Refined Decay Estimate and Analyticity of Solutions to the Linear Heat Equation in Homogeneous Besov Spaces

Ozawa, Tohru
Department of Applied Physics, Waseda University

Takeuchi, Taiki
Faculty of Science and Engineering,Waseda University

https://hdl.handle.net/2324/7329951

出版情報：Journal of Fourier Analysis and Applications. 29 (5), 2023-09-22. Springer
バージョン：

Journal of Fourier Analysis and Applications (2023) 29:61
https://doi.org/10.1007/s00041-023-10042-2

Refined Decay Estimate and Analyticity of Solutions to the Linear Heat Equation in Homogeneous Besov Spaces

Tohru Ozawa[1] · Taiki Takeuchi[2]

Received: 30 June 2022 / Revised: 4 June 2023 / Accepted: 17 August 2023

Abstract

The heat semigroup $\{T(t)\}_{t\geq 0}$ defined on homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is considered. We show the decay estimate of $T(t)f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ with an explicit bound depending only on the regularity index $\sigma > 0$ and space dimension n. It may be regarded as a refined result compared with that of the second author (Takeuchi in Partial Differ Equ Appl Math **4**:100174, 2021). As a result of the refined decay estimate, we also improve a lower bound estimate of the radius of convergence of the Taylor expansion of $T(\cdot)f$ in space and time. To refine the previous results, we show explicit pointwise estimates of higher order derivatives of the power function $|\xi|^\sigma$ for $\sigma \in \mathbb{R}$. In addition, we also refine the L^1-estimate of the derivatives of the heat kernel.

Keywords Heat semigroup · Decay estimate · Analyticity · Homogeneous Besov spaces

Mathematics Subject Classification Primary; 35A23, Secondary; 35A20 · 35B65 · 35K05 · 42B25 · 42B37

Communicated by Winfried Sickel.

✉ Taiki Takeuchi
kim28-egtea@fuji.waseda.jp

Tohru Ozawa
txozawa@waseda.jp

1 Department of Applied Physics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

2 Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Published online: 22 September 2023

1 Introduction

Consider the heat equation $\partial_t u = \Delta u$ in $(0, \infty) \times \mathbb{R}^n$ with initial condition $u(0) = f$ in $\mathbb{R}^n$, $n \geq 1$. Then, the solution u is given by $u(t) = T(t)f$ at least formally, where $\{T(t)\}_{t\geq 0}$ is the heat semigroup defined by $T(t) := \mathcal{F}^{-1} e^{-t|\xi|^2} \mathcal{F}$ for $t \geq 0$. Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively.

In this paper, we consider the function $u(t) := T(t)f$ in the case where f belongs to the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Our purpose is to show the decay estimate of $u(t) = T(t)f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ with an explicit bound depending only on the regularity index $\sigma > 0$ and space dimension n. In addition, we obtain an explicit lower bound of the radius of convergence of the Taylor expansion of $u(\cdot) = T(\cdot)f$ in space and time. These results may be regarded as an improved version compared with [23].

To refine the decay estimate of $\{T(t)\}_{t\geq 0}$, we show explicit pointwise estimates of higher order derivatives of the power function $|\xi|^\sigma$ for $\sigma \in \mathbb{R}$, where notice that $|\xi|^\sigma$ is the symbol of the fractional Laplacian $(-\Delta)^{\sigma/2} = \mathcal{F}^{-1}|\xi|^\sigma \mathcal{F}$. The pointwise estimates of $|\xi|^\sigma$ play an important role in giving an explicit bound appearing in the decay estimate of $\{T(t)\}_{t\geq 0}$. Moreover, we also refine the L^1-estimate of the derivatives of the heat kernel in [23, Proposition 5.4]. This improvement allows us to enlarge a lower bound estimate of the radius of convergence of the Taylor expansion.

Before stating our main results, let us define some function spaces. Take $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \leq |\xi| \leq 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$, and $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space. Then, for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ are defined by

$$
\begin{aligned}
\dot{B}^s_{p,q}(\mathbb{R}^n) &:= \left\{ f \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \;\middle|\; \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty \right\}, \\
\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} &:= \|\{2^{sj} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})},
\end{aligned}
$$

where $\mathscr{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{P}(\mathbb{R}^n)$ is the subspace of $\mathscr{S}'(\mathbb{R}^n)$ consisting of polynomials with complex coefficients, and $\{\varphi_j\}_{j\in\mathbb{Z}}$ is defined by $\varphi_j := \mathcal{F}^{-1}[\varphi(2^{-j}\xi)]$ for $j \in \mathbb{Z}$. In addition, we also define the Sobolev spaces $H^m(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \mid \|f\|_{H^m(\mathbb{R}^n)} := \sum_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} < \infty\}$ for $m \in \mathbb{N}$.

Our first result is the decay estimate of solutions to the heat equation with an explicit bound. In what follows, let Γ denote the gamma function and let $[a]$ denote an integer such that $[a] \leq a < [a] + 1$ for $a \in \mathbb{R}$. We also set constants $M_{\sigma,n}$ and K_n as follows:

$$
\begin{aligned}
M_{\sigma,n} &:= \max\left\{(\sigma/2)^{\sigma/2}, (25/2)^{n/8}(1/\sigma)\right\} \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^6}{\Gamma(n/2)} \right\}^{\sigma/2} (2\sigma+1)^{3n/8} K_n, \\
K_n &:= \frac{5 \cdot 2^{5/2}}{\log 2} \left\{ \frac{(2/\pi)^{n/2}[n/2+1]^2}{\Gamma(n/2+1)} \right\}^{3/8} \left\{ \frac{2([n/2+1]!)^3}{n} \right\}^{3n/(8n+8)} \|\varphi\|^{3/4}_{H^{[n/2+1]}(\mathbb{R}^n)}.
\end{aligned}
$$

Theorem 1.1 *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$, it holds that $T(t)f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for all $\sigma > 0$ and $t > 0$ with the estimate*

$$\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)} \leq M_{\sigma,n} t^{-\sigma/2} \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)}.$$

Remark 1.2 (i) The decay estimate of $T(t)f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ was first shown by Kozono-Ogawa-Taniuchi [16, Lemma 2.2]. However, it is unknown the dependence of a bound on n, p, s, and σ in [16, Lemma 2.2]. Then, the second author [23, Lemma 5.1] refined the decay estimate by considering the dependence on the regularity index σ. In contrast to these results, we give an explicit bound depending only on σ and n. Moreover, we also refined the increasing rate of σ as compared with [23, Lemma 5.1].

(ii) Concerning the increasing rate of σ for fixed n, we have

$$M_{\sigma,n} = O(\sigma^{-1}) \quad \text{as } \sigma \to +0, \quad M_{\sigma,n} = O(\sigma^{\sigma/2} C_n^\sigma \sigma^{3n/8}) \quad \text{as } \sigma \to \infty,$$

where $C_n := 2^{5/2}(n/(2e))^{n/4}\{\Gamma(n/2)\}^{-1/2}$. In addition, since the Stirling formula yields $\Gamma(n/2+1) \sim (4\pi/n)^{1/2}(n/(2e))^{n/2+1}$ and $[n/2+1]! \leq \Gamma(n/2+2) \sim (4\pi/n)^{1/2}(n/(2e))^{n/2+2}$, we obtain

$$\begin{aligned}
K_n &\sim \left\{ \frac{(2/\pi)^{n/2} n^2}{n^{-1/2}(n/(2e))^{n/2+1}} \cdot \frac{n^{-3/2}(n/(2e))^{3n/2+6}}{n} \right\}^{3/8} \|\varphi\|^{3/4}_{H^{[n/2+1]}(\mathbb{R}^n)} \\
&\sim \left\{ (2/\pi)^{n/2} (n/(2e))^{n+5} \right\}^{3/8} \|\varphi\|^{3/4}_{H^{[n/2+1]}(\mathbb{R}^n)} \\
&\sim n^{3n/8} (2\pi e^2)^{-3n/16} n^{15/8} \|\varphi\|^{3/4}_{H^{[n/2+1]}(\mathbb{R}^n)}.
\end{aligned}$$

By using the Stirling formula again, we have

$$(n/(2e))^{n/2}\{\Gamma(n/2)\}^{-1} \sim (n/(4\pi))^{1/2}, \tag{1.1}$$

which implies that the increasing rate of n for fixed σ is given by

$$M_{\sigma,n} = O(n^{3n/8} C_\sigma^n n^{(2\sigma+15)/8} \|\varphi\|^{3/4}_{H^{[n/2+1]}(\mathbb{R}^n)}) \quad \text{as } n \to \infty,$$

where $C_\sigma := (25/2)^{1/8}(2\pi e^2)^{-3/16}(2\sigma+1)^{3/8}$.

(iii) The value $[n/2+1]$ appearing in K_n comes from the condition of $m \in \mathbb{N}$ such that $\int_{|x|>1} |x|^{-2m} dx < \infty$, namely, $2m \geq n+1$. This condition is used for the estimate of the Fourier multipliers. In addition, the powers appearing in K_n are related to $3/4$, which comes from the interpolation inequality on $\dot{B}^s_{p,q}(\mathbb{R}^n)$. However, the value $3/4$ is not essential in the estimate.

As a result of Theorem 1.1, we may show the two results on the analyticity of the bounded linear operator $T(\cdot)$ and the complex-valued function $T(\cdot)f$, respectively. In addition, we also give an explicit lower bound of the radius of convergence of the Taylor expansion. We first state the result on the analyticity of $T(\cdot)$. In the following, let $\mathcal{L}(\dot{B}^s_{p,\infty}(\mathbb{R}^n), \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n))$ denote the Banach space of all bounded linear operators from $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ onto $\dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$.

Theorem 1.3 *Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Then the operator $T(\cdot)$ is real analytic in $(0,\infty)$ as an $\mathcal{L}(\dot{B}^s_{p,\infty}(\mathbb{R}^n), \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n))$-valued function for all $\sigma > 0$. In addition, it holds that*

$$T(t) = \sum_{j=0}^{\infty} \frac{T^{(j)}(t_0)}{j!}(t-t_0)^j$$

in $\mathcal{L}(\dot{B}^s_{p,\infty}(\mathbb{R}^n), \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n))$ for all $t_0, t > 0$ satisfying

$$|t-t_0| < \left(\frac{2e}{n}\right)^{n/2} \frac{\Gamma(n/2)t_0}{2^{11}e}. \tag{1.2}$$

Next, we shall state the result on the analyticity of $T(\cdot)f$, namely, the space-time analyticity of solutions to the heat equation. Before doing so, we should mention the problem which the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ have. By recalling the definition of $\dot{B}^s_{p,q}(\mathbb{R}^n)$, we see that the function $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ is regarded as an element of the quotient space $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$, so it does not make sense to consider the value of $f(x) \in \mathbb{C}$ at $x \in \mathbb{R}^n$. However, we may overcome this problem by introducing a suitable realization. More precisely, we make use of the result given by Bourdaud [2, Theorem 4.1 (1)]. Then we have the following consequence:

Proposition 1.4 *Let $1 \le p \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$. Suppose that $s - n/p \notin \mathbb{N}$. Then there exists a unique continuous linear operator $\Phi : \dot{B}^s_{p,q}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ such that Φ has the following properties:*

(i) *It holds that $\{\Phi f + P \mid P \in \mathscr{P}(\mathbb{R}^n)\} = f$ for all $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$.*
(ii) *It holds that $(\Phi f)(\lambda\cdot) = \Phi(f(\lambda\cdot))$ for all $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\lambda > 0$.*

By regarding the element $\Phi f \in \mathscr{S}'(\mathbb{R}^n)$ as a function of $\dot{B}^s_{p,q}(\mathbb{R}^n)$, we may consider the value of $f(x) \in \mathbb{C}$. Therefore, we introduce the function space $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n) := \Phi\dot{B}^s_{p,q}(\mathbb{R}^n) = \{\Phi f \in \mathscr{S}'(\mathbb{R}^n) \mid f \in \dot{B}^s_{p,q}(\mathbb{R}^n)\}$ under the condition $s - n/p \notin \mathbb{N}$. Then for every $f \in \dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)$, we may take $g \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ such that $f = \Phi g$, so we may regard $\|f\|_{\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)} := \|g\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$. By using the notation $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^n)$, we may state our main result on the space-time analyticity of solutions to the heat equation.

Theorem 1.5 *Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Suppose that $s - n/p \notin \mathbb{N}$. Then for every $f \in \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n)$, the function $u(\cdot) := T(\cdot)f$ is real analytic in $(0,\infty) \times \mathbb{R}^n$. In addition, it holds that*

$$u(t,x) = \sum_{j,k=0}^{\infty} \sum_{|\alpha|=k} \frac{\partial_t^j \partial_x^\alpha u(t_0,x_0)}{\alpha! j!}(t-t_0)^j (x-x_0)^\alpha \tag{1.3}$$

for all $(t_0,x_0), (t,x) \in (0,\infty) \times \mathbb{R}^n$ satisfying (1.2).

Remark 1.6 (i) Concerning Theorem 1.5, the condition (1.2) states that the radius of convergence is independent of the spatial variables $x_0, x \in \mathbb{R}^n$. On the other hand, noting that (1.1), we have

$$\left(\frac{2e}{n}\right)^{n/2} \frac{\Gamma(n/2)t_0}{2^{11}e} \sim \frac{\pi^{1/2}t_0}{2^{10}en^{1/2}}.$$

Here, compared with the radius $\pi t_0/(2^{12}en) = O(n^{-1})$ of convergence in the preceding result [23, Theorem 3.3], we see that the decay rate as $n \to \infty$ is improved to $O(n^{-1/2})$. We also note that the corresponding ratio is $\pi^{1/2}t_0/(2^{10}en^{1/2}) \cdot 2^{12}en/(\pi t_0) = 4(n/\pi)^{1/2}$.

(ii) The second author showed that $\{T(t)\}_{t\geq 0}$ is a bounded analytic C_0-semigroup on $\dot{B}^s_{p,q}(\mathbb{R}^n)$, provided $1 \leq p, q < \infty$ [23, Theorem 3.6]. Namely, $T(\cdot)$ can be extended analytically on some sector Σ in the complex plane $\mathbb{C}$. By Theorem 1.3, we also see that $T(\cdot)$ can be extended analytically on

$$\Sigma := \left\{ t \in \mathbb{C} \setminus \{0\} \;\middle|\; |\arg t| < \sin^{-1}\left(\left(\frac{2e}{n}\right)^{n/2} \frac{\Gamma(n/2)}{2^{11}e} \right) \right\}$$

in a similar manner to the proof of [23, Theorem 3.6].

Remark 1.7 (i) In this paper, we use the definition of the homogeneous Besov spaces which is based on the sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$. On the other hand, we may define the homogeneous Besov spaces in another way instead of using $\{\varphi_j\}_{j\in\mathbb{Z}}$; By taking a suitable non-negative function $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and setting $\varphi_\lambda := \mathcal{F}^{-1}[\varphi(\lambda\xi)]$ for $\lambda > 0$, we may introduce the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \left\{ \int_0^\infty (\lambda^{-s}\|\varphi_\lambda * f\|_{L^p(\mathbb{R}^n)})^q \frac{d\lambda}{\lambda} \right\}^{1/q},$$

which is equivalent to that of our definition. Moreover, this norm has the property such that $\|f(\lambda\cdot)\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \lambda^{s-n/p}\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ for all $\lambda > 0$, so we may expect that the computations in this paper will become simpler. However, since our results focus on giving an explicit bound, we have to pay attention to using another definition of the function spaces, i.e., another norm. In particular, to compare with the previous result of the second author [23], we use the same definition as in [23].

(ii) For recent results on the analyticity of solutions to the heat equation, Dong-Zhang [5] and Zhang [25] showed the time analyticity in the case of complete non-compact Riemannian manifolds $\mathbf{M}$. Han-Hua-Wang [8] also considered the case of weighted graphs G. In addition, Zeng [24] showed the time analyticity of solutions to the biharmonic heat equation and the heat equation with potentials in the case of $\mathbf{M}$. Concerning the nonlinear problems, there are recent contributions on the analyticity of solutions to the Navier-Stokes system [1, 4, 17], Schrödinger equation [9–11, 19, 20], Burgers equation [12, 13], and primitive equations [6].

This paper is organized as follows: We give some notations and key estimates in the next section. In Sect. 3, we show the explicit estimates of the differential operators $(-\Delta)^{\sigma/2}$ and ∂_x^α on $\dot{B}^s_{p,q}(\mathbb{R}^n)$. Moreover, we prove Theorem 1.1, i.e., the decay estimate of $T(t)f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ with an explicit bound depending only on the regularity index $\sigma > 0$ and space dimension n. Section 4 is devoted to the estimate of a lower bound of the radius of convergence (1.2) of the Taylor expansion, namely, the proof of Theorems 1.3 and 1.5. In Appendix, we show explicit pointwise estimates of higher order derivatives of the power function $|\xi|^\sigma$ for $\sigma \in \mathbb{R}$.

2 Preliminaries

In this section, let us give some notations used throughout this paper. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. Then, for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$, and $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, where $\binom{\alpha_j}{\beta_j}$ denotes a binomial coefficient. We also set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $[a]$ denote an integer such that $[a] \le a < [a] + 1$ for $a \in \mathbb{R}$. The surface area of the unit ball in $\mathbb{R}^n$ is denoted by $\omega_{n-1} := 2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$, where Γ is the gamma function, i.e., $\Gamma(a) := \int_0^\infty \lambda^{a-1} e^{-\lambda} d\lambda$ with $a > 0$. Concerning the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$, we use the following definitions

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(\xi)d\xi$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. It is well-known that $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to the operators from $\mathscr{S}'(\mathbb{R}^n)$ onto itself. Moreover, let $\mathscr{S}_\infty(\mathbb{R}^n) := \{f \in \mathscr{S}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} f(x)g(x)dx = 0 \text{ for all } g \in \mathscr{P}(\mathbb{R}^n)\}$ and let $(-\Delta)^{\sigma/2} : \mathscr{S}_\infty(\mathbb{R}^n) \to \mathscr{S}_\infty(\mathbb{R}^n)$, $(-\Delta)^{\sigma/2} := \mathcal{F}^{-1}|\xi|^\sigma \mathcal{F}$ for $\sigma \in \mathbb{R}$. Then $(-\Delta)^{\sigma/2}$ is well-defined. Noting that $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ and $\mathscr{S}'_\infty(\mathbb{R}^n)$ are homeomorphic [22, Theorem 2.25], we see that $(-\Delta)^{\sigma/2}$ can be extended to the operator from $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$ onto itself. In what follows, the family $\{T(t)\}_{t\ge 0}$ of the operators denotes the heat semigroup, i.e., $T(t) := \mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}$ for $t \ge 0$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ denote the function which is taken in the definition of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ unless explicitly stated otherwise. In addition, for a function space $X(\mathbb{R}^n)$ defined on $\mathbb{R}^n$, we abbreviate $X := X(\mathbb{R}^n)$.

Next, we state explicit pointwise estimates of higher order derivatives of the power function $|\xi|^\sigma$ for $\sigma \in \mathbb{R}$.

Proposition 2.1 *Let $\sigma \in \mathbb{R}$. Then it holds that*

$$|\partial_\xi^\gamma |\xi|^\sigma| \le (\max\{2, |\sigma|\} + 1)^{|\gamma|}|\gamma|!|\xi|^{\sigma-|\gamma|}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{N}_0^n$.

The proof of Proposition 2.1 is given in Appendix. By using Proposition 2.1, we may show the following estimates, which play an important role in giving an explicit bound in Theorem 1.1:

Proposition 2.2 *Let $\sigma \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$, and $m \in \mathbb{N}$. Then it holds that*

$$\max_{|\gamma| \le m,\, 1/2 \le |\xi| \le 2} |\partial_\xi^\gamma |\xi|^\sigma| \le 2^{|\sigma - m|}(\max\{2, |\sigma|\} + 1)^m m!, \tag{2.1}$$

$$\max_{|\gamma| \le m,\, 1/2 \le |\xi| \le 2} |\partial_\xi^\gamma \xi^\alpha| \le 2^{|\alpha| - m}(\max\{2, |\alpha|\})^m. \tag{2.2}$$

Proof In the case $m \le \sigma$, we have $|\gamma| \le \sigma$ for all $\gamma \in \mathbb{N}_0^n$ satisfying $|\gamma| \le m$, which gives that $\max_{1/2 \le |\xi| \le 2} |\xi|^{\sigma - |\gamma|} = 2^{\sigma - |\gamma|}$. Thus we see by Proposition 2.1 that

$$\begin{aligned}
\max_{|\gamma| \le m,\, 1/2 \le |\xi| \le 2} |\partial_\xi^\gamma |\xi|^\sigma| &\le \max_{|\gamma| \le m} (\max\{2, |\sigma|\} + 1)^{|\gamma|} m! \cdot 2^{\sigma - |\gamma|} \\
&= \max_{|\gamma| \le m} 2^\sigma (\max\{1, |\sigma|/2\} + 1/2)^{|\gamma|} m! \\
&= 2^\sigma (\max\{1, |\sigma|/2\} + 1/2)^m m! \\
&= 2^{\sigma - m} (\max\{2, |\sigma|\} + 1)^m m!.
\end{aligned}$$

In the case $\sigma \le m$, we have $\max_{1/2 \le |\xi| \le 2} |\xi|^{\sigma - |\gamma|} = 2^{\sigma - |\gamma|}$ for all $\gamma \in \mathbb{N}_0^n$ satisfying $|\gamma| \le \sigma$. Thus we obtain

$$\max_{1/2 \le |\xi| \le 2} |\partial_\xi^\gamma |\xi|^\sigma| \le 2^{\sigma - m}(\max\{2, |\sigma|\} + 1)^m m!$$

in the same way as above. If $\sigma \le |\gamma| \le m$, then it holds that $\max_{1/2 \le |\xi| \le 2} |\xi|^{\sigma - |\gamma|} = 2^{|\gamma| - \sigma} \le 2^{m - \sigma}$, which yields

$$\max_{1/2 \le |\xi| \le 2} |\partial_\xi^\gamma |\xi|^\sigma| \le 2^{m - \sigma}(\max\{2, |\sigma|\} + 1)^m m!.$$

Hence we obtain (2.1). Concerning the estimate (2.2), assume that $\gamma \le \alpha$. Then we see that $\partial_\xi^\gamma \xi^\alpha = \binom{\alpha}{\gamma} \xi^{\alpha - \gamma}$ for all $\xi \in \mathbb{R}^n$. Since it holds that

$$\binom{k}{l} = \prod_{j=1}^{l} \frac{k - j + 1}{j} \le \prod_{j=1}^{l} \frac{k}{j} \le k^l$$

for all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ satisfying $l \le k$, we have $|\partial_\xi^\gamma \xi^\alpha| = \binom{\alpha}{\gamma} |\xi^{\alpha - \gamma}| \le |\xi|^{|\alpha| - |\gamma|}(\max\{1, |\alpha|\})^{|\gamma|}$, which implies that

$$\begin{aligned}
\max_{1/2 \le |\xi| \le 2} |\partial_\xi^\gamma \xi^\alpha| &\le 2^{|\alpha| - |\gamma|}(\max\{2, |\alpha|\})^{|\gamma|} = 2^{|\alpha|}(\max\{1, |\alpha|/2\})^{|\gamma|} \\
&\le 2^{|\alpha|}(\max\{1, |\alpha|/2\})^m = 2^{|\alpha| - m}(\max\{2, |\alpha|\})^m,
\end{aligned}$$

provided $|\gamma| \le m$. In addition, if $\alpha < \gamma$, then we have $\partial_\xi^\gamma \xi^\alpha = 0$, which yields (2.2). This completes the proof of Proposition 2.2. □

Finally, we give the L^1-estimate of the derivatives of the heat kernel $G_t(x) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ for $(t, x) \in (0, \infty) \times \mathbb{R}^n$. The following proposition is regarded as an improved estimate in [23, Proposition 5.4]:

Proposition 2.3 *It holds that*

$$\|(-\Delta)^N G_t\|_{L^1} \le \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2N}{\Gamma(n/2)t} \right\}^N$$

for all $N \in \mathbb{N}$ *and* $t > 0$.

Proof Since $\Delta G_t = \partial_t G_t = (4\pi t)^{-n/2} t^{-1} \{|x|^2/(4t) - n/2\} e^{-|x|^2/(4t)}$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$, we have

$$\begin{aligned}\|\Delta G_t\|_{L^1} &= \frac{1}{(4\pi t)^{n/2} t} \int_{\mathbb{R}^n} \left| \frac{|x|^2}{4t} - \frac{n}{2} \right| e^{-|x|^2/(4t)} dx \\ &= \frac{\omega_{n-1}}{(4\pi t)^{n/2} t} \int_0^\infty \left| \frac{r^2}{4t} - \frac{n}{2} \right| e^{-r^2/(4t)} r^{n-1} dr\end{aligned}$$

in the case $n \ge 2$. Thus we see by $\omega_{n-1} = 2\pi^{n/2} \{\Gamma(n/2)\}^{-1}$ that

$$\|\Delta G_t\|_{L^1} = \frac{2}{(4t)^{n/2} \Gamma(n/2) t} \int_0^\infty \left| \frac{r^2}{4t} - \frac{n}{2} \right| e^{-r^2/(4t)} r^{n-1} dr.$$

In addition, the substitution $r = (4t)^{1/2} \lambda$ yields

$$\begin{aligned}&\|\Delta G_t\|_{L^1} \\ &= \frac{2}{\Gamma(n/2)t} \int_0^\infty \left| \lambda^2 - \frac{n}{2} \right| e^{-\lambda^2} \lambda^{n-1} d\lambda \\ &= \frac{2}{\Gamma(n/2)t} \left\{ \int_{\sqrt{n/2}}^\infty \left(\lambda^2 - \frac{n}{2} \right) e^{-\lambda^2} \lambda^{n-1} d\lambda - \int_0^{\sqrt{n/2}} \left(\lambda^2 - \frac{n}{2} \right) e^{-\lambda^2} \lambda^{n-1} d\lambda \right\} \\ &= \frac{2}{\Gamma(n/2)t} \left(\left[-\frac{\lambda^n}{2} e^{-\lambda^2} \right]_{\sqrt{n/2}}^\infty - \left[-\frac{\lambda^n}{2} e^{-\lambda^2} \right]_0^{\sqrt{n/2}} \right) = \left(\frac{n}{2e}\right)^{n/2} \frac{2}{\Gamma(n/2)t}\end{aligned}$$

for all $t > 0$. Here we notice that the above identity is still valid in the case $n = 1$. Therefore, since $G_{t+\tau} = G_t * G_\tau$ for all $t, \tau > 0$, the Young convolution inequality implies that

$$\|(-\Delta)^N G_t\|_{L^1} \le \|\Delta G_{t/N}\|_{L^1}^N = \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2N}{\Gamma(n/2)t} \right\}^N$$

for all $N \in \mathbb{N}$ and $t > 0$. This proves Proposition 2.3. □

3 Explicit Estimates of Some Operators on the Homogeneous Besov Spaces

In this section, we shall show the explicit estimates of differential operators $(-\Delta)^{\sigma/2}$ and ∂_x^α on $\dot{B}^s_{p,q}$. We also prove the refined decay estimate of $\{T(t)\}_{t \ge 0}$, namely, Theorem 1.1.

3.1 Estimates of Differential Operators

Let us set

$$\begin{cases} \widetilde{M}_{\sigma,n} := 2^{2|\sigma|-(n/(n+2))(|\sigma|-|\sigma-n/2-1|)}(\max\{2, |\sigma|\}+1)^{n/2}\widetilde{K}_n, \\ \widetilde{M}_{\alpha,n} := 2^{2|\alpha|-n/2}(\max\{2, |\alpha|\})^{n/2}([n/2+1]!)^{-n/(n+1)}\widetilde{K}_n, \\ \widetilde{K}_n := \dfrac{2[n/2+1]}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}}\left\{\dfrac{2([n/2+1]!)^3}{n}\right\}^{n/(2n+2)} \|\varphi\|_{H^{[n/2+1]}} \end{cases} \tag{3.1}$$

and show the following theorem:

Theorem 3.1 *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}^s_{p,q}$, it holds that $(-\Delta)^{\sigma/2} f \in \dot{B}^{s-\sigma}_{p,q}$ for all $\sigma \in \mathbb{R}$ and $\partial_x^\alpha f \in \dot{B}^{s-|\alpha|}_{p,q}$ for all $\alpha \in \mathbb{N}_0^n$ with the estimates*

$$\|(-\Delta)^{\sigma/2} f\|_{\dot{B}^{s-\sigma}_{p,q}} \le 3\widetilde{M}_{\sigma,n}\|f\|_{\dot{B}^s_{p,q}}, \quad \|\partial_x^\alpha f\|_{\dot{B}^{s-|\alpha|}_{p,q}} \le 3\widetilde{M}_{\alpha,n}\|f\|_{\dot{B}^s_{p,q}}.$$

To show Theorem 3.1, we consider the symbols $|\xi|^\sigma$ and $(i\xi)^\alpha$ of the Fourier multiplier operators $(-\Delta)^{\sigma/2} = \mathcal{F}^{-1}|\xi|^\sigma\mathcal{F}$ and $\partial_x^\alpha = \mathcal{F}^{-1}(i\xi)^\alpha\mathcal{F}$. We begin with giving the L^1-estimate of the function $\mathcal{F}^{-1} f$, where the estimate will be used to obtain the estimate of the Fourier multipliers.

Proposition 3.2 *Let $m \in \mathbb{N}$ satisfy $2m \ge n+1$. For every $f \in H^m$, it holds that $\mathcal{F}^{-1} f \in L^1$ with the estimate*

$$\|\mathcal{F}^{-1} f\|_{L^1} \le \frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)} \|f\|_{L^2}^{1-n/(2m)}\|(-\Delta)^{m/2} f\|_{L^2}^{n/(2m)}.$$

Proof Let $R > 0$ be arbitrary. Then, by the Cauchy-Schwarz inequality and Parseval identity $\|\mathcal{F} f\|_{L^2} = (2\pi)^{n/2}\|f\|_{L^2}$, we have

$$\begin{aligned} \int_{|x|\le R} |\mathcal{F}^{-1}[f](x)|dx &\le \left(\int_{|x|\le R} dx\right)^{1/2}\left(\int_{|x|\le R} |\mathcal{F}^{-1}[f](x)|^2 dx\right)^{1/2} \\ &\le \left(\omega_{n-1}\int_0^R r^{n-1}dr\right)^{1/2} \|\mathcal{F}^{-1} f\|_{L^2} \\ &= \frac{\omega_{n-1}^{1/2}}{(2\pi)^{n/2}}\left(\frac{R^n}{n}\right)^{1/2} \|f\|_{L^2}. \end{aligned}$$

In addition, since $m \in \mathbb{N}$ satisfies $2m \geq n+1$, it holds by the Cauchy-Schwarz inequality and $\mathcal{F}^{-1}[f](x) = (2\pi)^{-n}\mathcal{F}[f](-x)$ that

$$\begin{aligned}
&\int_{|x|>R} |\mathcal{F}^{-1}[f](x)|dx \\
&\leq \left(\int_{|x|>R} |x|^{-2m} dx\right)^{1/2} \left(\int_{|x|>R} |x|^{2m}|\mathcal{F}^{-1}[f](x)|^2 dx\right)^{1/2} \\
&\leq \left(\omega_{n-1}\int_R^\infty r^{-2m}\cdot r^{n-1}dr\right)^{1/2} (2\pi)^{-n}\left(\int_{\mathbb{R}^n} |x|^{2m}|\mathcal{F}[f](-x)|^2 dx\right)^{1/2} \\
&= \left(\omega_{n-1}\left[-\frac{r^{-(2m-n)}}{2m-n}\right]_R^\infty\right)^{1/2} (2\pi)^{-n/2}\|\mathcal{F}^{-1}[|x|^m\mathcal{F}f]\|_{L^2} \\
&= \frac{\omega_{n-1}^{1/2}}{(2\pi)^{n/2}}\left(\frac{R^{-(2mn)}}{2m-n}\right)^{1/2} \|(-\Delta)^{m/2}f\|_{L^2}.
\end{aligned}$$

Hence, by setting

$$R := \left(\frac{2m-n}{n}\right)^{1/(2m)}\left(\frac{\|(-\Delta)^{m/2}f\|_{L^2}}{\|f\|_{L^2}}\right)^{1/m},$$

we observe that

$$\begin{aligned}
&\|\mathcal{F}^{-1}f\|_{L^1} \\
&\leq \frac{\omega_{n-1}^{1/2}}{(2\pi)^{n/2}}\left\{\frac{1}{n^{1/2}}\left(\frac{2m-n}{n}\right)^{n/(4m)} + \frac{1}{(2mn)^{1/2}}\left(\frac{n}{2m-n}\right)^{1/2-n/(4m)}\right\} \\
&\quad\times \|f\|_{L^2}^{1-n/(2m)}\|(-\Delta)^{m/2}f\|_{L^2}^{n/(2m)} \\
&\leq \frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)} \|f\|_{L^2}^{1-n/(2m)}\|(-\Delta)^{m/2}f\|_{L^2}^{n/(2m)},
\end{aligned}$$

which completes the proof of Proposition 3.2. □

In the following, we consider the multiplier ρ, where $\rho(\xi) := |\xi|^\sigma$ or $\rho(\xi) := (i\xi)^\alpha$. Let us give the following estimate, which may be regarded as an improved version compared with [23, Proposition 4.3]:

Proposition 3.3 *Let $\sigma \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$, and $m \in \mathbb{N}$. Then it holds that*

$$\begin{aligned}
&\|(-\Delta)^{m/2}[\rho(2^k\xi)\varphi]\|_{L^2} \\
&\leq \begin{cases} 2^{\sigma k+|\sigma-m|+m/2}(\max\{2,|\sigma|\}+1)^m(m!)^{3/2}\|\varphi\|_{H^m} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha|k+|\alpha|-m/2}(\max\{2,|\alpha|\})^m(m!)^{1/2}\|\varphi\|_{H^m} & \text{if } \rho(\xi) = (i\xi)^\alpha \end{cases}
\end{aligned}$$

for all $k \in \mathbb{Z}$.

Proof Notice that we have

$$\|(-\Delta)^{m/2} f\|_{L^2} = \left(\sum_{|\beta|=m} \frac{m!}{\beta!} \|\partial_x^\beta f\|_{L^2}^2 \right)^{1/2} \tag{3.2}$$

for all $f \in H^m$. In the case $\rho(\xi) = |\xi|^\sigma$, since $\operatorname{supp}\varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \le |\xi| \le 2\}$, for all $\beta \in \mathbb{N}_0^n$ satisfying $|\beta| = m$, we see by the Leibniz rule and Proposition 2.2 that

$$\begin{aligned}
|\partial_\xi^\beta\{|2^k\xi|^\sigma\varphi(\xi)\}| &\le 2^{\sigma k} \sum_{\gamma\le\beta} \binom{\beta}{\gamma} |\partial_\xi^\gamma |\xi|^\sigma| |\partial_\xi^{\beta-\gamma}\varphi(\xi)| \\
&\le 2^{\sigma k + |\sigma - m|} (\max\{2, |\sigma|\} + 1)^m m! \sum_{\gamma\le\beta} \binom{\beta}{\gamma} |\partial_\xi^{\beta-\gamma}\varphi(\xi)|,
\end{aligned}$$

which yields

$$\|\partial_\xi^\beta[\rho(2^k\xi)\varphi]\|_{L^2} \le 2^{\sigma k + |\sigma - m|} (\max\{2, |\sigma|\} + 1)^m m! \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2}.$$

Hence it holds by (3.2) that

$$\begin{aligned}
&\|(-\Delta)^{m/2}[\rho(2^k\xi)\varphi]\|_{L^2} \\
&\le 2^{\sigma k + |\sigma - m|} (\max\{2, |\sigma|\} + 1)^m (m!)^{3/2} \left\{ \sum_{|\beta|=m} \frac{1}{\beta!} \left(\sum_{\gamma\le\beta} \binom{\beta}{\gamma} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2} \right)^2 \right\}^{1/2}.
\end{aligned}$$

Here we note that

$$\begin{aligned}
&\sum_{|\beta|=m} \frac{1}{\beta!} \left(\sum_{\gamma\le\beta} \binom{\beta}{\gamma} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2} \right)^2 \\
&= \sum_{|\beta|=m} \frac{1}{\beta!} \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2} \sum_{\gamma'\le\beta} \binom{\beta}{\gamma'} \|\partial_\xi^{\beta-\gamma'}\varphi\|_{L^2} \\
&\le \sum_{|\beta|=m} \sum_{\gamma\le\beta} \frac{1}{\gamma!(\beta-\gamma)!} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2} \sum_{\gamma'\le\beta} 2^m \|\partial_\xi^{\gamma'}\varphi\|_{L^2} \\
&\le 2^m \sum_{|\beta|=m} \sum_{\gamma\le\beta} \|\partial_\xi^{\beta-\gamma}\varphi\|_{L^2} \|\varphi\|_{H^m} = 2^m \|\varphi\|_{H^m}^2,
\end{aligned}$$

which implies the desired estimate. In the case $\rho(\xi) = (i\xi)^\alpha$, for all $\beta \in \mathbb{N}_0^n$ satisfying $|\beta| = m$, we also see by the Leibniz rule and Proposition 2.2 that

$$\begin{aligned}|\partial_\xi^\beta\{(2^k i\xi)^\alpha\varphi(\xi)\}| &\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\partial_\xi^\gamma(2^k i\xi)^\alpha||\partial_\xi^{\beta-\gamma}\varphi(\xi)|\\ &\le 2^{|\alpha|k+|\alpha|-m}(\max\{2,|\alpha|\})^m\sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\partial_\xi^{\beta-\gamma}\varphi(\xi)|.\end{aligned}$$

Thus we have the desired estimate in the same way. This completes the proof of Proposition 3.3. □

By combining the estimates in Propositions 3.2 and 3.3, we may show the following estimate, which provides the proof of Theorem 3.1 immediately. Here we set

$$\widetilde{K}_{n,m} := \frac{2(m/n)}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)}(m!)^{(3n)/(4m)}\|\varphi\|_{H^m}.$$

Proposition 3.4 *Let $\sigma\in\mathbb{R}$ and $\alpha\in\mathbb{N}_0^n$ and let $m\in\mathbb{N}$ satisfy $2m\ge n+1$. Then it holds that*

$$\begin{aligned}&\|\mathcal{F}^{-1}[\rho(2^k\xi)\varphi]\|_{L^1}\\ &\le\begin{cases}2^{\sigma k+|\sigma|-(n/(2m))(|\sigma|-|\sigma-m|)}(\max\{2,|\sigma|\}+1)^{n/2}\widetilde{K}_{n,m} & \text{if } \rho(\xi)=|\xi|^\sigma,\\ 2^{|\alpha|k+|\alpha|-n/2}(\max\{2,|\alpha|\})^{n/2}(m!)^{-n/(2m)}\widetilde{K}_{n,m} & \text{if } \rho(\xi)=(i\xi)^\alpha\end{cases}\end{aligned}$$

for all $k\in\mathbb{Z}$.

Proof We see by Proposition 3.2 that

$$\begin{aligned}&\|\mathcal{F}^{-1}[\rho(2^k\xi)\varphi]\|_{L^1}\\ &\le\frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)}\|\rho(2^k\xi)\varphi\|_{L^2}^{1-n/(2m)}\|(-\Delta)^{m/2}[\rho(2^k\xi)\varphi]\|_{L^2}^{n/(2m)}\end{aligned}$$

for all $k\in\mathbb{Z}$, provided $2m\ge n+1$. We also notice that

$$\|\rho(2^k\xi)\varphi\|_{L^2}\le\begin{cases}2^{\sigma k+|\sigma|}\|\varphi\|_{L^2} & \text{if } \rho(\xi)=|\xi|^\sigma,\\ 2^{|\alpha|k+|\alpha|}\|\varphi\|_{L^2} & \text{if } \rho(\xi)=(i\xi)^\alpha\end{cases}$$

by a direct computation with the aid of $\operatorname{supp}\varphi=\{\xi\in\mathbb{R}^n \mid 1/2\le|\xi|\le 2\}$. Therefore, in the case $\rho(\xi)=|\xi|^\sigma$, it holds by Proposition 3.3 that

$$\begin{aligned}&\|\mathcal{F}^{-1}[\rho(2^k\xi)\varphi]\|_{L^1}\\ &\le\frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)}(2^{\sigma k+|\sigma|}\|\varphi\|_{L^2})^{1-n/(2m)}\\ &\quad\times\left\{2^{\sigma k+|\sigma-m|+m/2}(\max\{2,|\sigma|\}+1)^m(m!)^{3/2}\|\varphi\|_{H^m}\right\}^{n/(2m)}\\ &\le\frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}}\left(\frac{n}{2m-n}\right)^{1-n/(4m)}\cdot 2^{\sigma k+|\sigma|-(n/(2m))(|\sigma|-|\sigma-m|)+n/4}\\ &\quad\times(\max\{2,|\sigma|\}+1)^{n/2}(m!)^{(3n)/(4m)}\|\varphi\|_{H^m}.\end{aligned}$$

Since $\omega_{n-1} = 2\pi^{n/2}\{\Gamma(n/2)\}^{-1} = n\pi^{n/2}\{\Gamma(n/2+1)\}^{-1}$, we have

$$\frac{2m\omega_{n-1}^{1/2}}{(2\pi)^{n/2}n^{3/2}} \cdot 2^{n/4} = \frac{2(m/n)}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}},$$

which implies the desired estimate. In the case $\rho(\xi) = (i\xi)^\alpha$, we may show the desired estimate in a similar manner. This proves Proposition 3.4. □

Corollary 3.5 *Let $\sigma \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$. Then it holds that*

$$\|\mathcal{F}^{-1}[\rho(2^k\xi)\varphi]\|_{L^1} \le \begin{cases} 2^{\sigma k-|\sigma|}\widetilde{M}_{\sigma,n} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha|k-|\alpha|}\widetilde{M}_{\alpha,n} & \text{if } \rho(\xi) = (i\xi)^\alpha \end{cases}$$

for all $k \in \mathbb{Z}$, where $\widetilde{M}_{\sigma,n}$ and $\widetilde{M}_{\alpha,n}$ are the same constants defined by (3.1).

Proof In Proposition 3.4, since we may take $m \in \mathbb{N}$ so that $n/2+1/2 \le m \le n/2+1$, we observe that

$$\begin{aligned} \widetilde{K}_{n,m} &\le \frac{(2/n)[n/2+1]}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}} \cdot \frac{n}{2m-n}\left\{\frac{(2m-n)(m!)^3}{n}\right\}^{n/(4m)} \|\varphi\|_{H^{[n/2+1]}} \\ &\le \frac{2[n/2+1]}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}}\left\{\frac{2([n/2+1]!)^3}{n}\right\}^{n/(2n+2)} \|\varphi\|_{H^{[n/2+1]}} = \widetilde{K}_n. \end{aligned}$$

Moreover, in the case $\rho(\xi) = |\xi|^\sigma$, we see that

$$2^{-(n/(2m))(|\sigma|-|\sigma-m|)} = \begin{cases} 2^{n/2} & \text{if } \sigma \in (-\infty, 0], \\ 2^{n/2-(2n\sigma)/(n+2)} & \text{if } \sigma \in (0, n/2+1) \text{ and } m = n/2+1, \\ 2^{-n/2} & \text{if } \sigma \in [n/2+1, \infty), \end{cases}$$

and

$$\begin{aligned} &2^{-(n/(2m))(|\sigma|-|\sigma-m|)} \\ &= \begin{cases} 2^{n/2-(2n\sigma)/(n+1)} & \text{if } \sigma \in (0, n/2+1/2) \text{ and } m = n/2+1/2, \\ 2^{-n/2} & \text{if } \sigma \in [n/2+1/2, n/2+1) \text{ and } m = n/2+1/2. \end{cases} \end{aligned}$$

Thus we obtain $2^{-(n/(2m))(|\sigma|-|\sigma-m|)} \le 2^{-(n/(n+2))(|\sigma|-|\sigma-n/2-1|)}$ for all $\sigma \in \mathbb{R}$, which yields the desired estimate. In the case $\rho(\xi) = (i\xi)^\alpha$, we may show the desired estimate in a similar manner. □

Proof of Theorem 3.1 First we notice that $g = \sum_{k\in\mathbb{Z}}(\varphi_k * g)$ for all $g \in L^p$ and $\varphi_j * \varphi_k = 0$ for all $j, k \in \mathbb{Z}$ such that $|j-k| \ge 2$ from the definition of $\{\varphi_j\}_{j\in\mathbb{Z}}$. Hence, we see by the Young convolution inequality that

$$\begin{aligned}\|\varphi_j * \mathcal{F}^{-1}[\rho\mathcal{F}f]\|_{L^p} &\le \sum_{k\in\mathbb{Z}} \|\varphi_k * \varphi_j * \mathcal{F}^{-1}[\rho\mathcal{F}f]\|_{L^p}\\ &\le \sum_{k=j-1}^{j+1} \|\varphi_j * f * \mathcal{F}^{-1}[\rho\varphi(2^{-k}\xi)]\|_{L^p}\\ &\le \|\varphi_j * f\|_{L^p} \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\rho\varphi(2^{-k}\xi)]\|_{L^1}\end{aligned}$$

for all $j \in \mathbb{Z}$. Since $\|\mathcal{F}^{-1}[\rho\varphi(2^{-k}\xi)]\|_{L^1} = \|\mathcal{F}^{-1}[\rho(2^k\xi)\varphi]\|_{L^1}$ holds for all $k \in \mathbb{Z}$ from simple substitutions, by applying Corollary 3.5 and using

$$\sum_{k=j-1}^{j+1} 2^{\sigma k} = 2^{\sigma(j-1)} + 2^{\sigma j} + 2^{\sigma(j+1)} = 2^{\sigma j}(2^{-\sigma} + 1 + 2^{\sigma}) \le 3 \cdot 2^{\sigma j + |\sigma|},$$

we obtain

$$\|\varphi_j * \mathcal{F}^{-1}[\rho\mathcal{F}f]\|_{L^p} \le \begin{cases} 3\widetilde{M}_{\sigma,n} \cdot 2^{\sigma j}\|\varphi_j * f\|_{L^p} & \text{if } \rho(\xi) = |\xi|^{\sigma},\\ 3\widetilde{M}_{\alpha,n} \cdot 2^{|\alpha| j}\|\varphi_j * f\|_{L^p} & \text{if } \rho(\xi) = (i\xi)^{\alpha}\end{cases}$$

for all $j \in \mathbb{Z}$. Thus we have the desired estimates. This completes the proof of Theorem 3.1. $\square$

3.2 Decay Estimate of the Heat Semigroup

In this subsection, we shall show the refined decay estimate of $T(t)f \in \dot{B}^{s+\sigma}_{p,1}$ for $f \in \dot{B}^{s}_{p,\infty}$, i.e., Theorem 1.1. To obtain an explicit bound, we make use of the L^1-estimate of the derivatives of the heat kernel given in Proposition 2.3. We begin with showing the following lemma:

Lemma 3.6 *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}^{s}_{p,q}$, it holds that $T(t)f \in \dot{B}^{s+2N}_{p,q}$ for all $N \in \mathbb{N}$ and $t > 0$ with the estimate*

$$\|T(t)f\|_{\dot{B}^{s+2N}_{p,q}} \le 2\left\{\left(\frac{n}{2e}\right)^{n/2} \frac{2^5 N}{\Gamma(n/2)t}\right\}^N (4N+2)^{n/2}\widetilde{K}_n\|f\|_{\dot{B}^{s}_{p,q}},$$

where $\widetilde{K}_n$ is the same constant defined by (3.1).

Proof Notice that $T(t)f = G_t * f$ holds for all $t > 0$, where $G_t(x) := (4\pi t)^{-n/2}e^{-|x|^2/(4t)}$ for $(t, x) \in (0,\infty) \times \mathbb{R}^n$. Therefore, in a similar manner to the proof of Theorem 3.1, we have $\|\varphi_j * (T(t)f)\|_{L^p} \le \|\varphi_j * f\|_{L^p} \sum_{k=j-1}^{j+1} \|\varphi_k * G_t\|_{L^1}$ for all $j \in \mathbb{Z}$ and $t > 0$. In addition, by the Young convolution inequality, we observe that

$$\begin{aligned}\|\varphi_k * G_t\|_{L^1} &= \|((-\Delta)^{-N}\varphi_k) * (-\Delta)^N G_t\|_{L^1} \\ &\le \|\mathcal{F}^{-1}[|\xi|^{-2N}\varphi(2^k\xi)]\|_{L^1}\|(-\Delta)^N G_t\|_{L^1} \\ &= \|\mathcal{F}^{-1}[|2^k\xi|^{-2N}\varphi]\|_{L^1}\|(-\Delta)^N G_t\|_{L^1}\end{aligned}$$

for all $N \in \mathbb{N}$, $k \in \mathbb{Z}$, and $t > 0$. Since it holds by Corollary 3.5 that

$$\begin{aligned}\|\mathcal{F}^{-1}[|2^k\xi|^{-2N}\varphi]\|_{L^1} &\le 2^{-2Nk+2N+n/2}(2N+1)^{n/2}\widetilde{K}_n \\ &= 2^{-2Nk+2N}(4N+2)^{n/2}\widetilde{K}_n,\end{aligned}$$

by applying Proposition 2.3 and using

$$\begin{aligned}\sum_{k=j-1}^{j+1} 2^{-2Nk} &= 2^{-2N(j-1)} + 2^{-2Nj} + 2^{-2N(j+1)} \\ &= 2^{-2Nj}(2^{2N} + 1 + 2^{-2N}) \le 2^{-2Nj+2N+1},\end{aligned}$$

we obtain

$$\|\varphi_j * (T(t)f)\|_{L^p} \le 2\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^5 N}{\Gamma(n/2)t}\right\}^N (4N+2)^{n/2}\widetilde{K}_n \cdot 2^{-2Nj}\|\varphi_j * f\|_{L^p}$$

for all $N \in \mathbb{N}$, $j \in \mathbb{Z}$, and $t > 0$. Thus we have the desired estimate. This proves Lemma 3.6. □

Notice that the decay estimate in Lemma 3.6 is valid in the case $N \in \mathbb{N}$. Then, we shall extend the decay estimate to the general regularity index $\sigma > 0$. To this end, we show the interpolation inequality of $\dot{B}^s_{p,q}$ with an explicit bound. By combining Lemma 3.6 and the interpolation inequality, we may show Theorem 1.1.

Proposition 3.7 *Let $1 \le p \le \infty$, $-\infty < s_0 < s_1 < \infty$, and $0 < \theta < 1$. For every $f \in \dot{B}^{s_0}_{p,\infty} \cap \dot{B}^{s_1}_{p,\infty}$, it holds that $f \in \dot{B}^s_{p,1}$ with the estimate*

$$\|f\|_{\dot{B}^s_{p,1}} \le \frac{1}{\log 2}\left(1 + \frac{1}{s_1 - s_0}\right)\frac{1}{\theta(1-\theta)}\|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,\infty}}\|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}},$$

where $s := (1-\theta)s_0 + \theta s_1$.

Proof Let $j_* \in \mathbb{Z}$ be arbitrary. Since $s = (1-\theta)s_0 + \theta s_1$, we obtain

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,1}} &= \left(\sum_{j \le j_*-1} + \sum_{j \ge j_*}\right) 2^{sj}\|\varphi_j * f\|_{L^p} \\ &\le \|f\|_{\dot{B}^{s_0}_{p,\infty}} \sum_{j \le j_*-1} 2^{(s_1-s_0)\theta j} + \|f\|_{\dot{B}^{s_1}_{p,\infty}} \sum_{j \ge j_*} 2^{-(s_1-s_0)(1-\theta)j} \\ &= \frac{2^{(s_1-s_0)\theta j_*}}{2^{(s_1-s_0)\theta} - 1}\|f\|_{\dot{B}^{s_0}_{p,\infty}} + \frac{2^{-(s_1-s_0)(1-\theta)j_*}}{1 - 2^{-(s_1-s_0)(1-\theta)}}\|f\|_{\dot{B}^{s_1}_{p,\infty}}.\end{aligned}$$

Hence, by taking $j_* \in \mathbb{Z}$ so that

$$\frac{\|f\|_{\dot{B}^{s_1}_{p,\infty}}}{\|f\|_{\dot{B}^{s_0}_{p,\infty}}} \le 2^{(s_1-s_0)j_*} \le 2^{s_1-s_0}\frac{\|f\|_{\dot{B}^{s_1}_{p,\infty}}}{\|f\|_{\dot{B}^{s_0}_{p,\infty}}},$$

we deduce that

$$\begin{aligned}\|f\|_{\dot{B}^{s}_{p,1}} \le{}& \frac{1}{2^{(s_1-s_0)\theta}-1}\left(2^{s_1-s_0}\frac{\|f\|_{\dot{B}^{s_1}_{p,\infty}}}{\|f\|_{\dot{B}^{s_0}_{p,\infty}}}\right)^{\theta}\|f\|_{\dot{B}^{s_0}_{p,\infty}}\\ &+\frac{1}{1-2^{-(s_1-s_0)(1-\theta)}}\left(\frac{\|f\|_{\dot{B}^{s_1}_{p,\infty}}}{\|f\|_{\dot{B}^{s_0}_{p,\infty}}}\right)^{-(1-\theta)}\|f\|_{\dot{B}^{s_1}_{p,\infty}}\\ ={}&\left(\frac{1}{1-2^{-(s_1-s_0)\theta}}+\frac{1}{1-2^{-(s_1-s_0)(1-\theta)}}\right)\|f\|^{1-\theta}_{\dot{B}^{s_0}_{p,\infty}}\|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}.\end{aligned}$$

Here, since it holds that $1/(1-2^{-\alpha\tau}) \le (\alpha\log 2)^{-1}(1+1/\tau)$ for all $0<\alpha<1$ and $0<\tau<\infty$, we observe that

$$\frac{1}{1-2^{-(s_1-s_0)\theta}}+\frac{1}{1-2^{-(s_1-s_0)(1-\theta)}} \le \frac{1}{\log 2}\left(1+\frac{1}{s_1-s_0}\right)\frac{1}{\theta(1-\theta)},$$

which implies the desired estimate. This proves Proposition 3.7. □

Proof of Theorem 1.1 First we consider the case of $\sigma \ge 2$. Then we may take $k \in \mathbb{N}$ and $0 \le \sigma_0 < 1$ so that $\sigma = \sigma_0 + k$ holds. Here we note that $k \ge 2$. In addition, by letting $\theta := \sigma/(2k)$, we have $0<\theta<1$ and

$$s+\sigma = s+2k\theta = (1-\theta)s+\theta(s+2k).$$

Hence we see by Proposition 3.7 that

$$\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}} \le \frac{1}{\log 2}\left(1+\frac{1}{2k}\right)\frac{1}{\theta(1-\theta)}\|T(t)f\|^{1-\theta}_{\dot{B}^{s}_{p,\infty}}\|T(t)f\|^{\theta}_{\dot{B}^{s+2k}_{p,\infty}}$$

for all $t>0$. Since $k\ge 2$ and $k \le \sigma < k+1$, we deduce that

$$\begin{aligned}\frac{1}{\log 2}\left(1+\frac{1}{2k}\right)\frac{1}{\theta(1-\theta)} &= \frac{2k(2k+1)}{\log 2}\frac{1}{\sigma(2k-\sigma)} \le \frac{2\sigma(2k+1)}{\log 2}\frac{1}{\sigma(2k-\sigma)}\\ &= \frac{2}{\log 2}\left(1+\frac{\sigma+1}{2k-\sigma}\right) \le \frac{2}{\log 2}\left(1+\frac{k+2}{k-1}\right)\\ &= \frac{2}{\log 2}\left(2+\frac{3}{k-1}\right) \le \frac{10}{\log 2}.\end{aligned}$$

Noting that $\theta = \sigma/(2k)$, we see by Lemma 3.6 and $\|T(t)f\|_{\dot{B}^s_{p,\infty}} \leq \|f\|_{\dot{B}^s_{p,\infty}}$ that

$$\begin{aligned}
&\|T(t)f\|^{1-\theta}_{\dot{B}^s_{p,\infty}} \|T(t)f\|^{\theta}_{\dot{B}^{s+2k}_{p,\infty}} \\
&\leq \left[2 \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^5 k}{\Gamma(n/2)t} \right\}^k (4k+2)^{n/2} \widetilde{K}_n \right]^{\theta} \|f\|_{\dot{B}^s_{p,\infty}} \\
&= \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^5 k}{\Gamma(n/2)t} \right\}^{\sigma/2} \{2(4k+2)^{n/2} \widetilde{K}_n\}^{\sigma/(2k)} \|f\|_{\dot{B}^s_{p,\infty}}.
\end{aligned}$$

Therefore, since $\sigma/(2k) \leq 1/2 + 1/(2k) \leq 3/4$, we obtain

$$\begin{aligned}
&\|T(t)f\|^{1-\theta}_{\dot{B}^s_{p,\infty}} \|T(t)f\|^{\theta}_{\dot{B}^{s+2k}_{p,\infty}} \\
&\leq \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^5 \sigma}{\Gamma(n/2)t} \right\}^{\sigma/2} \{2(4\sigma+2)^{n/2} \widetilde{K}_n\}^{3/4} \|f\|_{\dot{B}^s_{p,\infty}},
\end{aligned}$$

which yields

$$\begin{aligned}
&\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}} \\
&\leq \frac{10}{\log 2} \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^5 \sigma}{\Gamma(n/2)t} \right\}^{\sigma/2} \{2(4\sigma+2)^{n/2} \widetilde{K}_n\}^{3/4} \|f\|_{\dot{B}^s_{p,\infty}}.
\end{aligned} \tag{3.3}$$

Next we consider the case of $0 < \sigma < 2$. Then, by letting $\theta := \sigma/4$, we have $0 < \theta < 1$ and

$$s + \sigma = s + 4\theta = (1-\theta)s + \theta(s+4).$$

Hence we see by Proposition 3.7 that

$$\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}} \leq \frac{1}{\log 2}\left(1 + \frac{1}{4}\right) \frac{1}{\theta(1-\theta)} \|T(t)f\|^{1-\theta}_{\dot{B}^s_{p,\infty}} \|T(t)f\|^{\theta}_{\dot{B}^{s+4}_{p,\infty}}$$

for all $t > 0$. Since $0 < \sigma < 2$, we deduce that

$$\frac{1}{\log 2}\left(1 + \frac{1}{4}\right) \frac{1}{\theta(1-\theta)} = \frac{1}{\log 2} \cdot \frac{5}{4} \cdot \frac{4^2}{\sigma(4-\sigma)} \leq \frac{10}{\sigma \log 2}.$$

Noting that $\theta = \sigma/4$, we see by Lemma 3.6 that

$$\begin{aligned}
\|T(t)f\|^{1-\theta}_{\dot{B}^s_{p,\infty}} \|T(t)f\|^{\theta}_{\dot{B}^{s+4}_{p,\infty}} &\leq \left[2 \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^6}{\Gamma(n/2)t} \right\}^2 10^{n/2} \widetilde{K}_n \right]^{\theta} \|f\|_{\dot{B}^s_{p,\infty}} \\
&= \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^6}{\Gamma(n/2)t} \right\}^{\sigma/2} (2 \cdot 10^{n/2} \widetilde{K}_n)^{\sigma/4} \|f\|_{\dot{B}^s_{p,\infty}}.
\end{aligned}$$

Therefore, since $\sigma/4 \le 1/2$, we obtain

$$\|T(t)f\|_{\dot{B}^s_{p,\infty}}^{1-\theta}\|T(t)f\|_{\dot{B}^{s+4}_{p,\infty}}^{\theta} \le \left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)t}\right\}^{\sigma/2}(2\cdot 10^{n/2}\widetilde{K}_n)^{1/2}\|f\|_{\dot{B}^s_{p,\infty}},$$

which yields

$$\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}} \le \frac{10}{\sigma\log 2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)t}\right\}^{\sigma/2}(2\cdot 10^{n/2}\widetilde{K}_n)^{1/2}\|f\|_{\dot{B}^s_{p,\infty}}. \tag{3.4}$$

Here, by the estimates (3.3) and (3.4), we observe that

$$\begin{aligned}\|T(t)f\|_{\dot{B}^{s+\sigma}_{p,1}} &\le \max\left\{(\sigma/2)^{\sigma/2}\{2(4\sigma+2)^{n/2}\widetilde{K}_n\}^{3/4}, (1/\sigma)(2\cdot 10^{n/2}\widetilde{K}_n)^{1/2}\right\}\\ &\quad\times\frac{10}{\log 2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)t}\right\}^{\sigma/2}\|f\|_{\dot{B}^s_{p,\infty}}\\ &\le \max\left\{(\sigma/2)^{\sigma/2}, \frac{10^{n/4}}{\sigma(4\sigma+2)^{3n/8}}\right\}\\ &\quad\times\frac{10}{\log 2}(4\sigma+2)^{3n/8}(2\widetilde{K}_n)^{3/4}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)t}\right\}^{\sigma/2}\|f\|_{\dot{B}^s_{p,\infty}}\end{aligned}$$

for all $\sigma > 0$ and $t > 0$. Since we have $10^{n/4}(4\sigma+2)^{-3n/8} \le (10^2/2^3)^{n/8} = (25/2)^{n/8}$ and since it holds that

$$\begin{aligned}&\frac{10}{\log 2}(4\sigma+2)^{3n/8}(2\widetilde{K}_n)^{3/4}\\ &= \frac{10}{\log 2}2^{3n/8}(2\sigma+1)^{3n/8}\left\{\frac{2^2[n/2+1]}{(2\pi)^{n/4}\{\Gamma(n/2+1)\}^{1/2}}\right\}^{3/4}\\ &\times\left\{\frac{2([n/2+1]!)^3}{n}\right\}^{3n/(8n+8)}\|\varphi\|_{H^{[n/2+1]}}^{3/4}\\ &= K_n(2\sigma+1)^{3n/8},\end{aligned}$$

we obtain the desired estimate. This completes the proof of Theorem 1.1. □

4 Radius of Convergence of the Taylor Expansion

In this section, we give the proof of Theorems 1.3 and 1.5. Namely, we calculate a lower bound of the radius (1.2) of convergence of the Taylor expansion appearing in Theorems 1.3 and 1.5. To this end, we make use of Theorems 1.1 and 3.1. We also notice that the following continuous embedding $\dot{\mathcal{B}}^{n/p}_{p,1} \subset L^\infty$, which yields the pointwise estimate of the remainder term of the Taylor expansion (1.3). We first give the proof of Theorem 1.5. Theorem 1.3 may be shown in a similar manner.

Proof of Theorem 1.5 We focus on a derivation of a lower bound (1.2) of the radius of convergence of the Taylor expansion (1.3). Assume that $s - n/p \notin \mathbb{N}$ and $f \in \dot{\mathcal{B}}^s_{p,\infty}$. Set $u(t,x) := (T(t)f)(x)$ for $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and consider the remainder term of the Lagrange form

$$\sum_{|\alpha_*|=k_*} \frac{1}{\alpha_*! j_*!} \partial_t^{j_*} \partial_x^{\alpha_*} u(t_*, x_*)(t-t_0)^{j_*}(x-x_0)^{\alpha_*}$$

of the Taylor expansion (1.3) at an arbitrary $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^n$, where $(t,x) \in (0,\infty) \times \mathbb{R}^n$ is around (t_0,x_0) and $(t_*, x_*) = (\theta t + (1-\theta)t_0, \theta x + (1-\theta)x_0)$ with some $0 < \theta < 1$. Since we consider the case where $k_* + j_*$ is sufficiently large, we may take $k_0, j_0 \in \mathbb{N}_0$ so that $k_0 \le k_*$, $j_0 \le j_*$, and $k_0 + 2j_0 > s - n/p$. Hence, by setting $k := k_* - k_0$ and $j := j_* - j_0$, we observe that

$$\begin{aligned}
&\left| \sum_{|\alpha_*|=k_*} \frac{1}{\alpha_*! j_*!} \partial_t^{j_*} \partial_x^{\alpha_*} u(t_*, x_*)(t-t_0)^{j_*}(x-x_0)^{\alpha_*} \right| \\
&\le \sum_{|\alpha|=k} \sum_{|\alpha_0|=k_0} \frac{1}{(\alpha+\alpha_0)!(j+j_0)!} \|\partial_t^{j+j_0} \partial_x^{\alpha+\alpha_0} u(t_*)\|_{L^\infty} |t-t_0|^{j+j_0} |x-x_0|^{k+k_0}.
\end{aligned}$$

Here, since $\dot{\mathcal{B}}^{n/p}_{p,1} \subset L^\infty$ and since $\partial_t^{j+j_0} \partial_x^{\alpha+\alpha_0} u(t_*) = \partial_x^{\alpha_0} \Delta^{j_0} (\partial_t^j \partial_x^\alpha u(t_*))$, we see by Theorem 3.1 that

$$\begin{aligned}
&\sum_{|\alpha_0|=k_0} \frac{1}{(\alpha+\alpha_0)!(j+j_0)!} \|\partial_t^{j+j_0} \partial_x^{\alpha+\alpha_0} u(t_*)\|_{L^\infty} \\
&\le \frac{C}{\alpha! j!} \sum_{|\alpha_0|=k_0} \|\partial_t^{j+j_0} \partial_x^{\alpha+\alpha_0} u(t_*)\|_{\dot{B}^{n/p}_{p,1}} \\
&\le \frac{C}{\alpha! j!} \|\partial_t^j \partial_x^\alpha u(t_*)\|_{\dot{B}^{k_0+2j_0+n/p}_{p,1}}
\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| = k$, where $C > 0$ is a constant depending on k_0 and j_0 but not on k_* and j_*. In addition, noting that $\sigma_0 := k_0 + 2j_0 + n/p - s > 0$ and $\partial_t^j \partial_x^\alpha u(t_*) = \partial_x^\alpha \Delta^j u(t_*)$, by combining Theorems 1.1 and 3.1, we deduce that

$$\begin{aligned}
\|\partial_t^j \partial_x^\alpha u(t_*)\|_{\dot{B}^{k_0+2j_0+n/p}_{p,1}} &\le C \widetilde{M}_{\alpha,n} \widetilde{M}_{2j,n} \|u(t_*)\|_{\dot{B}^{s+\sigma_0+|\alpha|+2j}_{p,1}} \\
&\le C \widetilde{M}_{\alpha,n} \widetilde{M}_{2j,n} M_{\sigma_0+|\alpha|+2j,n} t_*^{-(\sigma_0+|\alpha|+2j)/2} \|f\|_{\dot{B}^s_{p,\infty}}.
\end{aligned}$$

Concerning the constants depending on α and j, we have

$$\widetilde{M}_{\alpha,n} \widetilde{M}_{2j,n} \le C 2^{2|\alpha|+4j} (|\alpha|+1)^{n/2} (j+1)^{n/2},$$

where $C > 0$ is a constant independent of α and j. In addition, it holds that

$$\begin{aligned}
&M_{\sigma_0+|\alpha|+2j,n}\\
&\le C\left(\frac{\sigma_0+|\alpha|+2j}{2}\right)^{(\sigma_0+|\alpha|+2j)/2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)}\right\}^{(\sigma_0+|\alpha|+2j)/2}(|\alpha|+j)^{3n/8}\\
&\le C\left(\frac{\sigma_0+|\alpha|+2j}{2}\right)^{(|\alpha|+2j)/2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)}\right\}^{(|\alpha|+2j)/2}(|\alpha|+j)^{\sigma_0/2+3n/8},
\end{aligned}$$

provided that $|\alpha|+j$ is sufficiently large. Therefore, noting that $\sum_{|\alpha|=k}1/\alpha! = n^k/k!$, we obtain

$$\begin{aligned}
&\left|\sum_{|\alpha_*|=k_*}\frac{1}{\alpha_*!j_*!}\partial_t^{j_*}\partial_x^{\alpha_*}u(t_*,x_*)(tt_0)^{j_*}(x-x_0)^{\alpha_*}\right|\\
&\le C\sum_{|\alpha|=k}\frac{1}{\alpha!j!}\cdot\widetilde{M}_{\alpha,n}\widetilde{M}_{2j,n}M_{\sigma_0+k+2j,n}t_*^{-(\sigma_0+k+2j)/2}\|f\|_{\dot{B}^s_{p,\infty}}\cdot|t-t_0|^{j+j_0}|x-x_0|^{k+k_0}\\
&\le C\sum_{|\alpha|=k}\frac{1}{\alpha!j!}\cdot 2^{2k+4j}(\sigma_0/2+k/2+j)^{k/2+j}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^6}{\Gamma(n/2)t_*}\right\}^{k/2+j}\\
&\quad\times P(k,j)|t-t_0|^{j+j_0}|xx_0|^{k+k_0}\|f\|_{\dot{B}^s_{p,\infty}}\\
&\le C\frac{n^k}{k!j!}(\sigma_0/2+k/2+j)^{k/2+j}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{10}}{\Gamma(n/2)t_*}\right\}^{k/2+j}\\
&\times P(k,j)|t-t_0|^{j+j_0}|x-x_0|^{k+k_0}\|f\|_{\dot{B}^s_{p,\infty}},
\end{aligned}$$

where $P(k,j) := (k+j)^{\sigma_0/2+3n/8}(k+1)^{n/2}(j+1)^{n/2}$. In addition, since $(a+b)^{a+b} \le 2^{a+b}a^a b^b$ for $a, b \ge 0$ from [23, Proposition 5.3] and since $(1+a/b)^b \le e^a$ for $a, b > 0$, we have

$$\begin{aligned}
(\sigma_0/2+k/2+j)^{k/2+j} &= \left(1+\frac{\sigma_0/2}{k/2+j}\right)^{k/2+j}(k/2+j)^{k/2+j}\\
&\le e^{\sigma_0/2}\cdot 2^{k/2+j}(k/2)^{k/2}j^j\\
&\le C2^j k^{k/2}j^j.
\end{aligned}$$

Hence we see by $k^k/k! \le e^k$ that

$$\begin{aligned}
&\frac{n^k}{k!j!}(\sigma_0/2+k/2+j)^{k/2+j}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{10}}{\Gamma(n/2)t_*}\right\}^{k/2+j}|t-t_0|^{j+j_0}|x-x_0|^{k+k_0}\\
&\le C\frac{k^k}{k!}k^{-k/2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{10}n^2|x-x_0|^2}{\Gamma(n/2)t_*}\right\}^{k/2}\cdot\frac{j^j}{j!}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{11}|t-t_0|}{\Gamma(n/2)t_*}\right\}^j|t-t_0|^{j_0}|x-x_0|^{k_0}\\
&\le C\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{10}e^2n^2|x-x_0|^2}{k\Gamma(n/2)t_*}\right\}^{k/2}\left\{\left(\frac{n}{2e}\right)^{n/2}\frac{2^{11}e|t-t_0|}{\Gamma(n/2)t_*}\right\}^j|t-t_0|^{j_0}|x-x_0|^{k_0}.
\end{aligned}$$

Here we recall that $k_0, j_0 \in \mathbb{N}_0$ are the fixed indices and $k = k_* - k_0$ and $j = j_* - j_0$. Notice that for any $x, x_0 \in \mathbb{R}^n$ and $t_* > 0$, by taking $k_* \in \mathbb{N}$ sufficiently large, we have

$$\left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^{10} e^2 n^2 |x - x_0|^2}{k \Gamma(n/2) t_*} \right\}^{k/2} = k^{-k/4} \left\{ \left(\frac{n}{2e}\right)^{n/2} \frac{2^{10} e^2 n^2 |x - x_0|^2}{k^{1/2} \Gamma(n/2) t_*} \right\}^{k/2} \leq k^{-k/4}.$$

Hence, the remainder term of the Taylor expansion (1.3) converges to 0, provided that $|t - t_0|$ is sufficiently small so that

$$|t - t_0| < \left(\frac{2e}{n}\right)^{n/2} \frac{\Gamma(n/2) t_*}{2^{11} e}$$

holds. Once we see that (1.3) is valid for $t > 0$ satisfying the above condition, in the same way as above, we may show that (1.3) holds, provided (1.2). This completes the proof of Theorem 1.5. □

Proof of Theorem 1.3 We see that

$$\|\partial_t^j T(t) f\|_{\dot{B}^{s+\sigma}_{p,1}} \leq 3\widetilde{M}_{2j,n} \|T(t) f\|_{\dot{B}^{s+\sigma+2j}_{p,1}} \leq 3\widetilde{M}_{2j,n} M_{\sigma+2j,n} t^{-(\sigma+2j)/2} \|f\|_{\dot{B}^{s}_{p,\infty}}$$

for all $j \in \mathbb{N}_0$ and $\sigma > 0$ by virtue of Theorems 1.1 and 3.1, which yields $T(t) \in \mathcal{L}(\dot{B}^s_{p,\infty}, \dot{B}^{s+\sigma}_{p,1})$ with the estimate

$$\|T^{(j)}(t)\|_{\mathcal{L}(\dot{B}^s_{p,\infty}, \dot{B}^{s+\sigma}_{p,1})} \leq 3\widetilde{M}_{2j,n} M_{\sigma+2j,n} t^{-(\sigma+2j)/2}.$$

Therefore, in a similar manner to the proof of Theorem 1.5, we may show the desired result. This completes the proof of Theorem 1.3. □

Acknowledgements The authors would like to express our hearty thanks to the referees for their variable comments.

Funding The research of the first author was partially supported by JSPS KAKENHI Grant Numbers JP18KK0073 and JP19H00644. The research of the second author was partially supported by JSPS KAKENHI Grant Number JP22J12100.

Appendix A: Explicit Pointwise Estimates of Higher Order Derivatives of $|\xi|^\sigma$

In what follows, we give explicit pointwise estimates of higher order derivatives of the power function $|\xi|^\sigma$ for $\sigma \in \mathbb{R}$, i.e., the proof of Proposition 2.1. A direct calculation of the derivatives of the function $|\xi|^\sigma$ is troublesome, but the derivatives of the function $|\xi|^2$ are easy to handle. Here, since $|\xi|^\sigma = (|\xi|^2)^{\sigma/2}$ holds, we shall show the formula of higher order derivatives of composite functions $\psi(f(\cdot))$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$. We should note that our method of the proof of Proposition 2.1 is similar to that of Kajitani [15].

In addition, we also mention that the formula of higher order derivatives of composite functions $\psi(f(\cdot))$ is called the Faà di Bruno formula, and there are a lot of previous works giving a variant formula of the original one. For instance, Roman [21, Theorem 2] gave a new proof of the Faà di Bruno formula in the case of $f : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$. Gzyl [7, Section 2] considered the more general case, i.e., $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$. Constantine-Savits [3, Theorem 2.1] and Leipnik-Pearce [18, Theorem 4.2] also treated the case of $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}$. For more details in this direction, we refer to the survey on the history of the Faà di Bruno formula by Johnson [14]. We may reach the desired formula by considering the special case of the above results, but to avoid introducing complicated notations of indices, we give a simple proof of the corresponding formula by focusing only on the case of $f : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$. Our method of proof is based on the Taylor theorem. We note that such a method is mentioned by Meyer [14, Section 3]. See also [18] for the representation of the remainder term.

Proposition A.1 *Let $\psi \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R}^n)$. Then it holds that*

$$\partial_x^\alpha\{\psi(f(x))\} = \alpha! \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \alpha^{(1)},\dots,\alpha^{(k)}\neq 0}} \frac{1}{k!}(\partial_\tau^k \psi)(f(x)) \prod_{l=1}^{k} \frac{\partial_x^{\alpha^{(l)}} f(x)}{\alpha^{(l)}!}$$

for all $\alpha \in \mathbb{N}_0^n \setminus \{0\}$ and $x \in \mathbb{R}^n$.

Proof Let $x, x_0 \in \mathbb{R}^n$ and $m \in \mathbb{N}$ be arbitrary. By the Taylor theorem, we have

$$f(x) - f(x_0) = \sum_{1\le|\alpha|\le m} \frac{1}{\alpha!}\partial_x^\alpha f(x_0)(x-x_0)^\alpha + R_m^{(0)}(x),$$
$$\lim_{x\to x_0} \frac{R_m^{(0)}(x)}{|x-x_0|^m} = 0,$$

which implies that

$$\{f(x)-f(x_0)\}^k=\left\{\sum_{1\le|\alpha|\le m}\frac{1}{\alpha!}\partial_x^\alpha f(x_0)(x-x_0)^\alpha\right\}^k+R_{m,k}^{(1)}(x),$$
$$\lim_{x\to x_0}\frac{R_{m,k}^{(1)}(x)}{|x-x_0|^{m+k-1}}=0$$

for all $k\in\mathbb{N}$. By the Taylor theorem again, we also obtain

$$\psi(f(x))-\psi(f(x_0))=\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\{f(x)-f(x_0)\}^k+R_m^{(2)}(f(x)),$$
$$\lim_{\tau\to f(x_0)}\frac{R_m^{(2)}(\tau)}{|\tau-f(x_0)|^m}=0.$$

Here, since $f(x)-f(x_0)=\nabla f(x_*)\cdot(x-x_0)$ with some $x_*\in\mathbb{R}^n$, we notice that

$$\begin{aligned}\limsup_{x\to x_0}\frac{R_m^{(2)}(f(x))}{|x-x_0|^m}&=\limsup_{x\to x_0}\frac{R_m^{(2)}(f(x))}{|f(x)-f(x_0)|^m}\frac{|f(x)-f(x_0)|^m}{|x-x_0|^m}\\&\le\limsup_{x\to x_0}\frac{R_m^{(2)}(f(x))}{|f(x)-f(x_0)|^m}|\nabla f(x_*)|^m=0.\end{aligned}$$

Moreover, let us set $R_m^{(3)}(x):=\sum_{k=1}^m(k!)^{-1}(\partial_\tau^k\psi)(f(x_0))R_{m,k}^{(1)}(x)$. Then, by a direct computation, we observe that

$$\begin{aligned}&\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\{f(x)-f(x_0)\}^k\\&=\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\left\{\sum_{1\le|\alpha|\le m}\frac{1}{\alpha!}\partial_x^\alpha f(x_0)(x-x_0)^\alpha\right\}^k+R_m^{(3)}(x)\\&=\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\prod_{l=1}^k\left\{\sum_{1\le|\alpha^{(l)}|\le m}\frac{1}{\alpha^{(l)}!}\partial_x^{\alpha^{(l)}}f(x_0)(x-x_0)^{\alpha^{(l)}}\right\}+R_m^{(3)}(x)\\&=\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\sum_{1\le|\alpha^{(1)}|,\dots,|\alpha^{(k)}|\le m}\left\{\prod_{l=1}^k\frac{\partial_x^{\alpha^{(l)}}f(x_0)}{\alpha^{(l)}!}\right\}(x-x_0)^{\alpha^{(1)}+\cdots+\alpha^{(k)}}+R_m^{(3)}(x)\\&=\sum_{k=1}^m\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\sum_{1\le|\alpha|\le m}\sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha\\\alpha^{(1)},\dots,\alpha^{(k)}\ne0}}\left\{\prod_{l=1}^k\frac{\partial_x^{\alpha^{(l)}}f(x_0)}{\alpha^{(l)}!}\right\}(x-x_0)^\alpha+R_m^{(3)}(x)\\&=\sum_{1\le|\alpha|\le m}\left\{\sum_{k=1}^{|\alpha|}\sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha\\\alpha^{(1)},\dots,\alpha^{(k)}\ne0}}\frac{1}{k!}(\partial_\tau^k\psi)(f(x_0))\prod_{l=1}^k\frac{\partial_x^{\alpha^{(l)}}f(x_0)}{\alpha^{(l)}!}\right\}(x-x_0)^\alpha+R_m^{(3)}(x).\end{aligned}$$

Therefore, it holds that

$$\begin{aligned}
&\psi(f(x)) - \psi(f(x_0))\\
&= \sum_{1\le|\alpha|\le m} \left\{ \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^{(1)} + \cdots + \alpha^{(k)} = \alpha \\ \alpha^{(1)}, \ldots, \alpha^{(k)} \neq 0}} \frac{1}{k!}(\partial_\tau^k \psi)(f(x_0)) \prod_{l=1}^{k} \frac{\partial_x^{\alpha^{(l)}} f(x_0)}{\alpha^{(l)}!} \right\} (x-x_0)^\alpha\\
&+ R_m^{(2)}(f(x)) + R_m^{(3)}(x).
\end{aligned}$$

Since $\lim_{x\to x_0}\{R_m^{(2)}(f(x)) + R_m^{(3)}(x)\}|x-x_0|^{-m} = 0$, the uniqueness of coefficients of the Taylor expansion yields

$$\frac{1}{\alpha!}\partial_x^\alpha\{\psi(f(x_0))\} = \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \alpha^{(1)}, \ldots, \alpha^{(k)}\neq 0}} \frac{1}{k!}(\partial_\tau^k\psi)(f(x_0)) \prod_{l=1}^{k} \frac{\partial_x^{\alpha^{(l)}} f(x_0)}{\alpha^{(l)}!},$$

which completes the proof of Proposition A.1. □

Proposition A.2 *It holds that*

$$\sum_{k=1}^{|\alpha|} \theta^k \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \alpha^{(1)}, \ldots, \alpha^{(k)}\neq 0}} \prod_{l=1}^{k} \frac{|\alpha^{(l)}|!}{\alpha^{(l)}!} = \frac{|\alpha|!}{\alpha!}\theta(1+\theta)^{|\alpha|-1}$$

for all $\alpha \in \mathbb{N}_0^n \setminus \{0\}$ *and* $\theta > 0$.

Proof Set $\mathscr{U}_\theta := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n < 1/(1+\theta)\}$ and define the functions $\psi : (-\infty, 1 + 1/\theta) \to (0, \infty)$ and $f : \mathscr{U}_\theta \to (0, \infty)$ by

$$\psi(\tau) := \frac{1}{1-\theta\tau/(1+\theta)}, \quad f(x) := \frac{1}{1-(x_1+\cdots+x_n)},$$

respectively. Since we have

$$\begin{aligned}
\psi(f(x)) &= \frac{\{f(x)\}^{-1}}{\{f(x)\}^{-1} - \theta/(1+\theta)} = \frac{1-(x_1+\cdots+x_n)}{1-(x_1+\cdots+x_n)-\theta/(1+\theta)}\\
&= 1 + \frac{\theta/(1+\theta)}{1/(1+\theta)-(x_1+\cdots+x_n)},
\end{aligned}$$

it holds that

$$\partial_x^\alpha\{\psi(f(x))\}\big|_{x=0} = \frac{|\alpha|!\theta/(1+\theta)}{\{1/(1+\theta)-(x_1+\cdots+x_n)\}^{|\alpha|+1}}\bigg|_{x=0} = |\alpha|!\theta(1+\theta)^{|\alpha|}.$$

In addition, we deduce that

$$\partial_\tau^k \psi(\tau)\Big|_{\tau=1} = \frac{k!\{\theta/(1+\theta)\}^k}{\{1-\theta\tau/(1+\theta)\}^{k+1}}\Bigg|_{\tau=1} = k!\theta^k(1+\theta),$$
$$\partial_x^\beta f(x)\big|_{x=0} = \frac{|\beta|!}{\{1-(x_1+\cdots+x_n)\}^{|\beta|+1}}\Bigg|_{x=0} = |\beta|!$$

for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$. Therefore, by applying Proposition A.1 with $x = 0$, we obtain

$$\begin{aligned}|\alpha|!\theta(1+\theta)^{|\alpha|} &= \alpha! \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \alpha^{(1)},\ldots,\alpha^{(k)}\neq 0}} \frac{1}{k!} \cdot k!\theta^k(1+\theta) \prod_{l=1}^k \frac{|\alpha^{(l)}|!}{\alpha^{(l)}!} \\ &= \alpha!(1+\theta)\sum_{k=1}^{|\alpha|} \theta^k \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(k)}=\alpha \\ \alpha^{(1)},\ldots,\alpha^{(k)}\neq 0}} \prod_{l=1}^k \frac{|\alpha^{(l)}|!}{\alpha^{(l)}!},\end{aligned}$$

which yields the desired estimate. This proves Proposition A.2. □

Proof of Proposition 2.1 By setting $\psi(\tau) := \tau^{\sigma/2}$ and $f(\xi) := |\xi|^2$, we have $|\xi|^\sigma = \psi(f(\xi))$. Since $\partial_\tau^k\psi(\tau) = \{\prod_{l=0}^{k-1}(\sigma/2-l)\}\tau^{\sigma/2-k}$ for all $k \in \mathbb{N}$, Proposition A.1 gives that

$$\partial_\xi^\gamma |\xi|^\sigma = \gamma! \sum_{k=1}^{|\gamma|} \sum_{\substack{\gamma^{(1)}+\cdots+\gamma^{(k)}=\gamma \\ \gamma^{(1)},\ldots,\gamma^{(k)}\neq 0}} \frac{1}{k!} \left\{\prod_{l=0}^{k-1}(\sigma/2-l)\right\} |\xi|^{\sigma-2k} \prod_{l=1}^k \frac{\partial_\xi^{\gamma^{(l)}}|\xi|^2}{\gamma^{(l)}!}.$$

Here, by a direct calculation, we have

$$\partial_\xi^\beta |\xi|^2 = \begin{cases} 0 & \text{if } |\beta| \geq 3, \\ 0 \text{ or } 2 & \text{if } |\beta| = 2, \\ 2\xi^\beta & \text{if } |\beta| = 1, \end{cases}$$

which yields $|\partial_\xi^\beta|\xi|^2| \leq 2|\xi|^{2-|\beta|}$. Thus we obtain

$$\left|\prod_{l=1}^k \frac{\partial_\xi^{\gamma^{(l)}}|\xi|^2}{\gamma^{(l)}!}\right| \leq 2^k|\xi|^{2k-(|\gamma^{(1)}|+\cdots+|\gamma^{(k)}|)} \prod_{l=1}^k \frac{1}{\gamma^{(l)}!} = 2^k|\xi|^{2k-|\gamma|} \prod_{l=1}^k \frac{1}{\gamma^{(l)}!}$$

for all $k \in \mathbb{N}$, provided $\gamma^{(1)} + \cdots + \gamma^{(k)} = \gamma$. In addition, if $|\sigma| \leq 2$, then it holds that

$$\left|\prod_{l=0}^{k-1}(\sigma/2 - l)\right| \leq \prod_{l=0}^{k-1}(|\sigma/2| + l) \leq \prod_{l=0}^{k-1}(1 + l) = k!.$$

If $|\sigma| \geq 2$, then it holds that

$$\left|\prod_{l=0}^{k-1}(\sigma/2 - l)\right| \leq \prod_{l=0}^{k-1}(|\sigma/2| + l) \leq \prod_{l=0}^{k-1}|\sigma/2|(1 + l) = |\sigma/2|^k k!,$$

which implies that $|\prod_{l=0}^{k-1}(\sigma/2 - l)| \leq (\max\{1, |\sigma/2|\})^k k!$ for all $\sigma \in \mathbb{R}$. Hence, we see by Proposition A.2 that

$$\begin{aligned}
&|\partial_\xi^\gamma |\xi|^\sigma| \\
&\leq \gamma! \sum_{k=1}^{|\gamma|} \sum_{\substack{\gamma^{(1)} + \cdots + \gamma^{(k)} = \gamma \\ \gamma^{(1)}, \ldots, \gamma^{(k)} \neq 0}} \frac{1}{k!} \cdot (\max\{1, |\sigma/2|\})^k k! |\xi|^{\sigma - 2k} \cdot 2^k |\xi|^{2k - |\gamma|} \prod_{l=1}^{k} \frac{1}{\gamma^{(l)}!} \\
&\leq \gamma! |\xi|^{\sigma - |\gamma|} \sum_{k=1}^{|\gamma|} \sum_{\substack{\gamma^{(1)} + \cdots + \gamma^{(k)} = \gamma \\ \gamma^{(1)}, \ldots, \gamma^{(k)} \neq 0}} (\max\{2, |\sigma|\})^k \prod_{l=1}^{k} \frac{|\gamma^{(l)}|!}{\gamma^{(l)}!} \\
&= \gamma! |\xi|^{\sigma - |\gamma|} \cdot \frac{|\gamma|!}{\gamma!} (\max\{2, |\sigma|\})(\max\{2, |\sigma|\} + 1)^{|\gamma| - 1} \\
&\leq (\max\{2, |\sigma|\} + 1)^{|\gamma|} |\gamma|! |\xi|^{\sigma - |\gamma|},
\end{aligned}$$

which completes the proof of Proposition 2.1. □

References

1. Bae, H.: Analyticity of solutions to the barotropic compressible Navier-Stokes equations. J. Differ Equ **269**(2), 1718–1743 (2020)
2. Bourdaud, G.: Realizations of homogeneous Besov and Lizorkin-Triebel spaces. Math. Nachr. **286**(5–6), 476–491 (2013)
3. Constantine, G.M., Savits, T.H.: A multivariate Faá di Bruno formula with applications. Trans. Am. Math. Soc. **348**(2), 503–520 (1996)
4. Dong, H., Pan, X.: Time analyticity for inhomogeneous parabolic equations and the Navier-Stokes equations in the half space. J. Math. Fluid Mech. 22, no. 4, Paper No. 53, 20 (2020)
5. Dong, H., Zhang, Q.S.: Time analyticity for the heat equation and Navier-Stokes equations. J. Funct. Anal. 279, no. 4, 108563, 15 (2020)
6. Giga, Y., Gries, M., Hieber, M., Hussein, A., Kashiwabara, T.: Analyticity of solutions to the primitive equations. Math. Nachr. **293**(2), 284–304 (2020)
7. Gzyl, H.: Multidimensional extension of Faa di Bruno's formula. J. Math. Anal. Appl. **116**(2), 450–455 (1986)

8. Han, F., Hua, B., Wang, L.: Time analyticity of solutions to the heat equation on graphs. Proc. Am. Math. Soc. **149**(6), 2279–2290 (2021)
9. Hoshino, G.: Global well-posedness and analytic smoothing effect for the dissipative nonlinear Schrödinger equations. J. Dyn. Differ. Equ. **31**(4), 2339–2351 (2019)
10. Hoshino, G.: Space-time analytic smoothing effect for the cubic nonlinear Schrödinger equations without pseudoconformal invariance. Partial Differ. Equ. Appl. 3, no. 1, Paper No. 14, 21 (2022)
11. Hoshino, G., Ozawa, T.: Analytic smoothing effect for nonlinear Schrödinger equation with quintic nonlinearity. J. Math. Anal. Appl. **419**(1), 285–297 (2014)
12. Iwabuchi, T.: Analyticity and large time behavior for the Burgers equation and the quasi-geostrophic equation, the both with the critical dissipation. Ann. Inst. H. Poincaré Anal. Non Linéaire **37**(4), 855–876 (2020)
13. Iwabuchi, T.: On analyticity up to the boundary for critical quasi-geostrophic equation in the half space. Commun. Pure Appl. Anal. **21**(4), 1209–1224 (2022)
14. Johnson, W.P.: The curious history of Faà di Bruno's formula. Am. Math. Mon. **109**(3), 217–234 (2002)
15. Kajitani, K.: Global real analytic solutions of the Cauchy problem for linear partial differential equations. Commun. Partial Differ. Equ. **11**(13), 1489–1513 (1986)
16. Kozono, H., Ogawa, T., Taniuchi, Y.: Navier-Stokes equations in the Besov space near L^∞ and BMO. Kyushu J. Math. **57**(2), 303–324 (2003)
17. Kozono, H., Okada, A., Shimizu, S.: Characterization of initial data in the homogeneous Besov space for solutions in the Serrin class of the Navier-Stokes equations. J. Funct. Anal. 278, no. 5, 108390, 47 (2020)
18. Leipnik, R.B., Pearce, C.E.M.: The multivariate Faà di Bruno formula and multivariate Taylor expansions with explicit integral remainder term. ANZIAM J. **48**(3), 327–341 (2007)
19. Ogawa, T., Sato, T.: Analytic smoothing effect for system of nonlinear Schrödinger equations with general mass resonance. Hiroshima Math. J. **50**(1), 73–84 (2020)
20. Ozawa, T., Yamauchi, K.: Remarks on analytic smoothing effect for the Schrödinger equation. Math. Z. **261**(3), 511–524 (2009)
21. Roman, S.: The formula of Faà di Bruno. Am. Math. Mon. **87**(10), 805–809 (1980)
22. Sawano, Y.: Theory of Besov Spaces, Developments in Mathematics, vol. 56. Springer, Singapore (2018)
23. Takeuchi, T.: Space-time analytic smoothing effect of the heat semigroup defined on homogeneous Besov spaces. Partial Differ. Equ. Appl. Math. **4**, 100174 (2021)
24. Zeng, C.: Time analyticity of the biharmonic heat equation, the heat equation with potentials and some nonlinear heat equations. Commun. Pure Appl. Anal. **21**(3), 749–783 (2022)
25. Zhang, Q.S.: A note on time analyticity for ancient solutions of the heat equation. Proc. Am. Math. Soc. **148**(4), 1665–1670 (2020)