On arithmetic Dijkgraaf-Witten theory

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On 3-dimensional foliated dynamical systems and Hilbert type reciprocity law

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Dedicated to Professor Christopher Deninger

Abstract. We introduce a geometric analog of the Hilbert symbol and show a Hilbert type reciprocity law for a 3-dimensional foliated dynamical system (FDS^3 for short). This answers the question posed by Deninger. For this, we employ the theory of smooth Deligne cohomology and the integration theory of Deligne cohomology classes. We also present a structure theorem for an FDS^3, which yields a classification of FDS^3's, and we construct concrete examples of FDS^3's for each type of the classification.

Introduction

In his monumental work and program on dynamical study of number theoretical zeta functions, Deninger pointed out that there are striking analogies between arithmetic schemes and foliated dynamical systems, namely, smooth manifolds equipped with 1-codimensional foliation and transversal flow satisfying certain conditions (cp. [11, 12, 13, 14, 15, 16, 17]; see also [30, 31]). In particular, a 3-dimensional foliated dynamical system, which is called an FDS^3 for short, may be regarded as a geometric analog of an arithmetic curve, where closed orbits (knots) correspond to closed points (finite primes). So Deninger's program fits and refines the analogies between knots and primes, 3-manifolds and number rings in arithmetic topology [33].

Following these analogies, Deninger asked if one can show a 3-dimensional geometric analog of the Hilbert reciprocity law in number theory. The main result of this paper is to answer his question by establishing a Hilbert type reciprocity law for an FDS^3. We also show a structure theorem for an FDS^3, which yields a classification of FDS^3's. We then construct concrete examples of FDS^3's for each type of the classification. This may be interesting because, before our work, most known examples of FDS^3 in the literature on Deninger's program were only closed surface bundles over S^1 with the bundle foliation and suspended flow. Let us describe our results more precisely in the following.

Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be a 3-dimensional foliated dynamical system, called an FDS^3 for short. Namely, M is a connected, closed smooth 3-manifold, $\mathcal{F}$ is a complex foliation by Riemann surfaces on M, ϕ is a smooth dynamical system on M. These data must satisfy the following conditions: there is a set of finitely many compact leaves $\mathcal{P}_{\mathfrak{S}}^{\infty} = \{L_1^{\infty}, \ldots, L_r^{\infty}\}$, which may be empty, such that, for any i and t, we have $\phi^t(L_i^{\infty}) = L_i^{\infty}$ and that any orbit of ϕ is transverse to leaves in $M_0 := M \setminus \bigcup_{i=1}^{r} L_i^{\infty}$, and ϕ^t maps any leaf to a leaf for each t (cp. Definition 1.5 below). Let $\mathcal{P}_{\mathfrak{S}}$ be the set of closed orbits in M_0, and let $\overline{\mathcal{P}_{\mathfrak{S}}} = \mathcal{P}_{\mathfrak{S}} \cup \mathcal{P}_{\mathfrak{S}}^{\infty}$. Note that $\mathcal{P}_{\mathfrak{S}}$ (resp. $\mathcal{P}_{\mathfrak{S}}^{\infty}$) may be regarded as an analog of the set $\mathcal{P}_k$ of finite primes (resp. the set $\mathcal{P}_k^{\infty}$ of infinite primes) of a global field (namely, the function field of an arithmetic curve) k and $\overline{\mathcal{P}_{\mathfrak{S}}}$ corresponds to the set $\overline{\mathcal{P}_k}$ of all primes of k.

The Hilbert reciprocity law is one of deep results in class field theory for a global field k. It has the form of a product formula $\prod_{\mathfrak{p} \in \overline{\mathcal{P}_k}} \{a, b\}_{\mathfrak{p}} = 1$, where $\{\,,\}_{\mathfrak{p}}$ stands for the Hilbert symbol in the completion of k at $\mathfrak{p}$ and $a, b \in k^{\times}$ (cp. [1, Chap. 12, 4], [35, Chap. IV, § 9]). When the global field k is replaced by the function field of a closed Riemann surface R, we have a similar reciprocity law

$$\prod_{P \in R} \{f, g\}_P = 1, \tag{1}$$

where f, g are meromorphic functions on R (not constantly zero) and $\{f, g\}_P$ is the tame symbol at P defined by

$$\{f, g\}_P = (-1)^{\mathrm{ord}_P(f)\,\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}(P)$$

(cp. [38, Chap. III, § 1, 4]). By Cauchy's theorem, the similar reciprocity law (1) can be stated as the following summation formula (cp. [10]):

$$\begin{cases} \displaystyle\sum_{P \in R} \langle f, g\rangle_P = 0 \mod \mathbb{Z}(2), \\ \displaystyle\langle f, g\rangle_P = \int_C \log(f)\, d\log(g) - \log(g(Q)) \int_C d\log(f), \end{cases} \tag{2}$$

where C is a small loop on R around P based at Q and $\mathbb{Z}(2) := (2\pi\sqrt{-1})^2\mathbb{Z}$. In [3, 4, 10], Beilinson, Bloch and Deligne interpreted the tame symbol $\{f, g\}_P$ (or $\langle f, g\rangle_P$) using holomorphic Deligne cohomology and showed the reciprocity law in a conceptual manner. Our method to obtain a geometric analog for an FDS^3 of the Hilbert reciprocity law is to generalize Beilinson–Bloch–Deligne's work to the case of an FDS^3. For this, we employ the theory of smooth Deligne cohomology and the integration theory of Deligne cohomology classes, which were studied by Brylinski [5, 6] and by Gawedzki [20] and Gomi-Terashima [22, 42]. Our result can be stated as follows. For $\gamma \in \overline{\mathcal{P}_{\mathfrak{S}}}$ and FDS^3-meromorphic functions f, g (cp. Definition 1.8 below), we introduce the local symbol $\langle f, g\rangle_{\gamma}$

and show the reciprocity law in the form similar to (2),

$$\sum_{\gamma \in \overline{\mathcal{P}_{\mathfrak{S}}}} \langle f, g \rangle_\gamma = 0 \mod \Lambda_{\mathfrak{S}}(3),$$

where $\Lambda_{\mathfrak{S}}$ is the period group of $\mathfrak{S}$ and $\Lambda_{\mathfrak{S}}(3) := (2\pi\sqrt{-1})^3 \Lambda_{\mathfrak{S}}$ (cp. Theorem 5.7 below). For a closed orbit $\gamma \in \mathcal{P}_{\mathfrak{S}}$, we give an explicit integral formula for the local symbol $\langle f, g \rangle_\gamma$,

$$\langle f, g \rangle_\gamma = \int_{T(\gamma)} \log(f)\, d\log(g) \wedge \omega_{\mathfrak{S}} - \int_{\mathfrak{m}} d\log(f) \int_{\mathfrak{l}} \log(g) \omega_{\mathfrak{S}} \mod \Lambda_{\mathfrak{S}}(3),$$

where $\omega_{\mathfrak{S}}$ is the canonical 1-form of $\mathfrak{S}$ (cp. Definition 1.10 below), and $\mathfrak{m}, \mathfrak{l}$ are a meridian and a longitude, respectively, on the boundary $T(\gamma)$ of a tubular neighborhood of γ (cp. Theorem 5.3 below). This formula may be regarded as a generalization of the integral formula for the tame symbol in (2) to the case of an FDS^3. Our result is also a generalization of Stelzig's work for the case of surface bundles over S^1 (see [40]).

We note that our result may indicate the possibility to develop an idèlic theory for FDS^3's. For this line of study, we mention the recent works by Niibo and Ueki [37] and by Mihara [32] on idèlic class field theory for 3-manifolds in arithmetic topology. In fact, our original desire was to refine and deepen the analogies between knots and primes, 3-manifolds and number rings in arithmetic topology [33], in the context of FDS^3's.

Before our work, most known examples of FDS^3's in the literature on Deninger's program were only closed surface bundles over S^1 with the bundle foliation and suspended flow, besides Álvarez-López's example in [17, p. 10]. In this paper, we show a structure theorem for an FDS^3, called a decomposition theorem (cp. Theorem 2.7 below), which yields a classification of FDS^3's according to the sets of transverse and non-transverse compact leaves (cp. Corollary 2.9). Our decomposition theorem may remind us of the JSJ decomposition of a 3-manifold [27, 28]. We then construct concrete examples of FDS^3's for each type of our classification. Furthermore, we show that any closed smooth 3-manifold admits a structure of an FDS^3, by using an open book decomposition, which is a peculiar property for the 3-dimensional case (cp. Example 3.7 below). In view of the analogy with an arithmetic curve which has countably infinitely many finite primes, we give examples of FDS^3's such that $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. We note that related results with our classification are described in a recent preprint [2].

The contents of this paper are organized as follows. In Section 1, we introduce a 3-dimensional foliated dynamical system (FDS^3) and some basic notions. In Section 2, we show a decomposition theorem for an FDS^3, which yields a classification of FDS^3's. In Section 3, for each type of the classification, we construct concrete examples of FDS^3's. In Section 4, we recall the theory of smooth Deligne cohomology for an FDS^3 and the integration theory of Deligne cohomology classes. Section 5 is concerned with the main result. We introduce local symbols of FDS^3-meromorphic functions along a closed orbit or along a non-transverse compact leaf, and show a Hilbert type reciprocity law, by using the results given in Section 4.

1. 3-dimensional foliated dynamical systems

In this section, following Deninger and Kopei [11, 12, 13, 14, 15, 16, 17, 30, 31]), we introduce the notion of a 3-dimensional foliated dynamical system, called an FDS^3 for short, and prepare some basic notions and properties. Although a foliated dynamical system can be introduced in any odd dimension, we consider only the 3-dimensional case in this paper since we are concerned with analogies with arithmetic curves. For general materials in foliation theory, we refer to [8, 41].

We begin to recall the notion of a complex foliation by Riemann surfaces (cp. [21]).

Definition 1.1. Let M be a smooth 3-manifold whose boundary ∂M may be nonempty. A 2-*dimensional smooth foliation* $\mathcal{F}$ on M is defined by a family of immersed 2-dimensional manifolds $\{L_a\}_{a\in A}$ satisfying the following conditions:
(1) $M = \bigsqcup_{a\in A} L_a$ (disjoint union);
(2) there is a system of foliated local coordinates $(U_i;\varphi_i)_{i\in I}$, where U_i is a foliated open subset of M and $\varphi_i : U_i \xrightarrow{\approx} \varphi_i(U_i) \subset \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is a diffeomorphism ($\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} \mid t \geq 0\}$) such that if $U_i \cap L_a$ is nonempty, $\varphi_i(U_i \cap L_a) = \mathrm{Int}(D) \times \{c_a\}$, where $\mathrm{Int}(D)$ is the interior of a 2-disc $D \subset \mathbb{R}^2$ and $c_a \in \mathbb{R}_{\geq 0}$.

Here each L_a is called a *leaf* of the foliation $\mathcal{F}$. We call the pair $(M, \mathcal{F})$ simply a *foliated* 3-*manifold*.

We call a foliation $\mathcal{F}$ on M a *complex foliation by Riemann surfaces* and the pair $(M, \mathcal{F})$ a *foliated* 3-*manifold by Riemann surfaces* if we require further the following conditions:
(3) each leaf L_a is a Riemann surface, namely, L_a has a complex structure;
(4) the above condition (2) with replacing $\mathbb{R}^2$ (leaf coordinate) by $\mathbb{C}$;
(5) $(\varphi_j \circ \varphi_i^{-1})(z,t) = (f_{ij}(z,t), g_{ij}(t))$ for $(z,x) \in \varphi_i(U_i \cap U_j) \subset \mathbb{C} \times \mathbb{R}_{\geq 0}$, where f_{ij} is holomorphic in z and g_{ij} is independent of z.

Remark 1.2. (1) By Frobenius' theorem, the following notions are equivalent if M is closed (cp. [8, 1.3], [41, § 28]):
- 2-dimensional smooth foliation on M,
- 2-plane field distribution E (rank 2 subbundle of the tangent bundle TM) which is involutive,
- 2-plane field distribution E which is completely integrable.

(2) If the tangent bundle to a 2-dimensional smooth foliation $\mathcal{F}$ is orientable, $\mathcal{F}$ admits a structure of a complex foliation (cp. [25, Rem. 1.2], [26, Lem. A.3.1]). In fact, each orientable 2-dimensional leaf is equipped with a Riemannian metric and a smooth almost complex structure. By the parametrized Newlander–Nirenberg's theorem [36], it follows that there is a system of complex foliated local coordinate as in conditions (4) and (5) in Definition 1.1.

Definition 1.3. For foliated 3-manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a *morphism* $(M, \mathcal{F}) \to (M', \mathcal{F}')$ is defined to be a smooth map $f : M \to M'$ such that f maps any leaf L of $\mathcal{F}$ into a leaf L' of $\mathcal{F}'$. When $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are foliated 3-manifolds by Riemann surfaces, we require further $f|_L : L \to L'$ to be holomorphic for any leaf L of $\mathcal{F}$. In terms of local coordinates, this is equivalent to saying that, for any $p \in M$, there are foliated local coordinates (U, φ) and (U', φ') around p and $f(p)$, respectively, such that $(\varphi' \circ f|_U \circ \varphi^{-1})(z, t) = (f^t(z), t)$, where $f^t(z)$ is holomorphic for fixed t.

An *isomorphism* $(M, \mathcal{F}) \xrightarrow{\sim} (M', \mathcal{F}')$ of foliated 3-manifolds (resp. foliated 3-manifolds by Riemann surfaces) is a diffeomorphism $f : M \to M'$ such that $f|_L : L \to L'$ is diffeomorphic (resp. biholomorphic) for any leaf L of $\mathcal{F}$. We identify $(M, \mathcal{F})$ with $(M', \mathcal{F}')$ if there is an isomorphism between them.

Definition 1.4. A *smooth dynamical system* (or *smooth flow*) on M is defined by a smooth action ϕ of $\mathbb{R}$ on M, namely, a smooth map $\phi : \mathbb{R} \times M \to M$ such that $\phi^t := \phi(t, \cdot\,)$ is a diffeomorphism of M for each $t \in \mathbb{R}$ which satisfies $\phi^0 = \mathrm{id}_M$, $\phi^{t+t'} = \phi^t \circ \phi^{t'}$ for any $t, t' \in \mathbb{R}$.

Now we introduce the main object in this paper (cp. [31]).

Definition 1.5. We define a 3-dimensional *foliated dynamical system* by a triple $\mathfrak{S} = (M, \mathcal{F}, \phi)$, where
(1) M is a connected, closed smooth 3-manifold,
(2) $\mathcal{F}$ is a complex foliation by Riemann surfaces on M,
(3) ϕ is a smooth dynamical system on M,
and these data must satisfy the following conditions:
(i) there are a finite number of compact leaves $L_1^\infty, \dots, L_r^\infty$, which may be empty, such that, for any i and t, we have $\phi^t(L_i^\infty) = L_i^\infty$ and that any orbit of the flow ϕ is transverse to leaves in $M \setminus \bigcup_{i=1}^r L_i^\infty$;
(ii) for each $t \in \mathbb{R}$, the diffeomorphism ϕ^t of M maps any leaf to a leaf.
In this paper, a 3-dimensional foliated dynamical system is called an FDS^3 for short.

Throughout this paper, we shall use the following notations. For an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$, we set

$$
\begin{aligned}
\mathcal{F}^c &:= \text{the set of all compact leaves},\\
\mathcal{P}_{\mathfrak{S}}^\infty &:= \text{the set of non-transverse compact leaves}\\
&= \{L_1^\infty, \dots, L_r^\infty\} \quad \text{(this set may be empty)},\\
L^\infty &:= \bigcup_{i=1}^r L_i^\infty,\\
M_0 &:= M \setminus L^\infty,\\
\mathcal{P}_{\mathfrak{S}} &:= \text{the set of closed orbits of the flow } \phi \text{ which are transverse}\\
&\ \text{to leaves in } M_0,\\
\overline{\mathcal{P}_{\mathfrak{S}}} &:= \mathcal{P}_{\mathfrak{S}} \sqcup \mathcal{P}_{\mathfrak{S}}^\infty.
\end{aligned}
\tag{3}
$$

Remark 1.6. As Deninger and Kopei suggested (cp. [11, 12, 13, 14, 15, 16, 30, 31]), an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ may be regarded as a geometric analog of a compact arithmetic curve, namely, a smooth proper algebraic curve over a finite field or $\overline{\mathrm{Spec}(\mathcal{O}_k)} = \mathrm{Spec}(\mathcal{O}_k) \cup \mathcal{P}_k^\infty$ for the ring $\mathcal{O}_k$ of integers of a number field k, where $\mathcal{P}_k^\infty$ is the set of infinite primes of k. The set $\mathcal{P}_{\mathfrak{S}}$ corresponds to the set of closed points (finite primes) of C or $\mathrm{Spec}(\mathcal{O}_k)$. When $\mathcal{P}_{\mathfrak{S}}^\infty$ is nonempty, it corresponds to $\mathcal{P}_k^\infty$. So the analogy is closer if $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. We give such examples in Section 3.

Definition 1.7. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ and $\mathfrak{S}' = (M', \mathcal{F}', \phi')$ be FDS^3's. An *FDS^3-morphism* $\mathfrak{S} \to \mathfrak{S}'$ is defined by a morphism of foliated 3-manifolds by Riemann surfaces (cp. Definition 1.3) such that $f : M \to M'$ commutes with the flows, namely, $f \circ \phi^t = \phi'^t \circ f$ for any $t \in \mathbb{R}$.

An *FDS^3-isomorphism* $\mathfrak{S} \xrightarrow{\sim} \mathfrak{S}'$ is an FDS^3-morphism which is an isomorphism of foliated 3-manifolds by Riemann surfaces. We identify $\mathfrak{S}$ with $\mathfrak{S}'$ if there is an FDS^3-isomorphism between them.

Definition 1.8. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3. An *FDS^3-meromorphic function* on $\mathfrak{S}$ is defined to be a smooth map $f : M_0 \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ satisfying the following conditions:

(i) f restricted to any leaf is a meromorphic function;
(ii) the zeros and poles of f lie along finitely many closed orbits.

For an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$, let $T\mathcal{F}$ denote the subbundle of the tangent bundle TM_0 whose total space is the union of the tangent spaces of leaves, and let $\dot{\phi}^t = \frac{d}{dt}\phi^t$ be the vector field on M_0 which generates the flow ϕ.

Lemma 1.9. *For an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$, there is the unique closed smooth 1-form ω on M_0 satisfying*

$$\text{(C)} \qquad \omega|_{T\mathcal{F}} = 0, \quad \omega(\dot{\phi}^t) = 1.$$

More precisely, let $(M, \mathcal{F}, \phi)$ be a triple satisfying (1), (2), (3) *and* (i) *in Definition 1.5. Then there is the unique smooth 1-form ω on M_0 satisfying* (C), *and condition* (ii) *is equivalent to that ω is closed.*

Proof. Let $(M, \mathcal{F}, \phi)$ be a triple satisfying (1), (2), (3) and (i) in Definition 1.5. Then we can take a smooth 1-form ω on M_0 such that, for each foliated coordinate $(z,t) \in U$, $\omega|_U = h(z,t)\,dt$. Writing $\dot{\phi}^t = a(z,t)\partial_z + b(z,t)\partial_{\overline{z}} + c(z,t)\partial_t$, we have $\omega(\dot{\phi}^t) = c(z,t)h(z,t)$, and so $h(z,t)$ is uniquely determined by $\omega(\dot{\phi}^t) = 1$. This yields the first assertion.

Now, assume condition (ii). For each point $p \in M_0$, there is a foliated local coordinate $(z,t) \in U$ such that $\phi^s(z,t) = (z, t+s)$. So there is a system of foliated local coordinates $(U_i; \varphi_i)_{i \in I}$ of M_0 such that if $U_i \cap U_j$ is not empty, $(\varphi_j \circ \varphi_i^{-1})(z_i, t_i) = (f_{ij}(z,t), t + c_{ij})$ for some constant $c_{ij} \in \mathbb{R}$. Hence $\omega|_{U_i} = dt_i$, and so ω is closed.

Conversely, assume that $d\omega = 0$. We may take each U_i is simply connected so that $\omega|_{U_i} = dt_i$ by Poincaré's lemma. Then we can follow the above argument in the reverse way to obtain condition (ii). □

Definition 1.10. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3. We call the smooth closed 1-form in Lemma 1.9 the *canonical* 1*-form* of $\mathfrak{S}$ and denote it by $\omega_{\mathfrak{S}}$. The de Rham cohomology class of $\omega_{\mathfrak{S}}$ defines the *period homomorphism*

$$[\omega_{\mathfrak{S}}] : H_1(M_0; \mathbb{Z}) \to \mathbb{R}; \quad [\ell] \mapsto \int_\ell \omega_{\mathfrak{S}},$$

and the *period group* of $\mathfrak{S}$ is defined by the image of $[\omega_{\mathfrak{S}}]$, which we denote by $\Lambda_{\mathfrak{S}}$.

The following lemma will be used in the examples of Section 3.

Lemma 1.11. *Let $\mathfrak{S}_1 = (M, \mathcal{F}, \phi_1)$ and $\mathfrak{S}_2 = (M, \mathcal{F}, \phi_2)$ be FDS^3's. Then we have $\omega_{\mathfrak{S}_1} = \omega_{\mathfrak{S}_2}$ if and only if $(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F}$ for any $p \in M_0$.*

Proof. Since $\omega_{\mathfrak{S}_i}|_{T\mathcal{F}} = 0, \omega_{\mathfrak{S}_i}(\dot{\phi}_i^t) = 1$ for $i = 1, 2$, we have

$$\begin{aligned}(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F} \quad \text{for any } p \in M_0 &\iff \omega_{\mathfrak{S}_1}(\dot{\phi}_1^t - \dot{\phi}_2^t) = 0 \\ &\iff \omega_{\mathfrak{S}_1}(\dot{\phi}_2^t) = \omega_{\mathfrak{S}_1}(\dot{\phi}_1^t) = 1 \\ &\iff \omega_{\mathfrak{S}_1}|_{T\mathcal{F}} = 0, \omega_{\mathfrak{S}_1}(\dot{\phi}_2^t) = 1 \\ &\iff \omega_{\mathfrak{S}_2} = \omega_{\mathfrak{S}_1}.\end{aligned}$$

□

2. A decomposition theorem and a classification of FDS^3's

In this section, we give a decomposition theorem for an FDS^3, which yields a classification of FDS^3's. We start with some preparations about the holonomy of a leaf and Tischler's theorem.

2.1. **Foliations without holonomy and Tischler's theorem.** First, we recall the notion of the holonomy group of a leaf (cp. [8, 2.2, 2.3], [41, § 22]). Let $(M, \mathcal{F})$ be a foliated 3-manifold, and let L be a leaf of $\mathcal{F}$. Choose $p \in L$, and let $[c] \in \pi_1(L, p)$ be represented by a loop $c : [0, 1] \to L$ with $c(0) = c(1) = p$. We may choose a subdivision $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ and a chain of foliated charts $(U_0; \varphi_0), \ldots, (U_m; \varphi_m)$ such that $c([t_i, t_{i+1}]) \subset U_i$ for $0 \leq i \leq m$. Then we have

$$(\varphi_{i+1} \circ \varphi_i^{-1})(z, t) = (f_{i,i+1}(z, t), g_{i,i+1}(t))$$

for some smooth functions $f_{i,i+1}, g_{i,i+1}$. We set

$$h_c := g_{m-1,m} \circ \cdots \circ g_{0,1},$$

which is a local homeomorphism of $\mathbb{R}$ fixing 0. Let G be the group of germs of local homeomorphisms of $\mathbb{R}$ fixing 0, and let $\widehat{h_c} \in G$ be the germ of h_c. The correspondence $[c] \mapsto \widehat{h_c}$ gives a well-defined homomorphism $\Psi : \pi_1(L, p) \to G$. The *holonomy group* $\mathcal{H}(L)$ of L is defined by the image of Ψ, which is independent, up to conjugation, of all choices. A foliated 3-manifold $(M, \mathcal{F})$ is said to be *without holonomy* if the holonomy group $\mathcal{H}(L)$ is trivial for any leaf L of $\mathcal{F}$.

The next theorem, due to Candel and Conlon, is an important ingredient for our decomposition theorem given in the following subsection. For the proof, we refer to [8, 9.1]. For a foliated 3-manifold $(M, \mathcal{F})$, a subset X of M is said to be *$\mathcal{F}$-saturated* if X is a union of leaves of $\mathcal{F}$. The following theorem is due to G. Hector.

Theorem 2.2 ([23]). *Let $(M, \mathcal{F})$ be a foliated 3-manifold with M being closed. Let X be a connected $\mathcal{F}$-saturated open subset of M. Assume that the foliated 3-manifold $(X, \mathcal{F}|_X)$ is without holonomy. Then one of the following holds.*
(1) *X is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_X$ is the bundle foliation.*
(2) *Each leaf of $\mathcal{F}|_X$ is dense in X.*

Next, we recall Tischler's theorem [43] and its generalization [9].

Theorem 2.3 ([43, Thm. 1], [9, Thm. 2.1]). *Let $(M, \mathcal{F})$ be a foliated 3-manifold with M being closed. Let X be an $\mathcal{F}$-saturated open set of M. Assume that there is a non-vanishing, closed 1-form ω on X such that $\mathrm{Ker}(\omega)$ defines $\mathcal{F}|_X$. Then X is a surface bundle over S^1.*

Remark 2.4. A non-vanishing, closed 1-form assumed in Theorem 2.2 is approximated (in the C^∞-topology) by the 1-form $\varpi^*(d\theta)$, where $\varpi : X \to S^1$ is the fibration and $d\theta$ is an angular 1-form on S^1 (cp. *ibid.*).

2.5. **A decomposition theorem.** Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3. Suppose that $X_1, \dots, X_d$ are connected components of $M_0 := M \setminus L^\infty$. Each $(X_a, \mathcal{F}|_{X_a})$ is a foliated 3-manifold for $1 \le a \le d$.

Lemma 2.6. *The foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is without holonomy.*

Proof. Let L be any leaf of $\mathcal{F}|_{X_a}$. Since ϕ^t maps any leaf to a leaf, there is a system foliated charts $\{(U_i; \varphi_i)\}$ which covers L and satisfies

$$\varphi_i(\phi^t(p)) = (z_i(p), t)$$

for any $p \in L$. Therefore, if $U_i \cap U_j$ is nonempty, we have

$$(\varphi_i \circ \varphi_j^{-1})(z_j, t) = (z_i(\varphi_j^{-1}(z_j, t)), t),$$

and so the holonomy group $\mathcal{H}(L)$ is trivial. Hence $(X_a, \mathcal{F}|_{X_a})$ is without holonomy. □

Theorem 2.7 (A decomposition theorem). *Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3, and let X_a be a connected component of $M_0 := M \setminus L^\infty$. The foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is one of the following:*
(1) *X_a is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_{X_a}$ is the bundle foliation;*
(2) *X_a is a surface-bundle over S^1 and any leaf in $\mathcal{F}|_{X_a}$ is dense in X_a.*

Proof. By Definition 1.5 of an FDS^3, X_a is a connected $\mathcal{F}$-saturated open subset of M. By Theorem 2.2, (1) X_a is a surface bundle over S^1 or an open interval, and $\mathcal{F}|_X$ is the bundle foliation, or (2) each leaf of $\mathcal{F}|_X$ is dense in X. For case (2), X is a surface bundle over S^1 by Lemma 1.11 and Theorem 2.3. □

Remark 2.8. For an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$, if M is cut along L^∞, then M_0 is decomposed into connected components $M_0 = X_1 \sqcup \cdots \sqcup X_d$, where each foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ has the structure described in Theorem 2.7 and the flow $\phi|_{X_a}$ is transverse to leaves of $\mathcal{F}|_{X_a}$. This is the reason that we call Theorem 2.7 a *decomposition theorem.* It may remind us of the JSJ decomposition of a 3-manifold [27, 28].

Theorem 2.7 can be restated as the following classification of FDS^3's. For the notations, see (3).

Corollary 2.9 (A classification)**.** *An FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is classed as one of the following types.*

(1) *$\mathcal{F} = \mathcal{F}^c$ and $\mathcal{P}^\infty_{\mathfrak{S}}$ is empty. Then M is a surface bundle over S^1 and $\mathcal{F}$ is the bundle foliation.*
(2) *$\mathcal{F}^c$ is empty. Then M is a surface bundle over S^1 and any leaf of $\mathcal{F}$ is dense in M.*
(3) *$\mathcal{P}^\infty_{\mathfrak{S}}$ is a nonempty (finite) set. Let X_a be a connected component of M_0. Then the foliated 3-manifold $(X_a, \mathcal{F}|_{X_a})$ is one of the following:*
    (a) *X_a is a surface bundle over S^1, and $\mathcal{F}|_{X_a}$ is the bundle foliation;*
    (b) *X_a is a surface bundle over an open interval, and $\mathcal{F}|_{X_a}$ is the bundle foliation;*
    (c) *X_a is a surface-bundle over S^1, and any leaf in $\mathcal{F}|_{X_a}$ is dense in X_a.*

We note that the class of FDS^3's of bundle foliation over S^1 is characterized by the period group. Although this may be known (cp. [7, 9.3], [19, 2.1]), we give a proof in the following, for the sake of readers.

Proposition 2.10. *Let $\mathfrak{S}$ be an FDS^3. If $\mathfrak{S}$ is of type I or of type III-1, then the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$. Conversely, if the $\Lambda_{\mathfrak{S}}$ has rank one (namely, $\Lambda_{\mathfrak{S}} \simeq \mathbb{Z}$) and M_0 is connected, then $\mathfrak{S}$ is of type I or of type III-1.*

Proof. Suppose that $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is an FDS^3 of type I or of type III-1. Then, for any connected component X_a of M_0, there is a fibration $\varpi_a : X_a \to S^1$. Let $d\theta$ be the angular 1-form on S^1 such that $\int_{S^1} d\theta = 1$. Then the canonical 1-form $\omega_{\mathfrak{S}}$ is given by $\omega_{\mathfrak{S}}|_{X_a} = \varpi_a^*(d\theta)$ for any a. For $[\ell] \in H_1(X_a; \mathbb{Z})$, we have

$$[\omega_{\mathfrak{S}}]([\ell]) = \int_{(\varpi_a)_*(\ell)} d\theta = \text{the degree of } \varpi_*(\ell) \text{ on } S^1$$

for any a, and hence $\Lambda_{\mathfrak{S}} = \mathbb{Z}$. Conversely, suppose that $\mathfrak{S} = (M, \mathcal{F}, \phi)$ is an FDS^3 with M_0 being connected and $\Lambda_{\mathfrak{S}} = \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}^\times$ so that $[\omega_{\mathfrak{S}}](H_1(M_0; \mathbb{Z})) = \lambda\mathbb{Z}$. Fix a base point $p_0 \in M_0$, and define the map $\varpi : M_0 \to S^1$ by

$$\varpi(p) := \exp\biggl(\frac{2\pi i}{\lambda} \int_\gamma \omega_{\mathfrak{S}}\biggr),$$

where γ is a path from p_0 to p. Then we see easily that $\lambda\varpi^*(d\theta) = \omega_{\mathfrak{S}}$. By Definition 1.10 of $\omega_{\mathfrak{S}}$, ϖ is a fibration and $\mathcal{F}$ consists of fibers of φ. Hence $\mathfrak{S}$ is of type I or of type III-1. ☐

Finally, we note that an FDS^3 of type III-2 has no transverse closed orbits.

Proposition 2.11. *Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3 of type III-2. Then $\mathcal{P}_{\mathfrak{S}}$ is empty and $\Lambda_{\mathfrak{S}} = \{0\}$.*

Proof. Let X_a be a connected component of M_0. We may identify $(X_a, \mathcal{F}|_{X_a})$ with $(L \times (0,1), \{L \times \{t\}\}_{t\in(0,1)})$, where $L \in \mathcal{F}|_{X_a}$. Let $\varpi : X_a \to (0,1)$ be the projection. For any closed curve $\ell : S^1 \to X_a$, $\varpi \circ \ell : S^1 \to (0,1)$ has the maximum μ, and so ℓ is not transverse to the leaf $L \times \{\mu\}$. Hence $\mathcal{P}_{\mathfrak{S}}$ is empty.

Next, since X_a is homotopy equivalent to the leaf $L \times \{\frac{1}{2}\}$, ℓ is homotopic to a closed curve ℓ' in $L \times \{\frac{1}{2}\}$. Since $\omega_{\mathfrak{S}}|_{T\mathcal{F}} = 0$, we have $[\omega_{\mathfrak{S}}]([\ell]) = [\omega_{\mathfrak{S}}]([\ell']) = 0$. Hence $\Lambda_{\mathfrak{S}} = \{0\}$. □

3. EXAMPLES OF FDS^3'S

In this section, we construct concrete examples of FDS^3's for each type in Corollary 2.9.

I. We give an example of an FDS^3 of type I. Note that any smooth surface bundle over S^1 is obtained by the mapping torus of a surface diffeomorphism.

Example 3.1 (Mapping torus and pseudo-Anosov flow). Let R_g be a connected, closed smooth surface of genus $g \geq 1$, and let φ be a diffeomorphism of R_g. Let M be the *mapping torus* $M(R_g, \varphi)$ defined by

$$M(R_g, \varphi) := (R_g \times [0,1])/(z,1) \sim (\varphi(z), 0).$$

Then the projection

$$\varpi : M \to S^1 = \mathbb{R}/\mathbb{Z}; \quad \varpi([z,s]) = s \pmod{\mathbb{Z}}$$

is a fibration, where each fiber $\varpi^{-1}(\theta)$ over $\theta \in S^1$ is diffeomorphic to R_g. The set of fibers $\mathcal{F} := \{\varpi^{-1}(\theta)\}_{\theta\in S^1}$ defines a 2-dimensional foliation, the bundle foliation. Since $T\mathcal{F}$ is orientable, $\mathcal{F}$ admits a complex foliation structure (cp. Remark 1.2 (2)). Let ϕ be the suspension flow defined by

$$\phi^t([z,s]) := [z, s+t].$$

Then $\mathfrak{S} := (M, \mathcal{F}, \phi)$ forms an FDS^3. Suppose further that the diffeomorphism φ is of pseudo-Anosov type (cp. [7, 1.11]). Note that φ is an Anosov diffeomorphism when R_g is a torus. Then, since φ has countably infinite periodic points, $P_{\mathfrak{S}}$ is a countably infinite set. By Proposition 2.11, the canonical 1-form $\omega_{\mathfrak{S}}$ is $\varpi^*(d\theta)$ and the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$.

Remark 3.2. An FDS^3 of type I may be regarded as an analog of a smooth proper algebraic curve C over a finite field $\mathbb{F}_q$, where the 2-dimensional foliation corresponds to the geometric fiber $C \otimes \overline{\mathbb{F}_q}$ and the monodromy φ corresponds to the Frobenius automorphism in $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

II. We construct two examples of FDS^3's of type II. First, we give a method to construct countably infinitely many closed orbits using the horseshoe map. Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3 and $\gamma \in \mathcal{P}_{\mathfrak{S}}$. We say that the flow ϕ is *of contraction type* around γ if there is a tubular neighborhood $V = D \times S^1$ of γ, where D ($\subset \mathbb{C}$) is a 2-disc centered at 0 and $\gamma = \{0\} \times S^1$, such that, for any $t \in \mathbb{R}$ and $p \in \gamma$, $\phi^t(V) \subset V$ and

$$\phi^t_{D,p} := \phi^t|_{D\times\{p\}} : D = D \times \{p\} \xrightarrow{\phi^t} \phi^t(D) \times \{\phi^t(p)\} \subset D$$

is a contraction map, namely, $T_z(\phi^t_{D,p})$ has the eigenvalues λ_1, λ_2 satisfying $0 < |\lambda_1|, |\lambda_2| < 1$ for $z \in \mathrm{Int}(D) \setminus \{0\}$. Let $h : \mathrm{Int}(D) \to \mathrm{Int}(D)$ be the *horseshoe diffeomorphism* [39], and let $U = M(\mathrm{Int}(D), h)$ be the mapping torus of h equipped with the suspension flow ϕ_h. Note that there is a diffeomorphism $\psi : U \xrightarrow{\cong} \mathrm{Int}(V)$.

Lemma 3.3. *Notations being as above, let us replace* $(\mathrm{Int}(V), \phi|_{\mathrm{Int}(V)})$ *by* (U, ϕ_h) *via* ψ *so that the resulting* 3*-manifold* $M_{\gamma,h}$ *is equipped with a new smooth flow* $\phi_{\gamma,h}$ *satisfying* $\phi_{\gamma,h} = \phi_h$ *in* U *and* $\phi_{\gamma,h} = \phi$ *in* $M \setminus \mathrm{Int}(V)$*. We define the foliation* $\mathcal{F}_{\gamma,h}$ *on* $M_{\gamma,h}$ *by the foliation* $\mathcal{F}$ *on* M *via* ψ*. Then the triple* $\mathfrak{S}_{\gamma,h} := (M_{\gamma,h}, \mathcal{F}, \phi_{\gamma,h})$ *forms an FDS*^{3}*, and the flow* $\phi_{\gamma,h}$ *has countably infinitely many closed orbits around* γ*. Moreover, the canonical forms of* $\mathfrak{S}$ *and* $\mathfrak{S}_{\gamma,h}$ *are the same,* $\omega_{\mathfrak{S}} = \omega_{\mathfrak{S}_{\gamma,h}}$*.*

Proof. That $(M_{\gamma,h}, \mathcal{F}_{\gamma,h}, \phi_{\gamma,h})$ forms an FDS^3 is easily seen by the construction. The latter property follows from the fact that the horseshoe map has countably infinitely many periodic points [29, Cor. 2.5.1]. For any $p \in \gamma$, we may assume, by the change of parameters, that there is $t_0 > 0$ such that

$$\begin{aligned} \phi^{t_0}|_{\mathrm{Int}(D)\times\{p\}} &: \mathrm{Int}(D) \times \{p\} \to \phi^{t_0}(\mathrm{Int}(D)) \times \{p\}, \\ \phi_h^{t_0}|_{\mathrm{Int}(D)\times\{p\}} &: \mathrm{Int}(D) \times \{p\} \to \phi_h^{t_0}(\mathrm{Int}(D)) \times \{p\} \end{aligned}$$

are first return diffeomorphisms. Since these are isotopic and an isotopy is realized as a flow in $\mathrm{Int}(V)$, we see that the canonical 1-forms are unchanged. □

Example 3.4. The following example of a foliated 3-manifold was considered in [7, 4.2] and [26, A.5] (see also Álvarez-López's example in [17, p. 10]). Let T^2 be the 2-dimensional torus $\mathbb{C}/\mathbb{Z}^2$ so that the fundamental group $\pi_1(T^2)$ is generated by the homotopy classes of a meridian and a longitude, say $\mathfrak{m}$ and $\mathfrak{l}$, respectively. Let $\rho : \pi_1(T^2) \to \mathbb{R}$ be a given homomorphism such that $\rho(\mathfrak{m}) \notin \mathbb{Q}$ or $\rho(\mathfrak{l}) \notin \mathbb{Q}$, and we set $\overline{\rho}(g) := \rho(g) \bmod \mathbb{Z} \in S^1$ for $g \in \pi_1(T^2)$. Let M be the quotient 3-manifold of $\mathbb{C} \times S^1$ by the action of $\pi_1(T^2)$,

$$M := (\mathbb{C} \times S^1)/\pi_1(T^2),$$

where $\pi_1(T^2)$ acts on the universal cover $\mathbb{C}$ of T^2 as the monodromy and on S^1 by $\theta \mapsto \theta + \overline{\rho}(g)$ for $\theta \in S^1$ and $g \in \pi_1(T^2)$. Let L_θ denote the image in M of $\mathbb{C} \times \{\theta\}$ in M. Then $\mathcal{F} := \{L_\theta\}_{\theta \in S^1}$ forms a 2-dimensional foliation on M. Here we see that

$$L_\theta = \begin{cases} S^1 \times \mathbb{R} & \text{if } \rho(\mathfrak{m}) \notin \mathbb{Q},\ \rho(\mathfrak{l}) \in \mathbb{Q} \text{ or } \rho(\mathfrak{m}) \in \mathbb{Q},\ \rho(\mathfrak{l}) \notin \mathbb{Q}, \\ \mathbb{R}^2 & \text{if } \rho(\mathfrak{m}), \rho(\mathfrak{l}) \notin \mathbb{Q} \text{ and } \rho(\mathfrak{m})/\rho(\mathfrak{l}) \notin \mathbb{Q}, \end{cases}$$

and that any leaf L_θ is dense in M. Since $T\mathcal{F}$ is orientable, $\mathcal{F}$ admits a complex foliation structure (Remark 1.2 (2)). Let ϕ_1 be the flow defined by

$$\phi_1^t([z,\theta]) := [z, \theta + \overline{t}],$$

where $\overline{t} := t \bmod \mathbb{Z}$. Then we obtain an FDS^3 $\mathfrak{S}_1 := (M, \mathcal{F}, \phi_1)$ of type II. The canonical 1-form $\omega_{\mathfrak{S}_1}$ is given by $\omega_{\mathfrak{S}_1}|_{T\mathcal{F}} = 0$ and $\omega_{\mathfrak{S}_1}|_{TS^1} = d\theta$, and the period group $\Lambda_{\mathfrak{S}_1} = \mathbb{Z} + \rho(\mathfrak{m})\mathbb{Z} + \rho(\mathfrak{l})\mathbb{Z}$. For this example, however, any orbit of the flow ϕ_1 is closed with period 1, and hence $P_{\mathfrak{S}}$ is uncountable.

In order to obtain an FDS^3 $\mathfrak{S}$ with countably infinite $\mathcal{P}_{\mathfrak{S}}$, we interpret the above $(M, \mathcal{F})$ from a different view and define a new dynamical system. In fact, the natural map $\mathbb{C} \times S^1 \to T^2 \times S^1$ induces the diffeomorphism

$$M \xrightarrow{\approx} T^3 = (\mathbb{R}/\mathbb{Z})^3.$$

Then the leaf L_θ is given by

$$L_\theta := \{(\theta_1, \theta_2, \theta - \rho(\mathfrak{m})\theta_1 - \rho(\mathfrak{l})\theta_2) \mid \theta_1, \theta_2 \in S^1\}$$

for $p = (\theta_1, \theta_2, \theta_3) \in T^3$. Let $V_{\mathfrak{m}} := (1, 0, -\rho(\mathfrak{m}))$, $V_{\mathfrak{l}} := (0, 1, -\rho(\mathfrak{l}))$ be vector fields on T^3, and we define the smooth dynamical system ϕ_2 on T^3 by the equation

$$\begin{aligned} \frac{d}{dt}\phi_2^t(p) &:= (0,0,1) + \sin(2\pi\theta_1)V_{\mathfrak{m}} + \sin(2\pi\theta_2)V_{\mathfrak{l}} \\ &= \big(\sin(2\pi\theta_1), \sin(2\pi\theta_2), 1 - \rho(\mathfrak{m})\sin(2\pi\theta_1) - \rho(\mathfrak{l})\sin(2\pi\theta_2)\big). \end{aligned}$$

Then we obtain an FDS^3 $\mathfrak{S}_2 := (T^3, \mathcal{F}, \phi_2)$ of type II. As $V_{\mathfrak{m}}$ and $V_{\mathfrak{l}}$ are tangent to $\mathcal{F}$ and so $(\dot{\phi}_1^t - \dot{\phi}_2^t)_p \in T_p\mathcal{F}$ for $p \in M = T^3$, we have $\omega_{\mathfrak{S}_2} = \omega_{\mathfrak{S}_1}, \Lambda_{\mathfrak{S}_2} = \Lambda_{\mathfrak{S}_1}$ by Lemma 1.11. We see that $\mathcal{P}_{\mathfrak{S}_2}$ consists of the following four closed orbits:

$$\begin{cases} \gamma_1 = \{(0,0,\theta) \,|\, \theta \in S^1\}, \\ \gamma_2 = \{(\frac{1}{2}, 0, \theta) \mid \theta \in S^1\}, \quad \gamma_2' = \{(0, \frac{1}{2}, \theta) \mid \theta \in S^1\}, \\ \gamma_3 = \{(\frac{1}{2}, \frac{1}{2}, \theta) \mid \theta \in S^1\}, \end{cases}$$

and that ϕ_2 is of contracting type around γ_3. We replace a tubular neighborhood of γ_3 by the suspension of the horseshoe map. By Lemma 3.3, we obtain an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type II such that $\mathcal{P}_{\mathfrak{S}}$ is countably infinite, and the canonical 1-form $\omega_{\mathfrak{S}}$ and the period group $\Lambda_{\mathfrak{S}}$ are the same as $\omega_{\mathfrak{S}_i}$ and $\Lambda_{\mathfrak{S}_i}$ $(i = 1, 2)$, respectively.

Example 3.5. Let $T^2 := (\mathbb{R}/\mathbb{Z})^2$ be the 2-dimensional torus, and let $\varphi_A : T^2 \to T^2$ be the linear Anosov diffeomorphism of T^2 defined by the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}.$$

Then there are two fixed points $p_1 = (0,0)$ and $p_2 = (\frac{1}{2}, 0)$ of φ_A. For each p_i, we remove p_i from T^2 and glue $(T_{p_i}(T^2) \setminus \{0\})/\mathbb{R}_+ = S^1$ on it. Then we obtain a twice-punctured torus T_*^2 with $\partial T_*^2 = S_1^1 \sqcup S_2^1$, $S_i^1 = S^1$, and the diffeomorphism $\varphi_A^* : T_*^2 \to T_*^2$.

Let $M^* = M(T^2_*, \varphi^*_A)$ be the mapping torus of φ^*_A, which has two boundary components, say T_1 and T_2, which correspond to p_1 and p_2, respectively. Let $\mathcal{F}^* = \{L_\theta\}_{\theta \in S^1}$ be the bundle foliation of the fibration $\varpi_* : M^* \to S^1$, and let ϕ_* be the suspension flow of φ^*_A. For each i, $\mathcal{F}^*|_{T_i}$ defines a foliation on T_i whose leaves are S^1. The 1-form $\omega_* := \varpi_*^{-1}(d\theta)$ satisfies $\omega_*|_{T\mathcal{F}^*} = 0, \omega_*(\dot{\phi}^t_*) = 1$.

Finally, let $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, and consider the diffeomorphism $\psi_\lambda : T_1 \to T_2$ defined by sending L_θ to $L_{\theta+\lambda}$. We define M to be the 3-manifold obtained from M^* by gluing T_1 and T_2 via ψ_λ. In fact, M is a Σ_2-bundle over S^1. The foliation $\mathcal{F}$ and the flow ϕ on M are induced by $\mathcal{F}^*$ and ϕ_*, and $\mathcal{F}$ has a complex structure. Then any leaf in $\mathcal{F}$ is dense in M by the way of the gluing. Thus we obtain the FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type II. Further, since ϕ comes from the suspension of the Anosov diffeomorphism, $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. The canonical 1-form $\omega_{\mathfrak{S}}$ is induced by ω_* and the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z} + \lambda\mathbb{Z}$.

III. We construct examples of FDS^3's of type III. This type of FDS^3 may be regarded as an analog of a number ring $\overline{\mathrm{Spec}(\mathcal{O}_k)} = \mathrm{Spec}(\mathcal{O}_k) \cup \mathcal{P}^\infty_k$ in the respect that $\mathcal{P}^\infty_{\mathfrak{S}}$ corresponds to the set $\mathcal{P}^\infty_k$ of infinite primes of a number field k (cp. Remark 1.6).

III-1. We give two examples of FDS^3's of type III-1.

Example 3.6 (Reeb foliation on S^3 and the horseshoe flow)**.** Let M be the 3-sphere S^3. Let $M := S^3 = V_1 \cup V_2$ be the Heegaard splitting of genus one, where V_i is a solid torus $D^2 \times S^1$ (see [24, Chap. 2]). Consider the *Reeb foliation* $\mathcal{F}_i$ on each V_i and so the 2-dimensional foliation $\mathcal{F}$ on S^3 by getting $\mathcal{F}_1$ and $\mathcal{F}_2$ together, where any leaf is diffeomorphic to $\mathbb{R}^2$ besides the only compact leaf $L^\infty = \partial V_1 = \partial V_2$ (see [8, 1.1], [41, § 1]). We define the dynamical system ϕ as follows.

First, we consider a flow ϕ_1 on M such that any orbit of ϕ_1 is transverse to leaves in $\mathcal{F} \setminus L^\infty$ and $\phi^t_1(L^\infty) = L^\infty$ for $t \in \mathbb{R}$. Then there is only one closed orbit $\gamma_i = \{0\} \times S^1$ in each V_i (0 being the center of D^2). In fact, the flow ϕ_1 on $\mathrm{Int}(V_i)$ is the suspension flow of a contraction map $\varphi_i : \mathrm{Int}(D^2) \ni z \mapsto az \in \mathrm{Int}(D^2)$ $(0 < a < 1)$, and so ϕ_1 is of contraction type around γ_i.

Next, we replace the contraction map φ_i by the horseshoe map h and the flow ϕ_1 around γ_i by the suspension of h. Let ϕ be the resulting flow on all of M. By Lemma 3.3, we have an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-1 such that $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set and $\mathcal{P}^\infty_{\mathfrak{S}}$ consists of the only non-transverse compact leaf L^∞. We have the decomposition of $M_0 = S^3 \setminus L^\infty$ into connected components

$$M_0 = \mathrm{Int}(V_1) \sqcup \mathrm{Int}(V_2),$$

and each $(\mathrm{Int}(V_i), \mathcal{F}|_{\mathrm{Int}(V_i)})$ $(i = 1, 2)$ is the bundle foliation of the $\mathrm{Int}(D^2)$-bundle over S^1. By Proposition 2.10, the canonical 1-form $\omega_{\mathfrak{S}}$ restricted on each $\mathrm{Int}(V_i)$ is the pullback of the angular 1-form of S^1 under the fibration, and so the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$.

In view of the analogy in arithmetic topology [33, Chap. 3], the 3-sphere S^3 may be regarded as an analog of $\overline{\mathrm{Spec}(\mathbb{Z})} = \mathrm{Spec}(\mathbb{Z}) \cup \mathcal{P}^\infty_{\mathbb{Q}}$, where $\mathcal{P}^\infty_{\mathbb{Q}}$ consists of the only infinite prime of $\mathbb{Q}$.

Example 3.7 (Open book decomposition)**.** Let M be a closed 3-manifold. It is known that M contains a fibered link $L = K_1 \cup \cdots \cup K_r$, namely, there is a fibration $\varpi : M \setminus \mathrm{Int}(V(L)) \to S^1$, where $\mathrm{Int}(V(L)) = \bigsqcup_{i=1}^r \mathrm{Int}(V(K_i))$ is the interior of a tubular neighborhood of L and any fiber of ϖ is a surface with r boundary components. We have the foliation on $M \setminus \mathrm{Int}(V(L))$ by tubularizing the fibers of ϖ around $\partial V(L) = \bigsqcup_{i=1}^r \partial\, \mathrm{Int}(V(K_i))$ (see [8, Ex. 3.3.11]). The structure this induces on M is called an *open book decomposition* [7, Ex. 4.11]. We fill in $V(L)$ with the Reeb component to obtain the foliation $\mathcal{F}$ on all of M. We define the flow on $M \setminus \mathrm{Int}(V(L))$ by the suspension of the monodromy φ of the fibration ϖ. We suppose that φ is of pseudo-Anosov type (for example, this is the case if L is a hyperbolic link). The flow on $\mathrm{Int}(V(L))$ is defined to be the one transverse to any leaf of the Reeb foliation. Thus we have an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-1 such that $\mathcal{P}_{\mathfrak{S}}^{\infty} = \{\partial V(K_1), \ldots, \partial V(K_r)\}$ and $\mathcal{P}_{\mathfrak{S}}$ is a countably infinite set. We have the decomposition of $M_0 = M \setminus \partial V(L)$ into connected components

$$M_0 = \bigsqcup_{i=1}^{r} \mathrm{Int}(V(K_i)) \sqcup (M \setminus V(L)).$$

Here $\mathrm{Int}(V(K_i)) = \mathbb{R}^2 \times S^1$, and $\mathcal{F}|_{\mathrm{Int}(V(K_i))}$ is the bundle foliation over S^1, and $M \setminus V(L)$ is a surface bundle over S^1, and $\mathcal{F}|_{M \setminus V(L)}$ is also the bundle foliation over S^1. By Proposition 2.10, the canonical 1-form $\omega_{\mathfrak{S}}$ restricted on $\mathrm{Int}(V(K_i))$ or $M \setminus V(L)$ is the pullback of the angular 1-form of S^1 under the fibration, and so the period group $\Lambda_{\mathfrak{S}} = \mathbb{Z}$.

Remark 3.8. Example 3.7 shows that any closed smooth 3-manifold M admits a structure of an FDS^3 with nonempty $P_{\mathfrak{S}}^{\infty}$. So the notion of an FDS^3 is generic in this sense. Moreover, the fibration $\varpi : M \setminus \mathrm{Int}(V(L)) \to S^1$ above induces the surjective homomorphism $\pi_1(M \setminus L) \to \mathbb{Z}$, and hence M has a $\mathbb{Z}$-covering ramified over L. This may be analog to the fact that any number field k has a $\mathbb{Z}_l$-extension ramified over primes $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ lying above (l), where l is a prime number.

III-2. We give an example of an FDS^3 of type III-2.

Example 3.9. Let $T^3 := (\mathbb{R}/\mathbb{Z})^3 = \{(\theta_1, \theta_2, \theta_3) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}$ be the 3-dimensional torus, and let $\mathcal{F} = \{T^2 \times \{\theta_3\}\}_{\theta_3 \in \mathbb{R}/\mathbb{Z}}$ be the linear foliation which admits a complex structure. We define the smooth dynamical system ϕ on T^3 by

$$\frac{d}{dt}\phi^t(p) := (\cos(2\pi\theta_3), 0, \sin(2\pi\theta_3))$$

for $p = (\theta_1, \theta_2, \theta_3)$. Then we obtain an FDS^3 $(T^3, \mathcal{F}, \phi)$ which is equipped with $\mathcal{P}_{\mathfrak{S}}^{\infty} = \{L_1^{\infty}, L_2^{\infty}\}$, where

$$L_1^{\infty} := \{(\theta_1, \theta_2, 0) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}, \quad L_2^{\infty} := \{(\theta_1, \theta_2, \tfrac{1}{2}) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}.$$

We have the decomposition of $M_0 := T^3 \setminus (L_1^\infty \cup L_2^\infty)$ into connected components

$$\begin{aligned} M_0 &= X_1 \sqcup X_2, \\ X_1 &:= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1,\theta_2 \in \mathbb{R}/\mathbb{Z},\ 0 < \theta_3 < \tfrac{1}{2}\}, \\ X_2 &:= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1,\theta_2 \in \mathbb{R}/\mathbb{Z},\ \tfrac{1}{2} < \theta_3 < 1\}. \end{aligned}$$

Here each X_a $(a = 1, 2)$ is a T^2-bundle over an open interval, and $\mathcal{F}|_{X_a}$ is the bundle foliation. The canonical 1-form $\omega_\mathfrak{S}$ is given by $\omega_\mathfrak{S}|_{X_a} = \operatorname{cosec}(2\pi\theta_3)\, d\theta_3$ and the period group $\Lambda_\mathfrak{S} = \{0\}$ by Proposition 2.11.

III-3. Finally, we give an example of an FDS^3 of type III-3.

Example 3.10. Let $T^3 := (\mathbb{R}/\mathbb{Z})^3 = \{(\theta_1,\theta_2,\theta_3) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}$. Fix an irrational number $\rho \in \mathbb{R} \setminus \mathbb{Q}$. Let ω_0 be the smooth 1-form on T^3 defined by

$$\omega_0 := \sin(2\pi\theta_3)(d\theta_1 + \rho\, d\theta_2) + d\theta_3.$$

Since we see $\omega_0 \wedge d\omega_0 = 0$, $\operatorname{Ker}(\omega_0)$ defines the foliation $\mathcal{F}$ on T^3 by Frobenius' theorem (cp. Remark 1.2 (1)). Let ϕ be the smooth dynamical system defined by

$$\frac{d}{dt}\phi_1^t(p) := (1, 0, 0)$$

for $p = (\theta_1,\theta_2,\theta_3)$. Then we obtain an FDS^3 $\mathfrak{S}_1 := (T^3, \mathcal{F}, \phi_1)$ which is equipped with $\mathcal{P}_\mathfrak{S}^\infty = \{L_1^\infty, L_2^\infty\}$, where

$$L_1^\infty := \{(\theta_1,\theta_2,0) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}, \quad L_2^\infty := \{(\theta_1,\theta_2,\tfrac{1}{2}) \mid \theta_i \in \mathbb{R}/\mathbb{Z}\}.$$

We have the decomposition of $M_0 := T^3 \setminus (L_1^\infty \cup L_2^\infty)$ into connected components

$$\begin{aligned} M_0 &= X_1 \sqcup X_2, \\ X_1 &:= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1,\theta_2 \in \mathbb{R}/\mathbb{Z},\ 0 < \theta_3 < \tfrac{1}{2}\}, \\ X_2 &:= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1,\theta_2 \in \mathbb{R}/\mathbb{Z},\ \tfrac{1}{2} < \theta_3 < 1\}. \end{aligned}$$

Here any leaf of $\mathcal{F}|_{X_a}$ is dense in X_a for each $a = 1, 2$. So $\mathfrak{S}_1$ is of type III-3. Let $\omega_{\mathfrak{S}_1}$ be the smooth 1-form defined by

$$\omega_{\mathfrak{S}_1} := \operatorname{cosec}(2\pi\theta_3)\omega_0 = d\theta_1 + \rho\, d\theta_2 + \operatorname{cosec}(2\pi\theta_3)\, d\theta_3.$$

Then we see

$$\operatorname{Ker}(\omega_{\mathfrak{S}_1}) = \operatorname{Ker}(\omega_0), \quad \omega_\mathfrak{S}(\dot{\phi}^t) = 1,$$

and so $\omega_{\mathfrak{S}_1}$ is indeed the canonical 1-form of $\mathfrak{S}_1$. Let γ_a^1, γ_a^2 be closed curves in X_a defined by

$$\gamma_a^1 := \Big\{\Big(t, 0, \frac{2a-1}{4}\Big) \;\Big|\; t \in \mathbb{R}/\mathbb{Z}\Big\}, \quad \gamma_a^2 := \Big\{\Big(0, t, \frac{2a-1}{4}\Big) \;\Big|\; t \in \mathbb{R}/\mathbb{Z}\Big\}.$$

Then we see that $H_1(X_a, \mathbb{Z})$ is generated by the homology classes of γ_a^1, γ_a^2 and that $[\omega_\mathfrak{S}]([\gamma_a^1]) = 1$, $[\omega_\mathfrak{S}]([\gamma_a^2]) = \rho$. Hence the period group $\Lambda_\mathfrak{S}$ is $\mathbb{Z} + \rho\mathbb{Z}$. For this example $\mathfrak{S}_1$, any orbit in M_0 is closed, and so $\mathcal{P}_\mathfrak{S}$ is uncountable. However, as in Example 3.4, we can change ϕ_1 to obtain an FDS^3 having countably infinitely many closed orbits in M_0 as follows. Let $V_2 := (-\rho, 1, 0)$,

$V_3 := (-1, 0, \sin(2\pi\theta_3))$ be vector fields on T^3, and we consider the smooth dynamical system ϕ_2 defined by

$$\begin{aligned}\frac{d}{dt}\phi_2^t(p) &:= (1,0,0) + \sin(2\pi\theta_2)V_2 + \cos(2\pi\theta_3)V_3 \\ &= \big(1 - \rho\sin(2\pi\theta_2) - \cos(2\pi\theta_3), \sin(2\pi\theta_2), \tfrac{1}{2}\sin(4\pi\theta_3)\big).\end{aligned}$$

Then we obtain an FDS^3 $\mathfrak{S}_2 := (T^3, \mathcal{F}, \phi_2)$ of type III-3. As V_2 and V_3 are tangent to $\mathcal{F}$ and so $(\dot\phi_1^t - \dot\phi_2^t)_p \in T_p\mathcal{F}$ for $p \in T^3$, we have $\omega_{\mathfrak{S}_2} = \omega_{\mathfrak{S}_1}$, $\Lambda_{\mathfrak{S}_2} = \Lambda_{\mathfrak{S}_1}$ by Lemma 1.11. We see that $\mathcal{P}_{\mathfrak{S}_2}$ consists of the following four closed orbits in M_0:

$$\begin{aligned}\gamma_1 &= \{(\theta_1, 0, \tfrac{1}{4}) \mid \theta_1 \in S^1\}, \quad \gamma_2 = \{(\theta_1, 0, \tfrac{3}{4}) \mid \theta_1 \in S^1\}, \\ \gamma_3 &= \{(\theta_1, \tfrac{1}{2}, \tfrac{1}{4}) \mid \theta_1 \in S^1\}, \quad \gamma_4 = \{(\theta_1, \tfrac{1}{2}, \tfrac{3}{4}) \mid \theta_1 \in S^1\},\end{aligned}$$

and that ϕ_2 is of contracting type around γ_3 and γ_4. We replace a tubular neighborhood of γ_3 or γ_4 by the suspension of the horseshoe map. By Lemma 3.3, we obtain an FDS^3 $\mathfrak{S} = (M, \mathcal{F}, \phi)$ of type III-3 such that $\mathcal{P}_{\mathfrak{S}}$ is countably infinite, and the canonical form $\omega_{\mathfrak{S}}$ and the period group $\Lambda_{\mathfrak{S}}$ are the same as $\omega_{\mathfrak{S}_i}$ and $\Lambda_{\mathfrak{S}_i}$ $(i = 1, 2)$, respectively.

4. SMOOTH DELIGNE COHOMOLOGY AND INTEGRATION THEORY FOR FDS^3'S

In this section, we recall the theory of smooth Deligne cohomology for an FDS^3 and the integration theory of Deligne cohomology classes. For general materials on smooth Deligne cohomology, we consult [5]. For the integration theory of Deligne cohomology classes, we refer to [20, 22, 42].

4.1. **Smooth Deligne cohomology.** Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3. Let X be a submanifold of M_0 obtained by removing some finitely many closed orbits.

Let $\mathcal{A}^i$ denote the sheaf of $\mathbb{C}$-valued smooth i-forms on X. For example, an element of $\mathcal{A}^1(U)$ is given in terms of a foliated local coordinate $(z, x) \in U$ by

$$f_1(z,x)\,dz + f_2(z,x)\,d\overline{z} + f_3(z,x)\,dx,$$

where $f_i(z, x)$'s are $\mathbb{C}$-valued smooth functions on U.

Let Λ be a subgroup of the additive group $\mathbb{R}$. For a nonnegative integer n, we set $\Lambda(n) := (2\pi\sqrt{-1})^n\Lambda$.

Definition 4.2. Let n be an integer with $1 \leq n \leq 3$. We define the *smooth Deligne complex* $\Lambda(n)_{\mathscr{D}}$ on X by

$$\Lambda(n)_{\mathscr{D}} : \Lambda(n) \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{n-1},$$

where $\Lambda(n)$ is put in degree 0 and d denotes the differential. For an integer $q \geq 0$, the q-th *smooth Deligne cohomology group* with coefficients in $\Lambda(n)_{\mathscr{D}}$ is defined to be the q-th hypercohomology group of the complex $\Lambda(n)_{\mathscr{D}}$, denoted by $H^q_{\mathscr{D}}(M; \Lambda(n))$,

$$H^q_{\mathscr{D}}(X; \Lambda(n)) := \mathbb{H}^q(X; \Lambda(n)_{\mathscr{D}}).$$

In particular, when Λ is the period group $\Lambda_{\mathfrak{S}}$, we call $H^q_{\mathscr{D}}(X;\Lambda_{\mathfrak{S}}(n))$ the *FDS*^{3}*-Deligne cohomology groups* of $\mathfrak{S}$.

We compute the smooth Deligne cohomology groups as Čech hypercohomology groups of an open covering $\mathcal{U}=\{U_a\}_{a\in I}$ of X with coefficients in $\Lambda(n)_{\mathscr{D}}$,

$$H^q_{\mathscr{D}}(M;\Lambda(n))=\mathbb{H}^q(\mathcal{U};\Lambda(n)_{\mathscr{D}}),$$

where the open covering $\mathcal{U}$ is taken so that all nonempty intersections $U_{a_0\cdots a_j}:=U_{a_0}\cap\cdots\cap U_{a_j}$ are contractible. So a Čech cocycle representing an element of $H^n_{\mathscr{D}}(M;\Lambda(n))$ is of the form

$$(\lambda_{a_0\dots a_n},\theta^0_{a_0\dots a_{n-1}},\dots,\theta^{n-1}_{a_0})\in C^n(\Lambda(n))\oplus C^{n-1}(\mathcal{A}^0)\oplus\cdots\oplus C^0(\mathcal{A}^{n-1})$$

which satisfies the cocycle condition

$$\delta(\theta^0_{a_0\dots a_{n-1}})+(-1)^n\lambda_{a_0\dots a_n}=0,$$
$$\delta(\theta^i_{a_0\dots a_{n-1-i}})+(-1)^{n-i}\,d\theta^{i-1}_{a_0\dots a_{n-i}}=0\quad(i\geq 1),$$

where δ is the Čech differential with respect to the open covering $\mathcal{U}$.

Example 4.3. Let f be an FDS^3-meromorphic function on $\mathfrak{S}$ whose zeros and poles are lying along closed orbits $\gamma_1,\dots,\gamma_N$. Let $X:=M_0$. Let $\log_a f$ denote a branch of $\log f$ on U_a. Then the Čech cocycle

$$(n_{a_0a_1},\log_{a_0}f),\quad n_{a_0a_1}=(\delta\log f)_{a_0a_1}\in\mathbb{Z}(1),$$

determines the cohomology class of $H^1_{\mathscr{D}}(X;\mathbb{Z}(1))$, by which we denote $c(f)$.

Example 4.4. Let $\omega_{\mathfrak{S}}$ be the canonical 1-form of $\mathfrak{S}$ (cp. Definition 1.10). Fix a base point $p_0\in X$. For each $U_a(a\in I)$, we choose a point $p_a\in U_a$ and a path γ_a in U_a from p_0 to p_a. For $p\in U_a$, we set

$$f_{\omega_{\mathfrak{S}},a}(p):=2\pi\sqrt{-1}\int_{\gamma_p\cdot\gamma_a}\omega_{\mathfrak{S}},$$

where γ_p is a path from p_a to p inside U_a. Since U_a is contractible and $\omega_{\mathfrak{S}}$ is closed, $f_{\omega_{\mathfrak{S}},a}$ is a smooth function on U_a which is independent of the choice of γ_p. Then the Čech cocycle

$$(\lambda_{a_0a_1},f_{\omega_{\mathfrak{S}},a_0}),\quad \lambda_{a_0a_1}=(\delta f_{\omega_{\mathfrak{S}}})_{a_0a_1}\in\Lambda_{\mathfrak{S}}(1),$$

defines the cohomology class of $H^1_{\mathscr{D}}(X;\Lambda_{\mathfrak{S}}(1))$, by which we denote $c(\omega_{\mathcal{S}})$. The class $c(\omega_{\mathcal{S}})$ is independent of the choices of p_a, γ_a and γ_p. In fact, let p'_a, γ'_a and γ'_p be different choices of a point in U_a, a path from p_0 to p'_a and a path from p'_a to p, respectively, and let $f'_{\omega_{\mathfrak{S}}}$ and $\lambda'_{a_0a_1}$ be defined as above using p'_a,γ'_a and γ'_p. Since $df'_{\omega_{\mathfrak{S}},a}=df_{\omega_{\mathfrak{S}},a}=\omega_{\mathfrak{S}}$, $f'_{\omega_{\mathfrak{S}},a}-f_{\omega_{\mathfrak{S}},a}=\lambda_a$ is a constant on U_a. Then we have

$$\lambda_a=f'_{\omega_{\mathfrak{S}},a}(p)-f_{\omega_{\mathfrak{S}},a}(p)=2\pi\sqrt{-1}\int_{(\gamma_p\cdot\gamma_a)^{-1}\cdot(\gamma'_p\cdot\gamma'_a)}\omega_{\mathfrak{S}}\in\Lambda_{\mathfrak{S}}(1)$$

and

$$\begin{aligned}\lambda'_{a_0a_1} - \lambda_{a_0a_1} &= (\delta f'_{\omega_{\mathfrak{S}}})_{a_0a_1} - (\delta f'_{\omega_{\mathfrak{S}}})_{a_0a_1} \\ &= f'_{\omega_{\mathfrak{S}},a_1} - f'_{\omega_{\mathfrak{S}},a_0} - (f_{\omega_{\mathfrak{S}},a_1} - f_{\omega_{\mathfrak{S}},a_0}) \\ &= (f'_{\omega_{\mathfrak{S}},a_1} - f_{\omega_{\mathfrak{S}},a_1}) - (f'_{\omega_{\mathfrak{S}},a_0} - f_{\omega_{\mathfrak{S}},a_0}) \\ &= \lambda_{a_1} - \lambda_{a_0} = \delta(\lambda)_{a_0a_1},\end{aligned}$$

and hence $c(\omega_{\mathcal{S}})$ is independent of the choices of p_a, γ_a and γ_p, but it depends on the choice of p_0.

Definition 4.5. For $1 \le n \le 3$, we define the *n-curvature homomorphism*

$$\Omega : H^n_{\mathscr{D}}(X;\Lambda(n)) \to \mathcal{A}^n(X) \quad \text{by} \quad \Omega(c)|_{U_a} := d\theta_a^{n-1}$$

for $c = [(\lambda_{a_0\ldots a_n}, \ldots, \theta_{a_0}^{n-1})]$.

When Λ is a subring of $\mathbb{R}$, the smooth Deligne cohomology groups are equipped with the cup product, which is induced by the product on the smooth Deligne complexes

$$\Lambda(n)_{\mathscr{D}} \otimes \Lambda(n')_{\mathscr{D}} \to \Lambda(n+n')_{\mathscr{D}} \tag{4}$$

defined by

$$x \cup y = \begin{cases} xy, & \deg(x) = 0, \\ x \wedge dy, & \deg(x) > 0 \text{ and } \deg(y) = n', \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For our purpose, we extend the product (4) for the case where Λ is a subring of $\mathbb{R}$ and Λ' is a Λ-submodule of $\mathbb{R}$ as follows. Namely, by the same formula as in (5), we have the product

$$\Lambda(n)_{\mathscr{D}} \otimes \Lambda'(n')_{\mathscr{D}} \to \Lambda'(n+n')_{\mathscr{D}}$$

which induces the cup product on the smooth Deligne cohomology groups

$$H^n_{\mathscr{D}}(X;\Lambda(n)) \otimes H^{n'}_{\mathscr{D}}(X;\Lambda'(n')) \to H^{n+n'}_{\mathscr{D}}(X;\Lambda'(n+n')). \tag{6}$$

4.6. **Integration theory.** As in Subsection 4.1, let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3, and let X be a submanifold of M_0 obtained by removing some finitely many closed orbits. Let n be an integer with $1 \le n \le 3$, and let $c \in H^n_{\mathscr{D}}(X;\Lambda(n))$. Let Y be an $(n-1)$-dimensional closed submanifold of X. We shall define a paring $\int_Y c$, which takes values in $\mathbb{C}$ mod $\Lambda(n)$ as follows.

First, we fix an open covering $\mathcal{U} = \{U_a\}_{a\in I}$ of X such that all nonempty intersections $U_{a_0\cdots a_j} := U_{a_0} \cap \cdots \cap U_{a_j}$ are contractible and choose a Čech representative cocycle $(\lambda_{a_0\ldots a_n}, \theta^0_{a_0\ldots a_{n-1}}, \ldots, \theta^{n-1}_{a_0})$ of c. Second, we choose a smooth finite triangulation $K = \{\sigma\}$ of Y and an index map $\iota : K \to I$ satisfying $\sigma \subset U_{\iota(\sigma)}$. For $i = 0, \ldots, n-1$, we define the set $F_K(i)$ of flags of simplices

$$F_K(i) := \{\vec{\sigma} = (\sigma^{n-1-i}, \ldots, \sigma^{n-1}) \mid \sigma^j \in K, \dim \sigma^j = j, \\ \sigma^{n-1-i} \subset \cdots \subset \sigma^{n-1}\}. \tag{7}$$

Then we define the *integral* of c over Y by

$$(8)\quad \int_Y c := \sum_{i=0}^{n-1} \sum_{\vec{\sigma}\in F_K(i)} (-1)^i \int_{\sigma^{n-1-i}} \theta^{n-1-i}_{\iota(\sigma^{n-1})\iota(\sigma^{n-2})\dots\iota(\sigma^{n-1-i})} \mod \Lambda(n),$$

which is proved to be independent of all choices [22, Thm. 3.4 (i)]).

The following Stokes-type formula was shown by Gawedzki [20] when Y is 2-dimensional. We refer to [22, 42] for more general statements and proofs.

Theorem 4.7 (cp. [20], [22, Thm. 3.4 (ii)], [42, Prop. 5.5])**.** *If there is an n-dimensional submanifold Z of X whose boundary is $\partial Z = Y$, we have*

$$\int_Y c = \int_Z \Omega_n(c) \mod \Lambda(n).$$

5. Local symbols and Hilbert type reciprocity law

In this section, we introduce a local symbol by using the integral of a certain FDS-Deligne cohomology class along a torus and show the Hilbert type reciprocity law.

5.1. **Local symbols.** Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be an FDS^3. Let f and g be FDS^3-meromorphic functions on $\mathfrak{S}$ whose zeros and poles lie along $\gamma_1, \dots, \gamma_N \in \mathcal{P}_{\mathfrak{S}}$. We set $X := M_0 \setminus \bigcup_{i=1}^N \gamma_i$. For $\gamma \in \overline{\mathcal{P}_{\mathfrak{S}}}$, let $V(\gamma)$ denote a tubular neighborhood of γ, and we denote by $T(\gamma)$ the boundary of $V(\gamma)$.

As in Example 4.3, we have the smooth Deligne cohomology classes

$$\begin{aligned} c(f) &= [(m_{a_0a_1}, \log_{a_0} f)], \\ c(g) &= [(n_{a_0a_1}, \log_{a_0} g)] \in H^1_{\mathscr{D}}(X; \mathbb{Z}(1)), \end{aligned}$$

and as in Example 4.4, we have the FDS^3-Deligne cohomology class

$$c(\omega_{\mathfrak{S}}) = [(\lambda_{a_0a_1}, f_{\omega_{\mathfrak{S}},a_0})] \in H^1_{\mathscr{D}}(X; \Lambda_{\mathfrak{S}}(1)).$$

By the product in (6) applied to the case that $\Lambda = \mathbb{Z}$ and $\Lambda' = \Lambda_{\mathfrak{S}}$, we have the third FDS^3-Deligne cohomology class

$$c(f) \cup c(g) \cup c(\omega_{\mathfrak{S}}) \in H^3_{\mathscr{D}}(X; \Lambda_{\mathfrak{S}}(3)).$$

Definition 5.2. We define the *local symbol* $\langle f, g\rangle_\gamma$ of f, g along γ by

$$\langle f, g\rangle_\gamma := \int_{T(\gamma)} c(f) \cup c(g) \cup c(\omega_{\mathfrak{S}}) \mod \Lambda_{\mathfrak{S}}(3).$$

We note that the integral of the right-hand side is finite since $T(\gamma)$ is compact and that it is independent of a choice of $V(\gamma)$ by the Stokes theorem.

Theorem 5.3. *Notations being as above, the FDS^3-Deligne cohomology class $c(f) \cup c(g) \cup c(\omega_{\mathfrak{S}})$ is represented by the Čech cocycle*

$$(m_{a_0a_1} n_{a_1a_2} \lambda_{a_2a_3}, m_{a_0a_1} n_{a_1a_2} f_{\omega_{\mathfrak{S}},a_0a_1a_2}, m_{a_0a_1} \log_{a_1} g\, \omega_{\mathfrak{S}}, \log_{a_0} f d\log g \wedge \omega_{\mathfrak{S}}).$$

For $\gamma \in \mathcal{P}_{\mathfrak{S}}$, the local symbol $\langle f, g\rangle_\gamma$ is given by

$$\langle f, g\rangle_\gamma = \int_{T(\gamma)} \log(f)\, d\log(g) \wedge \omega_{\mathfrak{S}} - \int_{\mathfrak{m}} d\log(f) \int_{\mathfrak{l}} \log(g)\omega_{\mathfrak{S}} \mod \Lambda_{\mathfrak{S}}(3),$$

where $\mathfrak{m}$ and $\mathfrak{l}$ denote a meridian and longitude on $T(\gamma)$, respectively.

Proof. The first assertion follows from the definition (5) of the cup product on Deligne cohomology groups. We set for simplicity

$$\begin{cases} \theta^0_{a_0a_1a_2} = m_{a_0a_1} n_{a_1a_2} f_{\omega_{\mathfrak{S}}, a_0a_1a_2}, \\ \theta^1_{a_0a_1} = m_{a_0a_1} \log_{a_1} g\, \omega_{\mathfrak{S}}, \\ \theta^2_{a_0} = \log_{a_0} f d\log g \wedge \omega_{\mathfrak{S}}. \end{cases}$$

To prove the second formula for the local symbol, we make suitable choices of a triangulation K of $T(\gamma)$, an open covering $\mathcal{U}$, representatives of Deligne cohomology classes $c(f)$, $c(g)$, $c(\omega_{\mathfrak{S}})$, and the index map ι. First, let $K = \{\sigma\}$ be a triangulation of the 2-dimensional torus $T(\gamma)$ by eighteen triangles as in Figure 1.

We choose an open covering $\mathcal{U} = \{U_a\}_{a\in I}$ of X such that nine U_a's in $\mathcal{U}$ cover $T(\gamma)$ and are indexed as in Figure 2.

We define the index map $\iota : K \to I$ in the manner that one vertex, three edges and two triangles in U_a are sent to $a \in I$ as in Figure 3.

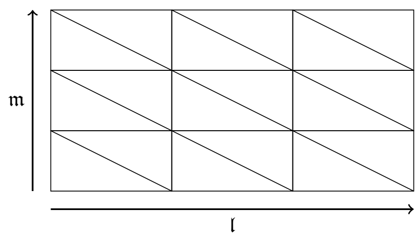

FIGURE 1. A triangulation of the 2-dimensional torus $T(\gamma)$.

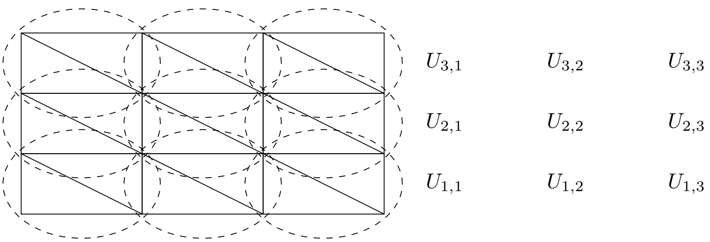

FIGURE 2. An open covering $\mathcal{U} = \{U_a\}_{a\in I}$ of X.

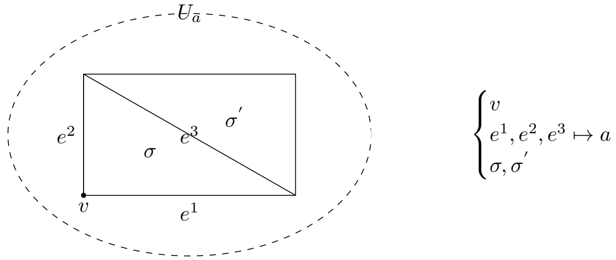

Figure 3. Three edges and two triangles in U_a are sent to $a \in I$.

We choose the representatives of $c(f)$, $c(g)$ and $c(\omega_{\mathfrak{S}})$ as follows:

$$c(f) = [(m_{a_0a_1}, \log_{a_0} f)], \quad m_{a_0a_1} = (\delta \log f)_{a_0a_1},$$

$$m_{a_0a_1} = \begin{cases} \int_{\mathfrak{m}} d\log f, & a_0 = (3,i),\ a_1 = (1,j), \\ -\int_{\mathfrak{m}} d\log f, & a_0 = (1,i),\ a_1 = (3,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$c(g) = [(n_{a_0a_1}, \log_{a_0} g)], \quad n_{a_0a_1} = (\delta \log g)_{a_0a_1},$$

$$n_{a_0a_1} = \begin{cases} \int_{\mathfrak{m}} d\log g, & a_0 = (3,i),\ a_1 = (1,j), \\ -\int_{\mathfrak{m}} d\log g, & a_0 = (1,i),\ a_1 = (3,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$c(\omega_{\mathfrak{S}}) = [(\lambda_{a_0a_1}, f_{\omega_{\mathfrak{S}},a_0})], \quad \lambda_{a_0a_1} = (\delta f_{\omega_{\mathfrak{S}}})_{a_0a_1},$$

$$\lambda_{a_0a_1} = \begin{cases} \int_{\mathfrak{l}} \omega_{\mathfrak{S}}, & a_0 = (i,3),\ a_1 = (j,1), \\ -\int_{\mathfrak{l}} \omega_{\mathfrak{S}}, & a_0 = (i,1),\ a_1 = (j,3), \\ 0 & \text{otherwise,} \end{cases}$$

where $i, j \in \{1,2,3\}$.

Let $F_K(i)$ be the set of flags of simplices for $i = 0,1,2$ as in (7). By (8) and Definition 5.2, we have

$$\langle f, g\rangle_\gamma = \sum_{\vec\sigma \in F_K(0)} \int_{\sigma^2} \theta^2_{\iota(\sigma^2)} - \sum_{\vec\sigma \in F_K(1)} \int_{\sigma^1} \theta^1_{\iota(\sigma^2)\iota(\sigma^1)} + \sum_{\vec\sigma \in F_K(2)} \int_{\sigma^0} \theta^0_{\iota(\sigma^2)\iota(\sigma^1)\iota(\sigma^0)}. \tag{9}$$

For the first term of the right-hand side of (9), we have

$$\begin{aligned} \sum_{\vec\sigma \in F_K(0)} \int_{\sigma^2} \theta^2_{\iota(\sigma^2)} &= \sum_{a \in I} \sum_{\sigma^2 \subset U_a} \int_{\sigma^2} \log_a f d\log g \wedge \omega_{\mathfrak{S}} \\ &= \int_{T(\gamma)} \log f d\log g \wedge \omega_{\mathfrak{S}}. \end{aligned} \tag{10}$$

For the second term of the right-hand side of (9), we note that $m_{\iota(\sigma^2)\iota(\sigma^1)} = 0$ unless $(\iota(\sigma^2), \iota(\sigma^1)) = ((3,i),(1,i))$ for $i = 1,2,3$, by our choices of m_{ab}'s and ι. Hence we have

$$\text{(11)} \quad \sum_{\vec{\sigma}\in F_K(1)} \int_{\sigma^1} \theta^1_{\iota(\sigma^2)\iota(\sigma^1)} = \sum_{a\in I} \sum_{\sigma^1\subset\sigma^2\subset U_a} \int_{\sigma^1} m_{\iota(\sigma^2)\iota(\sigma^1)} \log_{\iota(\sigma^1)} g\,\omega_{\mathfrak{S}}$$
$$= \sum_{i=1}^{3} \sum_{\iota(\sigma^1)\subset U_{3,i}} \int_{\sigma^1} m_{(3,i)(1,i)} \log_{\iota(\sigma^1)} g\,\omega_{\mathfrak{S}}$$
$$= \int_{\mathfrak{m}} d\log f \int_{\mathfrak{l}} \log g\,\omega_{\mathfrak{S}}.$$

For the third term of the right-hand side of (9), we note that

$$m_{\iota(\sigma^2)\iota(\sigma^1)} n_{\iota(\sigma^1)\iota(\sigma^0)} = 0$$

for any flags, by our choices of m_{ab}'s, n_{ab}'s and ι. Hence we have

$$\text{(12)} \quad \sum_{\vec{\sigma}\in F_K(2)} \int_{\sigma^0} \theta^0_{\iota(\sigma^2)\iota(\sigma^1)\iota(\sigma^0)} = 0.$$

Getting (9)–(12) together, we obtain the second formula for $\langle f, g\rangle_\gamma$. □

Let R_g be a closed Riemann surface of genus g and $P \in R_g$. For meromorphic functions f and g on R_g, which are not constantly zero, let $\{f, g\}_P$ denote the tame symbol at P defined by

$$\{f,g\}_P = (-1)^{\mathrm{ord}_P(f)\,\mathrm{ord}_P(g)} \frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}(P).$$

Corollary 5.4. *Let $\mathfrak{S} = (M, \mathcal{F}, \phi)$ be the FDS^3 as in Example 3.1, where M is the mapping torus $M(R_g, \varphi)$ for a closed Riemann surface R_g and a diffeomorphism φ of R_g. Let $P = [z, s] \in M$, and let γ_z be the closed orbit containing P, namely, $\gamma_z = \{[z, s] \mid 0 \le s \le 1\}$. Then we have*

$$\langle f, g\rangle_{\gamma_z} = 2\pi\sqrt{-1} \int_0^1 \log(\{f, g\}_P)\, ds \mod \mathbb{Z}(3).$$

Proof. By Cauchy's theorem, we have the following formula for the tame symbol (cp. [10]):

$$\{f,g\}_P = \exp\left(\frac{1}{2\pi\sqrt{-1}} \int_{\mathfrak{m}} \log(f)\, d\log(g) - \log(g(Q)) \int_{\mathfrak{m}} d\log(f) \right),$$

where the meridian $\mathfrak{m}$ is a small loop around P based at Q on the fiber $R_g = \varpi^{-1}(s)$. Since $\omega_{\mathfrak{S}} = \varpi^*(ds)$ and $\Lambda_{\mathfrak{S}} = \mathbb{Z}$, we have

$$2\pi\sqrt{-1} \int_0^1 \log(\{f,g\}_P)\, ds = \int_{T(\gamma_z)} \log(f)\, d\log(g) \wedge \omega_{\mathfrak{S}}$$
$$- \int_{\mathfrak{m}} d\log(f) \int_{\mathfrak{l}} \log(g) \omega_{\mathfrak{S}} \mod \mathbb{Z}(3),$$

which yields the assertion by Theorem 5.3. □

Remark 5.5. (1) Brylinski and McLaughlin showed a formula for the holonomy of a certain gerbe along a torus [6, Thm. 3.6]. We note that their formula has a form similar to ours above and that their method of the computation is different from ours.

(2) Bloch pointed out to us that the formula in Theorem 5.3 has a form similar to a formula for the regulator of an elliptic curve (cp. [3, 4.2], [18, 1.10]).

(3) For a surface bundle over S^1, Stelzig [40] introduced a local symbol by the right-hand side of the formula in Corollary 5.4. So our result generalizes his to arbitrary FDS^3.

5.6. **Hilbert type reciprocity law.** We keep the same notations as in Subsection 5.1. We show a geometric analog for our local symbol of the reciprocity law for the Hilbert symbol in a global field.

Theorem 5.7. *We have*

$$\sum_{\gamma\in\overline{\mathcal{P}_{\mathfrak{S}}}}\langle f,g\rangle_\gamma = 0 \mod \Lambda_{\mathfrak{S}}(3).$$

Proof. We set

$$Z := X\setminus\Big(\bigcup_{i=1}^{r} V(\gamma_i^\infty)\cup\bigcup_{i=1}^{N} V(\gamma_i)\Big).$$

Noting $\partial Z = \bigcup_{i=1}^{r} T(\gamma_i^\infty)\cup\bigcup_{i=1}^{N} T(\gamma_i)$, we have, by Theorem 4.7,

$$\begin{aligned}\sum_{\gamma\in\overline{\mathcal{P}_{\mathfrak{S}}}}\langle f,g\rangle_\gamma &= \sum_{i=1}^{r}\langle f,g\rangle_{\gamma_i^\infty}+\sum_{i=1}^{N}\langle f,g\rangle_{\gamma_i}\\ &=\sum_{i=1}^{r}\int_{T(\gamma_i^\infty)} c(f)\cup c(g)\cup c(\omega_{\mathfrak{S}})\\ &\qquad +\sum_{i=1}^{N}\int_{T(\gamma_i)} c(f)\cup c(g)\cup c(\omega_{\mathfrak{S}}) \mod \Lambda_{\mathfrak{S}}(3)\\ &=\int_{\partial Z} c(f)\cup c(g)\cup c(\omega_{\mathfrak{S}}) \mod \Lambda_{\mathfrak{S}}(3)\\ &=\int_{Z}\Omega_3(c(f)\cup c(g)\cup c(\omega_{\mathfrak{S}})) \mod \Lambda_{\mathfrak{S}}(3).\end{aligned}$$

By Definition 4.5 and Theorem 5.3, we have

$$\Omega_3(c(f)\cup c(g)\cup c(\omega_{\mathfrak{S}}))|_{U_a} = d(\log_a f\, d\log g\wedge\omega_{\mathfrak{S}}).$$

Taking a foliated local coordinate (z,x) on U_a, we have

$$\begin{aligned}d(\log_a f\, d\log g\wedge\omega_{\mathfrak{S}}) &= d(\log_a f\, d\log g)\wedge\omega_{\mathfrak{S}} \quad (\text{since } \omega_{\mathfrak{S}} \text{ is closed})\\ &= d\log f\wedge d\log g\wedge\omega_{\mathfrak{S}}\\ &= \Big(\frac{f_z}{f}\,dz+\frac{f_x}{f}\,dx\Big)\wedge\Big(\frac{g_z}{g}\,dz+\frac{g_x}{g}\,dx\Big)\wedge\omega_{\mathfrak{S}}\\ &\qquad (\text{since } \log f,\ \log g \text{ are holomorphic on } U_a)\end{aligned}$$

$$
\begin{aligned}
&= \frac{f_z g_x - f_x g_z}{fg}\, dz \wedge dx \wedge \omega_{\mathfrak{S}} \\
&= 0 \quad (\text{by } \omega_{\mathfrak{S}} = h(x)\, dx),
\end{aligned}
$$

and hence the assertion follows. □

Remark 5.8. (1) Our method to introduce local symbols and to show the reciprocity law may be regarded as a generalization to an FDS^3 of Deligne, Bloch and Beilinson's interpretation of the tame symbol on a Riemann surface using the holomorphic Deligne cohomology [3, 4, 10].

(2) It would be interesting to generalize our local symbol to multiple symbols $\langle f_1, \dots, f_n \rangle_\gamma$ for several FDS^3-meromorphic functions f_i by using the Massey products in the smooth Deligne cohomology and the iterated integrals, as the tame symbol on a Riemann surface was generalized to polysymbols in [34].

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