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<https://hdl.handle.net/2324/7347458>

出版情報 : pp.1-17, 2023-10-09

バージョン :



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Finding a Minimum Spanning Tree with a Small Non-Terminal Set

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Abstract

In this paper, we study the problem of finding a minimum weight spanning tree that contains each vertex in a given subset V_{NT} of vertices as an internal vertex. This problem, called MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, includes s - t HAMILTONIAN PATH as a special case, and hence it is NP-hard. In this paper, we first observe that NON-TERMINAL SPANNING TREE, the unweighted counterpart of MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, is already NP-hard on some special graph classes. Moreover, it is $W[1]$ -hard when parameterized by clique-width. In contrast, we give a $3k$ -vertex kernel and $O^*(2^k)$ -time algorithm, where k is the size of non-terminal set V_{NT} . The latter algorithm can be extended to MINIMUM WEIGHT NON-TERMINAL SPANNING TREE with the restriction that each edge has a polynomially bounded integral weight. We also show that MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable parameterized by the number of edges in the subgraph induced by the non-terminal set V_{NT} , extending the fixed-parameter tractability of MINIMUM WEIGHT NON-TERMINAL SPANNING TREE to a more general case. Finally, we give several results for structural parameterization.

2012 ACM Subject Classification Theory of computation \rightarrow Parameterized complexity and exact algorithms

Keywords and phrases Spanning tree, Fixed-parameter algorithms, Parameterized complexity, Kernelization

Digital Object Identifier 10.4230/LIPIcs...

Funding *Tesshu Hanaka*: JSPS KAKENHI Grant Number JP21K17707, JP21H05852, JP22H00513, JP23H04388, and JST CRONOS Grant Number JPMJCS24K2

Yasuaki Kobayashi: JSPS KAKENHI Grant Numbers JP20H00595 and JP23H03344

1 Introduction

The notion of spanning trees plays a fundamental role in graph theory, and the minimum weight spanning tree problem is arguably one of the most well-studied combinatorial optimization problems. Numerous variants of this problem are studied in the literature (e.g. [24, 25, 40]). In this paper, we consider MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, which is defined as follows. In a (spanning) tree T , leaf vertices, which are of degree 1, are called *terminals*, and other vertices are called *non-terminals*. In MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, we are given an edge-weighted graph $G = (V, E, w)$ with $w: E \rightarrow \mathbb{R}^+$ and a set of designated non-terminals $V_{NT} \subseteq V$. The goal of this problem is to find a minimum weight spanning tree T of G subject to the condition that each vertex in V_{NT} is of degree at least 2 in T . We also consider the unweighted variant of MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, which we call NON-TERMINAL SPANNING TREE: Given a graph $G = (V, E)$ and $V_{NT} \subseteq V$, the goal is to determine whether G has a spanning tree T such that each vertex in V_{NT} is of degree at least 2 in T .

MINIMUM WEIGHT NON-TERMINAL SPANNING TREE was firstly introduced by Zhang and Yin [44]. Unlike the minimum weight spanning tree problem, MINIMUM WEIGHT



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Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

NON-TERMINAL SPANNING TREE is NP-hard [44]. Nakayama and Masuyama [33] observed that NON-TERMINAL SPANNING TREE is also NP-hard as s - t HAMILTONIAN PATH, the problem of finding a Hamiltonian path between specified vertices s and t , is a special case of NON-TERMINAL SPANNING TREE: When $V_{\text{NT}} = V \setminus \{s, t\}$, any solution of NON-TERMINAL SPANNING TREE is a Hamiltonian path between s and t . Nakayama and Masuyama [33, 34, 35, 36] devised polynomial-time algorithms for NON-TERMINAL SPANNING TREE on several classes of graphs, such as cographs, outerplanar graphs, and series-parallel graphs.

1.1 Our contribution

In this paper, we study MINIMUM WEIGHT NON-TERMINAL SPANNING TREE and NON-TERMINAL SPANNING TREE from the viewpoint of parameterized complexity.

On the positive side, we first give a $3k$ -vertex kernel for NON-TERMINAL SPANNING TREE, where k is the number of vertices in V_{NT} . We also give polynomial kernelizations for NON-TERMINAL SPANNING TREE with respect to vertex cover number and max leaf number. For MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, we give an $O^*(2^k)$ -time algorithm¹ when the weight of edges is integral and upper bounded by a polynomial in the input size. Moreover, we design an $O^*(2^\ell)$ -time algorithm for MINIMUM WEIGHT NON-TERMINAL SPANNING TREE for an arbitrary edge weight, where ℓ is the number of edges in the subgraph induced by V_{NT} . Note that this parameter ℓ is “smaller” than k in the sense that $\ell \leq \binom{k}{2}$ but k could be arbitrarily large even when $\ell = 0$. Finally, we show that the property of being a spanning tree that has all vertices in V_{NT} as internal vertices is expressed by an MSO_2 formula, which implies that MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable when parameterized by treewidth.

On the negative side, we observe that NON-TERMINAL SPANNING TREE is NP-hard even on planar bipartite graphs of maximum degree 3, strongly chordal split graphs, and chordal bipartite graphs. Moreover, we observe that NON-TERMINAL SPANNING TREE is $\text{W}[1]$ -hard when parameterized by cliquewidth, and it cannot be solved in time $2^{o(n)}$ unless the Exponential Time Hypothesis fails, where n is the number of vertices in the input graph. In contrast to the fact that NON-TERMINAL SPANNING TREE admits a polynomial kernelization with respect to vertex cover number, we show that it does not admit a polynomial kernelization with respect to vertex integrity under some complexity-theoretic assumption.

1.2 Related work

The minimum weight spanning tree problem is a fundamental graph problem. There are many variants of this problem, such as MINIMUM DIAMETER SPANNING TREE [25], DEGREE-BOUNDED SPANNING TREE [24], DIAMETER-BOUNDED SPANNING TREE [21], and MAX LEAF SPANNING TREE [40].

MAX INTERNAL SPANNING TREE is highly related to NON-TERMINAL SPANNING TREE. Given a graph G and an integer k , the task of MAX INTERNAL SPANNING TREE is to find a spanning tree T of G that has at least k internal vertices. The difference between NON-TERMINAL SPANNING TREE and MAX INTERNAL SPANNING TREE is that the former designates internal vertices in advance whereas the latter does not. Since MAX INTERNAL

¹ The O^* notation suppresses a polynomial factor of the input size.

SPANNING TREE is a generalization of HAMILTONIAN PATH, it is NP-hard. For restricted graph classes, the problem can be solved in polynomial time, such as interval graphs [30], cacti, block graphs, cographs, and bipartite permutation graphs [41]. Binkele-Raible et al. [3] give an $O^*(3^n)$ -time algorithm for MAX INTERNAL SPANNING TREE and then Nederlof [37] improves the running time to $O^*(2^n)$, where n is the number of vertices of the input graph. For the fixed-parameter tractability, Fomin et al. [15] show that MAX INTERNAL SPANNING TREE admits a $3k$ -vertex kernel. Then Li et al. [29] improve the size of the kernel by giving a $2k$ -vertex kernel. The best known deterministic $O^*(4^k)$ -time algorithm is obtained by combining a $2k$ -vertex kernel with an $O^*(2^n)$ -time algorithm, while there is a randomized $O^*(\min\{3.455^k, 1.946^n\})$ -time algorithm [4]. For approximation algorithms, Li et al. [31] propose a $3/4$ -approximation algorithm and Chen et al. [8] improve the approximation ratio to $13/17$.

MAX LEAF SPANNING TREE is a “dual” problem of MAX INTERNAL SPANNING TREE, where the goal is to compute a spanning tree that has leaves as many as possible. This problem is highly related to CONNECTED DOMINATING SET and well-studied in the context of parameterized complexity [10, 40].

The *designated leaves* version of MAX LEAF SPANNING TREE is called TERMINAL SPANNING TREE. In this problem, given a graph $G = (V, E)$ and terminal set $W \subseteq V$, the task is to find a spanning tree T such that every vertex in W is a leaf in T . Unlike NON-TERMINAL SPANNING TREE, it can be solved in polynomial time by finding a spanning tree in the subgraph induced by $V \setminus W$ and adding vertices in W as leaves.

In the context of graph theory, NON-TERMINAL SPANNING TREE has already discussed in [13, 26]. More precisely, the paper [13] gives a sufficient condition that for a graph G , a vertex set S , and a function $f: S \rightarrow \mathbb{N}$, G has a spanning tree such that the degree of each vertex $v \in S$ is at least $f(v)$ in the spanning tree. Király [26] gives the shorter proofs of the sufficient condition and proposes a polynomial-time algorithm for checking the condition.

2 Preliminaries

In this paper, we use several basic concepts and terminology in parameterized complexity. We refer the reader to [10].

Graphs. Let $G = (V, E)$ be an undirected graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G , respectively. We use n to denote the number of vertices in G . For $X \subseteq V$, we denote by $G[X]$ the subgraph induced by X . For $v \in V$ and $X \subseteq V$, $G - v$ and $G - X$ denote $G[V \setminus \{v\}]$ and $G[V \setminus X]$, respectively. We also define $G - e = (V, E \setminus \{e\})$ and $G + e = (V, E \cup \{e\})$. For $v \in V$, we let $N_G(v)$ denote the set of neighbors of v and $N_G[v] = N_G(v) \cup \{v\}$. This notation is extended to sets: for $X \subseteq V$, we let $N_G[X] = \bigcup_{v \in X} N_G[v]$ and $N_G(X) = N_G[X] \setminus X$. When the subscript G is clear from the context, we may omit it.

A *forest* is a graph having no cycles. If a forest is connected, it is called a *tree*. In a forest F , a vertex is called a *leaf* if it is of degree 1 in F , and called an *internal* vertex otherwise.

► **Definition 1.** For a vertex set $X \subseteq V$ and a forest F , we say F is admissible for X if each $v \in X$ is an internal vertex in F .

In NON-TERMINAL SPANNING TREE, given a non-terminal set V_{NT} , the goal is to determine whether there is an admissible spanning tree for V_{NT} in G . Throughout the paper, we denote $k = |V_{\text{NT}}|$ and $\ell = |E(G[V_{\text{NT}}])|$, and assume that $k \leq n - 2$. We also define an

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optimization version of NON-TERMINAL SPANNING TREE. In MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, we are additionally given an edge weight function $w: E(G) \rightarrow \mathbb{R}$, the goal is to find an admissible spanning tree T for V_{NT} in G minimizing its total weight $w(E(T)) = \sum_{e \in E(T)} w(e)$. We also assume that the input graph G is connected as otherwise, our problems are trivially infeasible. The following easy proposition is useful to extend a forest into a spanning tree.

► **Proposition 2.** *Let G be a connected graph and F be a forest in G . Then there is a spanning tree that contains all edges in F .*

Graph parameters. A set S of vertices is called a *vertex cover* if every edge has at least one endpoint in S . The vertex cover number $\text{vc}(G)$ of G is defined by the size of a minimum vertex cover in G . The *vertex integrity* $\text{vi}(G)$ of G is the minimum integer p satisfying that there exists $S \subseteq V$ such that $|S| + \max_{H \in \text{cc}(G-S)} |V(H)| \leq p$, where $\text{cc}(G-S)$ is the set of connected components of $G-S$. The *max leaf number* $\text{ml}(G)$ of G is the maximum integer q such that there exists a spanning tree having q leaves in G . For the definition of treewidth $\text{tw}(G)$, pathwidth $\text{pw}(G)$, and treedepth $\text{td}(G)$, we refer the reader to the books [38, 10]. It is well-known that these graph parameters have the following relationship.

► **Proposition 3** ([1, 14, 22]). *For every graph G , it holds that $\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1 \leq \text{vi}(G) - 1 \leq \text{vc}(G)$ and $\text{tw}(G) \leq \text{pw}(G) \leq 2\text{ml}(G)$.*

Matroids. Let U be a finite set. A pair $\mathcal{M} = (U, \mathcal{B})$ with $\mathcal{B} \subseteq 2^U$ is called a *matroid*² if the following axioms are satisfied:

- \mathcal{B} is nonempty;
- For $X, Y \in \mathcal{B}$ with $X \neq Y$ and for $x \in X \setminus Y$, there is $y \in Y \setminus X$ such that $(X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

It is not hard to verify that all the sets in \mathcal{B} have the same cardinality.

There are many combinatorial objects that can be represented as matroids. Let $H = (V, E)$ be a graph. If \mathcal{B}_H consists of all subsets of edges, each of which forms a spanning tree in H , then the pair (E, \mathcal{B}_H) is a matroid, which is called a *graphic matroid*. We are also interested in another matroid. Let U be a finite set and let $\{U_1, U_2, \dots, U_t\}$ be a partition of U : $U_i \cap U_j = \emptyset$ for $i \neq j$ and $\bigcup_{1 \leq i \leq t} U_i = U$. For $1 \leq i \leq t$, let ℓ_i and u_i be two non-negative integers with $\ell_i \leq u_i$. For a non-negative integer r , we define a set \mathcal{B}_r consisting of all size- r subsets $U' \subseteq U$ such that $\ell_i \leq |U' \cap U_i| \leq u_i$ for $1 \leq i \leq t$. Then, the pair (U, \mathcal{B}_r) is a matroid unless $\mathcal{B}_r = \emptyset$.

► **Proposition 4.** *If $\mathcal{B}_r \neq \emptyset$, then (U, \mathcal{B}_r) is a matroid.*

Proof. It suffices to show that \mathcal{B}_r satisfies the second axiom of matroids. Let $X, Y \in \mathcal{B}_r$ with $X \neq Y$. For $1 \leq i \leq t$, let $x_i = |X \cap U_i|$ (resp. $y_i = |Y \cap U_i|$). Let $x \in X \setminus Y$ and assume that $x \in U_i$. Suppose first that $x_i \leq y_i$. Since $x \in X \setminus Y$ and $|X \cap U_i| \leq |Y \cap U_i|$, there is $y \in U_i$ such that $y \in Y \setminus X$. Then, we have $Z = (X \setminus \{x\}) \cup \{y\} \in \mathcal{B}_r$ as $|Z \cap U_j| = |X \cap U_j|$ for $1 \leq j \leq t$. Suppose next that $x_i > y_i$. As $\sum_{1 \leq j \leq k} x_j = \sum_{1 \leq j \leq k} y_j = r$, there is an index i' with $x_{i'} < y_{i'}$. This implies that $U_{i'}$ contains an element $y \in Y \setminus X$. Then, $Z = (X \setminus \{x\}) \cup \{y\} \in \mathcal{B}_r$ as $\ell_i \leq y_i \leq x_i - 1 = |Z \cap U_i|$, $|Z \cap U_{i'}| = x_{i'} + 1 \leq y_{i'} \leq u_{i'}$, and $|Z \cap U_j| = |X \cap U_j|$ for all $j' \neq i, i'$. ◀

² This definition is equivalent to that defined by the family of *independent sets* [39].

Let $\mathcal{M}_1 = (U, \mathcal{B}_1)$ and $\mathcal{M}_2 = (U, \mathcal{B}_2)$ be matroids. It is well known that there is a polynomial-time algorithm to check whether $\mathcal{B}_1 \cap \mathcal{B}_2$ is empty when set U and an *independence oracle* is given as input (e.g., [19]), where an independence oracle for a matroid $\mathcal{M} = (U, \mathcal{B})$ is a black-box procedure that given a set $U' \subseteq U$, returns true if there is a set $B \in \mathcal{B}$ that contains U' . Moreover, we can find a minimum weight common base in two matroids in polynomial time.

► **Theorem 5** ([19]). *Let U be a finite set and let $w: U \rightarrow \mathbb{R}$. Given two matroids $\mathcal{M}_1 = (U, \mathcal{B}_1)$ and $\mathcal{M}_2 = (U, \mathcal{B}_2)$, we can compute a set $X \in \mathcal{B}_1 \cap \mathcal{B}_2$ minimizing $w(X)$ in polynomial time, provided that the polynomial-time independence oracles of \mathcal{M}_1 and \mathcal{M}_2 are given as input.*

Finally, we observe that there are polynomial-time independence oracles for graphic matroids and (U, \mathcal{B}_r) . By Proposition 2, the independence oracle for the graphic matroid defined by a connected graph G returns true for given $F \subseteq E(G)$ if and only if F is a forest. The following lemma gives an independence oracle for (U, \mathcal{B}_r) .

► **Lemma 6.** *There is a polynomial-time algorithm that given a set $U' \subseteq U$, determines whether there is a set $U'' \in \mathcal{B}_r$ with $U' \subseteq U'' \subseteq U$, that is, $|U''| = r$ and $\ell_i \leq |U'' \cap U_i| \leq u_i$ for all $1 \leq i \leq k$.*

Proof. The problem can be reduced to that of finding a degree-constrained subgraph of an auxiliary bipartite graph. To this end, we construct a bipartite graph H as follows. The graph H consists of two independent sets A and B , where each vertex a_i of A corresponds to a block U_i of the partition and each vertex b_e of B corresponds to an element $e \in U \setminus U'$. For each $e \in U \setminus U'$, H contains an edge between b_e and a_i , where i is the unique index with $e \in U_i$. Then, we define functions $d^\ell: A \cup B \rightarrow \mathbb{N}$ and $d^u: A \cup B \rightarrow \mathbb{N}$ as:

$$d^\ell(v) = \begin{cases} 0 & \text{if } v \in B \\ \ell_i - |U' \cap U_i| & \text{if } v \in A \end{cases} \quad d^u(v) = \begin{cases} 1 & \text{if } v \in B \\ u_i - |U' \cap U_i| & \text{if } v \in A \end{cases}$$

Then, we claim that there is a set U'' satisfying the condition in the statement of this lemma if and only if H has a subgraph H' such that $d^\ell(v) \leq d_{H'}(v) \leq d^u(v)$ for all $v \in V(H)$, where $d_{H'}(v)$ is the degree of v in H' , and H' contains exactly $r - |U'|$ edges. From a feasible set $U'' \subseteq U$, we construct a subgraph H' in such a way that for each $b_e \in B$ we take the unique edge (a_i, b_e) if $e \in U'' \setminus U'$. Clearly, this subgraph H' contains $r - |U'|$ edges. Moreover, H' satisfies the degree condition as

$$d_{H'}(a_i) = |(U'' \setminus U') \cap U_i| = |U'' \cap U_i| - |U' \cap U_i| \geq \ell_i - |U' \cap U_i| = d^\ell(a_i),$$

and, analogously, we have $d_{H'}(a_i) \leq d^u(a_i)$. This transformation is reversible and hence the converse direction is omitted here.

Given a graph H , an integer r' , and the degree bounds d^ℓ, d^u , there is a polynomial-time algorithm that finds a subgraph H' with r' edges satisfying $d^\ell(v) \leq d_{H'}(v) \leq d^u(v)$ for $v \in V(H)$ [20, 42]. Hence, the lemma follows. ◀

3 Kernelization

3.1 Linear kernel for k

In this subsection, we give a linear vertex kernel of NON-TERMINAL SPANNING TREE when parameterized by $k = |V_{\text{NT}}|$. Let G be a graph with a non-terminal set $V_{\text{NT}} \subseteq V(G)$. The following easy lemma is a key to our results.

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► **Lemma 7.** *There is an admissible spanning tree for V_{NT} in G if and only if there is an admissible forest for V_{NT} in $G[N[V_{\text{NT}}]]$.*

Proof. Let T be an admissible spanning tree for V_{NT} in G . We remove vertices in $V(G) \setminus N[V_{\text{NT}}]$ from T . Since this does not change the degree of any vertex in V_{NT} , we have an admissible forest in $G[N[V_{\text{NT}}]]$.

Conversely, suppose that there is an admissible forest F for V_{NT} in $G[N[V_{\text{NT}}]]$. This forest is indeed a forest in G . By Proposition 2, there is a spanning tree of G that contains F as a subgraph. This spanning tree is admissible for V_{NT} in G as the degree of any vertex in V_{NT} is at least 2 in F . ◀

Let G' be the graph obtained from G by deleting all the vertices in $V(G) \setminus N[V_{\text{NT}}]$, adding a vertex r , and connecting r to all the vertices in $N(V_{\text{NT}})$. From the assumption that G is connected, G' is also connected. By Lemma 7, we can immediately obtain the following corollary.

► **Corollary 8.** *There is an admissible spanning tree for V_{NT} in G if and only if there is an admissible spanning tree for V_{NT} in G' .*

Due to Corollary 8, we can “safely” reduce G to G' , that is, G has an admissible spanning tree for V_{NT} if and only if G' does. The graph G' may still have an arbitrary number of vertices as the size of $N_{G'}(V_{\text{NT}})$ cannot be upper bounded by a function in k . To reduce this part, we apply the expansion lemma [18].

► **Definition 9.** *Let $H = (A \cup B, E_H)$ be a bipartite graph. For positive integer q , $M \subseteq E_H$ is called a q -expansion of A into B if it satisfies the following:*

- every vertex in A is incident to exactly q edges in M .
- there are exactly $q|A|$ vertices in B , each of which is incident to exactly one edge in M .

► **Lemma 10** (Expansion lemma [18, 43]). *Let q be a positive integer and $H = (A \cup B, E_H)$ be a bipartite graph such that:*

- $|B| \geq q|A|$, and
- there are no isolated vertices in B .

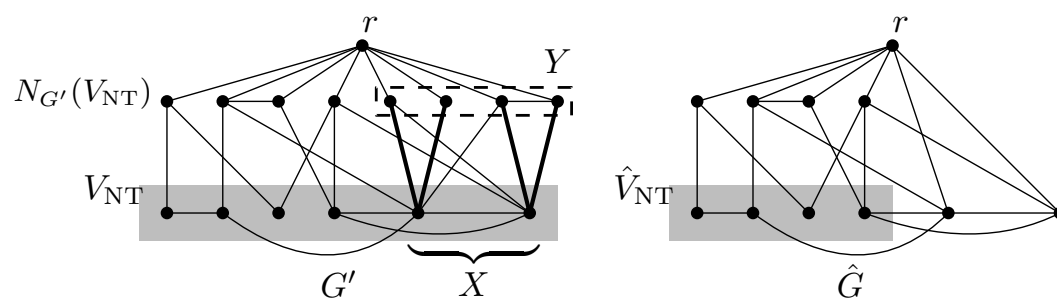
Then there are non-empty vertex sets $X \subseteq A$ and $Y \subseteq B$ such that:

- $N_H(Y) \subseteq X$, and
- there is a q -expansion $M \subseteq E_H$ of X into Y .

Moreover, one can find such X , Y , and M in polynomial time in the size of H .

Suppose that $N_{G'}(V_{\text{NT}})$ has at least $2k$ vertices. Let $H = (V_{\text{NT}} \cup N_{G'}(V_{\text{NT}}), E_H)$ be the bipartite graph obtained from $G'[N_{G'}[V_{\text{NT}}]]$ by deleting all edges between vertices in V_{NT} and those between vertices in $N_{G'}(V_{\text{NT}})$. As $N_{G'}(V_{\text{NT}})$ has no isolated vertices in H , by Lemma 10, there exists a 2-expansion $M \subseteq E_H$ of X into Y for some non-empty sets $X \subseteq V_{\text{NT}}$ and $Y \subseteq N_{G'}(V_{\text{NT}})$ such that $N_H(Y) \subseteq X$. We then construct a smaller instance $(\hat{G}, \hat{V}_{\text{NT}})$ of NON-TERMINAL SPANNING TREE as follows. We first delete all vertices in Y from G' and add an edge between r and x for each $x \in X$. The graph obtained in this way is denoted by \hat{G} . Then we set $\hat{V}_{\text{NT}} = V_{\text{NT}} \setminus X$. See Figure 1 for an illustration.

Observe that \hat{G} is also connected. This follows from the fact that every vertex in $(N_{G'}(V_{\text{NT}}) \cup X) \setminus Y$ is adjacent to r and every vertex v in \hat{V}_{NT} is adjacent to some vertex of $(N_{G'}(V_{\text{NT}}) \cup X) \setminus Y$ in \hat{G} as otherwise $v \in \hat{V}_{\text{NT}}$ is adjacent to a vertex in Y , which violates the fact that $N_H(Y) \subseteq X$.



■ **Figure 1** The construction of \hat{G} . The shaded areas indicate the vertices of V_{NT} and \hat{V}_{NT} . The bold lines are edges in a 2-expansion of X into Y .

► **Lemma 11.** *There exists an admissible spanning tree for V_{NT} in G' if and only if there exists an admissible spanning tree for \hat{V}_{NT} in \hat{G} .*

Proof. Suppose that there exists an admissible spanning tree T for V_{NT} in G' . Since T is admissible for V_{NT} , for every $v \in V_{NT} \setminus X$, T has at least two edges incident to v . These edges are contained in \hat{G} as there are no edges between Y and $V_{NT} \setminus X$. Thus, there is an admissible forest for \hat{V}_{NT} in \hat{G} , consisting of these two incident edges for each $v \in V_{NT} \setminus X$, and by Proposition 2, \hat{G} has an admissible spanning tree for \hat{V}_{NT} .

Conversely, let \hat{T} be an admissible spanning tree for \hat{V}_{NT} in \hat{G} . To construct a spanning tree of G' , we select all edges of \hat{T} incident to \hat{V}_{NT} . These edges are also contained in G' . Moreover, we select all edges in the 2-expansion M of X into Y . Since there are no edges between \hat{V}_{NT} and Y in G' , the subgraph consisting of selected edges does not have cycles. Moreover, as every vertex of V_{NT} is of degree at least 2 in the subgraph, it is an admissible forest for V_{NT} in G' . By Proposition 2, there is an admissible spanning tree for V_{NT} in G' . ◀

Our kernelization is described as follows. From a connected graph G , we first construct the graph G' by deleting the vertices in $V(G) \setminus N[V_{NT}]$ and adding r adjacent to each vertex in $N(V_{NT})$. By Corollary 8, (G, V_{NT}) is a yes-instance of NON-TERMINAL SPANNING TREE if and only if so is (G', V_{NT}) . If G' has at most $3k$ vertices, we are done. Suppose otherwise. As $V(G') \setminus N_{G'}[V_{NT}] = \{r\}$ and $|V_{NT}| = k$, we have $|N_{G'}(V_{NT})| \geq 2k$. We apply the expansion lemma to G' and then obtain \hat{G} and \hat{V}_{NT} . By Lemma 11, (G, V_{NT}) is a yes-instance of NON-TERMINAL SPANNING TREE if and only if so is (\hat{G}, \hat{V}_{NT}) . We repeatedly apply these reduction rules as long as the reduced graph \hat{G} has at least $3|\hat{V}_{NT}| + 1$ vertices. Therefore, we have the following theorem.

► **Theorem 12.** *NON-TERMINAL SPANNING TREE admits a $3k$ -vertex kernel.*

3.2 Linear kernel for vertex cover number

In this subsection, we show that NON-TERMINAL SPANNING TREE admits a linear kernel when parameterized by vertex cover number. We first apply an analogous transformation used in the previous subsection. By Lemma 7, there is an admissible spanning tree for V_{NT} in G if and only if there is an admissible forest for V_{NT} in $G[N[V_{NT}]]$. We remove all vertices of $V(G) \setminus N[V_{NT}]$ and then add a vertex r that is adjacent to every vertex in $N(V_{NT})$. As seen in the previous subsection, G has an admissible spanning tree for V_{NT} if and only if the obtained graph has. Thus, in the following, G consists of three vertex sets V_{NT} , $N(V_{NT})$, and $\{r\}$, where r is adjacent to every vertex in $N(V_{NT})$. Furthermore, we delete all the edges

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within $G[N(V_{\text{NT}})]$, which is safe by Lemma 7. Thus, $N(V_{\text{NT}})$ forms an independent set in G . Let S be a vertex cover of G and $I = V \setminus S$. As G is obtained from a subgraph of the original graph by adding a vertex r and deleting edges within $G[N(V_{\text{NT}})]$, we have $|S| \leq \tau + 1$, where τ is the vertex cover number of the original graph. Suppose that $|V_{\text{NT}} \cap I| \geq |S|$. Then we conclude that G has no admissible spanning tree for V_{NT} .

▷ **Claim 13.** If $|V_{\text{NT}} \cap I| \geq |S|$, there is no admissible spanning tree for V_{NT} .

Proof. Suppose that T is an admissible spanning tree for V_{NT} in G . Then T has at least $2|V_{\text{NT}} \cap I|$ edges between $V_{\text{NT}} \cap I$ and S . Consider the subforest T' of T induced by $(V_{\text{NT}} \cap I) \cup S$. We have

$$2|V_{\text{NT}} \cap I| \leq |E(T')| \leq |V_{\text{NT}} \cap I| + |S| - 1.$$

Thus, we have $|V_{\text{NT}} \cap I| \leq |S| - 1$. ◀

Suppose next that $2|V_{\text{NT}} \cap S| \leq |N(V_{\text{NT}}) \cap I|$. We define a bipartite graph $H = ((V_{\text{NT}} \cap S) \cup (N(V_{\text{NT}}) \cap I), E_H)$, where E_H is the set of edges in G , each of which has an endpoint in $V_{\text{NT}} \cap S$ and the other in $N(V_{\text{NT}}) \cap I$. As $2|V_{\text{NT}} \cap S| \leq |N(V_{\text{NT}}) \cap I|$ and each vertex in $N(V_{\text{NT}}) \cap I$ has a neighbor in $V_{\text{NT}} \cap S$, there is a 2-expansion of $X \subseteq V_{\text{NT}} \cap S$ into $Y \subseteq N(V_{\text{NT}}) \cap I$ in H . By applying the same reduction rule used in Lemma 11, the obtained instance $(\hat{G}, \hat{V}_{\text{NT}})$ satisfies the following claim.

▷ **Claim 14.** Suppose that G satisfies $2|V_{\text{NT}} \cap S| \leq |N_G(V_{\text{NT}}) \cap I|$. Then G has an admissible spanning tree for V_{NT} if and only if \hat{G} has an admissible spanning tree for \hat{V}_{NT} .

The proof of this claim is analogous to that in Lemma 11. Let us note that S remains a vertex cover of \hat{G} .

Our kernelization is formalized as follows. Let G be an input graph. Let S be a vertex cover of G with $|S| \leq 2 \cdot \text{vc}(G)$ and let $I = V(G) \setminus S$. Such a vertex cover S can be computed in polynomial time by a well-known 2-approximation algorithm for the minimum vertex cover problem. Then our kernel (G', V'_{NT}) is obtained by exhaustively applying these reduction rules by Claims 13 and 14 to G . Let $S' = (V(G') \cap S) \cup \{r\}$ be a vertex cover of G' and let $I' = V(G') \setminus S'$. After these reductions, we have

$$\begin{aligned} |V(G')| &= |S'| + |I'| \\ &\leq |S'| + |V'_{\text{NT}} \cap I'| + |N(V'_{\text{NT}}) \cap I'| \\ &\leq |S'| + |V'_{\text{NT}} \cap I'| + 2|V_{\text{NT}} \cap S'| - 1 && \text{(by Claim 14)} \\ &\leq |S'| + |S'| - 1 + 2|V_{\text{NT}} \cap S'| - 1 && \text{(by Claim 13)} \\ &= 4|S'| - 2 && \text{(by } |S'| = |S| + 1) \\ &= 4|S| + 2, \end{aligned}$$

yielding the following theorem.

► **Theorem 15.** *NON-TERMINAL SPANNING TREE admits a $(8\text{vc}(G) + 2)$ -vertex kernel.*

3.3 Quadratic kernel for max leaf number

We show that NON-TERMINAL SPANNING TREE admits a polynomial kernel when parameterized by max leaf number. Thanks to the following lemma, we can suppose that the input graph has at most $4\text{ml}(G) - 2$ vertices of degree at least 3.

► **Lemma 16** ([28, 14, 6]). *Every graph G with max leaf number $\text{ml}(G)$ is a subdivision of a graph with at most $4\text{ml}(G) - 2$ vertices. In particular, G has at most $4\text{ml}(G) - 2$ vertices of degree at least 3.*

Thus, we only have to reduce the number of vertices of degree at most 2. Let $V_i \subseteq V$ be the set of vertices of degree i and $V_{\geq i} = \bigcup_{j \geq i} V_j$. In the following, we say that a reduction rule is *safe* if the instance (G', V'_{NT}) obtained by applying the reduction rule is equivalent to the original instance (G, V_{NT}) : G has an admissible spanning tree for V_{NT} if and only if G' has an admissible spanning tree for V'_{NT} .

In the following, we assume that each vertex in V_{NT} has degree at least 2, as otherwise the instance is clearly infeasible. Let v be a vertex of degree 1. By the assumption, we have $v \notin V_{\text{NT}}$. If v has a neighbor that is not in V_{NT} , we can safely delete v . Otherwise, v has a neighbor u , which belongs to V_{NT} . Then we delete v and set $V_{\text{NT}} := V_{\text{NT}} \setminus \{u\}$. This is safe because the degree of u is at least 2 in any spanning tree of G .

Suppose that G has two adjacent vertices $u, v \in V_2$ of degree 2. Let x (resp. y) be the other neighbor of u (resp. v). If both u and v are contained in V_{NT} , we contract edge $e = \{u, v\}$ and include the corresponding vertex uv in V_{NT} . Since $\{x, u\}, \{u, v\}, \{v, y\}$ must be contained in any admissible spanning tree for V_{NT} , this reduction is also safe. Suppose that neither of u and v is contained in V_{NT} . If $\{u, v\}$ is a bridge in G , both $\{x, u\}$ and $\{v, y\}$ are also bridges in G , meaning that every admissible spanning tree for V_{NT} in G contains all of these bridges. Thus, we can contract edge $e = \{u, v\}$, which is safe. Otherwise, $G - e$ remains connected. If G has an admissible spanning tree T for V_{NT} containing e , forest $T - e$ is admissible for V_{NT} . Then, by Proposition 2, there is an admissible spanning tree for V_{NT} in $G - e$. Thus, we can safely remove edge e from G . In $G - e$, the degrees of u and v are exactly one. Therefore, we can further remove u, v safely.

From the above reductions, we can assume that G does not have consecutive vertices u, v of degree 2, both of which are in V_{NT} or not in V_{NT} . Let p, q, r, s be four consecutive vertices of degree 2 in G . Let x (resp. y) be the other neighbor of p (resp. s). Without loss of generality, we assume that $p, r \in V_{\text{NT}}$ and $q, s \notin V_{\text{NT}}$. Let G' be the graph obtained from G by contracting edge $\{p, q\}$ to a vertex pq and $V'_{\text{NT}} = V_{\text{NT}} \setminus \{p, q\} \cup \{pq\}$. Since any admissible spanning tree for V_{NT} in G contains all the edges $\{x, p\}, \{p, q\}$, and $\{q, r\}$, we can reduce the instance (G, V_{NT}) to (G', V'_{NT}) safely.

Hence, when the above reductions are applied exhaustively, G has neither a vertex of degree 1 nor consecutive four vertices of degree 2.

Now we show that the reduced graph G' has $O(\text{ml}(G)^2)$ vertices. As our reduction rules either delete a vertex of degree 1, delete an edge between vertices of degree 2, or contract an edge between vertices of degree 2, they do not newly introduce a vertex of degree at least 3. This implies that the set of vertices of degree at least 3 in G' is a subset of $V_{\geq 3}$. Moreover, as G' has no vertices of degree 1, we conclude that G' is a subdivision of a graph H with at most $4\text{ml}(G) - 2$ vertices. Since this subdivision is obtained from H by subdividing each edge with at most three times, G' contains $O(\text{ml}(G)^2)$ vertices.

► **Theorem 17.** *NON-TERMINAL SPANNING TREE admits a quadratic vertex-kernel when parameterized by max leaf number.*

3.4 Kernel lower bound for vertex integrity

In this subsection, we show that NON-TERMINAL SPANNING TREE does not admit a polynomial kernel when parameterized by vertex integrity unless $\text{NP} \subseteq \text{coNP/poly}$.

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► **Theorem 18.** *NON-TERMINAL SPANNING TREE does not admit a polynomial kernel when parameterized by vertex integrity unless $NP \subseteq coNP/poly$.*

Proof. We construct an AND-cross-composition [5, 12] from s - t HAMILTONIAN PATH. For q instances $(G_i = (V_i, E_i), s_i, t_i)$ of s - t HAMILTONIAN PATH and a vertex r , we connect r to s_i by an edge $\{r, s_i\}$ for $1 \leq i \leq q$. Let G be the constructed graph and let $V_{NT} = \bigcup_{1 \leq i \leq q} V_i \setminus \{t_i\}$. Then it is easy to see that every G_i has an s_i - t_i Hamiltonian path if and only if G has an admissible spanning tree for V_{NT} . Since the vertex integrity of G is at most $\max_{1 \leq i \leq q} |V(G_i)| + 1$, the theorem holds. ◀

We remark that NON-TERMINAL SPANNING TREE is significantly different from s - t HAMILTONIAN PATH with respect to kernelization complexity.

► **Remark 19.** s - t HAMILTONIAN PATH admits a polynomial kernel when parameterized by vertex integrity while it does not admit a polynomial kernel when parameterized by treedepth unless $NP \subseteq coNP/poly$.

Proof. We first show that s - t HAMILTONIAN PATH admits a polynomial kernel when parameterized by vertex integrity. Let $vi(G)$ be the vertex integrity of G . By a polynomial-time $O(\log vi(G))$ approximation algorithm [23], we can obtain a vertex set S such that $|S| + \max_{H \in cc(G-S)} |V(H)| = O(vi(G) \log vi(G))$ in polynomial time. If the number of connected components of $G - S$ is at least $|S| + 2$, we immediately conclude that the instance is infeasible because any Hamiltonian path has to go through at least one vertex in S from a connected component to another one. Otherwise, the number of connected components is at most $|S| + 1$. Since $|S| + \max_{H \in cc(G-S)} |V(H)| = O(vi(G) \log vi(G))$, the number of vertices in G is at most $O(vi(G)^2 \log^2 vi(G))$.

To complement this positive result, we then show that the problem does not admit a polynomial kernel when parameterized by treedepth by showing an AND-cross-composition from s - t HAMILTONIAN PATH. For q instances $(G_i = (V_i, E_i), s_i, t_i)$ of s - t HAMILTONIAN PATH, we connect in series G_i and G_{i+1} by identifying t_i as s_{i+1} for $1 \leq i \leq q - 1$. Let G be the constructed graph. By taking the center vertex (i.e., $s_{\lceil q/2 \rceil + 1}$) of G as a separator recursively, we can observe that the treedepth of G is at most $\lceil \log_2 q \rceil + \max_{1 \leq i \leq q} |V(G_i)|$. It is to see that G has an s_1 - t_q Hamiltonian path if and only if every G_i has an s_i - t_i Hamiltonian path. Thus, the theorem holds. ◀

4 Fixed-Parameter Algorithms

4.1 Parameterization by k

By Theorem 12, NON-TERMINAL SPANNING TREE is fixed-parameter tractable parameterized by $k = |V_{NT}|$. A trivial brute force algorithm on a $3k$ -vertex kernel yields a running time bound $2^{O(k^2)} + n^{O(1)}$, where n is the number of vertices in the input graph $G = (V, E)$.³ However, this cannot be applied to the weighted case, namely MINIMUM WEIGHT NON-TERMINAL SPANNING TREE. In this subsection, we give an $O^*(2^k)$ -time algorithm for MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, provided that the weight function w is (positive) integral with $\max_{v \in V} w(v) = n^{O(1)}$. The algorithm runs in (pseudo-)polynomial space. Our algorithm is based on the Inclusion-Exclusion principle, which is quite useful to

³ A slightly non-trivial dynamic programming algorithm yields a better running time bound $O^*(8^k)$.

design exact exponential algorithms [17, 10], and counts the number of admissible spanning trees for V_{NT} in G .

► **Theorem 20** (Inclusion-Exclusion principle). *Let U be a finite set and let $A_1, \dots, A_t \subseteq U$. Then, the following holds:*

$$\left| \bigcap_{i \in \{1, \dots, t\}} A_i \right| = \sum_{X \subseteq \{1, \dots, t\}} (-1)^{|X|} \left| \bigcap_{i \in X} \overline{A_i} \right|,$$

where $\overline{A_i} = U \setminus A_i$ and $\bigcap_{i \in \emptyset} \overline{A_i} = U$.

Let $G = (V, E)$ be a graph with $w: E \rightarrow \mathbb{N}$ and let $W = \max_{v \in V} w(v)$. For $0 \leq q \leq (n-1)W$, let \mathcal{T}_G^q be the set of all spanning trees of G with weight exactly q . By a weighted counterpart of Kirchhoff's matrix tree theorem [27], we can count the number of spanning trees in \mathcal{T}_G^q efficiently.

► **Theorem 21** ([7]). *There is an algorithm that, given an edge-weighted graph $G = (V, E)$ with $w: E \rightarrow \{1, 2, \dots, W\}$ for some $W \in \mathbb{N}$ and an integer q , computes the number of spanning trees T of G with $w(T) = q$. Moreover, this algorithm runs in time $(n+W)^{O(1)}$ with space $(n+W)^{O(1)}$, where $n = |V|$.*

For $v \in V_{\text{NT}}$, let $\mathcal{A}_v^q \subseteq \mathcal{T}_G^q$ be the set of spanning trees of G with weight q that have v as an internal node. Clearly, the number of admissible spanning trees for V_{NT} with weight q is $|\bigcap_{v \in V_{\text{NT}}} \mathcal{A}_v^q|$. Thus, we can solve MINIMUM WEIGHT NON-TERMINAL SPANNING TREE by checking if $|\bigcap_{v \in V_{\text{NT}}} \mathcal{A}_v^q| > 0$ for each q . Due to Theorem 20, it suffices to compute $|\bigcap_{v \in X} \overline{\mathcal{A}_v^q}|$ for each $X \subseteq V_{\text{NT}}$, where $\overline{\mathcal{A}_v^q} = \mathcal{T}_G^q \setminus \mathcal{A}_v^q$. Here, $|\bigcap_{v \in X} \overline{\mathcal{A}_v^q}|$ is equal to the number of spanning trees $T \in \mathcal{T}_G^q$ such that every vertex in X is a leaf of T .

► **Lemma 22.** *Given $X \subseteq V$, we can compute $|\bigcap_{v \in X} \overline{\mathcal{A}_v^q}|$ in time $(n+W)^{O(1)}$.*

Proof. Assume that $|V| \geq 3$. Then, every leaf $w \in X$ of a spanning tree T in $\bigcap_{v \in X} \overline{\mathcal{A}_v^q}$ has exactly one neighbor in $V \setminus X$. Moreover, the subtree of T obtained by removing all vertices in X is a spanning tree of $G[V \setminus X]$. Thus, the following equality holds:

$$\left| \bigcap_{v \in X} \overline{\mathcal{A}_v^q} \right| = \sum_{0 \leq q' \leq q} |\mathcal{T}_{G[V \setminus X]}^{q'}| \cdot |\mathcal{M}^{q-q'}(X, V \setminus X)|.$$

In the above equality, for $Y \subseteq X$ and $0 \leq j \leq q$, we denote by $\mathcal{M}^j(Y, V \setminus X)$ the collection of edge subsets $M \subseteq E \cap (Y \times (V \setminus X))$ with $w(M) = j$ such that each vertex in Y is incident to exactly one edge in M . By Theorem 21, we can compute $|\mathcal{T}_{G[V \setminus X]}^{q'}|$ in $(n+W)^{O(1)}$ time. Thus, it suffices to compute $|\mathcal{M}^{q-q'}(X, V \setminus X)|$ in time $(n+W)^{O(1)}$ as well. This can be done by dynamic programming described as follows. Let $X = \{x_1, x_2, \dots, x_p\}$. For $0 \leq i \leq p$ and $0 \leq j \leq q - q'$, we define $m^j(i) = |\mathcal{M}^j(\{x_1, \dots, x_i\}, V \setminus X)|$. Clearly, $m^0(0) = 1$, $m^j(0) = 0$ for $j > 0$, and $m^j(p) = |\mathcal{M}^j(X, V \setminus X)|$ for $j \geq 0$. For $i \geq 1$, it is easy to verify that

$$m^j(i) = \sum_{e \in E \cap (\{x_i\} \times (V \setminus X))} m^{j-w(e)}(i-1),$$

where we define $m^{j'}(i-1) = 0$ for negative integer j' . We can evaluate $m^j(i)$ in time $(n+W)^{O(1)}$ by dynamic programming, and hence the lemma follows. ◀

► **Theorem 23.** *MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is solvable in time $O^*(2^k)$ and polynomial space when the edge weight function w is integral with $\max_{v \in V} w(v) = n^{O(1)}$.*

4.2 Parameterization by ℓ

In this subsection, we give a fixed-parameter algorithm for MINIMUM WEIGHT NON-TERMINAL SPANNING TREE with respect to the number of edges ℓ in $G[V_{\text{NT}}]$. Note that ℓ is a “smaller” parameter than k in the sense that $\ell \leq \binom{k}{2}$, while k can be arbitrary large even if $\ell = 0$. Note also that the algorithm described in this subsection works for MINIMUM WEIGHT NON-TERMINAL SPANNING TREE with an arbitrary edge weight function, whereas the algorithm in the previous subsection works only for bounded integral weight functions.

We first consider the case where $\ell = 0$. To this end, we construct a partition of $E(G)$ as follows. Let $V_{\text{NT}} = \{v_1, v_2, \dots, v_k\}$. For each $v \in V_{\text{NT}}$, we let E_v be the set of edges that are incident to v in G and let $R = E(G) \setminus (E_{v_1}, E_{v_2}, \dots, E_{v_k})$. Since $G[V_{\text{NT}}]$ is edge-less, $\{E_{v_1}, E_{v_2}, \dots, E_{v_k}, R\}$ is a partition of $E(G)$. Clearly, a spanning tree T of G is admissible for V_{NT} if and only if T contains at least two edges from E_v for every $v \in V_{\text{NT}}$. This condition can be represented by the intersection of the following two matroids \mathcal{M}_1 and \mathcal{M}_2 . $\mathcal{M}_1 = (E, \mathcal{B}_g)$ is just a graphic matroid of G and $\mathcal{M}_2 = (E, \mathcal{B}_{n-1})$ is defined as follows: \mathcal{B}_{n-1} consists of all edge subsets $F \subseteq E(G)$ with $|F| = n - 1$ such that for $v \in V_{\text{NT}}$, it holds that $|F \cap E_v| \geq \ell_v := 2$. Thus, a set of edges F forms an admissible spanning tree for V_{NT} if and only if $F \in \mathcal{B}_g \cap \mathcal{B}_{n-1}$. By Proposition 4, \mathcal{M}_2 is a matroid. By Theorem 5 and Lemma 6, we can find a minimum weight admissible spanning tree of G for V_{NT} in polynomial time.

This polynomial-time algorithm can be extended to an $O^*(2^\ell)$ -time algorithm for general $\ell \geq 0$. For each $F \subseteq E(G[V_{\text{NT}}])$, we compute a minimum weight admissible spanning tree T for V_{NT} such that $E(T) \cap E(G[V_{\text{NT}}]) = F$. To this end, we first check whether F has no cycle. If F has a cycle, there is no such an admissible spanning tree for V_{NT} . Otherwise, we modify the matroids \mathcal{M}_1 and \mathcal{M}_2 as follows. Let G' be the multigraph obtained from G by deleting edges in $E(G[V_{\text{NT}}]) \setminus F$ and then contracting edges in F . Note that the edge set of G' corresponds to $E \setminus E(G[V_{\text{NT}}])$ and hence $\mathcal{E} = \{E_{v_1} \setminus E(G[V_{\text{NT}}]), \dots, E_{v_k} \setminus E(G[V_{\text{NT}}]), R \setminus E(G[V_{\text{NT}}])\}$ is a partition of $E(G')$. Moreover, it is easy to see that every spanning tree of G can be modified to a spanning tree of G' by contracting all edges in F and vice-versa. Now, we define $\mathcal{M}'_1 = (E(G'), \mathcal{B}_{g'})$ as the graphic matroid of G' . For $v \in V_{\text{NT}}$, we let $\ell'_v = \max(0, 2 - d_F(v))$, where $d_F(v)$ is the number of edges incident to v in F . We denote by \mathcal{M}'_2 a pair $(E(G'), \mathcal{B}'_{n-|F|-1})$, where $\mathcal{B}'_{n-|F|-1}$ is the family of edge sets $F' \subseteq E(G')$ with $|F'| = n - |F| - 1$ such that for $v \in V_{\text{NT}}$, it holds that $|F' \cap (E_v \setminus E(G[V_{\text{NT}}]))| \geq \ell'_v$. This pair is indeed a matroid as \mathcal{E} is a partition of $E(G')$. Then, every spanning tree T satisfying $E(T) \cap E(G[V_{\text{NT}}]) = F$ is admissible for V_{NT} if and only if the edge set of T belongs to the intersection of the following modified matroids \mathcal{M}'_1 and \mathcal{M}'_2 .

\mathcal{M}'_2 is a matroid consisting a pair $(E(G) \setminus E(G[V_{\text{NT}}]), \mathcal{B}'_{n-|F|-1})$, where $\mathcal{B}'_{n-|F|-1}$ is Thus, every common base in $\mathcal{B}'_g \cap \mathcal{B}'_{n-|F|-1}$ corresponds to an edge set $F' \subseteq E(G) \setminus E(G[V_{\text{NT}}])$ such that $F' \cup F$ forms an admissible spanning tree for V_{NT} in G . Again, by Theorem 5 and Lemma 6, we can compute a minimum weight admissible spanning tree T for V_{NT} in G with $E(T) \cap E(G[V_{\text{NT}}]) = F$ in polynomial time, which yields the following theorem.

► **Theorem 24.** *MINIMUM WEIGHT NON-TERMINAL SPANNING TREE can be solved in time $O^*(2^\ell)$ and polynomial space.*

As a straightforward consequence of the above theorem, MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable parameterized by the number k of non-terminals.

► **Corollary 25.** *MINIMUM WEIGHT NON-TERMINAL SPANNING TREE can be solved in time $O^*(2^{\binom{k}{2}})$ and polynomial space.*

4.3 Parameterization by tree-width

Nakayama and Masuyama [34, 36] propose polynomial-time algorithms for NON-TERMINAL SPANNING TREE on several subclasses of bounded tree-width graphs, such as outerplanar graphs and series-parallel graphs. In this section, we show that NON-TERMINAL SPANNING TREE can be solved in linear time on bounded tree-width graphs.

The property of being an admissible spanning tree for V_{NT} can be expressed by a formula in Monadic Second Order Logic, which will be discussed below. By the celebrated work of Courcelle [9] and its optimization version [2], MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable when parameterized by tree-width.

► **Theorem 26.** *MINIMUM WEIGHT NON-TERMINAL SPANNING TREE can be solved in linear time on bounded tree-width graphs.*

Proof. Let $\text{NTST}(F)$ be an MSO_2 formula that is true if and only if $F \subseteq E(G)$ forms an admissible spanning tree for V_{NT} . $\text{NTST}(F)$ can be expressed as follows:

$$\begin{aligned} \text{NTST}(F) &:= \text{connE}(F) \wedge \text{acyclic}(F) \wedge \forall v \in V, \exists e \in F \text{ inc}(v, e) \\ &\quad \wedge \forall v \in V_{\text{NT}}, \exists e_1, e_2 \in F ((e_1 \neq e_2) \wedge \text{inc}(v, e_1) \wedge \text{inc}(v, e_2)) \\ \text{acyclic}(F) &:= \forall F' \subseteq F (F' \neq \emptyset \implies \exists v \in V \text{ deg1}(v, F')). \end{aligned}$$

$\text{acyclic}(F)$ is an auxiliary formula that is true if and only if F is acyclic in G . Here, $\text{connE}(F)$ means that the subgraph induced by an edge set F is connected and $\text{deg1}(v, F')$ means that the degree of vertex v is exactly 1 in the subgraph induced by an edge set F' , which can be formulated in MSO_2 [10]. By the optimization version of Courcelle's theorem [2], MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable when parameterized by tree-width. ◀

5 Hardness results

In this section, we observe that s - t HAMILTONIAN PATH is NP-hard even on several restricted classes of graphs, which immediately implies the NP-hardness of NON-TERMINAL SPANNING TREE as well. The results in this section follow from the following observation. Let G be a graph and let v be an arbitrary vertex in G . Let G' be a graph obtained from G by adding a new vertex v' and an edge between v' and w for each $w \in N(v)$. In other words, v and v' are false twins in G' . Then, the following proposition is straightforward.

► **Proposition 27.** *Suppose that G has at least three vertices. Then G has a Hamiltonian cycle if and only if G' has a Hamiltonian path between v and v' .*

Proof. Let C be a Hamiltonian cycle of G and let w be one of the two vertices adjacent to v in C . As w is adjacent to v' in G' , $C - \{v, w\} + \{v', w\}$ is a Hamiltonian path between v and v' in G' . Conversely, let P be a Hamiltonian path between v and v' in G' . Let w and w' be the vertices of G' that are adjacent to v and v' in P , respectively. As P is a Hamiltonian path of length at least 3, w and w' must be distinct. Moreover, both vertices are adjacent to v in G , implying that $P - \{v', w'\} + \{v, w'\}$ is a Hamiltonian cycle of G . ◀

This proposition leads to polynomial-time reductions from HAMILTONIAN CYCLE to s - t HAMILTONIAN PATH on several classes of graphs. As HAMILTONIAN CYCLE is NP-hard even on strongly chordal split graphs and chordal bipartite graphs [32], the following corollary follows.

► **Corollary 28.** *s-t HAMILTONIAN PATH is NP-hard even on strongly chordal split graphs and chordal bipartite graphs.*

Furthermore, since v and v' are twins, the clique-width of G' is the same as the clique-width of G . As HAMILTONIAN CYCLE is $W[1]$ -hard when parameterized by clique-width [16], we obtain the following corollary.

► **Corollary 29.** *s-t HAMILTONIAN PATH is $W[1]$ -hard when parameterized by clique-width.*

In [11], de Melo, de Figueiredo, and Souza show that *s-t HAMILTONIAN PATH* is NP-hard even on planar graphs of maximum degree 3. The following theorem summarizes the above facts.

► **Theorem 30.** *NON-TERMINAL SPANNING TREE is NP-hard even on planar bipartite graphs of maximum degree 3, strongly chordal split graphs, and chordal bipartite graphs. Furthermore, it is $W[1]$ -hard when parameterized by clique-width.*

Furthermore, it is known that HAMILTONIAN CYCLE cannot be solved in time $O^*(2^{o(n)})$ unless Exponential Time Hypothesis (ETH) fails [10]. Thus, we immediately obtain the following theorem.

► **Theorem 31.** *NON-TERMINAL SPANNING TREE cannot be solved in time $O^*(2^{o(n)})$ unless ETH fails.*

6 Conclusion

In this paper, we studied NON-TERMINAL SPANNING TREE and MINIMUM WEIGHT NON-TERMINAL SPANNING TREE from the viewpoint of parameterized complexity. We showed that NON-TERMINAL SPANNING TREE admits a linear vertex kernel with respect to the number of non-terminal vertices k , as well as polynomial kernels with respect to vertex cover number and max leaf number. For the weighted counterpart, namely MINIMUM WEIGHT NON-TERMINAL SPANNING TREE, we give an $O^*(2^k)$ -time algorithm for graphs with polynomially-bounded integral edge weight and $O^*(2^\ell)$ -time algorithm for graphs with arbitrary edge weight, where ℓ is the number of edges in the subgraph induced by non-terminals. We proved that MINIMUM WEIGHT NON-TERMINAL SPANNING TREE is fixed-parameter tractable when parameterized by tree-width whereas it is $W[1]$ -hard when parameterized by clique-width.

As future work, we are interested in whether NON-TERMINAL SPANNING TREE can be solved in time $O^*((2 - \epsilon)^k)$ or $O^*((2 - \epsilon)^\ell)$ for some $\epsilon > 0$. Also, it would be worth considering other structural parameterizations, such as cluster deletion number or modular-width. Another possible direction is to consider the problem of finding a spanning tree that maximizes the number of internal vertices in V_{NT} . This problem simultaneously generalizes our problem and MAX INTERNAL SPANNING TREE by setting $V = V_{NT}$. It would be interesting to explore the (parameterized) approximability of this problem.

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