A Lyapunov-Based Method of Reducing Activation Functions of Recurrent Neural Networks for Stability Analysis

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A Lyapunov-based Method of Reducing Activation Functions of Recurrent Neural Networks for Stability Analysis

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***Abstract*— This paper proposes a Lyapunov-based method of reducing the number of activation functions of a recurrent neural network (RNN) for its stability analysis. To the best of the authors' knowledge, no method has been presented for pruning RNNs with respecting their stability properties. We are the first to present an effective solution method for this important problem in the control community and machine learning community. The proposed reduction method follows the intuitive policy: compose a reduced RNN by removing some activation functions whose "magnitudes" with respect to their weighted actions are "small" in some sense, and analyze its stability to guarantee the stability of the original RNN. Moreover, we theoretically justify this policy by proving several theorems that are applicable to general reduction methods. In addition, we propose a method of rendering the proposed reduction method less conservative, on the basis of semidefinite programming. The effectiveness of the proposed methods is demonstrated on a numerical example.**

***Index Terms*— Model/controller reduction, neural networks, stability of nonlinear systems.**

I. INTRODUCTION

A. Background and Motivation

1) Recurrent Neural Networks and its Stability Analysis: A recurrent neural network (RNN) is one of the basic architectures of deep neural networks (DNNs). Due to its feedback mechanism, RNNs are able to imitate the behaviors of dynamical systems and hence effective, for instance, for time series analysis. However, the feedback mechanism may render the RNN unstable unless the nonlinear activation functions and the weights of edges are appropriately chosen. Therefore, the stability analysis of RNNs has been regarded as an important issue in the machine learning field [1]–[3].

2) Integral Quadratic Constraint Approach to Stability Analysis: Control theoretic approaches to the analysis and synthesis of DNNs have recently attracted great attention, especially in the stability analysis of RNNs [4], [5] and performance analysis of feedback control systems driven by neural networks (NNs) [6], [7]. In particular, the integral quadratic constraint (IQC) approach [7], [8] is frequently employed for these analyses. This approach enables us to capture the behavior of nonlinear activation functions by multipliers and reduce the analysis problems into semidefinite programming problems (SDPs). However, since its computational burden rapidly grows as the number of activation functions increases, the approach is not applicable to large-scale NNs in practice.

3) Model Reduction of Neural Networks: The *Model reduction* is an approach to approximating a large-scale NN by a smaller-scale NN. This forks into two directions: the dimension reduction removes a part of the neurons from the original NN [9], and the *pruning* removes a part of the weight parameters [10]. Our concern in this paper is to remove a part of the activation functions, which corresponds to pruning all the weights multiplied to the part. However, to the best of the authors' knowledge, no method has been presented for pruning RNNs with respecting their stability properties.

B. Contribution

This paper proposes a Lyapunov-based method of RNN model reduction, in the sense of reducing the number of activation functions, for the stability analysis of the original RNN. We are the first to present an effective solution method for this important problem. We theoretically justify the intuitive policy: compose a reduced RNN by removing some activation functions whose "magnitudes" with respect to their weighted actions are "small" in some sense, and analyze its stability to guarantee the stability of the original RNN. First, for a reduced RNN obtained by removing arbitrary activation functions, we derive a sufficient condition to guarantee the stability of the original RNN from that of the reduced one. Moreover, we show that if the original RNN is stable (but we do not know it beforehand), then the reduced RNN obtained by removing sufficiently "small" activation functions always meets the above sufficient condition. These results justifies the above policy. After that, for ReLU-RNNs, we propose an intuitive procedure of model reduction based on the derived theorems. In the above sufficient condition, we need to compute an upper-bound estimate of the "magnitude" of the removed activation functions. Hence, for ReLU-RNNs, we propose a method of computing a less conservative upper-bound estimate on the basis of SDP. The proposed theorems and methods are

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demonstrated on a numerical example. We finally note that, since the proposed stability analysis method is based on the feedback system representation of an RNN which is composed of a linear system and a nonlinearity gathering the activation functions, the method can also be applied to general NN-driven feedback control systems.

II. PRELIMINARIES

A. Notation

Throughout this paper, $\mathbf{R}$ denotes the field of real numbers. In this subsection, suppose we are given positive integers m and n. The n-dimensional real space is denoted by $\mathbf{R}^n$. The set of $n \times m$ real matrices is denoted by $\mathbf{R}^{n\times m}$. For a matrix $M \in \mathbf{R}^{n\times m}$, let $M^\top$ denote its transpose. For matrices $M \in \mathbf{R}^{n\times n}$ and $N \in \mathbf{R}^{n\times m}$, the expression $(*)^\top MN$ is a shorthand for $N^\top MN$. The set of positive integers is denoted by $\mathbf{Z}_{++}$. The set $\mathcal{N}_m$ denotes $\{1, 2, \dots, m\} \subset \mathbf{Z}_{++}$. For a finite set $\Lambda \subset \mathbf{Z}_{++}$, the mapping $\kappa_\Lambda : \mathcal{N}_{|\Lambda|} \to \Lambda$ is defined by letting $\kappa_\Lambda(i)$ denote the i-th smallest number in Λ for each $i \in \mathcal{N}_{|\Lambda|}$.

For a vector $v \in \mathbf{R}^m$, its Euclidean norm is denoted by $\|v\|$. The (Euclidean) induced norm of a (possibly nonlinear) mapping $\Psi : \mathbf{R}^m \to \mathbf{R}^n$ is defined by $\|\Psi\| := \sup_{v\in\mathbf{R}^m\setminus\{0\}} \frac{\|\Psi(v)\|}{\|v\|}$. For a matrix $M \in \mathbf{R}^{n\times m}$, the norm $\|M\|$ means the (Euclidean) induced norm of the linear mapping $\mathbf{R}^m \ni v \mapsto Mv \in \mathbf{R}^n$, and $\|M\|_\mathrm{F}$ denotes the Frobenius matrix norm, i.e., $\|M\|_\mathrm{F} := \sqrt{\sum_{\substack{1\le i\le n\\ 1\le j\le m}} M_{ij}^2}$, where M_{ij} is the (i, j) entry of M.

The set of $n \times n$ real symmetric matrices is denoted by $\mathbb{S}^n$. For $P \in \mathbb{S}^n$, we write $P \succ 0$ (resp. $P \prec 0$) to denote that P is positive (resp. negative) definite. The set of $n \times n$ diagonal matrices (resp. the set of $n \times n$ diagonal matrices whose every diagonal entry is strictly positive) is denoted by $\mathbb{D}^n$ (resp. $\mathbb{D}^n_{++}$).

B. Target System and Reduced model

The target system of this paper is the general model of continuous-time RNN [2], [3] described as

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bp(t) + B_\mathrm{in} w(t), \\ q(t) = Cx(t), \\ z(t) = C_\mathrm{out} x(t), \\ p(t) = \Phi(q(t)), \end{cases} \tag{1}$$

with the weight matrices $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $B_\mathrm{in} \in \mathbf{R}^{n\times n_w}$, $C \in \mathbf{R}^{m\times n}$, and $C_\mathrm{out} \in \mathbf{R}^{n_z\times n}$, where t is the continuous time parameter, x is the n-dimensional state vector representing the hidden states, w is the n_w-dimensional exogenous input vector, z is the n_z-dimensional output vector, and $q := [q_1, \dots, q_m]^\top$ and $p := [p_1, \dots, p_m]^\top$ are the m-dimensional signals respectively input to and output from the (static) nonlinear activation operator $\Phi : \mathbf{R}^m \to \mathbf{R}^m$. This operator is defined by $\Phi : [q_1, \dots, q_m]^\top \mapsto [\phi_1(q_1), \dots, \phi_m(q_m)]^\top$, where each ϕ_i $(i \in \mathcal{N}_m)$ is the locally Lipschitz function on $\mathbf{R}$ that represents the corresponding nonlinear activation function of the RNN. Suppose that $\phi_i(0) = 0$ for all $i \in \mathcal{N}_m$. Since this system is time-invariant, the initial time is assumed to be 0. We call m the *activation degree* of the RNN, which represents the number of the RNN's activation functions.

Let us given the desired activation degree $m_\mathrm{r} \in \mathcal{N}_{m-1}$ of the reduced RNN model to be designed. In this paper, we aim at selecting the index set $\Lambda_\mathrm{c} \subset \mathcal{N}_m$ $(|\Lambda_\mathrm{c}| = m - m_\mathrm{r} =: l)$ of the activation functions to be removed, to compose the reduced RNN model

$$\Sigma_\mathrm{r} : \begin{cases} \dot{x}(t) = Ax(t) + B_\mathrm{r} p_\mathrm{r}(t) + B_\mathrm{in} w(t), \\ q_\mathrm{r}(t) = C_\mathrm{r} x(t), \\ z(t) = C_\mathrm{out} x(t), \\ p_\mathrm{r}(t) = \Phi_\mathrm{r}(q_\mathrm{r}(t)), \end{cases} \tag{2}$$

where q_r and p_r are the m_r-dimensional signals respectively input to and output from the reduced activation operator defined by $\Phi_\mathrm{r} := [\phi_{\kappa_{\Lambda_\mathrm{r}}(1)}, \dots, \phi_{\kappa_{\Lambda_\mathrm{r}}(m_\mathrm{r})}]^\top$ with $\Lambda_\mathrm{r} := \mathcal{N}_m \setminus \Lambda_\mathrm{c}$. The reduced weight matrices are defined by

$$B_\mathrm{r} := \begin{bmatrix} b_{\kappa_{\Lambda_\mathrm{r}}(1)} & \cdots & b_{\kappa_{\Lambda_\mathrm{r}}(m_\mathrm{r})} \end{bmatrix} \text{ and } C_\mathrm{r} := \begin{bmatrix} c_{\kappa_{\Lambda_\mathrm{r}}(1)} \\ \vdots \\ c_{\kappa_{\Lambda_\mathrm{r}}(m_\mathrm{r})} \end{bmatrix},$$

where b_i and c_i $(i \in \mathcal{N}_m)$ are the i-th column and row vectors of B and C, respectively.

Since we are concerned with the internal stability, in the subsequent sections, we set $w = 0$ in (1) and (2), and consider the following autonomous systems respectively corresponding to Σ and Σ_r:

$$\Sigma^\mathrm{aut} : \ \dot{x}(t) = Ax(t) + B\Phi\left(Cx(t)\right), \tag{3}$$
$$\Sigma^\mathrm{aut}_\mathrm{r} : \ \dot{x}(t) = Ax(t) + B_\mathrm{r}\Phi_\mathrm{r}\left(C_\mathrm{r}x(t)\right). \tag{4}$$

Clearly, systems (3) and (4) have their equilibrium points at the origins $(x = 0)$ of their respective state spaces.

Our requirement on the reduced RNN (4) is that the stability of the original RNN (3) can be concluded through the stability analysis of the reduced model (4). In the following sections, we will derive a constructive sufficient condition for composing the reduced RNN that meets this requirement.

In what follows, the phrase "model reduction" means selecting Λ_c, i.e., the index set of the activation functions to be removed.

III. REVIEW OF LYAPUNOV STABILITY THEORY

A. Lyapunov Functions for Exponential Stability

Let us consider a general autonomous system of the form

$$\dot{x}(t) = f(x(t)) \tag{5}$$

where x is the n-dimensional state vector, and $f : \mathcal{D} \to \mathbf{R}^n$ is a locally Lipschitz mapping defined on a domain $\mathcal{D} \subset \mathbf{R}^n$ that contains the origin. Suppose that $f(0) = 0$.

Lemma 1 ([11, Th. 4.10][12, Th. 3.1 & 3.2]): Consider (5) and suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbf{R}$ satisfying

$$c_1 \|x\|^2 \le V(x) \le c_2 \|x\|^2, \tag{6a}$$
$$\frac{\partial V(x)}{\partial x} f(x) \le -c_3 \|x\|^2 \tag{6b}$$

for all $x \in \mathcal{D}$ and some positive constants c_1, c_2, and c_3. Then, the origin of (5) is exponentially stable (for short, ES). If the assumptions hold globally on $\mathbf{R}^n$, the origin is globally exponentially stable (for short, GES).

Under mild assumptions [11, Th. 4.14] [12, Cor. 3.2, Th. 3.11], the local (resp. global) exponential stability of the origin guarantees the existence of a continuously differentiable function V that satisfies (6a)–(6b) and additionally

$$\left\| \frac{\partial V(x)}{\partial x} \right\| \leq c_4 \|x\| \tag{6c}$$

for all $x \in \mathcal{D}$ (resp. $x \in \mathbf{R}^n$) and some positive constant c_4 [11, Th. 4.14]. Hence, in many cases, the local/global exponential stability is equivalent to the existence of a Lyapunov function V satisfying (6a)–(6c).

B. Stability of Perturbed Systems

Consider the perturbed system obtained by adding a perturbation term g to (5) as

$$\dot{x}(t) = \overline{f}(x(t)) := f(x(t)) + g(x(t)), \tag{7}$$

where $g : \mathcal{D} \to \mathbf{R}^n$ is locally Lipschitz on $\mathcal{D}$ and satisfies $g(0) = 0$. Moreover, assume that g satisfies the linear growth bound

$$\|g(x)\| \leq \gamma \|x\| \quad (\forall x \in \mathcal{D}), \tag{8}$$

where γ is a nonnegative constant. Note that (8) always hold by taking $\mathcal{D}$ to be a sufficiently small ball centered at the origin because of the Lipschitz continuity of g and $g(0) = 0$.

Lemma 2 ([11, Lem. 9.1]): Suppose that the origin of the nominal system (5) is ES and there exists a continuously differentiable function $V : \mathcal{D} \to \mathbf{R}$ satisfying (6a)–(6c). Moreover, assume that g satisfies (8) and

$$\gamma < \frac{c_3}{c_4}. \tag{9}$$

Then, the origin of the perturbed system (7) is ES. Moreover, if all assumptions hold globally in $\mathbf{R}^n$, then the origin of (7) is GES.

IV. MODEL REDUCTION

A. Sufficient Condition and Existence Theorem

First, for an arbitrarily chosen Λ_c, we give a sufficient condition to guarantee the stability of the original RNN (3) from that of the resulting reduced RNN (4).

Theorem 1 (Sufficient Condition): Given $\Lambda_c \subset \mathcal{N}_m$, assume that the origin of the reduced RNN (4) is ES (resp. GES) and there exists a continuously differentiable function V satisfying (6a)–(6c) on $\mathcal{D}$ (resp. $\mathbf{R}^n$) for (4). Moreover, assume that the mapping composed of the removed activation functions described as

$$g(x) := \sum_{i \in \Lambda_c} b_i \phi_i(c_i x) \tag{10}$$

satisfies (8) on $\mathcal{D}$ (resp. $\mathbf{R}^n$) and (9). Then, the origin of the original RNN (3) is ES (resp. GES).

Proof: We can see that (10) is the difference between the right-hand sides of the original and reduced RNNs (3) and (4). From this viewpoint, we regard the original RNN (3) and the reduced one (4) as the perturbed and nominal systems, respectively, as follows:

$$\Sigma^{\rm aut} : \ \dot{x}(t) = \overline{f}(x(t)) := f(x(t)) + g(x(t)), \tag{11}$$
$$\Sigma_{\rm r}^{\rm aut} : \ \dot{x}(t) = f(x(t)). \tag{12}$$

Applying Lemma 2 to (11), (12), and (10), we can prove the claim of theorem. ■

One question on this theorem is how probably there exists Λ_c that meets the sufficient condition. The following theorem gives an answer to this question. Let us define $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) := \min\{\gamma \text{ satisfying (8) with (10)}\}$.

Theorem 2 (Existence Theorem): Suppose that the origin of the original RNN (3) is ES (resp. GES) and there exists a continuously differentiable function $\overline{V}$ satisfying the counterpart of (6a)–(6c) for (3), i.e., $\overline{c}_1 \|x\|^2 \leq \overline{V}(x) \leq \overline{c}_2 \|x\|^2$, $\frac{\partial \overline{V}(x)}{\partial x}(Ax + B\Phi(Cx)) \leq -\overline{c}_3 \|x\|^2$, and $\left\| \frac{\partial \overline{V}(x)}{\partial x} \right\| \leq \overline{c}_4 \|x\|$ on $\mathcal{D}$ (resp. $\mathbf{R}^n$) for some positive constants $\overline{c}_1, \ldots, \overline{c}_4$. Then, there exists a positive constant γ_0 such that for any $\Lambda_c \subset \mathcal{N}_m$ satisfying $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) < \gamma_0$ (resp. $\gamma_{\mathbf{R}^n}^{\star}(\Lambda_c) < \gamma_0$), if such Λ_c exists, the resulting reduced RNN (4) meets all the assumptions of Theorem 1.

Proof: Let us prove the theorem for the local case because the global case can be proved in the same way. Conversely to Theorem 1, we regard the original RNN (3) and the reduced one (4) as the nominal and perturbed systems, respectively, as follows:

$$\Sigma^{\rm aut} : \ \dot{x}(t) = \overline{f}(x(t)), \tag{13}$$
$$\Sigma_{\rm r}^{\rm aut} : \ \dot{x}(t) = \overline{f}(x(t)) + \overline{g}(x(t)), \tag{14}$$

where $\overline{g}(x) := -g(x)$ defined with (10). Define $\gamma_0 := \frac{\overline{c}_3}{2\overline{c}_4} > 0$ and choose Λ_c satisfying $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) < \gamma_0$. Now, we want to show that the resulting reduced RNN (14) meets all the assumptions of Theorem 1. First, by applying Lemma 2 to systems (13) and (14), we find that the origin of the reduced RNN (14) is ES. Next, we must find a continuously differentiable function V satisfying (6a)–(6c) for the reduced RNN (14). This is achieved by setting $V := \overline{V}$, $c_i := \overline{c}_i$ $(i = 1, 2, 4)$ and $c_3 := \frac{\overline{c}_3}{2}$ because $\frac{\partial V(x)}{\partial x}\left(\overline{f}(x) + \overline{g}(x)\right) \leq -\overline{c}_3 \|x\|^2 + \overline{c}_4 \gamma_{\mathcal{D}}^{\star}(\Lambda_c) \|x\|^2$. The remaining condition we must prove is $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) < \frac{c_3}{c_4}$. Since $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) < \gamma_0$, this inequality holds. Therefore, all the assumptions of Theorem 1 are satisfied with this Λ_c. ■

Remark 1: The key condition to be proved in the last part of the proof of Theorem 2 is the inequality $\gamma_{\mathcal{D}}^{\star}(\Lambda_c) < \frac{1}{\overline{c}_4}(\overline{c}_3 - \overline{c}_4 \gamma_{\mathcal{D}}^{\star}(\Lambda_c))$. Therefore, counterintuitively, just taking $\gamma_0 := \frac{\overline{c}_3}{\overline{c}_4}$ does not work in Theorem 2.

The value of γ_0 in Theorem 2 cannot be known before the model reduction and stability analysis because we do not know the Lyapunov function $\overline{V}$ for the original RNN beforehand. However, Theorem 2 validates the qualitative claim: if the origin of the original RNN is ES/GES (but we do not know it beforehand), then with any Λ_c whose "magnitude" $\gamma_{\mathcal{D}}^{\star}(\Lambda_c)$ is sufficiently small, the resulting reduced RNN is always capable of guaranteeing the local/global exponential stability of the original RNN by using Theorem 1. This ensures the validity of using Theorem 1 for the stability analysis of large scale RNNs.

B. Reduction Procedure

Theorems 1 and 2 do not give any executable procedure of selecting Λ_c. Hence, in this subsection, we propose a model reduction procedure based on the underlying concept of Theorems 1 and 2: select Λ_c such that $\gamma^\star_{\mathcal{D}}(\Lambda_\mathrm{c})$ (or its upper-bound estimate) is as small as possible. We assume that the activation functions ϕ_i's are a common ReLU function

$$\phi : \mathbf{R} \ni q \mapsto \max\{0, q\} \in \mathbf{R} \tag{15}$$

as is common practice. Since the procedure proposed below is conservative, the obtained Λ_c will not necessarily satisfy the sufficient condition of Theorem 1. However, the procedure is executable and intuitively reasonable.

First, since $\phi(q) \le |q|$ holds for ReLU functions, the following inequality holds: $\|g(x)\| \le \sum_{i\in\Lambda_\mathrm{c}} \|b_i \phi_i(c_i x)\| \le \sum_{i\in\Lambda_\mathrm{c}} \|b_i\| \, |c_i x| \le \sum_{i\in\Lambda_\mathrm{c}} \|b_i\| \, \|c_i\| \, \|x\|$. Therefore, the bound (8) holds on $\mathcal{D} = \mathbf{R}^n$ by putting $\gamma = \overline{\gamma} := \sum_{i\in\Lambda_\mathrm{c}} \|b_i\| \, \|c_i\|$. This fact motivates us to remove the l smallest activation functions in terms of the magnitude of $\|b_i\| \, \|c_i\|$. From this observation, we propose the following model reduction procedure.

Procedure 1 (Model Reduction):

Input: RNN (1) and the desired activation degree m_r.
Output: index set Λ_c to be removed.

1: Compute the value of $\overline{\gamma}_i := \|b_i\| \, \|c_i\|$ for all $i \in \mathcal{N}_m$.
2: Select the l smallest values from the $\overline{\gamma}_i$'s.
3: Set Λ_c to be the index set of the selected $\overline{\gamma}_i$'s.

If the resulting reduced RNN satisfies the assumptions of Theorem 1, the local/global exponential stability of the origin of the original RNN will be guaranteed. Theorems 1 and 2 give a foundation of the validity of this intuitive and simple model-reduction procedure.

Remark 2: Theorems 1 and 2 are applicable to general reduction procedures including, but not limited to, Procedure 1.

V. LESS-CONSERVATIVE UPPER BOUND

A. Motivation and Setup

To make the condition (9) to hold for the selected Λ_c, the value of γ should be computed as small as possible. In other words, a less conservative upper-bound estimate of $\gamma^\star_{\mathcal{D}}(\Lambda_\mathrm{c})$ should be computed. Although one can put $\gamma = \overline{\gamma}$ as in subsection IV-B, it might be an overestimate. Therefore, we propose an SDP-based method of computing the upper-bound estimate less than or equal to $\overline{\gamma}$. The activation function ϕ_i's are assumed to be a common ReLU function (15) as done in subsection IV-B.

Now, let us re-describe (10) as $g(x) = \check{B}\Phi_l(\check{C}x)$, where $\Phi_l : \mathbf{R}^l \ni [\check{q}_1, \ldots, \check{q}_l]^\top \mapsto [\phi(\check{q}_1), \ldots, \phi(\check{q}_l)] \in \mathbf{R}^l$, and

$$\check{B} := \begin{bmatrix} \check{b}_1 & \cdots & \check{b}_l \end{bmatrix} \in \mathbf{R}^{n\times l}, \quad \check{b}_i := b_{\kappa_{\Lambda_\mathrm{c}}(i)} \ (i = 1, \ldots, l), \tag{16a}$$

$$\check{C} := \begin{bmatrix} \check{c}_1 \\ \vdots \\ \check{c}_l \end{bmatrix} \in \mathbf{R}^{l\times n}, \ \check{c}_i := c_{\kappa_{\Lambda_\mathrm{c}}(i)} \ (i = 1, \ldots, l). \tag{16b}$$

By definition, we see $\left\|\check{B} \circ \Phi_l \circ \check{C}\right\| = \gamma^\star_{\mathbf{R}^n}(\Lambda_\mathrm{c})$, where $\circ$ is the composition operator of mappings. Hence, in what follows, we compute a less conservative upper bound of $\left\|\check{B} \circ \Phi_l \circ \check{C}\right\|$.

B. Upper Bound Characterization

First, let us define the following property.

Definition 1: For given matrices $\check{B} \in \mathbf{R}^{n\times l}$ and $\check{C} \in \mathbf{R}^{l\times n}$, the pair $(\check{B}, \check{C})$ is said to be *balanced* if $\|\check{b}_i\| = \|\check{c}_i\|$ $(i = 1, \ldots, l)$.

In the following, we make the next assumption without loss of generality.

Assumption 1: We assume that the matrices $\check{B} \in \mathbf{R}^{n\times l}$ and $\check{C} \in \mathbf{R}^{l\times n}$ satisfy

(i) $\|\check{b}_i\| \, \|\check{c}_i\| > 0$ $(i = 1, \ldots, l)$;
(ii) The pair $(\check{B}, \check{C})$ is balanced.

To see the rationality of the first assumption, let us suppose $\|\check{b}_{i^*}\| = 0$ just for instance. Then, by defining

$$\widehat{B} := \begin{bmatrix} \check{b}_1 & \cdots & \check{b}_{i^*-1} & \check{b}_{i^*+1} & \cdots & \check{b}_l \end{bmatrix} \in \mathbf{R}^{n\times(l-1)},$$
$$\widehat{C} := \begin{bmatrix} \check{c}_1 \\ \vdots \\ \check{c}_{i^*-1} \\ \check{c}_{i^*+1} \\ \vdots \\ \check{c}_l \end{bmatrix} \in \mathbf{R}^{(l-1)\times n},$$

we can readily see that $\check{B} \circ \Phi_l \circ \check{C} = \widehat{B} \circ \Phi_{l-1} \circ \widehat{C}$. Therefore, if $\|\check{b}_{i^*}\| = 0$, then we can discard the i^*-th column of $\check{B}$ and the i^*-th row of $\check{C}$ from the outset. To see the rationality of the second assumption, we note that $\check{B} \circ \Phi_l \circ \check{C} = (\check{B}D) \circ \Phi_l \circ (D^{-1}\check{C})$ $\left(\forall D \in \mathbb{D}^l_{++}\right)$ holds for ReLU nonlinearities [13]. With this fact in mind, let us define

$$D_0 := \mathrm{diag}(d_1, \cdots, d_l) \in \mathbb{D}^l_{++}, \ d_i := \sqrt{\frac{\|\check{c}_i\|}{\|\check{b}_i\|}} \ (i = 1, \cdots, l),$$
$$\widehat{B} := \check{B}D_0 \in \mathbf{R}^{n\times l}, \ \widehat{C} := D_0^{-1}\check{C} \in \mathbf{R}^{l\times n}.$$

Then, we see that $\check{B} \circ \Phi_l \circ \check{C} = \widehat{B} \circ \Phi_l \circ \widehat{C}$ and the pair $(\widehat{B}, \widehat{C})$ is balanced with $\|\hat{b}_i\| = \|\hat{c}_i\| = \sqrt{\|\check{b}_i\| \, \|\check{c}_i\|}$ $(i = 1, \cdots, l)$. Therefore, we can assume $(\check{B}, \check{C})$ is balanced from the outset without loss of generality. Note that $\gamma^\star_{\mathbf{R}^n}(\Lambda_\mathrm{c})$ and $\overline{\gamma}$ are invariant under these transformations.

Since exact computation of $\left\|\check{B} \circ \Phi_l \circ \check{C}\right\|$ is hard, we focus on its upper bound computation. In view of $\|\phi\| = 1$ and hence $\|\Phi_l\| = 1$ for ReLU nonlinearities, we can obtain the next upper bounds:

$$\left\|\check{B} \circ \Phi_l \circ \check{C}\right\| \le \|\check{B}\| \, \|\check{C}\|,$$
$$\left\|\check{B} \circ \Phi_l \circ \check{C}\right\| = \left\|\sum_{i=1}^{l} \check{b}_i \circ \phi \circ \check{c}_i\right\| \le \sum_{i=1}^{l} \|\check{b}_i\| \, \|\check{c}_i\|.$$

Regarding these bounds, we can obtain the next result.

Proposition 1: For given $\check{B} \in \mathbf{R}^{n\times l}$ and $\check{C} \in \mathbf{R}^{l\times n}$ satisfying Assumption 1, we have $\|\check{B}\| \, \|\check{C}\| \le \sum_{i=1}^{l} \|\check{b}_i\| \, \|\check{c}_i\|$.

Proof: Note that $\|\check{B}\| \le \|\check{B}\|_\mathrm{F} = \sqrt{\sum_{i=1}^{l} \|\check{b}_i\|^2}$ and $\|\check{C}\| \le \|\check{C}\|_\mathrm{F} = \sqrt{\sum_{i=1}^{l} \|\check{c}_i\|^2}$. It follows that $\|\check{B}\| \, \|\check{C}\| \le \sqrt{\sum_{i=1}^{l} \|\check{b}_i\|^2} \sqrt{\sum_{i=1}^{l} \|\check{c}_i\|^2} = \sum_{i=1}^{l} \|\check{b}_i\|^2 = \sum_{i=1}^{l} \|\check{b}_i\| \, \|\check{c}_i\|$. This completes the proof. ■

Remark 3: For Proposition 1 to hold, it suffices that the pair $(\check{B}, \check{C})$ is balanced. Again, we note that $\overline{\gamma} = \sum_{i=1}^{l} \|\check{b}_i\| \|\check{c}_i\|$ naturally arises as an upper bound of $\gamma^{\star}_{\mathbf{R}^n}(\Lambda_{\rm c})$ in Procedure 1 to choose the index set $\Lambda_{\rm c}$. However, once $\Lambda_{\rm c}$ is chosen, Proposition 1 shows that $\|\check{B}\| \|\check{C}\|$ is a better (no worse) upper bound. In the next subsection, we further derive an SDP that enables us to obtain a better (no worse) upper bound than $\|\check{B}\| \|\check{C}\|$.

C. Upper Bound Computation by SDP

The next result, inspired by [14], forms an important basis for the SDP-based upper bound computation of $\gamma^{\star}_{\mathbf{R}^n}(\Lambda_{\rm c})$.

Proposition 2: Let us define $\mathbf{\Pi}^{\star} \subset \mathbb{S}^{2l}$ by

$$\mathbf{\Pi}^{\star} := \left\{ \Pi \in \mathbb{S}^{2l} : \begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix}^{\top} \Pi \begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix} \geq 0, \quad \forall \begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix} \in \mathbf{R}^{2l} \text{ s.t. } \check{p} = \Phi_l(\check{q}) \right\}.$$

Then, we have $\|\check{B} \circ \Phi_l \circ \check{C}\| \leq \gamma$ if there exists $\Pi \in \mathbf{\Pi}^{\star}$ such that

$$\begin{bmatrix} -\gamma^2 I_n & 0 \\ 0 & \check{B}^{\top}\check{B} \end{bmatrix} + \begin{bmatrix} \check{C} & 0 \\ 0 & I_l \end{bmatrix}^{\top} \Pi \begin{bmatrix} \check{C} & 0 \\ 0 & I_l \end{bmatrix} \preceq 0. \quad (17)$$

Proof: For an input-output pair (x, y) of the operator $\check{B} \circ \Phi_l \circ \check{C}$, let us define $\check{q} = \check{C}x$ and $\check{p} = \Phi_l(\check{q})$. Then we have $y = \check{B}\check{p}$. Multiplying $[x^{\top} \ \check{p}^{\top}]^{\top}$ from the right and its transpose from the left to (17), we obtain

$$-\gamma^2 \|x\|^2 + \|y\|^2 + \begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix}^{\top} \Pi \begin{bmatrix} \check{q} \\ \check{p} \end{bmatrix} \leq 0.$$

Since $\Pi \in \mathbf{\Pi}^{\star}$, we readily obtain $-\gamma^2 \|x\|^2 + \|y\|^2 \leq 0$. This inequality implies that $\|y\|^2 \leq \gamma^2 \|x\|^2$ for any $x \in \mathbf{R}^n$. This clearly shows that $\|\check{B} \circ \Phi_l \circ \check{C}\| \leq \gamma$ holds. ∎

In Proposition 2, it is of prime importance to employ a set of multipliers $\mathbf{\Pi} \subset \mathbf{\Pi}^{\star}$ that is numerically tractable and captures the input-output properties of ReLU Φ_l as accurately as possible. On this issue, the next results have been obtained.

Proposition 3 ([15, Th. 2]): Suppose Φ_l is ReLU and let us define the sets of multipliers $\mathbf{\Pi}_{\mathbb{COP}}, \mathbf{\Pi}_{\mathbb{NN}} \subset \mathbb{S}^{2l}$ by

$$\mathbf{\Pi}_{\mathbb{COP}} := \{\Pi \in \mathbb{S}^{2l} : \ \Pi = (*)^{\top} (Q + \mathcal{J}(J)) E, \quad J \in \mathbb{D}^l, \ Q \in \mathbb{COP}^{2l}\},$$

$$\mathbf{\Pi}_{\mathbb{NN}} := \{\Pi \in \mathbb{S}^{2l} : \ \Pi = (*)^{\top} (Q + \mathcal{J}(J)) E, \quad J \in \mathbb{D}^l, \ Q \in \mathbb{NN}^{2l}\},$$

$$E := \begin{bmatrix} -I_l & I_l \\ 0_l & I_l \end{bmatrix}, \ \mathcal{J}(J) := \begin{bmatrix} 0_{l,l} & J \\ * & 0_{l,l} \end{bmatrix},$$

where $\mathbb{COP}^{2l} \subset \mathbb{S}^{2l}$ and $\mathbb{NN}^{2l} \subset \mathbb{S}^{2l}$ are the copositive cone and the nonnegative cone [16], respectively. Then, we have $\mathbf{\Pi}_{\mathbb{NN}} \subset \mathbf{\Pi}_{\mathbb{COP}} \subset \mathbf{\Pi}^{\star}$.

Remark 4: The problem to determine whether a given matrix is copositive or not is a co-NP complete problem in general [17], and hence $\mathbf{\Pi}_{\mathbb{COP}}$ is numerically intractable. We therefore employ $\mathbf{\Pi}_{\mathbb{NN}}$ in this paper. Since the ReLU Φ_l is a (repeated) slope-restricted nonlinearity, we can also employ known multipliers such as (static) O'Shea-Zames-Falb multipliers [18], [19] and the multipliers proposed by Fazlyab et al. [14]. It has been shown that $\mathbf{\Pi}_{\mathbb{NN}}$ encompasses these known multipliers; see [15] for details.

We are now ready to state the main result of this section.

Theorem 3: Suppose Φ_l is ReLU. For given $\check{B} \in \mathbf{R}^{n \times l}$ and $\check{C} \in \mathbf{R}^{l \times n}$ that are not necessarily balanced, let us consider the SDP:

$$\gamma_{\rm SDP} := \inf_{\gamma, \Pi \in \mathbf{\Pi}_{\mathbb{NN}}} \gamma \text{ subject to (17).}$$

Then, we have $\|\check{B} \circ \Phi_l \circ \check{C}\| \leq \gamma_{\rm SDP} \leq \|\check{B}\| \|\check{C}\|$.

Proof: We prove $\gamma_{\rm SDP} \leq \|\check{B}\| \|\check{C}\|$. To this end, it suffices to show that there exists $\Pi \in \mathbf{\Pi}_{\mathbb{NN}}$ such that

$$\begin{bmatrix} -\nu_B^2 \nu_C^2 I_n & 0 \\ 0 & \check{B}^{\top}\check{B} \end{bmatrix} + \begin{bmatrix} \check{C} & 0 \\ 0 & I_l \end{bmatrix}^{\top} \Pi \begin{bmatrix} \check{C} & 0 \\ 0 & I_l \end{bmatrix} \preceq 0 \quad (18)$$

where $\nu_B := \|\check{B}\|$ and $\nu_C := \|\check{C}\|$. To prove (18), we first note $-\nu_C^2 I_n + \check{C}^{\top}\check{C} \preceq 0$ and hence $-\nu_B^2 \nu_C^2 I_n + \nu_B^2 \check{C}^{\top}\check{C} \preceq 0$ holds. By Schur complement argument, we thus obtain

$$\begin{bmatrix} -\nu_B^2 \nu_C^2 I_n & \nu_B^2 \check{C}^{\top} \\ * & -\nu_B^2 I_l \end{bmatrix} \preceq 0$$

or equivalently

$$\begin{bmatrix} -\nu_B^2 \nu_C^2 I_n & -\check{C}^{\top} J \\ * & J \end{bmatrix} \preceq 0$$

where $J := -\nu_B^2 I_l$. Since $J + \check{B}^{\top}\check{B} \preceq 0$, the above inequality implies

$$\begin{bmatrix} -\nu_B^2 \nu_C^2 I_n & -\check{C}^{\top} J \\ * & \check{B}^{\top}\check{B} + 2J \end{bmatrix} \preceq 0.$$

This clearly shows that (18) holds with $\Pi = (*)^{\top} \mathcal{J}(J) E \in \mathbf{\Pi}_{\mathbb{NN}}$. From Proposition 3, this completes the proof. ∎

Remark 5: Proof of Theorem 3 implies that the claim of the theorem is still valid even when we set $Q = 0$ in the search within $\mathbf{\Pi}_{\mathbb{NN}}$ to solve the SDP.

From Proposition 1 and Theorem 3, we obtain the following corollary.

Corollary 1: Suppose Φ_l is ReLU. If the pair $(\check{B}, \check{C})$ is balanced, we have $\gamma^{\star}_{\mathbf{R}^n}(\Lambda_{\rm c}) \leq \gamma_{\rm SDP} \leq \|\check{B}\| \|\check{C}\| \leq \overline{\gamma}$.

VI. Numerical Example

In this letter paper, we give a small-scale numerical example to demonstrate the process and effectiveness of main results. Large-scale problems should be examined in future publications. Set $n = 10$ and $m = 10$ of the original RNN (3). The activation functions are the common ReLU nonlinearity (15). We generated the coefficient matrices of (3) in a random manner, whose resulting values are:

$$A = \begin{bmatrix} -11.40 & -1.34 & 2.25 & -1.46 & 1.92 & 3.92 & 1.80 & -0.98 & -0.84 & 0.79 \\ -2.02 & -6.69 & -3.44 & 1.17 & 0.33 & -4.72 & -1.28 & -0.88 & -1.20 & 0.89 \\ -0.31 & 2.45 & -7.74 & 0.03 & -1.18 & 0.11 & 1.59 & 1.18 & -1.47 & 0.12 \\ 1.74 & -1.60 & 2.24 & -10.74 & 2.69 & 1.29 & 2.51 & 2.03 & -0.09 & 1.26 \\ 0.04 & 0.60 & 0.92 & 1.87 & -9.71 & -0.33 & 0.36 & -0.25 & 0.86 & -1.76 \\ 0.08 & 1.56 & 3.69 & 3.55 & -2.05 & -7.60 & 1.75 & 0.39 & 0.80 & 2.29 \\ -0.94 & -1.58 & -4.52 & 0.19 & 3.81 & -0.73 & -7.09 & -3.05 & 0.60 & -4.47 \\ 0.14 & 3.59 & -4.09 & -2.20 & -1.85 & -1.04 & -0.42 & -8.85 & 0.91 & 4.53 \\ -0.49 & 0.07 & -4.32 & 0.42 & 2.71 & 1.96 & 2.33 & 1.26 & -5.25 & -0.63 \\ 1.71 & -2.62 & 1.40 & -0.41 & 1.90 & 0.95 & -0.29 & -0.27 & -3.59 & -11.47 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.34 & -0.42 & 0.01 & 0.19 & 0.40 & -0.50 & -0.21 & 0.08 & -0.16 & 0.26 \\ -0.45 & 0.54 & 0.01 & -0.03 & -0.44 & -0.04 & -0.23 & -0.65 & -0.10 & 0.21 \\ -0.52 & -0.62 & -0.47 & -0.19 & -0.12 & -0.03 & -0.19 & -0.37 & -0.23 & -0.04 \\ -0.33 & -0.01 & -0.21 & -0.56 & -0.17 & -0.41 & -0.63 & 0.19 & -0.16 & -0.33 \\ -0.53 & 0.50 & -0.53 & -0.15 & 0.14 & 0.50 & 0.18 & 0.44 & -0.28 & 0.57 \\ -0.16 & -0.54 & -0.34 & -0.50 & -0.08 & -0.50 & -0.39 & 0.12 & 0.66 & -0.35 \\ -0.01 & 0.09 & -0.42 & -0.70 & -0.64 & -0.13 & -0.53 & -0.43 & -0.14 & 0.33 \\ -0.30 & -0.47 & 0.01 & 0.04 & -0.47 & -0.48 & -0.37 & -0.49 & -0.89 & 0.02 \\ 0.33 & -0.55 & -0.02 & 0.12 & 0.36 & -0.06 & -0.44 & 0.11 & -0.09 & -0.46 \\ 0.21 & 0.16 & -0.15 & 0.18 & 0.28 & 0.29 & -0.22 & -0.84 & -0.17 & -0.18 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.26 & -0.31 & -0.02 & 0.03 & 0.20 & -0.29 & 0.15 & 0.17 & -0.45 & -0.08 \\ -0.47 & -0.26 & -0.36 & 0.22 & 0.38 & -0.50 & 0.22 & -0.21 & 0.11 & -0.33 \\ 0.38 & 0.47 & 0.17 & 0.50 & -0.08 & 0.11 & -0.12 & -0.05 & -0.47 & -0.19 \\ 0.06 & 0.26 & -0.30 & 0 & 0.77 & -0.22 & -0.42 & -0.71 & 0.25 & 0.16 \\ 0.08 & -0.18 & 0.11 & -0.22 & 0.20 & 0.29 & -0.41 & -0.02 & 0.32 & 0.22 \\ -0.16 & -0.19 & 0.02 & -0.54 & 0.34 & -0.28 & 0.24 & 0.08 & -0.09 & 0.02 \\ -0.23 & 0.25 & 0.77 & -0.18 & -0.16 & -0.21 & 0.50 & 0.37 & 0.03 & 0.09 \\ 0.47 & -0.06 & -0.38 & 0.06 & 0.53 & 0.31 & -0.09 & 0.13 & -0.47 & 0.50 \\ -0.14 & 0.03 & -0.12 & 0 & 0.01 & -0.09 & -0.14 & 0.11 & 0.02 & 0.60 \\ -0.31 & 0.27 & -0.37 & 0.19 & -0.51 & -0.21 & 0.41 & -0.08 & 0.45 & -0.09 \end{bmatrix}.$$

We aim at composing a reduced RNN of activation degree $m_{\mathrm{r}} = 6$ and guarantee the stability of the original RNN through Theorem 1.

Following Procedure 1, we selected the removed activation functions as $\Lambda_{\mathrm{c}} = \{1, 5, 6, 9\}$ and composed the reduced RNN (4). For this index set, we composed the matrices (16) and solved the SDP of Theorem 3 with transforming $\check{B}$ and $\check{C}$ into the balanced ones. The obtained solution was $\gamma_{\mathrm{SDP}} = 1.2561$. Then, to apply Theorem 1, we must find a Lyapunov function V satisfying (6a)–(6c) for the reduced RNN. To this end, we employed the IQC approach [8] with the set of static multipliers $\mathbf{\Pi}_{\mathbb{NN}}$ given in Proposition 3 and obtained the quadratic Lyapunov function $V(x) = x^\top P x$ with the positive definite matrix

$$P = \begin{bmatrix} 0.65 & -0.15 & 0.15 & 0.08 & 0.08 & 0.36 & 0.01 & -0.03 & -0.07 & 0.09 \\ -0.15 & 1.05 & 0.07 & -0.13 & -0.21 & -0.50 & -0.42 & -0.01 & -0.26 & 0.06 \\ 0.15 & 0.07 & 1.76 & -0.05 & -0.42 & 0.24 & -0.07 & 0.09 & -0.96 & 0.46 \\ 0.08 & -0.13 & -0.05 & 0.81 & 0.34 & 0.40 & 0.31 & -0.11 & 0.29 & -0.14 \\ 0.08 & -0.21 & -0.42 & 0.34 & 1.46 & 0.21 & 0.49 & -0.28 & 0.76 & -0.18 \\ 0.36 & -0.50 & 0.24 & 0.40 & 0.21 & 1.65 & 0.63 & 0.04 & 0.40 & 0.15 \\ 0.01 & -0.42 & -0.07 & 0.31 & 0.49 & 0.63 & 1.59 & 0.07 & 0.56 & -0.29 \\ -0.03 & -0.01 & 0.09 & -0.11 & -0.28 & 0.04 & 0.07 & 0.87 & -0.09 & 0.24 \\ -0.07 & -0.26 & -0.96 & 0.29 & 0.76 & 0.40 & 0.56 & -0.09 & 1.87 & -0.47 \\ 0.09 & 0.06 & 0.46 & -0.14 & -0.18 & 0.15 & -0.29 & 0.24 & -0.47 & 0.95 \end{bmatrix} \times 10^{-1}.$$

The constants were obtained as $c_3 = 1$ and $c_4 = 0.7917$, yielding $c_3/c_4 = 1.2630$. Therefore, all the assumptions of Theorem 1 were satisfied globally on $\mathbf{R}^n$, which leads to the global exponential stability of the origin of original RNN (3), as desired. It should be noted that we also obtained $\|\check{B}\| \, \|\check{C}\| = 1.5326$ and $\overline{\gamma} = 3.3392$, by which (9) is not satisfied in contrast to γ_{SDP}. This shows the effectiveness of the SDP computation of Theorem 3.

Besides these results, introducing the matrix variable Q in Proposition 3 causes another issue that the number of scalar variables of the SDP rapidly increases as $l = |\Lambda_{\mathrm{c}}|$ increases, which may conflict to our motivation. To avoid this issue, we may use only the matrix variable J, whose scalar variables increases only linearly in l, to compose the multipliers in Proposition 3 (see Remark 5). We computed the solution of the SDP with this treatment and obtained $\gamma_{\mathrm{SDP},J} = 1.2562$, which satisfies $\gamma_{\mathrm{SDP},J} < c_3/c_4$ as well as γ_{SDP}. Even with this loose (and conservative) SDP computation, we could conclude the global exponential stability of the origin of (3).

VII. Conclusion

We proposed a Lyapunov-based method of reducing the number of activation functions of an RNN for stability analysis. The proposed theorems and methods were demonstrated on a numerical example.

Some future works can be suggested: for effective stability analysis, instability criterion should be derived; since this paper presented only the theoretical results and the demonstration on a simple example as an early stage of study, the proposed methods should be examined on large-scale real-life RNNs; the proposed methodology should be extended to discrete-time RNN models; and model reduction methods for preserving input-to-output properties should be proposed.

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