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Paper:

Innovation Sharing Distributed Kalman Filter with Packet Loss

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This study investigates the problem of distributed state estimation. A distributed Kalman filter algorithm is proposed, in which sensors exchange their innovations. A detailed analysis is conducted for the case of two sensor networks, demonstrating that the proposed algorithm outperforms the case where each sensor runs a conventional Kalman filter without communication. The upper bounds of error covariance matrices are also derived in the case of packet loss. Numerical examples verify the effectiveness of the proposed algorithm.

Keywords: distributed Kalman filter, packet dropout, multisensor network

1. Introduction

State estimation is a method used to estimate the internal states of a dynamic system based on available measurement data. Accurate determination of these states is crucial for effective feedback control. However, practical implementations often face challenges leading to inaccuracies or the inability to obtain internal states. For instance, sensor measurement noise can result in inaccurate readings, and certain states may not be directly measured by the sensors. To address these issues, several classical state-estimation algorithms have been developed, including the Kalman filter, H_∞ filter, particle filter, and Luenberger observer.

However, these state estimation algorithms typically rely on a single sensor, which limits the amount of information gathered. Compared to single-sensor filtering, multisensor filtering enables more comprehensive information acquisition and enhances robustness. With improvements in computing capabilities, research interest in state estimation using multiple sensors has increased. In 2005, Olfati-Saber and Shamma proposed consensus filters for multisensor networks [1]. Distributed Kalman filters incorporating consensus filters have been proposed in [2, 3], while distributed state observers have studied discrete-time deterministic systems in [4, 5] and continuous-time deterministic systems in [6, 7].

In general, sensors exchange information through networks. However, challenges such as unreliable networks, sensor failures, and harsh environmental conditions can lead to phenomena such as packet dropouts and communication delays, which can significantly impact sensor network performance. Consequently, extensive research has been conducted to address these communication issues. Studies on multisensor Kalman filtering in the presence of dropouts can be found in [8, 9], and research on multi-observers with random communication for continuous-time stochastic systems is discussed in [10]. Furthermore, [11] investigates the upper bound of the expectation of the covariance matrix for a distributed Kalman filter exchanging predicted values between sensors in scenarios involving packet dropouts.

Motivated by the discussion above, we propose a new distributed Kalman filter that exchanges innovations between sensors. This approach differs from the conventional settings of existing distributed filters [12–14], which typically enable sensors to exchange estimated or predicted values. Each time, a sensor sends its predicted values and measurements, along with timestamps, to its neighboring sensors only once. We demonstrate that the proposed method, which utilizes the exchange of innovations, provides a more precise estimation than the conventional Kalman filter, a single-sensor filtering method, as well as some existing distributed filtering methods. Subsequently, we consider the situation in which packet dropouts occur when the proposed Kalman filter works and study the performance of the proposed distributed Kalman filter in this case.

This study makes three contributions. Firstly, we propose a new distributed Kalman filter and validate its improvement in estimation accuracy compared to the conventional Kalman filter through derivation and simulation. Secondly, we demonstrate that the proposed distributed Kalman filter can achieve consensus between sensors, enabling the sensors to obtain equally accurate measurements. Thirdly, we study situations in which packet dropouts may occur and derive the upper bounds of the covariance matrix.

2. Problem Formulation

The topological structure of the sensor network in this study is represented by $G = (V, E)$, where V represents all sensor nodes in the network and E represents all possible communication channels between any two sensor nodes.

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A communication channel between two sensor nodes indicates that they can exchange information with each other. Sensor nodes that share communication channels with a specific sensor node are called neighboring nodes. In this study, we denote N_i as the sensor node set composed of all neighboring nodes of sensor node i and sensor node i itself. For any sensor node $i \in V$, we define the number of sensor nodes in N_i as m_i and all sensor nodes in N_i from 1 to m_i, respectively, ensuring that sensor node i is numbered as m_i. To express this accurately, we stipulate that the subscript i represents the i-th sensor node in the sensor network, and the subscripts i and j represent the j-th neighboring sensors of sensor i.

Consider the following linear, discrete-time stochastic system:

$$x(t) = Ax(t-1) + w(t-1), \quad (1)$$

$$y_i(t-1) = H_i x(t-1) + v_i(t-1), \quad (2)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, and $y_i(t) \in \mathcal{R}^m$ is the measurement vector of sensor node i. $w(t) \in \mathcal{R}^n$ and $v_i(t) \in \mathcal{R}^m$ are the dynamic-model noise and sensor node i's measurement noise, respectively. A and H_i are $n \times n$ and $m \times n$ known constant matrices.

In this paper, all the derivations are based on the following assumptions:

Assumption 1: $w(t)$ and $v_i(t)$ are uncorrelated zero-mean white Gaussian noises. The covariance matrices $w(t)$ and $v_i(t)$ are $E[w(t)w(t)^\top] = Q$ and $E[v_i(t)v_i(t)^\top] = R_i$, respectively.

Assumption 2: The initial state $x(0)$ is uncorrelated with $w(t)$ and $v_i(t)$.

For system (1), we propose the following distributed Kalman filter for each sensor node:

$$\hat{x}_i(t|t) = \hat{x}_i(t|t-1) + \sum_{l \in N_i} k_{il}\left[y_l(t) - H_l\hat{x}_l(t|t-1)\right] \quad (3)$$

where $\hat{x}_i(t|t-1)$ is the predicted value and $\hat{x}_i(t|t)$ is the estimated value. k_{il} is the Kalman gain matrix of the innovation from sensor node l with appropriate dimensions. For convenience of derivation, we define $K_i = [k_{i1}, k_{i2}, \ldots, k_{im_i}]$.

The purpose of this study was to design a distributed Kalman filter with the structure shown in Eq. (3). In the following sections, we first show the iterative process of the proposed distributed Kalman filter and then use derivation and simulation to confirm its effectiveness.

3. Design of Distributed Kalman Filter

This section presents the iterative process of the proposed distributed Kalman filter with structure (3).

The following lemma will be used to prove our main results.

Lemma 1— [15]: Some formulas for calculating the partial derivative of the trace of a matrix are given as

$$\frac{\partial}{\partial X}\operatorname{tr}\{AX^\top\} = A$$

$$\frac{\partial}{\partial X}\operatorname{tr}\{XA\} = A^\top$$

$$\frac{\partial}{\partial X}\operatorname{tr}\{XAX^\top\} = X(A + A^\top)$$

where A and X are matrices with appropriate dimensions.

Theorem 1: Given the initial estimated value $\hat{x}_i(0|0)$, initial error covariance matrix $P_i(0|0)$, and initial error cross-covariance matrix $P_{il}(0|0)$, the estimate for each iteration is obtained as follows:

Prediction:

$$\hat{x}_i(t|t-1) = A\hat{x}_i(t-1|t-1) \quad (4)$$

$$P_i(t|t-1) = AP_i(t-1|t-1)A^\top + Q \quad (5)$$

$$P_{ij}(t|t-1) = AP_{ij}(t-1|t-1)A^\top + Q \quad (6)$$

Update:

$$K_i = S_i(T+R)^{-1} \quad (7)$$

$$\hat{x}_i(t|t) = \hat{x}_i(t|t-1) + \sum_{j \in N_i} k_{ij}\left[y_j(t) - H_j\hat{x}_j(t|t-1)\right] \quad (8)$$

$$P_i(t|t) = P_i(t|t-1) - S_i(T+R)^{-1}S_i^\top \quad (9)$$

$$P_{ij}(t|t) = P_{ij}(t|t-1) - S_i(T+R)^{-1}S_j^\top \quad (10)$$

where

$$S_i = \left[P_{i1}(t|t-1)H_1^\top, P_{i2}(t|t-1)H_2^\top, \ldots, P_{im_i}(t|t-1)H_{m_i}^\top\right]$$

$$T = \begin{bmatrix} H_1P_{11}H_1^\top & H_1P_{12}H_2^\top & \cdots & H_1P_{1m_i}H_{m_i}^\top \\ H_2P_{21}H_1^\top & H_2P_{22}H_2^\top & \cdots & H_2P_{2m_i}H_{m_i}^\top \\ \vdots & \vdots & \ddots & \vdots \\ H_{m_i}P_{m_i1}H_1^\top & H_{m_i}P_{m_i2}H_2^\top & \cdots & H_{m_i}P_{m_im_i}H_{m_i}^\top \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{m_i} \end{bmatrix}.$$

Proof: We define the estimation $\widetilde{x}_i(t|t)$ and prediction errors $\widetilde{x}_i(t|t-1)$ of sensor node i at time t as:

$$\widetilde{x}_i(t|t) = x(t) - \hat{x}_i(t|t), \quad (11)$$

$$\widetilde{x}_i(t|t-1) = x(t) - \hat{x}_i(t|t-1), \quad (12)$$

where $x(t)$ is the actual value at time t.

Substituting Eqs. (1) and (4) into Eq. (12), we obtain

$$\widetilde{x}_i(t|t-1) = A\widetilde{x}_i(t-1|t-1) + w(t-1). \quad (13)$$

Under Assumptions 1 and 2, using the definitions $P_i(t|t-1) = E[\widetilde{x}_i(t|t-1)\widetilde{x}_i(t|t-1)^\top]$ and $P_{ij}(t|t-1) = E[\widetilde{x}_i(t|t-1)\widetilde{x}_j(t|t-1)^\top]$, we can easily obtain Eqs. (5) and (6), respectively.

Substituting Eq. (3) into Eq. (11), we obtain

$$\widetilde{x}_i(t|t) = \widetilde{x}_i(t|t-1) - \sum_{l \in N_i} k_{il}[H_l\widetilde{x}_l(t|t-1) + v_l]. \quad (14)$$

Under Assumptions 1 and 2, substituting Eq. (14) into definition $P_i(t|t) = E[\widetilde{x}_i(t|t)\widetilde{x}_i(t|t)^\top]$, we have

$$\begin{aligned}P_i(t|t) &= E\left[\widetilde{x}_i(t|t)\widetilde{x}_i(t|t)^\top\right]\\ &= P_i(t|t-1) - \sum_{l\in N_i} P_{il}(t|t-1)H_l^\top k_{il}^\top\\ &\quad - \sum_{l\in N_i} k_{il}H_l P_{li}(t|t-1)\\ &\quad + \sum_{m\in N_i}\sum_{n\in N_i} k_{im}H_m P_{mn}(t|t-1)H_n^\top k_{in}^\top\\ &\quad + \sum_{l\in N_i} k_{il}R_l k_{il}^\top\\ &= P_i(t|t-1) - S_iK_i^\top - K_iS_i^\top\\ &\quad + K_iTK_i^\top + K_iRK_i^\top.\end{aligned}$$

Similarly,

$$\begin{aligned}P_{ij}(t|t) &= E\left[\widetilde{x}_i(t|t)\widetilde{x}_j(t|t)^\top\right]\\ &= P_{ij}(t|t-1) - S_iK_j^\top - K_iS_j^\top\\ &\quad + K_iTK_j^\top + K_iRK_j^\top.\end{aligned}$$

The optimal Kalman gain for the proposed distributed Kalman filter is given by the K_i that minimizes $\mathrm{tr}\{P_i(t|t)\}$. To achieve this, let $\frac{\partial}{\partial K_i}\mathrm{tr}\{P_i(t|t)\} = 0$, then we obtain the following equation:

$$-S_i - S_i + K_i(T + T^\top) + K_i(R + R^\top) = 0. \quad \quad (15)$$

Because T and R are symmetric matrices, we obtain Eq. (7). Finally, by substituting $K_i = S_i(T+R)^{-1}$ and $K_j = S_j(T+R)^{-1}$ into $P_i(t|t)$ and $P_{ij}(t|t)$, Eqs. (9) and (10) are obtained. ■

4. Analysis of Distributed Kalman Filter

In this section, we present and prove some properties of the proposed distributed Kalman filter.

In this paper, the classical Kalman filter, which estimates the states of a plant using the following iterative algorithm [16], is referred to as the "conventional" Kalman filter:

Prediction:

$$\hat{x}_i(t|t-1) = A\hat{x}_i(t-1|t-1) \quad \quad (16)$$

$$P_i(t|t-1) = AP_i(t-1|t-1)A^\top + Q \quad \quad (17)$$

Update:

$$K = P_i(t|t-1)H^\top(HP_i(t|t-1)H^\top + R)^{-1} \quad \quad (18)$$

$$\hat{x}_i(t|t) = \hat{x}_i(t|t-1) + K(y - H\hat{x}_i(t|t-1)) \quad \quad (19)$$

$$P_i(t|t) = (I - KH)P_i(t|t-1) \quad \quad (20)$$

The following lemmas will be used for the proof of this section.

Lemma 2— [17]: Define the modified algebraic Riccati equation for the conventional Kalman filter as follows:

$$g(X) = AXA^\top + Q - AXH^\top(HXH^\top + R)^{-1}HXA^\top$$

where A and H are a state matrix and a measurement matrix, respectively; Q and R are covariance matrices of dynamic model noise and measurement noise; and X is a covariance matrix of predicted values. Then $g(X_1) \le g(X_2)$ if $X_1 \le X_2$.

Lemma 3— [17]: Consider a sequence $P_{t+1} = g(P_t)$ with the initial value P_0. For any initial value $P_0 \ge 0$, the sequence will always converge, and the limits will be the same.

The following theorem shows that the proposed distributed Kalman filter performs better than the conventional Kalman filter under the same initial conditions. For convenience of derivation, we define $P_i^c(t|t)$ as the error covariance matrix of the estimated value at time t when a conventional Kalman filter is used.

Theorem 2: With the same initial error covariance matrices $P_i(0|0)$, we have $P_i(t|t) \le P_i^c(t|t)$ for $i \in V$.

Proof: When calculating the estimate of sensor i, we can combine all innovations received by sensor i into one with the subscript l_i as follows:

$$\begin{aligned}\hat{x}_i(t|t) &= \hat{x}_i(t|t-1) + k_i\left[y_i - H_i\hat{x}_i(t|t-1)\right]\\ &\quad + K_{l_i}\left[Y_{l_i} - H_{l_i}\hat{X}_{l_i}(t|t-1)\right]\end{aligned}$$

where

$$\hat{X}_{l_i}(t|t-1) = \begin{bmatrix}\hat{x}_{i,1}(t|t-1)\\ \vdots\\ \hat{x}_{i,m_i-1}(t|t-1)\end{bmatrix},$$

$$Y_{l_i}(t) = \begin{bmatrix}y_{i,1}(t)\\ \vdots\\ y_{i,m_i-1}(t)\end{bmatrix},$$

$$H_{l_i} = \begin{bmatrix}H_{i,1} & & \\ & \ddots & \\ & & H_{i,m_i-1}\end{bmatrix},$$

and the error covariance matrix of $Y_{l_i}(t)$ is

$$R_{l_i} = \begin{bmatrix}R_{i,1} & & \\ & \ddots & \\ & & R_{i,m_i-1}\end{bmatrix}.$$

We then define

$$\begin{aligned}T + R &= \begin{bmatrix}H_iP_iH_i^\top + R_i & H_iP_{il_i}H_{l_i}^\top\\ H_{l_i}P_{l_ii}H_i^\top & H_{l_i}P_{l_i}H_{l_i}^\top + R_{l_i}\end{bmatrix},\\ &=: \begin{bmatrix}M_{ii} & M_{il_i}\\ M_{l_ii} & M_{l_il_i}\end{bmatrix},\end{aligned}$$

then by using the Schur complement [18] for $(T+R)^{-1}$, we obtain

$$\begin{aligned}
&P_i(t|t)\\
&= P_i(t|t-1) - S_i(T+R)^{-1}S_i^\top\\
&= P_i(t|t-1) - [P_i(t|t-1)H_i^\top, P_{il_i}(t|t-1)H_{l_i}^\top]\\
&\quad \times (T+R)^{-1}[P_i(t|t-1)H_i^\top, P_{il_i}(t|t-1)H_{l_i}^\top]^\top\\
&= P_i - (P_iH_i^\top M_{ii}^{-1}H_iP_i\\
&\quad + P_iH_i^\top M_{ii}^{-1}M_{il_i}(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}M_{l_ii}M_{ii}^{-1}H_iP_i\\
&\quad - P_iH_i^\top M_{ii}^{-1}M_{il_i}(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}H_{l_i}P_{l_ii}\\
&\quad - P_{il_i}H_{l_i}^\top(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}M_{l_ii}M_{ii}^{-1}H_iP_i\\
&\quad + P_{il_i}H_{l_i}^\top(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}H_{l_i}P_{l_ii})\\
&= P_i - P_iH_i^\top M_{ii}^{-1}H_iP_i\\
&\quad - (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}\\
&\quad \times (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)^\top\\
&= P_i - P_iH_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_i\\
&\quad - (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}\\
&\quad \times (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)^\top.
\end{aligned}$$

Conversely, the error covariance matrices of sensor i using the conventional Kalman filter algorithm are given as

$$P_i^c(t|t) = P_i - P_iH_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_i.$$

When the estimated error covariance matrices of sensor i at time $t-1$ are the same in the two situations, then we can get the estimated error covariance matrix of the two cases at time t with the following relationship:

$$\begin{aligned}
&P_i^c(t|t) - P_i(t|t)\\
&= (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)(M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}\\
&\quad \times (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)^\top.
\end{aligned}$$

Now we will show that $P_i^c(t|t) - P_i(t|t)$ is a positive definite matrix. First, we have

$$\begin{aligned}
&M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i}\\
&= H_{l_i}(P_{l_i} - P_{l_ii}H_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_{il_i})H_{l_i}^\top + R_{l_i}.
\end{aligned}$$

Therefore, we only need to prove that $P_{l_i} - P_{l_ii}H_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_{il_i}$ is a positive semi-definite matrix. Note that $(H_iP_iH_i^\top + R_i)^{-1} \leq (H_iP_iH_i^\top)^{-1}$. Then, we obtain

$$\begin{aligned}
&P_{l_i} - P_{l_ii}H_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_{il_i}\\
&\geq P_{l_i} - P_{l_ii}H_i^\top(H_iP_iH_i^\top)^{-1}H_iP_{il_i}.
\end{aligned}$$

In this case, we only need to prove $P_{l_i} - P_{l_ii}H_i^\top(H_iP_iH_i^\top)^{-1}H_iP_{il_i} \geq 0$, which is equivalent to prove

$$\begin{bmatrix} H_iP_iH_i^\top & H_iP_{il_i}\\ P_{l_ii}H_i^\top & P_{l_i}\end{bmatrix} = \begin{bmatrix} H_i & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} P_i & P_{il_i}\\ P_{l_ii} & P_{l_i}\end{bmatrix}\begin{bmatrix} H_i & 0\\ 0 & I\end{bmatrix}^\top \geq 0$$

where we used the Schur complement. Define the random vector $\widetilde{x}^u = [\widetilde{x}_i(t|t-1); \widetilde{x}_{l_i}(t|t-1)] \in \mathcal{R}^{nm_i}$, then obviously

$$\begin{bmatrix} P_i & P_{il_i}\\ P_{l_ii} & P_{l_i}\end{bmatrix} = E\left[\widetilde{x}^u(t|t-1)\widetilde{x}^u(t|t-1)^\top\right] \geq 0.$$

Therefore, we can determine that $P_i^c(t|t) - P_i(t|t) > 0$. Then, we can further get $P_i(t+1|t) < P_i^c(t+1|t)$. For sensor i, we have

$$\begin{aligned}
&P_i(t+1|t+1)\\
&= P_i(t+1|t) - P_iH_i^\top(H_iP_iH_i^\top + R_i)^{-1}H_iP_i\\
&\quad - (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)\\
&\quad \times (M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}\\
&\quad \times (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)^\top\\
&< P_i^c(t+1|t) - P_i^cH_i^\top(H_iP_i^cH_i^\top + R_i)^{-1}H_iP_i^c\\
&\quad - (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)\\
&\quad \times (M_{l_il_i} - M_{l_ii}M_{ii}^{-1}M_{il_i})^{-1}\\
&\quad \times (P_iH_i^\top M_{ii}^{-1}M_{il_i} - P_{il_i}H_{l_i}^\top)^\top\\
&< P_i^c(t+1|t) - P_i^cH_i^\top(H_iP_i^cH_i^\top + R_i)^{-1}H_iP_i^c\\
&= P_i^c(t+1|t+1)
\end{aligned}$$

where Lemma 2 is used. This implies that with the same initial error covariance matrices $P_i(0|0)$, the proposed distributed Kalman filter will always perform better. ∎

Remark 1: When packet loss occurs in a sensor network, each sensor cannot receive any information from the other sensors. Therefore, each sensor can only perform single-sensor filtering, that is, using a conventional Kalman filter for state estimation. When considering packet dropout, Theorem 2 will still hold. This is because Theorem 2 and Lemma 2 show that when $P_i \leq P_i^c$, this will still be true, regardless of whether packet dropout occurs at the next moment.

The following theorem shows the consensus of the proposed distributed Kalman filter.

Theorem 3: Assume that the connectivity graph of the sensor network is undirected and complete, and no packet loss occurs. Define the number of sensors in the sensor network as m. Then, for the error covariance matrices $P_i(t|t)$ and estimated values $\hat{x}_i(t|t)$ given in Eqs. (9) and (8), respectively, there exists a matrix $\widetilde{P}$ and a vector $\widetilde{x}$ such that

$$\lim_{t\to\infty} P_1(t|t) = \lim_{t\to\infty} P_2(t|t) = \cdots = \lim_{t\to\infty} P_m(t|t) = \widetilde{P}$$
$$\lim_{t\to\infty} \hat{x}_1(t|t) = \lim_{t\to\infty} \hat{x}_2(t|t) = \cdots = \lim_{t\to\infty} \hat{x}_m(t|t) = \widetilde{x}$$

for any initial conditions $P_i(0|0) \geq 0$, $i = 1, 2, \ldots, m$.

Proof: If all sensors are considered as a whole, we can combine the algorithms of all sensors in the error covariance matrix as

Prediction:

$$P^u(t|t-1) = A^uP^u(t-1|t-1)A^{u\top} + Q^u$$

Update:

$$\begin{aligned}
P^u(t|t) &= P^u(t|t-1) - P^u(t|t-1)H^{u\top}\\
&\quad \times(H^uP^u(t|t-1)H^{u\top} + R^u)^{-1}H^uP^u(t|t-1)
\end{aligned}$$

where

$$P^u(t|t) = \begin{bmatrix} P_1(t|t) & \cdots & P_{1m}(t|t) \\ \vdots & \ddots & \vdots \\ P_{m1}(t|t) & \cdots & P_m(t|t) \end{bmatrix},$$

$$A^u = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix},$$

$$H^u = \begin{bmatrix} H_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_m \end{bmatrix},$$

$$Q^u = \begin{bmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{bmatrix},$$

$$R^u = \begin{bmatrix} R_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_m \end{bmatrix}.$$

Then we obtain

$$\begin{aligned} P^u(t+1|t) = {} & A^u P^u(t|t-1) A^{u\top} + Q^u - A^u P^u(t|t-1) \\ & \times H^{u\top} (H^u P^u(t|t-1) H^{u\top} + R^u)^{-1} \\ & \times H^u P^u(t|t-1) A^{u\top}, \end{aligned}$$

which has the same form as $g(X)$ in Lemma 2. By using Lemma 3, we obtain the conclusion that for any $P^u(1|0)$, $P^u(t+1|t)$ always converges to the same limit as $t \to \infty$.

Now let us consider a particular initial condition for $P^u(1|0)$, with $P_i(1|0) = P_j(1|0) = P_{ij}(1|0) = P_{ji}(1|0)$ for $i, j \in V$. Using Eqs. (5), (6), (9), and (10), it is easy to see that $P_i(t+1|t) = P_j(t+1|t) = P_{ij}(t+1|t) = P_{ji}(t+1|t)$ for any t. Because the limit of $P^u(t+1|t)$ is the same for any initial condition $P^u(1|0)$, this implies that $P^u(t+1|t)$ has this property $P_i(t+1|t) = P_j(t+1|t) = P_{ij}(t+1|t) = P_{ji}(t+1|t)$ at its limit regardless of the initial condition. Then, it is clear that $P_i(t|t) = P_j(t|t) = P_{ij}(t|t) = P_{ji}(t|t)$ is satisfied by Eqs. (5) and (6).

We then prove the consensus of $\hat{x}_i(t|t)$. We first define random vector $\widetilde{e}_{ij}(t|t)$ as

$$\begin{aligned} \widetilde{e}_{ij}(t|t) &= \hat{x}_i(t|t) - \hat{x}_j(t|t) \\ &= (x(t|t) - \hat{x}_j(t|t)) - (x(t|t) - \hat{x}_i(t|t)) \\ &= \widetilde{x}_j(t|t) - \widetilde{x}_i(t|t), \end{aligned}$$

for $i, j \in V$. Because the means of $\widetilde{x}_j(t|t)$ and $\widetilde{x}_i(t|t)$ are both zero vectors, the mean of $\widetilde{e}_{ij}(t|t)$ is also a zero vector. Above, we have proved that $P_i(t|t) = P_j(t|t) = P_{ij}(t|t) = P_{ji}(t|t)$ for $i, j \in V$. Therefore, the cross-covariance matrix of $\widetilde{e}_{ij}$ is a zero matrix, which means $\widetilde{e}_{ij}$ is a zero vector for $i, j \in V$. Consequently, it is clear that $\lim_{t\to\infty} \hat{x}_1(t|t) = \lim_{t\to\infty} \hat{x}_2(t|t) = \cdots = \lim_{t\to\infty} \hat{x}_m(t|t) = \widetilde{x}$. ■

Remark 2: When considering packet dropout, Theorem 3 no longer holds because, without communication between sensors, each sensor will have varying estimation accuracy based on its individual characteristics.

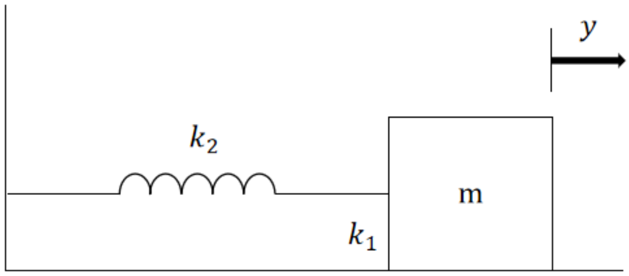

Fig. 1. Mechanical system.

The following theorem provides an upper bound on the error covariance matrix of each sensor when using the proposed distributed Kalman filter under packet loss conditions. We assume that when a sensor experiences packet loss, it can only rely on its own innovation for state estimation.

Theorem 4: If each sensor uses Eqs. (16)–(20) for state estimation when packet dropout occurs, then for time-invariant systems (1) and (2), the solution $P_i(t+1|t)$ of Eq. (5) is bounded with $P_i(t+1|t) \le P_i^b(t+1|t)$, where

$$\begin{aligned} & P_i^b(t+1|t) \\ & = A P_i^b(t|t-1) A^\top + Q - A P_i^b(t|t-1) H_i^\top \\ & \quad \times (H_i P_i^b(t|t-1) H_i^\top + R_i)^{-1} H_i P_i^b(t|t-1) A^\top. \end{aligned}$$

Proof: Let $P_i^a(t+1|t)$ represent the actual predicted error covariance matrix when packet dropout occurs. We assume that packet dropout occurs for the first time for sensor i at time t. According to Theorem 2, $P_i^a(t|t-1) \le P_i^b(t|t-1)$. Next, consider the case where the error covariance matrix is maximized; that is, packet dropout occurs continuously after time t. Then, based on Lemma 2, we can always obtain $P_i^a < P_i^b$ after time t. Therefore, the sequence P_i^b serves as the upper bound on the error covariance matrix of sensor i in the case of a possible packet dropout. ■

5. Numerical Simulation

The mechanical system shown in **Fig. 1** is used to demonstrate the effectiveness of the proposed distributed Kalman filter. In this system, the mass is subjected to friction with the ground, and the tension of a spring is characterized by coefficients k_1 and k_2 for viscous friction and spring constant, respectively. The state of the mass is described as $x(t) = [p(t), q(t)]^\top$, where $p(t)$ and $q(t)$ are the displacement and velocity of the mass, respectively.

By setting $m = 1$, $k_1 = 0.6$, and $k_2 = 0.4$, and applying forward Euler discretization, the discrete-time state-space equation can be given by

$$x(t) = \begin{bmatrix} 1 & 1 \\ -0.5 & 0.4 \end{bmatrix} x(t-1) + w(t-1).$$

Suppose there are two sensors, in which the first sensor measures the displacement of the mass while the second sensor measures both the displacement and veloc-

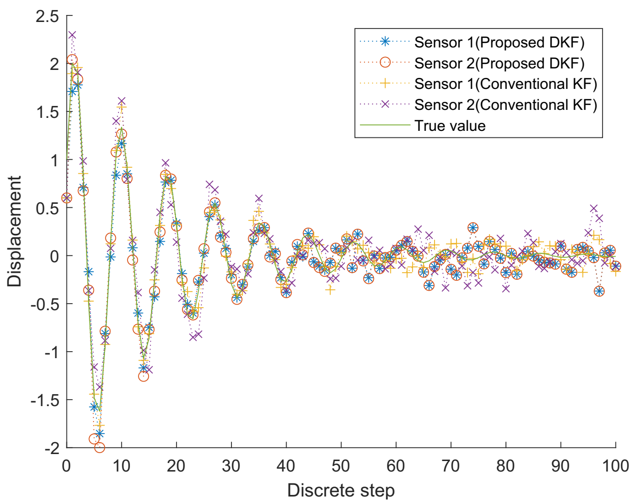

Fig. 2. True values and estimates of displacements by the proposed DKF and conventional KF.

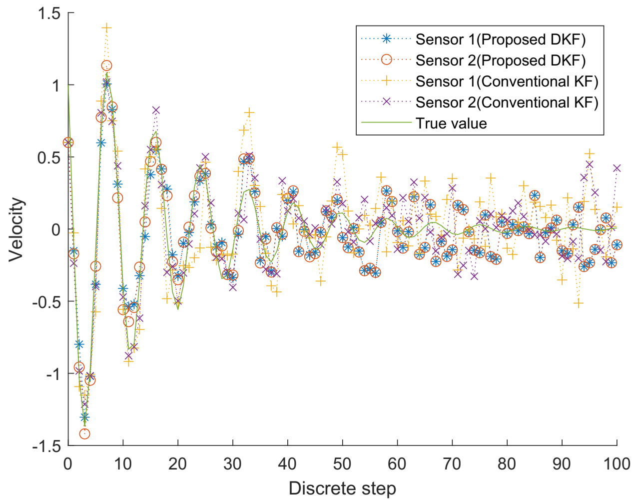

Fig. 3. True values and estimates of velocity by the proposed DKF and conventional KF.

ity. Then, the measurement models of the two sensors are given as

$$y_1(t-1) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t-1) + v_1(t-1),$$

$$y_2(t-1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t-1) + v_2(t-1).$$

The covariance matrices of $w(t)$, $v_1(t)$, and $v_2(t)$ are

$$Q = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix},$$

$$R_1 = 0.06,$$

$$R_2 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}.$$

respectively. The initial states and error covariance matrices of two sensors are

$$x_1 = x_2 = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}.$$

5.1. Comparisons Between the Proposed DKF and Conventional KF

In this subsection, we compare the proposed distributed Kalman filter and conventional Kalman filter based on the trace of the error covariance matrices. Note that when using the conventional Kalman filter, each sensor independently uses Eqs. (16)–(20) without communicating with the other sensor.

Figures 2 and **3** show the estimates of displacement and velocity using the proposed distributed Kalman filter and conventional Kalman filter, respectively. From **Figs. 2** and **3**, we can observe that as the state estimation progresses, the estimated values from the two sensors converge when using the proposed approach, verifying the consensus of the estimated values as proposed in Theorem 3. **Figs. 4** and **5** show the comparison results between the proposed distributed Kalman filter and the conventional Kalman filter. From **Figs. 4** and **5**, we can observe that for both sensors 1 and 2, the proposed distributed Kalman filter consistently outperforms the conventional Kalman filter, which aligns with the findings of Theorem 2.

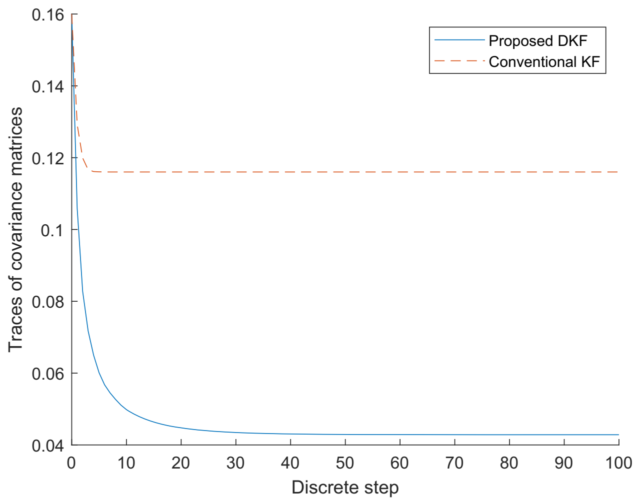

Fig. 4. Comparison of the proposed distributed Kalman filter and conventional Kalman filter on sensor 1.

5.2. Consensus of the Proposed DKF

In this subsection, we present a consensus regarding the proposed distributed Kalman filter. The numerical simulation results are shown in **Fig. 6**. As shown in **Fig. 6**, although the traces of the error covariance matrices of the two sensors are initially different, the traces of the error covariance matrices of the two sensors converge to the same value as the state estimation continues. This simulation result is consistent with the conclusions drawn from Theorem 3.

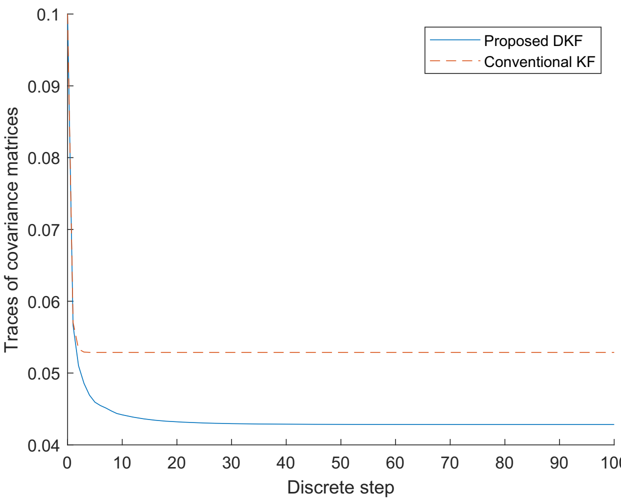

Fig. 5. Comparison of the proposed distributed Kalman filter and conventional Kalman filter on sensor 2.

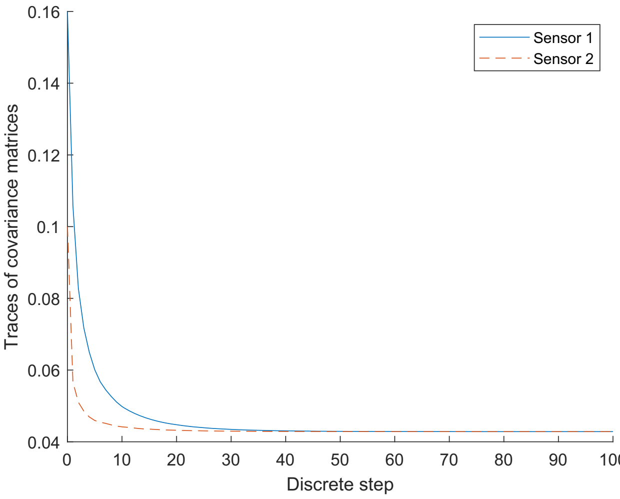

Fig. 6. Traces of error covariance matrices of two sensors when using the proposed distributed Kalman filter.

5.3. Upper Bound of the Proposed DKF

In this subsection, we demonstrate the upper bounds of the proposed distributed Kalman filter under the scenario where both sensors may experience packet losses at any given moment. **Figs. 7** and **8** show the situation with a packet loss rate of 0.2 for the two sensors, while **Figs. 9** and **10** show the situation with a packet loss rate of 0.8. From **Figs. 7–10**, we can observe that the proposed upper bound holds for both sensors, consistent with the findings of Theorem 4.

5.4. Comparison with Existing Distributed Filtering

In this subsection, we compare our approach with the distributed filtering method proposed in [12]. The filtering performance is evaluated based on the average of the absolute values of the errors across all sensors. **Figs. 11** and **12** show the estimates of displacement and velocity

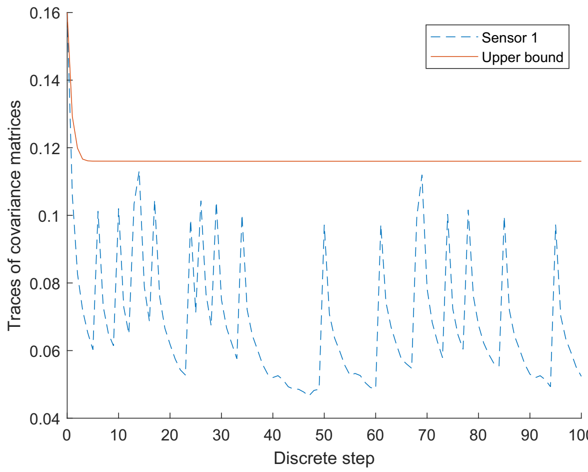

Fig. 7. Traces of error covariance matrices of sensor 1 when using the proposed distributed Kalman filter in the case where the dropout rate is 0.2.

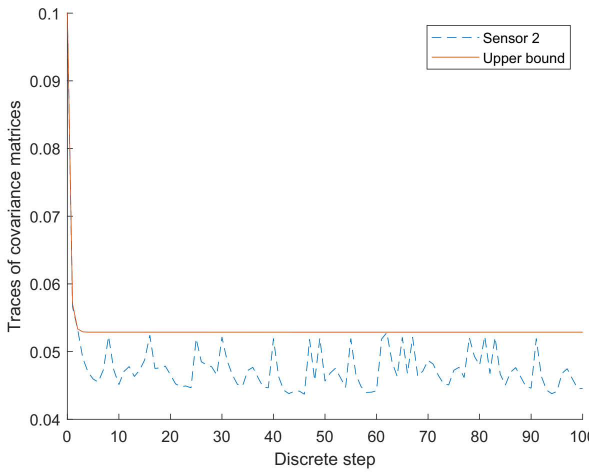

Fig. 8. Traces of error covariance matrices of sensor 2 when using the proposed distributed Kalman filter in the case where the dropout rate is 0.2.

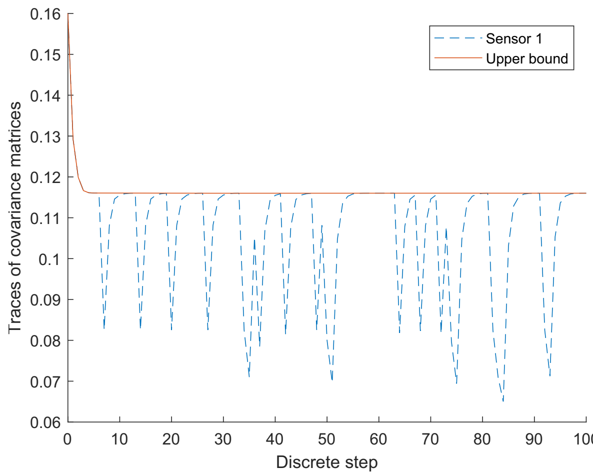

Fig. 9. Traces of error covariance matrices of sensor 1 when using the proposed distributed Kalman filter in the case where the dropout rate is 0.8.

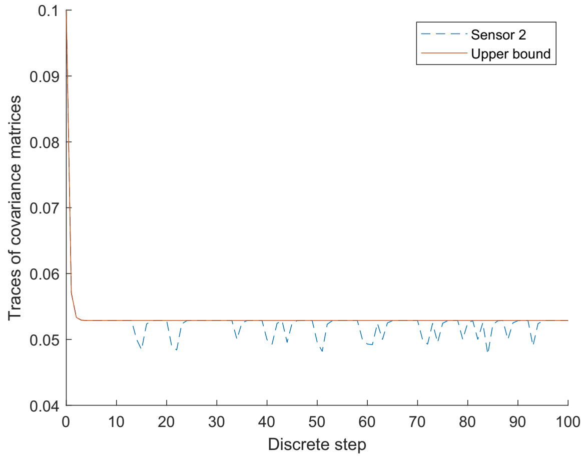

Fig. 10. Traces of error covariance matrices of sensor 2 when using the proposed distributed Kalman filter in the case where the dropout rate is 0.8.

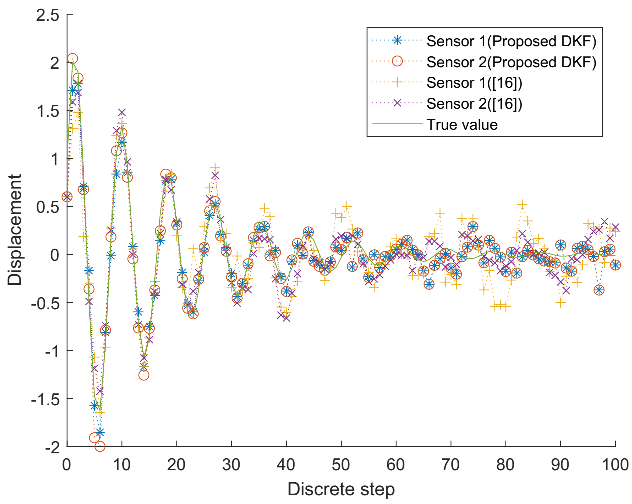

Fig. 11. True values and estimates of displacements by the proposed DKF and approach proposed in [12].

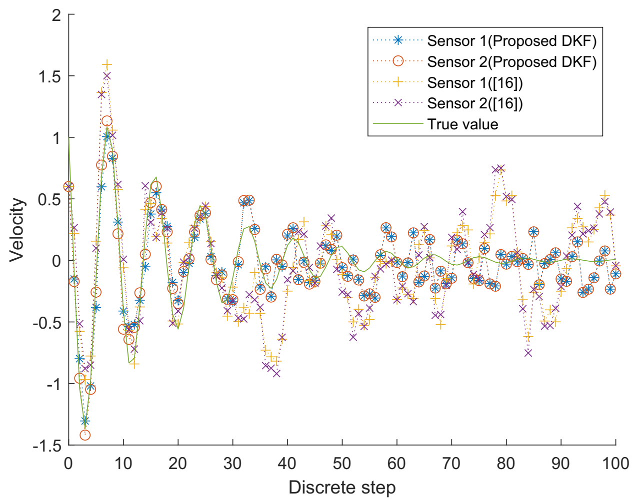

Fig. 12. True values and estimates of velocity by the proposed DKF and approach proposed in [12].

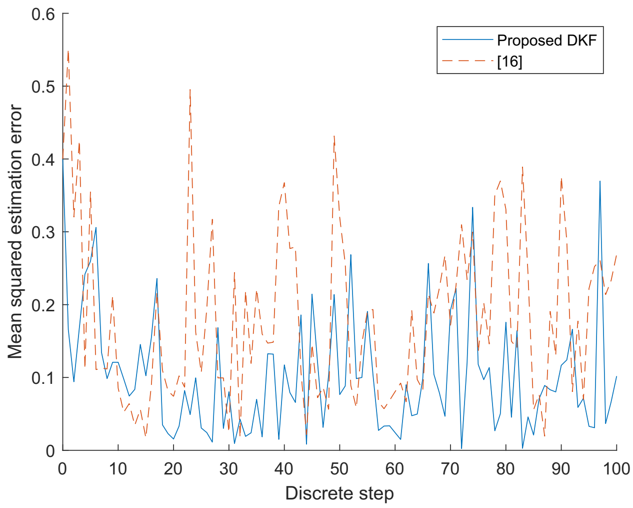

Fig. 13. Comparison of mean squared error of displacement for the proposed DKF and approach proposed in [12].

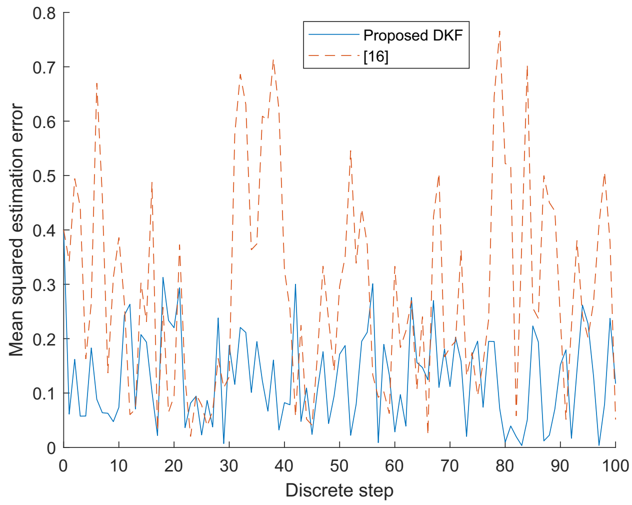

Fig. 14. Comparison of mean squared error of velocity for the proposed DKF and approach proposed in [12].

using the proposed distributed Kalman filter and the existing distributed filtering method from [12], respectively. Additionally, **Figs. 13** and **14** show their corresponding mean squared errors. The results show that, compared to the approach proposed in [12], the estimation error of the proposed distributed Kalman filter is smaller, which shows the advantage of our proposed approach.

6. Conclusion

A distributed Kalman filter, which involves sensors exchanging innovations, was proposed for linear discrete-time systems. A detailed analysis was performed on the two sensor networks. In particular, it has been shown that the proposed distributed Kalman filter outperforms the case where each sensor independently runs the conventional Kalman filter. Additionally, consensus between the two sensors was guaranteed. Furthermore, an upper bound for the error covariance matrix of each sensor was derived to account for packet dropouts during communication between the sensors.

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