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On the Number of Solutions Generated by the Dual Simplex Method

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Abstract

In this short paper, we give an upper bound for the number of different basic feasible solutions generated by the dual simplex method with Dantzig's rule for LP. The bound is comparable with the bound given by Kitahara and Mizuno [3] for the primal simplex method. We apply the result to the maximum flow problem and get a strong polynomial bound.

Keywords: Dual simplex method, Linear programming, Basic feasible solutions, Maximum flow problem.

1. Introduction

The simplex method for a linear programming problem (LP) is developed by Dantzig [1]. In spite of its practical efficiency, a good upper bound for the number of iterations when applied to general LPs has not been known for years. A strong obstacle is a pessimistic result shown by Klee and Minty [5]. They show that the simplex method needs an exponential number of iterations for a special LP.

Kitahara and Mizuno [3] give a bound for the number of different basic feasible solutions (BFSs) generated by the primal simplex method with Dantzig's rule. They extend the analysis by Ye [6] for the Markov Decision Problem to general LPs. The bound they get is

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil,$$

where m is the number of constraints, n is the number of variables, δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs, respectively, and $\lceil a\rceil$ is the smallest integer bigger than $a \in \Re$. When the primal problem is nondegenerate, it becomes a bound for the number of iterations. Kitahara et al. [2] improve the bound to

$$(n-m)\lceil\min\{m, n-m\}\frac{\gamma}{\delta}\log(\min\{m, n-m\}\frac{\gamma}{\delta})\rceil.$$

Kitahara and Mizuno [4] show that these bounds are almost tight in the sense that there are no upper bounds smaller than γ/δ.

In this paper, we analyze the dual simplex method with Dantzig's rule and obtain an upper bound for the number of dual BFSs. The bound is

$$m\lceil\min\{m, n-m\}\frac{\gamma_D}{\delta_D}\log(\min\{m, n-m\}\frac{\gamma_D}{\delta_D})\rceil,$$

where δ_D and γ_D are the minimum and the maximum values of all the positive elements of dual BFSs, respectively. When we apply the bound to the maximum flow problem, it becomes

$$(|V|+|E|)\lceil(|E|-|V|+1)\log(|E|-|V|+1)\rceil,$$

where E is a set of edges and V is a set of nodes. This is strong polynomial and better than the one obtained from the primal problem in [2].

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2. The Dual Simplex Method for LP

We consider the standard form of LP

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}, \\ & \boldsymbol{x}\geq \boldsymbol{0}, \end{array} \tag{1}$$

where $\boldsymbol{A}\in\Re^{m\times n}$, $\boldsymbol{b}\in\Re^m$, and $\boldsymbol{c}\in\Re^n$ are data and $\boldsymbol{x}\in\Re^n$ is a variable vector. The dual of (1) is

$$\begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{y}, \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y}+\boldsymbol{s}=\boldsymbol{c}, \\ & \boldsymbol{s}\geq \boldsymbol{0}, \end{array} \tag{2}$$

where $(\boldsymbol{y},\boldsymbol{s})\in\Re^m\times\Re^n$ is a variable vector. Throughout this paper, we assume that $\mathrm{rank}(\boldsymbol{A})=m$, the problems (1) and (2) have optimal solutions, and an initial BFS of (2) is available.

Given a set of indices $B\subset\{1,\dots,n\}$, we split the constraint matrix $\boldsymbol{A}^T$, the constraint vector $\boldsymbol{c}$, and the variable vector $\boldsymbol{s}$ according to B and $N=\{1,\dots,n\}-B$ like

$$\boldsymbol{A}^T=\begin{bmatrix}\boldsymbol{A}_B^T\\ \boldsymbol{A}_N^T\end{bmatrix},\ \boldsymbol{c}=\begin{bmatrix}\boldsymbol{c}_B\\ \boldsymbol{c}_N\end{bmatrix},\ \boldsymbol{s}=\begin{bmatrix}\boldsymbol{s}_B\\ \boldsymbol{s}_N\end{bmatrix}.$$

Then the dual problem (2) is written as

$$\begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{y}, \\ \text{subject to} & \boldsymbol{A}_B^T\boldsymbol{y}+\boldsymbol{s}_B=\boldsymbol{c}_B, \\ & \boldsymbol{A}_N^T\boldsymbol{y}+\boldsymbol{s}_N=\boldsymbol{c}_N, \\ & \boldsymbol{s}_B\geq\boldsymbol{0},\boldsymbol{s}_N\geq\boldsymbol{0}. \end{array} \tag{3}$$

Let B be a basis and $N=\{1,\dots,n\}-B$ be a nonbasis. Then the matrix $\boldsymbol{A}_B$ is nonsingular and a dual basic solution for B and N is

$$\boldsymbol{y}=(\boldsymbol{A}_B^T)^{-1}\boldsymbol{c}_B,\ \boldsymbol{s}_B=\boldsymbol{0},\ \boldsymbol{s}_N=\boldsymbol{c}_N-\boldsymbol{A}_N^T(\boldsymbol{A}_B^T)^{-1}\boldsymbol{c}_B.$$

If $\boldsymbol{s}_N\geq\boldsymbol{0}$, the solution is a BFS. We often refer to $(\boldsymbol{s}_B,\boldsymbol{s}_N)$ as a dual BFS associated with B and N. Let δ_D and γ_D be the minimum and the maximum values of all the positive elements of dual BFSs. Thus we have

$$\delta_D\leq s_j\leq\gamma_D \text{ if } s_j\neq 0$$

for any BFS $(\boldsymbol{s}_B,\boldsymbol{s}_N)$ and any $j\in\{1,\dots,n\}$. The values of δ_D and γ_D depend on $\boldsymbol{A}$ and $\boldsymbol{c}$, but not on $\boldsymbol{b}$.

Let B^t be the basis of the t-th iteration of the dual simplex method and $N^t=\{1,\dots,n\}-B^t$. The problem (3) can be written as

$$\begin{array}{ll} \max & \boldsymbol{b}^T(\boldsymbol{A}_{B^t}^T)^{-1}\boldsymbol{c}_{B^t}-\bar{\boldsymbol{b}}_{B^t}^T\boldsymbol{s}_{B^t}, \\ \text{subject to} & \boldsymbol{s}_{N^t}=\boldsymbol{c}_{N^t}-\boldsymbol{A}_{N^t}^T(\boldsymbol{A}_{B^t}^T)^{-1}\boldsymbol{c}_{B^t}+\bar{\boldsymbol{A}}_{N^t}^T\boldsymbol{s}_{N^t}, \\ & \boldsymbol{s}_{B^t}\geq\boldsymbol{0},\ \boldsymbol{s}_{N^t}\geq\boldsymbol{0}, \end{array}$$

where $\bar{\boldsymbol{b}}_{B^t}=\boldsymbol{A}_{B^t}^{-1}\boldsymbol{b}$ and $\bar{\boldsymbol{A}}_{N^t}=\boldsymbol{A}_{B^t}^{-1}\boldsymbol{A}_{N^t}$. This is called a dictionary of the problem (2) for the basis B^t. The coefficient vector $\bar{\boldsymbol{b}}_{B^t}$ of $\boldsymbol{s}_{B^t}$ in the objective function is called the reduced cost vector. If $\bar{\boldsymbol{b}}_{B^t}\geq\boldsymbol{0}$, the current solution is optimal. Otherwise we conduct a pivot. We choose one dual basic variable (entering variable) s_j $(j\in B^t)$ and increase the variable until one dual nonbasic variable (leaving variable) s_j $(j\in N^t)$ becomes zero. Then we exchange the two variables. We adopt Dantzig's rule for selecting an entering variable, that is, we choose an index

$$j^t=\arg\min_{j\in B^t}\bar{b}_j.$$

Set $\Delta^t=-\min_{j\in B^t}\bar{b}_j=-\bar{b}_{j^t}$, which is the absolute value of the minimum reduced cost.

3. Analysis of the Dual Simplex Method

In this section, we analyze the dual simplex method with Dantzig's rule. The analysis in this section is similar to the one in Kitahara and Mizuno [3]. The next result corresponds to Theorem 3.1 in [3].

Theorem 3.1. *Let z^* be the optimal value of (2) and $\{(\boldsymbol{y}^t,\boldsymbol{s}^t)\}$ be the sequence of iterates generated by the dual simplex method with Dantzig's rule. If $\boldsymbol{s}^{t+1}\neq\boldsymbol{s}^t$, then we have*

$$z^*-\boldsymbol{b}^T\boldsymbol{y}^{t+1}\leq\left(1-\frac{\delta_D}{\min\{m,n-m\}\gamma_D}\right)(z^*-\boldsymbol{b}^T\boldsymbol{y}^t).$$

Proof. Let B^* be a basis at an optimal BFS $(\boldsymbol{y}^*,\boldsymbol{s}^*)$ and set $N^*=\{1,\dots,n\}-B^*$. Then we have that

$$\begin{array}{ll} z^* & =\boldsymbol{b}^T\boldsymbol{y}^* \\ & =\boldsymbol{b}^T(\boldsymbol{A}_{B^t}^T)^{-1}\boldsymbol{c}_{B^t}-\bar{\boldsymbol{b}}_{B^t}^T\boldsymbol{s}_{B^t}^* \\ & =\boldsymbol{b}^T\boldsymbol{y}^t-\sum_{j\in B_t\cap N^*}\bar{b}_j s_j^* \\ & \leq\boldsymbol{b}^T\boldsymbol{y}^t+\min\{m,n-m\}\Delta^t\gamma_D, \end{array}$$

where the last inequality follows from $|B^t|=m$, $|N^*|=n-m$, $-\bar{b}_j\leq\Delta^t$, and $s_j^*\leq\gamma_D$. Rewriting the above inequality, we have

$$\Delta^t\geq\frac{z^*-\boldsymbol{b}^T\boldsymbol{y}^t}{\min\{m,n-m\}\gamma_D} \tag{4}$$

Let s_{j^t} be the entering variable chosen at the t-th iteration by Dantzig's rule. Since $\boldsymbol{s}^{t+1} \neq \boldsymbol{s}^t$, we have $s^{t+1}_{j^t} \neq 0$, which implies $s^{t+1}_{j^t} \geq \delta_D$ from the definition of δ_D. We see that

$$\begin{aligned} \boldsymbol{b}^T\boldsymbol{y}^{t+1} - \boldsymbol{b}^T\boldsymbol{y}^t &= \Delta^t s^{t+1}_{j^t} \\ &\geq \Delta^t \delta_D \\ &\geq \tfrac{(z^* - \boldsymbol{b}^T\boldsymbol{y}^t)\delta_D}{\min\{m, n-m\}\gamma_D}, \end{aligned}$$

where the last inequality follows from (4). Now it is easy to obtain the desired inequality. □

Lemma 3.1. *Let $(\boldsymbol{y}^t, \boldsymbol{s}^t)$ be the t-th iterate generated by the simplex method with Dantzig's rule. If $(\boldsymbol{y}^t, \boldsymbol{s}^t)$ is not optimal, there exists $\bar{j} \in B^* \cap N^t$ such that $s^t_{\bar{j}} > 0$ and*

$$s_{\bar{j}} \leq \min\{m, n-m\} \frac{z^* - \boldsymbol{b}^T\boldsymbol{y}}{z^* - \boldsymbol{b}^T\boldsymbol{y}^t} s^t_{\bar{j}}.$$

for any dual feasible solution $(\boldsymbol{y}, \boldsymbol{s})$.

Proof. Let $\boldsymbol{x}^*$ be an optimal primal BFS. There exists an index $\bar{j} \in B^* \cap N^t$ such that

$$z^* - \boldsymbol{b}^T\boldsymbol{y}^t = (\boldsymbol{x}^*)^T\boldsymbol{s}^t = \sum_{j \in B^* \cap N^t} x^*_j s^t_j \leq |B^* \cap N^t| x^*_{\bar{j}} s^t_{\bar{j}}$$

and $s^t_{\bar{j}} > 0$. So we have that

$$x^*_{\bar{j}} s^t_{\bar{j}} \geq \tfrac{1}{\min\{m,n-m\}}(z^* - \boldsymbol{b}^T\boldsymbol{y}^t)$$

or

$$x^*_{\bar{j}} \geq \frac{1}{\min\{m, n-m\} s^t_{\bar{j}}}(z^* - \boldsymbol{b}^T\boldsymbol{y}^t).$$

The desired result follow from this inequality and

$$x^*_{\bar{j}} s_{\bar{j}} \leq \sum_{j=1}^n x^*_j s_j = z^* - \boldsymbol{b}^T\boldsymbol{y}$$

for any dual feasible solution $(\boldsymbol{y}, \boldsymbol{s})$. □

Lemma 3.2. *Let $(\boldsymbol{y}^t, \boldsymbol{s}^t)$ be the t-th iterate generated by the simplex method with Dantzig's rule. Assume that $(\boldsymbol{y}^t, \boldsymbol{s}^t)$ is not optimal. Then there exist an index $\bar{j} \in B^*$ satisfying the following two conditions*

- $s^t_{\bar{j}} > 0$,
- *if $\lceil \min\{m, n-m\} \frac{\gamma_D}{\delta_D} \log(\min\{m, n-m\} \frac{\gamma_D}{\delta_D}) \rceil$ different BFSs are generated after the t-th iteration, then $s_{\bar{j}}$ becomes zero and stays zero thereafter.*

Proof. Let $\bar{j}$ be the index specified in Lemma 3.1. Let $l \geq t+1$ and let $\bar{l}$ be the number of different BFSs appearing between $(t+1)$-th and l-th iterations. Then we have from Theorem 3.1 and Lemma 3.1 that

$$\begin{aligned} s^l_{\bar{j}} &\leq \min\{m, n-m\} \tfrac{z^* - \boldsymbol{b}^T\boldsymbol{y}^l}{z^* - \boldsymbol{b}^T\boldsymbol{y}^t} s^t_{\bar{j}} \\ &\leq \min\{m, n-m\}(1 - \tfrac{\delta_D}{\min\{m,n-m\}\gamma_D})^{\bar{l}} s^t_{\bar{j}} \\ &\leq \min\{m, n-m\}(1 - \tfrac{\delta_D}{\min\{m,n-m\}\gamma_D})^{\bar{l}} \gamma_D. \end{aligned}$$

where the last inequality follows from the definition of γ_D. Thus if $\bar{l} > \min\{m, n-m\} \frac{\gamma_D}{\delta_D} \log(\min\{m, n-m\} \frac{\gamma_D}{\delta_D})$, $s^l_{\bar{j}}$ is less than δ_D, that is, $s^l_{\bar{j}} = 0$ from the definition of δ_D. □

Note that the event described in Lemma 3.2 occurs at most once for each index $j \in B^*$. From this observation, we get the following result.

Theorem 3.2. *When we apply the dual simplex method with Dantzig's rule for LP (2) having optimal solutions, we encounter at most*

$$m\lceil \min\{m, n-m\} \frac{\gamma_D}{\delta_D} \log(\min\{m, n-m\} \frac{\gamma_D}{\delta_D}) \rceil$$

different BFSs.

If the dual problem (2) is nondegenerate, $(\boldsymbol{y}^{t+1}, \boldsymbol{s}^{t+1}) \neq (\boldsymbol{y}^t, \boldsymbol{s}^t)$ for any t. Thus the bound in Theorem 3.2 becomes a bound for the number of iterations.

Corollary 3.1. *Assume the dual problem (2) has optimal solutions and it is nondegenerate. Then the dual simplex method with Dantzig's rule terminates in at most*

$$m\lceil \min\{m, n-m\} \frac{\gamma_D}{\delta_D} \log(\min\{m, n-m\} \frac{\gamma_D}{\delta_D}) \rceil$$

iterations.

4. Maximum Flow Problem

In this section, we apply the result in the previous section to the maximum flow problem and show a strongly polynomial bound for the number of BFSs generated by the dual simplex method.

In the maximum flow problem, we find the maximum flow f from a source s to a terminal t satisfying capacity constraints in a graph. Let $G = (V, E)$ be a graph, where V is a set of nodes and E is a set of directed arcs. Then the maximum flow problem is formulated as

$$\begin{array}{ll} \max & f, \\ \text{subject to} & \sum_{j:(s,j)\in E} x_{sj} - \sum_{j:(j,s)\in E} x_{js} = f, \\ & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = 0 \\ & \qquad\qquad\qquad\qquad \text{for } i \neq s,t, \\ & \sum_{j:(t,j)\in E} x_{tj} - \sum_{j:(j,t)\in E} x_{jt} = -f, \\ & 0 \leq x_{ij} \leq u_{ij} \text{ for each } (i,j) \in E. \end{array} \tag{5}$$

We assume that each capacity u_{ij} is integral. The dual problem of the maximum flow problem is known as the minimum cut problem, in which all the elements of BFSs are 0 or 1. Thus we have $\delta_D = \gamma_D = 1$. When we rewrite the problem (5) in the primal standard form by introducing slack variables, we have $(|V| + |E|)$ equality constraints and $(2|E| + 1)$ variables. Thus we obtain the following result from Theorem 3.2.

Corollary 4.1. *When we apply the dual simplex method with Dantzig's rule to the maximum flow problem, we encounter at most*

$$(|V| + |E|)\lceil(|E| - |V| + 1)\log(|E| - |V| + 1)\rceil$$

different BFSs.

Kitahara et al. [2] show that the primal simplex method with Dantzig's rule for the maximum flow problem generates at most

$$(|E|-|V|+1)\lceil(|E|-|V|+1)u_{max}\log(|E|-|V|+1)u_{max})\rceil$$

different BFSs, where $u_{max} = \max_{(i,j)\in E} u_{ij}$. Clearly the bound in Corollary 4.1 is better than this bound when u_{max} is not small.

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