DIFFERENTIABLE MAPS ON LINKS OF COMPLEX ISOLATED HYPERSURFACE SINGULARITIES

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Dedicated to Professor Goo Ishikawa on the occasion of his retirement

Abstract. We consider links of complex isolated hypersurface singularities in $\mathbb{C}^{n+1}$ and study differentiable maps defined by restricting holomorphic functions to the links. We give an explicit example in which such a restriction gives a fold map into the plane $\mathbb{C} = \mathbb{R}^2$ whose singular value set consists of concentric circles.

1. Introduction

Let us consider an isolated singularity of a complex surface in $\mathbb{C}^3$. Its link is a closed connected orientable 3–dimensional manifold [11], and it is known to be a graph manifold [12, 13]. Here, a *graph manifold* is a closed orientable 3–dimensional manifold obtained from some S^1–bundles over compact surfaces with boundary attached along their torus boundaries.

On the other hand, graph manifolds can be characterized by using their C^∞ stable maps into $\mathbb{R}^2$ [14, 15]. Recall that smooth generic maps of manifolds of dimension ≥ 2 into $\mathbb{R}^2$ have folds and cusps as their singularities [9, 17, 18], and C^∞ stable maps, in general, have such singularities. In [8], it has been proved that a closed orientable 3–dimensional manifold is a graph manifold if and only if it admits a round fold map into $\mathbb{R}^2$, where a round fold map is a smooth map that has only folds as its singularities and its restriction to the singular point set is an embedding onto a family of concentric circles in $\mathbb{R}^2$ (for details, see §2).

Then, a natural question arises: can we construct an explicit round fold map into $\mathbb{R}^2 = \mathbb{C}$ on a link of an isolated surface singularity? Recall that, by a celebrated result due to Mather [10], if we restrict a real linear projection $\mathbb{C}^3 \to \mathbb{R}^2$ to the link of a surface singularity, then generically it is a C^∞ stable map. However, it has, in general, cusps and its singular value set might be very complicated. So, an explicit construction seems to be a challenging problem.

In this paper, we consider the link of the hypersurface singularity in $\mathbb{C}^{n+1}$ defined by the polynomial

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^2 + z_2^2 + \cdots + z_{n+1}^2$$

and construct an explicit round fold map on its link by restricting a complex linear function on $\mathbb{C}^{n+1}$. Although the example is very simple and the proof consists of tedious, but elementary computations, this seems to be a first example in which a simple generic map is constructed explicitly.

The paper is organized as follows. In §2, we first formulate our motivating problem, then we give some definitions and preliminary results for determining the singular point set of the relevant smooth map into the plane. In §3, we consider

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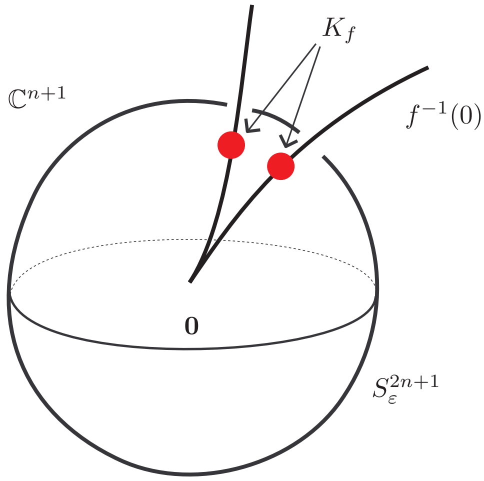

FIGURE 1. Link of an isolated hypersurface singularity

the explicit example as mentioned above and show that the restriction of a complex linear function to the hypersurface singularity link gives a round fold map into the plane and we also identify the indices of the fold loci.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. For a map $\varphi : M \to N$ between manifolds, we denote by $S(\varphi)$ the set of points in M where the differential of f does not have maximal rank $\min\{\dim M, \dim N\}$. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds.

2. PRELIMINARIES

2.1. **Problem.** Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$, $n \geq 1$, be a holomorphic function germ with an isolated critical point at the origin. In the following, we abuse the notation and denote a representative of the germ by the same letter. It is known that for a sufficiently small $\varepsilon > 0$,

$$K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$$

is a smooth closed orientable $(n-2)$–connected $(2n-1)$–dimensional manifold, called the *link* of f, where S_ε^{2n+1} is the sphere of radius ε centered at the origin in $\mathbb{C}^{n+1}$ [11] (see Fig. 1). Note also that the diffeomorphism type of K_f and also its isotopy class in S_ε^{2n+1} do not depend on the choice of ε as long as it is sufficiently small.

Now let $g : U \to \mathbb{C}$ be an arbitrary holomorphic function defined on an open neighborhood of the origin $\mathbf{0}$ in $\mathbb{C}^{n+1}$, and set $h = g|_{K_f} : K_f \to \mathbb{C} = \mathbb{R}^2$, where we take $\varepsilon > 0$ so small that K_f is contained in U. The problem that we consider in the present paper is the following.

Problem 2.1. Study the smooth map h from a global singularity theoretical viewpoint.

2.2. **Round fold maps.** We will see that in a special case, such a map h as in Problem 2.1 is a round fold map as defined below.

Definition 2.2. Let M be a smooth m–dimensional manifold and let $\varphi : M \to \mathbb{R}^p$, $m \geq p \geq 1$, be a smooth map. A singular point $q \in M$ of φ is called a *fold* if φ can be written as in the following normal form with respect to some local coordinates around q and $\varphi(q)$:

$$(x_1, x_2, \dots, x_m) \mapsto (x_1, x_2, \dots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_m^2).$$

Let λ denote the number of negative signs appearing in the last quadratic form as above. Then, the integer $\min\{\lambda, m-p+1-\lambda\}$ is well-defined and is called the *absolute index* of the fold q. A fold is *definite* if its absolute index is equal to zero, otherwise it is *indefinite*. If φ has only fold as its singularities, then it is called a *fold map* [1].

Note that if φ is a fold map, then its singular point set $S(\varphi)$ is a $(p-1)$–dimensional closed submanifold of M and the restriction $\varphi|_{S(\varphi)} : S(\varphi) \to \mathbb{R}^p$ is a codimension one immersion.

Definition 2.3. A smooth map $\varphi : M \to \mathbb{R}^p$ is called a *round fold map* if it is a fold map, each connected component of $S(\varphi)$ is diffeomorphic to the standard $(p-1)$–dimensional sphere, and the map $\varphi|_{S(\varphi)}$ is an embedding whose image is isotopic to a family of concentric spheres. For details, see [5, 6, 7].

Remark 2.4. Let M be a smooth closed m–dimensional manifold with $m \geq 2$. As mentioned in §1, a generic smooth map $\varphi : M \to \mathbb{R}^2$ has folds and cusps as its singularities. In this case, the singular point set $S(\varphi)$ is a regular 1–dimensional submanifold of M and φ restricted to $S(\varphi) \setminus \{\text{cusps}\}$ is an immersion. It is known that φ is C^∞ stable if and only if the immersion has normal crossings and for each cusp point q, we have $\varphi^{-1}(\varphi(q)) \cap S(\varphi) = \{q\}$. In particular, if $\varphi|_{S(\varphi)}$ is injective, then φ is C^∞ stable.

For a fold map $\varphi : M \to \mathbb{R}^p$ of a smooth closed m–dimensional manifold M with $m \geq p$, it is C^∞ stable if and only if the immersion $\varphi|_{S(\varphi)}$ has normal crossings (see [3]). Therefore, round fold maps are always C^∞ stable.

2.3. **Inner products.** In the following, for

$$\mathbf{u} = (u_1, u_2, \dots, u_{n+1}),\ \mathbf{v} = (v_1, v_2, \dots, v_{n+1}) \in \mathbb{C}^{n+1},$$

their *Hermitian inner product* is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{j=1}^{n+1} u_j \bar{v}_j \in \mathbb{C}.$$

On the other hand, if we regard $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ and set $u_j = u_j^0 + \sqrt{-1}u_j^1$, $v_j = v_j^0 + \sqrt{-1}v_j^1$ with $u_j^0, u_j^1, v_j^0, v_j^1 \in \mathbb{R}$, then their *real inner product* is defined as

$$(\mathbf{u}, \mathbf{v})_{\mathbb{R}} = \sum_{j=1}^{n+1} (u_j^0 v_j^0 + u_j^1 v_j^1) \in \mathbb{R}.$$

Then, the following is easily verified.

Lemma 2.5. *We always have* $(\mathbf{u}, \mathbf{v})_{\mathbb{R}} = \mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle$.

Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open set $U \subset \mathbb{C}^{n+1}$. For $\mathbf{z}_0 \in U$, we set

$$\overline{\mathbf{grad}}\, f(\mathbf{z}_0) := \left(\overline{\frac{\partial f}{\partial z_1}(\mathbf{z}_0)}, \overline{\frac{\partial f}{\partial z_2}(\mathbf{z}_0)}, \dots, \overline{\frac{\partial f}{\partial z_{n+1}}(\mathbf{z}_0)} \right).$$

We warn the reader that this notation is different from that used in [11].

Lemma 2.6. *Let $y \in \mathbb{C}$ be a regular value of f and set $M = f^{-1}(y) \subset \mathbb{C}^{n+1}$. Then, the complex tangent space $T_{\mathbf{z}_0}M$ of the complex manifold M at $\mathbf{z}_0 \in M$ can be identified with the $\mathbb{C}$–vector space*

$$\overline{\mathbf{grad}}\, f(\mathbf{z}_0)^{\perp_{\mathbb{C}}} = \{\mathbf{v} \in \mathbb{C}^{n+1} \mid \langle \mathbf{v}, \overline{\mathbf{grad}}\, f(\mathbf{z}_0)\rangle = 0\}.$$

Proof. Let $\mathbf{p}(t) = (p_1(t), p_2(t), \dots, p_{n+1}(t))$ $(t \in (-\delta, \delta), \delta > 0)$ be a C^∞ curve in M with $\mathbf{p}(0) = \mathbf{z}_0$. Since $f(\mathbf{p}(t)) = y$ is constant, we have

$$\begin{aligned} \frac{d}{dt} f(\mathbf{p}(t))\bigg|_{t=0} &= \sum_{j=1}^{n+1} \frac{\partial f}{\partial z_j}(\mathbf{z}_0)\frac{dp_j}{dt}(0) \\ &= \langle \mathbf{p}'(0), \overline{\mathbf{grad}}\, f(\mathbf{z}_0)\rangle = 0. \end{aligned}$$

Therefore, we have

$$T_{\mathbf{z_0}}M \subset \overline{\mathbf{grad}}\, f(\mathbf{z_0})^{\perp_{\mathbb{C}}}.$$

Since the two vector spaces over $\mathbb{C}$ have the same dimension, we get the desired result. $\square$

2.4. **Singular point set.** Now, let us go back to the situation of Subsection 2.1: $f : (\mathbb{C}^{n+1}, \mathbf{0}) \to (\mathbb{C}, 0)$ is a holomorphic function germ with an isolated critical point at the origin, $K = K_f = f^{-1}(0) \cap S^{2n+1}_\varepsilon$ is the link associated with f for a sufficiently small $\varepsilon > 0$, $g : U \to \mathbb{C}$ is a holomorphic function defined on an open neighborhood U of the origin $\mathbf{0}$ in C^{n+1}, and consider the smooth map $h : K \to \mathbb{C} = \mathbb{R}^2$ defined by $h = g|_K$. Note that h is a smooth map of a smooth closed $(2n-1)$–dimensional manifold K into $\mathbb{R}^2$. In the following, we assume $n \geq 2$. We denote by $S(h)$ $(\subset K)$ the set of singular points of h.

Lemma 2.7. *For $\mathbf{z}_0 \in K$, we have $\mathbf{z}_0 \in S(h)$ if and only if the three vectors $\overline{\mathbf{grad}}\, f(\mathbf{z}_0)$, $\overline{\mathbf{grad}}\, g(\mathbf{z}_0)$, and $\mathbf{z}_0$ in $\mathbb{C}^{n+1}$ are linearly dependent over $\mathbb{C}$.*

Proof. For $\mathbf{z}_0 \in K$, since $f^{-1}(0)$ and S^{2n+1}_ε intersect transversely [11], by Lemma 2.6 and its proof, we have

$$T_{\mathbf{z}_0}K = \overline{\mathbf{grad}}\, f(\mathbf{z}_0)^{\perp_{\mathbb{C}}} \cap \mathbf{z}_0^{\perp_{\mathbb{R}}},$$

where

$$T_{\mathbf{z}_0}S^{2n+1}_\varepsilon = \mathbf{z}_0^{\perp_{\mathbb{R}}} = \{\mathbf{v} \in \mathbb{C}^{n+1} \mid (\mathbf{v}, \mathbf{z}_0)_{\mathbb{R}} = 0\}.$$

Then, we see easily the following equivalences:

$$\begin{aligned} & \mathbf{z}_0 \in S(h) = S(g|_K) \\ \iff & \dim_{\mathbb{R}} \operatorname{Ker}(dg_{\mathbf{z}_0}|_{T_{\mathbf{z}_0}K}) \geq 2n-2 \\ \iff & \dim_{\mathbb{R}}((\overline{\mathbf{grad}}\, f(\mathbf{z}_0)^{\perp_{\mathbb{C}}} \cap \mathbf{z_0}^{\perp_{\mathbb{R}}}) \cap \overline{\mathbf{grad}}\, g(\mathbf{z}_0)^{\perp_{\mathbb{C}}}) \geq 2n-2 \\ \iff & \dim_{\mathbb{R}}((\overline{\mathbf{grad}}\, f(\mathbf{z}_0)^{\perp_{\mathbb{C}}} \cap \overline{\mathbf{grad}}\, g(\mathbf{z}_0)^{\perp_{\mathbb{C}}}) \cap \mathbf{z_0}^{\perp_{\mathbb{R}}}) \geq 2n-2. \end{aligned}$$

Note that $\overline{\mathbf{grad}}\, f(\mathbf{z}_0)$ is non-zero by our assumption, while $\overline{\mathbf{grad}}\, g(\mathbf{z}_0)$ might be zero. Then, the above condition, in turn, is equivalent to (2.1) or (2.2) as described below:

$$\overline{\mathbf{grad}}\, f(\mathbf{z}_0) \text{ and } \overline{\mathbf{grad}}\, g(\mathbf{z}_0) \text{ are linearly dependent over } \mathbb{C}, \text{ or} \tag{2.1}$$

$$V := \overline{\mathbf{grad}}\, f(\mathbf{z}_0)^{\perp_{\mathbb{C}}} \cap \overline{\mathbf{grad}}\, g(\mathbf{z}_0)^{\perp_{\mathbb{C}}} \subset \mathbf{z}_0^{\perp_{\mathbb{R}}}. \tag{2.2}$$

If (2.2) holds, then $\forall \mathbf{v} \in V$, we have $0 = (\mathbf{v}, \mathbf{z}_0)_{\mathbb{R}} = \operatorname{Re}\langle \mathbf{v}, \mathbf{z}_0\rangle$. As V is a vector space over $\mathbb{C}$ and we have $0 = (-\sqrt{-1}\mathbf{v}, \mathbf{z}_0)_{\mathbb{R}} = \operatorname{Re}\langle -\sqrt{-1}\mathbf{v}, \mathbf{z}_0\rangle = \operatorname{Im}\langle \mathbf{v}, \mathbf{z}_0\rangle$. Hence, we have $\langle \mathbf{v}, \mathbf{z}_0\rangle = 0$ and hence $\mathbf{v} \in \mathbf{z}_0^{\perp_{\mathbb{C}}}$. Therefore, (2.2) implies $V \subset \mathbf{z}_0^{\perp_{\mathbb{C}}}$. Note that the converse obviously holds. Furthermore, $V \subset \mathbf{z}_0^{\perp_{\mathbb{C}}}$ is equivalent to that $\mathbf{z}_0$ is a linear combination of $\overline{\mathbf{grad}}\, f(\mathbf{z}_0)$ and $\overline{\mathbf{grad}}\, g(\mathbf{z}_0)$ over $\mathbb{C}$. This completes the proof. $\square$

Remark 2.8. Kamiya [4, Theorem 1] obtains a general result in the real C^∞ setting similar to the above lemma. The above lemma can be considered to be a refined version of Kamiya's result in our situation involving links of complex isolated singularities.

When $n = 2$, the condition described in Lemma 2.7 is equivalent to that the 3×3 complex matrix consisting of the three column vectors ${}^T\overline{\mathbf{grad}}\, f(\mathbf{z}_0)$, ${}^T\overline{\mathbf{grad}}\, g(\mathbf{z}_0)$, and ${}^T\mathbf{z}_0$ satisfies

$$\det\left({}^T\overline{\mathbf{grad}}\, f(\mathbf{z}_0) \quad {}^T\overline{\mathbf{grad}}\, g(\mathbf{z}_0) \quad {}^T\mathbf{z}_0\right) = 0,$$

where "T" means the transpose. Therefore, we have the following.

Proposition 2.9. *For $n = 2$, the singular point set $S(h)$ of the smooth map $g|_K = h : K \to \mathbb{R}^2$ is given by*

$$S(h) = \{\mathbf{z} \in K \mid \det\left({}^T\overline{\mathbf{grad}}\, f(\mathbf{z}) \quad {}^T\overline{\mathbf{grad}}\, g(\mathbf{z}) \quad {}^T\mathbf{z}\right) = 0.\}$$

3. Explicit construction

Let us consider the polynomial

$$f(\mathbf{z}) = z_1^2 + z_2^2 + \cdots + z_{n+1}^2,$$

which defines a simple singularity of type A_1, where $\mathbf{z} = (z_1, z_2, \ldots, z_{n+1})$. For the moment, we assume $n \geq 2$. Setting $\varepsilon = 1$, its link is given by

$$K_f = \{\mathbf{z} \in \mathbb{C}^{n+1} \mid f(\mathbf{z}) = 0, ||\mathbf{z}||^2 = 1\}.$$

Furthermore, let us consider the holomorphic function $g : \mathbb{C}^{n+1} \to \mathbb{C}$ defined by

$$g(\mathbf{z}) = z_1 + \frac{\sqrt{-1}}{2} z_2.$$

In fact, the choice of g was made so that we get the desired result as follows; however, according to the authors' intuition, to find such an explicit example is not an easy task.

Remark 3.1. It is known that the $(2n-1)$–dimensional manifold K_f is diffeomorphic to the total space of the unit tangent sphere bundle over S^n. When $n = 2$, it is the total space of the S^1–bundle over S^2 with Euler number 2, and is diffeomorphic to the real projective space $\mathbb{R}P^3$, or the lens space $L(2, 1)$. For general n, the singularity of f at the origin is called the simple singularity of type A_1, and in a certain sense, it can be considered to be the simplest complex singularity.

The main result of this paper is the following.

Theorem 3.2. *The smooth map $h = g|_{K_f} : K_f \to \mathbb{C} = \mathbb{R}^2$ is a round fold map. The singular point set $S(h)$ consists of two circles; one consists of definite folds and the other consists of indefinite folds of absolute index $n - 1$.*

For the image $h(S(h))$ of the singular point set, see Fig. 2.

Proof. First, by Lemma 2.7, for $\mathbf{z} \in K_f$, we have $\mathbf{z} \in S(h)$ if and only if

$$\operatorname{rank}\left({}^T\overline{\mathbf{grad}}\, f(\mathbf{z}) \quad {}^T\overline{\mathbf{grad}}\, g(\mathbf{z}) \quad {}^T\mathbf{z}\right) \leq 2, \tag{3.1}$$

where

$$\begin{aligned} \overline{\mathbf{grad}}\, f(\mathbf{z}) &= 2(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{n+1}), \\ \overline{\mathbf{grad}}\, g(\mathbf{z}) &= (1, -\sqrt{-1}/2, 0, 0, \ldots, 0), \\ \mathbf{z} &= (z_1, z_2, \ldots, z_{n+1}). \end{aligned}$$

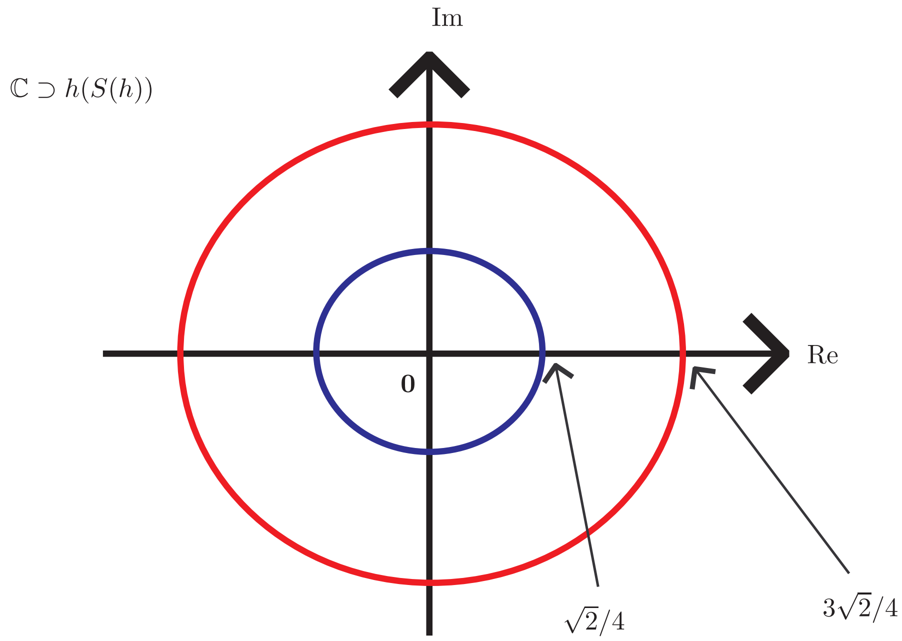

FIGURE 2. Singular value set $h(S(h))$ of h

Suppose $\mathbf{z} \in S(h)$, then we have, for $j \geq 3$,

$$\det \begin{pmatrix} 2\bar{z}_1 & 1 & z_1 \\ 2\bar{z}_2 & -\sqrt{-1}/2 & z_2 \\ 2\bar{z}_j & 0 & z_j \end{pmatrix} = 4\sqrt{-1}\,\mathrm{Im}(z_2\bar{z}_j) - 2\,\mathrm{Im}(z_1\bar{z}_j) = 0,$$

which implies that $z_1\bar{z}_j$ and $z_2\bar{z}_j$ are real numbers. Furthermore, for $j \geq 4$, we have

$$\det \begin{pmatrix} 2\bar{z}_1 & 1 & z_1 \\ 2\bar{z}_3 & 0 & z_3 \\ 2\bar{z}_j & 0 & z_j \end{pmatrix} = 4\sqrt{-1}\,\mathrm{Im}(z_3\bar{z}_j) = 0,$$

which implies that $z_3\bar{z}_j$ is a real number. Now suppose that $z_3 \neq 0$. Then, the above conditions imply that $z_1, z_2, z_4, z_5, \ldots, z_{n+1}$ are all real multiples of z_3. Since $f(\mathbf{z}) = z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0$, this implies that $z_3 = 0$, which is a contradiction. Hence, we must have $z_3 = 0$. By similar arguments, we see that $z_j = 0$ for all $j \geq 3$.

Consequently, since $\mathbf{z} \in K_f$, we have

$$S(h) \subset \{(z_1, z_2, 0, \ldots, 0) \in \mathbb{C}^{n+1} \mid z_1 = \pm\sqrt{-1}z_2, |z_1| = |z_2| = 1/\sqrt{2}\}.$$

On the other hand, if $\mathbf{z}$ belongs to the set on the right hand side above, then we see easily that (3.1) holds. Therefore, we have

$$S(h) = \{(z_1, z_2, 0, \ldots, 0) \in \mathbb{C}^{n+1} \mid z_1 = \pm\sqrt{-1}z_2, |z_1| = |z_2| = 1/\sqrt{2}\} \cong S^1 \cup S^1.$$

Furthermore, as

$$h(z_1, z_2, 0, \ldots, 0) = z_1 + \frac{\sqrt{-1}}{2}z_2 = \begin{cases} 3\sqrt{-1}z_2/2, \\ -\sqrt{-1}z_2/2, \end{cases}$$

we see that $h|_{S(h)} : S(h) \to \mathbb{C}$ is an embedding and that its image is a family of two concentric circles as depicted in Fig. 2.

Let us now show that h is a fold map. Set $L := [0, \infty) \times \{0\} \subset \mathbb{R}^2 = \mathbb{C}$. Then, $Q = h^{-1}(L)$ is a compact $(2n-2)$–dimensional manifold with non-empty boundary. Let us consider the smooth function $\psi = h|_Q : Q \to L = [0, \infty)$.

Lemma 3.3. *The function ψ is a Morse function with exactly two critical points with indices $2n-2$ and $n-1$.*

Proof. Note that by the above computation about $S(h)$, we see that the critical point set of ψ coincides with $S(h) \cap Q = \{q, q'\}$, where

$$\begin{aligned} q &= \left(\sqrt{2}/2, -\sqrt{-1}\sqrt{2}/2, 0, 0, \ldots, 0\right), \\ q' &= \left(\sqrt{2}/2, \sqrt{-1}\sqrt{2}/2, 0, 0, \ldots, 0\right). \end{aligned}$$

Let us first calculate the Hessian at the critical point q.

We easily see that

$$\mathbf{z}' = (z_3, z_4, \ldots, z_{n+1})$$

can be chosen as local coordinates around q for Q. More precisely, we consider the parametrization

$$\Phi(\mathbf{z}') = \left(\frac{\sqrt{2}}{2} + a(\mathbf{z}'), -\frac{\sqrt{2}}{2}\sqrt{-1} + b(\mathbf{z}')\sqrt{-1}, z_3, \ldots, z_{n+1}\right),$$

where $a(\mathbf{z}')$ and $b(\mathbf{z}')$ are complex functions of class C^∞ with $a(\mathbf{0}) = b(\mathbf{0}) = 0$ determined by the following:

$$\left(\frac{\sqrt{2}}{2} + a(\mathbf{z}')\right)^2 - \left(-\frac{\sqrt{2}}{2} + b(\mathbf{z}')\right)^2 + z_3^2 + \cdots + z_{n+1}^2 = 0, \tag{3.2}$$

$$\begin{aligned} &\left(\frac{\sqrt{2}}{2} + a(\mathbf{z}')\right)\left(\frac{\sqrt{2}}{2} + \overline{a(\mathbf{z}')}\right) + \left(-\frac{\sqrt{2}}{2} + b(\mathbf{z}')\right)\left(-\frac{\sqrt{2}}{2} + \overline{b(\mathbf{z}')}\right) \\ &\qquad + z_3\bar{z}_3 + \cdots + z_{n+1}\bar{z}_{n+1} = 1, \end{aligned} \tag{3.3}$$

$$\operatorname{Im} h(\Phi(\mathbf{z}')) = \operatorname{Im} a(\mathbf{z}') - \frac{1}{2}\operatorname{Im} b(\mathbf{z}') = 0. \tag{3.4}$$

Under this parametrization, by differentiating each of the equations (3.2), (3.3) and (3.4) twice, after a tedious but elementary calculation, we see that, for $3 \le j \le n+1$,

$$\begin{pmatrix} \frac{\partial^2 h}{\partial z_j^2}(q) & \frac{\partial^2 h}{\partial z_j \partial \bar{z}_j}(q) \\ \frac{\partial^2 h}{\partial \bar{z}_j \partial z_j}(q) & \frac{\partial^2 h}{\partial \bar{z}_j^2}(q) \end{pmatrix} = \begin{pmatrix} -1/4 & -3/4 \\ -3/4 & -1/4 \end{pmatrix}.$$

Furthermore, all the other entries of the Hessian of h with respect to

$$(z_3, \bar{z}_3, z_4, \bar{z}_4, \ldots, z_{n+1}, \bar{z}_{n+1})$$

are zero.

Then, for

$$x_j = \frac{z_j + \bar{z}_j}{2}, \; y_j = \frac{z_j - \bar{z}_j}{2\sqrt{-1}},$$

we have

$$\begin{pmatrix} \frac{\partial^2 h}{\partial x_j^2}(q) & \frac{\partial^2 h}{\partial x_j \partial y_j}(q) \\ \frac{\partial^2 h}{\partial y_j \partial x_j}(q) & \frac{\partial^2 h}{\partial y_j^2}(q) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, the Hessian of h with respect to $(x_3, y_3, x_4, y_4, \ldots, x_{n+1}, y_{n+1})$ is nondegenerate and it has $2n-2$ negative eigenvalues.

For the critical point $q' = \left(\sqrt{2}/2, \sqrt{-1}\sqrt{2}/2, 0, 0, \ldots, 0\right) \in S(h) \cap Q$, we can similarly calculate the Hessian and in this case, the number of negative eigenvalues

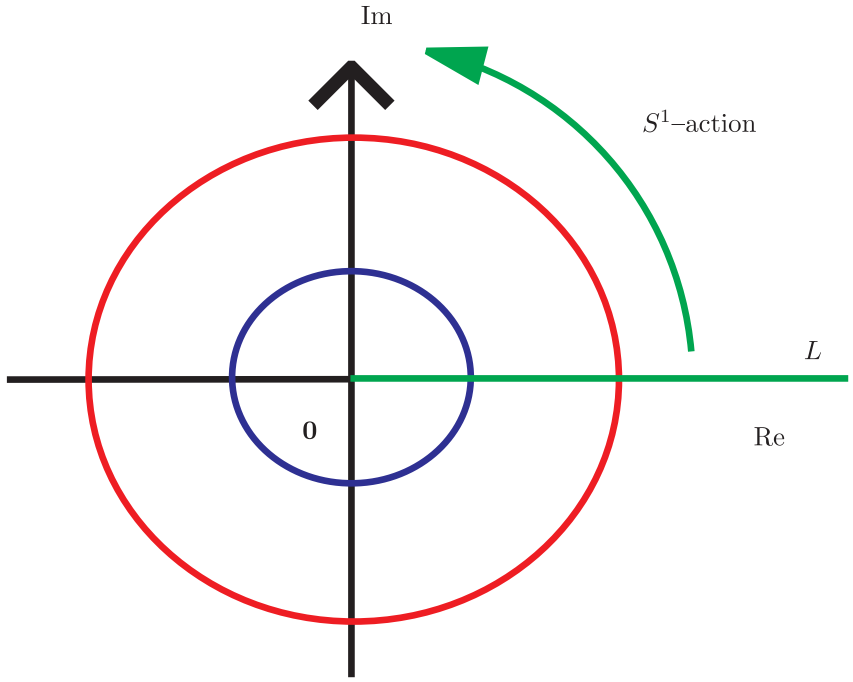

FIGURE 3. S^1–action

of each (2×2)–matrix is equal to 1 and hence the index of the critical point is equal to $n-1$. This completes the proof of Lemma 3.3. $\square$

Now, we see that the Lie group $S^1 = \{\alpha \in \mathbb{C} \,|\, |\alpha| = 1\}$ acts differentiably on K_f and on $\mathbb{C}$ from left by the natural scalar multiplication (recall that f is a homogeneous polynomial):

$$\begin{aligned}
&\alpha \in S^1, (z_1, z_2, \ldots, z_{n+1}) \in K_f, z \in \mathbb{C},\\
&\alpha \cdot (z_1, z_2, \ldots, z_{n+1}) = (\alpha z_1, \alpha z_2, \ldots, \alpha z_{n+1}),\\
&\alpha \cdot z = \alpha z.
\end{aligned}$$

For the action on $\mathbb{C}$, refer to Fig. 3.

As h is a restriction of a linear function, h is equivariant with respect to the S^1–actions:

$$h(\alpha \cdot \mathbf{z}) = \alpha \cdot h(\mathbf{z}).$$

In particular, the singular point set $S(h)$ is S^1–invariant.

Furthermore, by the S^1–action, L sweeps out $\mathbb{C}$ and it covers all the singular values. This, together with the fact that $h|_{S(h)}$ is an embedding, implies that h is a fold map.

Since its singular value set consists of two concentric circles, it is a round fold map. This completes the proof of Theorem 3.2. $\square$

Remark 3.4. When $n = 1$, $K_f \subset S^3$ is, in fact, a Hopf link. In this case, $h = g|_{K_f}$ is an embedding into $\mathbb{C}$ and its image is as depicted in Fig. 2.

As a corollary of our main theorem, we have the following, immediately.

Corollary 3.5. *Let $\eta : \mathbb{C} \to \mathbb{R}$ be an arbitrary nonzero real linear function. Then, $\eta \circ h : K_f \to \mathbb{R}$ is a Morse function with exactly four critical points of indices* 0*,* $n-1$*,* n*, and* $2n-1$.

The above corollary can be proved by using some results concerning the critical points of such a composite function, obtained in [2].

As a further challenge related to Problem 2.1, it would be an interesting problem to construct explicit examples of C^∞ stable maps on links of isolated hypersurface singularities defined by, for example, Brieskorn–Pham type polynomials.

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