Torus links T_<2s,2t> and (s,t)-log VOA

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TORUS LINKS $T_{2s,2t}$ AND (s,t)-LOG VOA

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Abstract. We reveal an intimate connection between the torus link $T_{2s,2t}$ and the logarithmic (s,t) VOA. We show that the singlet character of (s,t)-log VOA at the root of unity coincides with the Kashaev invariant and that it has a property of the quantum modularity. Also shown is that the tail of the N-colored Jones polynomial gives the character. Furthermore we propose a geometric method to computer the character.

1. Introduction

Quantum invariants of knots and 3-manifolds are fascinating topics from both physics and mathematics. Recent studies reveal intriguing connections with geometry, number theory, and representation theory.

From a geometric side, a key object is the Kashaev invariant $\langle K\rangle_N$ [18], which is believed to have a structure of hyperbolic geometry in a large N limit via the volume conjecture

$$\lim_{N\to\infty}\frac{2\pi}{N}\log|\langle K\rangle_N| = \mathrm{Vol}(S^3\setminus K), \tag{1.1}$$

where Vol denotes a hyperbolic volume. It is well known [23] that the Kashaev invariant $\langle K\rangle_N$ for a knot K is a specific value of the N-colored Jones polynomial $J_N(q;K)$, which is a $\mathcal{U}_q(\mathfrak{sl}_2)$ knot invariant with N-dimensional irreducible representation;

$$\langle K\rangle_N = J_N(\zeta_N;K), \qquad \zeta_N = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}. \tag{1.2}$$

Through extensive studies on the Kashaev invariant, a notion of the quantum modular form were proposed [29]. A typical example of the quantum modular form is the Kontsevich–Zagier series [28], which was generalized to those corresponding to the Kashaev invariant for the torus knot $T_{2,2t+1}$ [14]. These results suggest that the quantum invariant of knots and 3-manifolds has an intimate connection with a q-series, which has a similar property with mock modular forms [9, 19].

Such a q-series is reminiscent of the character of logarithmic conformal field theories. See [3] where studied was a relationship between the WRT invariant for 3-manifolds and the character of VOA. Therein the character of (s,t)-log VOA explicitly given in [6] plays a role. Later in [2] the character of the singlet $(1,t)$-log VOA was identified with a tail of the colored Jones polynomial for torus link $T_{2,2t}$ which was proved to exist for alternating link [4]. This indicates that not only the WRT invariant but the quantum knot invariant could have a connection with the character of VOA.

The purpose of this letter is to study a relationship between the colored Jones polynomial for torus link $T_{2s,2t}$ and the character of (s,t)-log VOA. In Sections 2 and 4, we introduce Laurent polynomials as a family of the colored Jones polynomial $J_N(q;T_{2s,2t})$. In Section 5, we shall show that they coincide with the singlet characters of (s,t)-log VOA at the root of unity. We also discuss that the tail of $J_N(q;T_{2s,2t})$

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also gives the character. Section 3 is devoted to a quick review of properties of modular forms and their Eichler integrals. In Section 6, we propose a geometrical method to calculate the characters of irreducible modules of (s,t)-log VOA using the Atiyah–Bott formula [1].

2. Colored Jones Polynomials for $T_{2s,2t}$

We assume that s and t are positive coprime integers. The torus knot $T_{s,t}$ has a braid group presentation $(\sigma_1\sigma_2\dots\sigma_{s-1})^t$. Here σ_i denotes the generators of the Artin braid group satisfying the braid relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } |i-j|>1.$$

The N-colored Jones polynomial for the 0-framing torus knot $T_{s,t}$ was given in [22] based on [24] as

$$J_N(q;T_{s,t}) = \frac{q^{\frac{1}{4}st(1-N^2)}}{q^{\frac{N}{2}}-q^{-\frac{N}{2}}} \sum_{r=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left(q^{str^2-(s+t)r+\frac{1}{2}} - q^{str^2-(s-t)r-\frac{1}{2}} \right). \tag{2.1}$$

Here the invariant $J_N(q;K)$ is normalized so that $J_N(q;\text{unknot})=1$.

We have interests in the 2-component torus link $T_{2s,2t}$, which has a braid group presentation $(\sigma_1\sigma_2\dots\sigma_{2s-1})^{2t}$ as in Fig. 1. Therein we have used the braid relation to see that it is a cabling of the torus knot $T_{s,t}$. As is shown in Fig. 2, the braid σ_i^2 corresponds to the twist which is a central element of the ribbon category. Then, by replacing the braid σ_i^2 by the twists in Fig. 1, the N-colored Jones polynomial for $T_{2s,2t}$ can be given by use of $J_N(q;T_{s,t})$ in (2.1) as

$$J_N(q;T_{2s,2t}) = \frac{q^{st(1-N^2)}}{q^{\frac{N}{2}}-q^{-\frac{N}{2}}} \sum_{j=0}^{N-1}\sum_{k=-j}^{j} \left(q^{stk^2-(s+t)k+\frac{1}{2}} - q^{stk^2-(s-t)k-\frac{1}{2}} \right). \tag{2.2}$$

Here both two components of the link are assigned the N-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. We note that the colored HOMFLY polynomial for torus link are given in terms of the Schur function [20].

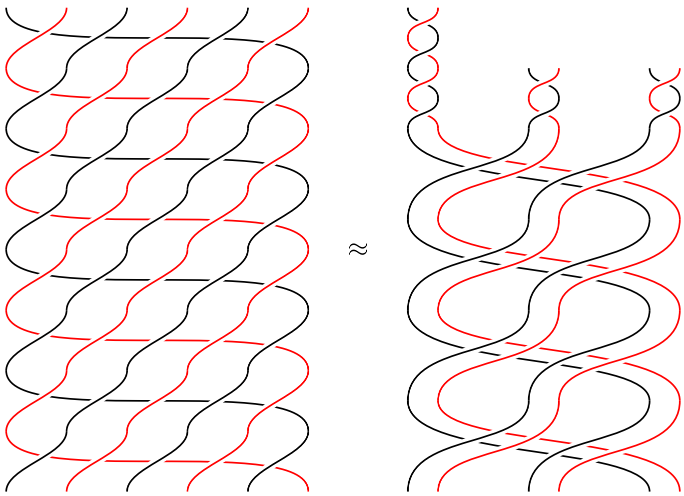

FIGURE 1. A braid group presentation for the torus link $T_{6,8}$. The second component of the link is in red. The right hand side is an isotopic diagram, which shows that $T_{6,8}$ is a cabling of the torus knot $T_{3,4}$.

One sees that, for a sufficiently large N, the tail of the colored Jones polynomial stabilize, and the polynomial is read as

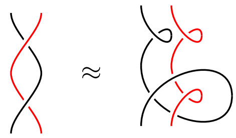

FIGURE 2. An isotopy of σ_i^2.

$$J_N(q;T_{2s,2t}) = N q^{\frac{N-1}{2}+st(1-N^2)} \times \left(1 - q - \frac{N-1}{N} q^{(s-1)(t-1)} + \frac{N-1}{N} q^{st+s-t} + \frac{N-1}{N} q^{st-s+t} - \frac{N-1}{N} q^{(s+1)(t+1)} + \dots \right). \quad (2.3)$$

We will give a proof later in (5.8).

For our later use, we recall that [9]

$$J_N(q;T_{2,2p}) = \frac{q^{p(1-N^2)}}{q^{\frac{N}{2}} - q^{-\frac{N}{2}}} \sum_{j=0}^{N-1} q^{pj(j+1)} \left(q^{j+\frac{1}{2}} - q^{-j-\frac{1}{2}} \right) \quad (2.4)$$

3. Modular Forms and Eichler Integrals

3.1 Unary Theta Series

We introduce periodic functions with mean values zero as follows.

$$\psi_{2p}^{(a)}(k) = \begin{cases} \pm 1, & \text{for } k = \pm a \mod 2p, \\ 0, & \text{otherwise,} \end{cases} \quad (3.1)$$

$$\chi_{2st}^{(n,m)}(k) = \begin{cases} 1, & \text{for } k = \pm(nt - ms) \mod 2st, \\ -1, & \text{for } k = \pm(nt + ms) \mod 2st, \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

Here s and t are coprime positive integers. We assume that $0 < a < p$, and $0 < n < s$, $0 < m < t$. See that $\chi_{2st}^{(n,m)}(k) = \chi_{2st}^{(s-n,t-m)}(k)$. The unary theta series are defined by

$$\Psi_p^{(a)}(\tau) = \frac{1}{2} \sum_{k \in \mathbb{Z}} k\, \psi_{2p}^{(a)}(k)\, q^{\frac{k^2}{4p}}, \quad (3.3)$$

$$\Phi_{s,t}^{(n,m)}(\tau) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \chi_{2st}^{(n,m)}(k)\, q^{\frac{k^2}{4st}}, \quad (3.4)$$

where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ for $\tau \in \mathbb{H}$. The q-series, $\Psi_p^{(a)}(\tau)$ and $\Phi_{s,t}^{(n,m)}(\tau)$, are vector-valued modular forms with weight $\frac{3}{2}$ and $\frac{1}{2}$, respectively. We have

$$\Psi_p^{(a)}(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{\frac{3}{2}} \sum_{b=1}^{p} \sqrt{\frac{2}{p}} \sin\left(\frac{ab}{p}\pi\right) \Psi_p^{(b)}\left(-\tfrac{1}{\tau}\right), \quad (3.5)$$

$$\Psi_p^{(a)}(\tau+1) = \mathrm{e}^{\frac{a^2}{2p}\pi \mathrm{i}}\, \Psi_p^{(a)}(\tau),$$

$$\Phi_{s,t}^{(n,m)}(\tau) = \sqrt{\frac{\mathrm{i}}{\tau}} \sum_{n',m'}{}' S(s,t)_{n,m}^{n',m'}\, \Phi_{s,t}^{(n',m')}\left(-\tfrac{1}{\tau}\right), \quad (3.6)$$

$$\Phi_{s,t}^{(n,m)}(\tau+1) = \mathrm{e}^{\frac{(nt-ms)^2}{2st}\pi \mathrm{i}} \Phi_{s,t}^{(n,m)}(\tau),$$

where $\Sigma'_{n',m'}$ means that n' and m' runs over a $\frac{1}{2}(s-1)(t-1)$-dimensional space, and

$$S(s,t)_{n,m}^{n',m'} = \sqrt{\frac{8}{st}} (-1)^{nm'+n'm+1} \sin\left(nn'\frac{t}{s}\pi\right) \sin\left(mm'\frac{s}{t}\pi\right). \quad (3.7)$$

The weight 0 modular forms, $\frac{\Psi_P^{(a)}(\tau)}{[\eta(\tau)]^3}$ and $\frac{\Phi_{s,t}^{(n,m)}(\tau)}{\eta(\tau)}$ where $\eta(\tau)$ denotes the Dedekind η-function, are characters of the $A_1^{(1)}$ conformal field theory and the Virasoro algebra $\mathrm{Vir}_{(s,t)} = \mathcal{W}_2(s,t)$ respectively.

3.2 Eichler Integrals

Following [9, 14, 19, 28], we introduce the Eichler integrals of the vector modular forms (3.3) and (3.4) as

$$\widetilde{\Psi}_p^{(a)}(\tau) = \sum_{k=0}^{\infty} \psi_{2p}^{(a)}(k)\, q^{\frac{k^2}{4p}}, \tag{3.8}$$

$$\widetilde{\Phi}_{s,t}^{(n,m)}(\tau) = -\frac{1}{2}\sum_{k=0}^{\infty} k\, \chi_{2st}^{(n,m)}(k)\, q^{\frac{k^2}{4st}}. \tag{3.9}$$

Limiting values when $\tau \downarrow \frac{1}{N}$ for $N \in \mathbb{Z}_{>0}$ were given in [9, 14] as

$$\widetilde{\Psi}_p^{(a)}\left(\tfrac{1}{N}\right) = -\sum_{k=1}^{2pN} \psi_{2p}^{(a)}(k)\, \mathrm{e}^{\frac{k^2}{2pN}\pi\mathrm{i}}\, B_1\left(\tfrac{k}{2pN}\right), \tag{3.10}$$

$$\widetilde{\Phi}_{s,t}^{(n,m)}\left(\tfrac{1}{N}\right) = \frac{stN}{2}\sum_{k=1}^{2stN} \chi_{2st}^{(n,m)}(k)\, \mathrm{e}^{\frac{k^2}{2stN}\pi\mathrm{i}}\, B_2\left(\tfrac{k}{2stN}\right), \tag{3.11}$$

where $B_n(x)$ is the n-th Bernoulli polynomials, $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$. We note that these limiting values were closely related with the Kashaev invariant $\langle T_{2,2p}\rangle_N$ and $\langle T_{s,t}\rangle_N$ respectively [9, 14];

$$\langle T_{2,2p}\rangle_N = -pN\zeta_N^{\frac{3p^2-1}{4p}}\ \widetilde{\Psi}_p^{(p-1)}\left(\tfrac{1}{N}\right), \tag{3.12}$$

$$\langle T_{s,t}\rangle_N = \zeta_N^{\frac{s^2t^2-s^2-t^2}{4st}}\ \widetilde{\Phi}_{s,t}^{(s-1,1)}\left(\tfrac{1}{N}\right). \tag{3.13}$$

These follow from (2.1) and (2.4). See also [10, 12, 13] for a relationship with the WRT invariant for Seifert manifolds.

Asymptotic expansions of the Kashaev invariants $\langle T_{2,2p}\rangle_N$ and $\langle T_{s,t}\rangle_N$ in $N \to \infty$ follow from [9, 14]

$$\widetilde{\Psi}_p^{(a)}\left(\tfrac{1}{N}\right) + \sqrt{\frac{N}{\mathrm{i}}}\sum_{b=1}^{p-1}\sqrt{\frac{2}{p}}\sin\left(\frac{ab}{p}\pi\right)\left(1-\frac{b}{p}\right)\mathrm{e}^{-\frac{b^2}{2p}\pi\mathrm{i}N} \simeq \sum_{k=0}^{\infty}\frac{L\left(-2k, \psi_{2p}^{(a)}\right)}{k!}\left(\frac{\pi\mathrm{i}}{2pN}\right)^k, \tag{3.14}$$

$$\widetilde{\Phi}_{s,t}^{(n,m)}\left(\tfrac{1}{N}\right) + \left(\frac{N}{\mathrm{i}}\right)^{\frac{3}{2}}\sum_{n',m'}{}' S(s,t)_{n,m}^{n',m'}\phi_{s,t}(n',m')\,\mathrm{e}^{-\frac{(n't-m's)^2}{2st}\pi\mathrm{i}N} \simeq \frac{-1}{2}\sum_{k=0}^{\infty}\frac{L\left(-2k-1, \chi_{2st}^{(n,m)}\right)}{k!}\left(\frac{\pi\mathrm{i}}{2stN}\right)^k, \tag{3.15}$$

where

$$\phi_{s,t}(n,m) = \begin{cases}(s-n)m, & nt > ms,\\ n(t-m), & nt < ms.\end{cases} \tag{3.16}$$

These prove the quantum modularity [29] of the Eichler integrals (3.8) and (3.9).

4. Kashaev Invariant of $T_{2s,2t}$

As a family of the N-colored Jones polynomial (2.2) for $T_{2s,2t}$, we define

$$\mathcal{J}_N(q;\, {}^{(n,m)}_{2s,2t}) = \frac{1}{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}\sum_{c=0}^{N-1}\sum_{r=-c}^{c}\left(q^{str^2-(nt+ms)r+\frac{mn}{2}} - q^{str^2+(nt-ms)r-\frac{mn}{2}}\right). \tag{4.1}$$

We have $\mathcal{J}_N(q;\, {}^{(n,m)}_{2s,2t}) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ due to

$$\sum_{k=0}^{N-1}\left(\zeta_N^{\frac{(2stk-(nt+ms))^2}{4st}} - \zeta_N^{\frac{(2stk+(nt-ms))^2}{4st}}\right) = 0. \tag{4.2}$$

One sees that

$$\mathcal{J}_N(q;\, {}^{(1,1)}_{2s,2t}) = q^{-st(1-N^2)}J_N(q; T_{2s,2t}), \tag{4.3}$$

and that the Kashaev invariant (1.2) for $T_{2s,2t}$ is given as

$$\langle T_{2s,2t}\rangle_N = J_N(\zeta_N; T_{2s,2t}) = \zeta_N^{\ st}\mathcal{J}_N(\zeta_N; \tbinom{(1,1)}{2s,2t}). \tag{4.4}$$

We shall confirm the quantum modularity of the Kashaev invariant $\langle T_{2s,2t}\rangle_N$. At the N-th root of unity ζ_N, the Laurent polynomial (4.1) reduces to

$$\begin{aligned}\mathcal{J}_N(\zeta_N; \tbinom{(n,m)}{2s,2t}) &= \frac{1}{N}\sum_{c=0}^{N-1}\sum_{r=-c}^{c} f(r) \\ &= f(0) + \frac{1}{N}\sum_{k=1}^{N-1}\left\{(N-k)\, f(k) + k\, f(k-N)\right\},\end{aligned} \tag{4.5}$$

where for brevity we mean

$$f(r) = \left(str^2-(nt+ms)r+\frac{mn}{2}\right)\zeta_N^{\ str^2-(nt+ms)r+\frac{mn}{2}} - \left(str^2+(nt-ms)r-\frac{mn}{2}\right)\zeta_N^{\ str^2+(nt-ms)r-\frac{mn}{2}}.$$

Then we get

$$\begin{aligned}&\zeta_N^{\frac{(nt)^2+(ms)^2}{4st}}\mathcal{J}_N(\zeta_N; \tbinom{(n,m)}{2s,2t}) \\ &= \sum_{k=0}^{N-1}\left\{\left(stk(N-k)+\frac{mn}{2}\right)\zeta_N^{\frac{(2stk-(nt+ms))^2}{4st}} - \left(stk(N-k)-\frac{mn}{2}\right)\zeta_N^{\frac{(2stk+(nt-ms))^2}{4st}}\right\} \\ &= \sum_{k=0}^{N-1}\Bigg\{\left(-stN^2B_2\left(\tfrac{2stk-(nt+ms)}{2stN}\right)-(nt+ms)N\,B_1\left(\tfrac{2stk-(nt+ms)}{2stN}\right)\right)\zeta_N^{\frac{(2stk-(nt+ms))^2}{4st}} \\ &\qquad - \left(-stN^2B_2\left(\tfrac{2stk+(nt-ms)}{2stN}\right)+(nt-ms)N\,B_1\left(\tfrac{2stk+(nt-ms)}{2stN}\right)\right)\zeta_N^{\frac{(2stk+(nt-ms))^2}{4st}}\Bigg\},\end{aligned}$$

where we have used (4.2) in the last equality. Recalling (3.10) and (3.11), we conclude that the $\mathcal{J}_N(\zeta_N; \tbinom{(n,m)}{2s,2t})$ can be written as a sum of limiting values of the Eichler integrals;

$$\frac{1}{N}\zeta_N^{\frac{(nt)^2+(ms)^2}{4st}}\mathcal{J}_N(\zeta_N; \tbinom{(n,m)}{2s,2t}) = -\widetilde{\Phi}_{s,t}^{(n,m)}\left(\tfrac{1}{N}\right) - \frac{nt-ms}{2}\,\widetilde{\Psi}_{st}^{(nt-ms)}\left(\tfrac{1}{N}\right) + \frac{nt+ms}{2}\,\widetilde{\Psi}_{st}^{(nt+ms)}\left(\tfrac{1}{N}\right). \tag{4.6}$$

As a consequence of (3.14) and (3.15), we obtain the quantum modularity of $\mathcal{J}_N(q; \tbinom{(n,m)}{2s,2t})$.

5. log VOA

In this section, we consider the case of $\mathfrak{sl}_2$ of the lattice VOA. We denote α and ϖ by the simple root and the fundamental weight, respectively. For a VOA or its module M, ch_q means $\mathrm{Tr}_M\, q^{L_0-\frac{c}{24}}$, and $\mathrm{ch}_{q,z} = \mathrm{Tr}_M\, q^{L_0-\frac{c}{24}} z^h$.

5.1 (s,t)-log VOA

Let us consider the lattice VOA $V_{\sqrt{st}Q}$ associated with the rescaled root lattice $\sqrt{st}Q = \sqrt{2st}\mathbb{Z}$. The irreducible modules of $V_{\sqrt{st}Q}$ are given by $V_{n,m}^{+} = V_{\sqrt{st}Q+\alpha_{n,m}}$ and $V_{n,m}^{-} = V_{\sqrt{st}(Q-\varpi)+\alpha_{n,m}}$, where for $1\le n\le s$ and $1\le m\le t$, set

$$\alpha_{n,m} := \frac{-t(n-1)+s(m-1)}{\sqrt{st}}\varpi, \qquad \Delta_{n,m,k} := \frac{(ms-nt+stk)^2}{4st}. \tag{5.1}$$

We note that

$$\Delta_{n,m,k} = \Delta_{-n,-m,-k} = \Delta_{s+n,t+m,k}. \tag{5.2}$$

Let $\mathcal{L}$ be the Virasoro algebra at the central charge $c = 1-6\frac{(s-t)^2}{st}$. The Virasoro VOA $\mathrm{Vir}_{s,t} = U(\mathcal{L})|0\rangle$ is a sub VOA of the Heisenberg VOA $V^{h=0}_{\sqrt{st}Q}$. Then the conformal weight of $e^{\sqrt{st}k\varpi+\alpha_{n,m}}$ is $\Delta_{n,m,k}+\frac{c}{24}$ for the central charge $c = 1-6\frac{(s-t)^2}{st}$. In particular, we have

$$\mathrm{ch}_q\, V^+_{n,m} = \sum_{k\in\mathbb{Z}} \frac{q^{\Delta_{n,m,2k}}}{\eta(\tau)}, \qquad \mathrm{ch}_q\, V^-_{n,m} = \sum_{k\in\mathbb{Z}} \frac{q^{\Delta_{n,m,2k+1}}}{\eta(\tau)}. \tag{5.3}$$

To define the (s,t)-log VOA and its irreducible module, we need the short screening operators

$$\mathcal{Q}^{[n]}_+ \colon V^\pm_{n,m} \to V^\pm_{s-n,m}, \qquad \mathcal{Q}^{[m]}_- \colon V^\pm_{n,m} \to V^\pm_{n,t-m} \tag{5.4}$$

in [27, Definition 3.23]. Then the (s,t)-log VOA $\mathcal{K}^+_{1,1}$ is defined by $\mathcal{K}^+_{1,1} = \ker \mathcal{Q}^{[1]}_+ \cap \ker \mathcal{Q}^{[1]}_-$ [6, 27].

5.2 Characters and Kashaev Invariant

It is known that there are $2st + \frac{1}{2}(s-1)(t-1)$ irreducible modules of the (s,t)-log VOA $\mathcal{K}^+_{1,1}$. The characters of irreducible modules $\mathcal{X}^\pm_{n,m} = \mathrm{im}\, \mathcal{Q}^{[s-n]}_+ \cap \mathrm{im}\, \mathcal{Q}^{[t-m]}_-$ were computed in [6] as

$$\mathrm{ch}_q\, \mathcal{X}^+_{n,m} = \frac{1}{\eta(\tau)} \sum_{k\in\mathbb{Z}} k^2 \left(q^{\frac{(2stk-nt-ms)^2}{4st}} - q^{\frac{(2stk-nt+ms)^2}{4st}} \right), \tag{5.5}$$

$$\mathrm{ch}_q\, \mathcal{X}^-_{n,m} = \frac{1}{\eta(\tau)} \sum_{k\in\mathbb{Z}} k(k+1) \left(q^{\frac{(2stk+st-nt-ms)^2}{4st}} - q^{\frac{(2stk+st-nt+ms)^2}{4st}} \right). \tag{5.6}$$

Amongst others, the singlet character was explicitly written in [3] as

$$\begin{aligned} \eta(\tau)\ \mathrm{ch}_q(\mathcal{X}^+_{n,m})^{h=0} &= \sum_{k\in\mathbb{Z}} |k| \left(q^{\frac{(2stk-nt-ms)^2}{4st}} - q^{\frac{(2stk-nt+ms)^2}{4st}} \right) \\ &= \frac{1}{st}\left(\widetilde{\Phi}^{(n,m)}_{s,t}(\tau) + \frac{nt-ms}{2} \widetilde{\Psi}^{(nt-ms)}_{st}(\tau) - \frac{nt+ms}{2} \widetilde{\Psi}^{(nt+ms)}_{st}(\tau) \right). \end{aligned} \tag{5.7}$$

We point out that, in view of (4.6), the Laurent polynomial (4.1) at the N-th root of unity coincides with a limiting value of the singlet character (5.7) up to multiples.

Theorem 5.1. *The Kashaev invariant $\langle T_{2s,2t}\rangle_N$ is a limiting value of the character of (s,t)-log VOA $\mathrm{ch}_q(\mathcal{X}^+_{1,1})^{h=0}$ (up to the Dedekind η-function).*

Note that in [3] discussed was a relationship between the singlet character and the WRT invariants for 4-fibered Seifert manifolds. See [11] for quantum modularity. See also [8, 21].

5.3 Characters and Tail of the Colored Jones Polynomial

The relationship between the singlet characters (5.7) and the Laurent polynomial (4.1) can also be seen in different manner. As a generalization of (2.3), we have the following correspondence.

Theorem 5.2. *The tail of the N-colored Jones polynomial coincides with the characters of the (s,t)-log VOA;*

$$\lim_{N\to\infty} \left(q^{\frac{(nt)^2+(ms)^2}{4st}-\frac{N}{2}} \mathcal{J}_N(q;\, {}^{(n,m)}_{2s,2t}) - N\, \Phi^{(n,m)}_{s,t}(\tau) \right) = \eta(\tau)\ \mathrm{ch}_q(\mathcal{X}^+_{n,m})^{h=0}. \tag{5.8}$$

Proof. This can be proved as follows. We have

$$
\begin{aligned}
&\left(q^{\frac{N}{2}}-q^{-\frac{N}{2}}\right)\ \mathcal{J}_N(q;\ {}^{(n,m)}_{2s,2t})\\
&=q^{-\frac{m^2s^2+n^2t^2}{4st}}\sum_{c=0}^{N-1}\sum_{r=-c}^{c}\left(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{n,m,-2r}}\right)\\
&=q^{-\frac{m^2s^2+n^2t^2}{4st}}\Bigg(N(q^{\Delta_{s-n,m,1}}-q^{\Delta_{n,m,0}})\\
&\qquad+\sum_{r=1}^{N-1}(N-r)(q^{\Delta_{s-n,m,2r+1}}-q^{\Delta_{n,m,2r}}+q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{n,m,-2r}})\Bigg)\\
&=q^{-\frac{m^2s^2+n^2t^2}{4st}}\Bigg(N(q^{\Delta_{s-n,m,1}}-q^{\Delta_{n,m,0}})\\
&\qquad+\sum_{r=1}^{N-1}(N-r)(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{s-n,t-m,-2r}}-q^{\Delta_{n,m,-2r}}+q^{\Delta_{n,t-m,-2r-1}})\Bigg)\\
&=q^{-\frac{m^2s^2+n^2t^2}{4st}}\left(N(q^{\Delta_{s-n,m,1}}-q^{\Delta_{n,m,0}})+(q;q)_\infty\sum_{r=1}^{N-1}\frac{(N-r)}{r}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1}\right)\\
&=N\left(q^{\frac{mn}{2}}-q^{-\frac{mn}{2}}\right)+q^{-\frac{m^2s^2+n^2t^2}{4st}}\ \eta(\tau)\sum_{r=1}^{N-1}\frac{(N-r)}{r}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1},
\end{aligned}
$$

where the third equality follows from (5.2), and

$$
\operatorname{ch}_q\mathcal{J}_{n,t-m,2k-1}=\frac{1}{\eta(\tau)}k(q^{\Delta_{s-n,m,-2k+1}}-q^{\Delta_{s-n,t-m,-2k}}-q^{\Delta_{n,m,-2k}}+q^{\Delta_{n,t-m,-2k-1}}) \tag{5.9}
$$

is the character of irreducible $L(c_{s,t},0)$-module $\mathcal{J}_{n,t-m,2k-1}$ generated by $e^{-(k-1)\sqrt{st}\alpha+\alpha_{n,m}}$ (see [3, (3.42)]). Then we have

$$
\begin{aligned}
&q^{\frac{m^2s^2+n^2t^2}{4st}-\frac{N}{2}}\left(q^N-1\right)\ \mathcal{J}_N(q;\ {}^{(n,m)}_{2s,2t})\\
=&Nq^{\Delta_{n,m,0}}\left(q^{mn}-1\right)+\eta(\tau)\sum_{r=1}^{N-1}\frac{(N-r)}{r}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1}\\
=&N\left(q^{\Delta_{n,m,0}}(q^{mn}-1)+\eta(\tau)\sum_{r=1}^{N}\frac{1}{r}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1}\right)-\eta(\tau)\sum_{r=1}^{N}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1}\\
=&N\left(q^{\Delta_{n,m,0}}(q^{mn}-1)+\sum_{r=1}^{N}(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{s-n,t-m,-2r}}-q^{\Delta_{n,m,-2r}}+q^{\Delta_{n,t-m,-2r-1}})\right)\\
&\quad-\eta(\tau)\sum_{r=1}^{N}\operatorname{ch}_q\mathcal{J}_{n,t-m,2r-1}
\end{aligned}
$$

and thus we get

$$
\begin{aligned}
&\lim_{N\to\infty}\left(q^{\frac{m^2s^2+n^2t^2}{4st}-\frac{N}{2}}\mathcal{J}_N(q;\ {}^{(n,m)}_{2s,2t})-N\Phi^{(n,m)}_{s,t}(\tau)\right)\\
&\qquad=\lim_{N\to\infty}N\Bigg(-\sum_{r=1}^{N}\left(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{s-n,t-m,-2r}}-q^{\Delta_{n,m,-2r}}+q^{\Delta_{n,t-m,-2r-1}}\right)\\
&\qquad\qquad-q^{\Delta_{n,m,0}}(q^{mn}-1)-\Phi^{(n,m)}_{s,t}(\tau)\Bigg)+\eta(\tau)\operatorname{ch}_q(\mathcal{X}^+_{n,m})^{h=0}.
\end{aligned}
$$

Because

$$
\Phi^{(n,m)}_{s,t}(\tau)=-\sum_{r\geq1}\left(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{s-n,t-m,-2r}}-q^{\Delta_{n,m,-2r}}+q^{\Delta_{n,t-m,-2r-1}}\right)-q^{\Delta_{n,m,0}}(1-q^{mn})
$$

we have

$$\lim_{N\to\infty} N\left(-\sum_{r=1}^{N}\left(q^{\Delta_{s-n,m,-2r+1}}-q^{\Delta_{s-n,t-m,-2r}}-q^{\Delta_{n,m,-2r}}+q^{\Delta_{n,t-m,-2r-1}}\right)\right.$$
$$\left.-q^{\Delta_{n,m,0}}(q^{mn}-1)-\Phi_{s,t}^{(n,m)}(\tau)\right)=0.$$

and (5.8) is proved. □

Our result (5.8) is motivated by [2] where discussed was a relationship between the tail of the colored Jones polynomial for the torus link $T_{2,2t}$ and the singlet $(1,t)$-log VOA. We also note that in [16, 17] the tail of the colored $\mathfrak{sl}_r$ polynomial for $T_{s,t}$ gives the character of $\mathcal{W}_r(s,t)$.

Remark 5.3. *We should note that the characters (5.5) and (5.6) also appear as a tail when we consider a 3-component torus link $T_{3s,3t}$. We can apply the same method with Section 2 to obtain*

$$J_N(q;T_{3s,3t})=\frac{q^{\frac{9}{4}st(1-N^2)}}{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}\sum_{b=0}^{N-1}\sum_{\substack{|2b-N+1|\le c\le 2b+N-1\\ c+N:odd}}\sum_{r=-\frac{c}{2}}^{\frac{c}{2}}\left(q^{str^2-(s+t)r+\frac{1}{2}}-q^{str^2-(s-t)r-\frac{1}{2}}\right). \tag{5.10}$$

When we define a family of Laurent polynomials by

$$\mathcal{J}_N(q;\,{}^{(n,m)}_{3s,3t})=\frac{1}{q^{\frac{N}{2}}-q^{-\frac{N}{2}}}\sum_{b=0}^{N-1}\sum_{\substack{|2b-N+1|\le c\le 2b+N-1\\ c+N:odd}}\sum_{r=-\frac{c}{2}}^{\frac{c}{2}}\left(q^{str^2-(ms+nt)r+\frac{mn}{2}}-q^{str^2-(ms-nt)r-\frac{mn}{2}}\right), \tag{5.11}$$

we get (5.5) and (5.6) by the similar computations

- *for odd N*
$$\lim_{N\to\infty}\left(q^{\frac{(ms)^2+(nt)^2}{4st}-\frac{N}{2}}\mathcal{J}_N(q;\,{}^{(n,m)}_{3s,3t})-\frac{3N^2+1}{4}\Phi_{s,t}^{(n,m)}(\tau)\right)=\eta(\tau)\operatorname{ch}_q\mathcal{X}_{n,m}^{+}, \tag{5.12}$$
- *for even N*
$$\lim_{N\to\infty}\left(q^{\frac{(ms)^2+(nt)^2}{4st}-\frac{N}{2}}\mathcal{J}_N(q;\,{}^{(n,m)}_{3s,3t})+\frac{3N^2}{4}\Phi_{s,t}^{(s-n,m)}(\tau)\right)=\eta(\tau)\operatorname{ch}_q\mathcal{X}_{n,m}^{-}. \tag{5.13}$$

6. log VOA and Atiyah–Bott formula

We explain a geometrical method to calculate the character of the irreducible modules $\mathcal{X}_{n,m}^{\pm}$ of $\mathcal{K}_{1,1}^{+}$ using the Atiyah–Bott formula.

6.1 Geometric construction of $(1,t)$-log VOA and Atiyah–Bott formula

The geometric construction of (s,t)-log VOA for $s=1$ was proposed in [7] and given a rigorous mathematical proof in [25, 26]. That is, the $(1,t)$-log VOA [1] is given by the space of global sections

$$H^0(G\times_B V_{\sqrt{t}Q}) \tag{6.1}$$

of the homogeneous vector bundle $G\times_B V_{\sqrt{t}Q}$ over the flag variety G/B, where Q is the root lattice of G and $V_{\sqrt{t}Q}$ is the lattice VOA associated with the rescaled root lattice $\sqrt{t}Q$. B is the (lower) Borel subgroup of G. Furthermore, for an irreducible module $V_{\sqrt{t}Q+\lambda}$ over $V_{\sqrt{t}Q}$, $H^0(G\times_B V_{\sqrt{t}Q+\lambda})$ is an $H^0(G\times_B V_{\sqrt{t}Q})$-module (where $\lambda=-\sqrt{t}\lambda_0+\lambda_t$ and λ_0 is a minuscule weight). By the main results of [25, 26], for λ such that $(\sqrt{t}\lambda_t+\rho,\theta)\le t$ where ρ and θ are respectively the Weyl vector and highest

[1] In other literature, it is often represented by the symbol $W(t)_Q$ or $W_{\sqrt{t}Q}$.

root, $H^0(G\times_B V_{\sqrt{t}Q+\lambda})$ is irreducible as $H^0(G\times_B V_{\sqrt{t}Q})$-module and $H^k(G\times_B V_{\sqrt{t}Q+\lambda})=0$ for $k>0$. In particular, by using the Atiyah–Bott fixed point formula [1]

$$\sum_{k\geq 0}(-1)^k \operatorname{ch}_{q,z} H^k(G\times_B V)=\sum_{\beta\in P_+}\operatorname{ch}_z L(\beta)\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q V^{h=\sigma\circ\beta}, \tag{6.2}$$

where $\operatorname{ch}_z L(\beta)$ is the Weyl character formula of the irreducible module $L(\beta)$ of $\mathfrak{g}$ with highest weight β, we obtain the character formula

$$\operatorname{ch}_{q,z} H^0(G\times_B V_{\sqrt{t}Q+\lambda})=\sum_{\beta\in P_+}\operatorname{ch}_z L(\beta)\sum_{\sigma\in W}(-1)^{l(\sigma)}\frac{q^{\frac{1}{2}|-\sqrt{t}\sigma(\beta+\rho)+\lambda_t+\frac{1}{\sqrt{t}}\rho|^2}}{\eta(\tau)^{\operatorname{rank}\mathfrak{g}}}. \tag{6.3}$$

The singlet $(1,t)$-log VOA is given by $H^0(G\times_B V_{\sqrt{t}Q})^{h=0}$ and $H^0(G\times_B V_{\sqrt{t}Q+\lambda})^{h=\gamma}$ are its modules. By using the corollary of the Atiyah–Bott fixed point formula

$$\operatorname{ch}_q H^0(G\times_B V)^{h=\gamma}=\sum_{\beta\in P_+}m_{\beta,\gamma}\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q V^{h=\sigma\circ\beta}, \tag{6.4}$$

where $m_{\beta,\gamma}$ is the Kostant multiplicity, the character of $H^0(G\times_B V_{\sqrt{t}Q+\lambda})^{h=\gamma}$ is given by

$$\operatorname{ch}_q H^0(G\times_B V_{\sqrt{t}Q+\lambda})^{h=\gamma}=\sum_{\beta\in P_+}m_{\beta,\gamma}\sum_{\sigma\in W}(-1)^{l(\sigma)}\frac{q^{\frac{1}{2}|-\sqrt{t}\sigma(\beta+\rho)+\lambda_t+\frac{1}{\sqrt{t}}\rho|^2}}{\eta(\tau)^{\operatorname{rank}\mathfrak{g}}}. \tag{6.5}$$

From now on, we consider the case of $\mathfrak{g}=\mathfrak{sl}_2$. In this case, all λ satisfies $(\sqrt{t}\lambda_0+\rho,\theta)\leq t$, and thus $H^0(G\times_B V_{\sqrt{t}Q+\lambda})$ is irreducible and $H^k(G\times_B V_{\sqrt{t}Q+\lambda})=0$ for all λ and $k>0$. To simplify the discussion, we consider the case $V^+_{n,m}=V_{\sqrt{st}Q+\alpha_{n,m}}$ (another case is similar).

Let us check that the character $H^0(G\times_B V_{\sqrt{t}Q})$ coincides with the character of $(1,t)$-log VOA. The irreducible modules of $(1,t)$-log VOA is given by

$$H^0(G\times_B V_{\sqrt{t}Q+\alpha_m})\quad(1\leq m\leq t), \tag{6.6}$$

where $\alpha_m=\frac{m-1}{\sqrt{t}}\varpi$. The character of the irreducible module $V_{\sqrt{t}Q+\alpha_m}$ of the lattice VOA $V_{\sqrt{t}Q}$ is

$$\operatorname{ch}_{q,z} V_{\sqrt{t}Q+\alpha_m}=\sum_{k\in\mathbb{Z}}\operatorname{ch}_{q,z}\pi_{\alpha_m+2k\sqrt{t}\varpi}=\sum_{k\in\mathbb{Z}}z^{(\alpha,\alpha_m+2k\varpi)}\frac{q^{\Delta_{m,2k}}}{\eta(\tau)}, \tag{6.7}$$

where $\Delta_{m,k}=\frac{(m-t+kt)^2}{4t}$ is the conformal weight of $e^{\alpha_{m,k}}\in\pi_{\alpha_m+k\sqrt{t}\varpi}$ plus $\frac{c}{24}$. Note that we have

$$\operatorname{ch}_q(V_{\sqrt{t}Q+\alpha_m})^{h=\sigma_1\circ(2k\varpi)}=\operatorname{ch}_q(V_{\sqrt{t}(Q-\varpi)+\alpha_{t-m}})^{h=(2k+1)\varpi} \tag{6.8}$$

because of $|\beta|^2=|\sigma(\beta)|^2$. Then we obtain the character of $H^0(G\times_B V_{\sqrt{t}Q+\alpha_m})^{h=0}$ as

$$\begin{aligned}
&\operatorname{ch}_q H^0(G\times_B V_{\sqrt{t}Q+\alpha_m})^{h=0}\\
=&\sum_{\beta\in P_+}m_{\beta,0}\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q(V_{\sqrt{t}Q+\alpha_m})^{h=\sigma\circ\beta}\\
=&\sum_{k\geq 0}\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q(V_{\sqrt{t}Q+\alpha_m})^{h=\sigma\circ(2k\varpi)}\\
=&\sum_{k\geq 0}\Big(\operatorname{ch}_q(V_{\sqrt{t}Q+\alpha_m})^{h=2k\varpi}-\operatorname{ch}_q(V_{\sqrt{t}(Q-\varpi)+\alpha_{t-m}})^{h=(2k+1)\varpi)}\Big)\\
=&\frac{1}{\eta(\tau)}\sum_{k\geq 0}\left(q^{\Delta_{m,-2k}}-q^{\Delta_{t-m,-2k-1}}\right),
\end{aligned} \tag{6.9}$$

where the first and third equalities follow from (6.4) and (6.8) respectively. In fact, it coincides with $q^{-\frac{c}{24}}\tilde{\operatorname{ch}}_{W(2,2t-1)}(q)$ in [2, Section 7] for $m=1$.

6.2 Geometric construction of (s,t)-log VOA and Atiyah–Bott formula

For the case of $s \geq 2$, the (s,t)-log VOA and its irreducible modules are constructed and studied algebraically [6, 27], but not yet geometrically. The second author conjectured that the irreducible modules $\mathcal{X}^{\pm}_{n,m}$ over (s,t)-log VOA is given by studying "$H^0(G\times_B H^0(G\times_B V^{\pm}_{n,m}))$". In the following, we propose a method to compute the characters of $\mathcal{X}^{\pm}_{n,m}$ in Section 5.2.

We recall some results on $\mathrm{Vir}_{s,t}$ and $\mathcal{K}^{+}_{1,1}$ following [27]. We fix n and m as $1 \leq n < s$, $1 \leq m < t$ and $k \in \mathbb{Z}$. We denote $L_{n,m,k}$ by the irreducible $\mathrm{Vir}_{s,t}$-module with the lowest conformal weight $\Delta_{n,m,k}$ (5.1). Hereafter we use k, k, k, k as the unique simple quotient given by the irreducible $U(\mathcal{L})$-modules $L_{s-n,m,-k}$, $L_{n,m,-k}$, $L_{s-n,t-m,-k}$, $L_{n,t-m,-k}$, respectively. Then it is known [5, 27] (see also [15]) that the socle sequence of $V^{+}_{n,m}$ as $\mathrm{Vir}_{s,t}$-module is given by Fig. 3, and that the irreducible $\mathcal{K}_{1,1}$-module $\mathcal{X}^{+}_{n,m} = \operatorname{im}\mathcal{Q}^{[s-n]}_{+} \cap \operatorname{im}\mathcal{Q}^{[t-m]}_{-}$ is the $\mathrm{Vir}_{s,t}$-submodule of $V^{+}_{n,m}$ which consists of all k in Figure 3. It was shown [27] that $X_{n,m,+} = \operatorname{im}\mathcal{Q}^{[s-n]}_{+} \subseteq V^{+}_{n,m}$ given in Figure 4 has the B-module structure defined by the Frobenius homomorphism (the H-action is given by $h = -\frac{1}{\sqrt{st}}(\alpha_{(0)} - (\alpha, \alpha_{n,m}))$). Under the B-module structure, $\mathcal{X}^{+}_{n,m}$ is regarded as the maximal G-submodule of $X_{n,m,+}$. In the same manner as the case of $(1,t)$-log VOA [25, Lemma 4.19], the map

$$H^0(G\times_B X_{n,m,+}) \hookrightarrow X_{n,m,+}, \quad f \mapsto f(\mathrm{id}_{G/B}) \tag{6.10}$$

sends $H^0(G\times_B X_{n,m,+})$ to the maximal G-submodule of $X_{n,m,+}$. Therefore we can regard $\mathcal{X}^{+}_{n,m} \simeq H^0(G\times_B X_{n,m,+})$ and set

$$\tilde{H}^0(G\times_B V^{+}_{n,m}) := X_{n,m,+}. \tag{6.11}$$

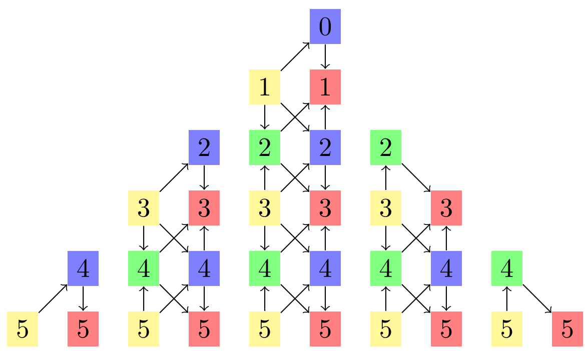

FIGURE 3. The socle sequence of $V^{+}_{n,m}$ (at $h = 4\varpi, 2\varpi, 0, -2\varpi, -4\varpi$). We denote $a \to b$ as $b \in U(\mathcal{L})a$.

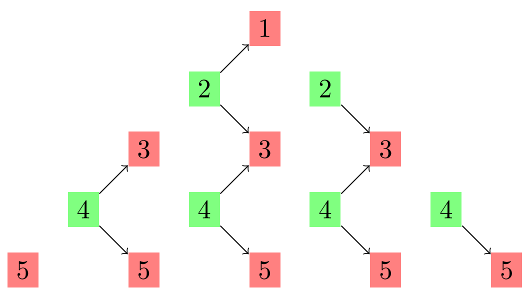

FIGURE 4. The socle sequence of $X_{n,m,+} = \operatorname{im}\mathcal{Q}^{[s-n]}_{+}$ is depicted at $h = 4\varpi, 2\varpi, 0, -2\varpi, -4\varpi$. The maximal G-submodule $\mathcal{X}^{+}_{n,m} = H^0(G\times_B X_{n,m,+})$ consists of all k in $X_{n,m,+}$.

We introduce the $\mathrm{Vir}_{s,t}$-modules $\tilde{V}^{\pm}_{n,m}$ to relate $X_{n,m,+}$ with the form $H^0(G\times_B -)$.

Definition 6.1. *Let $\tilde{V}^{+}_{n,m}$ and $\tilde{V}^{-}_{n,m}$ be the* $\mathrm{Vir}_{s,t}$*-modules given by the socle sequence in Fig. 5 and Fig. 6, respectively. They are B-modules by inclusions and projections. Precisely $\tilde{V}^{\pm}_{n,m}$ are defined as follows.*

- $(\tilde{V}^{+}_{n,m})^{h=k\varpi\geq 0}$ *is the subspace of*
$$(\tilde{V}^{+}_{n,m})^{h=0} := \operatorname{im} Q_{-}^{[m]} \oplus ((V^{+}_{s-n,t-m})^{h=0} / \operatorname{im} Q_{-}^{[m]}) \tag{6.12}$$
such that $(\tilde{V}^{+}_{n,m})^{h=k\varpi\geq 0} \simeq (\,k+1 \longleftarrow k+2 \longrightarrow \cdots) \oplus (\,k \longleftarrow k+1 \longrightarrow \cdots)$.
- $(\tilde{V}^{+}_{n,m})^{h=-k\varpi<0}$ *is the quotient of* $(\tilde{V}^{+}_{n,m})^{h=0}$ *such that* $(\tilde{V}^{+}_{n,m})^{h=-k\varpi<0} \simeq (\,k \longrightarrow k+1 \longleftarrow \cdots) \oplus (\,k-1 \longrightarrow k \longleftarrow \cdots)$.
- $(\tilde{V}^{-}_{n,m})^{h=k\varpi>0}$ *is the subspace of*
$$(\tilde{V}^{-}_{n,m})^{h=-\varpi} := \ker Q_{-}^{[m]} \oplus ((V^{+}_{s-n,m})^{h=0} / \ker Q_{-}^{[m]}) \tag{6.13}$$
such that $(\tilde{V}^{-}_{n,m})^{h=k\varpi} \simeq (\,k \longleftarrow k+1 \longrightarrow \cdots) \oplus (\,k+1 \longleftarrow k+2 \longrightarrow \cdots)$.
- $(\tilde{V}^{-}_{n,m})^{h=k\varpi<0}$ *is the quotient of* $(\tilde{V}^{+}_{n,m})^{h=-\varpi}$ *such that* $(\tilde{V}^{-}_{n,m})^{h=-k\varpi<0} \simeq (\,k-1 \longrightarrow k \longleftarrow \cdots) \oplus (\,k \longrightarrow k+1 \longleftarrow \cdots)$.

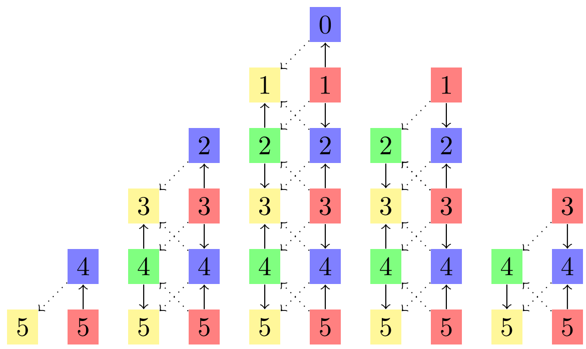

FIGURE 5. The socle sequence of $\tilde{V}^{+}_{n,m}$ (at $h = 4\varpi, 2\varpi, 0, -2\varpi, -4\varpi$). Here $a \cdots\!\!\rightarrow b$ means that $a \to b$ in $(V^{+}_{s-n,t-m})^{h=0}$, but not in $\tilde{V}^{+}_{n,m}$.

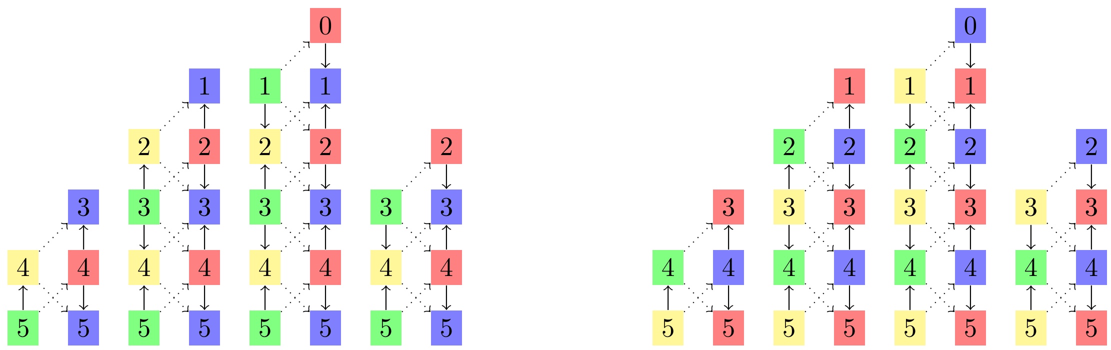

FIGURE 6. The socle sequences of $\tilde{V}^{-}_{n,m}$ (left) and $\tilde{V}^{-}_{s-n,m}$ (right) are depicted at $h = 3\varpi, \varpi, -\varpi, -3\varpi$.

The key observation is that $\tilde{V}^{+}_{n,m}$ in Fig. 5 and $X_{n,m,+}$ in Fig. 4 have the same shape if we regard $(\,k \cdots\!\!\rightarrow k+1)$ and $(\,k \cdots\!\!\rightarrow k+1)$ in Fig. 5 as one component each. This is intended to treat $\tilde{V}^{+}_{n,m}$ and $X_{n,m,+}$ (or $V_{\sqrt{t}Q+\alpha_m}$ in the previous subsection appearing in the $(1,t)$-log VOA setting) as if they were the same. The following proposition is essential for the computation of the character of $\mathcal{X}^{\pm}_{r,s}$.

Proposition 6.2. *The B-modules $\tilde{H}^0(G \times_B V^{+}_{n,m})$ (6.11) and $\tilde{V}^{+}_{n,m}$ in Def. 6.1 satisfy the following (the same results holds if Q is changed to $Q - \varpi$).*

(1) *For $\beta \in \mathbb{Z}_{\geq 0}\varpi$, we have*

$$\operatorname{ch}_q \tilde{H}^0(G\times_B V^+_{n,m})^{h=\sigma_1\circ\beta} = \operatorname{ch}_q H^0(G\times_B \tilde{V}^-_{n,t-m})^{h=\beta+\varpi}, \tag{6.14}$$

$$\operatorname{ch}_q(\tilde{V}^+_{n,m})^{h=\sigma_1\circ\beta} = \operatorname{ch}_q(\tilde{V}^-_{s-n,m})^{h=\beta+\varpi}. \tag{6.15}$$

(2) *For $k>0$ and $\beta \in \mathbb{Z}_{\geq 0}\varpi$, we have*

$$\operatorname{ch}_q \tilde{H}^0(G\times_B V^+_{n,m})^{h=\beta} = \operatorname{ch}_q H^0(G\times_B \tilde{V}^-_{s-n,m})^{h=\beta+\varpi}, \tag{6.16}$$

$$\operatorname{ch}_q(\tilde{V}^+_{n,m})^{h=k\varpi} = \operatorname{ch}_q(V^+_{n,m})^{h=k\varpi}. \tag{6.17}$$

(3) *We have the cohomology vanishings*

$$H^1(G\times_B \tilde{H}^0(G\times_B V^+_{n,m})) = H^1(G\times_B \tilde{V}^+_{n,m}) = 0. \tag{6.18}$$

Proof. (1) By comparing Fig. 5 with Fig. 6, we obtain (6.15). In the same manner, (6.14) is also proved.

(2) By Fig. 3 and Fig. 5, we have (6.17). The socle sequence of $H^0(G\times_B \tilde{V}^-_{s-n,m})$ (namely, maximal G-submodule of $\tilde{V}^-_{s-n,m}$) is given by Fig. 7. Then we obtain (6.16) by comparing with Fig. 4.

(3) The cohomology vanishing (6.18) follows in the same manner as the case of $(1,t)$-log VOA [25, Lemma 4.10].

□

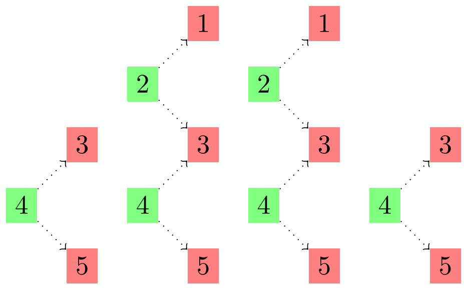

FIGURE 7. The socle sequence of $H^0(G\times_B \tilde{V}^-_{s-n,m})$ is given at $h = 3\varpi, \varpi, -\varpi, -3\varpi$.

Theorem 6.3.

$$\operatorname{ch}_q(\mathcal{X}^{\pm}_{n,m})^{h=0} = H^0(G\times_B \tilde{H}^0(G\times_B V^{\pm}_{n,m}))^{h=0}. \tag{6.19}$$

Proof. As a consequence of Prop. 6.2, we can apply the Atiyah–Bott formula (6.4). The proof is as follows.

$$\begin{aligned}
&\operatorname{ch}_q H^0(G\times_B \tilde{H}^0(G\times_B V^+_{n,m}))^{h=0}\\
\overset{(6.4)}{=}& \sum_{\beta\in P_+} m_{\beta,0} \sum_{\sigma\in W} (-1)^{l(\sigma)} \operatorname{ch}_q \tilde{H}^0(G\times_B V^+_{n,m})^{h=\sigma\circ\beta}\\
=& \sum_{k\geq 0}\sum_{\sigma\in W} (-1)^{l(\sigma)} \operatorname{ch}_q \tilde{H}^0(G\times_B V^+_{n,m})^{h=\sigma\circ 2k\varpi}\\
\overset{(6.14)}{=}& \sum_{k\geq 0}\left(\operatorname{ch}_q \tilde{H}^0(G\times_B V^+_{n,m})^{h=2k\varpi} - \operatorname{ch}_q \tilde{H}^0(G\times_B V^-_{n,t-m})^{h=(2k+1)\varpi}\right)\\
\overset{(6.16)}{=}& \sum_{k\geq 0}\left(\operatorname{ch}_q H^0(G\times_B \tilde{V}^-_{s-n,m})^{h=(2k+1)\varpi} - \operatorname{ch}_q H^0(G\times_B \tilde{V}^+_{s-n,t-m})^{h=(2k+2)\varpi}\right)\\
\overset{(6.4)}{=}& \sum_{k\geq 0}\left(\sum_{\beta\in P_+} m_{\beta,(2k+1)\varpi} \sum_{\sigma\in W} (-1)^{l(\sigma)} \operatorname{ch}_q(\tilde{V}^-_{s-n,m})^{h=\sigma\circ\beta}\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{\beta\in P_+} m_{\beta,(2k+2)\varpi} \sum_{\sigma\in W} (-1)^{l(\sigma)} \operatorname{ch}_q(\tilde V^+_{s-n,t-m})^{h=\sigma\circ\beta}\Bigg)\\
&= \sum_{k\geq 0}\Bigg(\sum_{k'\geq 0}\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q(\tilde V^-_{s-n,m})^{h=\sigma\circ(2k+2k'+1)\varpi} - \sum_{k'\geq 0}\sum_{\sigma\in W}(-1)^{l(\sigma)}\operatorname{ch}_q(\tilde V^+_{s-n,t-m})^{h=\sigma\circ(2k+2k'+2)\varpi}\Bigg)\\
&\overset{(6.15)}{=} \sum_{k,k'\geq 0}\Big(\Big(\operatorname{ch}_q(\tilde V^-_{s-n,m})^{h=(2k+2k'+1)\varpi} - \operatorname{ch}_q(\tilde V^+_{n,m})^{h=(2k+2k'+2)\varpi}\Big)\\
&\qquad - \Big(\operatorname{ch}_q(\tilde V^+_{s-n,t-m})^{h=(2k+2k'+2)\varpi} - \operatorname{ch}_q(\tilde V^-_{n,t-m})^{h=(2k+2k'+3)\varpi}\Big)\Big)\\
&\overset{(6.17)}{=} \sum_{k,k'\geq 0}\Big(\Big(\operatorname{ch}_q(\tilde V^-_{s-n,m})^{h=(2k+2k'+1)\varpi} - \operatorname{ch}_q(\tilde V^+_{n,m})^{h=(2k+2k'+2)\varpi}\Big)\\
&\qquad - \Big(\operatorname{ch}_q(\tilde V^+_{s-n,t-m})^{h=(2k+2k'+2)\varpi} - \operatorname{ch}_q(\tilde V^-_{n,t-m})^{h=(2k+2k'+3)\varpi}\Big)\Big)\\
&= \frac{1}{\eta(\tau)}\sum_{k,k'\geq 0}\left(q^{\Delta_{s-n,m,-2k-2k'-1}} - q^{\Delta_{s-n,t-m,-2k-2k'-2}} - q^{\Delta_{n,m,-2k-2k'-2}} + q^{\Delta_{n,t-m,-2k-2k'-3}}\right).
\end{aligned}
$$

By (5.2), this coincides with $\operatorname{ch}_q(\mathcal{X}^+_{n,m})^{h=0}$ (see [3, (3.43),(3.52)]).

We can compute the character $\operatorname{ch}_q(\mathcal{X}^-_{n,m})^{h=0}$ in the same manner. □

Remark 6.4. *The characters* (5.5) *and* (5.6) *are given by*

$$
\operatorname{ch}_q(\mathcal{X}^\pm_{n,m}) = \operatorname{ch}_q H^0(G\times_B \tilde H^0(G\times_B V^\pm_{n,m})), \tag{6.20}
$$

which follow from

$$
\begin{aligned}
&\operatorname{ch}_{q,z} H^0(G\times_B \tilde H^0(G\times_B V^+_{n,m}))\\
=&\frac{1}{\eta(\tau)}\sum_{k,k'\geq 0}\operatorname{ch}_z L(2k)\left(q^{\Delta_{s-n,m,-2k-2k'-1}} - q^{\Delta_{s-n,t-m,-2k-2k'-2}} - q^{\Delta_{n,m,-2k-2k'-2}} + q^{\Delta_{n,t-m,-2k-2k'-3}}\right).
\end{aligned}\tag{6.21}
$$

We have obtained the character by introducing $(\tilde V^\pm_{n,m})^{h=k\varpi}$ in Definition 6.1. Another method would be to disregard the commutativity of the B-action with the Virasoro action except for the conformal grading instead of taking $(\tilde V^\pm_{n,m})^{h=k\varpi}$ as a Fock space. There remains for a future work to introduce screening operators that define such B-action. We hope to report on a geometrical construction of the characters of (s,t)-log VOA for $\mathfrak{g}$, and on a relationship with $\mathfrak{g}$-quantum invariant for torus link $T_{rs,rt}$ as a generalization of [17].

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