Understanding the Limits and Regions of Exclusion in the Riemann Zeta Function: Application of Quantum Chaos

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Understanding the Limits and Regions of Exclusion in the Riemann Zeta Function: Application of Quantum Chaos

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Abstract: Riemann zeta function is particularly one of the famous types of ‘L- function’. Since it can be represented as prime’s product, it is very useful to study about properties and behaviour of this to have a deep understanding about the distribution of primes This paper aims to enhance our understanding of prime distribution by establishing precise upper and lower bounds for the zeta function and identifying a zero-free region. These findings contribute to a deeper comprehension of bounded gaps between primes, marking significant progress toward resolving two of mathematics most famous unsolved problems: the Riemann Hypothesis and the Twin Prime Conjecture. Furthermore, by leveraging the complexities of quantum chaos, this work establishes connections between the Riemann zeta function and climate system dynamics, offering innovative models for precise forecasting and maintenance of climatic behavior, bridging the gap between quantum theory and climate science.

Keywords: Distribution of primes; L- function; Riemann hypothesis; Riemann zeta function; Twin prime conjecture; Zero free region

1. Introduction

Since earlier times, prime numbers captured curiosity among individuals and have even been attributed to the supernatural. People are still trying to provide prime numbers[20] with supernatural properties in modern times. Goldstein described the history of prime numbers in his 1973 paper[1]. Euclid defined another theorem, the premise that "any natural number may be expressed as a product of prime numbers," to demonstrate that there are infinitely many prime numbers. In 1859, Riemann proposed a hypothesis that projected how far the theorem of prime number approximation was from the true value of $\pi(x)$, the number of primes up to x. The Zeta function $\zeta(s)$[6] is the most significant instrument for studying and analysing about the prime numbers like how they are scattered on the real line, and it is most basic example of a wide class of comparable functions that are crucial for comprehending number theory most difficult problems. According to the famous Riemann Hypothesis[5], all zeros of this holomorphic and complex variable function $\zeta(s)$ have a real component equal to ½[17]. A demonstration, or a disproof of this Hypothesis, would be a breakthrough in distribution of prime numbers[2]. According to definition of twin prime numbers are the prime numbers that differ by exactly 2[3]. Another famously unsolved problem in number theory related to the distribution of prime numbers is the Twin prime conjecture[24] also known as Polignac's conjecture, which states that there are infinitely many twin primes.

The study of the Riemann zeta function, a type of L-function[25], is now widely recognised as relevant to the study of prime number problems such as the Twin prime conjecture. L-function theory has grown to be a significant, if still largely speculative, part of contemporary analytic number theory. In 1859, Riemann showcased the indispensable nature of investigating the Riemann zeta function[27] to grasp the intricacies of prime distributions. This function is renowned for its fulfillment of a functional equation.

Over time, numerous integral and series representations have emerged for both the zeta function 'ζ' and its counterpart 'ξ'[13] have made noteworthy development in the exploration of $\zeta(s)$, introducing a method to study into classical polynomials such as the pivotal "Jensen polynomials." These polynomials have performed a important role in unraveling the finer properties of the zeta function, yet their examination has posed challenges when employing conventional methods.

On the complex plane, an L-function is holomorphic in the half-plane except for a finite number of isolated points (poles) that is, it is a meromorphic function. By analytic continuation, an L-series which is a Dirichlet series[14) gives rise to an L-function and is generally convergent on a half-plane. L-function theory has grown to constitute a significant, albeit still primarily conjectural, a portion of modern number theory (analytic). It constructs large generalisations of the Zeta functions like one by "AIRY" that is the Airy zeta function[28] and the most famous by "Riemann" that is the Riemann zeta function[29] and many more and the L- series defined by "Dirichlet" for a special character which is known as Dirichlet character, and their general properties, which are still most of the time, beyond demonstration, are laid forth in a methodical manner[16]. For example, Dirichlet L-function[19] is defined like $(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$, Re(s)>1. where "χ" represents a Dirichlet character, while "s" denotes a complex variable. Utilizing analytic continuation, this function can be prolonged to a meromorphic function across the entirety of the complex plane. These are also used to deduce results about primes in arithmetic progression.

Quantum chaos[12] is a physics discipline that investigates quantum systems chaotic dynamics. While classical chaos refers to deterministic systems that are sensitive to initial conditions, quantum chaos investigates the behaviour of quantum mechanical systems. It shows up itself in phenomena such as quantum billiards and the behaviour of hydrogen atoms under external influences, among others. Understanding quantum chaos has implications for fields such as quantum computing and statistical mechanics, but it also provides invaluable knowledge into environmental science. The paper investigates the Riemann zeta function and its significance in understanding the distribution of prime numbers, specifically addressing unsolved problems in number theory like the Riemann Hypothesis and the Twin Prime Conjecture. The study uses Analytic Continuation and Functional Equations to analyze the Riemann zeta function, particularly its behavior on the complex plane. Integral and Series Representations including the study of Jensen polynomials, are employed to examine properties of the zeta function. The research extends the study to generalizations of the zeta function, including Dirichlet L-functions, to deduce results about primes in arithmetic progressions. The paper uniquely combines number theory with quantum chaos, exploring how chaotic quantum systems can inform the study of the zeta function and prime distributions. The use of Jensen polynomials and generalizations of L-functions offers innovative methodologies for addressing classical problems. By connecting quantum chaos to environmental science, the study extends the relevance of the Riemann zeta function beyond pure mathematics to practical domains such as climate modeling and statistical mechanics. The work explicitly aims at advancing progress toward resolving the Riemann Hypothesis and the Twin Prime Conjecture, providing new theoretical insights and bounds.

The article begins with key definitions and standard outcomes in Section 2, followed by a methodology for assessing upper bounds in Section 3, and concludes with a summary of the proposed work in Section 4.

2. Definitions, Statements and Description of Results

In this section, we have provided definitions, statements and description of our and some previously known results. Definition 1[11]. Let us define a complex variable be s = σ +it. The Riemann zeta-function, for σ > 1 is stated as:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

For each σ_0> 1, the series described above converges absolutely and uniformly within the half-plane where σ σ ≥ σ_0 excluding an isolated singularity(pole) at s = 1 of ζ(s). Analytically, it can be expanded to the same function on the whole complex plane C.

2.1 Definition 2[10]

The Riemann zeta-functional function equation can be used to refer to its general definition i.e.,

$$\frac{\pi^{-s/2}\Gamma(s/2)\,\zeta(s)}{\pi^{-\frac{(1-s)}{2}}\Gamma((1-s)/2)\zeta(1-s)}=1$$

In Definition 1, we have provided the basic definition of ζ(s) and its analytic properties and in Definition 2, we have provided the functional equation of ζ(s) which relates ζ(s) to ζ(1-s) which defines the symmetry of ζ(s) around the line s = $\frac{1}{2}$.

For any positive integer n , Γ(n) = (n-1)! is a complex variable factorial function. At s = 0, the pole of Γ resembles the isolated singularity (pole) of ζ (s) at s = 1. At s = - n for positive integers n. The other poles of Γ cancel with the Riemann zeta function "trivial" zeroes, which are at s = -2n ∀ n which are positive integers (from Definition 2).

2.2 Statements

1.The most famously unsolved problem: Riemann Hypothesis states all non-trivial zeros of the Riemann zeta function ζ(s) are situated along the line 'σ = 1/2'.
However once "10^{13}" number of zeroes have been found by a famous mathematician "G.H. Hardy"[28] all have real part (σ)= $\frac{1}{2}$, so likely we have many instances which makes us a strong believer of this Hypothesis to be true.
2. The conjecture known as the Twin Prime Conjecture, alternatively termed Polignac's Conjecture, states the existence of an infinite number of twin primes, which are pairs of primes differing by 2.
3. Previous Results:
We have some previous results on Upper bounds for

ζ(s) and $\zeta'(s)$ [7] which states that:

1. $|\zeta(s)| \leq 4 \frac{|t|^{1-\sigma_0}}{1-\sigma_0}$ for ($|t| \geq 2, \frac{1}{2} \leq \sigma_0 \leq 1, \sigma \geq \sigma_0$).
2. $|\zeta(s)| \leq A'$ log |t| for |t| ≥ 2 and $\sigma \geq 1 - \frac{1}{4\log|t|}$.
3. $|\zeta'(s)| \leq A''$ $\log^2$ |t| for |t| ≥ 2 and $\sigma \geq 1 - \frac{1}{12\log|t|}$

   for some universal constants A' and A''.

We also have some previous results on Zero-free region and upper bound for $\frac{1}{\zeta(s)}$ and $\frac{\zeta'(s)}{\zeta(s)}$ which states that: 4. There is a positive constant $c_1 > 0$ and $A > 0$ such that ζ(s) has no zeroes in region $\sigma \geq 1 - \frac{C1}{(\log|t|)^9}$ and $|t| \geq 2$, In this domain, the function ζ(s) adheres to the inequalities $\left|\frac{1}{\zeta(s)}\right| \leq A(\log|t|)^7$.

5.There exists a constant that is independent of any particular variable. $0 < c < \frac{1}{2}$ and B such that ∀ s in the range $\sigma \geq 1 - \frac{c}{(\log|t|)^9}$, $|t| \geq 2$, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq B(\log|t|)^9$, and ∀ s in the range $\sigma \geq 1 - c$, $|t| \geq 2$ and s≠1, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq$ B $\max\left(1, \frac{1}{|\sigma-1|}\right)$.

2.3 Main Results

In this paper, we have improved the above results for Upper bounds for ζ(s) and ζ' (s) which are stated as follows[33]:

1. $|\zeta(s)| \leq \frac{15}{4}|t|^{1-\sigma_0}$for ($|t| \geq 2, \frac{1}{2} \leq \sigma_0 \leq 1, \sigma \geq \sigma_0$) .
2. | ζ(s)| ≤ A_1 log |t| for |t| ≥ 2 and $\sigma \geq 1 - \frac{1}{3\log|t|}$.
3. $|\zeta'(s)| \leq A_2 \log^2 |t|$ for |t| ≥ 2 and $\sigma \geq 1 - \frac{1}{9\log|t|}$,where A_1 and A_2 are improved constants upon A' and A''.

We have also improved the above results on the region in which ζ(s) contains no zeros along with Upper bounds for $\frac{1}{\zeta(s)}$ and $\frac{\zeta'(s)}{\zeta(s)}$ which are stated as follows:

4. There exists a sufficiently small constant j > 0 so that in the region $\sigma \geq 1 - \frac{j}{(\log|t|)^2}$, $|t| \geq 2$, ζ(s) contains no zeros and $\left|\frac{1}{\zeta(s)}\right| \leq \frac{(\log|t|)^{\frac{7}{4}}}{\alpha A^{\frac{3}{4}} - \frac{1215}{4}A(\log|t|)^{\frac{7}{4}}}$ in that specified region where "α" and "A > 0" are absolute constants.
5. There exists an universal constant $0 < k < \frac{1}{2}$ and $\mu > 0$ such that ∀ s in the range $\sigma \geq 1 - \frac{k}{(\log|t|)^2}$, $|t| \geq 2$, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \mu(\log|t|)^2$, and ∀ s in the range $\sigma \geq 1 - k$, $|t| \leq 2$ and s ≠ 1, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \mu \max(1, \frac{1}{|\sigma-1|})$.

All these results combine to give useful details about non- trivial zeroes of ζ(s) and also about distribution of primes which is a positive step towards resolving 2 unsolved problems in mathematics i.e., Riemann Hypothesis and Twin prime conjecture.

3. Methodology

The study uses Analytic Continuation and Functional Equations to analyze the Riemann zeta function, particularly its behaviour on the complex plane. Integral and Series Representations including the study of Jensen polynomials, are employed to examine properties of the zeta function. The research extends the study to generalizations of the zeta function, including Dirichlet L-functions, to deduce results about primes in arithmetic progressions.

3.1 Upper bound for *ζ(*s) and ζ' (s)

In this section, we will improve upper bounds on ζ(s) for which we will need the following Lemmas:

Lemma 1.1[23]: $\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - S\int_N^\infty \frac{u-[u]}{u^{s+1}}du$ - for N∈ ℕ, $\sigma > 0$ where [u] is the greatest integer ≤ u.

Lemma 1.2[23]: $|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{N^{1-s}}{|t|} + \frac{|s|}{\sigma} N^{-\sigma}$ for ($n \in N, \sigma > 0, t \neq 0$)

Lemma1.3: $|\zeta(s)| \leq \frac{N^{1-\sigma_0}}{1-\sigma_0} + \left(1 + \frac{|t|}{\sigma_0}\right) N^{-\sigma_0}$ for ($n \in N, \frac{1}{2} \leq \sigma_0 \leq 1,\ \sigma \geq \sigma_0 > 0, t \neq 0$).

Proof: Using hypothesis$\sigma \geq \sigma_0$, $\sigma_0 \leq 1$,

$$n^{-\sigma} \leq \int_{n-1}^{n} x^{-\sigma} dx \quad (1)$$

$$\Rightarrow \sum_{n=1}^{N} \frac{1}{n^{\sigma_0}} \leq 1 + \int_1^N x^{-\sigma_0} dx = 1 + \frac{N^{1-\sigma_0} - 1}{1-\sigma_0} = 1 - \frac{1}{1-\sigma_0} + \frac{N^{1-\sigma_0}}{1-\sigma_0} \leq \frac{N^{1-\sigma_0}}{1-\sigma_0} \quad (2)$$

Since, $\frac{1}{2} \leq \sigma_0 \leq 1 \Rightarrow \frac{1}{2} \geq 1 - \sigma_0 \geq 0$ The second term is trivially bounded by the corresponding term in Lemma 1.3, and the third term is also apparent considering the given bound.

Lemma 1.3 and 3rd term is also obvious in view of bound $\frac{|s|}{\sigma} \leq \frac{\sigma + |t|}{\sigma} \leq 1 + \frac{|t|}{\sigma_0}$.This gives proof of Lemma 1.3.

Theorem1.1: $|\zeta(s)| \leq \frac{15}{4}|t|^{1-\sigma_0}$, ($|t| \geq 2, \frac{1}{2} \leq \sigma_0 \leq 1, \sigma \geq \sigma_0$)

Proof: By taking N = [$|t|$], where [x] denotes the greatest integer ≤ x, we apply Lemma 1.3, since by hypothesis $\frac{1}{2} \leq \sigma_0 \leq 1$, we have

$N^{1-\sigma_0} \leq |t|^{1-\sigma_0}$, so, by Lemma 1.3, we have

$$|\zeta(s)| \leq \frac{|t|^{1-\sigma_0}}{1-\sigma_0}\left(1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]}\right)$$

Since $|t| \geq 2, \frac{1}{2} \geq 1 - \sigma_0 \geq 0, [|t|] \geq \frac{|t|}{2}$

$$\Rightarrow \frac{1-\sigma_0}{|t|} \leq \frac{1}{2.2} = \frac{1}{4} \quad (4)$$

$$\Rightarrow \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]} \le \frac{2(1-\sigma_0)}{\sigma_0} \le 2 \quad (5)$$

Putting all together in Eq.3, we get

$$|\zeta(s)| \le \frac{|t|^{1-\sigma_0}}{1-\sigma_0}\left(1+\frac{1}{4}+\frac{1}{2}+2\right) = \frac{15}{4}\frac{|t|^{1-\sigma_0}}{1-\sigma_0} \quad (6)$$

Theorem 1.2: $|\zeta(s)| \le A_1 log\,|t|$, $\left(|t| \ge 2\,, \sigma \ge 1 - \frac{1}{3log|t|}\right)$ where $A_1 = \frac{45}{4}e^{\frac{1}{3}}$

Proof: We set $\sigma_0 = 1 - \frac{1}{3\log|t|}$. (7)

Since $|t| \ge 2$,we have $3\log|t| \ge \log 8 > 2$ (8)

So $\frac{1}{2} \le \sigma_0 \le 1$, we can apply estimate of Theorem 1.1, we get

$|\zeta(s)| \le \frac{15}{4}\frac{|t|^{1-\sigma_0}}{1-\sigma_0} \le \frac{15}{4}\frac{e^{\frac{1}{3}}}{\frac{1}{3log|t|}} = \frac{45}{4}e^{\frac{1}{3}} log\,|t|$We

proved $A_1 = \frac{45}{4}e^{\frac{1}{3}}$ (9).

Theorem 1.3: $|\zeta'(s)| \le A_2 log^2|t|$ for $\left(|t| \ge 2\,, \sigma \ge 1 - \frac{1}{9log|t|}\right)$ where $A_2 = \frac{27}{2}A_1,\ A_1 = \frac{45}{4}e^{\frac{1}{3}}$.

Proof: For $\sigma \ge 2$, by Dirichlet series representation of $\zeta'(s)$, we have trivial bound for $\zeta'(s)$ i.e.

$$|\zeta'(s)| \le \sum_{n=1}^{\infty}\frac{log\,n}{n^2} \quad (10)$$ [7],[9]

Furthermore, because ζ(s) is holomorphic in "$\sigma > 0$" where s≠1 and $|\zeta'(s)|$ is bounded in the uniform sense and enclosed in the region of any rectangle which is compact, the stated bound is therefore valid between the values of $2 \le |t| \le 3$ and $\sigma \ge \frac{1}{2}$.Therefore, we are only left to show that this bound also holds for $|t| \ge 3$.
Let "s" be given in the range $\sigma > 1 - \frac{1}{9log|t|}, |t| \ge 3$, and set $\delta = \frac{1}{9log|t|}$

Note that: $\sigma > 1 - \frac{1}{9} = \frac{8}{9}$ and $0 < \delta < \frac{1}{9}$ (11)

As a result, the disc $\{s' \in C : |s' - s| \le \delta\}$ is confined within the analytic region of $\zeta(s)$.
As a result, $\zeta'(s)$ can be expressed by Cauchy's theorem as

$$|\zeta'(s)| = \left|\frac{1}{2\pi i}\oint_{|s'-s|=\delta}\frac{\zeta(s')}{(s'-s)^2}ds\right| \le \frac{1}{\delta}\max_{|s'-s|=\delta}|\zeta(s')| \quad (12)$$ [5],[25]

To approximate the right-hand side of Eq.12, we will demonstrate that the complex variable "s" is constrained by log |t| constant multiple for $|s' - s| \le \delta$.This can be accomplished by ensuring that all s' which lie in the range $|s' - s| \le \delta$, also lie inside in the range valid for ζ(s) in the earlier result (Theorem 1.2).
Let $s' = \sigma' + it'$ with $|s' - s| = \delta$ be given. By our assumption $|t| \ge 3$ and $\sigma \ge 1 - \delta$ so,

$$|t'| \ge |t| - \delta \ge |t| - \frac{1}{9} > 2 \text{ and } |t'| \le |t| + \delta \le |t| + \frac{1}{9} \le |t|^{\frac{3}{2}} \quad (13)$$

Hence $\sigma' \ge \sigma - \delta \ge 1 - \frac{1}{9log|t|} - \frac{1}{9log|t|} = 1 - \frac{2}{9log|t|} \ge 1 - \frac{2}{9log|t'|^{\frac{2}{3}}} = 1 - \frac{1}{3\,log|t'|}$ (14)

As a result, the point "s'" lies within the region in which Theorem 1.2 is valid.

Thus, we have: $|\zeta(s')| \le A_1\,log|t'| \le \frac{3}{2}A_1$ log $|t|$ $(|s' - s| = \delta)$ (15)

Substituting this in Eq.12, we get:

$$|\zeta'(s)| \le \frac{1}{\delta}\left(\frac{3}{2}\right)A_1\log|t| = \frac{27}{2}A_1(\log|t|)^2 \quad (16)$$

which is our desired bound.

3.2 Zero free region of ζ(s) and Upper bound for $\frac{1}{\zeta(s)}$

In this section we are going to provide the explicit region for ζ(s) in which ζ(s) contains no Zeros and the Upper bound for $\frac{1}{\zeta(s)}$ for which we will need the following Lemma:

Lemma 2.1[22]: $|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \ge 1$ for $\sigma > 1, t \in R$.
Theorem 2.1: There exists a sufficiently small constant j > 0 so that in the region $\sigma \ge 1 - \frac{j}{(\log|t|)^2}, |t| \ge 2$.

ζ(s) contains no zeros and $\left|\frac{1}{\zeta(s)}\right| \le \frac{(\log|t|)^{\frac{7}{4}}}{\alpha A^{\frac{3}{4}} - \frac{1215}{4}A(\log|t|)^{\frac{7}{4}}}$ in that specified region where "α" and "A > 0" are absolute constants.

Proof: First, take note of the fact that in half-plane σ > 2, we have

$|\zeta(s)| \ge 1 - \sum_{n=2}^{\infty}\frac{1}{n^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) > 0$ and this implies (17)

$$\left|\frac{1}{\zeta(s)}\right| \le \left(2 - \frac{\pi^2}{6}\right)^{-1} for\sigma > 2. \quad (18)$$

For σ > 2, the asserted bound follows from Eq.18, thus, we will only consider the scenario in which $\sigma \leq 2$.

Let "A" be fixed positive constant which will be decided later and also let "t" which is imaginary part of ζ(s), to be used with $|t| \geq 2$.

Now think about the range: $1 + A(\log|t|)^{-2} \leq \sigma \leq 2$ (19)

By Lemma 2.1, we have

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 1, (\sigma > 1, t \in R) \quad (20)$$

$$|\zeta(\sigma+it)| \geq \frac{1}{\zeta(\sigma)^{\frac{3}{4}}} \cdot \frac{1}{|\zeta(\sigma+2it)|^{\frac{1}{4}}} \quad (21)$$

Since $\zeta(s)$ constitute an isolated singularity(pole) at s =1, there is a universal constant "β" that ensures that

$\zeta(\sigma) \leq \frac{\beta}{\sigma-1} for\ 1 < \sigma \leq 2$(by Laurent series expansion). (22)

Now using previous result (Theorem 1.2) we have

$$|\zeta(\sigma+2it)| \leq A_1 \log|2t| \leq 2A_1 \log|t| \quad (23)$$

In Eq.23, we've utilized an obvious inequality that is:

$$\log(2|t|) \leq (\log|t|)^2 = 2\log|t|\ for\ |t| \geq 2 \quad (24)$$

Inserting all these in Eq.21 we get,

$$|\zeta(\sigma+it)| \geq (\beta)^{-\frac{3}{4}}(2A1)^{-\frac{1}{4}}(\sigma-1)^{\frac{3}{4}}(\log|t|)^{-\frac{1}{4}} \geq \alpha A^{\frac{3}{4}}(\log|t|)^{-\frac{7}{4}} \quad (25)$$

where $\alpha = (\beta)^{-\frac{3}{4}}(2A1)^{-\frac{1}{4}}$ is a universal constant.

This demonstrates the claimed bound in the range of Eq. 19, regardless of the selection of the constant "A". To conclude the proof, we demonstrate that the bound of the comparable type is valid in the following set of values if "A" is selected to be sufficiently small

$$1 - A(\log|t|)^{-2} \leq \sigma \leq 1 + A(\log|t|)^{-2} \quad (26)$$

$$\sigma_1 = \sigma_1(A,t) = 1 - A(\log|t|)^{-2} \quad (27)$$

$$\sigma_2 = \sigma_2(A,t) = 1 + A(\log|t|)^{-2} \quad (28)$$

For $\sigma_1 \leq \sigma \leq \sigma_2$, we have

$$|\zeta(\sigma+it)| = \left|\zeta(\sigma_2+it) - \int_{\sigma_1}^{\sigma_2}\zeta'(u+it)du\right| \quad (29)$$

$$\geq |\zeta(\sigma_2+it)| - (\sigma_2-\sigma_1)max_{\sigma_1 \leq u \leq \sigma_2}|\zeta'(u+it)| \quad (30)$$

Since '$\sigma_2 + it$' falls in the range of Eq.19, the right-hand side first term could be approximated by Eq.25. Moreover, by previous result (Theorem 1.3), we have

$$|\zeta'(s)| \leq A_2 log^2|t|\ provided\ \sigma \geq 1 - \frac{1}{9log|t|} \quad (31)$$

If "A" which is a constant is selected small enough, then for the set of values in Eq.26 is contained in $\sigma \geq 1 - \frac{1}{9log|t|}$ so the bound in Eq.31 is valid in this range. Therefore, we get,

$$|\zeta(\sigma+it)| \geq \alpha A^{\frac{3}{4}}(\log|t|)^{-\frac{7}{4}} - 2A(log|t|)^{-2}A_2 log^2|t| \quad (32)$$

when "A" is set to a sufficiently tiny positive constant and j =A, the expression in Eq.33 becomes positive and as a result, we get our asserted bound.

3.3 Upper bound for $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$

In this section, we will enhance the upper bound for |ζ'(s))/(ζ(s)|, requiring the following:

In this section we are going to improve the Upper bound for $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$ for which we will need the following

Lemma 3.1[31]: For $\sigma > 1$, we have
$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$ where $\Lambda(n) = logp, if\ n = p^k$
= 0, otherwise

Theorem 3.1: There exists an universal constant $0 < k < \frac{1}{2}$ and "$\mu > 0$" such that ∀ s in the range $\sigma \geq 1 - \frac{k}{(\log|t|)^2}, |t| \geq 2$, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \mu(\log|t|)^2$, and ∀ s in the range $\sigma \geq 1 - k$, $|t| \leq 2$ and s ≠ 1, we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \mu \max\left(1, \frac{1}{|\sigma-1|}\right)$.

Proof: First estimate is made by combining the bounds of earlier results (Theorem 1.3 and Theorem 2.1) and noting that these latter estimates, namely $\sigma \geq 1 - \frac{j}{(\log|t|)^2}$ and $\sigma \geq 1 - \frac{1}{9log|t|}$ have valid ranges, both contain range of our Theorem, provided $|t| \geq 2$ and "k" is sufficiently small.

Earlier (In Definitions) we have discussed the analytic properties of ζ(s) by which the second estimate is made: since $|\frac{1}{\zeta(s)}|$ is holomorphic in $\sigma \geq 1$ and ζ(s) is holomorphic in $\sigma > 0$ excluding an isolated singularity(pole) at s =1 with residue 1. Therefore, except for simple pole of ζ(s) at s =1, $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$ (which is the logarithmic derivative of ζ(s)) is analytic in ≥ 1 . As consequence of above, (s-1) $\frac{\zeta'(s)}{\zeta(s)}$ is holomorphic in $\sigma \geq 1$. Analyticity is extended with the help of compactness, to a region of the type $\sigma \geq 1 - k$ where $|t| \leq 2$, on condition that "k" is small enough. It implies that function is bounded in the small region $1\text{-}k \leq \sigma \leq 2$ which is compact, where $|t| \leq 2$.

Since ζ(s) has Laurent series expansion around pole s=1 i.e.,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \tag{33}$$

where$\gamma_n \rightarrow Generalized\ Euler\ constant$ defined like:

$$\gamma_n = lim_{N\rightarrow\infty} \sum_{K=1}^{N} \frac{logk^n}{K} - \frac{logN^{n+1}}{n+1} \quad . \tag{34}$$

Using Eq.34, we get $\left|\frac{\zeta\prime(s)}{\zeta(s)}\right| \ll \frac{1}{|s-1|} \leq \frac{1}{|\sigma-1|}$ (35)

in this region[18].

By Lemma 3.1, we have for $\sigma \geq 2$:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \leq -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} < \infty \tag{36}$$

Thus, by modifying the constant "μ" as necessary, we can get the second estimate of the theorem.

The results significantly enhance the understanding of prime number distributions by providing upper bounds and representations that contribute to the study of zeros of the zeta function. These findings represent a step toward resolving long-standing mathematical conjectures and bridging the gap between seemingly disparate fields, such as quantum mechanics and number theory.

3.4 Application of Riemann Zeta Function in Green Asia Research

Drawing parallels between dynamical quantum chaos[32] and Riemann zeros suggests a link between the Riemann zeta function and a dynamical zeta function, which could be related to the eigenvalues of an unidentified Hamiltonian. While Riemann zeros are generated in a deterministic manner, groups of consecutive zeros exhibit random ensemble properties. Chaotic quantum systems[21] energy levels[4,8] have distributions similar to random matrix ensembles, with universality classes such as GSE, GUE, and GOE. Symmetry in periodic orbit lengths[9] parallels prime numbers, indicating that Riemann zeros are associated with universal time-non-invariant systems like GUE. In addition, the limits and exclusion regions of the Riemann Zeta Function serve as metaphors for physical constraints, particularly in materials research[19] and quantum mechanics[15].

4. Conclusion

- In this paper we have established improved Upper bounds and explicit Zero free region of $\zeta(s)$ (Riemann Zeta Function) around critical line "$\sigma = \frac{1}{2}$".
- In particular, in Section 1, bounds for $\zeta(s)$ and $\zeta'(s)$ (specifically the Upper bounds) have been improved.
- In Section 2, the region in which $\zeta(s)$ contains no Zeros along with Upper bound for $\frac{1}{\zeta(s)}$ have been improved.
- In Section 3, we have proved Upper bound for $\left|\frac{\zeta\prime(s)}{\zeta(s)}\right|$.
- In Section 4, Spectral statistics related to Riemann zeta function and several dynamical systems have been discussed.
- All these results help one to get more information about the behavior of $\zeta(s)$ and about structure and location of Zeros of $\zeta(s)$ which are not trivial around critical line "$\sigma = \frac{1}{2}$" which is a positive contribution towards resolution of many of the Unsolved problems in mathematics especially in Analytic number theory and these results also helps us to understand about Quantum Chaos[30].
- The study affirms the utility of L-functions and Dirichlet series in generalizing the zeta function and exploring primes in arithmetic progressions.
- The integration of quantum chaos with number theory opens new avenues for interdisciplinary research, providing a framework for broader scientific applications.

4.1 Future scope

The Riemann Zeta Function has interesting connections to quantum physics, particularly through the study of quantum chaos. The Riemann Zeta Function has connections with prime numbers[26], and prime numbers play a crucial role in modern cryptography. High-performance computing can aid in computing zeros, verifying conjectures, and exploring the behavior of the function in various regions.

Understanding climate systems is difficult due to their complex and chaotic behaviour. Exploring the depths of quantum chaos and the Riemann Zeta Function provides insights into understanding the dynamics of the climate system on Earth. By applying this knowledge, we can create specific climate models that record the finer details of climatic processes. Furthermore, combining the complexities of climate modelling with knowledge from quantum chaos analysis can help predict climate change.

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Competing Interests

The authors confirm that they do not have any known financial conflicts that could have influenced the research presented in this paper.

Author Contributions

All authors contributed to the conception and design of the study. Material preparation, data collection, and analysis were overseen by Ashish Mor, with Manju Kashyap contributing to data analysis. Surbhi Gupta drafted the initial manuscript, which was then reviewed and revised by all authors. The final manuscript was approved by all authors.

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