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Dynamics of infectious diseases in predator–prey populations: A stochastic model, sustainability, and invariant measure

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ABSTRACT

This paper introduces an innovative model for infectious diseases in predator–prey populations. We not only prove the existence of global non-negative solutions but also establish essential criteria for the system's decline and sustainability. Furthermore, we demonstrate the presence of a Borel invariant measure, adding a new dimension to our understanding of the system. To illustrate the practical implications of our findings, we present numerical results. With our model's comprehensive approach, we aim to provide valuable insights into the dynamics of infectious diseases and their impact on predator–prey populations.

1. Introduction

The predator–prey relationship is one of the fundamental interactions in ecosystems and serve as the building blocks of large food chains and food webs. Several mathematical models have been proposed to describe this relationship with specialist and generalist predators. One such model, originally introduced by Leslie [1,2], has attracted considerable attention from researchers. This model takes the form of a system of differential equations:

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)[a_1 - b_1 x(t)] - p(x)y(t), \\ \dfrac{dy(t)}{dt} = y(t)\left[a_2 - b_2 \dfrac{y(t)}{x(t)}\right]. \end{cases} \tag{1.1}$$

Here, $x(t)$ and $y(t)$ denote the densities of the prey and predator populations at time t, respectively. The prey and predator populations grow logistically with intrinsic growth rates of a_1 and a_2, respectively, and with environmental carrying capacities of $\frac{a_1}{b_1}$ and $\frac{a_2 x}{b_2}$. The function $p(x)$ denotes intake rate of predator.

When $p(x) = \frac{ax}{b+x}$, it is called a Holling type II functional response [3]. Here, parameters $\frac{a}{b}$ and $\frac{1}{ab}$ respectively mean encounter rate of a predator with prey and handling time taken by the predator for a prey individual.

When $p(x) = \frac{ax}{b+y+x}$, it is known as the Beddington–DeAngelis functional response [4,5]. Here, a is the maximal consumption rate of predator, and b is a measure of abundance of prey and predator relative to the environment in which they interact. In [6], authors considered a deterministic predator–prey model with Beddington–DeAngelis functional response and generalist predators

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)[a_1 - b_1 x(t)] - \dfrac{ax(t)y(t)}{b + y(t) + x(t)}, \\ \dfrac{dy(t)}{dt} = y(t)\left[a_2 - b_2 \dfrac{y(t)}{c + x(t)}\right]. \end{cases} \tag{1.2}$$

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and its stochastic version. Here, the per-capita growth rate of generalist predator population is modified to $a_2 - b_2 \frac{y(t)}{c+x(t)}$ in (1.2) based upon the assumption that the effective focal prey dependent carrying capacity is proportional to the sum of favorite prey population and alternative implicit food source. The effective focal prey dependent carrying capacity for the generalist predators is equal to $\frac{a_2(c+x)}{b_2}$. Therefore, it prevents the decline of net generalist predators growth rate when the focal prey species is rare. This feature is realistic from an ecological standpoint, as generalist predators typically survive on an assortment of prey, giving themselves a sort of environmental buffer when the focal resource begin to run short. For example, wolves are generalist predators that can prey on various animals, including deer, rabbits, and mice, demonstrating the ability to adapt the abrupt change in available prey densities.

It is worth noting that the environmental carrying capacities of predators in Eq. (1.1) denoted by $\frac{a_2 x}{b_2}$ and in Eq. (1.2), denoted by $\frac{a_2(c+x)}{b_2}$, have lower limits of 0 and $\frac{a_2 c}{b_2}$, respectively. However, these capacities do not have an upper bound that is independent of the initial values. As a consequence, in cases where the initial population of prey is very large, the carrying capacity of generalist predators can also be very high.

On the other hand, in the real world, infectious diseases are a common occurrence in populations. One of the most well-known models for infectious diseases is the SIR model, which was introduced by Kernmack and McKendrick [7]. This model is described by a system of differential equations,:

$$\begin{cases} \dfrac{dS(t)}{dt} = -\dfrac{\beta}{N} S(t)I(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta}{N} S(t)I(t) - \gamma I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t). \end{cases} \tag{1.3}$$

The population is divided into three compartments:: “susceptible” $S(t)$, “infected” $I(t)$, and “recovered” $R(t)$, where $N = S(t) + I(t) + R(t)$ remains constant over time. The function $S(t)$ represents the number of individuals who have not been infected with the disease or who are still susceptible to it. The function $I(t)$ represents the number of infected individuals who can infect the susceptible individuals. The function $R(t)$ denotes the number of individuals who have recovered from the disease, including those who have died and can no longer be infected or transmit the disease to others. The parameter $\beta > 0$ represents the average number of individuals that an infected person can infect per day, while $\gamma > 0$ denotes the recovery rate, $\frac{1}{\gamma}$ measures the average recovery time. The SIR model assumes a short-lived outbreak and does not consider natural birth and death rates. Moreover, it assumes that recovered individuals acquire lifetime immunity and are immune to the disease.

Hence, the consideration of infectious diseases' impact on predator–prey populations holds significance in real-world ecological systems. Various researchers have investigated the dynamics of infectious diseases in predator–prey models. For instance, Pierre et al. [8] developed a deterministic model in which the predator species is affected by a disease, while Xiao and Chen [9] analyzed a predator–prey model with disease in the prey population. Eilersen et al. [10] investigated a deterministic model with one predator and two prey species, in which one prey species carries a disease. In contrast, Li and Wang [11] studied a classical stochastic predator–prey model with disease in the predator population. Notably, their models assume that the predator species declines if there is no prey, i.e., the predators are specialist and they have no alternative food source.

In this paper, we investigate a predator–prey model that includes an infectious disease in the predator species and exhibits logistic growth. We assume that the environmental carrying capacity of predators is bounded both above and below by constants. Unlike the SIR model (1.3) introduced earlier, we assume that any recovered predator individual can be infected again, and the infection rate is the same as that for susceptible individuals.

Let $x(t)$ denote the prey density, $S(t)$ denote the density of susceptible predators, and $I(t)$ denote the density of infected predators at time t. Our model is described by the following system of stochastic differential equations:

$$\begin{cases} dx(t) = \left[x(t)\{a_1 - b_1 x(t)\} - \dfrac{ax(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)} \right] dt + \sigma_1 x(t) dW_1(t), \\ dS(t) = \left[S(t)\left(a_2 - \left\{ \dfrac{1}{c+x(t)} + h \right\} \{S(t)+I(t)\} \right) - \beta(S,I)S(t)I(t) + \gamma I(t) \right] dt \\ \qquad + \sigma_2 S(t) dW_2(t), \\ dI(t) = [\beta(S,I)S(t)I(t) - \alpha I(t)]dt + \sigma_3 I(t) dW_3(t) \end{cases} \tag{1.4}$$

coupled with non-negative initial values $(x(0), y(0), z(0))$. As stated above, we assume that the upper bound of the environmental carrying capacity of predator is a constant, say $\frac{a_2}{h}$, which is the maximal value of the fraction between two terms a_2 and $\frac{1}{c+x(t)} + h$ in (1.4). In addition, we suppose that the rate of infection $\beta(S,I)$ depends on the densities of susceptible and infected predators, and given by

$$\beta(S,I) = \frac{\beta^*}{S(t)+I(t)+k},$$

where $\frac{\beta^*}{k}$ is the maximal infection rate. The parameter $\gamma > 0$ is the recovery rate of infected predators. Therefore, the term $\gamma I(t)$ in (1.4) shows the contribution of infected population to the growth of the susceptible population. The constant α is the sum of the recovery rate γ and the mortality α^* of infected predators. Here, $\alpha^* > 0$ represents the probability of a predator dying after being infected.

Furthermore, in (1.4), the parameters a_1, a_2, and α^* are influenced by white noise, which can be modeled as a generalized derivative of Brownian motion:

$$a_1 \to a_1 + \sigma_1 dW_1(t),\ a_2 \to a_2 + \sigma_2 dW_2(t),\ \alpha^* \to \alpha^* - \sigma_3 dW_3(t), \tag{1.5}$$

where $W_i(t), i = 1,2,3$ are independent Brownian motions defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. Here, the positive constants σ_i represent the intensity of the noise. This is reasonable because the growth rate and mortality of populations are subject to environmental factors that are often random and unpredictable. Other parameters in (1.4) have the same meanings as in the models above. We suppose that all parameters in the model are positive constants.

The present paper is organized as follows: In the next section, we prove the existence of global positive solutions for the stochastic model. Section 3 provides sufficient conditions for the decline of each predator and prey species. We give a sustainability condition for this system in Section 4. In Section 5, we prove the existence of a Borel invariant measure. Section 6 presents numerical illustrations to validate the analytical findings. Finally, the paper concludes with a summary of our key findings in Section 7.

2. Global solutions

In this section, we prove the existence of unique global non-negative solutions to (1.4) and provide some properties of the solutions.

Theorem 2.1. *Let $(x_0, S_0, I_0) \in \overline{\mathbb{R}^3_+}$. Then, there exists a unique global solution $(x(t), S(t), I(t))$ of (1.4), such that $(x(0), S(0), I(0)) = (x_0, S_0, I_0)$ and $(x(t), S(t), I(t)) \in \overline{\mathbb{R}^3_+}$ a.s. Furthermore, if $(x_0, S_0, I_0) > 0$, then $(x(t), S(t), I(t)) \in \mathbb{R}^3_+$ a.s.*

Proof. Since all functions on the right-hand side of (1.4) are locally Lipschitz continuous on $\overline{\mathbb{R}^3_+}$, there is a unique local solution $(x(t), S(t), I(t))$ defined on an interval $[0, \tau)$, where τ is a stopping time. We know that [12,13], if $\mathbb{P}\{\tau < \infty\} > 0$ then τ is an explosion time on $\tau < \infty$, i.e., $\lim_{t\to\tau}(|x(t)| + |S(t)| + |I(t)|) = \infty$ or $\lim_{t\to\tau}[S(t) + I(t) + x(t) + b] = 0$ a.s. on $\tau < \infty$.

To establish the existence of global solutions, we must demonstrate that $\tau = \infty$ a.s. To do so, we will examine five cases.

Case 1: $x_0 = S_0 = I_0 = 0$. This is a trivial case because the initial densities of all three populations are zero, which implies that they will remain zero for all time. Thus, we have $x(t) = S(t) = I(t) = 0$ a.s. for $0 < t < \infty$.

Case 2: $x_0 > 0, S_0 > 0, I_0 = 0$. By uniqueness, we have $I(t) = 0$ a.s. for all $t \in [0, \tau)$. Therefore, the system of the first two equations in (1.4) reduces to a stochastic Holling–Tanner type model. The existence of global positive solution $(x(t), S(t))$ can be found in [6].

Case 3: $x_0 = 0, S_0 > 0, I_0 > 0$.

By uniqueness, $x(t) = 0$ a.s. for all $t \in [0, \tau)$. Therefore, the system of the two first equations in (1.4) is a SIR model with logistic growth. Similarly to Case 2, the existence of global positive solution $(S(t), I(t))$ is given in [12].

Case 4: $x_0 > 0, S_0 > 0, I_0 > 0$. Let $k_0 > 0$ be a positive integer such that x_0, S_0, I_0 lie in the interval $[\frac{1}{k_0}, k_0]$.

Denote

$$H_k = \left[\frac{1}{k}, k\right] \times \left[\frac{1}{k}, k\right] \times \left[\frac{1}{k}, k\right], \qquad k = 1, 2, \ldots$$

then $\cup_{k=k_0}^{\infty} = \mathbb{R}^3_+$. We define a sequence $\{\tau_k\}_{k=k_0}^{\infty}$ of stopping times by

$$\tau_k = \inf\{0 < t < \tau; (x(t), S(t), I(t)) \notin H_k\},$$

with the convention $\inf \emptyset = \infty$. It is clear that this sequence is non-decreasing. Therefore, there exists a limit of this sequence, called τ_∞, and

$$\tau_\infty = \lim_{k\to\infty} \tau_k \leqslant \tau \qquad a.s.$$

Thus, to prove that $\tau = \infty$, we prove that $\tau_\infty = \infty$ a.s. Indeed, suppose on the contrary that $\tau_\infty < \infty$. Then, there exists $T > 0$ and $0 < \varepsilon < 1$ such that

$$\mathbb{P}\{\tau_\infty < T\} > \varepsilon.$$

Consider a positive increasing continuous function V defined by

$$\begin{aligned} V(x, S, I) &= x^2 + S^2 + I^2 - \log x - \log S - \log I \\ &= V_1 + V_2 + V_3, \quad x > 0, S > 0, I > 0, \end{aligned}$$

where

$$V_1 = x^2 - \log x, V_2 = S^2 - \log S, V_3 = I^2 - \log I. \tag{2.1}$$

Then,

$$dV = dV_1 + dV_2 + dV_3.$$

Taking Itô formula into V_1 first, we have

$$dV_1 = \left[\mu\left(2x - \frac{1}{x}\right) + \frac{1}{2}\sigma_1^2 x^2\left(2 + \frac{1}{x^2}\right)\right]dt + \sigma_1 x\left(2x - \frac{1}{x}\right)dW_1,$$

where $\mu = x(t)(a_1 - b_1x(t)) - \frac{ax(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)}$. Thus,

$$\begin{aligned}
dV_1 =& \Big[[x(t)(a_1 - b_1x(t)) - \frac{ax(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)}](2x(t) - \frac{1}{x(t)}) \\
&+ \frac{1}{2}\sigma_1^2x^2(t)(2 + \frac{1}{x^2(t)})\Big]dt + \sigma_1x(t)(2x(t) - \frac{1}{x(t)})dW_1 \\
=& \{2x^2(t)(a_1 - b_1x(t)) - (a_1 - b_1x(t)) - \frac{2ax^2(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)} + \frac{a(S(t)+I(t))}{b+S(t)+I(t)+x(t)} \\
&+ \sigma_1^2x^2(t) + \frac{1}{2}\sigma_1^2\}dt + \sigma_1x(t)(2x(t) - \frac{1}{x(t)})dW_1 \\
\leqslant& \{2a_1x^2(t) - (a_1 - b_1x(t)) + \frac{a(S(t)+I(t))}{b+S(t)+I(t)+x(t)} + \sigma_1^2x^2(t) + \frac{1}{2}\sigma_1^2\}dt \\
&+ \sigma_1x(t)(2x(t) - \frac{1}{x(t)})dW_1.
\end{aligned} \tag{2.2}$$

Since $x(t)$, $S(t)$, $I(t)$ are positive for $t \in [0, \tau_\infty)$, there exist M_1, M_2 such that

$$\begin{aligned}
&2a_1x^2(t) - (a_1 - b_1x(0)) + \frac{a(S(t)+I(t))}{b+S(t)+I(t)+x(t)} + \sigma_1^2x^2(t) + \frac{1}{2}\sigma_2^2 \\
&\leqslant 2a_1x^2(t) - (a_1 - b_1x(t)) + a + \sigma_1^2x^2(t) + \frac{1}{2}\sigma_2^2 \\
&\leqslant M_1V_1 + M_2.
\end{aligned}$$

Thus,

$$dV_1(t) \leqslant (M_1V_1 + M_2)dt + \sigma_1x(2x - \frac{1}{x})dW_1, \qquad t \in [0, \tau_\infty). \tag{2.3}$$

Taking the same argument to V_2, V_3, we have

$$\begin{aligned}
dV_2 =& [[S(t)(a_2 - (\frac{1}{x(t)+e} + h)(S(t)+I(t))) - \beta(S,I)I(t)S(t) + \gamma I(t)](2S(t) - \frac{1}{S(t)}) \\
&+ \frac{1}{2}\sigma_2^2S^2(t)(2 + \frac{1}{S^2(t)})]dt + (2\sigma_2S^2(t) - \sigma_2)dW_2 \\
=& [2S^2(t)\{a_2 - (\frac{1}{x(t)+e} + h)(S(t)+I(t))\} - 2\beta(S,I)I(t)S^2(t) + 2\gamma I(t)S(t) \\
&- \{a_2 - (\frac{1}{x(t)+e} + h)(S(t)+I(t))\} + \beta(S,I)I(t) - \gamma\frac{I(t)}{S(t)} + \sigma_2^2S^2(t) + \frac{1}{2}\sigma_2^2]dt \\
&+ (2\sigma_2S^2(t) - \sigma_2)dW_2 \\
\leqslant& [2a_2S^2(t) + 2\gamma I(t)S(t) + (\frac{1}{x(t)+e} + h)(S(t)+I(t)) + \beta(S,I)I(t) + \sigma_2^2S^2(t) + \frac{1}{2}\sigma_2^2]dt \\
&+ (2\sigma_2S^2(t) - \sigma_2)dW_2,
\end{aligned}$$

and

$$\begin{aligned}
dV_3 =& [(\beta(S,I)S(t)I(t) - \alpha I(t))(2I(t) - \frac{1}{I(t)}) + \sigma_3^2I^2(t) + \frac{1}{2}\sigma_3^2]dt \\
&+ (2\sigma_3I^2(t) - \sigma_3)dW_3 \\
=& [2\beta(S,I)S(t)I^2(t) - 2(\alpha-\gamma)I^2(t) - 2\gamma I^2(t) - \beta(S,I)S(t) + \alpha + \sigma_3^2I^2(t) + \frac{1}{2}\sigma_3^2]dt \\
&+ (2\sigma_3I^2(t) - \sigma_3)dW_3.
\end{aligned} \tag{2.4}$$

Since $\beta(S,I) = \frac{\beta^*}{S(t)+I(t)+k}$ where k is a constant, $\beta(S,I)S(t) \leqslant \beta^*$, and $\beta(S,I)I(t) \leqslant \beta^*$. There exist $M_3, M_4 > 0$ such that

$$\begin{aligned}
&2a_2S^2(t) + 2\gamma I(t)S(t) + (\frac{1}{x(t)+e} + h)(S(t)+I(t)) + \beta(S,I)I(t) + \sigma_2^2S^2(t) + \frac{1}{2}\sigma_2^2 \\
&\leqslant M_3V_2 + \frac{1}{2}M_3V_3 + M_4,
\end{aligned}$$

and

$$2\beta(S,I)S(t)I^2(t) - 2(\alpha-\gamma)I^2(t) - 2\gamma I^2(t) - \beta(S,I)S(t) + \alpha + \sigma_3^2I^2(t) + \frac{1}{2}\sigma_3^2 \leqslant \frac{1}{2}M_3V_3 + M_4.$$

As a consequence,

$$dV_2 \leqslant (M_3V_2 + \frac{1}{2}M_3V_3 + M_4)dt + [2\sigma_2S^2(t) - \sigma_2]dW_2(t), \tag{2.5}$$

and

$$dV_3 \leqslant (\frac{1}{2}M_3V_3 + M_4)dt + [2\sigma_3 I^2(t) - \sigma_3]dW_3(t). \tag{2.6}$$

From (2.3), (2.5) and (2.6), we have

$$dV \leqslant (K_1V + K_2)dt + \sigma_1 x(t)(2x(t) - \frac{1}{x(t)})dW_1(t) + (2\sigma_2 S^2(t) - \sigma_2)dW_2(t) + (2\sigma_3 I^2(t) - \sigma_3)dW_3(t),$$

where $K_1 = 3\max\{M_1, M_3\}$, $K_2 = 3\max\{M_2, M_4\}$.

Therefore, we have

$$\int_0^{t\wedge\tau_k} dV \leqslant \int_0^{t\wedge\tau_k} (K_1V + K_2)\,ds + \int_0^{t\wedge\tau_k} \sigma_1 x(s)(2x(s) - \frac{1}{x(s)})\,dW_1 + \int_0^{t\wedge\tau_k} (2\sigma_2 S^2(s) - \sigma_2)\,dW_2 + \int_0^{t\wedge\tau_k} (2\sigma_3 I^2(s) - \sigma_3)\,dW_3, \qquad t \in [0, T].$$

Taking expectation in the two sides of the above inequality, we obtain

$$\mathbb{E}\int_0^{t\wedge\tau_k} dV \leqslant \mathbb{E}\int_0^{t\wedge\tau_k} (K_1V + K_2)\,dt + \mathbb{E}\int_0^{t\wedge\tau_k} \sigma_1 x(s)(2x(s) - \frac{1}{x(s)})\,dW_1 + \mathbb{E}\int_0^{t\wedge\tau_k} (2\sigma_2 S^2(s) - \sigma_2)\,dW_2 + \mathbb{E}\int_0^{t\wedge\tau_k} (2\sigma_3 I^2(s) - \sigma_3)\,dW_3, \qquad t \in [0, T].$$

Then,

$$\begin{aligned}\mathbb{E}[V(x(t\wedge\tau_k), S(t\wedge\tau_k), I(t\wedge\tau_k)) - V(x_0, S_0, I_0)] &\leqslant K_2\mathbb{E}(t\wedge\tau_k) + \mathbb{E}\int_0^{t\wedge\tau_k} K_1V\,dt \\ &\leqslant K_2T + \int_0^{t\wedge\tau_k} \mathbb{E}K_1V\,dt, \qquad t \in [0, T].\end{aligned}$$

According to the Gronwall inequality, we can draw a conclusion that

$$\mathbb{E}V(x_{t\wedge\tau_k}, S_{t\wedge\tau_k}, I_{t\wedge\tau_k}) \leqslant [V(x_0, S_0, I_0) + K_2T]e^{K_1t}, \qquad t \in [0, T].$$

Thus,

$$\mathbb{E}V(x_{T\wedge\tau_k}, S_{T\wedge\tau_k}, I_{T\wedge\tau_k}) \leqslant [V(x_0, S_0, I_0) + K_2T]e^{K_1T}. \tag{2.7}$$

On the other hand,

$$\begin{aligned}\mathbb{E}V(x(T\wedge\tau_k), S(T\wedge\tau_k), I(T\wedge\tau_k)) &\geqslant \mathbb{E}\mathbf{1}_{\{\tau_\infty<T\}}V(x(T\wedge\tau_k), S(T\wedge\tau_k), I(T\wedge\tau_k)) \\ &\geqslant \mathbb{E}\mathbf{1}_{\{\tau_\infty<T\}}V(x(\tau_k), S(\tau_k), I(\tau_k)) \\ &\geqslant \varepsilon\mathbb{E}V(x(\tau_k), S(\tau_k), I(\tau_k)),\end{aligned} \tag{2.8}$$

(because we have $\mathbb{P}(\tau_\infty < T) > \varepsilon$).

From (2.7) and (2.8), we have

$$\varepsilon\mathbb{E}V(x(\tau_k), S(\tau_k), I(\tau_k)) \leqslant [V(x_0, S_0, I_0) + K_2T]e^{K_1T} < \infty, \qquad t \in [0, T]. \tag{2.9}$$

According to the definition of τ_k, we find that $(x(\tau_k), S(\tau_k), I(\tau_k)) \in \partial H_k$. Therefore,

$$\begin{aligned}\mathbb{E}V(x(\tau_k), S(\tau_k), I(\tau_k)) &\geqslant \min\{K^2 - \log K, (\frac{1}{K})^2 - \log\frac{1}{K}\} \\ &= \min\{K^2 - \log K, (\frac{1}{K})^2 + \log K\}.\end{aligned}$$

Thus, taking $K \to \infty$ and using (2.9), we have $\infty \leq [V(x_0, S_0, I_0) + K_2T]e^{K_1T} < \infty$. This contradiction implies that $\tau_\infty = \infty$ a.s. Thus, $\tau = \infty$ a.s.

Case 5: $x_0 > 0, S_0 = 0, I_0 > 0$. Consider two stopping times $\tau_1^* = \inf\{0 < t < \tau : x(t) = 0\}$ and $\tau_2^* = \inf\{0 < t < \tau : I(t) = 0\}$ with the convention $\inf\emptyset = \infty$. Obviously, we have $x(t) > 0$ in $[0, \tau_1^*)$ and $I(t) > 0$ in $[0, \tau_2^*)$.

Putting $\tau^* = \min\{\tau_1^*, \tau_2^*\}$, we have $x(t) > 0$ and $I(t) > 0$ in $[0, \tau^*)$.

Firstly, we prove that $S(t) > 0$ in $(0, \tau^*)$. Consider the equations:

$$dS(t) = \{S(t)[a_2 - (\frac{1}{x(t)+e} + h)(S(t) + I(t))] - \beta(S, I)S(t)I(t) + \gamma I(t)\}dt + \sigma_2 S(t)dW_2(t),$$

and

$$d\underline{S}(t) = \{\underline{S}(t)[a_2 - (\frac{1}{x(t)+e} + h)(\underline{S}(t) + I(t))] - \beta(S, I)\underline{S}(t)I(t)\}dt + \sigma_2\underline{S}(t)dW_2(t),$$

where $\underline{S}(0) = S(0) = 0$. We have $\underline{S}(t) = 0$. Since $\gamma I(t) > 0$ on $[0, \tau_2^*)$, using the comparison theorem, we have $S(t) \geqslant 0$, $t \in [0, \tau_2^*)$.

Let us now show that $\mathbb{E}I(t)$ is bounded above. From

$$dI(t) = \{\beta(S,I)S(t)I(t) - \alpha I(t)\}dt + \sigma_3 I(t)dW_3(t),$$

we have

$$d\ln I(t) = \{\beta(S,I)S(t) - \alpha - \frac{1}{2}\sigma_3^2\}dt + \sigma_3 dW_3(t).$$

Taking the integration of both the hand sides, we have

$$\ln\frac{I(t)}{I_0} = \int_0^t [\beta(s)S(s) - \alpha - \frac{1}{2}\sigma_3^2]ds + \int_0^t \sigma_3\, dW_3(s).$$

Then,

$$\begin{aligned}\mathbb{E}I(t) &\leqslant I_0\mathbb{E}e^{\int_0^t[\beta(s)S(s)-\gamma-(\alpha-\gamma)-\frac{1}{2}\sigma_3^2]ds+\int_0^t\sigma_3 dW_3(s)}\\ &\leqslant I_0\mathbb{E}e^{\int_0^t[\beta^*-\alpha-\frac{1}{2}\sigma_3^2]ds+\int_0^t\sigma_3 dW_3(s)} \qquad t\in[0,\bar{T}],\end{aligned}$$

where $\bar{T}$ is a positive constant. Since $\mathbb{E}e^{\int_{t_0}^t \alpha(s)\,dW_s} = e^{\frac{1}{2}\int_{t_0}^t \alpha^2(s)ds}$ [14], we have

$$\mathbb{E}I(t) \le e^{(\beta^*-\alpha)t} \le \tilde{M} = \max\{e^{(\beta^*-\alpha)\bar{T}}, 1\}, \qquad t\in[0,\bar{T}].$$

Using this result, we now consider the boundedness of $S(t)$. Taking integration and expectation of the above equation for S, we have

$$\begin{aligned}\mathbb{E}S(t) &\leqslant \int_0^t \{a_2\mathbb{E}S(t) - h\mathbb{E}S^2(t)\}\,dt + \gamma\int_0^t \mathbb{E}(I(t))dt\\ &\leqslant \int_0^t \{a_2\mathbb{E}S(t) - h\mathbb{E}^2S(t)\}\,dt + \gamma\int_0^t \tilde{M}dt, \qquad t\in[0,\bar{T}].\end{aligned}$$

Therefore, $\mathbb{E}S(t) \le y(t), t\in[0,\bar{T}]$, where y is the solution of this ordinary differential equation:

$$\begin{cases} dy(t) = \{a_2 y(t) - hy^2(t) + \gamma\tilde{M}\}dt,\\ y(0) = S_0 > 0, \qquad t\in[0,\bar{T}].\end{cases} \tag{2.10}$$

It is easy see that there exists $\tilde{\tilde{M}} > 0$ such that $y(t) \le \tilde{\tilde{M}}$ in $[0,\bar{T}]$. Therefore, we have $\mathbb{E}S(t) \leqslant \tilde{\tilde{M}}$ in $[0,\bar{T}]$.

Let us now prove the global positivity of x, S, and I. We use the same method in Case 4 to show that $\tau^* = \infty$. Consider a positive integer k_0 such that x_0, I_0 lie in $[\frac{1}{k_0}, k_0]$. Define

$$H_k^2 = [\frac{1}{k}, k]\times[\frac{1}{k}, k],$$

and

$$\tau_k = \inf\{0 < t < \tau^*; (x(t), I(t)) \notin H_k\}.$$

The increasing sequence $\{\tau_k\}_{k=k_0}^{\infty}$ has a limit τ_∞ satisfying

$$\tau_\infty = \lim_{k\to\infty}\tau_k \leqslant \tau^* \qquad a.s.$$

Therefore, it suffices to show that $\tau_\infty = \infty$ a.s. We suppose the contrary. Then, there exist $T > 0$, and $0 < \varepsilon < 1$, such that

$$\mathbb{P}\{\tau_\infty < T\} > \varepsilon.$$

Consider a positive function

$$H(x, I) = x^2 + I^2 - \log x - \log I = V_1 + V_3,$$

where V_1 and V_3 are defined by (2.1). By the Itô formula, we have

$$\begin{aligned}dV_1 =&[2x^2(t)(a_1 - b_1x(t)) - (a_1 - b_1x(t)) - \frac{2ax^2(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)} + \frac{a(S(t)+I(t))}{b+S(t)+I(t)+x(t)}\\ &+ \sigma_1^2x^2(t) + \frac{1}{2}\sigma_1^2]dt + \sigma_1x(t)[2x(t) - \frac{1}{x(t)}]dW_1,\end{aligned}$$

and

$$\begin{aligned}dV_3 =&[2\beta(S,I)S(t)I^2(t) - 2(\alpha-\gamma)I^2(t) - 2\gamma I^2(t) - \beta(S,I)S(t) + (\alpha-\gamma) + \gamma + \sigma_3^2I^2(t)\\ &+ \frac{1}{2}\sigma_3^2]dt + (2\sigma_3I^2(t) - \sigma_3)dW_3.\end{aligned}$$

Using the same arguments in Case 4 (see (2.3) and (2.6)), it is clear that there exist $K_1, K_2 > 0$ such that

$$dV_1 \leqslant (K_1V_1 + K_2)dt + \sigma_1x(t)(2x(t) - \frac{1}{x(t)})dW_1, \qquad t\in[0,\tau_k),$$

and

$$dV_3 \leqslant [(K_1V_3 + K_2)]dt + \sigma_3 I(t)(2I(t) - \frac{1}{I(t)})dW_3, \qquad t \in [0, \tau_k).$$

Thus, we have

$$dH(t) \leqslant [K_1H(t) + 2K_2]dt + \sigma_1 x(t)(2x(t) - \frac{1}{x(t)})dW_1 + \sigma_3 I(t)(2I(t) + \frac{1}{I(t)})dW_3.$$

Therefore,

$$\int_0^{t\wedge\tau_k} dH \leqslant \int_0^{t\wedge\tau_k} (K_1H(s) + 2K_2)\,ds + \int_0^{t\wedge\tau_k} \sigma_1 x(s)(2x(s) - \frac{1}{x(s)})\,dW_1 + \int_0^{t\wedge\tau_k} (2\sigma_3 I^2(s) - \sigma_3)\,dW_3, \qquad t \in [0, T].$$

Taking expectation in the two sides, we obtain that

$$\mathbb{E}\int_0^{t\wedge\tau_k} dH \leqslant \mathbb{E}\int_0^{t\wedge\tau_k} (K_1H(s) + K_2)\,ds + \mathbb{E}\int_0^{t\wedge\tau_k} \sigma_1 x(s)(2x(s) - \frac{1}{x(s)})\,dW_1 + \mathbb{E}\int_0^{t\wedge\tau_k} (2\sigma_3 I^2(s) - \sigma_3)\,dW_3.$$

Thus

$$\begin{aligned}\mathbb{E}[V(x(t\wedge\tau_k), I(t\wedge\tau_k)) - V(x_0, I_0)] &\leqslant K_2(t\wedge\tau_k) + \mathbb{E}\int_0^{t\wedge\tau_k} K_1H\,ds \\ &\leqslant K_2T + \int_0^{t\wedge\tau_k} \mathbb{E}K_1H\,ds.\end{aligned}$$

According to the Gronwall inequality, we can draw a conclusion that

$$\begin{aligned}\mathbb{E}H(x(t\wedge\tau_k), I(t\wedge\tau_k)) &\leqslant [H(x_0, I_0) + K_2T]e^{K_1t} \\ &\leqslant [H(x_0, I_0) + K_2T]e^{K_1T}, \qquad t \in [0, T].\end{aligned}$$

Using the same arguments in Case 4, we arrive at a contradiction that

$$\infty \leq [H(x_0, I_0) + K_2T]e^{K_1T} < \infty.$$

Therefore $\tau_\infty = \infty$ a.s., and then $\tau^* = \infty$ a.s.

Thanks to Cases 1∼5, the proof is complete. □

In the next theorem, we show boundedness in expectation for x, S, I. Boundedness in expectation means that, on average, both predator and prey population densities will remain within a predictable range at all future time, in the presence of random environmental fluctuations.

Theorem 2.2. *Let $(x(t), S(t), I(t))$ be the solution of (1.4) with initial data $(x_0, S_0, I_0) \in \overline{\mathbb{R}^3_+}$. Then*

(i) For $1 \leqslant \theta < \infty$, there exists $\alpha_\theta > 0$ such that

$$\mathbb{E}x^\theta(t) \leqslant \alpha_\theta, \quad t \in [0, \infty).$$

(ii) If $\alpha - \beta^* > 0$, then for any $1 \leqslant \theta < 1 + \frac{2(\alpha - \max\{\beta^*, \gamma\})}{\sigma_3^2}$, there exists a constant $\bar{\alpha}_\theta > 0$ such that

$$\mathbb{E}S^\theta(t) + \mathbb{E}I^\theta(t) \leq \bar{\alpha}_\theta, \quad t \in [0, \infty).$$

Proof. Let us first prove (i). From

$$dx(t) = \left[x(t)\{a_1 - b_1x(t)\} - \frac{ax(t)(S(t) + I(t))}{b + S(t) + I(t) + x(t)}\right]dt + \sigma_1 x(t)dW_1(t),$$

and the Itô formula, we have for $\theta \geq 1$

$$\begin{aligned}dx^\theta(t) =&[\theta x^\theta(t)\{a_1 - b_1x(t)\} - \frac{a\theta x^\theta(t)(S(t) + I(t))}{b + S(t) + I(t) + x(t)} + \frac{1}{2}\theta(\theta - 1)\sigma_1^2 x^\theta(t)]dt + \sigma_1\theta x^\theta(t)dW_1(t), \\ \leqslant&[\theta x^\theta(t)\{a_1 - b_1x(t)\} + \frac{1}{2}\theta(\theta - 1)\sigma_1^2 x^\theta(t)]dt + \sigma_1\theta x^\theta(t)dW_1(t).\end{aligned}$$

Taking integration and expectation in the two sides, we obtain

$$\mathbb{E}x^\theta(t) \leqslant \int_0^t [a_1\theta\mathbb{E}x^\theta(s) - b_1\theta\mathbb{E}x^{\theta+1}(s) + \frac{1}{2}\theta(\theta - 1)\mathbb{E}x^\theta(s)]\,ds + \mathbb{E}\int_0^t \theta\sigma_1 x^\theta(s)\,dW_1.$$

By using the Hölder inequality, we have

$$\mathbb{E}x^{\theta}(t) \leqslant \int_0^t [a_1\theta\mathbb{E}x^{\theta}(s) - b_1\theta(\mathbb{E}x^{\theta}(s))^{\frac{\theta+1}{\theta}} + \frac{1}{2}\theta(\theta-1)\mathbb{E}x^{\theta}(s)]ds.$$

It is easy to see that there exists a constant $C(\theta) > 0$ such that

$$a_1\theta\mathbb{E}x^{\theta}(s) - b_1\theta(\mathbb{E}x^{\theta}(s))^{\frac{\theta+1}{\theta}} + \frac{1}{2}\theta(\theta-1)\mathbb{E}x^{\theta}(s) < C(\theta) - \mathbb{E}x^{\theta}(s).$$

Thus,

$$d\mathbb{E}x^{\theta}(t) \leqslant [C(\theta) - \mathbb{E}x^{\theta}(t)]dt.$$

By the comparison theorem, $\mathbb{E}x^{\theta}(t) \leqslant y(t)$ where y is the solution to this ordinary differential equation

$$\begin{cases} dy = [C(\theta) - y]dt, \\ y(0) = \mathbb{E}x^{\theta}(0). \end{cases}$$

It is easy to see that there exists $\alpha_\theta > 0$ such that $y(t) \leqslant \alpha_\theta$ for every $t \geq 0$. Thus,

$$\mathbb{E}x^{\theta}(t) \leq \alpha_\theta, \quad t \in [0, \infty).$$

Next, let us prove (ii). From the second and third equations of (1.4) and the It$\hat{o}$ formula, we have

$$\begin{aligned} dS^{\theta}(t) =& [\theta S^{\theta}(t)(a_2 - \{\frac{1}{x(t)+e} + h\}\{S(t)+I(t)\}) - \theta\beta(S,I)S^{\theta}(t)I(t) + \theta\gamma S^{\theta-1}(t)I(t)]dt \\ &+ \frac{1}{2}\theta(\theta-1)\sigma_2^2 S^{\theta}(t)dt + \theta\sigma_2 S^{\theta}(t)dW_2, \end{aligned}$$

and

$$dI^{\theta}(t) = \Big[\theta I^{\theta-1}(t)[\beta(S,I)S(t)I(t) - \alpha I(t)] + \frac{1}{2}\theta(\theta-1)\sigma_3^2 I^{\theta}(t)\Big]dt + \theta\sigma_3 I^{\theta}(t)dW_3.$$

Taking integration and expectation of the two sides, we obtain that

$$\mathbb{E}S^{\theta}(t) \leqslant \int_0^t [\theta a_2\mathbb{E}S^{\theta}(s) - h\theta\mathbb{E}S^{\theta+1}(s) + \theta\gamma\mathbb{E}(S^{\theta-1}(s)I(s))]\,ds + \int_0^t \frac{1}{2}\theta(\theta-1)\sigma_2^2\mathbb{E}S^{\theta}(s)\,ds,$$

and

$$\mathbb{E}I^{\theta}(t) \leqslant \int_0^t [\theta\beta^*\mathbb{E}(I^{\theta-1}(s)S(s)) - \gamma\theta\mathbb{E}I^{\theta}(s) - (\alpha-\gamma)\theta\mathbb{E}I^{\theta}(s) + \frac{1}{2}\theta(\theta-1)\sigma_3^2\mathbb{E}I^{\theta}(s)]\,ds.$$

By using the Hölder inequality, we then have

$$d\mathbb{E}S^{\theta}(t) \leqslant \Big[\theta a_2\mathbb{E}S^{\theta}(t) - h\theta\{\mathbb{E}S^{\theta}(t)\}^{\frac{\theta+1}{\theta}} + \theta\gamma\{\mathbb{E}S^{\theta}(t)\}^{\frac{\theta-1}{\theta}}\{\mathbb{E}I^{\theta}(t)\}^{\frac{1}{\theta}} + \frac{1}{2}\theta(\theta-1)\sigma_2^2\mathbb{E}S^{\theta}(t)\Big]dt,$$

and

$$d\mathbb{E}I^{\theta}(t) \leqslant \Big[\theta\beta^*\{\mathbb{E}(I^{\theta}(t))\}^{\frac{\theta-1}{\theta}}\{\mathbb{E}S^{\theta}(t)\}^{\frac{1}{\theta}} - \gamma\theta\mathbb{E}I^{\theta}(t) - (\alpha-\gamma)\theta\mathbb{E}I^{\theta}(t) + \frac{1}{2}\theta(\theta-1)\sigma_3^2\mathbb{E}I^{\theta}(t)\Big]dt.$$

Since for any $x, y > 0$,

$$\theta\gamma x^{\frac{\theta-1}{\theta}}y^{\frac{1}{\theta}} + \theta\beta^* y^{\frac{\theta-1}{\theta}}x^{\frac{1}{\theta}} \leqslant \theta\max\{\beta^*,\gamma\}(x+y),$$

we have

$$\begin{aligned} d(\mathbb{E}S^{\theta}(t) + \mathbb{E}I^{\theta}(t)) \leq & \Big[\ \theta a_2\mathbb{E}S^{\theta}(t) - h\theta\{\mathbb{E}S^{\theta}(t)\}^{\frac{\theta+1}{\theta}} + \frac{1}{2}\theta(\theta-1)\sigma_2^2\mathbb{E}S^{\theta}(t) - \alpha\theta\mathbb{E}I^{\theta}(t) \\ & + \frac{1}{2}\theta(\theta-1)\sigma_3^2\mathbb{E}I^{\theta}(t) + \theta\max\{\beta^*,\gamma\}\{\mathbb{E}S^{\theta}(t) + \mathbb{E}I^{\theta}(t)\}\ \Big]\ dt \\ \leqslant & \Big[\bar{C}(\theta) - \mathbb{E}S^{\theta}(t) + [\frac{1}{2}(\theta-1)\sigma_3^2 - \alpha + \max\{\beta^*,\gamma\}]\theta\mathbb{E}I^{\theta}(t)\Big]dt, \end{aligned}$$

where $\bar{C}(\theta)$ is a positive constant only depending on θ and model parameters. Because

$$\frac{1}{2}(\theta-1)\sigma_3^2 - \alpha + \max\{\beta^*,\gamma\} < 0,$$

we have

$$d(\mathbb{E}S^{\theta}(t) + \mathbb{E}I^{\theta}(t)) \leq [\bar{C}(\theta) - \kappa\{\mathbb{E}S^{\theta}(t) + \mathbb{E}I^{\theta}(t)\}]dt,$$

where

$$\kappa = \min\{1, \alpha - \max\{\beta^*,\gamma\} - \frac{1}{2}(\theta-1)\sigma_3^2\} > 0.$$

Using the comparison theorem again for this differential inequality, we imply the boundedness of $\mathbb{E}S^{\theta}(t) + \mathbb{E}I^{\theta}(t)$. The proof is complete. □

3. Decline of species

This section presents sufficient conditions leading to the decline of predator or prey populations due to the environmental driving forces. These declines are indicated by either both susceptible predator $S(t)$ and infected predator $I(t)$ approaching zero as time approaches infinity, or susceptible prey $x(t)$ reaching zero at large time limit. Generally, such declines occur when the intensity of the environmental forcing is high.

In ecological systems, the environmental driving forces act as a source of noise, influencing the intrinsic rates of species. For instance, the birth rate of insects can be significantly affected by random fluctuation in temperature from one season to another season. These fluctuations can lead to substantial differences in insect birth rates between breeding seasons. Laboratory experimental studies have demonstrated that temperature fluctuations directly impact metabolic rates and reproductive success in insects [15]. Beside the random fluctuation in temperature, research on North American white-tailed deer reveals how random fluctuations in winter food availability affect their pregnancy rates and the survival of fawn deer [16]. As highlighted in biological researches, such environmental driving forces can significantly alter the long-term dynamics of the interacting species within natural environment by influencing the demographic rates and lead to survival risk [17]. Over extended periods of time, this leads to substantial changes in population levels, potentially shifting populations from a state of sustained coexistence to extinction. These changes are driven by the impact of stochastic variability on demographic parameters within the system, such as birth rate and mortality.

First, we investigate the decline of predator species. We show that the high intensities of noise affecting the intrinsic growth rate of susceptible predators and the mortality of infected predators lead to decline of both susceptible and infected predator populations.

Theorem 3.1. *Let $(x(t), S(t), I(t))$ be the solution of (1.4) with $(x(0), S(0), I(0)) = (x_0, S_0, I_0) \in \overline{\mathbb{R}^3_+} \setminus (0,0,0)$. Assume that*

$$\max\{\beta^* - \alpha, a_2\} < \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}.$$

Then, when $t \to \infty$, S(t) and I(t) converge to 0 almost surely.

Proof. The method in this proof is similar with [18]. Define a function $P(S, I)$ by

$$P(S, I) = \epsilon S + I,$$

where $\epsilon > 0$ is a constant such that

$$\epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})} < 0. \tag{3.1}$$

We have

$$d \ln P(S(t), I(t)) = \mathcal{L}(\ln P(S(t), I(t)))dt + \frac{1}{P(S(t), I(t))}\left[\epsilon\sigma_2 S(t)dW_2 + \sigma_3 I(t)dW_3\right],$$

where

$$\begin{aligned}
&\mathcal{L}(\ln P(S(t), I(t)))\\
=&\frac{\epsilon}{P(S(t), I(t))}\left[S(t)\big(a_2 - (\frac{1}{x(t)+e} + h)\{S(t) + I(t)\}\big] - \beta(S, I)S(t)I(t) + \gamma I(t)\right]\\
&+ \frac{1}{P(S(t), I(t))}\left[\beta(S, I)S(t)I(t) - \alpha I(t)\right]\\
&- \frac{\epsilon^2}{2P^2(S(t), I(t))}\sigma_2^2 S^2(t) - \frac{1}{2P^2(S(t), I(t))}\sigma_3^2 I^2(t)\\
=&Q_1 - Q_2,
\end{aligned}$$

in which

$$\begin{aligned}
Q_1 =&\frac{\epsilon}{P(S(t), I(t))}\left[S(t)\big(a_2 - (\frac{1}{x(t)+e} + h)\{S(t) + I(t)\}\big) - \beta(S, I)S(t)I(t) + \gamma I(t)\right]\\
&+ \frac{1}{P(S(t), I(t))}\left[\beta(S, I)S(t)I(t) - \alpha I(t)\right],
\end{aligned}$$

and

$$Q_2 = \frac{\epsilon^2}{2P^2(S(t), I(t))}\sigma_2^2 S^2(t) + \frac{1}{2P^2(S(t), I(t))}\sigma_3^2 I^2(t).$$

We have

$$\begin{aligned}
Q_1 \le&\frac{\epsilon}{P(S(t), I(t))}\left[a_2 S(t) - \beta(S, I)S(t)I(t) + \gamma I(t)\right] + \frac{1}{P(S(t), I(t))}\left[\beta(S, I)S(t)I(t) - \alpha I(t)\right]\\
\leqslant&\frac{\epsilon}{P(S(t), I(t))}\left[a_2 S(t) - \beta(S, I)S(t)I(t) + \gamma I(t)\right] + \frac{1}{P(S(t), I(t))}\left[\beta^* I(t) - \alpha I(t)\right]\\
=&\frac{\epsilon}{P(S(t), I(t))}I(t)\left[\gamma - \beta(S, I)S(t)\right] + \frac{1}{P(S(t), I(t))}\left[\epsilon a_2 S(t) + (\beta^* - \alpha)I(t)\right]\\
\leqslant&\frac{\epsilon}{P(S(t), I(t))}I(t)|\gamma - \beta(S, I)S(t)| + \frac{1}{P(S(t), I(t))}\left[\epsilon a_2 S(t) + (\beta^* - \alpha)I(t)\right].
\end{aligned}$$

Since $\beta(S,I)S(t) \in [0,\beta^*]$,

$$|\gamma - \beta(S,I)S(t)| \leqslant \max\{|\beta^* - \gamma|, \gamma\}.$$

In addition,

$$\frac{I(t)}{P(S(t),I(t))} = \frac{I(t)}{\epsilon S(t) + I(t)} \leqslant 1$$

Thus,

$$\begin{aligned} Q_1 &\leqslant \epsilon \max\{|\beta^* - \gamma|, \gamma\} + \frac{1}{P}\left[\epsilon a_2 S(t) + (\beta^* - \alpha) I(t)\right] \\ &\leqslant \epsilon \max\{|\beta^* - \gamma|, \gamma\} + \frac{1}{P}\max\{a_2, \beta^* - \alpha\}\left[\epsilon S(t) + I(t)\right] \\ &\leqslant \epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\}. \end{aligned}$$

For Q_2, from

$$P^2(S(t),I(t)) = (\epsilon S(t) + I(t))^2 \leqslant (\epsilon^2\sigma_2^2 S^2(t) + \sigma_3^2 I^2(t))(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}),$$

we have

$$Q_2 = \frac{\epsilon^2}{2P^2}\sigma_2^2 S^2(t) + \frac{1}{2P^2}\sigma_3^2 I^2(t) \geqslant \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}.$$

Therefore, we obtain that

$$\begin{aligned} d\ln P(S(t),I(t)) \leqslant &\Big[\epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}\Big]dt \\ &+ \frac{1}{P(S(t),I(t))}\left[\epsilon\sigma_2 S(t) dW_2 + \sigma_3 I(t) dW_3\right]. \end{aligned}$$

Taking the integral on $[0,t]$, we have

$$\begin{aligned} \ln P(S(t),I(t)) \leqslant &\ln P(S(0),I(0)) + \Big[\epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}\Big]t \\ &+ \int_0^t \frac{1}{P(S(s),I(s))}\left[\epsilon\sigma_2 S(s)\,dW_2 + \sigma_3 I(s)\,dW_3\right]. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\ln P(S(t),I(t))}{t} \leqslant &\frac{\ln P(S(0),I(0))}{t} + \Big[\epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}\Big] \\ &+ \frac{1}{t}\int_0^t \frac{1}{P(S(s),I(s))}\left[\epsilon\sigma_2 S(s)\,dW_2 + \sigma_3 I(s)\,dW_3\right]. \end{aligned}$$

Let us consider two real-valued continuous martingales vanishing at $t = 0$:

$$B_t^{(2)} = \int_0^t \frac{\epsilon\sigma_2 S(s)}{P(S(s),I(s))}\,dW_2(s), \quad B_t^{(3)} = \int_0^t \frac{\sigma_3 I(s)}{P(S(s),I(s))}\,dW_3(s).$$

They have quadratic form given by

$$\langle B_t^{(2)}\rangle = \int_0^t \frac{\xi_1^2\sigma_2^2 S^2(s)}{P^2(S(s),I(s))}\,ds \leq \sigma_2^2 t, \qquad \langle B_t^{(3)}\rangle = \int_0^t \frac{\xi_2^2\sigma_3^2 I^2(s)}{P^2(S(s),I(s))}\,ds \leq \sigma_3^2 t.$$

Using the strong law of large numbers for martingale [12], we have

$$\lim_{t\to\infty} \frac{B_t^{(i)}}{t} = 0 \qquad a.s. \qquad (i = 2,3).$$

Thus, we obtain that

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln P(S(t),I(t))}{t} \leqslant &\lim_{t\to\infty} \frac{\ln P(S(0),I(0))}{t} + \Big[\ \epsilon \max\{|\beta^* - \gamma|, \gamma\} \\ &+ \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})}\ \Big] \\ \leqslant &\epsilon \max\{|\beta^* - \gamma|, \gamma\} + \max\{\beta^* - \alpha, a_2\} - \frac{1}{2(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2})} \qquad a.s. \end{aligned}$$

It then follows from (3.1) that

$$\limsup_{t\to\infty} \frac{\ln P(S(t), I(t))}{t} < 0 \qquad a.s.$$

As a consequence, we have

$$\limsup_{t\to\infty} \frac{\ln S(t)}{t} < 0, \qquad \limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0 \qquad a.s.$$

These imply that

$$\lim_{t\to\infty} S(t) = 0, \qquad \lim_{t\to\infty} I(t) = 0 \qquad a.s.,$$

i.e. the population of predator declines. □

Next, we explore the decline of prey species. Similarly, we show that when the intensity of noise (environmental driving forces) affecting the intrinsic growth rate of prey species exceeds the square root of twice this rate, the population density of the prey species starts declining. Specifically, we have the following theorem.

Theorem 3.2. *Let* $(x(t), S(t), I(t))$ *be the solution of* (1.4) *with* $(x(0), S(0), I(0)) = (x_0, S_0, I_0) \in \overline{\mathbb{R}^3_+} \setminus (0,0,0)$. *Assume that* $a_1 - \frac{1}{2}\sigma_1^2 < 0$. *Then, when* $t \to \infty$, *x(t) converges to 0 almost surely.*

Proof. From

$$dx(t) = \left\{x(t)\left[a_1 - b_1 x(t)\right] - \frac{ax(t)(S(t)+I(t))}{b+S(t)+I(t)+x(t)}\right\}dt + \sigma_1 x(t) dW_1(t)$$

we have

$$\begin{aligned} d\ln x(t) &= \left\{(a_1 - b_1 x(t)) - \frac{a(S(t)+I(t))}{b+S(t)+I(t)+x(t)} - \frac{1}{2}\sigma_1^2\right\}dt + \sigma_1 dW_1(t) \\ &\leqslant (a_1 - \frac{1}{2}\sigma_1^2)dt + \sigma_1 dW_1(t). \end{aligned}$$

Therefore,

$$\frac{\ln x(t) - \ln x_0}{t} \leqslant (a_1 - \frac{1}{2}\sigma_1^2) + \frac{\sigma_1 W_1}{t} \qquad a.s.$$

Because $\lim_{t\to\infty} \frac{\sigma_1 W_1}{t} = 0$, taking $t \to \infty$ in the later inequality yields

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leqslant a_1 - \frac{1}{2}\sigma_1^2 < 0.$$

As a consequence, $\lim_{t\to\infty} x(t) = 0$ a.s. The proof is complete. □

4. Sustainability of species

In this section, we present some sufficient conditions for the sustainability of the system (1.4). Sustainability entails that over time, the averages of the prey density x and the susceptible predator density S do not approach zero, while the infected predator density I tends to zero. A precise definition for sustainability is provided below.

Definition 4.1. The system (1.4) is sustainable if, for every initial value $(x(0), S(0), I(0)) \in \overline{\mathbb{R}^3_+} \setminus (0,0,0)$, the solution $(x(t), S(t), I(t))$ satisfies

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds > 0, \qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds > 0, \qquad \lim_{t\to\infty} I(t) = 0 \qquad a.s.$$

To show the sustainability of the system, we use the following lemma.

Lemma 4.1 ([6,19]). *Let* $\{B_i(t)\}_{i=1,2,\ldots,N}$ *be sequence of standard Brownian motions and* $x \in C[\overline{\mathbb{R}}_+, \mathbb{R}^0_+]$ *a random process on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) If there are positive constants μ, λ, *and a random time* $T > 0$ *such that*

$$\ln x(t) \leqslant \lambda t - \mu \int_0^t x(s)\,ds + \sum_{i=1}^n \beta_i B_i(t)$$

for $t \geqslant T$ *a.s., where* β_i *are constants, then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds \leqslant \frac{\lambda}{\mu} \qquad a.s.$$

(ii) If there are positive constants μ, λ, and a random time $T > 0$ such that

$$\ln x(t) \geqslant \lambda t - \mu \int_0^t x(s)\,ds + \sum_{i=1}^{n} \beta_i B_i(t)$$

for $t \geqslant T$ a.s., where β_i are constants, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds \geqslant \frac{\lambda}{\mu} \quad a.s.$$

We are now ready to state our main results in this section.

Theorem 4.1. *Let $(x(t), S(t), I(t))$ be the solution of (1.4) with initial value $(x_0, S_0, I_0) \in \overline{\mathbb{R}^3_+} \setminus (0,0,0)$. Assume that*

$$\begin{cases} a_1 - a - \frac{1}{2}\sigma_1^2 > 0, \\ a_2 - \beta^* - \frac{1}{2}\sigma_2^2 > 0, \\ \beta^* - \alpha - \frac{1}{2}\sigma_3^2 < 0. \end{cases} \tag{4.1}$$

Then,

$$\lim_{t\to\infty} I(t) = 0 \quad a.s.$$

Furthermore, there exists $\varepsilon > 0$ which is independent of initial value such that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds > \varepsilon, \qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds > \varepsilon.$$

Proof. We firstly show the extinction of infected population, i.e. $\lim_{t\to\infty} I(t) = 0$. From

$$dI(t) = \left[\beta(S,I)S(t)I(t) - \alpha I(t)\right]dt + \sigma_3 I(t) dW_3(t),$$

we have

$$\begin{aligned} d\ln I(t) &= (\beta(S,I)S(t) - \alpha - \frac{1}{2}\sigma_3^2)dt + \sigma_3 dW_3(t) \\ &\leqslant (\beta^* - \alpha - \frac{1}{2}\sigma_3^2)dt + \sigma_3 dW_3(t). \end{aligned}$$

Then,

$$\frac{\ln I(t) - \ln I_0}{t} \leqslant (\beta^* - \alpha - \frac{1}{2}\sigma_3^2) + \frac{\sigma_3 W_3(t)}{t} \qquad a.s.$$

Thus,

$$\lim_{t\to\infty} \frac{\ln I(t)}{t} \leqslant \beta^* - \alpha - \frac{1}{2}\sigma_3^2 < 0 \qquad a.s.$$

As a result, we obtain that

$$\lim_{t\to\infty} I(t) = 0 \quad a.s.$$

Secondly, we consider $S(t)$. Since

$$dS(t) = \Big(S(t)\big[a_2 - (\frac{1}{x(t)+e} + h)\{S(t) + I(t)\}\big] - \beta(S,I)S(t)I(t) + \gamma I(t)\Big)dt + \sigma_2 S(t) dW_2(t),$$

we have

$$\begin{aligned} d\ln S(t) &= \big[a_2 - (\frac{1}{x(t)+e} + h)(S(t) + I(t)) - \beta(S,I)I(t) + \gamma\frac{I(t)}{S(t)} - \frac{1}{2}\sigma_2^2\big]dt + \sigma_2 dW_2(t) \\ &\geqslant \big[a_2 - (\frac{1}{x(t)+e} + h)(S(t) + I(t)) - \beta(S,I)I(t) - \frac{1}{2}\sigma_2^2\big]dt + \sigma_2 dW_2(t) \\ &\geqslant \big[a_2 - (\frac{1}{e} + h)(S(t) + I(t)) - \beta^* - \frac{1}{2}\sigma_2^2\big]dt + \sigma_2 dW_2(t). \end{aligned}$$

Therefore,

$$\frac{\ln S(t) - \ln S_0}{t} \geqslant (a_2 - \beta^* - \frac{1}{2}\sigma_2^2) - \frac{1}{t}\int_0^t (\frac{1}{e} + h)S(s)\,ds - \frac{1}{t}\int_0^t (\frac{1}{e} + h)I(s)\,ds + \frac{\sigma_2 W_2(t)}{t} \qquad a.s.$$

Since $\lim_{t\to\infty} I(t) = 0$ a.s. and $a_2 - \beta^* - \frac{1}{2}\sigma_2^2 > 0$, there exists a random $T > 0$ such that for all $t \geqslant T$

$$\frac{1}{t}\int_0^t (\frac{1}{e} + h)I(s)\,ds < \frac{1}{2}(a_2 - \beta^* - \frac{1}{2}\sigma_2^2) \quad a.s.$$

Thus, for all $t > T$,

$$\begin{aligned}\frac{\ln S(t) - \ln S_0}{t} &\geqslant (a_2 - \beta^* - \frac{1}{2}\sigma_2^2) - \frac{1}{t}\int_0^t (\frac{1}{e} + h)S(s)\,ds - \frac{1}{2}(a_2 - \beta^* - \frac{1}{2}\sigma_2^2) + \frac{\sigma_2 W_2(t)}{t} \\ &= \frac{1}{2}(a_2 - \beta^* - \frac{1}{2}\sigma_2^2) - \frac{1}{t}\int_0^t (\frac{1}{e} + h)S(s)\,ds + \frac{\sigma_2 W_2(t)}{t},\end{aligned} \tag{4.2}$$

or equivalently

$$\ln z(t) \geqslant \frac{1}{2}(a_2 - \beta^* - \frac{1}{2}\sigma_2^2)t - (\frac{1}{e} + h)S(0)\int_0^t z(s)\,ds + \sigma_2 W_2(t),$$

where $z(t) = \frac{S(t)}{S(0)}$.

Using Lemma 4.1 for the latter inequality, we obtain that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t z(s)\,ds \geqslant \frac{a_2 - \beta^* - \frac{1}{2}\sigma_2^2}{2(\frac{1}{e} + h)S(0)} > 0 \qquad a.s.$$

Therefore,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S(s)\,ds \geqslant \frac{a_2 - \beta^* - \frac{1}{2}\sigma_2^2}{2(\frac{1}{e} + h)} > 0 \qquad a.s.$$

Finally, we consider $x(t)$. From

$$dx(t) = \left\{x(t)(a_1 - b_1 x(t)) - \frac{ax(t)(S(t) + I(t))}{b + S(t) + I(t) + x(t)}\right\}dt + \sigma_1 x(t)dW_1(t),$$

we have

$$\begin{aligned}d\ln x(t) &= \left\{a_1 - b_1 x(t) - \frac{a(S(t) + I(t))}{b + S(t) + I(t) + x(t)} - \frac{1}{2}\sigma_1^2\right\}dt + \sigma_1 dW_1(t) \\ &\geqslant \left\{a_1 - b_1 x(t) - a - \frac{1}{2}\sigma_1^2\right\}dt + \sigma_1 dW_1(t).\end{aligned}$$

By using the same argument as above, we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\,ds \geqslant \frac{a_1 - a - \frac{1}{2}\sigma_1^2}{b_1} > 0 \qquad a.s.$$

The proof is complete. □

Remark 4.1. Since $\alpha = \gamma + \alpha^*$, the third condition in (4.1) is equivalent to $\gamma + \alpha^* + \frac{\sigma_3^2}{2} > \beta^*$. Recall that γ and α^* represent the recovery rate and mortality of infected predators, respectively. Meanwhile, σ_3 is the intensity of noise (random environmental driving forces) influencing the mortality of infected predators, and β^* is proportional to the maximal infection rate of susceptible predators. This condition suggests that high recovery rates γ, high mortality rates α^*, high intensity of noise σ_3, or a low maximum infection rate β^* can lead to a decline of infected predators ($I \to 0$) over time. Intuitively, a high recovery rate, high mortality, or low maximum infection rate will naturally decrease the infected predator population. Additionally, high intensity of noise can amplify the mortality rate randomly, leading to population decline. This highlights that the random environmental driving forces can eradicate the infected predator population.

Similar arguments apply to the first and second conditions in (4.1). A high intrinsic growth rate a_1 of prey, a low maximum consumption rate a of predators, or low environmental variability σ_1 in consumption rate leads to a sustainable prey population. Likewise, a high intrinsic growth rate a_2 of susceptible predators, a low maximum infection rate, or low environmental variability σ_2 in susceptible predator growth rate fosters a sustainable susceptible predator population.

5. Existence of borel invariant measure

In this section, we show the existence of a Borel invariant measure for the Itô process $(x(t), S(t), I(t))$ defined by the system (1.4) on the domain $\mathbb{R}_+^3$. The existence of an invariant measure in the predator–prey system (1.4) carries several biological implications. If this measure corresponds to a probability distribution, it represents an stationary distribution for the system. Specifically, if the probability distribution of initial values (initial population densities) is identical to the invariant measure, the probability distribution of solutions at any time will also match the invariant measure. This implies that, with a probability of one, the solution trajectories of prey and susceptible predator populations neither tend towards zero nor blow up. Instead, these two components of the system coexist over time, while the infected predator population approaches zero, maintaining a dynamic equilibrium. Even if the probability distribution of initial values differs from the invariant measure, but the latter is stable in the sense that the probability distribution of solutions converges in probability to that measure, long-term coexistence of the two components and decline of the infected predator population still occur. In such scenarios, fluctuations in population densities resulting from stochastic environmental factors or random events are dampened over time, leading to a relatively stationary state. This regulatory mechanism helps prevent population crashes or runaway growth, thereby maintaining balance within the ecosystem.

Let us denote by $P(.,.,.,.,.)$ the transition probability of $(x(t),S(t),I(t))$, i.e.

$$P(t,\xi,\eta,\varsigma,\mathcal{K})=\mathbb{P}\{(x(t),S(t),I(t))\in\mathcal{K};(x(0),S(0),I(0))=(\xi,\eta,\varsigma)\}$$

for $0\leqslant t<\infty,(\xi,\eta,\varsigma)\in\overline{\mathbb{R}^3_+}$ and $\mathcal{K}\in\mathcal{B}(\overline{\mathbb{R}^3_+})$.

Following [20,21], we have

(i) $P(t,x,S,I,.)$ induces a strongly continuous semi-group $\{P_t\}_{0\leqslant t<\infty}$ of operators on the space $C_B(\overline{\mathbb{R}^3_+})$ of bounded continuous functions:

$$P_tf(x,S,I)=\int_{\mathbb{R}^3_+}f(\xi,\eta,\varsigma)P(t,x,S,I,d\xi d\eta d\varsigma),\qquad f\in C_B(\overline{\mathbb{R}^3_+}).$$

(ii) $P(t,\xi,\eta,\varsigma,.)$ induces a positive contraction $[.P_t]$ on the space $M(\overline{\mathbb{R}^3_+},\mathcal{B}(\overline{\mathbb{R}^3_+}))$ of finite signed measures:

$$(\mathcal{K})=\int_{\mathbb{R}^3_+}P(t,\xi,\eta,\varsigma,\mathcal{K})\mu(d\xi_1d\eta_1d\varsigma_1),\qquad \mu\in M(\overline{\mathbb{R}^3_+},\mathcal{B}(\overline{\mathbb{R}^3_+})),\mathcal{K}\in\mathcal{B}(\overline{\mathbb{R}^3_+}).$$

Definition 5.1. A Borel measure ν on $\overline{\mathbb{R}^3_+}$ (i.e. a positive measure that is finite on any compact set of $\overline{\mathbb{R}^3_+}$) is said to be invariant with respect to $\{P_t\}_{0\leqslant t<\infty}$ if for $0\leqslant t<\infty$ and $\mathcal{K}\in\mathcal{B}(\overline{\mathbb{R}^3_+})$,

$$(\mathcal{K})=\nu(\mathcal{K}).$$

The following result is well-known.

Lemma 5.1 ([20]). *Let X be a locally compact perfectly normal topological space. Let $\{Q_t\}_{0\leqslant t<\infty}$ be a strongly continuous semi-group on $C_B(X)$ generated by a transition probability on $(X,\mathcal{B}(X))$. If there exists a non-negative function g in the space $C_0(X)$ of continuous functions with compact support such that*

$$\int_0^\infty Q_tg(t)\,dt=\infty,\qquad x\in X,$$

then there exists a Borel invariant measure for $\{Q_t\}_{0\leqslant t<\infty}$.

Utilizing Lemma 5.1, we establish that if the system (1.4) is sustainable and the sum of the recovery rate and the mortality of infected predators exceeds the maximal infection rate multiplied by a constant, then a Borel invariant measure exists. Specifically, we state the following theorem:

Theorem 5.1. *Assume that sustainable condition (4.1) holds true. If $\alpha>\beta^*$, then $\{P_t\}_{0\leqslant t<\infty}$ has a Borel invariant measure on $\overline{\mathbb{R}^3_+}$.*

Proof. It follows from Theorem 2.2 that there exists $\kappa>0$ such that

$$\sup_{t\geq0}\mathbb{E}x(t)\leq\kappa,\quad \sup_{t\geq0}\mathbb{E}S(t)\leqslant\kappa,\quad \sup_{t\geq0}\mathbb{E}I(t)\leqslant\kappa.$$

Let $\varepsilon\in(0,1)$ and $M=\frac{4\kappa}{\varepsilon}$. The Markov inequality provides that for every $t\geq0$,

$$\begin{cases}\mathbb{P}\{x(t)>M\}\leqslant\frac{\mathbb{E}x(t)}{M}\leq\frac{\kappa}{M}=\frac{\varepsilon}{4},\\ \mathbb{P}\{S(t)>M\}\leqslant\frac{\mathbb{E}S(t)}{M_2}\leq\frac{\kappa}{M}=\frac{\varepsilon}{4},\\ \mathbb{P}\{I(t)>M\}\leqslant\frac{\mathbb{E}I(t)}{M_3}\leq\frac{\kappa}{M}=\frac{\varepsilon}{4},\end{cases}\tag{5.1}$$

and therefore

$$\begin{cases}\inf_{t\geq0}\mathbb{P}\{0\leqslant x(t)\leqslant M\}\geqslant1-\frac{\varepsilon}{4},\\ \inf_{t\geq0}\mathbb{P}\{0\leqslant S(t)\leq M\}\geq1-\frac{\varepsilon}{4},\\ \inf_{t\geq0}\mathbb{P}\{0\leqslant I(t)\leqslant M\}\geq1-\frac{\varepsilon}{4}.\end{cases}\tag{5.2}$$

Define a set K by

$$K=[0,M]\times[0,M]\times[0,M].$$

Using (5.1) and (5.2), we have for every $t\geq0$,

$$\begin{aligned}\mathbb{P}\{(x(t),S(t),I(t))\in K\}=&\mathbb{P}\{0\leqslant I(t)\leqslant M_3\}-\mathbb{P}\{x(t)>M,0\leqslant S(t)\leqslant M,0\leqslant I(t)\leqslant M\}\\ &-\mathbb{P}\{0\leqslant x(t)\leqslant M,S(t)>M,0\leqslant I(t)\leqslant M\}\\ &-\mathbb{P}\{x(t)>M,S(t)>M,0\leqslant I(t)\leqslant M\}\\ \geq&(1-\frac{\varepsilon}{4})-\frac{\varepsilon}{4}-\frac{\varepsilon}{4}-\frac{\varepsilon}{4}=1-\varepsilon.\end{aligned}\tag{5.3}$$

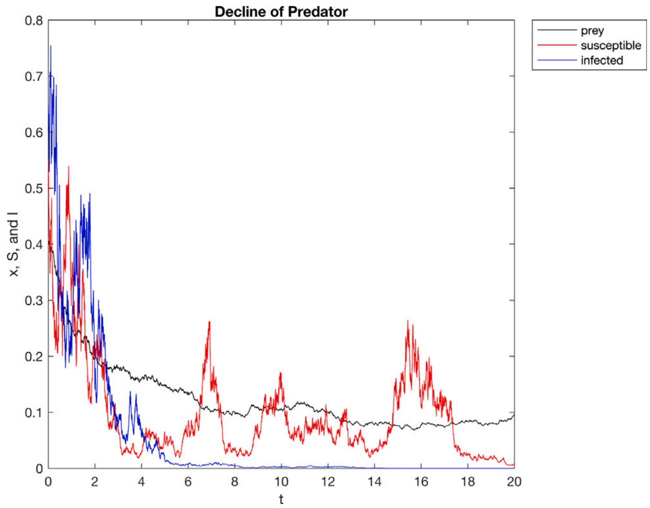

Fig. 6.1. Decline of predator with $a = 0.6$, $b = 0.8$, $a_1 = 0.1$, $a_2 = 0.2$, $b_1 = 0.5$, $c = 0.9$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.85$, $\sigma_3 = 0.95$, $\gamma = 0.3$, $\alpha = 0.8$, $x(0) = 0.4$, $S(0) = 0.5$, and $I(0) = 0.6$.

Consider a non-negative function $g \in C_0(\overline{\mathbb{R}^3_+})$ such that

$$g(x,S,I) = \begin{cases} 1, & (x,S,I) \in K, \\ 0, & (x,S,I) \notin \overline{\mathbb{R}^3_+} \setminus K_1, \end{cases}$$

where $K_1 \supset K$ is a bounded set of $\overline{\mathbb{R}^3_+}$. By using (5.3), we have

$$\begin{aligned}\int_0^t P_s g(x,S,I)\,ds &= \int_0^t \int_{\overline{\mathbb{R}^3_+}} g(\xi,\eta,\varsigma) P(s,x,S,I,d\xi d\eta d\varsigma)\,ds \\ &\geqslant \int_0^t \int_K g(\xi,\eta,\varsigma) P(s,x,S,I,d\xi d\eta d\varsigma)\,ds \\ &= \int_0^t \mathbb{P}\{(x(s),S(s),I(s)) \in K\}\,ds \to \infty \qquad \text{when} \quad t \to \infty.\end{aligned}$$

Thanks to Lemma 5.1, we conclude that there exists a Borel invariant measure ν for $\{P_t\}_{0\leqslant t<\infty}$ such that $\nu(K) > 0$. The proof is complete. □

6. Numerical examples

This section exhibits some numerical examples for possibility of decline and sustainability of the model (1.4). For the computations, we used a scheme of order 1.5 (see, e.g., [22]).

6.1. Decline

We set the parameters as follows: $a = 0.6$, $b = 0.8$, $a_1 = 0.1$, $a_2 = 0.2$, $b_1 = 0.5$, $c = 0.9$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.85$, $\sigma_3 = 0.95$, $\gamma = 0.3$, $\alpha = 0.4$. We take the initial values of x, S, and I to be 0.4, 0.5, and 0.6, respectively. We use $N = 5000$ and $T = 20$. These parameter values satisfy the conditions in Theorem 3.1.

Fig. 6.1 illustrates the decline of the predator species, with S and I both tending to 0 as $t \to \infty$.

Next, we change the values of the noise intensity and α while keeping all other parameters the same as in Fig. 6.1. The new parameter values are $\sigma_1 = 0.9, \sigma_2 = 0.2, \sigma_3 = 0.1, \alpha = 0.4$ and they satisfy the conditions in Theorem 3.2. Fig. 6.2 illustrates the decline of the prey species, where x tends to 0.

6.2. Sustainability

We set the parameters as follows: $a = 0.6$, $b = 2$, $a_1 = 2.7$, $a_2 = 1.2$, $b_1 = 0.1$, $c = 1$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\gamma = 0.3$, $\alpha = 1.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\sigma_3 = 0.4$, and take the initial values $x(0) = 6$, $S(0) = 10$, and $I(0) = 2$. Furthermore, we choose $N = 500$ and $T = 20$. Note that these parameter values satisfy the conditions of Theorem 4.1.

Fig. 6.3 displays sample trajectories of x, S, and I in the phase space and in time. We calculate N points in the time interval $(0, T]$ with a step size of $\frac{T}{N}$.

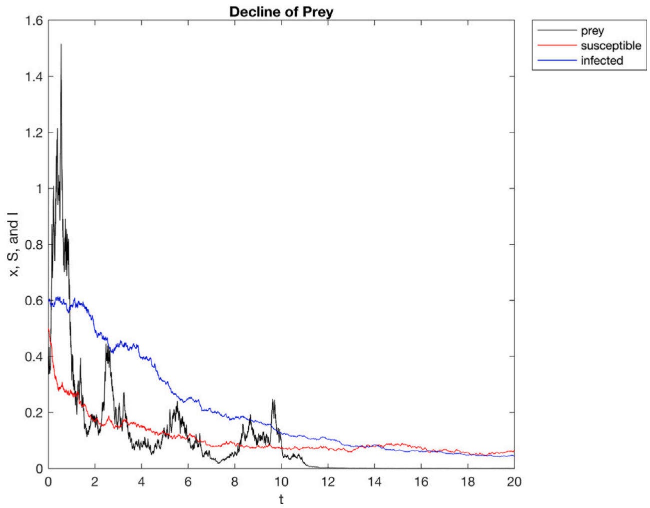

Fig. 6.2. Decline of prey with $a = 0.6$, $b = 0.8$, $a_1 = 0.1$, $a_2 = 0.2$, $b_1 = 0.5$, $c = 0.9$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.9, \sigma_2 = 0.2, \sigma_3 = 0.1$, $\gamma = 0.3$, $\alpha = 0.4$, $x(0) = 0.4$, $S(0) = 0.5$, and $I(0) = 0.6$.

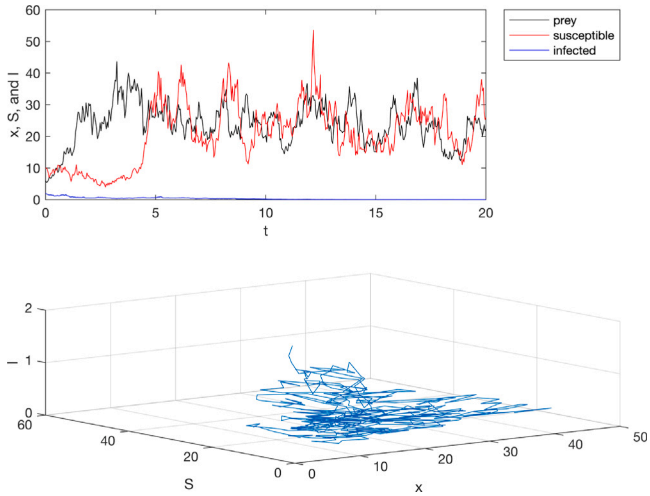

Fig. 6.3. Sample trajectories in time (above) and in the phase space (below) with $a = 0.6$, $b = 2$, $a_1 = 2.7$, $a_2 = 1.2$, $b_1 = 0.1$, $c = 1$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\sigma_3 = 0.4$, $\gamma = 0.3$, $\alpha = 1.1$, and initial values $x(0) = 6$, $S(0) = 10$, and $I(0) = 2$.

To explore the sustainability of the system, we set new values for N and T, as $N = 5000$ and $T = 50$. We keep the other parameters the same as before. We define the time-averaged quantities

$$x^*(t) = \frac{1}{t}\int_0^t x(s)ds, \quad S^*(t) = \frac{1}{t}\int_0^t S(s)ds, \quad t \in [0, \infty),$$

with a convention that $x^*(0) = x(0)$ and $S^*(0) = S(0)$. Fig. 6.4 presents a sample trajectory of the three processes x^*, S^*, and I. It is observed that x^* and S^* are bounded below by some positive constant, while I converges to zero.

To visualize the support of the invariant measure, we generate 10 000 samples of the process (x, S, I) and plot their values at time T and $T - \frac{T}{N}$ in Fig. 6.5. The parameter values are the same as in Fig. 6.4.

7. Conclusions

In this paper, we have considered a prey–predator model with infected predator population and studied the dynamical behavior under environmental driving forces. The predator population is assumed to be generalist in nature as they have an alternative implicit food source. The predator growth rate primarily follow the Leslie–Gower formulation but an modification is introduced to model the resource independent intra-specific competition rate explicitly. This modified formulation has an advantage — large

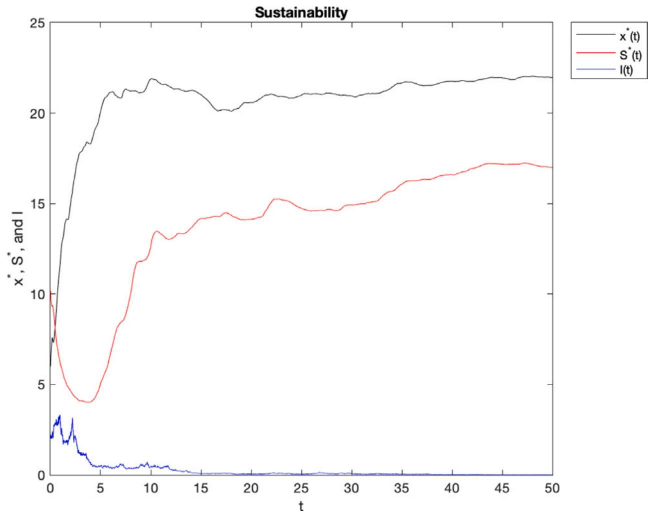

Fig. 6.4. Sample trajectories of x^*, S^*, and I with $a = 0.6$, $b = 2$, $a_1 = 2.7$, $a_2 = 1.2$, $b_1 = 0.1$, $c = 1$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\sigma_3 = 0.4$, $\gamma = 0.3$, $\alpha = 1.1$, and initial values $x(0) = 6$, $S(0) = 10$, and $I(0) = 2$.

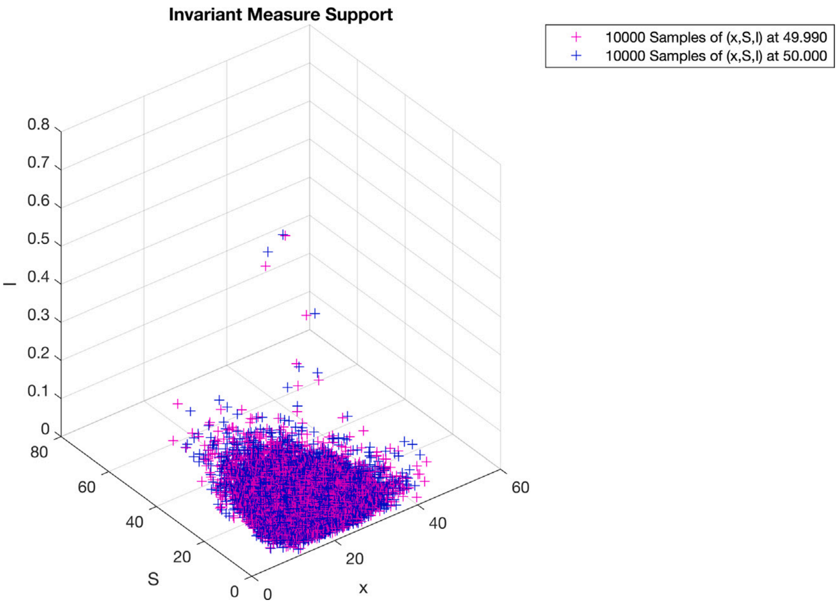

Fig. 6.5. Distribution of (x, S, I) with $a = 0.6$, $b = 2$, $a_1 = 2.7$, $a_2 = 1.2$, $b_1 = 0.1$, $c = 1$, $h = 0.01$, $\beta^* = 1$, $k = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\sigma_3 = 0.4$, $\gamma = 0.3$, $\alpha = 1.1$, and initial values $x(0) = 6$, $S(0) = 10$, and $I(0) = 2$.

density of favorable food source cannot eliminate the strength of intra-specific competition. Mathematically, the modified specialist predator's growth rate helps to prove the global existence of solution for the stochastic differential equation model.

Our study has provided analytical conditions for the sustainability of both prey and predator populations, as well as for the decline of each species. Additionally, we have explored the existence of a Borel invariant measure.

First, the analysis of sustainability conditions offers valuable insights into the long-term viability of prey and predator populations within their respective ecosystems. Understanding the factors that contribute to population sustainability is essential for effective conservation efforts and the preservation of balanced ecological systems.

Second, our investigation of decline conditions for each species offers valuable information on the factors that can lead to population decreases or even local extinctions. Identifying these conditions helps identify potential threats or vulnerabilities in the ecosystem, enabling targeted conservation strategies to mitigate population decline and preserve biodiversity.

Finally, the discovery of a Borel invariant measure adds to our understanding of the long-term behavior of prey–predator populations and contributes to assessing the stability and resilience of the ecosystem.

In conclusion, our research holds significance for informing conservation strategies, ecosystem management, and enhancing our understanding of the intricate dynamics within ecological systems.

CRediT authorship contribution statement

Yujie Gao: Writing – review & editing, Writing – original draft, Visualization, Validation, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Malay Banerjee:** Writing – review & editing, Validation, Conceptualization. **Ton Viet Ta:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors declare that the manuscript has no associated data.

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