INFINITELY MANY INVEX FUNCTIONS WITHOUT CONVEXITY

Shiraishi, Shunsuke
Faculty of Applied Information Science, Hiroshima Institute of Technology

Obata, Tsuneshi
Otemon Gakuin University

Yokoyama, Kazunori
Faculty of Economics, University of Toyama

https://doi.org/10.5109/7358369

出版情報：Bulletin of informatics and cybernetics. 57 (2), pp.1-14, 2025. 統計科学研究会
バージョン：
権利関係：

INFINITELY MANY INVEX FUNCTIONS WITHOUT CONVEXITY

by

Shunsuke Shiraishi, **Tsuneshi Obata** and **Kazunori Yokoyama**

Reprinted from the Bulletin of Informatics and Cybernetics
Research Association of Statistical Sciences, Vol.57, No. 2

FUKUOKA, JAPAN
2025

Bulletin of Informatics and Cybernetics, Vol. 57, No. 2, 2025

INFINITELY MANY INVEX FUNCTIONS WITHOUT CONVEXITY

By

Shunsuke Shiraishi[*], **Tsuneshi Obata**[†] and **Kazunori Yokoyama**[‡]

Abstract

Invex functions form a broader class of differentiable functions than convex functions, ensuring global optimality at stationary points. However, the theoretical properties of invex functions, particularly those that drop convexity, remain underexplored. In this paper, we investigate these properties, providing examples that include both smooth and nonsmooth cases, and demonstrate the existence of infinitely many invex functions that are not necessarily convex. An illustrative example using a characteristic polynomial highlights the critical role of consistency in a 4th-order pairwise comparison matrix in determining invexity. Our findings expand the theoretical understanding of invexity and suggest potential enhancements in optimization techniques, enabling the analysis of a broader class of functions beyond convexity.

Key Words and Phrases: Invex function, convex function, AHP, nonsmooth function, optimization

1. Preliminary results

Optimization problems often rely on convexity to ensure global optimality; however, many real-world problems involve functions that do not satisfy convexity. Although invex functions generalize convexity and provide global optimality for stationary points, there remains a gap in understanding their broader theoretical properties and practical implications. In particular, the distinction between invexity and convexity requires further exploration, particularly in cases involving nonsmooth or higher-dimensional functions.

Understanding the properties of invex functions has significant implications for optimization theory, as it opens avenues for solving problems that cannot be addressed using convexity alone. This study is motivated by the need to expand the theoretical framework of invexity and to explore its potential applications, such as in mathematical economics and other fields where convexity limitations arise.

[*] Faculty of Applied Information Science, Hiroshima Institute of Technology, 2-1-1 Miyake, Saeki-ku, Hiroshima 731-5193, Japan. tel +81–82–921–4138 s.shiraishi.wx@it-hiroshima.ac.jp, shunke.shira@gmail.com

[†] Otemon Gakuin University, 2-1-15 Nishiai, Ibaraki, Osaka 567-8502, Japan. tel +81–72–665–5194 t-obata@haruka.otemon.ac.jp

[‡] Faculty of Economics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan. tel +81–76–445–6468 kazu@eco.u-toyama.ac.jp

This paper aims to investigate the properties of invex functions by 1. Constructing examples of invex functions, including both smooth and nonsmooth cases. 2. Demonstrating the existence of infinitely many invex functions that are not convex. 3. Highlighting the role of 4th-order pairwise comparison matrices in determining invexity. Through these objectives, the study seeks to expand the theoretical understanding of invexity and its potential applications.

We begin with the notion of convex functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex (see Tiel (1984))[1], if it satisfies

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in \mathbb{R}^n, \forall \lambda \in [0,1].$$

There are many generalizations of the concept of convex functions. In this paper, we focus on a well-known generalization of convex functions, called invex functions. For other generalizations of convexity, see Auslender (1976), Avriel (1976), Fukushima (2001). We do not consider further generalizations of invexity in this study (for definitions of generalized invexity, see, for example, Kuk et al. (2001)). However, generalized invexity has faced some criticism by Zălinescu (2014); hence, we do not focus on it in this study.

Invex functions have been applied to problems in nonlinear optimization programming, where significant results have been achieved under the assumption of invexity alone (not generalized invexity). For more details, see Ben-Israel and Mond (1986), Das et al. (2018), Hanson (1981), Jeyakumar and Mond (1992), Obata and Shiraishi (1997), Mishra and Giorgi (2008), Shiraishi (1998), Tanaka (1990), Tanaka et al. (1989) and the references therein. Recent research has explored the application of invex functions in the study of neural networks, see Sapkota and Bhattarai (2021). For a generalization of invex functions and their applications to optimization problems, see Wang and Feng (2024) and the references therein. However, the original invexity continues to be an appealing and active area of research. In this section, we revisit the definition of invex functions for smooth and nonsmooth cases, along with their basic properties.

1.1. Smooth invex functions

For smooth convex functions, the following theorems are essential tools in optimization theory. As for the following result, see Theorem 4.32 and Corollary 4.37 in Avriel (1976) and

THEOREM 1.1. *Let f be a convex function on $\mathbb{R}^n$. Then, every local minimum is a global minimum over $\mathbb{R}^n$.*

Since a local minimizer $x^* \in \mathbb{R}^n$ satisfies the first-order condition,

$$\nabla f(x^*) = 0,$$

we immediately obtain the following result:

THEOREM 1.2. *Let f be a differentiable convex function on $\mathbb{R}^n$. Then,*

$$\nabla f(x^*) = 0$$

if and only if f attains its global minimum at x^*.

[1] For extended valued functions, consult the definitions of convexity in Auslender (1976), Fukushima (2001), Rockafellar (1970)

Broadly speaking, every stationary point of a smooth convex function is a global minimizer. A natural question arises: Is a function for which every stationary point is a global minimizer necessarily convex? The answer is no. A function for which every stationary point is a global minimizer is called an invex function. The (smooth) invex function is formally defined as follows (see, Ben-Israel and Mond (1986), Hanson (1981)):

DEFINITION 1.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be invex if there exists a mapping $\eta(x, u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(u) \geq [\eta(x, u)]^\top \nabla f(u) \quad \forall x, u \in \mathbb{R}^n.$$

The following theorem (Theorem 1 in Ben-Israel and Mond (1986)) answers the aforementioned question.

THEOREM 1.4. *A function f is invex if and only if every stationary point is a global minimum.*

In Sections 2.1. and 3., we show that there exist smooth invex functions that are not necessarily convex. Moreover, in Section 4., we demonstrate that there are infinitely many invex functions that do not exhibit convexity.

The following theorem is used to determine whether a given function is not convex (see, Theorem 4.30 in Avriel (1976)).

THEOREM 1.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Then, f is convex if and only if the Hessian $\nabla^2 f$ of f is positive semidefinite at every point.*

REMARK. We can determine that a given function is non-convex if the Hessian of the function is negative at some point. This procedure can also be applied to nonsmooth functions. If the Hessian of the function is negative at a smooth point, then the function under consideration is non-convex.

1.2. Nonsmooth invex functions

Tanaka (1990), Tanaka et al. (1989) generalized the notion of invexity for nonsmooth functions. Throughout the sequel, let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous.

DEFINITION 1.6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz near x if there exists a neighborhood U of x and a nonnegative scalar K such that

$$|f(y) - f(y')| \leq K\|y - y'\|$$

for all points $y, y' \in U$.

For example, the absolute-value function $y \mapsto |y|$ is Lipschitz continuous. From the triangle inequality, we have

$$\begin{aligned} |y| &= |y - y' + y'| \\ &\leq |y - y'| + |y'|. \end{aligned}$$

Thus,

$$|y| - |y'| \leq |y - y'|.$$

The following assertion follows directly from the definition. See Theorem 4.6.14 in Sohrab (2014).

LEMMA 1.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz near x. Then, the additive function $f + g$ is also locally Lipschitz near x. If, in addition, both are bounded near x, then the product function $f \cdot g : \mathbb{R}^n \to \mathbb{R}$ is also locally Lipschitz near x.*

PROOF. By the assumption of the lemma, there exists a neighborhood U of x and scalars $L, K > 0$ such that for all $y, y' \in U$, the following inequalities hold:

$$\begin{aligned} |f(y) - f(y')| &\leq L\|y - y'\|, \\ |g(y) - g(y')| &\leq K\|y - y'\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} |(f(y) + g(y)) - (f(y') + g(y'))| &\leq |f(y) - f(y')| + |g(y) - g(y')| \\ &\leq L\|y - y'\| + K\|y - y'\|. \end{aligned}$$

Moreover, if the functions are locally bounded, there exists a neighborhood U of x and scalars $M, N > 0$ such that for all $y, y' \in U$, we have

$$\begin{aligned} |f(y')| &\leq M, \\ |g(y)| &\leq N. \end{aligned}$$

Thus, we have

$$\begin{aligned} |f(y)g(y) - f(y')g(y')| &= |f(y)g(y) - f(y')g(y) + f(y')g(y) - f(y')g(y')| \\ &\leq |(f(y) - f(y'))g(y) + f(y')(g(y) - g(y'))| \\ &\leq LN\|y - y'\| + KM\|y - y'\|. \end{aligned}$$

The following lemmas are trivial.

LEMMA 1.8. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz near x. Then, the composite function $f \circ g(x) = f(g(x))$ is also locally Lipschitz near x.*

LEMMA 1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and $g : \mathbb{R}^m \to \mathbb{R}^n$ be an affine map, where $g(y) = Ay + b$ and A is an $m \times n$ matrix. Then, the composite function $f \circ g(y) = f(Ay + b)$ is locally Lipschitz.*

Clarke (1983) defined the generalized directional derivative and the generalized gradient for a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

DEFINITION 1.10. The generalized directional derivative of f at x in the direction d is defined by

$$f^\circ(x; d) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + td) - f(y)}{t}.$$

The generalized gradient of f at x is defined by

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f^\circ(x; d) \geq \langle \xi, d \rangle \forall d \in \mathbb{R}^n\}.$$

THEOREM 1.11 PROPOSITION 2.3.2 IN CLARKE (1983). *Let f be locally Lipschitz near x. If f attains a local minimum at x, then $0 \in \partial f(x)$.*

The following theorems are due to Proposition 2.1.2 and Theorem 2.5.1 in Clarke (1983).

THEOREM 1.12. *Let f be locally Lipschitz near x. Then,*

(a) *$\partial f(x)$ is a nonempty convex compact set.*

(b) *For every d in $\mathbb{R}^n$, one has*

$$f^\circ(x;d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial f(x)\}.$$

THEOREM 1.13. *Let f be locally Lipschitz near x, and suppose S is any set of Lebesgue measure 0 in $\mathbb{R}^n$. Then,*

$$\partial f(x) = co\{\lim \nabla f(x_i) \mid x_i \to x, x_i \notin S, x_i \notin \Omega_f\},$$

where Ω_f denotes the set of points at which f fails to be differentiable.

Let $f'(x;d)$ be the one-sided directional derivative, defined by

$$f'(x;d) = \lim_{t\downarrow 0} \frac{f(x+td) - f(x)}{t}.$$

Note that both $f^\circ(x;d)$ and $f'(x;d)$ are positively homogeneous in d, see Clarke (1983), Rockafellar (1970). The regularity in Clarke's sense is defined in terms of the one-sided directional derivative.

DEFINITION 1.14. A function f is said to be regular at x if for all d in $\mathbb{R}^n$, the one-sided directional derivative exists and satisfies

$$f^\circ(x;d) = f'(x;d), \quad \forall d \in \mathbb{R}^n.$$

DEFINITION 1.15. Assume that a function f is regular at each x. The function f is said to be invex if there exists a mapping $\eta(x,u) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(u) \geq f'(u;\eta(x,u)), \quad \forall x, u \in \mathbb{R}^n.$$

The following theorem is the variant for nonsmooth functions of Theorem 1.4, see Theorem 2.1 of Tanaka et al. (1989).

THEOREM 1.16. *Let f be locally Lipschitz and regular at each point. Then, f is invex if and only if every point u such that $0 \in \partial f(u)$ is a global minimum of f.*

For a convex function f, the generalized gradient is the same as the subdifferential in the sense of convex analysis. Every convex function has the property that all stationary points are global minima. We consider the converse of this assertion. Is a function for which every stationary point is a global minimum necessarily convex?

In Section 2.3., we show that there exists a nonsmooth invex function without convexity.

2. Examples of one variable

2.1. Smooth functions of one variable

We begin with a simple example (Exercise 5.15 in Shiraishi (2014)).

Example 2.1. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f_1(x) = xe^x.$$

Then, $f_1(x)$ is invex but not convex.

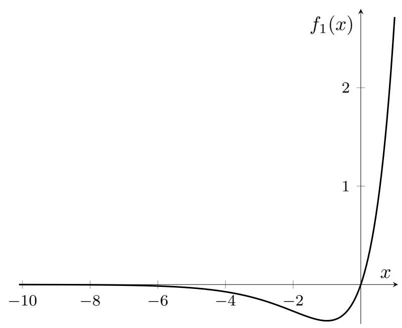

Figure 1: Graph of $f_1(x)$

Proof. The first-order derivative of $f_1(x)$ is

$$f_1'(x) = e^x(x + 1).$$

The first-order condition implies that $x = -1$ is the point where $f_1(x)$ achieves its global minimum. Therefore, $f_1(x)$ is invex.

The second-order derivative of $f_1(x)$ is

$$f_1''(x) = e^x(x + 2).$$

The second derivative $f_1''(x)$ is negative for $x < -2$. Therefore, $f_1(x)$ is not convex.

The next example is a slight modification of the characteristic polynomial of the 4th-order pairwise comparison matrix of the analytic hierarchy process (AHP)[2] , see Obata and Shiraishi (2021).

Example 2.2. Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f_2(x) = x^4 - 4x^3 - 64x$$

Then, $f_2(x)$ is invex but not convex.

[2] For details, see Brunelli (2014), Kułakowski (2021).

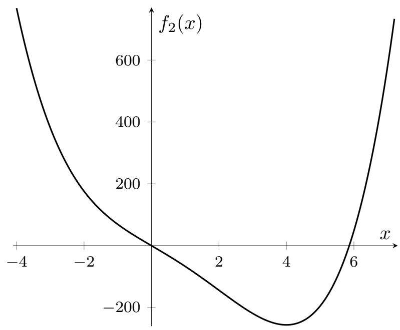

Figure 2: Graph of $f_2(x)$

Proof. If we take the first-order derivative of $f_2(x)$, we get

$$f_2'(x) = 4x^3 - 12x^2 - 64 = 4(x-4)(x^2+x+4).$$

Since $x^2+x+4 > 0$, the first-order condition implies that $x = 4$ is the unique stationary point of $f_2(x)$. The function $f_2(x)$ attains its global minimum at $x = 4$; hence, $f_2(x)$ is invex.

Next, we take the second-order derivative of $f_2(x)$, which is

$$f_2''(x) = 12x^2 - 24x = 12x(x-2).$$

The second derivative $f_2''(x)$ takes negative values when $0 < x < 2$. Hence, $f_2(x)$ is not convex.

Remark. If we define a 4×4 matrix A by

$$A = \begin{pmatrix} 1 & 4 & 1/2 & 1/7 \\ 1/4 & 1 & 1/3 & 2 \\ 2 & 3 & 1 & 3 \\ 7 & 1/2 & 1/3 & 1 \end{pmatrix},$$

then, we have the following characteristic polynomial $P_A(\lambda)$ of A:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda E - A) \\ &= \lambda^4 - 4\lambda^3 - \frac{5389}{84}\lambda + \frac{5225}{336} \\ &= \lambda^4 - 4\lambda^3 - 64.1547619\lambda + 15.55059524, \end{aligned}$$

where E denotes the identity matrix. The indicated polynomial function is a source of $f_2(x)$ in Example 2.2.

2.2. Characteristic polynomial and invexity

In this subsection, we deal with the pairwise comparison matrix of order 4:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 1/a_{12} & 1 & a_{23} & a_{24} \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & 1/a_{24} & 1/a_{34} & 1 \end{pmatrix}.$$

The characteristic polynomial $P_A(\lambda) = \det(\lambda E - A)$ of A has the following form (Obata and Shiraishi (2021)):

THEOREM 2.3. $P_A(\lambda) = \lambda^4 - 4\lambda^3 + c_3\lambda + \det A$, *where*

$$c_3 = \sum_{i<k<j} \Big(2 - \frac{a_{ik}a_{kj}}{a_{ij}} - \frac{a_{ij}}{a_{ik}a_{kj}}\Big).$$

We remark that $c_3 \leq 0$. The pairwise comparison matrix is said to be consistent if $a_{ik}a_{kj} = a_{ij}$ for all i, k, j. If this condition is not satisfied, we call the matrix A inconsistent. The consistency of the pairwise comparison matrix is determined by c_3 (see, Shiraishi et al. (1998)).

THEOREM 2.4. *The pairwise comparison matrix A is consistent if and only if $c_3 = 0$. In this case, we have $P_A(\lambda) = \lambda^4 - 4\lambda^3$.*

We can assert that the characteristic polynomial of the 4th-order pairwise comparison matrix is invex if it is inconsistent.

THEOREM 2.5. *The characteristic polynomial $P_A(\lambda)$ of the 4th-order pairwise comparison matrix A is invex if and only if A is inconsistent.*

PROOF. (If) Let A be inconsistent. Then, by Theorem 2.4, we have $c_3 < 0$. Taking the derivative of $P_A(\lambda)$, we get $P'_A(\lambda) = 4\lambda^3 - 12\lambda^2 + c_3$. Since $P'_A(0) = c_3 < 0$ and $\lim_{\lambda\to\infty} P'_A(\lambda) = \infty$, the equation $P'_A(\lambda) = 0$ has a unique root $\lambda^* > 0$, which is a stationary point of $P_A(\lambda)$, as shown in Figure 3. Hence, $P'_A(\lambda) = 4\lambda^2(\lambda - 3) + c_3 < 0$ for $\lambda < \lambda^*$, and $P'_A(\lambda) > 0$ for $\lambda > \lambda^*$. Therefore, $P_A(\lambda)$ attains a strict minimum at λ^*. Thus, $P_A(\lambda)$ is invex.

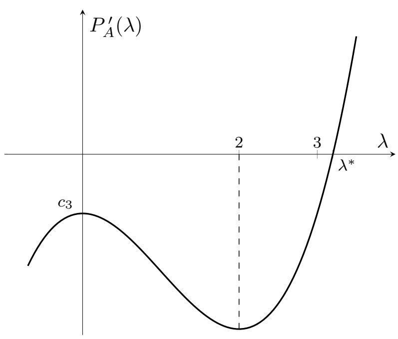

Figure 3: Graph of $P'_A(\lambda) = 4\lambda^3 - 12\lambda^2 + c_3$

(Only if) We prove by using the contrapositive. If A is consistent, then by Theorem 2.4, we have $c_3 = 0$ and $P_A(\lambda) = \lambda^4 - 4\lambda^3$. The function has two stationary points, $\lambda = 0, 4$ and $\lambda = 0$ is not a global minimum. Hence, $P_A(\lambda)$ is not invex. See Figure 4.

A 4th-order pairwise comparison matrix based on Saaty's discrete scale has a total of 17^6 = 24,137,569 possible matrices. Among these, only 343 are consistent (Obata and Shiraishi (2021)). As a result, the characteristic polynomials of 4th-order

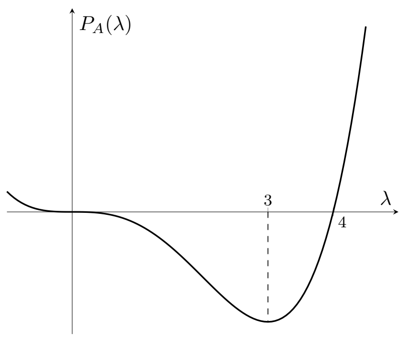

Figure 4: Graph of $P_A(\lambda) = \lambda^4 - 4\lambda^3$

pairwise comparison matrices generate a huge number of invex functions, specifically 24,137,226. However, not all of these are necessarily distinct functions; some of them may coincide. Consequently, the total number will be smaller. In Chapter 4., we will proceed to construct an infinite number of invex functions.

2.3. Nonsmooth function of one variable

EXAMPLE 2.6. Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f_3(x) = |x|e^{(x-1)^2}.$$

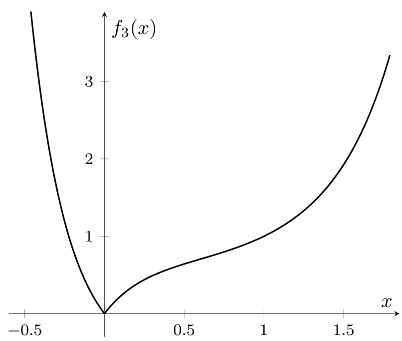

Figure 5: Graph of $f_3(x)$

LEMMA 2.7. *The function $f_3(x)$ is locally Lipschitz.*

PROOF. Since the functions $x \mapsto |x|$ and $x \mapsto e^{(x-1)^2}$ are locally Lipschitz and bounded, the assertion of the lemma follows from Lemma 1.7.

LEMMA 2.8. *The function $f_3(x)$ is regular.*

PROOF. If we take the derivative of $f_3(x)$ for $x > 0$ and $x < 0$, respectively, where the function is smooth, $f_3(x)$ has the following formulas.

$$f_3(x) = xe^{(x-1)^2} \quad \text{for} \quad x > 0,$$
$$f_3(x) = -xe^{(x-1)^2} \quad \text{for} \quad x < 0.$$

Then, we get the following expressions for the derivative:

$$f_3'(x) = e^{(x-1)^2}(2x^2 - 2x + 1) \quad \text{for} \quad x > 0,$$
$$f_3'(x) = -e^{(x-1)^2}(2x^2 - 2x + 1) \quad \text{for} \quad x < 0.$$

Hence, at the unique nonsmooth point $x = 0$ of $f_3(x)$, the generalized gradient is

$$\partial f_3(0) = \text{co}\{\lim f_3'(x_i) \mid x_i \to 0\} = \text{co}\{-e, e\} = [-e, e],$$
$$f_3^\circ(0; 1) = \max_{\xi \in [-e,e]} \xi \cdot 1 = e,$$
$$f_3^\circ(0; -1) = \max_{\xi \in [-e,e]} \xi \cdot (-1) = e.$$

Furthermore, we have

$$f_3'(0; 1) = \lim_{t \downarrow 0} \frac{te^{(t-1)^2}}{t} = e,$$
$$f'(0; -1) = \lim_{t \downarrow 0} \frac{te^{(-t-1)^2}}{t} = e.$$

Hence, we have $f_3^\circ(0; \pm 1) = f_3'(0; \pm 1)$. By the positive homogeneity of $f'(x; d)$ and $f^\circ(x; d)$, we conclude that $f_3^\circ(0; d) = f_3'(0; d)$ for all d. Thus, f_3 is regular.

PROPOSITION 2.9. *The function $f_3(x)$ is invex.*

PROOF. As we noted above lemma, the derivative of $f_3(x)$ for $x > 0$ and $x < 0$, respectively, where the function is smooth, is non zero, because

$$f_3'(x) = e^{(x-1)^2}\Big(2\big(x - \frac{1}{2}\big)^2 + \frac{1}{2}\Big) > 0 \quad \text{for} \quad x > 0,$$
$$f_3'(x) = -e^{(x-1)^2}\Big(2\big(x - \frac{1}{2}\big)^2 + \frac{1}{2}\Big) < 0 \quad \text{for} \quad x < 0.$$

At the unique nonsmooth point $x = 0$ of $f(x)$, the generalized gradient is

$$\partial f_3(0) = [-e, e].$$

Since $0 \in \partial f_3(0)$ is unique stationary point and $f_3(x)$ attains its global minimum at this point, we conclude that $f_3(x)$ is invex.

PROPOSITION 2.10. *The function $f_3(x)$ is not convex.*

PROOF. If we take the second derivative of $f_3(x)$ for $x > 0$, where the function is smooth, then we get

$$f_3''(x) = 2e^{(x-1)^2}(2x^3 - 4x^2 + 5x - 2) < 0,$$

for sufficiently small $x > 0$.

3. Example of two variables

EXAMPLE 3.1. Let $F_1 : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$F_1(x_1, x_2) = (x_1 + x_2)e^{x_1+x_2}.$$

Then, $F_1(x_1, x_2)$ is invex but not convex.

PROOF. If we take the partial derivatives of $F_1(x_1, x_2)$, we get

$$\frac{\partial F_1}{\partial x_1}(x_1, x_2) = (x_1 + x_2 + 1)e^{x_1+x_2},$$
$$\frac{\partial F_1}{\partial x_2}(x_1, x_2) = (x_1 + x_2 + 1)e^{x_1+x_2}.$$

The first-order condition implies $x_1 + x_2 = -1$. From Example 2.1, we know that for all $x = x_1 + x_2$ and $-1 = x_1^* + x_2^*$,

$$F_1(x_1, x_2) = f_1(x) \geq f_1(-1) = F_1(x_1^*, x_2^*).$$

Thus, $F_1(x_1, x_2)$ achieves the global minimum at all stationary points of $F_1(x_1, x_2)$, which confirms that $F_1(x_1, x_2)$ is invex. Now, consider $x_2 = 0$. In this case, $F_1(x_1, 0) = f_1(x_1)$. Since $f_1(x)$ is not convex, the function $F_1(x_1, x_2)$ is also not convex.

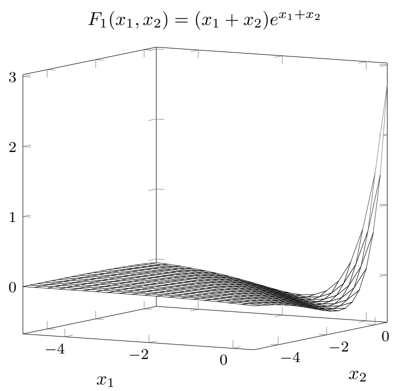

Figure 6: Graph of $F_1(x_1, x_2)$

4. Infinitely many invex functions

THEOREM 4.1. *There exist infinitely many invex functions that are not convex.*

PROOF. We consider the function $f_2(x)$ of Example 2.2, again. For non-negative $\alpha \geq 0$, let us consider the family of functions $F_2(x_1, x_2 : \alpha) : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F_2(x_1, x_2 : \alpha) = f_2(x_1) + \alpha x_2^2.$$

Evidently, $F_2(x_1, x_2 : \alpha)$ is invex but not convex for all $\alpha \geq 0$. Hence, the statement of the theorem holds.

According to this method, we can construct other infinitely many invex functions by employing the functions in Examples 2.1 and 2.6. Furthermore, we can construct infinitely many invex functions defined on $\mathbb{R}^n$. The function is given by

$$F(x:\alpha) = f_i(x_1) + \sum_{j=2}^{n} \alpha_j x_j^2,$$

where $i = 1, 2, 3$, $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_2, \ldots, \alpha_n)$, and $\alpha_j \geq 0$ for $j = 2, \ldots, n$.

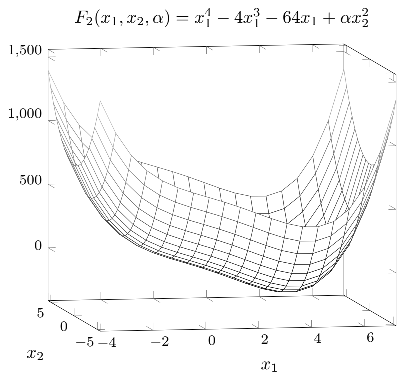

Figure 7: Graph of $F_2(x_1, x_2)$

5. Conclusion

Invex functions have been defined by restricting the domain of the functions (see, Kuk et al. (2001), Martinez-Legaz (2009)). Under such definitions, every polynomial function could be considered invex, which we find counterintuitive. We believe the significant feature of an invex function lies in its global properties. In this study, we demonstrated that the class of convex functions and the class of global invex functions defined on the entire space $\mathbb{R}^n$ are distinct.

Can one truly claim that the class of invex functions is substantially richer than the class of convex functions? Reflecting on our proof of Theorem 4.1, we must conclude, "No." The task of identifying practical, concrete examples remains an open avenue for future research. To the best of our knowledge, invexity has yet to find significant application in mathematical economics (see, Crouzeix (2003)). Therefore, exploring applications of invexity in economics is an important area for future investigation.

Acknowledgement

The authors would like to thank the reviewer and the editor for helpful comments that improved the paper. The authors gratefully acknowledge the Japan Society for the Promotion of Science, JSPS KAKENHI Grant Number 24K07950.

References

Auslender A. (1976). *Optimisation: Méthodes Numérique, Masson.*

Avriel M. (1976). *Nonlinear Programming: Analysis and Methods, Prentice-Hall.*

Ben-Israel A. and Mond B. (1986). What is Invexity? *The ANZIAM Journal*, **28**, 1–9.

Brunelli, M. (2014). *Introduction to the Analytic Hierarchy Process, Springer.*

Clarke F. H. (1983). *Optimization and Nonsmooth Analysis, John Wiley and Sons.*

Crouzeix J. P. (2003). La convexité Généralizée en Économie Mathématique, *ESAIM:Proceedings*, **13**, 31–40.

Das A. K., Jana R. and Deepmala (2018). Invex programming problems with equality and inequality constraints, *Transactions of A. Razmadze Mathematical Institute*, **172**, 361–371.

Fukushima M. (2001). *Fundamentals of Nonlinear Optimization, Asakura Shoten,* (in Japanese).

Hanson M. A. (1981). On sufficiency of the Kuhn-Tucker conditions. *Journal of Mathematical Analysis and Applications*, **80**, 545–550.

Jeyakumar V. and Mond B. (1992). On Generalized Convex Mathematical Programming. *The ANZIAM Journal*, **34**, 43–53.

Kuk H., Lee G. M. and Tanino T. (2001). Optimality and Duality for Nonsmooth Multiobjective Fractional Programming with Generalized Invexity. *Journal of Mathematical Analysis and Applications*, **262**, 365–375.

Kułakowski, K. (2021). *Understanding the Analytic Hierarchy Process, CRC Press.*

Martinez-Legaz J. E. (2009). What is invexity with respect to the same η? *Taiwanese Journal of Mathematics*, **13**, 753–755.

Obata T. and Shiraishi S. (1997). Optimality Conditions and Sensitivity Analysis of Nonlinear Optimization Problems under Invexity. *Reports of the Faculty of Engineering, Oita University*, **35**, 41–47, (in Japanese).

Obata T. and Shiraishi S. (2021). Computational study of characteristic polynomial of 4th order PCM in AHP. *Bulletin of Informatics and Cybernetics*, **53**, 1–12.

Mishra S. K. and Giorgi G. (2008). *Invexity and Optimization, Sprimger.*

Rockafellar R. T. (1970). *Convex Analysis, Princeton University Press.*

Sapkota S. and Bhattarai B. (2021). Input Invex Neural Network. `https://arxiv.org/abs/2106.08748`.

Shiraishi S. (1998). Sensitivity analysis of nonlinear programming problems via minimax functions. *In A. V. Fiacco (ed.):mathematical programming with data perturbations, Marcel Dekker*, 387–397.

Shiraishi S. (2014). *Workbook of Mathematics for Economic Analysis, Nippyo,* (in Japanese).

Shiraishi S., Obata T. and Daigo M. (1998). Properties of positive reciprocal matrix and their application to AHP. *Journal of Operations Society of Japan*, **41**, 404–414.

Sohrab H. H. (2014). *Basic Real Analysis: Second Edition, Springer.*

Tanaka Y. (1990). Note on generalized convex functions, *Journal of Optimization Theory and Applications*, **66**, 345–349.

Tanaka Y., Fukushima M. and Ibaraki T. (1989). On Generalized Pseudoconvex Functions, *Journal of Mathematical Analysis and Applications*, **144**, 342–355.

Tiel J. V. (1984). *Convex Analysis: An Introductory Text, John Wiley and Sons.*

Wang R. and Feng Q. (2024). Optimality and Duality of Semi-Preinvariant Convex Multi-Objective Programming Involving Generalized (F, α, ρ, d)–I–Type Invex Functions, *Mathematics*, **12**, 2599.

Zălinescu C. (2014). A critical view on invexity, *Journal of Optimization Theory and Applications*, **162**, 695–704.

Received: March 6, 2025
Revised: May 9, 2025
Accept: May 9, 2025