschetz-thimble

The deviation  $\Delta z_\sigma(\tau)$  satisfies the boundary condition  $\Delta z_\sigma(\tau_i) = \Delta z_\sigma(\tau_f) = 0$ . The integral contour  $\mathcal{C}_\sigma$  represents the linearization of  $\mathcal{J}_\sigma$  around  $z_\sigma$  and comprises downward flows satisfying the following flow equation:

$$\begin{aligned} \frac{\partial \overline{\Delta z_\sigma(\tau; u)}}{\partial u} &= - \int d\tau' \left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau; u) \delta z(\tau'; u)} \right]_{z_\sigma} \Delta z_\sigma(\tau'; u), \\ \frac{\partial \Delta z_\sigma(\tau; u)}{\partial u} &= - \int d\tau' \overline{\left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau; u) \delta z(\tau'; u)} \right]_{z_\sigma} \Delta z_\sigma(\tau'; u)}. \end{aligned}$$

We can expand  $\Delta z(\tau; u)$  as

$$\Delta z_\sigma(\tau; u) = \sum_n \underbrace{g_n^\sigma(u)}_{\in \mathbb{R}} \chi_n^\sigma(\tau),$$

where  $\chi_n^\sigma(\tau)$ s are the **flow eigenfunctions** which satisfy the **flow eigenequation** with the normalization condition  $\int_{\tau_i}^{\tau_f} d\tau \chi_m^\sigma(\tau) \chi_n^\sigma(\tau) = \delta_{mn}$  and the boundary condition  $\chi_n^\sigma(\tau_i) = \chi_n^\sigma(\tau_f) = 0$ . Then the flow eigenequations can be represented as

$$\begin{aligned} - \int d\tau' \left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau; u) \delta z(\tau'; u)} \right]_{z_\sigma} \chi_n^\sigma(\tau') &= \kappa_n^\sigma \overline{\chi_n^\sigma(\tau)}, \\ - \int d\tau' \overline{\left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau; u) \delta z(\tau'; u)} \right]_{z_\sigma} \chi_n^\sigma(\tau')} &= \kappa_n^\sigma \chi_n^\sigma(\tau), \end{aligned} \quad (\text{B.16})$$

where  $\kappa_n^\sigma$  is the flow eigenvalue and is real. Utilizing the expansion in flow eigenfunctions, the following relation can be obtained

$$\begin{aligned} \mathcal{I}(z) &\approx \mathcal{I}[z_\sigma] + \frac{1}{2} \int d\tau_1 d\tau_2 \Delta z_\sigma(\tau_1) \left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau_1) \delta z(\tau_2)} \right]_{z_\sigma} \Delta z_\sigma(\tau_2) \\ &= \mathcal{I}[z_\sigma] - \frac{1}{2} \sum_n \kappa_n^\sigma g_n^\sigma(u) g_n^\sigma(u). \end{aligned} \quad (\text{B.17})$$

Hence,  $Z_\sigma$  can be expressed as,

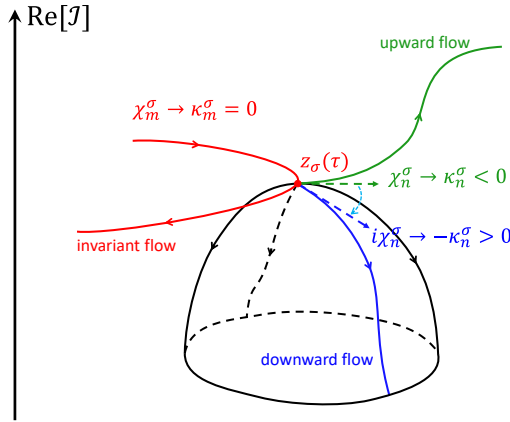
$$Z_\sigma = \int_{\mathcal{J}_\sigma} \mathcal{D}z e^{\mathcal{I}[z]} \approx |J_\chi^\sigma| e^{\mathcal{I}[z_\sigma]} \int \prod \frac{dg_n^\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2} \sum_n \kappa_n^\sigma g_n^\sigma(u) g_n^\sigma(u)}, \quad (\text{B.18})$$

where  $J_\chi^\sigma$  denotes the Jacobian matrix.

We can interpret the flow eigenfunctions  $\chi_n^\sigma(\tau)$  as the directions of a local coordinate system around  $z_\sigma$  in the configuration space.  $g_n^\sigma(u)$  is the coordinate along  $\chi_n^\sigma(\tau)$ . Referring to (B.17), if the flow eigenvalue  $\kappa_n^\sigma$  is positive, then the corresponding direction  $\chi_n^\sigma(\tau)$  is a decreasing direction of  $\mathcal{I}(z)$ . Conversely, if the flow eigenvalue  $\kappa_n^\sigma$  is negative, the corresponding direction  $\chi_n^\sigma(\tau)$  is an increasing direction of  $\mathcal{I}(z)$ . In this case, the integral of the corresponding  $g_n^\sigma$  in (B.18) will be infinite. However, if we replace  $\chi_n^\sigma(\tau)$  with  $\pm i\chi_n^\sigma(\tau)$ , the flow eigenequation of  $\pm i\chi_n^\sigma(\tau)$  becomes

$$-\int d\tau' \left[ \frac{\delta^2 \mathcal{I}[z]}{\delta z(\tau; u) \delta z(\tau'; u)} \right]_{z_\sigma} (\pm i\chi_n^\sigma(\tau')) = -\kappa_n^\sigma \overline{(\pm i\chi_n^\sigma(\tau))}.$$

Therefore,  $\pm i\chi_n^\sigma(\tau)$  also serves as a flow eigenfunction with a flow eigenvalue of  $-\kappa_n^\sigma$ , indicating a direction of change opposite to the original one. We can use  $\pm i\chi_n^\sigma(\tau)$  rather than  $\chi_n^\sigma(\tau)$  in the expansion flow eigenfunction, and the Jacobian determinant will have an extra factor  $\pm i$  compared with the one using  $\chi_n^\sigma(\tau)$ .



**Figure B.3.:** Three Kinds of Flows around the Saddle Point  $z_\sigma$  in Configuration Space

If the flow eigenvalue  $\kappa_n^\sigma$  is 0, then  $\mathcal{I}(z)$  doesn't change along the direction in the neighborhood of  $z_\sigma$ . An invariant flow may appear along this direction. Since Lefschetz-thimble consists of downward flows, there is no need to count invariant flow. However, all points on the invariant flow could also be solutions of  $\delta\mathcal{I}(z) = 0$  and make the same contribution to the integral. The calculation of the contribution

from these points is simple because, since the contribution from each point is the same, it is enough to calculate the length of the invariant flow in the configuration space. Assume the points on the invariant flow are parameterized as  $z_1^\sigma(\tau; u)$  and the flow eigenfunction with an eigenvalue 0 at  $z_1^\sigma(\tau; u)$  is  $\chi_1^\sigma(\tau, u)$ , then the length of invariant flow in configuration space can be calculated as :

$$L = \int dL = \int \left[ \int d\tau \chi_1^\sigma \frac{dz_1^\sigma(\tau; u)}{du} \right] \frac{du}{\sqrt{2\pi}}. \quad (\text{B.19})$$

This consideration stems from [67] and the result is consistent with the one obtained by the collective coordinate method (see [70]).

Now, consider  $\mathcal{I}[x(\tau)]$  to be

$$-\frac{1}{\hbar} S_E[x(\tau)] = -\frac{1}{\hbar} \int d\tau \left( \frac{1}{2} m \dot{x}^2 + V(x) \right).$$

Assume that we have found a real classical solution  $z_\sigma = x_\sigma(\tau)$  which obeys the EoM  $\delta S_E[x_\sigma] = 0$  and the boundary condition. Then the Gaussian approximation of the integral on Lefschetz-thimble Eq.(B.15) will be

$$Z_\sigma \approx e^{-\frac{1}{\hbar} S_E[x_\sigma]} \int_{\mathcal{C}_\sigma} \mathcal{D}\Delta z_\sigma e^{-\frac{2}{\hbar} \frac{m}{2} \int d\tau \Delta z_\sigma(\tau) \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_\sigma) \right] \Delta z_\sigma(\tau)}. \quad (\text{B.20})$$

The flow eigenequation Eq.(B.16), which will become

$$\begin{aligned} \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_\sigma) \right] \chi_n^\sigma(\tau) &= \kappa_n^\sigma \overline{\chi_n^\sigma(\tau)}, \\ \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_\sigma) \right] \overline{\chi_n^\sigma(\tau)} &= \kappa_n^\sigma \chi_n^\sigma(\tau). \end{aligned}$$

Since  $x_\sigma$  is real,  $\chi_n^\sigma(\tau)$  and  $\overline{\chi_n^\sigma(\tau)}$  satisfy the same flow eigenequation with the same eigenvalue. So  $\chi_n^\sigma(\tau) = \pm \overline{\chi_n^\sigma(\tau)}$ . Therefore, we can consider the real eigenequation

$$\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_\sigma) \right] \psi_n(\tau) = \lambda_n \psi_n(\tau).$$

For  $\lambda_n > 0$ , we can use  $\psi_n(\tau)$  as the eigenfunction to define  $\Delta z_\sigma$ . For  $\lambda_n < 0$ , we can use  $\pm i \psi_n(\tau)$  and the Jacobian determinant will have an extra factor  $\pm i$ . Eq.(B.20) becomes

$$Z_\sigma \approx e^{-\frac{1}{\hbar} S_E[x_\sigma]} |J_\psi^\sigma| \int \prod' \frac{dg_n^\sigma}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2} \sum_n \lambda_n g_n^\sigma(u) g_n^\sigma(u)} = \frac{|J_\psi^\sigma|}{\sqrt{|\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_\sigma) \right]|}},$$

where  $\det'$  means excluding zero eigenvalues.

## B.2. Functional Integral Stationary-phase Method

Consider the calculation of the following functional integral

$$G(T) \equiv \text{Tre}^{-i\hat{H}T} = \int dx_0 \mathcal{N} \int_{x(0)=x(T)=x_0} \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x(t)]}.$$

Before proceeding, it should be determined that which types of paths are included in the path integral at first. Naturally, all paths with endpoints at  $x_0$  should be included, as discussed in [88]. Then how to calculate it? According to [86], the dominant contribution to the following functional integral comes from those paths which are close to the classical periodic orbits  $\bar{x}(\tau)$ s with period  $T$ , a method referred to as the stationary-phase method. Therefore, when using stationary-phase method to calculate the path integral, we should contain the contributions from all classical periodic orbits  $\bar{x}(\tau)$ s. So, why do the classical periodic orbits  $\bar{x}(\tau)$  give rise to a stationary phase? This issue was first explained in [87]. The stationary phase implies that the action of those classical orbits remains unchanged with respect to the endpoints. Consequently,

$$0 = \frac{\partial}{\partial x_0} S[\bar{x}(t)] = m\dot{\bar{x}}(T) - m\dot{\bar{x}}(0)$$

can be derived. A more intuitive explanation is provided in [88].

Thus, only the classical fully periodic orbits  $\bar{x}(\tau)$  should be included when employing the stationary-phase method. After identifying one such orbit  $\bar{x}(\tau)$ ,  $G(T)$  can be expressed as

$$G(T) \approx e^{\frac{i}{\hbar} S[\bar{x}]} \int d\bar{x}(0) \mathcal{N} \int_{\Delta x(0)=0}^{\Delta x(T)=0} \mathcal{D}\Delta x(t) e^{\frac{i}{\hbar} \tilde{S}[\Delta x]},$$

where  $\tilde{S}[\Delta x] \equiv \int_0^T dt \left( \frac{1}{2} m \dot{\Delta x}^2 - \frac{1}{2} \Delta x^2 V''(\bar{x}) \right)$ . Now we introduce

$$\chi(t) = \Delta x(t) - \int_0^t \frac{\dot{\psi}(\mu)}{\psi(\mu)} \Delta x(\mu) d\mu$$

and its inverse

$$\Delta x(t) = \chi(t) + \psi(t) \int_0^t \frac{\dot{\psi}(\mu)}{\psi^2(\mu)} \chi(\mu) d\mu,$$

where  $\psi(\tau)$  satisfies  $\left[ -\partial_\tau^2 - \frac{1}{m} V''(\bar{x}) \right] \psi(\tau) = 0$ . Then two relations can be acquired as

$$\begin{aligned} \dot{\Delta x}(t) &= \dot{\chi}(t) + \dot{\psi}(t) \int_0^t \frac{\dot{\psi}(\mu)}{\psi^2(\mu)} \chi(\mu) d\mu + \frac{\dot{\psi}(t)}{\psi(t)} \chi(t) = \dot{\chi}(t) + \dot{\psi}(t) \frac{\Delta x(t)}{\psi(t)} \\ \ddot{\Delta x}(t) &= \ddot{\chi}(t) + \ddot{\psi}(t) \frac{\Delta x(t)}{\psi(t)} + \dot{\psi}(t) \frac{\dot{\chi}(t)}{\psi(t)}. \end{aligned}$$

The the functional integral over  $\Delta x(t)$  can be transformed to the functional integral over  $\chi(t)$  as follows:

$$\begin{aligned}
 \tilde{S}[\Delta x] &= \int_0^T dt \left( \frac{1}{2} m \dot{\Delta x}^2(t) - \frac{1}{2} \Delta x^2 V''(\bar{x}) \right) \\
 &= \int_0^T dt \left( -\frac{1}{2} m \Delta x(t) \ddot{\Delta x}(t) - \frac{1}{2} \Delta x^2 V''(\bar{x}) \right) \\
 &= \int_0^T dt \Delta x(t) \left( -\frac{1}{2} m \left( \ddot{\chi}(t) + \ddot{\psi}(t) \frac{\Delta x(t)}{\psi(t)} + \dot{\psi}(t) \frac{\dot{\chi}(t)}{\psi(t)} \right) - \frac{1}{2} \Delta x V''(\bar{x}) \right) \\
 &= \int_0^T dt \Delta x(t) \left( -\frac{1}{2} m \ddot{\chi}(t) - \frac{1}{2} m \dot{\psi}(t) \frac{\dot{\chi}(t)}{\psi(t)} \right),
 \end{aligned}$$

where the equation of  $\psi(t)$  was used. Then the  $\tilde{S}[\Delta x]$  becomes

$$\begin{aligned}
 \tilde{S}[\Delta x] &= \left[ -\frac{1}{2} m \dot{\chi}(t) \Delta x(t) \right]_0^T + \int_0^T dt \left( \frac{1}{2} m \dot{\chi}(t) \dot{\Delta x}(t) - \frac{1}{2} m \Delta x(t) \dot{\psi}(t) \frac{\dot{\chi}(t)}{\psi(t)} \right) \\
 &= \int_0^T dt \left( \frac{1}{2} m \dot{\chi}^2(t) + \frac{1}{2} m \dot{\chi}(t) \dot{\psi}(t) \frac{\Delta x(t)}{\psi(t)} - \frac{1}{2} m \Delta x(t) \dot{\psi}(t) \frac{\dot{\chi}(t)}{\psi(t)} \right) \\
 &= \int_0^T dt \frac{1}{2} m \dot{\chi}^2(t).
 \end{aligned}$$

Therefore, by inserting the functional Jacobian  $\left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right|$  and enforcing the boundary condition  $\Delta x(T) = 0$ ,  $G(T)$  can be written as

$$\begin{aligned}
 G(T) &= e^{iS[\bar{x}]} \int d\bar{x}(0) \mathcal{N} \int \mathcal{D}\chi(t) \left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right| e^{\frac{i}{\hbar} \int_0^T dt \frac{1}{2} m \dot{\chi}^2(t)} \delta(\Delta x(T)) \\
 &= e^{iS[\bar{x}]} \int d\bar{x}(0) \mathcal{N} \int \mathcal{D}\chi(t) \left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right| e^{\frac{i}{\hbar} \int_0^T dt \frac{1}{2} m \dot{\chi}^2(t)} \int \frac{d\alpha}{2\pi\hbar} e^{\frac{i}{\hbar} \alpha \Delta x(T)} \\
 &= e^{iS[\bar{x}]} \int d\bar{x}(0) \mathcal{N} \int \mathcal{D}\chi(t) \left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right| e^{\frac{i}{\hbar} \int_0^T dt \frac{1}{2} m \dot{\chi}^2(t)} \int \frac{d\alpha}{2\pi\hbar} e^{\frac{i}{\hbar} \alpha \psi(T) \int_0^T \frac{\dot{\chi}(t)}{\psi(t)} dt},
 \end{aligned}$$

where  $\frac{d}{dt} \left( \frac{\Delta x(t)}{\psi(t)} \right) = \frac{\dot{\chi}(t)}{\psi(t)}$  was used. According to [86], the map  $\Delta x(t) \rightarrow \chi(t)$  is Volterra integral equation and the functional Jacobian  $\left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right|$  is equal to  $\sqrt{\frac{\psi(T)}{\psi(0)}}$  consequently. The final result can then be expressed as

$$\begin{aligned}
 G(T) &= e^{iS[\bar{x}]} \int d\bar{x}(0) \mathcal{N} \int \mathcal{D}\chi(t) \left| \frac{\mathcal{D}\Delta x}{\mathcal{D}\chi} \right| e^{\frac{i}{\hbar} \int_0^T dt \frac{1}{2} m \dot{\chi}^2(t)} \int \frac{d\alpha}{2\pi\hbar} e^{\frac{i}{\hbar} \alpha \psi(T) \int_0^T \frac{\dot{\chi}(\mu)}{\psi(\mu)} d\mu} \\
 &= e^{iS[\bar{x}]} \int d\bar{x}(0) \sqrt{\frac{\psi(T)}{\psi(0)}} \mathcal{N} \int \mathcal{D}\chi(t) e^{\frac{i}{\hbar} \frac{1}{2} m \int_0^T dt \left( \dot{\chi}(t) + \frac{\alpha}{m} \frac{\psi(T)}{\psi(t)} \right)^2} \int \frac{d\alpha}{2\pi\hbar} e^{-\frac{i}{\hbar} \frac{1}{2} m \int_0^T dt \left( \frac{\alpha}{m} \frac{\psi(T)}{\psi(t)} \right)^2}.
 \end{aligned}$$

Since  $\chi(0) = \Delta x(0) = 0$  is fixed, the functional integral of  $\chi(t)$  can be expressed as the following canonical quantization form which equals to 1.

$$\mathcal{N} \int \mathcal{D}\chi(t) e^{\frac{i}{\hbar} \frac{1}{2} m \int_0^T dt \left( \dot{\chi}(t) + \frac{\alpha}{m} \frac{\psi(T)}{\psi(t)} \right)^2} = \int d\chi(T) \langle \chi(T) | e^{-\frac{\hat{P}^2}{2m}} | \chi(0) \rangle = 1$$

As a result,

$$\begin{aligned} G(T) &= e^{iS[\bar{x}]} \int d\bar{x}(0) \sqrt{\frac{\psi(T)}{\psi(0)}} \cdot 1 \cdot \frac{1}{2\pi\hbar} \sqrt{\frac{\pi}{\frac{i}{\hbar} \frac{1}{2} m \int_0^T dt \left( \frac{1}{m} \frac{\psi(T)}{\psi(t)} \right)^2}} \\ &= \sqrt{\frac{i}{2\pi\hbar}} e^{iS[\bar{x}]} \int d\bar{x}(0) \sqrt{-\frac{m}{\psi(0)\psi(T) \int_0^T dt \frac{1}{\psi^2(t)}}}. \end{aligned}$$

In [86], the general form of  $\psi(t)$  is given as

$$\psi(t) = \left( \alpha \int^t \frac{1}{\dot{x}^2(\mu)} d\mu + \beta \right) \dot{x}(t)$$

and

$$\int d\bar{x}(0) \sqrt{-\frac{m}{\psi(0)\psi(T) \int_0^T dt \frac{1}{\psi^2(t)}}} = \frac{e^{-i\pi T}}{\sqrt{\left| \frac{dT}{dE} \right|}}$$

In conclusion, the final result becomes

$$G(T) = \sqrt{\frac{i}{2\pi\hbar}} e^{iS[\bar{x}]} \frac{e^{-i\pi T}}{\sqrt{\left| \frac{dT}{dE} \right|}}.$$

## B.3. The Calculation of Functional Determinants

The problem of deriving a functional determinant can be transformed to solving the eigenequation under the corresponding initial conditions. We here simply present the result (see [103–111] for more details).

### B.3.1. Functional Determinants with Generalized Boundary Conditions

In order to calculate the functional determinant  $\det \left[ -\partial_\tau^2 + W^{(i)}(\tau) \right]$  defined over  $\left[ -\frac{T}{2}, \frac{T}{2} \right]$ , it needs to be specified first that the boundary conditions for the eigenfunctions  $\varphi_n^{(i)}$ s which satisfy the equation

$$\left[ -\partial_\tau^2 + W^{(i)}(\tau) \right] \varphi_n^{(i)} = \lambda_n^{(i)} \varphi_n^{(i)},$$

where  $\lambda_n^{(i)}$ s are eigenvalues. The determinant  $\det \left[ -\partial_\tau^2 + W^{(i)}(\tau) \right]$  can be expressed as the product of all  $\lambda_n^{(i)}$ s. The generalized boundary conditions can be written as

$$M \begin{pmatrix} \varphi_n^{(i)}(-\frac{T}{2}) \\ \dot{\varphi}_n^{(i)}(-\frac{T}{2}) \end{pmatrix} + N \begin{pmatrix} \varphi_n^{(i)}(\frac{T}{2}) \\ \dot{\varphi}_n^{(i)}(\frac{T}{2}) \end{pmatrix} = 0$$

by use of two matrix  $M$  and  $N$ . There are mainly four types of boundary conditions.

1. Dirichlet boundary conditions :  $\varphi_n^{(i)}(-\frac{T}{2}) = \varphi_n^{(i)}(\frac{T}{2}) = 0$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

2. Neumann boundary conditions

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3. Periodic boundary conditions

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. Antiperiodic boundary conditions

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Furthermore, the ratio of two functional determinants can be expressed as

$$\frac{\det[-\partial_\tau^2 + W^{(1)}(\tau)]}{\det[-\partial_\tau^2 + W^{(2)}(\tau)]} = \frac{\prod \lambda_n^{(1)}}{\prod \lambda_n^{(2)}} = \frac{\det \left( M + N \begin{pmatrix} u_1^{(1)}(\frac{T}{2}) & u_2^{(1)}(\frac{T}{2}) \\ \dot{u}_1^{(1)}(\frac{T}{2}) & \dot{u}_2^{(1)}(\frac{T}{2}) \end{pmatrix} \right)}{\det \left( M + N \begin{pmatrix} u_1^{(2)}(\frac{T}{2}) & u_2^{(2)}(\frac{T}{2}) \\ \dot{u}_1^{(2)}(\frac{T}{2}) & \dot{u}_2^{(2)}(\frac{T}{2}) \end{pmatrix} \right)}$$

by using two independent solutions  $u_1^{(i)}(\tau)$  and  $u_2^{(i)}(\tau)$  of the following equation

$$[-\partial_\tau^2 + W^{(i)}(\tau)] u_i^{(i)}(\tau) = 0.$$

$u_1^{(i)}(\tau)$  and  $u_2^{(i)}(\tau)$  should satisfy the following initial conditions

$$\begin{pmatrix} u_1^{(i)}(-\frac{T}{2}) & u_2^{(i)}(-\frac{T}{2}) \\ \dot{u}_1^{(i)}(-\frac{T}{2}) & \dot{u}_2^{(i)}(-\frac{T}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1. Dirichlet boundary conditions

$$\frac{\det[-\partial_\tau^2 + W^{(1)}(\tau)]}{\det[-\partial_\tau^2 + W^{(2)}(\tau)]} = \frac{u_2^{(1)}(\frac{T}{2})}{u_2^{(2)}(\frac{T}{2})}$$

2. Periodic boundary conditions

$$\frac{\det[-\partial_\tau^2 + W^{(1)}(\tau)]}{\det[-\partial_\tau^2 + W^{(2)}(\tau)]} = \frac{(1 - u_1^{(1)}(\frac{T}{2}))(1 - \dot{u}_2^{(1)}(\frac{T}{2})) - \dot{u}_1^{(1)}(\frac{T}{2})u_2^{(1)}(\frac{T}{2})}{(1 - u_1^{(2)}(\frac{T}{2}))(1 - \dot{u}_2^{(2)}(\frac{T}{2})) - \dot{u}_1^{(2)}(\frac{T}{2})u_2^{(2)}(\frac{T}{2})}$$

### B.3.2. Functional Determinants with Special Eigenvalue being Excluded

In this subsection, the method for calculating the functional determinant without a specific eigenvalue will be explained. From the previous explanation, we know that

$$\det \left[ -\partial_\tau^2 + W(\tau) \right] = \prod \lambda_n \propto \det \left( M + N \begin{pmatrix} u_1(\frac{T}{2}) & u_2(\frac{T}{2}) \\ \dot{u}_1(\frac{T}{2}) & \dot{u}_2(\frac{T}{2}) \end{pmatrix} \right).$$

If one wants to calculate the functional determinant excluding a specific eigenvalue  $\lambda_\alpha$ , the necessary step is to find the result of

$$\det' \left[ -\partial_\tau^2 + W(\tau) \right] = \prod_{n \neq \alpha} \lambda_n \propto \frac{1}{\lambda_\alpha} \det \left( M + N \begin{pmatrix} u_1(\frac{T}{2}) & u_2(\frac{T}{2}) \\ \dot{u}_1(\frac{T}{2}) & \dot{u}_2(\frac{T}{2}) \end{pmatrix} \right).$$

$\det'$  is used to represent the exclusion of  $\lambda_\alpha$ . Since the determinant defined under Dirichlet boundary conditions or periodic boundary conditions is most important in our research, only the calculation of  $\frac{1}{\lambda_\alpha} \det \left( M + N \begin{pmatrix} u_1(\frac{T}{2}) & u_2(\frac{T}{2}) \\ \dot{u}_1(\frac{T}{2}) & \dot{u}_2(\frac{T}{2}) \end{pmatrix} \right)$  under Dirichlet boundary conditions and periodic boundary conditions will be explained in this subsection.

Here, by introducing a parameter  $\lambda$  into the functional determinant, the previous determinant becomes

$$\det \left[ -\partial_\tau^2 + W(\tau) - \lambda \right] = \prod (\lambda_n - \lambda) \propto \det \left( M + N \begin{pmatrix} u_{1\lambda}(\frac{T}{2}) & u_{2\lambda}(\frac{T}{2}) \\ \dot{u}_{1\lambda}(\frac{T}{2}) & \dot{u}_{2\lambda}(\frac{T}{2}) \end{pmatrix} \right),$$

where  $u_{1\lambda}(\tau)$  and  $u_{2\lambda}(\tau)$  satisfy the same initial condition as before but obey the following equation

$$\left[ -\partial_\tau^2 + W(\tau) - \lambda \right] u_{i\lambda}(\tau) = 0.$$

One may think  $W(\tau)$  changes to  $W(\tau) - \lambda$  now.

The calculation of  $\frac{1}{\lambda_\alpha - \lambda} \det \left( M + N \begin{pmatrix} u_{1\lambda}(\frac{T}{2}) & u_{2\lambda}(\frac{T}{2}) \\ \dot{u}_{1\lambda}(\frac{T}{2}) & \dot{u}_{2\lambda}(\frac{T}{2}) \end{pmatrix} \right)$  under Dirichlet bound-

ary conditions and periodic boundary conditions will be introduced as following. The determinants defined by other boundary conditions can be derived similarly.

#### Dirichlet Boundary Conditions

The related determinant becomes

$$\det' \left[ -\partial_\tau^2 + W(\tau) - \lambda \right] = \prod_{n \neq \alpha} (\lambda_n - \lambda) \propto \frac{u_{2\lambda}(\frac{T}{2})}{\lambda_\alpha - \lambda}.$$

Since  $\varphi_\alpha(\tau)$  and  $u_{2\lambda}(\tau)$  are solutions of the following equations,

$$\left[-\partial_\tau^2 + W(\tau) - \lambda\right] u_{2\lambda}(\tau) = 0 \quad (\text{B.21})$$

$$\left[-\partial_\tau^2 + W(\tau)\right] \varphi_\alpha(\tau) = \lambda_\alpha \varphi_\alpha(\tau) \quad (\text{B.22})$$

one can multiply  $u_{2\lambda}(\tau)$  on both side of Eq.(B.22) and integrate with respect to  $\tau$  over the time interval  $\left[-\frac{\mathcal{T}}{2}, \frac{\mathcal{T}}{2}\right]$ , yielding

$$[\dot{u}_{2\lambda}(\tau)\varphi_\alpha(\tau) - u_{2\lambda}(\tau)\dot{\varphi}_\alpha(\tau)]_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} = (\lambda_\alpha - \lambda) \int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{2\lambda}(\tau)\varphi_\alpha(\tau).$$

By use of the boundary values of  $\varphi_\alpha(\tau)$  and  $u_{2\lambda}(\tau)$ , the following result can be easily obtained:

$$\frac{u_{2\lambda}(\frac{\mathcal{T}}{2})}{\lambda_\alpha - \lambda} = -\frac{\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{2\lambda}(\tau)\varphi_\alpha(\tau)}{\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2})}. \quad (\text{B.23})$$

Therefore, the ratio becomes

$$\frac{\det\left[-\partial_\tau^2 + W^{(1)}(\tau) - \lambda\right]}{\det'\left[-\partial_\tau^2 + W^{(2)}(\tau) - \lambda\right]} = \frac{\lambda_\alpha - \lambda}{u_2^{(2)}(\frac{\mathcal{T}}{2})} u_2^{(1)}(\frac{\mathcal{T}}{2}) = -\frac{\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2})}{\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{2\lambda}^{(2)}(\tau)\varphi_\alpha(\tau)} u_2^{(1)}(\frac{\mathcal{T}}{2}). \quad (\text{B.24})$$

The result is the same as the result derived by perturbation expansion in [110].

### Periodic Boundary Conditions

The determinant required to be calculated becomes

$$\det'\left[-\partial_\tau^2 + W(\tau) - \lambda\right] = \prod_{n \neq \alpha} (\lambda_n - \lambda) \propto \frac{\left(1 - u_{1\lambda}(\frac{\mathcal{T}}{2})\right) \left(1 - \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2})\right) - \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) u_{2\lambda}(\frac{\mathcal{T}}{2})}{\lambda_\alpha - \lambda}.$$

Using the eigenvalue equation of  $\varphi_\alpha(\tau)$ , we can derive two equations involving  $u_{1\lambda}(\tau)$  and  $u_{2\lambda}(\tau)$ .

$$[\dot{u}_{i\lambda}(\tau)\varphi_\alpha(\tau) - u_{i\lambda}(\tau)\dot{\varphi}_\alpha(\tau)]_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} = (\lambda_\alpha - \lambda) \int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{i\lambda}(\tau)\varphi_\alpha(\tau).$$

However, the boundary values differ from the case of Dirichlet boundary conditions. The following two equations can be derived:

$$\dot{u}_{1\lambda}(\frac{\mathcal{T}}{2})\varphi_\alpha(\frac{\mathcal{T}}{2}) + \left(1 - u_{1\lambda}(\frac{\mathcal{T}}{2})\right)\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2}) = (\lambda_\alpha - \lambda) \int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{1\lambda}(\tau)\varphi_\alpha(\tau), \quad (\text{B.25})$$

$$-\left(1 - \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2})\right)\varphi_\alpha(\frac{\mathcal{T}}{2}) - u_{2\lambda}(\frac{\mathcal{T}}{2})\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2}) = (\lambda_\alpha - \lambda) \int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau u_{2\lambda}(\tau)\varphi_\alpha(\tau). \quad (\text{B.26})$$

1. If  $\varphi_\alpha(\frac{\mathcal{T}}{2}) \neq 0$ , then we can try

$$\boxed{\text{Eq. (B.25)}} \cdot u_{2\lambda}(\frac{\mathcal{T}}{2}) + \boxed{\text{Eq. (B.26)}} \cdot \left(1 - u_{1\lambda}(\frac{\mathcal{T}}{2})\right) \quad (\text{B.27})$$

and obtain

$$\begin{aligned} & \frac{\left(1 - u_{1\lambda}(\frac{\mathcal{T}}{2})\right) \left(1 - \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2})\right) - \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) u_{2\lambda}(\frac{\mathcal{T}}{2})}{\lambda_\alpha - \lambda} \\ &= - \frac{\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau \begin{vmatrix} u_{1\lambda}(\tau) & u_{2\lambda}(\tau) \\ u_{1\lambda}(\frac{\mathcal{T}}{2}) - 1 & u_{2\lambda}(\frac{\mathcal{T}}{2}) \end{vmatrix} \varphi_\alpha(\tau)}{\varphi_\alpha(\frac{\mathcal{T}}{2})}. \end{aligned}$$

2. If  $\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2}) \neq 0$ , then we can try

$$\boxed{\text{Eq. (B.25)}} \cdot \left(1 - \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2})\right) + \boxed{\text{Eq. (B.26)}} \cdot \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) \quad (\text{B.28})$$

and obtain

$$\begin{aligned} & \frac{\left(1 - u_{1\lambda}(\frac{\mathcal{T}}{2})\right) \left(1 - \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2})\right) - \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) u_{2\lambda}(\frac{\mathcal{T}}{2})}{\lambda_\alpha - \lambda} \\ &= - \frac{\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau \begin{vmatrix} u_{1\lambda}(\tau) & u_{2\lambda}(\tau) \\ \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) & \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2}) - 1 \end{vmatrix} \varphi_\alpha(\tau)}{\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2})}. \end{aligned} \quad (\text{B.29})$$

$$= - \frac{\int_{-\frac{\mathcal{T}}{2}}^{\frac{\mathcal{T}}{2}} d\tau \begin{vmatrix} u_{1\lambda}(\tau) & u_{2\lambda}(\tau) \\ \dot{u}_{1\lambda}(\frac{\mathcal{T}}{2}) & \dot{u}_{2\lambda}(\frac{\mathcal{T}}{2}) - 1 \end{vmatrix} \varphi_\alpha(\tau)}{\dot{\varphi}_\alpha(\frac{\mathcal{T}}{2})}. \quad (\text{B.30})$$

Finally, by substituting the result into the previous ratio, the final expression is attained.

### B.3.3. Fluctuation Determinant of a Classical Solution

In this subsection, the calculation of the fluctuation determinant of a classical solution  $\bar{x}$  in Euclidean time will be introduced. The fluctuation operator can be written as  $[-\partial_\tau^2 + \frac{1}{m}V''(\bar{x})]$ . As shown in the previous subsection, once we know the corresponding  $u_1(\tau)$  and  $u_2(\tau)$ , we can obtain the result of a fluctuation determinant immediately.  $u_1(\tau)$  and  $u_2(\tau)$  are independent solutions of

$$\left[-\partial_\tau^2 + \frac{1}{m}V''(\bar{x})\right] u_i(\tau) = 0 \quad (\text{B.31})$$

and need satisfy the particular initial conditions. Therefore, we can find two independent solutions first and then combine them linearly to produce two new solutions that meet the initial conditions. Since  $\bar{x}$  is a classical solution, it must obey the equation of motion

$$-\ddot{\bar{x}} + \frac{1}{m}V'(\bar{x}) = 0. \quad (\text{B.32})$$

Now, we can take the partial derivative of the equation of motion with respect to a parameter  $\alpha$  which is explicitly contained in  $\bar{x}(\tau)$  but not in  $V(\bar{x})$  on both side. Then we get

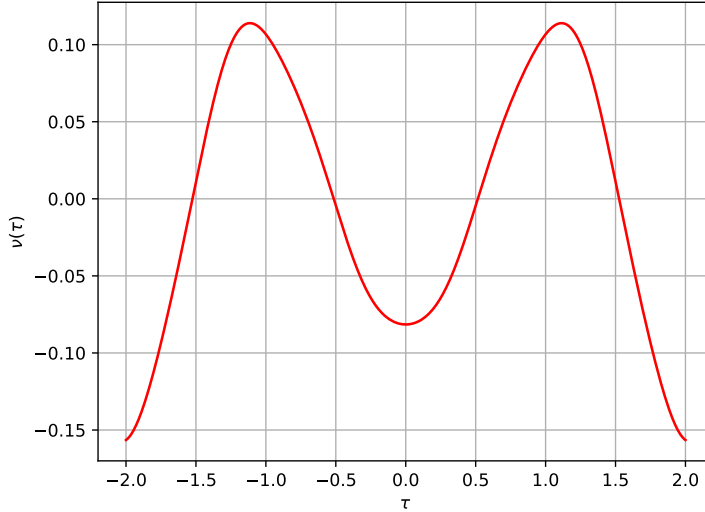
$$\left[ -\partial_\tau^2 + \frac{1}{m}V''(\bar{x}) \right] \partial_\alpha \bar{x} = 0.$$

So  $\partial_\alpha \bar{x}$  is just a solution to the eigenequation of the fluctuation operator with  $\lambda = 0$ . But pay attention, we can't assert that  $\partial_\alpha \bar{x}$  is the eigenfunction since  $\partial_\alpha \bar{x}$  may not satisfy the boundary condition.

When  $\alpha$  is chosen to be  $\tau$ ,  $\partial_\alpha \bar{x}$  becomes  $\dot{\bar{x}}(\tau)$ , which is the velocity of the classical solution  $\bar{x}(\tau)$ , as shown in [70].

If it is noticed that classical solutions exist for every energy in a given range, then it is natural to realize that  $\bar{x}$  also depends on the energy  $E$ . Therefore,  $\alpha$  can be chosen to be  $E$  as well (see [112]). Denote  $\nu_{\bar{x}}(\tau) \equiv m\partial_E \bar{x}$ , then  $\nu_{\bar{x}}(\tau)$  is another solution to the eigenequation with  $\lambda = 0$ <sup>1</sup>.

Before focusing on the function that meets the initial conditions, the properties of  $\nu_{\bar{x}}(\tau)$  will be discussed in more detail.



**Figure B.4.:**  $\nu_{\bar{x}}(\tau)$

<sup>1</sup>In some reference like [77],  $\frac{\partial \bar{x}(\tau)}{\partial \bar{x}(\tau_i)}$  is used as another solution.

First, the linear independence of  $\dot{\bar{x}}(\tau)$  and  $\nu_{\bar{x}}(\tau)$  can be proven immediately. Since  $\bar{x}$  is the classical solution in Euclidean time, its energy is conserved over Euclidean time  $\tau$  :

$$\frac{1}{2}m\dot{\bar{x}}^2 - V(\bar{x}) = E.$$

Taking the derivative of Eq.(B.32) over energy  $E$  gives

$$m\dot{\bar{x}}\partial_E\dot{\bar{x}} - V'(\bar{x})\partial_E\bar{x} = 1.$$

Using the equation of motion and the commutative property of partial derivatives, the following relation can be derived:

$$\dot{\bar{x}}(\tau)\dot{\nu}_{\bar{x}}(\tau) - \ddot{\bar{x}}(\tau)\nu_{\bar{x}}(\tau) = \begin{vmatrix} \dot{\bar{x}}(\tau) & \nu_{\bar{x}}(\tau) \\ \ddot{\bar{x}}(\tau) & \dot{\nu}_{\bar{x}}(\tau) \end{vmatrix} = 1. \quad (\text{B.33})$$

Therefore, the Wronskian of  $\dot{\bar{x}}(\tau)$  and  $\nu_{\bar{x}}(\tau)$  is a non-zero constant, which indicates that  $\dot{\bar{x}}$  and  $\nu_{\bar{x}}(\tau)$  are linearly independent.

Next, it must be emphasized that the periodicity of  $\nu_{\bar{x}}(\tau)$  is different from that of  $\bar{x}(\tau)$ . Even if  $\bar{x}(\tau)$  is periodic function with period of  $P$  ( $P$  doesn't need to be equal to  $\mathcal{T}$ ),  $\nu_{\bar{x}}(\tau)$  may not be periodic since the period  $P$  could also depend on the energy of classical orbit. As a result, considering  $\bar{x}(\tau) = \bar{x}(\tau + P)$ , the following relation holds

$$\nu_{\bar{x}}(\tau) = \nu_{\bar{x}}(\tau + P) + m\dot{\bar{x}}(\tau + P)\frac{dP}{dE}. \quad (\text{B.34})$$

Finally, the parity of  $\nu_{\bar{x}}(\tau)$  is the same as  $\bar{x}(\tau)$ . If  $\bar{x}(\tau)$  is an odd or even function with regard to the Euclidean time  $\tau$ , then  $\nu_{\bar{x}}(\tau)$  will exhibit the same parity.

Since two independent solutions have been found,  $u_1(\tau)$  and  $u_2(\tau)$  which corresponds to different initial conditions can be constructed as

$$u_1(\tau) = \dot{\bar{x}}(\tau)\dot{\nu}_{\bar{x}}(-\frac{\mathcal{T}}{2}) - \ddot{\bar{x}}(-\frac{\mathcal{T}}{2})\nu_{\bar{x}}(\tau) \quad (\text{B.35})$$

and

$$u_2(\tau) = \dot{\bar{x}}(-\frac{\mathcal{T}}{2})\nu_{\bar{x}}(\tau) - \dot{\bar{x}}(\tau)\nu_{\bar{x}}(-\frac{\mathcal{T}}{2}) \quad (\text{B.36})$$

with the help of the Wronskian.

Although [112] used  $\nu_{\bar{x}}(\tau)$  to construct the solutions satisfying the initial condition, [86] and [88] used  $\dot{\bar{x}}(\tau)$  alone to attain a solution as

$$\psi(\tau) = \left( \alpha \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{\bar{x}}^2(\mu)} d\mu + \beta \right) \dot{\bar{x}}(\tau),$$

with the derivative being

$$\dot{\psi}(\tau) = \alpha \frac{1}{\dot{\bar{x}}(\tau)} + \left( \alpha \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{\bar{x}}^2(\mu)} d\mu + \beta \right) \ddot{\bar{x}}(\tau).$$

It is not different to prove that  $\psi(\tau)$  also satisfies Eq.(B.31). Furthermore, a function meeting arbitrary initial conditions can be constructed by adjusting the two parameters  $\alpha$  and  $\beta$ . For example, if a function satisfying the same initial condition as  $u_1(\tau)$  is required, then  $\alpha$  and  $\beta$  can be set as  $\alpha = -\ddot{x}(-\frac{\mathcal{T}}{2})$  and  $\beta = \frac{1}{\dot{x}(-\frac{\mathcal{T}}{2})}$  and  $\psi(\tau)$  becomes

$$\psi(\tau) = \left( -\ddot{x}(-\frac{\mathcal{T}}{2}) \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{x}^2(\mu)} d\mu + \frac{1}{\dot{x}(-\frac{\mathcal{T}}{2})} \right) \dot{x}(\tau).$$

If a function satisfying the same initial condition as  $u_2(\tau)$  is required, then  $\psi(\tau)$  should be

$$\psi(\tau) = \dot{x}(-\frac{\mathcal{T}}{2}) \dot{x}(\tau) \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{x}^2(\mu)} d\mu.$$

There is only one issue : if  $\dot{x}(\tau)$  becomes zero at some time  $\tau_0$  in the time interval, then the integral  $\int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{x}^2(\mu)} d\mu$  could be infinite once  $\tau$  exceeds  $\tau_0$ , which leads to the requirement of using analytic continuation.

Now, consider  $\frac{\nu_{\bar{x}}(\tau)}{\dot{x}(\tau)}$ . The time derivative is

$$\frac{d}{d\tau} \frac{\nu_{\bar{x}}(\tau)}{\dot{x}(\tau)} = \frac{\dot{\nu}_{\bar{x}}(\tau) \dot{x}(\tau) - \ddot{x}(\tau) \nu_{\bar{x}}(\tau)}{\dot{x}^2(\tau)} = \frac{1}{\dot{x}^2(\tau)},$$

where the Wronskian of  $\dot{x}(\tau)$  and  $\nu_{\bar{x}}(\tau)$  was used. As a result,  $\frac{\nu_{\bar{x}}(\tau)}{\dot{x}(\tau)}$  can be expressed as the integral over time  $\tau$  as

$$\frac{\nu_{\bar{x}}(\tau)}{\dot{x}(\tau)} = \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{x}^2(\mu)} d\mu,$$

where the lower bound can be used to server as the constant of integration. Therefore,

$$\nu_{\bar{x}}(\tau) = \dot{x}(\tau) \int_{-\frac{\mathcal{T}}{2}}^{\tau} \frac{1}{\dot{x}^2(\mu)} d\mu,$$

which has the same structure of  $\psi(\tau)$ .

### B.3.4. Functional Determinants at Finite Temperature

In this subsection, some relations of the fluctuation determinants of the classical solutions will be derived. The fluctuation determinant of the false vacuum solution  $\det \left[ -\partial_{\tau}^2 + \frac{1}{m} V''(x_F^{\beta}) \right]$  and the bounce solution  $\det \left[ -\partial_{\tau}^2 + \frac{1}{m} V''(x_B^{\beta}) \right]$  will be used to normalize.

1. For the false vacuum solution  $x_F^\beta(\tau) = x_F$ ,  $\frac{1}{m}V''(x_F^\beta)$  is just equal to a constant  $\omega_F^2$  that does not depend on the time  $\tau$ . It is easy to obtain the corresponding  $u_1(\tau)$  and  $u_2(\tau)$  as

$$u_1(\tau) = \cosh\left(\omega_F(\tau + \frac{\beta\hbar}{2})\right)$$

and

$$u_2(\tau) = \frac{1}{\omega_F} \sinh\left(\omega_F(\tau + \frac{\beta\hbar}{2})\right).$$

2. For the bounce solution  $x_B^\beta(\tau)$ , the corresponding  $u_1(\tau)$  and  $u_2(\tau)$  are given by

$$u_1(\tau) = \dot{x}_B^\beta(\tau)\dot{\nu}_B^\beta(-\frac{\beta\hbar}{2}) - \ddot{x}_B^\beta(-\frac{\beta\hbar}{2})\nu_B^\beta(\tau)$$

and

$$u_2(\tau) = \dot{x}_B^\beta(-\frac{\beta\hbar}{2})\nu_B^\beta(\tau) - \dot{x}_B^\beta(\tau)\nu_B^\beta(-\frac{\beta\hbar}{2})$$

according to Eq.(B.35) and Eq.(B.36). Moreover, since  $\dot{x}_B^\beta(-\frac{\beta\hbar}{2}) = \dot{x}_B^\beta(+\frac{\beta\hbar}{2}) = 0$  and  $\ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \neq 0$  at finite temperature,  $u_2(\tau)$  can be expressed as

$$u_2(\tau) = \frac{\dot{x}_B^\beta(\tau)}{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})},$$

which indicates that  $\nu_B^\beta(-\frac{\beta\hbar}{2}) = -\frac{1}{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})}$  that is consistent with the Wronskian.

### Dirichlet Boundary Conditions

1. **Fluctuation determinant of bounce solution  $x_B^\beta(\tau)$**  : Since fluctuation determinant of Bounce solution has a zero eigenvalue with corresponding eigenfunction  $\frac{\dot{x}_B^\beta(\tau)}{\|\dot{x}_B^\beta(\tau)\|}$ , the zero eigenvalue should be excluded. By use of Eq.(B.24), the following relation without the zero eigenvalue can be obtained.

$$\frac{\det\left[-\partial_\tau^2 + \frac{1}{m}V''(x_F^\beta)\right]}{\det'\left[-\partial_\tau^2 + \frac{1}{m}V''(x_B^\beta)\right]} = -\frac{\frac{\ddot{x}_B^\beta(\frac{\beta\hbar}{2})}{\|\dot{x}_B^\beta(\tau)\|} \frac{\sinh(\omega_F\beta\hbar)}{\omega_F}}{\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \frac{\dot{x}_B^\beta(\tau)}{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})} \frac{\dot{x}_B^\beta(\tau)}{\|\dot{x}_B^\beta(\tau)\|}} = -\frac{\left[\ddot{x}_B^\beta(\frac{\beta\hbar}{2})\right]^2 \sinh(\omega_F\beta\hbar)}{\left\|\dot{x}_B^\beta(\tau)\right\|^2 \omega_F}. \quad (\text{B.37})$$

2. **Fluctuation determinant of shifted-bounce solution  $x_{sB}^\beta(\tau)$** : The related  $u_2(\tau)$  is given by

$$u_2(\tau) = \dot{x}_{sB}^\beta(-\frac{\beta\hbar}{2})\nu_{sB}^\beta(\tau) - \dot{x}_{sB}^\beta(\tau)\nu_{sB}^\beta(-\frac{\beta\hbar}{2}).$$

First, the fluctuation determinant of the Bounce solution  $\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]$  will be used to normalize. The corresponding result becomes

$$\begin{aligned}
 & \frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta) \right]}{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]} \\
 &= \frac{\frac{\dot{x}_B^\beta(\frac{\beta\hbar}{2})}{\|\dot{x}_B^\beta(\tau)\|}}{\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \frac{\dot{x}_B^\beta(\tau)}{\dot{x}_B^\beta(-\frac{\beta\hbar}{2})} \frac{\dot{x}_B^\beta(\tau)}{\|\dot{x}_B^\beta(\tau)\|}} \left( \dot{x}_{sB}^\beta(-\frac{\beta\hbar}{2}) \nu_{sB}^\beta(\frac{\beta\hbar}{2}) - \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \nu_{sB}^\beta(-\frac{\beta\hbar}{2}) \right) \\
 &= \frac{\left[ \dot{x}_B^\beta(\frac{\beta\hbar}{2}) \right]^2}{\|\dot{x}_B^\beta(\tau)\|^2} \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \left( \nu_{sB}^\beta(\frac{\beta\hbar}{2}) - \nu_{sB}^\beta(-\frac{\beta\hbar}{2}) \right) \\
 &= \frac{\left[ \dot{x}_B^\beta(\frac{\beta\hbar}{2}) \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \right]^2}{\|\dot{x}_B^\beta(\tau)\|^2} m \frac{d\beta\hbar}{dE_B}, \tag{B.38}
 \end{aligned}$$

where the relation  $\dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) = \dot{x}_{sB}^\beta(-\frac{\beta\hbar}{2})$  and Eq.(B.34) were used. Clearly, the ratio is positive except for cases where  $\dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) = 0$ , which reveals that the fluctuation operator also has at least one negative eigenvalue. Then, if the  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]$  is used to normalize, the corresponding result will be

$$\begin{aligned}
 & \frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_{sB}^\beta) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]} = \frac{\dot{x}_{sB}^\beta(-\frac{\beta\hbar}{2}) \nu_{sB}^\beta(\frac{\beta\hbar}{2}) - \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \nu_{sB}^\beta(-\frac{\beta\hbar}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)} \\
 &= \frac{\dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \left( \nu_{sB}^\beta(\frac{\beta\hbar}{2}) - \nu_{sB}^\beta(-\frac{\beta\hbar}{2}) \right)}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)} \\
 &= -\frac{m \left[ \dot{x}_{sB}^\beta(\frac{\beta\hbar}{2}) \right]^2 \frac{d\beta\hbar}{dE_B}}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)}. \tag{B.39}
 \end{aligned}$$

The same conclusion about the eigenvalues can be obtained.

3. **Fluctuation determinant of shot solution  $x_S^\beta(\tau)$ :** The corresponding  $u_2(\tau)$  is given by

$$u_2(\tau) = \dot{x}_S^\beta(-\frac{\beta\hbar}{2}) \nu_S^\beta(\tau) - \dot{x}_S^\beta(\tau) \nu_S^\beta(-\frac{\beta\hbar}{2}).$$

Then the ratio can be expressed as

$$\begin{aligned}
\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_S) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F) \right]} &= \frac{\dot{x}_S^\beta(-\frac{\beta\hbar}{2}) \nu_S^\beta(\frac{\beta\hbar}{2}) - \dot{x}_S^\beta(\frac{\beta\hbar}{2}) \nu_S^\beta(-\frac{\beta\hbar}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)} \\
&= -\frac{2\dot{x}_S^\beta(\frac{\beta\hbar}{2}) \nu_S^\beta(\frac{\beta\hbar}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)} \\
&= \frac{-\frac{\partial S_S}{\partial E_S}}{\frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)} \cdot \leftarrow \boxed{\text{Eq. (3.26)}} \quad (\text{B.40})
\end{aligned}$$

where we have used  $\dot{x}_S^\beta(\frac{\beta\hbar}{2}) = -\dot{x}_S^\beta(-\frac{\beta\hbar}{2})$ . The sign of  $\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_S) \right]$  depends on the sign of  $\frac{\partial S_S}{\partial E_S}$ .

### Periodic Boundary Conditions

When considering the ratio of two fluctuation determinants with periodic boundary conditions, both  $u_1$  and  $u_2$  are required. Here, the ratio between the fluctuation determinant of the bounce solution and the false vacuum solution will be derived.  $u_1$  and  $u_2$  relating to the fluctuation determinants of the false vacuum solution and the bounce solution have already been given in the previous part.

Since the time derivative of the zero eigenfunction of the bounce solution is not zero :  $\dot{\varphi}_0(-\frac{\beta\hbar}{2}) = \frac{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})}{\|\dot{x}_B^\beta(\tau)\|} \neq 0$  at finite temperature, Eq.(B.30) can be used to obtain the ratio without the zero eigenvalue as :

$$\begin{aligned}
&\frac{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\beta) \right]}{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_F^\beta) \right]} \\
&= \frac{\int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau \begin{vmatrix} \dot{x}_B^\beta(\tau) \dot{\nu}_B^\beta(-\frac{\beta\hbar}{2}) - \ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \nu_B^\beta(\tau) & \frac{\dot{x}_B^\beta(\tau)}{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})} \\ \ddot{x}_B^\beta(\frac{\beta\hbar}{2}) \dot{\nu}_B^\beta(-\frac{\beta\hbar}{2}) - \ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \dot{\nu}_B^\beta(\frac{\beta\hbar}{2}) & \frac{\ddot{x}_B^\beta(\frac{\beta\hbar}{2})}{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})} - 1 \end{vmatrix}}{\frac{\ddot{x}_B^\beta(-\frac{\beta\hbar}{2})}{\|\dot{x}_B^\beta(\tau)\|}} \left| \frac{\dot{x}_B^\beta(\tau)}{\|\dot{x}_B^\beta(\tau)\|} \right| \\
&= \frac{(1 - \cosh(\omega_F \beta\hbar)) (1 - \cosh(\omega_F \beta\hbar)) - \omega_F \cosh(\omega_F \beta\hbar) \frac{1}{\omega_F} \sinh(\omega_F \beta\hbar)}{\frac{(\dot{\nu}_B^\beta(-\frac{\beta\hbar}{2}) - \dot{\nu}_B^\beta(\frac{\beta\hbar}{2}))}{\ddot{x}_B^\beta(\frac{\beta\hbar}{2})} \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} d\tau [\dot{x}_B^\beta(\tau)]^2} \\
&= -\frac{2 - 2 \cosh(\omega_F \beta\hbar)}{m \frac{d\beta\hbar}{dE_B} \|\dot{x}_B^\beta(\tau)\|^2} \\
&= \frac{m \frac{d\beta\hbar}{dE_B} \|\dot{x}_B^\beta(\tau)\|^2}{2 - 2 \cosh(\omega_F \beta\hbar)}.
\end{aligned}$$

## C. Limit Relations

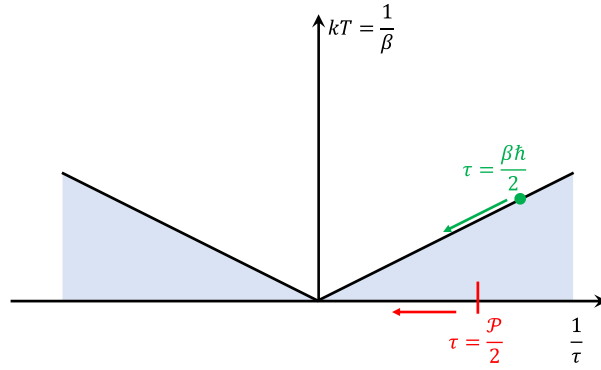


Figure C.1.: Two Types of Limits

There are two important limits to consider:

1. **Infinite time limit at zero temperature:** When focusing on the zero-temperature scenario, direct application of boundary values isn't feasible, as they either become zero or infinity. In order to complete the computation of the ratio of two determinants, the understanding of the behavior of  $u_2(\tau)$  at a large time  $\mathcal{T}$  is necessary.
2. **Zero-temperature limit of results obtained at finite temperature:** In this case, results at finite temperature are computed using boundary values. In order to take the zero-temperature limit, the relationships between those boundary values need to be established.

### C.1. Infinite Time Limit at Zero Temperature

Here, some important results at zero temperature will be introduced following [76].

### C.1.1. The Behavior of Bounce Solutions at Large Time

First, consider the behavior of the bounce solution  $x_B(\tau)$  at larger time. Since  $x_B(\tau)$  is a classical solution, the elapsed time can be written as the integral:

$$\begin{aligned}\tau &= - \int_{x_R}^{x_B(\tau)} \frac{1}{\sqrt{\frac{2}{m}(V(x_B) - V(x_F))}} dx_B \\ &= - \frac{1}{\omega_F} \int_{x_R}^{x_B(\tau)} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(x_B) - V(x_F))}} - \frac{1}{|x_B - x_F|} \right) dx_B + \frac{1}{\omega_F} \ln \frac{x_R - x_F}{x_B(\tau) - x_F},\end{aligned}$$

where  $\omega_F \equiv \sqrt{\frac{V''(x_F)}{m}}$ . Therefore, the following relation can be derived.

$$(x_B(\tau) - x_F) e^{\omega_F \tau} = (x_R - x_F) e^{\int_{x_B(\tau)}^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right)} \equiv \frac{\sqrt{S_B}}{\omega_F} A(\tau), \quad (\text{C.1})$$

where  $S_B$  is the action of bounce solution at zero temperature and  $A(\tau)$  is defined as

$$A(\tau) \equiv \frac{\omega_F}{\sqrt{S_B}} (x_R - x_F) e^{\int_{x_B(\tau)}^{x_R} d\bar{x} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right)}.$$

It is apparent that  $A(\tau)$  is finite for potentials related to our research when  $\tau \rightarrow \infty$ . Taking the derivative of Eq.(C.1) with respect to time leads to

$$\dot{x}_B(\tau) = \frac{-\omega_F \frac{\sqrt{S_B}}{\omega_F} A(\tau) e^{-\omega_F \tau}}{1 + e^{-\omega_F \tau} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(x_B(\tau)) - V(x_F))}} - \frac{1}{|x_B(\tau) - x_F|} \right) \frac{\sqrt{S_B}}{\omega_F} A(\tau)}.$$

When  $\tau \rightarrow \infty$ , it is easy to obtain

$$\dot{x}_B(\tau) \rightarrow -\sqrt{S_B} A(\tau) e^{-\omega_F \tau}. \quad (\text{C.2})$$

In fact, if you observe that  $A(\tau)$  becomes almost constant when  $\tau$  is large, Eq.(C.2) can be derived more easily. Since we are only interested in the limit  $\tau \rightarrow +\infty$ ,  $A(\tau)$  can just be replaced by  $A(\infty)$ . Therefore,  $\dot{x}_B(\tau) \rightarrow -\sqrt{S_B} A(\infty) e^{-\omega_F \tau}$  as  $\tau \rightarrow +\infty$ .

Then the behavior of the other solution  $\nu_B(\tau)$  at large time is simple.  $\nu_B(\tau)$  is another independent solution to the equation

$$\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B(\tau)) \right] \nu_B(\tau) = 0.$$

In order to satisfy the previous equation as well as the Wronskian

$$\dot{x}_B(\tau) \dot{\nu}_B(\tau) - \ddot{x}_B(\tau) \nu_B(\tau) = 1,$$

$\nu_B(\tau)$  should behave like

$$\nu_B(\tau) \rightarrow -\frac{1}{2\omega_F A(\infty) \sqrt{S_B}} e^{\omega_F \tau} \quad (\text{C.3})$$

as  $\tau \rightarrow +\infty$ .

### C.1.2. The Ratio of the Fluctuation Determinants

At zero temperature, to calculate the imaginary part of the energy, it is necessary to compute the ratio of the fluctuation determinants for the bounce solution  $x_B(\tau)$  and the false vacuum solution  $x_F(\tau) = x_F$ . One can consider the behavior of this ratio over a large time interval  $[-\frac{\mathcal{P}}{2}, \frac{\mathcal{P}}{2}]$ , and then take  $\mathcal{P}$  to infinity to get the final answer. First, choose a large  $\mathcal{P}$  and extract the bounce and false vacuum solutions over the interval  $[-\frac{\mathcal{P}}{2}, \frac{\mathcal{P}}{2}]$  from the entire solution. Then, the ratio of the fluctuation determinants for the partial solutions becomes

$$\frac{\det \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\mathcal{P}) \right]}{\det \left[ -\partial_\tau^2 + \omega_F^2 \right]} = \frac{u_2^\mathcal{P}(\frac{\mathcal{P}}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \mathcal{P})},$$

where  $u_2^\mathcal{P}(\tau)$  is defined as

$$u_2^\mathcal{P}(\tau) = \dot{x}_B^\mathcal{P}(-\frac{\mathcal{P}}{2}) \nu_B^\mathcal{P}(\tau) - \dot{x}_B^\mathcal{P}(\tau) \nu_B^\mathcal{P}(-\frac{\mathcal{P}}{2}).$$

The behaviors of  $x_B(\tau)$  and  $\nu_B(\tau)$  at larger time have already been derived as

$$\dot{x}_B(\tau) \rightarrow -\sqrt{S_B} A(\tau) e^{-\omega_F \tau} \quad (\text{C.4})$$

and

$$\nu_B(\tau) \rightarrow -\frac{1}{2\omega_F A(\infty) \sqrt{S_B}} e^{\omega_F \tau}.$$

Therefore,  $u_2(\frac{\mathcal{P}}{2})$  becomes a constant at large time when  $\mathcal{P}$  is large.

$$u_2^\mathcal{P}(\frac{\mathcal{P}}{2}) = \dot{x}_B^\mathcal{P}(-\frac{\mathcal{P}}{2}) \nu_B^\mathcal{P}(\frac{\mathcal{P}}{2}) - \dot{x}_B^\mathcal{P}(\frac{\mathcal{P}}{2}) \nu_B^\mathcal{P}(-\frac{\mathcal{P}}{2}) = -2\dot{x}_B^\mathcal{P}(\frac{\mathcal{P}}{2}) \nu_B^\mathcal{P}(\frac{\mathcal{P}}{2}) = -\frac{1}{\omega_F}.$$

It should be emphasized here that  $u_2^\mathcal{P}(\frac{\mathcal{P}}{2})$  is a non-zero constant with no  $\mathcal{P}$  dependence.

Since only part of the solution  $x_B^\mathcal{P}(\tau)$  are taken from the full solution  $x_B(\tau)$ , even if  $\dot{x}_B^\mathcal{P}(\tau)$  satisfy the eigenequation  $[-\partial_\tau^2 + \frac{1}{m} V''(x_B^\mathcal{P})] \dot{x}_B^\mathcal{P}(\tau) = 0$ , it doesn't satisfy the Dirichlet boundary conditions :

$$\dot{x}_B^\mathcal{P}(\pm \frac{\mathcal{P}}{2}) \neq 0.$$

Therefore, although  $[-\partial_\tau^2 + \frac{1}{m} V''(x_B^\mathcal{P}(\tau))]$  has a negative eigenvalue, which can be speculated from the ratio of determinants, it doesn't have a zero eigenvalue, which can also be seen from the ratio. While there is no zero eigenvalue, a small positive eigenvalue  $\lambda_1^\mathcal{P}$  does exist and approaches 0 as  $\mathcal{P}$  goes to infinity. The small positive eigenvalue  $\lambda_1^\mathcal{P}$  should be excluded from the ratio of determinants, which means the ratio actually needed to be calculated is

$$\frac{\det' \left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^\mathcal{P}(\tau)) \right]}{\det \left[ -\partial_\tau^2 + \omega_F^2 \right]} = \frac{1}{\lambda_1^\mathcal{P}} \frac{u_2(\frac{\mathcal{P}}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \mathcal{P})},$$

where  $\det'$  means excluding  $\lambda_1^{\mathcal{P}}$ .

$\lambda_1^{\mathcal{P}}$  can be calculated through the method proposed by [76]. Denoting the eigenfunction of  $\lambda_1^{\mathcal{P}}$  as  $\varphi_1^{\mathcal{P}}$ , then

$$\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^{\mathcal{P}}) \right] \varphi_1^{\mathcal{P}} = \lambda_1^{\mathcal{P}} \varphi_1^{\mathcal{P}}.$$

According to [76], when  $\mathcal{P}$  is large,  $\varphi_1^{\mathcal{P}}(\tau)$  can be written as

$$\varphi_1^{\mathcal{P}}(\tau) = u_2^{\mathcal{P}}(\tau) + \lambda_1^{\mathcal{P}} \varphi^{\mathcal{P}}(\tau) + \mathcal{O}\left((\lambda_1^{\mathcal{P}})^2\right).$$

At  $\mathcal{O}(\lambda_1^{\mathcal{P}})$ , the eigenequation is

$$\left[ -\partial_\tau^2 + \frac{1}{m} V''(x_B^{\mathcal{P}}) \right] \varphi^{\mathcal{P}}(\tau) = u_2^{\mathcal{P}}(\tau)$$

and the solution can be solved as

$$\varphi^{\mathcal{P}}(\tau) = - \int_{-\frac{\mathcal{P}}{2}}^{\tau} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}(\tau) - \dot{x}_B^{\mathcal{P}}(\tau) \nu_B^{\mathcal{P}}(\mu) \right) u_2^{\mathcal{P}}(\mu).$$

Therefore,

$$\varphi_1^{\mathcal{P}}(\tau) = u_2^{\mathcal{P}}(\tau) - \lambda_1^{\mathcal{P}} \int_{-\frac{\mathcal{P}}{2}}^{\tau} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}(\tau) - \dot{x}_B^{\mathcal{P}}(\tau) \nu_B^{\mathcal{P}}(\mu) \right) u_2^{\mathcal{P}}(\mu) + \mathcal{O}\left((\lambda_1^{\mathcal{P}})^2\right).$$

Since  $\varphi_1^{\mathcal{P}}(\tau)$  is the eigenfunction of  $\lambda_1^{\mathcal{P}}$ , it must satisfy the Dirichlet boundary conditions. From  $\varphi_1^{\mathcal{P}}(\frac{\mathcal{P}}{2}) = 0$ , the following relation can be obtain

$$\lambda_1^{\mathcal{P}} = \frac{u_2^{\mathcal{P}}(\frac{\mathcal{P}}{2})}{\int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) - \dot{x}_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \nu_B^{\mathcal{P}}(\mu) \right) u_2^{\mathcal{P}}(\mu)}.$$

Since  $\mathcal{P}$  is larger, using the parity of  $\dot{x}_B(\tau)$  and  $\nu_B(\tau)$ , it can easily derived that

$$\lambda_1^{\mathcal{P}} = \frac{\frac{1}{\omega_F}}{\int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu \left( \left[ \nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \dot{x}_B(\mu) \right]^2 - \left[ \dot{x}_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \nu_B^{\mathcal{P}}(\mu) \right]^2 \right)}.$$

Considering the behavior of  $\dot{x}_B(\tau)$  and  $\nu_B(\tau)$  at large time,  $\left[ \dot{x}_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \nu_B^{\mathcal{P}}(\mu) \right]^2$  is negligible compared to the first term, then

$$\lambda_1 = \lim_{\mathcal{P} \rightarrow \infty} \frac{\frac{1}{\omega_F}}{\int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu \left[ \nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \dot{x}_B(\mu) \right]^2} = \lim_{\mathcal{P} \rightarrow \infty} \frac{\frac{1}{\omega_F}}{\left[ \nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \right]^2 \|\dot{x}_B(\tau)\|^2} = 4m\omega_F A^2(\infty) e^{-\omega_F \mathcal{P}}.$$

We here explain the second method to derive  $\lambda_1$ .  $\frac{\dot{x}_B(\tau)}{\|\dot{x}_B(\tau)\|}$  is already known as the eigenfunction with zero eigenvalue when  $\mathcal{P}$  goes to infinity. So when  $\mathcal{P}$  is large but finite, the eigenfunction corresponding to  $\lambda_1^{\mathcal{P}}$  should be written as

$$\varphi_1^{\mathcal{P}}(\tau) = \frac{\dot{x}_B(\tau)}{\|\dot{x}_B(\tau)\|} + \lambda_1^{\mathcal{P}} \varphi^{\mathcal{P}}(\tau) + \mathcal{O}\left((\lambda_1^{\mathcal{P}})^2\right)$$

Solving the eigenequation at  $\mathcal{O}(\lambda_1^{\mathcal{P}})$  and the following relation can be obtained

$$\varphi^{\mathcal{P}}(\tau) = - \int_{\tau}^{\tau} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}(\tau) - \dot{x}_B^{\mathcal{P}}(\tau) \nu_B^{\mathcal{P}}(\mu) \right) \frac{\dot{x}_B(\mu)}{\|\dot{x}_B(\tau)\|},$$

where there is no need to determine the lower bound now. Since  $\varphi_1^{\mathcal{P}}(\tau)$  must satisfy the Dirichlet boundary conditions :

$$0 = \varphi_1^{\mathcal{P}}\left(-\frac{\mathcal{P}}{2}\right) = \frac{\dot{x}_B(-\frac{\mathcal{P}}{2})}{\|\dot{x}_B(\tau)\|} - \lambda_1 \int_{\tau}^{-\frac{\mathcal{P}}{2}} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}\left(-\frac{\mathcal{P}}{2}\right) - \dot{x}_B^{\mathcal{P}}\left(-\frac{\mathcal{P}}{2}\right) \nu_B^{\mathcal{P}}(\mu) \right) \frac{\dot{x}_B(\mu)}{\|\dot{x}_B(\tau)\|}$$

and

$$0 = \varphi_1^{\mathcal{P}}\left(\frac{\mathcal{P}}{2}\right) = \frac{\dot{x}_B(\frac{\mathcal{P}}{2})}{\|\dot{x}_B(\tau)\|} - \lambda_1 \int_{\tau}^{\frac{\mathcal{P}}{2}} d\mu \left( \dot{x}_B^{\mathcal{P}}(\mu) \nu_B^{\mathcal{P}}\left(\frac{\mathcal{P}}{2}\right) - \dot{x}_B^{\mathcal{P}}\left(\frac{\mathcal{P}}{2}\right) \nu_B^{\mathcal{P}}(\mu) \right) \frac{\dot{x}_B(\mu)}{\|\dot{x}_B(\tau)\|}.$$

Therefore,

$$\lambda_1^{\mathcal{P}} = \frac{\dot{x}_B(\frac{\mathcal{P}}{2}) - \dot{x}_B(-\frac{\mathcal{P}}{2})}{\nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu [\dot{x}_B^{\mathcal{P}}(\mu)]^2 - \dot{x}_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \left( \int_{\tau}^{-\frac{\mathcal{P}}{2}} d\mu \nu_B^{\mathcal{P}}(\mu) \dot{x}_B^{\mathcal{P}}(\mu) + \int_{\tau}^{\frac{\mathcal{P}}{2}} d\mu \nu_B^{\mathcal{P}}(\mu) \dot{x}_B^{\mathcal{P}}(\mu) \right)}.$$

The second term in the denominator is negligible compared to the first term when  $\mathcal{P}$  is large. Thus,

$$\lambda_1^{\mathcal{P}} = \frac{\dot{x}_B(-\frac{\mathcal{P}}{2}) - \dot{x}_B(\frac{\mathcal{P}}{2})}{\nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu [\dot{x}_B^{\mathcal{P}}(\mu)]^2} = \frac{2\dot{x}_B(-\frac{\mathcal{P}}{2})}{\nu_B^{\mathcal{P}}(\frac{\mathcal{P}}{2}) \int_{-\frac{\mathcal{P}}{2}}^{\frac{\mathcal{P}}{2}} d\mu \dot{x}_B^2(\mu)}.$$

After using the asymptotic forms of  $\dot{x}_B(\tau)$  and  $\nu_B(\tau)$  at large time, the same result can be derived as

$$\lambda_1 = 4m\omega_F A^2(\infty) e^{-\omega_F \mathcal{P}}.$$

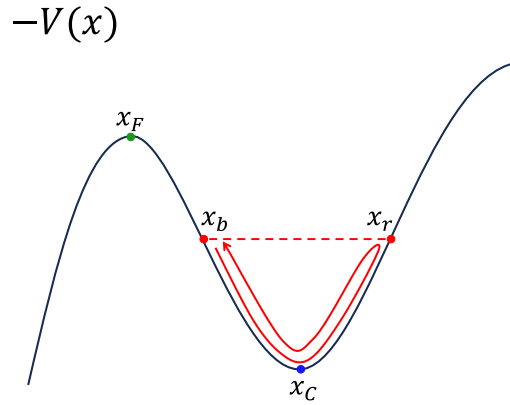
The ratio at zero temperature can be expressed as

$$\begin{aligned} \frac{\det' \left[ -\partial_{\tau}^2 + \frac{1}{m} V''(x_B) \right]}{\det \left[ -\partial_{\tau}^2 + \omega_F^2 \right]} &= \lim_{\mathcal{P} \rightarrow \infty} \frac{1}{\lambda_1} \frac{u_2(\frac{\mathcal{P}}{2})}{\frac{1}{\omega_F} \sinh(\omega_F \mathcal{P})} \\ &= \lim_{\mathcal{P} \rightarrow \infty} \frac{1}{4m\omega_F A^2(\infty) e^{-\omega_F \mathcal{P}}} \frac{-\frac{1}{\omega_F}}{\frac{1}{\omega_F} \sinh(\omega_F \mathcal{P})} \\ &= -\frac{1}{2m\omega_F A^2(\infty)}. \end{aligned}$$

## C.2. Zero-temperature Limit

In this section, we would like to present some limit relations obtained at finite temperature and then discuss the zero-temperature limit of these results. At finite temperature, the time interval is expressed as  $[-\frac{\beta\hbar}{2}, \frac{\beta\hbar}{2}]$ .

### C.2.1. Relation Relating to the Bounce Solution



**Figure C.2.:** Euclidean Potential

The end points and the turning point of the bounce solution are  $x_b$  and  $x_r$ . The relation between the period  $\beta\hbar$  and the endpoint  $x_b$  can be expressed as

$$\beta\hbar = 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_b))}} d\bar{x}. \quad (\text{C.5})$$

Here, the period is considered as a monotonic function of  $x_b$ .

When  $x_b$  approaches  $x_F$ ,  $V'(x_b)$  approaches zero while the period  $\beta\hbar$  approaches infinity. The period can be rewritten as

$$\begin{aligned} \beta\hbar = & 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_b))}} - \frac{1}{\sqrt{\frac{2}{m} (V'(x_b)(x - x_b) + \frac{1}{2}V''(x_b)(x - x_b)^2)}} d\bar{x} \\ & + 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m} (V'(x_b)(x - x_b) + \frac{1}{2}V''(x_b)(x - x_b)^2)}} d\bar{x} \end{aligned}$$

following the method used in calculation of the infinite time limit at zero temperature (see [113]). The integration in the additional term can be performed and the result

is

$$\begin{aligned}
 \text{A.T.} &= 2 \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m} \left( V'(x_b)(x - x_b) + \frac{1}{2} V''(x_b)(x - x_b)^2 \right)}} d\bar{x} \\
 &= 2 \sqrt{\frac{m}{V''(x_b)}} \ln \left| \frac{x_r - x_b + \frac{V'(x_b)}{V''(x_b)} + \sqrt{\left( x_r - x_b + \frac{V'(x_b)}{V''(x_b)} \right)^2 - \left[ \frac{V'(x_b)}{V''(x_b)} \right]^2}}{\left| \frac{V'(x_b)}{V''(x_b)} \right|} \right|.
 \end{aligned}$$

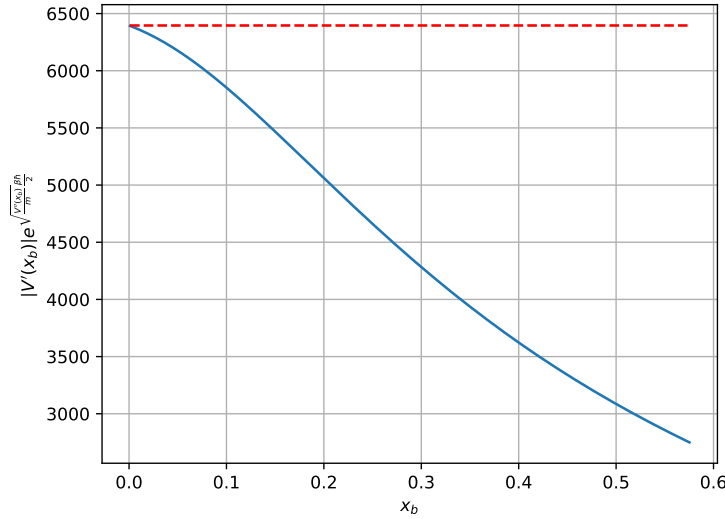
Rewrite both sides into exponential form and the following result can be obtained.

$$\begin{aligned}
 &|V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}} \\
 &= |V''(x_b)| \left| x_r - x_b + \frac{V'(x_b)}{V''(x_b)} + \sqrt{\left( x_r - x_b + \frac{V'(x_b)}{V''(x_b)} \right)^2 - \left[ \frac{V'(x_b)}{V''(x_b)} \right]^2} \right| \quad (\text{C.6})
 \end{aligned}$$

$$\cdot e^{\sqrt{\frac{V''(x_b)}{m}} \int_{x_b}^{x_r} \frac{1}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_b))}} - \frac{1}{\sqrt{\frac{2}{m} \left( V'(x_b)(x - x_b) + \frac{1}{2} V''(x_b)(x - x_b)^2 \right)}} d\bar{x}}. \quad (\text{C.7})$$

When  $x_b \rightarrow x_F$ ,  $\beta \hbar \rightarrow \infty$  and  $x_r \rightarrow x_R$ . Therefore, the relation at zero temperature limit becomes

$$\lim_{x_b \rightarrow x_F} |V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}} = 2 |V''(x_F)| |x_R - x_F| e^{\int_{x_F}^{x_R} \left( \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m} (V(\bar{x}) - V(x_F))}} - \frac{1}{|\bar{x} - x_F|} \right) d\bar{x}}}. \quad (\text{C.8})$$



**Figure C.3.:**  $|V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}}$

We here introduce the second method to derive Eq.(C.7). The integral Eq.(C.5) can be splitted into two parts:

$$\beta\hbar = 2 \int_{x_b}^{x_b+\Delta} \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx + 2 \int_{x_b+\Delta}^{x_r} \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx,$$

where  $\Delta$  is small. Since  $x_b \rightarrow x_F$ , the first can be calculated by expanding  $V(x)$  around  $x_F$  like

$$\begin{aligned} \text{FT} &= \lim_{x_b \rightarrow x_F} 2 \int_{x_b}^{x_b+\Delta} \frac{1}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx \\ &= \lim_{x_b \rightarrow x_F} 2 \int_{x_b}^{x_b+\Delta} \frac{1}{\sqrt{\frac{2}{m} \left( \frac{1}{2} V''(x_F)(x - x_F)^2 - \frac{1}{2} V''(x_F)(x_b - x_F)^2 \right)}} dx \\ &= \lim_{x_b \rightarrow x_F} \frac{2}{\sqrt{\frac{V''(x_F)}{m}}} \ln \frac{x_b + \Delta - x_F + \sqrt{(x_b + \Delta - x_F)^2 - (x_b - x_F)^2}}{x_b - x_F} \end{aligned}$$

By replacing the result into the total integral and rewriting both sides into exponential function, the following relation can be obtained

$$\begin{aligned} &\lim_{x_b \rightarrow x_F} e^{\sqrt{\frac{V''(x_F)}{m}} \frac{\beta\hbar}{2}} \\ &= \lim_{x_b \rightarrow x_F} e^{\int_{x_b+\Delta}^{x_r} \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx + \ln \frac{x_b + \Delta - x_F + \sqrt{(x_b + \Delta - x_F)^2 - (x_b - x_F)^2}}{x_b - x_F}} \\ &= \lim_{x_b \rightarrow x_F} \frac{e^{\int_{x_b+\Delta}^{x_r} \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} dx + \ln \left( x_b + \Delta - x_F + \sqrt{(x_b + \Delta - x_F)^2 - (x_b - x_F)^2} \right)}}{x_b - x_F}. \end{aligned}$$

The  $\ln$  function in the exponent on the right side can be expressed as

$$\ln \left( x_r - x_F + \sqrt{(x_r - x_F)^2 - (x_b - x_F)^2} \right) - \int_{x_b+\Delta}^{x_r} \frac{1}{\sqrt{(x - x_F)^2 - (x_b - x_F)^2}} dx.$$

As a result, the exponent on the right side becomes

$$\begin{aligned} &\int_{x_b}^{x_r} \left( \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} - \frac{1}{\sqrt{(x - x_F)^2 - (x_b - x_F)^2}} \right) dx \\ &\quad + \ln \left( x_r - x_F + \sqrt{(x_r - x_F)^2 - (x_b - x_F)^2} \right) \end{aligned}$$

and the right side can be expressed as

$$\frac{x_r - x_F + \sqrt{(x_r - x_F)^2 - (x_b - x_F)^2}}{x_b - x_F} e^{\int_{x_b}^{x_r} \left( \frac{\sqrt{\frac{V''(x_F)}{m}}}{\sqrt{\frac{2}{m}(V(x) - V(x_b))}} - \frac{1}{\sqrt{(x - x_F)^2 - (x_b - x_F)^2}} \right) dx}.$$

The final result is

$$\lim_{x_b \rightarrow x_F} (x_b - x_F)^2 e^{\sqrt{\frac{V''(x_F)}{m}} \beta \hbar} = 4 (x_R - x_F)^2 e^{2 \int_{x_F}^{x_R} \left( \frac{\omega_F}{\sqrt{\frac{2}{m}(V(x) - V(x_F))}} - \frac{1}{|x - x_F|} \right) dx},$$

which is the same as Eq.(C.8).

Since  $x_b \rightarrow x_F$  leading to an infinite  $\beta \hbar$ ,  $\frac{d\beta \hbar}{dx_b}$  should be infinite as well. The limit relation of  $\beta \hbar$  and  $x_b$  can be used to attain a relation relating to  $\frac{d\beta \hbar}{dx_b}$  by L'Hôpital's rule. Consider the ratio between  $\frac{\beta \hbar}{\ln V'(x_b)}$ , it is an  $\frac{\infty}{\infty}$  type limit and can be transform as

$$\lim_{x_b \rightarrow x_F} \frac{\beta \hbar}{\ln V'(x_b)} = \lim_{x_b \rightarrow x_F} \frac{\frac{d\beta \hbar}{dx_b}}{\frac{V''(x_b)}{V'(x_b)}} = \frac{1}{V''(x_F)} \lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta \hbar}{dx_b}.$$

With the help of the relation (C.8), it is apparent that

$$\lim_{x_b \rightarrow x_F} \frac{\beta \hbar}{\ln V'(x_b)} = \lim_{x_b \rightarrow x_F} \frac{\beta \hbar}{\ln \left( |V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}} \right) - \sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}} = -2 \sqrt{\frac{m}{V''(x_F)}}$$

since  $|V'(x_b)| e^{\sqrt{\frac{V''(x_b)}{m}} \frac{\beta \hbar}{2}}$  is finite. Therefore, the following limit relation can be obtained.

$$\lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta \hbar}{dx_b} = -2 \sqrt{m V''(x_F)}. \quad (\text{C.9})$$

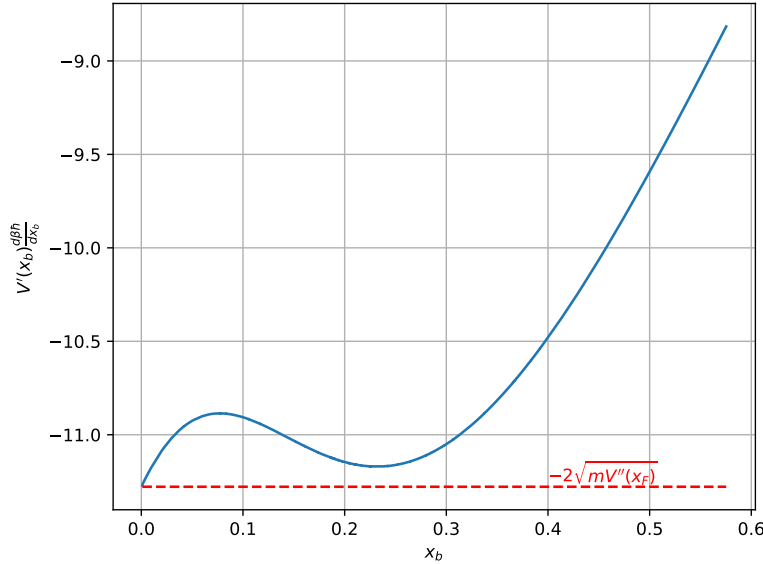


Figure C.4.:  $V'(x_b) \frac{d\beta \hbar}{dx_b}$

We here introduce another method to derive the limit relation of  $V'(x_b) \frac{d\beta\hbar}{dx_b}$ . By taking the derivative of both sides of Eq.(C.5) and using the relation of  $V'(x_b) = V'(x_r) \frac{dx_r}{dx_b}$ , we can get

$$\begin{aligned}
\frac{d\beta\hbar}{dx_b} &= 2 \lim_{\bar{x} \rightarrow x_r} \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} \frac{dx_r}{dx_b} - 2 \lim_{\bar{x} \rightarrow x_b} \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} \\
&\quad + 2 \int_{x_b}^{x_r} \frac{\frac{1}{2} \frac{2}{m} V'(x_b)}{\left[ \frac{2}{m}(V(\bar{x}) - V(x_b)) \right]^{\frac{3}{2}}} d\bar{x} \\
&= V'(x_b) 2 \left[ \frac{1}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} \frac{1}{V'(\bar{x})} \right]_{x_b}^{x_r} + 2 \int_{x_b}^{x_r} \frac{\frac{1}{2} \frac{2}{m} V'(x_b)}{\left[ \frac{2}{m}(V(\bar{x}) - V(x_b)) \right]^{\frac{3}{2}}} d\bar{x} \\
&= -2 \int_{x_b}^{x_r} \frac{V'(x_b)}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} \frac{V''(\bar{x})}{[V'(\bar{x})]^2} d\bar{x}. \tag{C.10}
\end{aligned}$$

Moreover, by making use of the equation of motion of bounce, Eq.(B.33) and  $\frac{d}{d\tau} \frac{\nu_B^\beta(\tau)}{\dot{x}_B^\beta(\tau)} = \frac{1}{[\dot{x}_B^\beta(\tau)]^2}$ , we can get

$$\begin{aligned}
\frac{d\beta\hbar}{dx_b} &= -2 \int_{-\frac{\beta\hbar}{2}}^0 \frac{m \ddot{x}_B^\beta(-\frac{\beta\hbar}{2})}{\dot{x}_B^\beta(\tau)} \frac{\frac{m \ddot{x}_B^\beta(\tau)}{\dot{x}_B^\beta(\tau)}}{[m \ddot{x}_B^\beta(\tau)]^2} \dot{x}_B^\beta(\tau) d\tau \\
&= -\ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \int_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} \frac{1}{\dot{x}_B^\beta(\tau)} \frac{\ddot{x}_B^\beta(\tau)}{[\ddot{x}_B^\beta(\tau)]^2} d\tau \\
&= -\ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \left[ -\frac{1}{\dot{x}_B^\beta(\tau)} \frac{1}{\ddot{x}_B^\beta(\tau)} - \frac{\nu_B^\beta(\tau)}{\dot{x}_B^\beta(\tau)} \right]_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} \\
&= \ddot{x}_B^\beta(-\frac{\beta\hbar}{2}) \left[ \frac{\nu_B^\beta(\tau)}{\ddot{x}_B^\beta(\tau)} \right]_{-\frac{\beta\hbar}{2}}^{\frac{\beta\hbar}{2}} \\
&= 2\dot{\nu}_B^\beta(\frac{\beta\hbar}{2}).
\end{aligned}$$

In conclusion, the following relation can be derived

$$\frac{d\beta\hbar}{dx_b} = -2 \int_{x_b}^{x_r} \frac{V'(x_b)}{\sqrt{\frac{2}{m}(V(\bar{x}) - V(x_b))}} \frac{V''(\bar{x})}{[V'(\bar{x})]^2} d\bar{x} = 2\dot{\nu}_B^\beta(\frac{\beta\hbar}{2}).$$

Since  $\dot{x}_B^\beta(\tau)$  and  $\nu_B^\beta(\tau)$  are solutions of  $[-\partial_\tau^2 + \frac{1}{m}V''(x_B^\beta(\tau))] \varphi(\tau) = 0$  and the original equation in the neighborhood of  $\tau = \frac{\beta\hbar}{2}$  is approximately  $[-\partial_\tau^2 + \omega_F^2] \varphi(\tau) = 0$  when  $\beta\hbar \rightarrow \infty$ ,  $\dot{x}_B^\beta(\tau)$  behaves like  $Ce^{-\omega_F\tau}$  and  $\nu_B^\beta(\tau)$  behaves like  $De^{\omega_F\tau}$ . Therefore, the behavior of  $\dot{\nu}_B^\beta(\tau)$  can be speculated as  $\dot{\nu}_B^\beta(\tau) \sim \omega_F \nu_B^\beta(\tau)$  when  $\beta\hbar \rightarrow \infty$ .

$V'(x_b) \frac{d\beta\hbar}{dx_b}$  can then be written as

$$\lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta\hbar}{dx_b} = \lim_{\beta\hbar \rightarrow \infty} 2m\ddot{x}_B\left(\frac{\beta\hbar}{2}\right) \dot{\nu}_B\left(\frac{\beta\hbar}{2}\right) = \lim_{\beta\hbar \rightarrow \infty} 2m\omega_F \dot{x}_B\left(\frac{\beta\hbar}{2}\right) \nu_B\left(\frac{\beta\hbar}{2}\right).$$

The relation between  $x_B^\beta(\tau)$  and  $\nu_B^\beta(\tau)$  has already been known as  $\dot{x}_B^\beta(\tau) \dot{\nu}_B^\beta(\tau) - \ddot{x}_B^\beta(\tau) \nu_B^\beta(\tau) = 1$  and  $\dot{x}_B^\beta(\frac{\beta\hbar}{2}) = 0$ . Then the following relation is easy to obtain.

$$\lim_{x_b \rightarrow x_F} V'(x_b) \frac{d\beta\hbar}{dx_b} = -2m\omega_F. \quad (\text{C.11})$$



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