Refined Interpolation Inequality in Besov Spaces With Applications to the Gagliardo-Nirenberg Inequality

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Refined interpolation inequality in Besov spaces with applications to the Gagliardo-Nirenberg inequality

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Abstract

We consider the interpolation inequality with respect to the regularity index $s \in \mathbb{R}$ in homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$. By choosing a general summability index $1 \le q \le \infty$ and estimating carefully, we reveal a precise representation of the constant appearing in the interpolation inequality. As an application of the refined interpolation inequality, we show a generalization of the Gagliardo-Nirenberg inequality in homogeneous Besov spaces given by Wadade (J Fourier Anal Appl. 2009;15(6):857-70).

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1 Introduction

In this paper, we consider the *interpolation inequality* in homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$, $n \ge 1$. To this end, we first give the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$; let us take $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \varphi = \{\xi \in \mathbb{R}^n \,|\, 1/2 \le |\xi| \le 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$, and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Here $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space. We also introduce the sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$ of functions by setting $\varphi_j := \mathcal{F}^{-1}[\varphi(2^{-j}\cdot)]$ for $j \in \mathbb{Z}$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Then, for $1 \le p \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by setting

$$\dot{B}^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \;\middle|\; \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty \right\}$$

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with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left\{ \sum_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)})^q \right\}^{1/q} & \text{if } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty, \end{cases} \tag{1.1}$$

where $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{P}(\mathbb{R}^n)$ denote the dual space of $\mathscr{S}(\mathbb{R}^n)$ and the subspace of $\mathscr{S}'(\mathbb{R}^n)$ consisting of polynomials with complex coefficients, respectively.

It is well known that the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ are characterized by the *real interpolation spaces* of the homogeneous Sobolev spaces;

$$\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n) = (\dot{H}^{s_0,p}(\mathbb{R}^n), \dot{H}^{s_1,p}(\mathbb{R}^n))_{\theta,q}, \tag{1.2}$$

where $s_0 < s_1$, $1 \le q \le \infty$, and $0 < \theta < 1$ [1, Theorem 6.4.5 and line 5 on p.150]. Here we remark that such a characterization (1.2) yields the equivalence of norms, but in this paper we agree that the norm $\|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ of the homogeneous Besov spaces is given by the definition (1.1) at any time; we will discuss representations of constants appearing in some inequalities, so we keep using the definition (1.1) of the norm $\|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ to avoid confusion. Similarly to the relation (1.2), it is also known that

$$\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n) = (\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n), \dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} \tag{1.3}$$

for all $q, q_0, q_1 \in [1,\infty]$ whenever $s_0 < s_1$ and $0 < \theta < 1$ with equivalence of norms. Since we may choose $q = 1$ and $q_0 = q_1 = \infty$ in (1.3), the usual property [2, Corollary 1.7] of the real interpolation spaces implies that

$$\|f\|_{\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,1}(\mathbb{R}^n)} \le C_0 \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,\infty}(\mathbb{R}^n)} \|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^n)} \tag{1.4}$$

for all $f \in \dot{B}^{s_0}_{p,\infty}(\mathbb{R}^n) \cap \dot{B}^{s_1}_{p,\infty}(\mathbb{R}^n)$. We also remark that the interpolation inequality (1.4) may be shown by a *direct computation* based on the definition of the homogeneous Besov spaces (cf. [3, Theorem 1], [4, Proposition 2.22]). Since the continuous embeddings $\dot{B}^s_{p,1}(\mathbb{R}^n) \subset \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ hold due to the embedding $l^1(\mathbb{Z}) \subset l^\infty(\mathbb{Z})$ of the sequence spaces, we may regard the inequality (1.4) as a *gain of regularities* in terms of the *summability index q* (See also, e.g., [5, Theorem 2], [6, Lemma 3.2], [7, Theorem 2.1], [8, Theorem 1.5], [9, Theorem 1.1], [10, Theorem 1.2], [11, Theorems 1.2 and 1.3]). Such an effect is obtained by choosing θ, s_0, and s_1 such that $0 < \theta < 1$ and $s_0 < s_1$, so in general we may *no longer expect* that (1.4) holds in the case of $\theta = 0$, $\theta = 1$, or $s_0 = s_1$. Thus we expect that the constant C_0 appearing in (1.4) *diverges* as the limit $\theta \to +0$, $\theta \to 1-0$, or $s_1 \to s_0 + 0$. In fact, the result of [4, Proposition 2.22] implies that the constant C_0 appearing in (1.4) behaves like $C_0 = O(\theta^{-1})$ as $\theta \to +0$, $C_0 = O((1-\theta)^{-1})$ as $\theta \to 1-0$, and $C_0 = O((s_1-s_0)^{-1})$ as $s_1 \to s_0 + 0$. Concerning this property, our interest in this paper is to reveal the *behaviors* of a constant appearing in the following interpolation inequality;

$$\|f\|_{\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n)} \le C_1 \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n)} \|f\|^{\theta}_{\dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n)},$$

where $1 \le q \le \infty$ and $q_0, q_1 \in [q, \infty]$. Although the above inequality is weaker than the original one (1.4), we may expect that the divergence rates of C_1 are *relaxed* under the condition $q > 1$, $q_0 < \infty$, or $q_1 < \infty$. In our main results, we give a *precise representation* of the constant C_1.

In addition, as an application of the refined interpolation inequality, we also show the following *Gagliardo-Nirenberg inequality*

$$\|f\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^n)} \le C_2 \|f\|^{p_0/p}_{\dot{B}^{s}_{p_0,q_0}(\mathbb{R}^n)} \|f\|^{1-p_0/p}_{\dot{B}^{s+n/p_1}_{p_1,q_1}(\mathbb{R}^n)}$$

for $p \in (p_0,\infty) \cap [p_1,\infty)$, $1 \le q \le \infty$, and $q_0, q_1 \in [q,\infty]$ and reveal the *behaviors* of C_2 as $p \to p_0+0$ or $p \to \infty$. We remark that a similar result has been shown by Wadade [9, Theorem 1.2], but our result may be regarded as a *generalization* of that of [9, Theorem 1.2]. We will discuss a more detailed comparison together with our main results in the next subsection.

1.1 Main results

Let us state our main results. First we give the refined interpolation inequality in homogeneous Besov spaces in the sense of revealing the *dependence* of the constant on the indices. We again note that the norm $\|\cdot\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^n)}$ appearing in this paper is given by (1.1) at any time.

Theorem 1.1. *Let* $1 \le p \le \infty$, $1 \le q \le \infty$, *and* $q_0, q_1 \in [q,\infty]$. *Suppose that* $s_0, s_1 \in \mathbb{R}$ *satisfy* $s_0 < s_1$ *and* $f \in \dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n) \cap \dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n)$. *Then it holds that* $f \in \dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n)$ *for all* $0 < \theta < 1$ *with the estimate*

$$\|f\|_{\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n)} \le M\left(s_1 - s_0, \theta, q, \frac{1}{q} - \frac{1}{q_0}, \frac{1}{q} - \frac{1}{q_1}\right) \|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n)} \|f\|^{\theta}_{\dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n)}, \tag{1.5}$$

where

$$M(\overline{s},\theta,q,\eta_0,\eta_1) := 2^{1/q} \left\{ \frac{1}{\theta \log 2} \left(1 + \frac{\eta_0}{\overline{s}}\right) \right\}^{(1-\theta)\eta_0} \left\{ \frac{1}{(1-\theta) \log 2} \left(1 + \frac{\eta_1}{\overline{s}}\right) \right\}^{\theta\eta_1}. \tag{1.6}$$

Remark 1.2. (i) As stated before, we may not expect that the inequality (1.5) is still valid in the case of $\theta = 0$, $\theta = 1$, or $s_0 = s_1$, so the constant appearing in (1.5) might *diverge* by taking the corresponding limits. By the definition (1.6) of the constant M, we obtain the following upper bounds of divergence rates

$$\|f\|_{\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n)} = \begin{cases} O(\theta^{-(1/q-1/q_0)}) & \text{as } \theta \to +0, \\ O((1-\theta)^{-(1/q-1/q_1)}) & \text{as } \theta \to 1-0, \\ O((s_1-s_0)^{-\{1/q-(1-\theta)/q_0-\theta/q_1\}}) & \text{as } s_1 \to s_0+0 \end{cases} \tag{1.7}$$

for all $f \in \dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n) \cap \dot{B}^{s_1}_{p,q_1}(\mathbb{R}^n)$. Thus we observe that if we choose $q = 1$ and $q_0 = q_1 = \infty$, then the corresponding rates coincide with those of [4, Proposition 2.22]. We also see that the remaining choices of q, q_0, and q_1 give *relaxations* of divergence rates as expected previously. Moreover, if $q = q_0$, then the first estimate of (1.7) *does not diverge* as $\theta \to +0$. Since the space $\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n)$ appearing on the left-hand side of (1.7) is given as $\dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n)$ in the case of $q = q_0$ and $\theta = 0$ and since this space coincides with the space appearing in the assumption $f \in \dot{B}^{s_0}_{p,q_0}(\mathbb{R}^n)$, we see that the boundedness as $\theta \to +0$ is derived *naturally.* We may obtain a similar boundedness as $\theta \to 1-0$ in the case of $q = q_1$. In addition, concerning the limit $s_1 \to s_0+0$, we have to assume $q = q_0 = q_1$ to ensure that the third estimate of (1.7) does not diverge.

(ii) We have already mentioned that the proof of the interpolation inequality (1.4) by a direct computation has been given in [3, Theorem 1] and [4, Proposition 2.22]. In particular, the result of [4, Proposition 2.22] implies that the constant C_0 appearing in (1.4) has the following representation;

$$C_0 = \frac{c}{\theta(1-\theta)(s_1 - s_0)}, \tag{1.8}$$

where $c > 0$ is a constant independent of s_0, s_1, and θ. After that, the authors [12, Proposition 3.7] obtained the explicit representation of the constant

$$C_0 = \frac{1}{\theta(1-\theta)\log 2}\left(1 + \frac{1}{s_1 - s_0}\right). \tag{1.9}$$

Our representation (1.9) is a little bit *different* from (1.8) in terms of the value of the limit $\lim_{s_1\to\infty} C_0$. Actually, we suspect that the representation (1.8) has a very *slight issue*; they have used the following estimate

$$\frac{1}{1 - 2^{-(1-\theta)(s_1-s_0)}} + \frac{1}{1 - 2^{-\theta(s_1-s_0)}} \le \frac{c}{\theta(1-\theta)(s_1 - s_0)}$$

for all $s_0 < s_1$ and $0 < \theta < 1$, but if $c > 0$ is independent of s_1, then the left-hand side tends to 2 and the right-hand side tends to 0 as $s_1 \to \infty$, respectively. Since this should be a contradiction, we have *modified* the representation (1.8) by adding 1 to part $1/(s_0 - s_1)$ in our preceding result [12, Proposition 3.7]. Likewise, the representation (1.6) in our main result is similar to (1.9). For more details of such a modification, see Proposition 2.1 below.

(iii) In this result, we assume that $q_0, q_1 \in [q, \infty]$, but actually the interpolation inequality (1.5) with the constant M replaced by some $C > 0$ is valid even if $q_0, q_1 \in [1, \infty]$. Remark that we aim to reveal the upper bounds of *divergence rates* of M, and hence the case of $q_0, q_1 \in [q, \infty]$ is more interesting. Otherwise, say, if $q_0 < q$, then we see by the continuous embedding $\dot{B}^s_{p,q_0}(\mathbb{R}^n) \subset \dot{B}^s_{p,q}(\mathbb{R}^n)$ that such a case is reduced to the case of $q_0 = q$. In addition, as mentioned above, the interpolation inequality (1.5) may be obtained by the relation (1.3) and the usual property [2, Corollary 1.7] of the real interpolation spaces. On the one hand (1.3) implies the relation due to the interpolation for the regularity index, but on the other hand it is also known that

$$\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,p}(\mathbb{R}^n) = (\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n))_{\theta,q}$$

for all $p_0, p_1, q_0, q_1 \in [1, \infty]$ whenever $s_0 < s_1$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1 = (1-\theta)/q_0 + \theta/q_1$ [1, Theorem 6.4.5 and line 5 on p.150], namely we see that

$$\|f\|_{\dot{B}^{(1-\theta)s_0+\theta s_1}_{p,p}(\mathbb{R}^n)} \le C\|f\|^{1-\theta}_{\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}\|f\|^{\theta}_{\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}$$

with some constant $C > 0$. It seems an interesting issue to reveal the dependence of the above constant $C > 0$ on the indices as well.

Next we give the Gagliardo-Nirenberg inequality [13, 14] in homogeneous Besov spaces which is obtained by applying Theorem 1.1.

Corollary 1.3. *Let $p_0, p_1 \in [1, \infty)$, $s \in \mathbb{R}$, $1 \le q \le \infty$, and $q_0, q_1 \in [q, \infty]$. Suppose that $f \in \dot{B}^s_{p_0,q_0}(\mathbb{R}^n) \cap \dot{B}^{s+n/p_1}_{p_1,q_1}(\mathbb{R}^n)$. Then it holds that $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ for all $p \in (p_0, \infty) \cap [p_1, \infty)$ with the estimate*

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \le C_n \left(\frac{1}{p_0} - \frac{1}{p}\right)^{-(1/q-1/q_0)} p^{1/q-1/q_1}\|f\|^{p_0/p}_{\dot{B}^s_{p_0,q_0}(\mathbb{R}^n)}\|f\|^{1-p_0/p}_{\dot{B}^{s+n/p_1}_{p_1,q_1}(\mathbb{R}^n)}, \tag{1.10}$$

where $C_n > 0$ is a constant depending only on n.

Remark 1.4. (i) Similarly to Theorem 1.1, we see that letting $p \to p_0 + 0$ or $p \to \infty$ yields the divergence of the constant appearing in (1.10). However, in the case of $q = q_0$ or $q = q_1$, the constant is still bounded even if we take the corresponding limits. In particular, concerning the divergence of the constant as the limit $p \to \infty$, we may describe such a phenomenon by noting the *limiting case* of the Sobolev embedding; since $\dot{B}^0_{p,1}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $1 \le p \le \infty$, setting $s = 0$ and $q = 1$ in (1.10) implies that

$$\|f\|_{L^p(\mathbb{R}^n)} = O(p^{1-1/q_1}) \quad \text{as } p \to \infty$$

for all $f \in \dot{B}^0_{p_0,q_0}(\mathbb{R}^n) \cap \dot{B}^{n/p_1}_{p_1,q_1}(\mathbb{R}^n)$. We remark that $\dot{H}^{n/p_1,p_1}(\mathbb{R}^n) \not\subset L^\infty(\mathbb{R}^n)$ for any $1 \le p_1 < \infty$ but $\dot{B}^{n/p_1}_{p_1,1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for all $1 \le p_1 \le \infty$. Therefore, it is a *natural result* that the constant diverges as $p \to \infty$ if $q_1 > 1$ but is still bounded if $q_1 = 1$.

(ii) As mentioned before, our result (1.10) may be regarded as a *generalization* of that of Wadade [9, Theorem 1.2]. In fact, he has shown the following inequality

$$\|f\|_{\dot{B}^0_{p,q}(\mathbb{R}^n)} \le C_n p^{1/q-1/q_1} \|f\|^{p_0/p}_{\dot{B}^0_{p_0,q}(\mathbb{R}^n)} \|f\|^{1-p_0/p}_{\dot{B}^{n/p_1}_{p_1,q_1}(\mathbb{R}^n)} \tag{1.11}$$

for all $p_0, p_1 \in [1, \infty)$, $\max\{p_0, p_1\} \le p < \infty$, and $1 \le q \le q_1 \le \infty$. By choosing $s = 0$ and $q_0 = q$ in (1.10), we see that (1.10) coincides with (1.11). Although we have to exclude the limiting case $p = p_0$ in our result unlike that of [9, Theorem 1.2], since setting $p = p_0$ in (1.11) implies a trivial inequality $\|f\|_{\dot{B}^0_{p_0,q}(\mathbb{R}^n)} \le C_n \|f\|_{\dot{B}^0_{p_0,q}(\mathbb{R}^n)}$, our restriction $p > p_0$ *does not* yield an essential disadvantage. Or rather, we may also take q_0 such that $q_0 > q$ and see the divergence rate as $p \to p_0 + 0$. It should be mentioned that Wadade [15, Theorem 1.1], [16, Theorem 1.4] has also shown the Gagliardo-Nirenberg inequality in Besov-type spaces which is closely related to our result.

1.2 Some previous works on the Gagliardo-Nirenberg inequality

At the end of this section, let us state the previous works on the Gagliardo-Nirenberg inequality with explicit dependence of constants with respect to the indices. Ogawa [17, Lemma 2] obtained

$$\|f\|_{L^p(\Omega)} \le \frac{1}{(4\pi)^{1/2-1/p}} \sqrt{\frac{p}{2}} \|f\|^{2/p}_{L^2(\Omega)} \|\nabla f\|^{1-2/p}_{L^2(\Omega)}$$

for all $2 \le p < \infty$ and $f \in H^1_0(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is an arbitrary domain. Ogawa and the first author [18, Theorem 2] showed

$$\|f\|_{L^p(\mathbb{R}^n)} \le \frac{1}{a_n^{1/2-1/p}} \sqrt{\frac{p}{2}} \|f\|^{2/p}_{L^2(\mathbb{R}^n)} \|(-\Delta)^{n/4} f\|^{1-2/p}_{L^2(\mathbb{R}^n)}$$

for all $2 \le p < \infty$ and $f \in H^{n/2}(\mathbb{R}^n)$, where $a_n := 2^n \pi^{n/2} \Gamma(n/2 + 1)$ and Γ denotes the gamma function. The case of the general domain $\Omega \subset \mathbb{R}^n$ is also discussed in [18, Theorem 2]. The first author [19, Proposition] gave

$$\|f\|_{L^p(\mathbb{R}^n)} \le C_{n,p_0} p^{1-1/p_0} \|f\|^{p_0/p}_{L^{p_0}(\mathbb{R}^n)} \|(-\Delta)^{n/(2p_0)} f\|^{1-p_0/p}_{L^{p_0}(\mathbb{R}^n)}$$

for all $1 < p_0 \le p < \infty$ and $f \in H^{n/p_0,p_0}(\mathbb{R}^n)$ with some constant $C_{n,p_0} > 0$ depending only on n and p_0. Del Pino and Dolbeault [20, Theorem 1], [21, Theorem 1.2] focused the following form

$$\|f\|_{L^p(\mathbb{R}^n)} \le K \|f\|^{1-\theta}_{L^{p_0}(\mathbb{R}^n)} \|\nabla f\|^{\theta}_{L^{p_1}(\mathbb{R}^n)}, \quad p := \frac{p_1(p_0 - 1)}{p_1 - 1}, \, \theta := \frac{(p_0 - p_1)n}{(p_0 - 1)\{np_1 - (n - p_1)p_0\}}$$

for all $1 < p_1 < n$, $p_1 < p_0 \le p_1(n-1)/(n-p_1)$, and $f \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{1,p_1}(\mathbb{R}^n)$ and obtained an explicit representation of the constant $K > 0$. They also showed that such a constant K is indeed optimal by giving a function f such that the above inequality becomes the equality. For the results on the best constants of Gagliardo-Nirenberg or Sobolev inequalities, we also refer to, e.g., Aubin [22], Talenti [23], Cordero-Erausquin-Nazaret-Villani [24], Cotsiolis-Tavoularis [25], and Cianchi-Fusco-Maggi-Pratelli [26]. Let us go back to generalizations of the Gagliardo-Nirenberg inequality. Kozono and Wadade [27, Theorem 2.1] showed

$$\|f\|_{L^p(\mathbb{R}^n)} \le \frac{C_n p_1}{p_1 - 1} p^{1-1/p_1} \|f\|_{L^{p_0}(\mathbb{R}^n)}^{p_0/p} \|(-\Delta)^{n/(2p_1)} f\|_{L^{p_1}(\mathbb{R}^n)}^{1-p_0/p}$$

for all $1 \le p_0 \le p < \infty$ and $1 < p_1 < \infty$ and $f \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{n/p_1,p_1}(\mathbb{R}^n)$ with some constant $C_n > 0$ depending only on n. They [27, Theorem 2.2] also obtained

$$\|f\|_{L^p(\mathbb{R}^n)} \le C_n p \|f\|_{L^{p_0}(\mathbb{R}^n)}^{p_0/p} \|f\|_{BMO(\mathbb{R}^n)}^{1-p_0/p}$$

for all $1 \le p_0 \le p < \infty$ and $f \in L^{p_0}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ as the limiting case of the above result, where $BMO(\mathbb{R}^n)$ denotes the space of functions of bounded mean oscillation. In addition, Wadade [9, Theorem 1.1] gave

$$\|f\|_{L^p(\mathbb{R}^n)} \le C_{n,p_1} p^{1-1/q_1} \|f\|_{L^{p_0}(\mathbb{R}^n)}^{p_0/p} \|f\|_{\dot{B}^{n/p_1}_{p_1,q_1}(\mathbb{R}^n)}^{1-p_0/p}$$

for all $1 \le p_0 \le p_1 < \infty$, $p_0 \le p < \infty$, and $1 \le q_1 \le \infty$ and $f \in L^{p_0}(\mathbb{R}^n) \cap \dot{B}^{n/q_1}_{p_1,q_1}(\mathbb{R}^n)$ with some constant $C_{n,p_1} > 0$ depending only on n and p_1. For the other versions of such results, we refer to, e.g., Nagayasu-Wadade [28] and Sawano-Wadade [29].

We also refer to some previous works on further generalizations of the Gagliardo-Nirenberg inequality in Besov-type spaces; Gerard, Meyer, and Oru [5, Theorem 2] showed

$$\|f\|_{L^{p_1/\theta}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n)}^{1-\theta} \|f\|_{\dot{H}^{s,p_1}(\mathbb{R}^n)}^{\theta}$$

for all $1 < p_1 < \infty$ and $s, \sigma \in (0,\infty)$ and $f \in \dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n) \cap \dot{H}^{s,p_1}(\mathbb{R}^n)$, where $\theta := \sigma/(s+\sigma) \in (0,1)$. Cohen, Dahmen, Daubechies, and DeVore [8, Theorem 1.5] established

$$\|f\|_{\dot{B}^{(1-\theta)s+\theta}_{p,p}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^{s}_{p_1,p_1}(\mathbb{R}^n)}^{1-\theta} \|f\|_{\dot{BV}(\mathbb{R}^n)}^{\theta}$$

for all $1 < p_1 \le \infty$, $s \in \mathbb{R} \setminus [1/p_1, 1]$, and $0 < \theta < 1$ and $f \in \dot{B}^{s}_{p_1,p_1}(\mathbb{R}^n) \cap \dot{BV}(\mathbb{R}^n)$, where p satisfies $1/p = (1-\theta)/p_1 + \theta$ and $\dot{BV}(\mathbb{R}^n)$ denotes the space of functions of bounded variation. For fairly recent contributions, Dao, Lam, and Lu [10, Theorem 1.2] obtained

$$\|f\|_{\dot{H}^{k,p_1/\theta}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n)}^{1-\theta} \|f\|_{\dot{H}^{m,p_1}(\mathbb{R}^n)}^{\theta}$$

for all $1 \le p_1 < \infty$, $\sigma \ge 0$, and $k, m \in \mathbb{N}$ such that $1 \le k < m$ and $f \in \dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n) \cap \dot{H}^{m,p_1}(\mathbb{R}^n)$, where $\theta := (k+\sigma)/(m+\sigma) \in (0,1)$. In addition, Dao [11, Theorems 1.2 and 1.3] extended the preceding result by using the Sobolev-Slobodeckij spaces; more precisely, he showed

$$\|f\|_{\dot{B}^{\alpha}_{p_1/\theta,p_1/\theta}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n)}^{1-\theta} \|f\|_{\dot{B}^{\beta}_{p_1,p_1}(\mathbb{R}^n)}^{\theta}$$

for all $1 \le p_1 \le \infty$, $\sigma \ge 0$, $\alpha \in (0,\infty) \setminus \mathbb{N}$, and $\beta \in (\alpha,\infty) \cap (1/p_1 - \sigma, \infty) \setminus \mathbb{N}$ and $f \in \dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n) \cap \dot{B}^{\beta}_{p_1,p_1}(\mathbb{R}^n)$, where $\theta := (\alpha+\sigma)/(\beta+\sigma) \in (0,1)$. The case $\alpha = 0$, $\alpha \in \mathbb{N}$, or $\beta \in \mathbb{N}$ is also discussed in [11, Theorems 1.2 and 1.3]. In these results, although we do not know any information about the dependence of the constant $C > 0$ on the indices, we may apply the inequalities under the assumption $f \in \dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^n)$ instead of the usual $L^p(\mathbb{R}^n)$-type condition. Finally, we also mention that Van Schaftingen [30] gave the result in this direction by considering the case of an open convex set $\Omega \subset \mathbb{R}^n$. The characterizations of the necessary and sufficient conditions of such inequalities have been established as well [31–33].

2 Proof of the main results

In what follows, we give the proof of our main results, Theorem 1.1 and Corollary 1.3. For simplicity, we will abbreviate $L^p := L^p(\mathbb{R}^n)$ and $\dot{B}^s_{p,q} := \dot{B}^s_{p,q}(\mathbb{R}^n)$.

2.1 Refined interpolation inequality: Proof of Theorem 1.1

Let us show Theorem 1.1. The proof relies only on simple calculations like the method in [3, Theorem 1] and [4, Proposition 2.22], namely, we *split* the series appearing in the definition of the homogeneous Besov spaces into two parts and apply the *Hölder inequalities*. As stated in Remark 1.2 (ii), we slightly *modify* the estimate used in [4, Proposition 2.22]. Although such an estimate is shown by a very simple calculation and we have already used it in our preceding result [12, Proposition 3.7], we start with its detailed proof since the estimate yields a *unique difference* from that of [4, Proposition 2.22].

Proposition 2.1. *There holds*

$$\sup_{t\in(0,\infty)} \left(\frac{1}{1-2^{-\alpha t}} \cdot \frac{t}{t+1} \right) = \frac{1}{\alpha \log 2}$$

for all $0 < \alpha < 1$.

Proof. By setting

$$\psi(t,\alpha) := 1 - 2^{-\alpha t} - \alpha t(t+1)2^{-\alpha t} \log 2$$

for $0 \le t < \infty$ and $0 < \alpha < 1$, we have

$$\begin{aligned} \partial_t \psi(t,\alpha) &= \alpha 2^{-\alpha t} \log 2 - \alpha(2t+1)2^{-\alpha t} \log 2 + \alpha^2 t(t+1)2^{-\alpha t}(\log 2)^2 \\ &= \alpha t 2^{-\alpha t} \log 2 \cdot (\alpha(t+1)\log 2 - 2), \end{aligned}$$

which yields

$$\partial_t \psi(t,\alpha) < 0 \text{ if } 0 < t < 2(\alpha \log 2)^{-1} - 1, \quad \partial_t \psi(t,\alpha) > 0 \text{ if } 2(\alpha \log 2)^{-1} - 1 < t < \infty.$$

In addition, since $\psi(0,\alpha) = 0$ and $\lim_{t\to\infty} \psi(t,\alpha) = 1$ for all $0 < \alpha < 1$, there exists a unique $2(\alpha \log 2)^{-1} - 1 < t_\alpha < \infty$ such that

$$\psi(t,\alpha) < 0 \text{ if } 0 < t < t_\alpha, \quad \psi(t,\alpha) > 0 \text{ if } t_\alpha < t < \infty.$$

Therefore, we see by

$$\begin{aligned}
&\partial_t\left(\frac{1}{1-2^{-\alpha t}}\cdot\frac{t}{t+1}\right)=-\frac{\alpha 2^{-\alpha t}\log 2}{(1-2^{-\alpha t})^2}\cdot\frac{t}{t+1}+\frac{1}{1-2^{-\alpha t}}\cdot\frac{1}{(t+1)^2}=\frac{\psi(t,\alpha)}{(1-2^{-\alpha t})^2(t+1)^2},\\
&\lim_{t\to+0}\left(\frac{1}{1-2^{-\alpha t}}\cdot\frac{t}{t+1}\right)=\lim_{t\to+0}\frac{t-0}{2^{-\alpha t}-1}\cdot\lim_{t\to+0}\left(-\frac{1}{t+1}\right)=\frac{1}{\partial_t(2^{-\alpha t})|_{t=0}}\cdot(-1)\\
&\qquad\qquad=-\left.\frac{1}{-\alpha 2^{-\alpha t}\log 2}\right|_{t=0}=\frac{1}{\alpha\log 2},\\
&\lim_{t\to\infty}\left(\frac{1}{1-2^{-\alpha t}}\cdot\frac{t}{t+1}\right)=1<\frac{1}{\alpha\log 2}
\end{aligned}$$

that the desired estimate holds. □

Proof of Theorem 1.1. Set $s := (1-\theta)s_0+\theta s_1$ for simplicity. Suppose that $f\neq 0$ in the sense of $\mathscr{S}'/\mathscr{P}$. First, we shall verify the case of $q=q_0=q_1=\infty$. In this case, since

$$2^{sj}\|\varphi_j*f\|_{L^p}=(2^{s_0j}\|\varphi_j*f\|_{L^p})^{1-\theta}\cdot(2^{s_1j}\|\varphi_j*f\|_{L^p})^{\theta}\le\|f\|^{1-\theta}_{\dot B^{s_0}_{p,\infty}}\|f\|^{\theta}_{\dot B^{s_1}_{p,\infty}}$$

for all $j\in\mathbb{Z}$, the desired estimate trivially holds.

Next, let us consider the case of $1\le q<\infty$ and $q_0,q_1\in(q,\infty)$. Then, since

$$\begin{aligned}
\|f\|^q_{\dot B^s_{p,q}}&=\sum_{j=-\infty}^{j_*-1}(2^{sj}\|\varphi_j*f\|_{L^p})^q+\sum_{j=j_*}^{\infty}(2^{sj}\|\varphi_j*f\|_{L^p})^q\\
&=\sum_{j=-\infty}^{j_*-1}2^{(s_1-s_0)\theta qj}(2^{s_0j}\|\varphi_j*f\|_{L^p})^q+\sum_{j=j_*}^{\infty}2^{-(s_1-s_0)(1-\theta)qj}(2^{s_1j}\|\varphi_j*f\|_{L^p})^q
\end{aligned}\tag{2.1}$$

for all $j_*\in\mathbb{Z}$, by taking $\lambda_0,\lambda_0',\lambda_1,\lambda_1'\in(1,\infty)$ so that

$$\frac{1}{\lambda_0}+\frac{1}{\lambda_0'}=\frac{1}{\lambda_1}+\frac{1}{\lambda_1'}=1\tag{2.2}$$

and applying the Hölder inequalities, we have

$$\begin{aligned}
\|f\|^q_{\dot B^s_{p,q}}&\le\left\{\sum_{j=-\infty}^{j_*-1}2^{(s_1-s_0)\theta qj\cdot\lambda_0'}\right\}^{1/\lambda_0'}\left\{\sum_{j=-\infty}^{j_*-1}(2^{s_0j}\|\varphi_j*f\|_{L^p})^{\lambda_0q}\right\}^{1/\lambda_0}\\
&\quad+\left\{\sum_{j=j_*}^{\infty}2^{-(s_1-s_0)(1-\theta)qj\cdot\lambda_1'}\right\}^{1/\lambda_1'}\left\{\sum_{j=j_*}^{\infty}(2^{s_1j}\|\varphi_j*f\|_{L^p})^{\lambda_1q}\right\}^{1/\lambda_1}\\
&\le\frac{2^{(s_1-s_0)\theta q(j_*-1)}}{(1-2^{-(s_1-s_0)\theta\lambda_0'q})^{1/\lambda_0'}}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0q}}+\frac{2^{-(s_1-s_0)(1-\theta)qj_*}}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1'q})^{1/\lambda_1'}}\|f\|^q_{\dot B^{s_1}_{p,\lambda_1q}}
\end{aligned}\tag{2.3}$$

for all $j_*\in\mathbb{Z}$. Here we have used the conditions $s_0<s_1$ and $0<\theta<1$. Hence, choosing $j_*\in\mathbb{Z}$ such that

$$2^{(s_1-s_0)q(j_*-1)}\le\frac{(1-2^{-(s_1-s_0)\theta\lambda_0'q})^{1/\lambda_0'}\|f\|^q_{\dot B^{s_1}_{p,\lambda_1q}}}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1'q})^{1/\lambda_1'}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0q}}}\le 2^{(s_1-s_0)qj_*},$$

we observe that

$$\begin{aligned}\|f\|^q_{\dot B^s_{p,q}} &\le \left\{ \frac{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{1/\lambda_0'}\|f\|^q_{\dot B^{s_1}_{p,\lambda_1 q}}}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0 q}}} \right\}^{\theta} \frac{1}{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{1/\lambda_0'}}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0 q}} \\ &\quad + \left\{ \frac{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0 q}}}{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{1/\lambda_0'}\|f\|^q_{\dot B^{s_1}_{p,\lambda_1 q}}} \right\}^{1-\theta} \frac{1}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'}}\|f\|^q_{\dot B^{s_1}_{p,\lambda_1 q}} \\ &= \frac{2}{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{(1-\theta)/\lambda_0'}(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{\theta/\lambda_1'}}\|f\|^{(1-\theta)q}_{\dot B^{s_0}_{p,\lambda_0 q}}\|f\|^{\theta q}_{\dot B^{s_1}_{p,\lambda_1 q}}.\end{aligned}$$

Since Proposition 2.1 yields

$$\frac{1}{1-2^{-\alpha t}} \le \frac{1}{\alpha\log 2}\left(1+\frac{1}{t}\right) \tag{2.4}$$

for all $0 < t < \infty$ and $0 < \alpha < 1$, we obtain

$$\begin{aligned}&\frac{2}{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{(1-\theta)/\lambda_0'}(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{\theta/\lambda_1'}} \\ &\le 2\left\{\frac{1}{\theta\log 2}\left(1+\frac{1}{(s_1-s_0)\lambda_0' q}\right)\right\}^{(1-\theta)/\lambda_0'}\left\{\frac{1}{(1-\theta)\log 2}\left(1+\frac{1}{(s_1-s_0)\lambda_1' q}\right)\right\}^{\theta/\lambda_1'}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\|f\|_{\dot B^s_{p,q}} \\ &\le 2^{1/q}\left\{\frac{1}{\theta\log 2}\left(1+\frac{1}{(s_1-s_0)\lambda_0' q}\right)\right\}^{(1-\theta)/(\lambda_0' q)}\left\{\frac{1}{(1-\theta)\log 2}\left(1+\frac{1}{(s_1-s_0)\lambda_1' q}\right)\right\}^{\theta/(\lambda_1' q)} \\ &\times \|f\|^{1-\theta}_{\dot B^{s_0}_{p,\lambda_0 q}}\|f\|^{\theta}_{\dot B^{s_1}_{p,\lambda_1 q}} \\ &= M\left(s_1-s_0,\theta,q,\frac{1}{\lambda_0' q},\frac{1}{\lambda_1' q}\right)\|f\|^{1-\theta}_{\dot B^{s_0}_{p,\lambda_0 q}}\|f\|^{\theta}_{\dot B^{s_1}_{p,\lambda_1 q}}\end{aligned}$$

from the definition (1.6). Since we have assumed that $1 \le q < \infty$ and $q_0, q_1 \in (q,\infty)$, by setting $\lambda_0 = q_0/q$ and $\lambda_1 = q_1/q$, we see that $\lambda_0, \lambda_1 \in (1,\infty)$. In addition, it holds by (2.2) that $1/\lambda_0' = 1 - q/q_0$ and $1/\lambda_1' = 1 - q/q_1$. Therefore, we obtain (1.5).

Next we consider the case of $1 \le q < q_0 < q_1 = \infty$. Since there holds (2.1) for all $j_* \in \mathbb{Z}$, by taking $\lambda_0, \lambda_0' \in (1,\infty)$ so that (2.2) and using the Hölder inequality, we observe that

$$\begin{aligned}\|f\|^q_{\dot B^s_{p,q}} &\le \left\{\sum_{j=-\infty}^{j_*-1} 2^{(s_1-s_0)\theta q j\cdot\lambda_0'}\right\}^{1/\lambda_0'}\left\{\sum_{j=-\infty}^{j_*-1}(2^{s_0 j}\|\varphi_j * f\|_{L^p})^{\lambda_0 q}\right\}^{1/\lambda_0} \\ &\quad + \sum_{j=j_*}^{\infty} 2^{-(s_1-s_0)(1-\theta)qj}\sup_{j_*\le j<\infty}(2^{s_1 j}\|\varphi_j * f\|_{L^p})^q \\ &\le \frac{2^{(s_1-s_0)\theta q(j_*-1)}}{(1-2^{-(s_1-s_0)\theta\lambda_0' q})^{1/\lambda_0'}}\|f\|^q_{\dot B^{s_0}_{p,\lambda_0 q}} + \frac{2^{-(s_1-s_0)(1-\theta)qj_*}}{1-2^{-(s_1-s_0)(1-\theta)q}}\|f\|^q_{\dot B^{s_1}_{p,\infty}}\end{aligned}$$

for all $j_* \in \mathbb{Z}$. Noting that we may obtain the same estimate as above by letting $\lambda_1 = \infty$ and $\lambda_1' = 1$ in (2.3), we see that the desired estimate (1.5) is still valid for the case of $q_1 = \infty$. Similarly, we also see that (1.5) holds even if $1 \le q < q_1 \le q_0 = \infty$.

Next we consider the case of $1 \le q = q_0 < q_1 < \infty$. We go back to (2.1) again and take $\lambda_1, \lambda_1' \in (1, \infty)$ satisfying (2.2). Then the Hölder inequality implies that

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}}^q &\le \sup_{-\infty<j\le j_*-1} 2^{(s_1-s_0)\theta qj} \sum_{j=-\infty}^{j_*-1} (2^{s_0 j}\|\varphi_j * f\|_{L^p})^q \\
&\quad + \left\{ \sum_{j=j_*}^{\infty} 2^{-(s_1-s_0)(1-\theta)qj\cdot\lambda_1'} \right\}^{1/\lambda_1'} \left\{ \sum_{j=j_*}^{\infty} (2^{s_1 j}\|\varphi_j * f\|_{L^p})^{\lambda_1 q} \right\}^{1/\lambda_1} \\
&\le 2^{(s_1-s_0)\theta q(j_*-1)} \|f\|_{\dot{B}^{s_0}_{p,q}}^q + \frac{2^{-(s_1-s_0)(1-\theta)qj_*}}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'}} \|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^q
\end{aligned}$$

for all $j_* \in \mathbb{Z}$. By choosing $j_* \in \mathbb{Z}$ such that

$$2^{(s_1-s_0)q(j_*-1)} \le \frac{\|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^q}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'} \|f\|_{\dot{B}^{s_0}_{p,q}}^q} \le 2^{(s_1-s_0)qj_*},$$

we obtain

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}}^q &\le \left\{ \frac{\|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^q}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'} \|f\|_{\dot{B}^{s_0}_{p,q}}^q} \right\}^{\theta} \|f\|_{\dot{B}^{s_0}_{p,q}}^q \\
&\quad + \left\{ \frac{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'} \|f\|_{\dot{B}^{s_0}_{p,q}}^q}{\|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^q} \right\}^{1-\theta} \frac{1}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{1/\lambda_1'}} \|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^q \\
&= \frac{2}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{\theta/\lambda_1'}} \|f\|_{\dot{B}^{s_0}_{p,q}}^{(1-\theta)q} \|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^{\theta q}.
\end{aligned}$$

In addition, we see by (2.4) that

$$\frac{2}{(1-2^{-(s_1-s_0)(1-\theta)\lambda_1' q})^{\theta/\lambda_1'}} \le 2 \left\{ \frac{1}{(1-\theta)\log 2} \left(1 + \frac{1}{(s_1-s_0)\lambda_1' q} \right) \right\}^{\theta/\lambda_1'},$$

which yields

$$\begin{aligned}
\|f\|_{\dot{B}^s_{p,q}} &\le 2^{1/q} \left\{ \frac{1}{(1-\theta)\log 2} \left(1 + \frac{1}{(s_1-s_0)\lambda_1' q} \right) \right\}^{\theta/(\lambda_1' q)} \|f\|_{\dot{B}^{s_0}_{p,q}}^{1-\theta} \|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^{\theta} \\
&= M\left(s_1 - s_0, \theta, q, 0, \frac{1}{\lambda_1' q} \right) \|f\|_{\dot{B}^{s_0}_{p,q}}^{1-\theta} \|f\|_{\dot{B}^{s_1}_{p,\lambda_1 q}}^{\theta}
\end{aligned}$$

from the definition (1.6). Hence, by setting $\lambda_1 = q_1/q$, we have $1 < \lambda_1 < \infty$ and $1/\lambda_1' = 1 - q/q_1$ from $q < q_1 < \infty$ and (2.2). Thus we observe that the desired estimate (1.5) is still valid for the case of $q = q_0$. We also see that (1.5) holds even if $1 \le q = q_1 \le q_0 < \infty$ in a similar way. Moreover, by combining the estimates in the above cases, we may show the desired estimate (1.5) for the case of $1 \le q = q_0 < q_1 = \infty$ or $1 \le q = q_1 < q_0 = \infty$. This completes the proof of Theorem 1.1. $\square$

2.2 Gagliardo-Nirenberg inequality: Proof of Corollary 1.3

Finally, we give the *Gagliardo-Nirenberg inequality* by combining Theorem 1.1 and the usual *Sobolev embeddings* in the homogeneous Besov spaces. Before showing Corollary 1.3, we shall verify that constants appearing in the Sobolev embeddings do not have *any effect* on our interest, i.e., the *divergence* of a constant appearing in the Gagliardo-Nirenberg inequality.

Proposition 2.2. *Let $1 \le p_0 \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$. Then for every $f \in \dot{B}^s_{p_0,q}$, it holds that $f \in \dot{B}^{s-n(1/p_0-1/p)}_{p,q}$ for all $p_0 \le p \le \infty$ with the estimate*

$$\|f\|_{\dot{B}^{s-n(1/p_0-1/p)}_{p,q}} \le 3 \cdot 2^{n(1/p_0-1/p)} \|\mathcal{F}^{-1}\varphi\|_{L^{(1-1/p_0+1/p)^{-1}}} \|f\|_{\dot{B}^s_{p_0,q}}, \tag{2.5}$$

where φ is the function appearing in the definition of the homogeneous Besov spaces. In particular, there holds

$$\|f\|_{\dot{B}^{s-n(1/p_0-1/p)}_{p,q}} \le C_n \|f\|_{\dot{B}^s_{p_0,q}}$$

with some constant $C_n > 0$ depending only on n.

Proof. Let $1 \le r \le \infty$ satisfy $1 + 1/p = 1/r + 1/p_0$. Then it holds by simple substitutions that

$$\|\varphi_k\|_{L^r} = \|\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)]\|_{L^r} = 2^{n(1-1/r)k}\|\mathcal{F}^{-1}\varphi\|_{L^r} = 2^{n(1/p_0-1/p)k}\|\mathcal{F}^{-1}\varphi\|_{L^{(1-1/p_0+1/p)^{-1}}}$$

for all $k \in \mathbb{Z}$. Here we note that the definition of $\{\varphi_j\}_{j\in\mathbb{Z}}$ yields $g = \sum_{k\in\mathbb{Z}}(\varphi_k * g)$ for all $g \in L^p$ and $\varphi_j * \varphi_k = 0$ for all $j, k \in \mathbb{Z}$ satisfying $|j-k| \ge 2$. Thus we have

$$\begin{aligned} \|\varphi_j * f\|_{L^p} &\le \sum_{k=j-1}^{j+1} \|\varphi_k * \varphi_j * f\|_{L^p} \le \|\varphi_j * f\|_{L^{p_0}} \sum_{k=j-1}^{j+1} \|\varphi_k\|_{L^r} \\ &= \|\varphi_j * f\|_{L^{p_0}} \sum_{k=j-1}^{j+1} 2^{n(1/p_0-1/p)k}\|\mathcal{F}^{-1}\varphi\|_{L^{(1-1/p_0+1/p)^{-1}}} \\ &\le 3 \cdot 2^{n(1/p_0-1/p)}\|\mathcal{F}^{-1}\varphi\|_{L^{(1-1/p_0+1/p)^{-1}}} \cdot 2^{n(1/p_0-1/p)j}\|\varphi_j * f\|_{L^{p_0}} \end{aligned}$$

for all $j \in \mathbb{Z}$, which implies that (2.5) holds. Moreover, since

$$\begin{aligned} 3 \cdot 2^{n(1/p_0-1/p)}\|\mathcal{F}^{-1}\varphi\|_{L^{(1-1/p_0+1/p)^{-1}}} &\le 3 \cdot 2^n \|\mathcal{F}^{-1}\varphi\|_{L^\infty}^{1/p_0-1/p}\|\mathcal{F}^{-1}\varphi\|_{L^1}^{1-1/p_0+1/p} \\ &\le 3 \cdot 2^n(\|\mathcal{F}^{-1}\varphi\|_{L^\infty} + \|\mathcal{F}^{-1}\varphi\|_{L^1}) \end{aligned}$$

and since the right-hand side depends only on n, the remaining assertion has been verified. □

Proof of Corollary 1.3. Since it holds by $p_0 < p < \infty$ that $0 < p_0/p < 1$, setting $\theta := 1 - p_0/p$ yields $0 < \theta < 1$. We also define s_0 and s_1 by setting

$$s_0 := s - n\left(\frac{1}{p_0} - \frac{1}{p}\right), \quad s_1 := s + \frac{n}{p}.$$

Then we see by $p_0 < \infty$ that $s_1 - s_0 = n/p_0 > 0$. Moreover, there holds

$$\begin{aligned} (1-\theta)s_0 + \theta s_1 &= \frac{p_0}{p} \cdot \left\{s - n\left(\frac{1}{p_0} - \frac{1}{p}\right)\right\} + \left(1 - \frac{p_0}{p}\right) \cdot \left(s + \frac{n}{p}\right) \\ &= \frac{p_0}{p}s - \frac{n}{p}\left(1 - \frac{p_0}{p}\right) + \left(1 - \frac{p_0}{p}\right)s + \frac{n}{p}\left(1 - \frac{p_0}{p}\right) \\ &= s. \end{aligned}$$

Since $p \in (p_0, \infty) \cap [p_1, \infty)$, by applying Proposition 2.2, we observe that

$$f \in \dot{B}^{s}_{p_0,q_0} \subset \dot{B}^{s-n(1/p_0-1/p)}_{p,q_0} = \dot{B}^{s_0}_{p,q_0}, \quad f \in \dot{B}^{s+n/p_1}_{p_1,q_1} \subset \dot{B}^{s+n/p}_{p,q_1} = \dot{B}^{s_1}_{p,q_1}$$

with the estimates

$$\|f\|_{\dot{B}^{s_0}_{p,q_0}} \leq C_n \|f\|_{\dot{B}^{s}_{p_0,q_0}}, \quad \|f\|_{\dot{B}^{s_1}_{p,q_1}} \leq C_n \|f\|_{\dot{B}^{s+n/p_1}_{p_1,q_1}},$$

where $C_n > 0$ is a constant depending only on n. Hence, we have $f \in \dot{B}^{s_0}_{p,q_0} \cap \dot{B}^{s_1}_{p,q_1}$ and we may apply Theorem 1.1 to obtain

$$\begin{aligned}\|f\|_{\dot{B}^{s}_{p,q}} &\leq M\left(\frac{n}{p_0}, 1-\frac{p_0}{p}, q, \frac{1}{q}-\frac{1}{q_0}, \frac{1}{q}-\frac{1}{q_1}\right) \|f\|^{p_0/p}_{\dot{B}^{s_0}_{p,q_0}} \|f\|^{1-p_0/p}_{\dot{B}^{s_1}_{p,q_1}} \\ &\leq C_n M\left(\frac{n}{p_0}, 1-\frac{p_0}{p}, q, \frac{1}{q}-\frac{1}{q_0}, \frac{1}{q}-\frac{1}{q_1}\right) \|f\|^{p_0/p}_{\dot{B}^{s}_{p_0,q_0}} \|f\|^{1-p_0/p}_{\dot{B}^{s+n/p_1}_{p_1,q_1}}.\end{aligned}$$

Here we assume that $1 \leq q < \infty$ and $q_0, q_1 \in (q, \infty]$. Then, since

$$\begin{aligned}\left\{\frac{1}{\log 2}\left(1-\frac{p_0}{p}\right)^{-1}\left(1+\frac{p_0}{n}\left(\frac{1}{q}-\frac{1}{q_0}\right)\right)\right\}^{(p_0/p)(1/q-1/q_0)} &\leq \left\{2\left(1-\frac{p_0}{p}\right)^{-1}(1+p_0)\right\}^{1/q-1/q_0} \\ &\leq 2^2\left(\frac{1}{p_0}-\frac{1}{p}\right)^{-(1/q-1/q_0)}, \\ \left\{\frac{1}{\log 2}\cdot\frac{p}{p_0}\left(1+\frac{p_0}{n}\left(\frac{1}{q}-\frac{1}{q_1}\right)\right)\right\}^{(1-p_0/p)(1/q-1/q_1)} &\leq \left\{\frac{2p}{p_0}(1+p_0)\right\}^{1/q-1/q_1} \\ &\leq 2^2 p^{1/q-1/q_1},\end{aligned}$$

we obtain

$$\begin{aligned}&M\left(\frac{n}{p_0}, 1-\frac{p_0}{p}, q, \frac{1}{q}-\frac{1}{q_0}, \frac{1}{q}-\frac{1}{q_1}\right) \\ &= 2^{1/q}\left\{\frac{1}{\log 2}\left(1-\frac{p_0}{p}\right)^{-1}\left(1+\frac{p_0}{n}\left(\frac{1}{q}-\frac{1}{q_0}\right)\right)\right\}^{(p_0/p)(1/q-1/q_0)} \\ &\times\left\{\frac{1}{\log 2}\cdot\frac{p}{p_0}\left(1+\frac{p_0}{n}\left(\frac{1}{q}-\frac{1}{q_1}\right)\right)\right\}^{(1-p_0/p)(1/q-1/q_1)} \\ &\leq 2^5\left(\frac{1}{p_0}-\frac{1}{p}\right)^{-(1/q-1/q_0)} p^{1/q-1/q_1},\end{aligned}$$

which yields the desired estimate. The remaining cases may be treated in a similar manner. This proves Corollary 1.3. □

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