Breakdown of C^∞-smoothing effects of solutions to the semilinear equation in the whole space

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BREAKDOWN OF C^∞-SMOOTHING EFFECTS OF SOLUTIONS TO THE SEMILINEAR HEAT EQUATION IN THE WHOLE SPACE

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ABSTRACT. The semilinear heat equation in the whole space $\mathbb{R}^n$ is considered, where the nonlinear terms are given as the forms $\pm|u|^\kappa$ or $\pm|u|^{\kappa-1}u$. In particular, we focus on global solutions for small initial data in $L^{(\kappa-1)n/2}(\mathbb{R}^n)$ under the conditions $1+2/n < \kappa < \infty$ and $\kappa \notin \mathbb{N}$. We reveal that the global solutions have a regularity nearly $C((0,\infty); C^{\kappa+2}(\mathbb{R}^n))$. Moreover, by taking special initial data, we show that the global solutions do not belong to C^∞-class in space. From these results, we see that the space $C^{\kappa+2}(\mathbb{R}^n)$ may be regarded as a threshold of the regularity in space. The proof relies on the estimates of higher order derivatives of the nonlinear terms $\pm|u|^\kappa$ and $\pm|u|^{\kappa-1}u$.

1. INTRODUCTION

In this paper, we consider the following initial value problem for the semilinear heat equation in the whole space $\mathbb{R}^n$, $n \geq 2$;

$$(1.1) \qquad \begin{cases} \partial_t u - \Delta u = \lambda |u|^{\kappa-l} u^l, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $u = u(t,x)$ is the unknown function and $u_0 = u_0(x)$ is the given initial data. In what follows, we focus on the case where u and u_0 are the real-valued (possibly sign-changing) functions. Moreover, $1 < \kappa < \infty$, $l = 0, 1$, and $\lambda \in \mathbb{R} \setminus \{0\}$ are the given constants.

The main purpose of this paper is to reveal the *smoothing effects* of global solutions u to (1.1) for sufficiently small initial data $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$, where $1+2/n < \kappa < \infty$. More precisely, by assuming that $\kappa \notin \mathbb{N}$, we show that the global solutions to (1.1) have a *regularity nearly* $C((0,\infty); C^{\kappa+2}(\mathbb{R}^n))$. In addition, we also show that the global solutions u *do not belong to C^∞-class* in space by taking special initial data u_0.

The semilinear heat equation (1.1) is one of the typical examples of nonlinear parabolic PDEs. The equation (1.1) was introduced by Fujita [13, 14] for study on the *blow-up phenomena* of solutions to the nonlinear parabolic PDEs, but it is also known that the equation (1.1) may be regarded as a mathematical model describing the combustion of the solid fuel [3]. Therefore, even now, various studies on the mathematical analysis of the equation (1.1) have been conducted. Fujita [13] considered the case where $\lambda = 1$ and both the initial data $u_0 \in BC^2(\mathbb{R}^n) \setminus \{0\}$ and the solution u are non-negative. And then he revealed the condition such that the solutions u to (1.1) exist *globally in time*; if $1 < \kappa < 1+2/n$, then (1.1) has no global solution. If $1+2/n < \kappa < \infty$, then (1.1) has a global solution for sufficiently small u_0. The essential exponent $1+2/n$ is called the *Fujita exponent*. Moreover, Hayakawa [19] and Kobayashi, Sirao, and Tanaka [23] showed that (1.1) has no global solution in the *critical case* $\kappa = 1+2/n$. Weissler [50, 51] also obtained the corresponding results to [13, 19, 23] in the case of $u_0 \in L^r(\mathbb{R}^n) \setminus \{0\}$. We also refer to [28, 49] for the results in this direction.

Here we note that in the results of [13, 19, 23, 28, 49–51], it is assumed that both u_0 and u are *non-negative*. In contrast to the above results, we will consider general u_0 and u, namely, *sign-changing* functions. In this case, the problems on the existence or non-existence of global solutions to (1.1)

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become more complicated. First we mention the results on the existence of global solutions to (1.1) with initial data u_0 in *scaling invariant spaces.* By following the method due to Kato [21], we may expect that if the initial data u_0 is sufficiently small, then (1.1) has a global solution u. For the known results in this direction, we refer to, e.g., [16, 25, 33, 34, 52]. Note that in these results, it is assumed the corresponding condition to $1 + 2/n < \kappa < \infty$, so we observe that a similar result is still valid under the condition $1 + 2/n < \kappa < \infty$. However, concerning the opposite case $1 < \kappa \le 1 + 2/n$, it is known that for any $1 < \kappa \le 1 + 2/n$, we may choose an initial data u_0 such that (1.1) has a *global sign-changing solution u*, unlike the case where u is non-negative. We refer to [7, 8, 38, 53] for such results. In addition, Mizoguchi and Yanagida [35, 37] considered the case where $n = 1$, $\lambda = 1$, and $l = 1$ and obtained a *new critical exponent* $1 + 2/(m + 1)$. Here m denotes the number of sign-changes of the initial data u_0. More precisely, if $1 < \kappa \le 1 + 2/(m + 1)$, then any solution to (1.1) with initial data u_0 blows up in finite time. If $1 + 2/(m + 1) < \kappa < \infty$, then we may take an initial data u_0 such that (1.1) has a global solution. For the other results on the blow-up phenomena of sign-changing solutions, we refer to, e.g., [5, 11, 27, 36, 42]. In this way, the studies on sign-changing solutions to (1.1) do not seem to be progressing enough due to the difficulty of the analysis in comparison with the case of non-negative solutions. Therefore, revealing the properties of sign-changing solutions to (1.1) such as the blow-up phenomena is one of the major issues. See also the book [41] written by Quittner and Souplet for a survey including the topic of the existence or non-existence of global solutions to the semilinear parabolic equations.

We recall that our aim in this paper is to reveal the smoothing effects of global solutions to (1.1), so next we shall mention the *parabolic smoothing effects.* In general, we may expect that solutions to the parabolic PDEs are *smooth* even if the initial data have a *singularity.* The classical arguments are written in the well-known book by Ladyženskaja, Solonnikov, and Ural′ceva [26]. In addition, there are various previous works on the *analyticity* of solutions to the nonlinear parabolic PDEs. We refer to [15, 22, 30, 46] for the well-known results. In particular, Masuda [31] and Grujić and Kukavica [17] considered the semilinear heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$;

$$\begin{cases} \partial_t u - \Delta u = u^\kappa, & t > 0, \, x \in \Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

where $\kappa \in \mathbb{N}$ satisfies $\kappa \ge 2$. Masuda [31] assumed $\kappa = 2$ and the Neumann boundary condition $\partial_\nu u|_{\partial\Omega} = 0$ and showed the analyticity of solutions u in time. Grujić and Kukavica [17] assumed the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and revealed the analyticity of solutions u in space and the estimate of analyticity radius. In addition, it is also known that solutions to several abstract parabolic equations are analytic by using the method of the *parameter trick,* which is given by Angenent [1]. The parameter trick is based on the maximal regularity theorem and the implicit function theorem, so this method may be applied to various nonlinear parabolic PDEs. For the applications of the parameter trick, we refer to [10, 12, 39, 40]. As stated above, a lot of studies on the smoothing effects of the parabolic PDEs have been conducted.

In contrast to the previous works, we show the *breakdown of C^∞-smoothing effects* of solutions to the semilinear heat equation (1.1). Let us consider why such a phenomenon occurs. A major difference between our case and those of Masuda [31] and Grujić and Kukavica [17] is the nonlinear term. They considered only the case where the nonlinear term is given by u^κ with $\kappa \in \mathbb{N}$, but we consider the following forms $|u|^\kappa$ or $|u|^{\kappa-1}u$ with $\kappa \notin \mathbb{N}$. So, the breakdown of C^∞-smoothing effects comes from the *structure of the nonlinear terms.* Here we consider the nonlinear term $|u|^\kappa$. The typical way to obtain the regularities of solutions is combining the smoothing effects of the analytic semigroup and the estimates of the nonlinear term. More precisely, if we obtain a solution $u \in C((0, \infty) \times \mathbb{R}^n)$ to (1.1), then the nonlinear term $|u|^\kappa$ of (1.1) may be regarded as a given function. Therefore, by applying the smoothing effects of the analytic semigroup to the given

function $|u|^\kappa \in C((0,\infty) \times \mathbb{R}^n)$, we see that u has a regularity nearly

$$C^1((0,\infty); C(\mathbb{R}^n)) \cap C((0,\infty); C^2(\mathbb{R}^n)).$$

Roughly speaking, since $\partial_t u$ and Δu may have similar regularities to those of $|u|^\kappa \in C((0,\infty) \times \mathbb{R}^n)$ from the original equation (1.1), we see a *gain of regularities* of u. In addition, we may expect that $|u|^\kappa$ also has *higher regularities* by using new regularities of u. Hence, by the *bootstrap argument*, we may expect that u is smooth. Here let us focus on the regularities of the *derivatives* of $|u|^\kappa$. By setting $h(\tau) := |\tau|^\kappa$ for $\tau \in \mathbb{R}$, we have $|u|^\kappa = h(u)$. We consider the derivative ∂_1 with respect to the first component of $x \in \mathbb{R}^n$. Then, since $\partial_\tau h(\tau) = \kappa \tau |\tau|^{\kappa-2}$, we have $\partial_1 |u|^\kappa = \kappa u |u|^{\kappa-2} \partial_1 u$ from the *chain rule* for the composite functions. Noting that we have to use the chain rule, we observe that the regularities of $|u|^\kappa$ *strongly rely* on those of the function h. Since $h \in C_{\mathrm{loc}}^\kappa(\mathbb{R}) \setminus C_{\mathrm{loc}}^{\kappa+\sigma}(\mathbb{R})$ for any small $\sigma > 0$ in the case of $\kappa \notin \mathbb{N}$, we may expect that $|u|^\kappa \in C((0,\infty); C_{\mathrm{loc}}^\kappa(\mathbb{R}^n))$ if u has sufficient regularities. Likewise, the chain rule is also valid for the derivative ∂_t with respect to the time variable t. Thus we have

$$|u|^\kappa \in C((0,\infty); C_{\mathrm{loc}}^\kappa(\mathbb{R}^n)) \cap C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); C^{[\kappa]-1}(\mathbb{R}^n)) \cap C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); C^{[\kappa]}(\mathbb{R}^n))$$

at least formally. Note that the sum of regularity indices of space and time coincides with κ, i.e., $(1 + \kappa - [\kappa]) + ([\kappa] - 1) = (\kappa - [\kappa]) + [\kappa] = \kappa$. Consequently, since $\partial_t u$ and Δu may have similar regularities to those of $|u|^\kappa$ from (1.1), we may expect that

$$\begin{aligned}
(1.2) \quad & u \in C^1((0,\infty); C_{\mathrm{loc}}^\kappa(\mathbb{R}^n)) \cap C((0,\infty); C_{\mathrm{loc}}^{\kappa+2}(\mathbb{R}^n)) \cap C_{\mathrm{loc}}^{2+\kappa-[\kappa]}((0,\infty); C^{[\kappa]-1}(\mathbb{R}^n)) \\
& \cap C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); C^{[\kappa]+1}(\mathbb{R}^n)) \cap C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); C^{[\kappa]+2}(\mathbb{R}^n)).
\end{aligned}$$

In addition, by using $h \notin C_{\mathrm{loc}}^{\kappa+\sigma}(\mathbb{R})$, we also expect that

$$u \notin C((0,\infty); C_{\mathrm{loc}}^{\kappa+2+\sigma}(\mathbb{R}^n)).$$

In our main results, we show that such an anticipation is true in some sense. However, we should emphasize that the property $h \notin C_{\mathrm{loc}}^{\kappa+\sigma}(\mathbb{R})$ is obtained by considering the *whole line* $\mathbb{R}$. In fact, we see that $h \in C^\infty(0,\infty)$ since $h(\tau) = \tau^\kappa$ for $\tau > 0$. This means that we expect the breakdown of C^∞-smoothing effects only in the case where we consider *sign-changing solutions* u to (1.1). It is also necessary to choose sign-changing initial data u_0 to obtain such a solution u.

1.1. **Main results.** Here let us state our main results in this paper. The main topic is the smoothing effects of solutions to the equation (1.1), but before considering such a topic, we recall the result on the existence of global solutions to (1.1). To this end, we rewrite the equation (1.1) as an integral form by using the heat semigroup $\{e^{t\Delta}\}_{t>0}$. Let us define $e^{t\Delta} := G_t *$ for $0 < t < \infty$, where we set $G_t(x) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ for $(t, x) \in (0,\infty) \times \mathbb{R}^n$. Noting that we will take the initial data $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$ under the condition $1 + 2/n < \kappa < \infty$, we shall consider the following integral equation:

$$(1.3) \qquad u(t, \cdot) = e^{t\Delta} u_0 + \lambda \int_0^t e^{(t-\tau)\Delta}(|u|^{\kappa-l} u^l)(\tau, \cdot) d\tau \quad \text{in } (0,\infty) \times L^{(\kappa-1)n/2}(\mathbb{R}^n).$$

The solution u to the integral equation (1.3) may be called a *mild solution* to the original equation (1.1). Here we begin with stating the following result given by, e.g., Weissler [51, Theorem 3 (b)] and Quittner and Souplet [41, Lemma 20.22 (i)], which ensures the existence of global mild solutions u to the semilinear heat equation (1.1) for sufficiently small initial data $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$:

Proposition 1.1. *Let* $1 + 2/n < \kappa < \infty$, $l = 0, 1$, *and* $\lambda \in \mathbb{R} \setminus \{0\}$. *There exists a constant* $0 < \delta < 1$ *such that if the initial data* $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$ *satisfies*

$$(1.4) \qquad\qquad \|u_0\|_{L^{(\kappa-1)n/2}(\mathbb{R}^n)} \leq \delta,$$

then the equation (1.3) *has a global solution* $u \in BC([0,\infty); L^{(\kappa-1)n/2}(\mathbb{R}^n))$.

Remark 1.2. (i) The condition $1 + 2/n < \kappa < \infty$ ensures that the integrability index of the space $L^{(\kappa-1)n/2}(\mathbb{R}^n)$ satisfies $1 < (\kappa-1)n/2 < \infty$. Here, the exponent $1 + 2/n$ is well-known as the Fujita exponent. Note that the space $L^{(\kappa-1)n/2}(\mathbb{R}^n)$ of the initial data is a *scaling invariant space* to (1.1). In fact, if u satisfies the first equation of (1.1), then $u_\eta(t,x) := \eta^{2/(\kappa-1)}u(\eta^2 t, \eta x)$ also satisfies the first equation of (1.1) for all $0 < \eta < \infty$. For such a function u_η, we observe that

$$\|u_\eta(0,\cdot)\|_{L^{(\kappa-1)n/2}(\mathbb{R}^n)} = \|u(0,\cdot)\|_{L^{(\kappa-1)n/2}(\mathbb{R}^n)}$$

for all $0 < \eta < \infty$.

(ii) We remark that the global solution u to (1.3) satisfies the original equation (1.1) in a *classical sense*. In addition, if we choose an arbitrary initial data $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$ without any smallness condition (1.4), then we obtain a *local* solution u to (1.1). As stated before, even if $1 < \kappa \leq 1+2/n$, there exists an initial data u_0 such that we may construct a global sign-changing solution to the equation (1.1). For more details, we refer to [7, 8, 38, 53]. In this paper, we focus on the global solutions u obtained in Proposition 1.1 under the condition $1 + 2/n < \kappa < \infty$. To reveal precise regularities of global solutions, we will use the smoothing effects of the heat semigroup. Therefore, we go back to the construction of global mild solutions, namely, we will state a more precise and refined statement of Proposition 1.1 and give its proof in Section 3.

(iii) In addition, the global solution u to (1.3) is *unique* under the suitable assumption of u. Such a consequence follows from the Banach fixed point theorem [50, Theorem 1]. Here we also note that there have been studies on the *unconditional uniqueness* and *non-uniqueness* of solutions to (1.3). More precisely, for the initial data u_0 belonging to a Banach space E, if a solution u to (1.3) such that $u \in BC([0,\infty); E)$ is unique, then we say that the unconditional uniqueness holds. Conversely, if the unconditional uniqueness does not hold, then we say that the non-uniqueness holds. For the solution u to (1.3) obtained in Proposition 1.1, we may observe that the unconditional uniqueness holds under the assumptions $n \geq 3$ and $n/(n-2) < \kappa < \infty$ by using the method of Brezis and Cazenave [6, Theorem 4]. Moreover, in the case of $n \geq 3$ and $\kappa = n/(n-2)$, Terraneo [47, Corollary 0.9] and Matos and Terraneo [32, Corollary 2] showed that the non-uniqueness holds. For the other works in this direction, we refer to, e.g., [9, 18, 20].

Since Proposition 1.1 ensures the existence of global mild solutions to (1.1), we shall focus on our main topic, the smoothing effects of solutions to (1.1). First we state that the global mild solutions u to (1.1) have a *regularity nearly* $C((0,\infty); C^{\kappa+2}(\mathbb{R}^n))$ under the condition $\kappa \notin \mathbb{N}$. Here, let $B_{r,q}^s(\mathbb{R}^n)$ and $\dot{B}_{r,q}^s(\mathbb{R}^n)$ denote the Besov spaces and the homogeneous Besov spaces, respectively.

Theorem 1.3 (Smoothing effects). *Let $l = 0,1$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $1 + 2/n < \kappa < \infty$ satisfies $\kappa \notin \mathbb{N}$. In addition, suppose that u is the global solution to* (1.3) *with the initial data $u_0 \in L^{(\kappa-1)n/2}(\mathbb{R}^n)$ obtained in Proposition 1.1. Then it holds that*

$$t^{1/(\kappa-1)}u \in BC([0,\infty); \dot{B}_{\infty,1}^0(\mathbb{R}^n)),$$

(1.5)
$$u \in \bigcap_{\theta,\sigma \in (0,1)} C_{\text{loc}}^{1-\theta}((0,\infty); (B_{\infty,1}^{\kappa+2\theta} \cap B_{(\kappa-1)n/2,1}^{[\kappa]+2\theta} \cap B_{\rho,1}^{\kappa-\sigma+2\theta})(\mathbb{R}^n)),$$

(1.6)
$$u \in C_{\text{loc}}^{2+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]-1}(\mathbb{R}^n)) \cap C_{\text{loc}}^{1-j+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]+1+j}(\mathbb{R}^n))$$

for $j = 0, 1$, where $\rho := (\kappa-[\kappa])^{-1}(\kappa-1)n/2$. Here, $[\kappa]$ denotes the integer satisfying $\kappa - 1 < [\kappa] \leq \kappa$.

Remark 1.4. (i) Note that $\dot{B}_{\infty,1}^0(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$ from [44, Theorem 3.21 and Remark 3.22]. By letting $\theta = 1/2$ in (1.5), we have $u \in C_{\text{loc}}^{1/2}((0,\infty); B_{\infty,1}^{\kappa+1}(\mathbb{R}^n))$. Noting that $B_{\infty,1}^{\kappa+1}(\mathbb{R}^n) \subset BUC^2(\mathbb{R}^n)$

from $\kappa > 1$, we observe that $u \in C((0, \infty); BUC^2(\mathbb{R}^n))$. In addition, the regularity (1.6) implies $u \in C^1((0, \infty); BUC(\mathbb{R}^n))$. Therefore, we see that u is a *classical solution* to (1.1). Note that it has already been known that the global solution u to (1.3) becomes a classical solution. A new contribution of Theorem 1.3 is to reveal *precise regularities* like (1.5) and (1.6).

(ii) The regularities (1.5) and (1.6) look like complicated, but we may expect that (1.6) holds from the relation (1.2). Concerning the regularity (1.5), we note that $B^{\kappa+2\theta}_{\infty,1}(\mathbb{R}^n) \subset C^{\kappa+2\theta}(\mathbb{R}^n)$. Therefore, by setting $\theta = 0$ and $\theta = 1$ in (1.5) formally, we see that the solutions u to (1.1) have a *regularity nearly*

$$C^1((0, \infty); C^\kappa(\mathbb{R}^n)) \cap C((0, \infty); C^{\kappa+2}(\mathbb{R}^n)),$$

which has appeared in (1.2) as well.

(iii) Moreover, comparing (1.2) with (1.5), we see that (1.5) contains the additional space

$$(B^{[\kappa]+2\theta}_{(\kappa-1)n/2,1} \cap B^{\kappa-\sigma+2\theta}_{\rho,1})(\mathbb{R}^n).$$

Here, the integrability index $(\kappa-1)n/2$ of the space $B^{[\kappa]+2\theta}_{(\kappa-1)n/2,1}(\mathbb{R}^n)$ coincides with that of the space $L^{(\kappa-1)n/2}(\mathbb{R}^n)$ of the initial data. Note that we *cannot* expect that $u(t, \cdot) \in L^r(\mathbb{R}^n)$ for $0 < t < \infty$ and $1 \le r < (\kappa-1)n/2$ due to the estimate of the heat semigroup in the *unbounded domain* $\mathbb{R}^n$. In addition, while $B^{[\kappa]+2\theta}_{(\kappa-1)n/2,1}(\mathbb{R}^n)$ comes from the usual Sobolev space $W^{[\kappa],(\kappa-1)n/2}(\mathbb{R}^n)$, the space $B^{\kappa-\sigma+2\theta}_{\rho,1}(\mathbb{R}^n)$ appears by considering the *Sobolev-Slobodeckij spaces*, which are regarded as a fractional version of the Sobolev spaces. The index $\rho = (\kappa - [\kappa])^{-1}(\kappa-1)n/2$ will appear in Lemma 4.5 (ii) which gives the estimate of the nonlinear term $|u|^{\kappa-l}u^l$ in the sense of the Sobolev-Slobodeckij spaces.

(iv) The reason why we assume that $\kappa \notin \mathbb{N}$ is due to the characteristic of the Hölder spaces and the smoothing effects of the analytic semigroup. If $\kappa \in \mathbb{N}$, then we expect that *different regularities* from (1.5) and (1.6) will be obtained.

Next, we shall state the breakdown of C^∞-smoothing effects. This means that the solutions u to (1.1) have smoothing effects in some sense, but the solutions u do not belong to C^∞-class in space in general. Let $C_0^\infty(\mathbb{R}^n)$ denote the set of all smooth functions with compact support.

Theorem 1.5 (Breakdown of C^∞-smoothing effects). *Let $l = 0, 1$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $1 + 2/n < \kappa < \infty$ satisfies $\kappa \notin \mathbb{N}$. Then there exist $u_0 \in C_0^\infty(\mathbb{R}^n)$ and $0 < T < 1$ such that*

$$(1.7) \qquad u(t, \cdot) \notin \bigcup_{0 < \sigma < 1 - (\kappa - [\kappa])} C^{\kappa+2+\sigma}_{\text{loc}}(\mathbb{R}^n)$$

for all $0 < t < T$, where u is the solution to (1.1) with the initial data u_0 obtained in Proposition 1.1. In particular, $u \notin C^\infty((0, \infty) \times \mathbb{R}^n)$.

Remark 1.6. (i) As mentioned in Remark 1.4 (ii), we see that the solutions u to (1.1) have the regularity nearly $C((0, \infty); C^{\kappa+2}(\mathbb{R}^n))$. Therefore, comparing this regularity with (1.7), we observe that $C^{\kappa+2}(\mathbb{R}^n)$ may be regarded as a *threshold* of the regularity in space of solutions u to (1.1).

(ii) If we assume that $\kappa \in \mathbb{N}$ and if $\kappa - l \in \mathbb{N}$ is an even number, then the nonlinear term $|u|^{\kappa-l}u^l$ appearing in (1.1) is given by u^κ. Note that Masuda [31] and Grujić and Kukavica [17] have treated the analyticity of solutions in such a case. Conversely, if $\kappa - l \in \mathbb{N}$ is an odd number, then it holds that $|u|^{\kappa-l}u^l = |u|u^{\kappa-1}$. Since the absolute value $|u|$ of u still remains, we expect the breakdown of C^∞-smoothing effects as well. However, in this case, it seems that the discussions will become more delicate.

(iii) The proof of Theorem 1.5 is strongly based on the property of solutions u such that $u(t_0, x_0) = 0$ and $\nabla u(t_0, x_0) \neq 0$ for some $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$. To this end, we choose *sign-changing* initial

data $u_0 \in C_0^\infty(\mathbb{R}^n)$. This means that we do not know whether the same property (1.7) is valid even if the initial data u_0 are *non-negative*.

This paper is organized as follows: In the next section, we give notations and function spaces used in this paper. We also recall the embeddings of the Besov spaces and the properties of the heat semigroup. Moreover, we state the key estimates for the terms $|\varphi|^{\kappa-l}\varphi^l$ with the power nonlinearity. Section 3 is devoted to the proof of the existence of global solutions to (1.3). In Section 4, we show the smoothing properties of solutions to (1.3), i.e., Theorem 1.3. Finally, in Section 5, we reveal the breakdown of C^∞-smoothing effects, namely, we prove Theorem 1.5. In Appendix, we give the proof of the key estimates and verify the smoothing properties of mild solutions to the heat equation.

2. Preliminaries

2.1. **Notations and function spaces.** In this subsection, we give notations and function spaces used in this paper. For $0 < \kappa < \infty$, let $[\kappa]$ denote the integer satisfying $\kappa - 1 < [\kappa] \le \kappa$. For $j \in \mathbb{N}$ such that $1 \le j \le n$, let ∂_j denote the derivative operator with respect to the j-th component of $x \in \mathbb{R}^n$. Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ be a multi-index. Then we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Let $C(\mathbb{R}^n)$ be the set of all continuous real-valued functions defined on $\mathbb{R}^n$. We also set $C^m(\mathbb{R}^n) := \{\varphi \in C(\mathbb{R}^n) \,|\, \partial_x^\alpha \varphi \in C(\mathbb{R}^n),\, 0 \le |\alpha| \le m\}$ for all $m \in \mathbb{N}$. In addition, define the function space $BC(\mathbb{R}^n) := (C \cap L^\infty)(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{BC(\mathbb{R}^n)} := \|\cdot\|_{L^\infty(\mathbb{R}^n)}$. Let $BUC(\mathbb{R}^n)$ denote the closed subspace of $BC(\mathbb{R}^n)$ such that all of $\varphi \in BUC(\mathbb{R}^n)$ are uniformly continuous on $\mathbb{R}^n$. We also define $BC^m(\mathbb{R}^n)$ and $BUC^m(\mathbb{R}^n)$ in a similar manner to the case of $C^m(\mathbb{R}^n)$, where the norm of $BC^m(\mathbb{R}^n)$ and $BUC^m(\mathbb{R}^n)$ is given by $\|\varphi\|_{BC^m(\mathbb{R}^n)} := \sum_{|\alpha| \le m} \|\partial_x^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)}$. In the case of a fractional index $0 < \gamma < 1$, we define the Hölder spaces $C^\gamma(\mathbb{R}^n) := \{\varphi \in BUC(\mathbb{R}^n) \,|\, \|\varphi\|_{C^\gamma(\mathbb{R}^n)} := \|\varphi\|_{L^\infty(\mathbb{R}^n)} + [\varphi]_{C^\gamma(\mathbb{R}^n)} < \infty\}$, where

$$[\varphi]_{C^\gamma(\mathbb{R}^n)} := \sup_{x_0, x_1 \in \mathbb{R}^n,\, x_0 \ne x_1} \frac{|\varphi(x_1) - \varphi(x_0)|}{|x_1 - x_0|^\gamma}.$$

In addition, we also set $C^{m+\gamma}(\mathbb{R}^n) := \{\varphi \in BUC^m(\mathbb{R}^n) \,|\, \|\varphi\|_{C^{m+\gamma}(\mathbb{R}^n)} := \sum_{|\alpha| \le m} \|\partial_x^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\alpha|=m} [\partial_x^\alpha \varphi]_{C^\gamma(\mathbb{R}^n)} < \infty\}$ for $m \in \mathbb{N}$. If $\varphi \in C^\gamma(K)$ holds for any compact subset $K \subset \mathbb{R}^n$, then we write $\varphi \in C_{\mathrm{loc}}^\gamma(\mathbb{R}^n)$. The same notation is used for the other function spaces as well.

Let $1 \le r < \infty$ and $m \in \mathbb{N} \cup \{0\}$. Then the Sobolev spaces $W^{m,r}(\mathbb{R}^n)$ are defined by $W^{m,r}(\mathbb{R}^n) := \{\varphi \in L^r(\mathbb{R}^n) \,|\, \|\varphi\|_{W^{m,r}(\mathbb{R}^n)} := \sum_{|\alpha| \le m} \|\partial_x^\alpha \varphi\|_{L^r(\mathbb{R}^n)} < \infty\}$. In addition, for $1 \le r \le \infty$, $0 \le s < \infty$, and $1 \le q \le \infty$, we define the Besov spaces $B_{r,q}^s(\mathbb{R}^n)$ and the homogeneous Besov spaces $\dot{B}_{r,q}^s(\mathbb{R}^n)$ by setting

$$B_{r,q}^s(\mathbb{R}^n) := \{\varphi \in \mathscr{S}'(\mathbb{R}^n) \,|\, \|\varphi\|_{B_{r,q}^s(\mathbb{R}^n)} := \|\{2^{sj}\|\Delta_j \varphi\|_{L^r(\mathbb{R}^n)}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})} < \infty\},$$

$$\dot{B}_{r,q}^s(\mathbb{R}^n) := \{\varphi \in \mathscr{S}'(\mathbb{R}^n) \,|\, \|\varphi\|_{\dot{B}_{r,q}^s(\mathbb{R}^n)} := \|\{2^{sj}\|\dot{\Delta}_j \varphi\|_{L^r(\mathbb{R}^n)}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})} < \infty\},$$

respectively, where $\mathscr{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$ and $\{\Delta_j\}_{j \in \mathbb{Z}}$ and $\{\dot{\Delta}_j\}_{j \in \mathbb{Z}}$ denote the dyadic blocks given in [2, Proposition 2.10 and Definitions 2.15 and 2.68].

Let $I \subset \mathbb{R}$ be an interval and let X be a Banach space. Then, let $C(I; X)$ denote the set of all continuous X-valued functions defined on I. The same notation is used for the other function spaces as well. In particular, the norm of the Hölder spaces $C^\gamma(I; X)$ is given by $\|f\|_{C^\gamma(I;X)} := \|f\|_{L^\infty(I;X)} + [f]_{C^\gamma(I;X)}$, where

$$[f]_{C^\gamma(I;X)} := \sup_{t_0, t_1 \in I,\, t_0 \ne t_1} \frac{\|f(t_1) - f(t_0)\|_X}{|t_1 - t_0|^\gamma}.$$

We also use the notation $C_{\mathrm{loc}}^{\gamma}(I; X)$, namely, if $f \in C^{\gamma}(K; X)$ holds for any compact subinterval $K \subset I$, then we write $f \in C_{\mathrm{loc}}^{\gamma}(I; X)$.

In what follows, we will use the following abbreviations: For a function space $X(\mathbb{R}^n)$ defined on the whole space $\mathbb{R}^n$, we set $X \coloneqq X(\mathbb{R}^n)$. For $1 \leq r \leq \infty$ and a Banach space X, we set $\|\cdot\|_{L^r(X)} \coloneqq \|\cdot\|_{L^r((0,\infty);X)}$. In addition, for $0 < \varepsilon < 1$, we also set $\|\cdot\|_{L_\varepsilon^r(X)} \coloneqq \|\cdot\|_{L^r((\varepsilon, \varepsilon^{-1}); X)}$. The same abbreviations are used for the other function spaces as well.

2.2. **Basic properties on the Besov spaces and the heat semigroup.** In this subsection, we recall basic properties on the Besov spaces $B_{r,q}^s$ and the heat semigroup $\{e^{t\Delta}\}_{t>0}$. In what follows, the inclusion $A \subset B$ implies the continuous embedding, namely, $A \subset B$ implies that there is some constant $C > 0$ satisfying $\|\varphi\|_B \leq C\|\varphi\|_A$ for all $\varphi \in A$. We begin with recalling the embeddings and characterizations of the Besov spaces $B_{r,q}^s$.

Proposition 2.1. *(i) Let $1 \leq r \leq \infty$, $0 \leq s < \infty$, $0 < \sigma < \infty$, and $1 \leq q_0 \leq q_1 \leq \infty$. Then it holds that $B_{r,1}^s \subset B_{r,q_0}^s \subset B_{r,q_1}^s \subset B_{r,\infty}^s$ and $B_{r,\infty}^{s+\sigma} \subset B_{r,1}^s$.*

(ii) Let $1 \leq r < \infty$ and $m \in \mathbb{N} \cup \{0\}$. Then it holds that $B_{r,1}^m \subset W^{m,r} \subset B_{r,\infty}^m$ and $B_{\infty,1}^m \subset BUC^m \subset B_{\infty,\infty}^m$. In addition, it holds that $\dot{B}_{r,1}^0 \subset L^r \subset \dot{B}_{r,\infty}^0$.

(iii) Let $1 \leq r < \infty$. Suppose that $0 < s < \infty$ satisfies $s \notin \mathbb{N}$. Then the Besov spaces $B_{r,r}^s$ are characterized as follows;

$$\sum_{|\alpha| \leq [s]} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial_x^\alpha \{\varphi(x_1) - \varphi(x_0)\}|^r}{|x_1 - x_0|^{(s-[s])r+n}} dx_0 dx_1 \leq C\|\varphi\|_{B_{r,r}^s}^r,$$

$$C^{-1}\|\varphi\|_{B_{r,r}^s}^r \leq \|\varphi\|_{W^{[s],r}}^r + \sum_{|\alpha|=[s]} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\partial_x^\alpha \{\varphi(x_1) - \varphi(x_0)\}|^r}{|x_1 - x_0|^{(s-[s])r+n}} dx_0 dx_1,$$

where $C > 0$ is a constant independent of φ. In addition, it holds that $B_{\infty,\infty}^s = C^s$ with the equivalence of norms.

(iv) Let $1 \leq r_0 \leq r_1 \leq \infty$, $0 \leq s < \infty$, and $1 \leq q \leq \infty$. Then it holds that

$$B_{r_0,q}^{s+n(1/r_0-1/r_1)} \subset B_{r_1,q}^s, \quad \dot{B}_{r_0,q}^{s+n(1/r_0-1/r_1)} \subset \dot{B}_{r_1,q}^s.$$

In particular, it holds that $B_{r,1}^\sigma \subset L^{(1/r-\sigma/n)^{-1}}$ for $1 \leq r < \infty$ and $0 \leq \sigma \leq n/r$.

Proof. The assertion of (i) is shown in [43, Propositions 2.2 and 2.3]. The assertion of (ii) may be verified by combining [43, Propositions 2.1, 2.2, 2.3, and 2.4 and Theorems 2.2 and 2.3] with [2, Proposition 2.39]. Concerning (iii), we may show from [48, Remark 4, p.189] and [43, Theorems 2.2, 2.3, and 2.7]. In addition we may verify (iv) by [2, Proposition 2.20] and [43, Theorem 2.5]. $\square$

Next we give the properties of the heat semigroup $\{e^{t\Delta}\}_{t>0}$.

Proposition 2.2. *(i) Let $1 \leq r_0 \leq \infty$ and $\varphi \in L^{r_0}$. Then it holds that $e^{t\Delta}\varphi \in W^{1,r_0}$ with the estimate*

$$\|\nabla^j e^{t\Delta}\varphi\|_{L^{r_0}} \leq Ct^{-j/2}\|\varphi\|_{L^{r_0}}$$

for $j = 0, 1$ and for all $0 < t < \infty$, where $C > 0$ is a constant independent of t and φ. In addition, if $r_0 < \infty$, then it holds that $e^{t\Delta}\varphi \in \dot{B}_{r_1,1}^0$ with the estimate

$$\|e^{t\Delta}\varphi\|_{\dot{B}_{r_1,1}^0} \leq Ct^{-(n/2)(1/r_0-1/r_1)}\|\varphi\|_{L^{r_0}}$$

for all $r_0 < r_1 \leq \infty$ and $0 < t < \infty$.

(ii) Let $1 \leq r_0 < \infty$. Then, for every $\varphi \in L^{r_0}$, it holds that

$$\lim_{t \to +0} t^{(n/2)(1/r_0-1/r_1)}\|e^{t\Delta}\varphi\|_{\dot{B}_{r_1,1}^0} = 0$$

for all $r_0 < r_1 < \infty$.

Proof. The first estimate of (i) is well-known. The second estimate of (i) is derived by [24, Lemma 2.2 (ii)] with the aid of Proposition 2.1 (ii) and (iv). The statement of (ii) may be obtained by the same argument as in [45, Proposition 3.10 (i)]. $\square$

2.3. **Key estimates for the terms with the power nonlinearity.** In this subsection, we state the key estimates for the terms $|\varphi|^{\kappa-l}\varphi^l$ with the power nonlinearity. These estimates play an essential role in the proof of our main results, Theorems 1.3 and 1.5. First we show the simple inequalities.

Proposition 2.3. *(i) Let $0 < \kappa \leq 1$ and $l = 0, 1$. Then it holds that*
$$||\tau_1|^{\kappa-l}\tau_1^l - |\tau_0|^{\kappa-l}\tau_0^l| \leq (2^{\kappa+1}+1)^l|\tau_1 - \tau_0|^\kappa$$
for all $\tau_0, \tau_1 \in \mathbb{R}$.

(ii) Let $1 < \kappa < \infty$ and $l = 0, 1$. Then it holds that
$$||\tau_1|^{\kappa-l}\tau_1^l - |\tau_0|^{\kappa-l}\tau_0^l| \leq \max\{\kappa, 2^{\kappa-2}\kappa\}(|\tau_1|^{\kappa-1} + |\tau_0|^{\kappa-1})|\tau_1 - \tau_0|$$
for all $\tau_0, \tau_1 \in \mathbb{R}$.

(iii) Let $1 < \kappa < \infty$, $l = 0, 1$, and $r, r_1 \in [1, \infty]$. Assume that $\max\{1, \kappa-1\} \leq r_0 \leq \infty$ satisfies $1/r = (\kappa-1)/r_0 + 1/r_1$. Then it holds that
$$|||\varphi_1|^{\kappa-l}\varphi_1^l - |\varphi_0|^{\kappa-l}\varphi_0^l\|_{L^r} \leq \max\{\kappa, 2^{\kappa-2}\kappa\}(\|\varphi_1\|_{L^{r_0}}^{\kappa-1} + \|\varphi_0\|_{L^{r_0}}^{\kappa-1})\|\varphi_1 - \varphi_0\|_{L^{r_1}}$$
for all $\varphi_0, \varphi_1 \in L^{r_0} \cap L^{r_1}$.

Proof. (i) Since the desired estimate is easily obtained in the case of $\tau_0 = 0$ or $\tau_1 = 0$, we assume that $\tau_0 \neq 0$ and $\tau_1 \neq 0$. In case $l = 0$, we see by $0 < \kappa \leq 1$ that

(2.1)
$$|\tau_1|^\kappa \leq (|\tau_1 - \tau_0| + |\tau_0|)^\kappa \leq |\tau_1 - \tau_0|^\kappa + |\tau_0|^\kappa$$

for all $\tau_0, \tau_1 \in \mathbb{R}$, which yields $||\tau_1|^\kappa - |\tau_0|^\kappa| \leq |\tau_1 - \tau_0|^\kappa$. In case $l = 1$, if $\tau_0, \tau_1 \in \mathbb{R}$ satisfy $|\tau_1 - \tau_0| \geq |\tau_0|/2$, then we see by (2.1) that
$$||\tau_1|^{\kappa-1}\tau_1 - |\tau_0|^{\kappa-1}\tau_0| \leq |\tau_1|^\kappa + |\tau_0|^\kappa \leq |\tau_1 - \tau_0|^\kappa + 2|\tau_0|^\kappa \leq |\tau_1 - \tau_0|^\kappa + 2 \cdot (2|\tau_1 - \tau_0|)^\kappa$$
$$= (2^{\kappa+1}+1)|\tau_1 - \tau_0|^\kappa.$$

If $\tau_1, \tau_0 \in \mathbb{R}$ satisfy $|\tau_1 - \tau_0| < |\tau_0|/2$, then it holds by $\tau_0 \neq 0$ that $|\tau_1\tau_0 - \tau_0^2| < \tau_0^2/2$. Since $-\tau_0^2/2 < \tau_1\tau_0 - \tau_0^2$, we have $\tau_0^2 < 2\tau_1\tau_0$, which yields $\tau_1\tau_0 > 0$. Hence, it holds by (2.1) that
$$||\tau_1|^{\kappa-1}\tau_1 - |\tau_0|^{\kappa-1}\tau_0| = ||\tau_1|^\kappa - |\tau_0|^\kappa| \leq |\tau_1 - \tau_0|^\kappa.$$
Consequently, we observe that $||\tau_1|^{\kappa-1}\tau_1 - |\tau_0|^{\kappa-1}\tau_0| \leq (2^{\kappa+1}+1)|\tau_1 - \tau_0|^\kappa$.

(ii) Since $1 < \kappa < \infty$, by applying the mean value theorem to the function $h(\tau) := |\tau|^{\kappa-l}\tau^l$ defined on $\mathbb{R}$, we observe that $|h(\tau_1) - h(\tau_0)| = |\partial_\tau h_0(\tau_*)||\tau_1 - \tau_0|$ holds for all $-\infty < \tau_1 < \tau_0 < \infty$ with some $\tau_1 < \tau_* < \tau_0$. Thus we see by $\partial_\tau h(\tau) = \kappa|\tau|^{\kappa-2+l}\tau^{1-l}$ and $|\tau_*| \leq |\tau_1| + |\tau_0|$ that
$$||\tau_1|^{\kappa-l}\tau_1^l - |\tau_0|^{\kappa-l}\tau_0^l| \leq \kappa(|\tau_1| + |\tau_0|)^{\kappa-1}|\tau_1 - \tau_0|$$
$$\leq \max\{\kappa, 2^{\kappa-2}\kappa\}(|\tau_1|^{\kappa-1} + |\tau_0|^{\kappa-1})|\tau_1 - \tau_0|$$
for all $\tau_1, \tau_0 \in \mathbb{R}$.

(iii) By the assertion of (ii) with the aid of the Hölder inequality, we deduce that
$$|||\varphi_1|^{\kappa-l}\varphi_1^l - |\varphi_0|^{\kappa-l}\varphi_0^l\|_{L^r} \leq \max\{\kappa, 2^{\kappa-2}\kappa\}|||\varphi_1|^{\kappa-1} + |\varphi_0|^{\kappa-1}\|_{L^{r_0/(\kappa-1)}}\|\varphi_1 - \varphi_0\|_{L^{r_1}}$$
$$\leq \max\{\kappa, 2^{\kappa-2}\kappa\}(\|\varphi_1\|_{L^{r_0}}^{\kappa-1} + \|\varphi_0\|_{L^{r_0}}^{\kappa-1})\|\varphi_1 - \varphi_0\|_{L^{r_1}}.$$

This proves Proposition 2.3. $\square$

The following lemma will be used to obtain the regularities of solutions u to (1.1) and to show the breakdown of C^∞-smoothing effects:

Lemma 2.4. *Let $1 < \kappa < \infty$ and $l = 0, 1$. Suppose that $m \in \mathbb{N}$ satisfies $1 \le m < \kappa$. Then the following statements hold:*

(i) It holds that

$$\sum_{1 \le |\alpha| \le m} |(\mathcal{R}_\alpha^{(l)} \varphi_1)(x) - (\mathcal{R}_\alpha^{(l)} \varphi_0)(x)|$$

$$\le C(1 + \|\varphi_1\|_{BC^m} + \|\varphi_0\|_{BC^m})^{\kappa + m - 1} \sum_{|\alpha| \le m} |\partial_x^\alpha \{\varphi_1(x) - \varphi_0(x)\}|$$

for all $x \in \mathbb{R}^n$ and $\varphi_0, \varphi_1 \in BC^m$, where

$$(2.2) \qquad \mathcal{R}_\alpha^{(l)} \varphi := \partial_x^\alpha(|\varphi|^{\kappa - l} \varphi^l) - \left\{ \prod_{j=1}^{|\alpha|} (\kappa + 1 - j) \right\} |\varphi|^{\kappa - 2|\alpha| + l} \varphi^{|\alpha| - l} \prod_{j=1}^n (\partial_j \varphi)^{\alpha_j}$$

and $C > 0$ is a constant independent of x, φ_0, and φ_1.

(ii) It holds that

$$\sum_{|\alpha| = m} |\partial_x^\alpha \{(|\varphi_1|^{\kappa - l} \varphi_1^l)(x) - (|\varphi_0|^{\kappa - l} \varphi_0^l)(x)\}|$$

$$\le C|(|\varphi_1|^{\kappa - 2m + l} \varphi_1^{m - l})(x) - (|\varphi_0|^{\kappa - 2m + l} \varphi_0^{m - l})(x)| \sum_{|\alpha| = 1} |\partial_x^\alpha \varphi_1(x)|^m$$

$$+ C(1 + \|\varphi_1\|_{BC^m} + \|\varphi_0\|_{BC^m})^{\kappa + m - 1} \sum_{|\alpha| \le m} |\partial_x^\alpha \{\varphi_1(x) - \varphi_0(x)\}|$$

for all $x \in \mathbb{R}^n$ and $\varphi_0, \varphi_1 \in BC^m$.

(iii) Let $0 < \varepsilon < 1$. Then it holds that

$$\sum_{|\alpha| = m - 1} |\partial_t \partial_x^\alpha \{(|f_1|^{\kappa - l} f_1^l)(t, x) - (|f_0|^{\kappa - l} f_0^l)(t, x)\}|$$

$$\le C|(|f_1|^{\kappa - 2m + l} f_1^{m - l})(t, x) - (|f_0|^{\kappa - 2m + l} f_0^{m - l})(t, x)||\partial_t f_1(t, x)| \sum_{|\alpha| = 1} |\partial_x^\alpha f_1(t, x)|^{m - 1}$$

$$+ C(1 + \|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa + m - 1}$$

$$\times \sum_{|\alpha| \le m - 1} \left(|\partial_x^\alpha \{f_1(t, x) - f_0(t, x)\}| + |\partial_t \partial_x^\alpha \{f_1(t, x) - f_0(t, x)\}| \right)$$

for all $(t, x) \in (\varepsilon, \varepsilon^{-1}) \times \mathbb{R}^n$ and $f_0, f_1 \in C^1((0, \infty); BC^{m-1})$, where $C > 0$ is a constant independent of ε, t, x, f_0, and f_1. Here $\| \cdot \|_{BC_\varepsilon^1(BC^{m-1})} := \| \cdot \|_{BC^1((\varepsilon, \varepsilon^{-1}); BC^{m-1})}$.

The proof of Lemma 2.4 is given in Appendix.

Remark 2.5. (i) For the assertion of Lemma 2.4 (i), by computing the derivatives of $|\varphi|^{\kappa - l} \varphi^l$, we see that

$$\partial_{i_0}(|\varphi|^{\kappa - l} \varphi^l) = \kappa |\varphi|^{\kappa - 2 + l} \varphi^{1 - l} \partial_{i_0} \varphi,$$

$$\partial_{i_1} \partial_{i_0}(|\varphi|^{\kappa - l} \varphi^l) = \kappa(\kappa - 1)|\varphi|^{\kappa - 4 + l} \varphi^{2 - l} (\partial_{i_1} \varphi) \partial_{i_0} \varphi + \kappa |\varphi|^{\kappa - 2 + l} \varphi^{1 - l} \partial_{i_1} \partial_{i_0} \varphi$$

for all $i_0, i_1 \in [1, n]$. Concerning the first term on the right-hand side of the above relations, we may expect that the corresponding term is given by

$$\left\{ \prod_{j=1}^{|\alpha|} (\kappa + 1 - j) \right\} |\varphi|^{\kappa - 2|\alpha| + l} \varphi^{|\alpha| - l} \prod_{j=1}^{n} (\partial_j \varphi)^{\alpha_j}$$

in the case of $\partial_x^\alpha(|\varphi|^{\kappa - l}\varphi^l)$. Here, note that the above term is the *worst term* in the sense of *regularities* of $|\varphi|^{\kappa - 2|\alpha| + l}\varphi^{|\alpha| - l}$. Therefore, the definition (2.2) of $\mathcal{R}_\alpha^{(l)}\varphi$ gives the *remainder terms* of $\partial_x^\alpha(|\varphi|^{\kappa - l}\varphi^l)$.

(ii) By assuming $m \neq \kappa$ in the assertions of Lemma 2.4, we avoid the case where the term $|\varphi|^{-(m-l)}\varphi^{m-l}$ appears. Moreover, concerning Lemma 2.4 (iii), we agree that $|\partial_x^\alpha f_1(t, x)|^{m-1} \equiv 1$ in the case of $m = 1$. Hence, the assertion of Lemma 2.4 (iii) is still valid for $m = 1$ even though we assume only $f_1 \in C^1((0, \infty); BC)$.

(iii) All of the statements of Proposition 2.3 and Lemma 2.4 may be extended to the case of the general domain $\Omega \subset \mathbb{R}^n$.

3. Construction of global mild solutions

As stated before, we see by Proposition 1.1 that the equation (1.3) has a global solution u under the condition (1.4). Although our aim in this paper is to reveal regularities of solutions to the semilinear heat equation (1.1), we begin with constructing global mild solutions in order to apply the smoothing effects of the heat semigroup.

Proposition 3.1 (Global mild solutions). *Let $1 + 2/n < \kappa < \infty$, $l = 0, 1$, and $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $u_0 \in L^{(\kappa-1)n/2}$ and $p_0, p_1 \in (1, \infty)$ satisfy*

$$(3.1) \qquad \max\left\{0, \frac{2}{(\kappa - 1)n}(2 - \kappa)\right\} < \frac{1}{p_0} < \frac{2}{(\kappa - 1)n},$$

$$(3.2) \qquad \max\left\{0, \frac{2}{(\kappa - 1)n} - \frac{\kappa - 1}{p_0}\right\} < \frac{1}{p_1} < \min\left\{\frac{2}{(\kappa - 1)n}, 1 - \frac{\kappa - 1}{p_0}, \frac{2}{n} - \frac{\kappa - 2}{p_0}\right\}$$

and set $p_2 := \infty$. Then the following assertions hold:

(i) There exists a constant $0 < \delta < 1$ independently of u_0 such that if (1.4), then the equation (1.3) has a global solution u satisfying

$$(3.3) \qquad u \in BC([0, \infty); L^{(\kappa-1)n/2}), \quad t^{1/(\kappa-1) - n/(2p_j)} u \in BC([0, \infty); \dot{B}_{p_j, 1}^0),$$

$$(3.4) \qquad \lim_{t \to +0} \left(t^{1/(\kappa-1) - n/(2p_j)} \|u(t, \cdot)\|_{\dot{B}_{p_j, 1}^0} + \|u(t, \cdot) - u_0\|_{L^{(\kappa-1)n/2}}\right) = 0$$

for $j = 0, 1, 2$. In addition, the following estimate

$$(3.5) \qquad \|u\|_{L^\infty(L^{(\kappa-1)n/2})} + \sum_{j=0}^{2} \sup_{0 < t < \infty} t^{1/(\kappa-1) - n/(2p_j)} \|u(t, \cdot)\|_{\dot{B}_{p_j, 1}^0} \leq C\|u_0\|_{L^{(\kappa-1)n/2}}$$

holds, where $C > 0$ is a constant independent of δ, u_0, and u.

(ii) Suppose that u is the global solution to (1.3) with the initial data u_0 obtained in (i). In addition, suppose that v is a global solution to (1.3) with an initial data $v_0 \in L^{(\kappa-1)n/2}$ satisfying

$\|v_0\|_{L^{(\kappa-1)n/2}} \leq \delta$. *Then it holds that*

(3.6)
$$\|u - v\|_{L^\infty(L^{(\kappa-1)n/2})} + \sum_{j=0}^{2} \sup_{0<t<\infty} t^{1/(\kappa-1)-n/(2p_j)}\|u(t,\cdot) - v(t,\cdot)\|_{\dot{B}^0_{p_j,1}}$$

$$\leq C\|u_0 - v_0\|_{L^{(\kappa-1)n/2}},$$

where $C > 0$ is a constant independent of δ, u_0, v_0, u, and v.

Remark 3.2. (i) As stated before Proposition 1.1, Weissler [51, Theorem 3 (b)] and Quittner and Souplet [41, Lemma 20.22 (i)] have already shown the existence of global solutions to (1.3) with the initial data $u_0 \in L^{(\kappa-1)n/2}$. Our result is regarded as a slightly improved version compared with [51, Theorem 3 (b)] and [41, Lemma 20.22 (i)] from the viewpoint of the space (3.3) of the solutions. More precisely, for the space (3.3) of the solutions, we note that $\dot{B}^0_{r,1} \subset L^r$ for all $1 \leq r \leq \infty$. Such regularities are obtained by Proposition 2.2 (i), namely, the estimate of the heat semigroup $\{e^{t\Delta}\}_{t>0}$ on homogeneous Besov spaces $\dot{B}^s_{r,q}$ given in [24, Lemma 2.2 (ii)].

(ii) We also recall Remark 1.2 (iii). Namely, the global solution u to (1.3) is unique under the suitable assumption of u. In particular, if $n \geq 3$ and $n/(n-2) < \kappa < \infty$, then a global solution u is unique by assuming only $u \in BC([0,\infty); L^{(\kappa-1)n/2}(\mathbb{R}^n))$ from [6, Theorem 4]. However, this assertion does not hold in the case of $\kappa = n/(n-2)$ from [32, Corollary 2].

The proof of Proposition 3.1 relies on the basic argument due to Kato [21]. More precisely, we will make use of the estimates of the heat semigroup and the Banach fixed point theorem. We also refer to [41, Lemma 20.22 (i)]. To this end, we first verify the conditions of the indices p_0 and p_1 appearing in the space (3.3) of the solutions.

Proposition 3.3. *(i) Let $1 + 2/n < \kappa < \infty$. Then there exists $1 < p_0 < \infty$ satisfying (3.1). In addition, there exists $1 < p_1 < \infty$ satisfying (3.2).*

(ii) In the statement of (i), let p satisfy $1/p = (\kappa - 1)/p_0 + 1/p_1$. Then it holds that $1 < p < \infty$ and

(3.7)
$$0 < \frac{n}{2p} - \frac{1}{\kappa - 1} < 1, \quad 0 < \frac{n}{2}\left(\frac{1}{p} - \frac{1}{p_j}\right) < 1$$

for $j = 0, 1$.

Proof. (i) First we note that $0 < 2\{(\kappa-1)n\}^{-1} < 1$ from the condition $1 + 2/n < \kappa < \infty$. Since we have

$$\frac{2}{(\kappa-1)n} - \frac{2}{(\kappa-1)n}(2-\kappa) = \frac{2}{n} > 0,$$

it holds that

$$\max\left\{0, \frac{2}{(\kappa-1)n}(2-\kappa)\right\} < \frac{2}{(\kappa-1)n} < 1,$$

which ensures the existence of $1 < p_0 < \infty$ satisfying (3.1). In addition, we see by (3.1) that

$$\frac{2}{(\kappa-1)n} - \left\{\frac{2}{(\kappa-1)n} - \frac{\kappa-1}{p_0}\right\} = \frac{\kappa-1}{p_0} > 0,$$

$$1 - \frac{\kappa-1}{p_0} > 1 - (\kappa-1) \cdot \frac{2}{(\kappa-1)n} = 1 - \frac{2}{n} \geq 0,$$

$$1 - \frac{\kappa-1}{p_0} - \left\{\frac{2}{(\kappa-1)n} - \frac{\kappa-1}{p_0}\right\} = 1 - \frac{2}{(\kappa-1)n} > 0,$$

$$\frac{2}{n} - \frac{\kappa-2}{p_0} - \left\{\frac{2}{(\kappa-1)n} - \frac{\kappa-1}{p_0}\right\} = \frac{1}{p_0} - \frac{2}{(\kappa-1)n}(2-\kappa) > 0$$

and

$$\frac{2}{n} - \frac{\kappa - 2}{p_0} > 0 \qquad\qquad\qquad\qquad \text{if } \kappa \in (1 + 2/n, 2],$$

$$\frac{2}{n} - \frac{\kappa - 2}{p_0} > \frac{2}{n} - (\kappa - 2) \cdot \frac{2}{(\kappa - 1)n} = \frac{2}{(\kappa - 1)n} > 0 \quad \text{if } \kappa \in (2, \infty).$$

Thus we obtain

$$\max\left\{0, \frac{2}{(\kappa - 1)n} - \frac{\kappa - 1}{p_0}\right\} < \min\left\{\frac{2}{(\kappa - 1)n}, 1 - \frac{\kappa - 1}{p_0}, \frac{2}{n} - \frac{\kappa - 2}{p_0}\right\} < 1,$$

which implies that we may take $1 < p_1 < \infty$ satisfying (3.2).

(ii) Since $1/p = (\kappa - 1)/p_0 + 1/p_1$, it holds by (3.2) that

$$1 - \frac{1}{p} = 1 - \frac{\kappa - 1}{p_0} - \frac{1}{p_1} > 1 - \frac{\kappa - 1}{p_0} - \left(1 - \frac{\kappa - 1}{p_0}\right) = 0.$$

Thus we have $1 < p < \infty$. Moreover, we obtain

$$\frac{n}{2p} - \frac{1}{\kappa - 1} = \frac{(\kappa - 1)n}{2p_0} - \frac{1}{\kappa - 1} + \frac{n}{2p_1},$$

$$\frac{n}{2}\left(\frac{1}{p} - \frac{1}{p_0}\right) = \frac{n}{2}\left(\frac{\kappa - 2}{p_0} + \frac{1}{p_1}\right), \quad \frac{n}{2}\left(\frac{1}{p} - \frac{1}{p_1}\right) = \frac{(\kappa - 1)n}{2p_0}.$$

Hence, by using (3.1) and (3.2), we observe that

$$\frac{(\kappa - 1)n}{2p_0} - \frac{1}{\kappa - 1} + \frac{n}{2p_1} > \frac{(\kappa - 1)n}{2p_0} - \frac{1}{\kappa - 1} + \frac{n}{2}\left\{\frac{2}{(\kappa - 1)n} - \frac{\kappa - 1}{p_0}\right\} = 0,$$

$$1 - \left\{\frac{(\kappa - 1)n}{2p_0} - \frac{1}{\kappa - 1} + \frac{n}{2p_1}\right\} > 1 - \frac{(\kappa - 1)n}{2} \cdot \frac{2}{(\kappa - 1)n} + \frac{1}{\kappa - 1} - \frac{n}{2} \cdot \frac{2}{(\kappa - 1)n} = 0,$$

$$\frac{n}{2}\left(\frac{\kappa - 2}{p_0} + \frac{1}{p_1}\right) > \frac{n}{2}\left\{\frac{\kappa - 2}{p_0} + \frac{2}{(\kappa - 1)n} - \frac{\kappa - 1}{p_0}\right\} = \frac{n}{2}\left\{\frac{2}{(\kappa - 1)n} - \frac{1}{p_0}\right\} > 0,$$

$$1 - \frac{n}{2}\left(\frac{\kappa - 2}{p_0} + \frac{1}{p_1}\right) > 1 - \frac{n}{2}\left(\frac{\kappa - 2}{p_0} + \frac{2}{n} - \frac{\kappa - 2}{p_0}\right) = 0,$$

$$1 - \frac{(\kappa - 1)n}{2p_0} > 0.$$

Thus we have (3.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In what follows, we show Proposition 3.1.

Proof of Proposition 3.1. (i) Define the function space X by setting

$$(3.8) \qquad X := \left\{u \in BC([0, \infty); L^{(\kappa - 1)n/2}) \,\middle|\, \begin{array}{l} t^{1/(\kappa - 1) - n/(2p_j)}u \in BC([0, \infty); \dot{B}_{p_j, 1}^0), \ j = 0, 1, 2, \\ \lim_{t \to +0} t^{1/(\kappa - 1) - n/(2p_j)}\|u(t, \cdot)\|_{\dot{B}_{p_j, 1}^0} = 0, \ j = 0, 1, 2 \end{array}\right\}.$$

In addition, we also set

$$[u]_{X_t} := \sum_{j=0}^{1} \sup_{0 < \tau < t} \tau^{1/(\kappa - 1) - n/(2p_j)}\|u(\tau, \cdot)\|_{\dot{B}_{p_j, 1}^0}, \qquad\qquad 0 < t < \infty,$$

$$(3.9) \qquad \mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot) := \int_0^t e^{(t - \tau)\Delta}(|u|^{\kappa - l}u^l - |v|^{\kappa - l}v^l)(\tau, \cdot)d\tau, \qquad\qquad 0 < t < \infty$$

for $u, v \in X$. Here we take p so that $1/p = (\kappa - 1)/p_0 + 1/p_1$. Then it holds by Proposition 3.3 (ii) that (3.7). Since $\dot{B}_{r,1}^0 \subset L^r$ for all $1 \le r \le \infty$ and since $\max\{1, \kappa - 1\} < p_0 < \infty$, Propositions 2.2 (i) and 2.3 (iii) yield

$$\|e^{t\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)\|_{L^{(\kappa-1)n/2}}$$
$$\le Ct^{1/(\kappa-1)-n/(2p)}\|(|u|^{\kappa-l}u^l)(\tau, \cdot) - (|v|^{\kappa-l}v^l)(\tau, \cdot)\|_{L^p}$$
$$\le Ct^{1/(\kappa-1)-n/(2p)}(\|u(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{p_1,1}^0}$$

for all $t, \tau \in (0, \infty)$ and $u, v \in X$. In the same way, we also deduce that

$$\|e^{t\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)\|_{\dot{B}_{p_j,1}^0}$$
$$\le Ct^{-(n/2)(1/p-1/p_j)}(\|u(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{p_1,1}^0}$$

for $j = 0, 1$. Moreover, there holds

$$\|e^{t\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)\|_{\dot{B}_{\infty,1}^0}$$
$$\le Ct^{-(\kappa-1)n/(2p_0)}\|(|u|^{\kappa-l}u^l)(\tau, \cdot) - (|v|^{\kappa-l}v^l)(\tau, \cdot)\|_{L^{p_0/(\kappa-1)}}$$
$$\le Ct^{-(\kappa-1)n/(2p_0)}(\|u(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{\infty,1}^0}.$$

Since

$$(3.10) \qquad \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau = B(\alpha, \beta)t^{\alpha+\beta-1}, \quad B(\alpha, \beta) := \int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1}d\tau < \infty$$

for all $0 < t < \infty$ and $\alpha, \beta \in (0, \infty)$, by combining the above estimates, (3.7), and (3.10), we observe that

$$\|\mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot)\|_{L^{(\kappa-1)n/2}}$$
$$(3.11) \quad \le C \int_0^t (t-\tau)^{1/(\kappa-1)-n/(2p)}(\|u(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{p_1,1}^0} d\tau$$
$$\le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u-v]_{X_t} \int_0^t (t-\tau)^{1/(\kappa-1)-n/(2p)}\tau^{(\kappa-1)n/(2p_0)-1}\tau^{n/(2p_1)-1/(\kappa-1)}d\tau$$
$$\le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u-v]_{X_t}$$

for all $0 < t < \infty$ and $u, v \in X$. Similarly, we obtain

$$\|\mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot)\|_{\dot{B}_{p_j,1}^0}$$
$$\le C \int_0^t (t-\tau)^{-(n/2)(1/p-1/p_j)}(\|u(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau, \cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{p_1,1}^0} d\tau$$
$$\le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u-v]_{X_t} \int_0^t (t-\tau)^{-(n/2)(1/p-1/p_j)}\tau^{(\kappa-1)n/(2p_0)-1}\tau^{n/(2p_1)-1/(\kappa-1)}d\tau$$
$$\le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u-v]_{X_t} \cdot t^{-1/(\kappa-1)+n/(2p_j)}$$

for $j = 0, 1$, which yields

$$(3.12) \qquad \|\mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot)\|_{L^{(\kappa-1)n/2}} + [\mathcal{I}_\kappa^{(l)}(u, v)]_{X_t} \le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u-v]_{X_t}.$$

In addition, it holds that

$$\left\| \int_{t/2}^{t} e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau,\cdot)d\tau \right\|_{\dot{B}_{\infty,1}^0}$$

$$\leq C \int_{t/2}^{t} (t-\tau)^{-(\kappa-1)n/(2p_0)}(\|u(\tau,\cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1} + \|v(\tau,\cdot)\|_{\dot{B}_{p_0,1}^0}^{\kappa-1})\|u(\tau,\cdot) - v(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0} d\tau$$

(3.13)
$$\leq C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1}) \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau,\cdot) - v(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0}$$

$$\times \int_{t/2}^{t} (t-\tau)^{-(\kappa-1)n/(2p_0)}\tau^{(\kappa-1)n/(2p_0)-1}\tau^{-1/(\kappa-1)}d\tau$$

$$\leq C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1}) \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau,\cdot) - v(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0} \cdot t^{-1/(\kappa-1)}.$$

Now we define the mapping Φ by setting

(3.14) $\quad (\Phi u)(t,\cdot) := e^{t\Delta}u_0 + \lambda \mathcal{I}_\kappa^{(l)}(u,0)(t,\cdot) = e^{t\Delta}u_0 + \lambda \int_0^t e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l)(\tau,\cdot)d\tau, \quad 0 < t < \infty$

for $u \in X$. Then the estimate (3.12) implies that

(3.15) $\qquad \|(\Phi u)(t,\cdot)\|_{L^{(\kappa-1)n/2}} \leq \|e^{t\Delta}u_0\|_{L^{(\kappa-1)n/2}} + C[u]_{X_t}^\kappa, \quad [\Phi u]_{X_t} \leq [e^{t\Delta}u_0]_{X_t} + C[u]_{X_t}^\kappa$

for all $0 < t < \infty$ and $u \in X$. In addition, noting that

$(\Phi u)(t,\cdot)$

$$= e^{(t/2)\Delta}e^{(t/2)\Delta}u_0 + \lambda \int_0^{t/2} e^{(t/2)\Delta}e^{(t/2-\tau)\Delta}(|u|^{\kappa-l}u^l)(\tau,\cdot)d\tau + \lambda \int_{t/2}^{t} e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l)(\tau,\cdot)d\tau$$

$$= e^{(t/2)\Delta}((\Phi u)(t/2,\cdot)) + \lambda \int_{t/2}^{t} e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l)(\tau,\cdot)d\tau$$

for all $0 < t < \infty$, we see by (3.13) and Proposition 2.2 (i) that

$\|(\Phi u)(t,\cdot)\|_{\dot{B}_{\infty,1}^0}$

$$\leq C(t/2)^{-n/(2p_0)}\|(\Phi u)(t/2,\cdot)\|_{L^{p_0}} + Ct^{-1/(\kappa-1)}[u]_{X_t}^{\kappa-1} \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0}$$

$$\leq Ct^{-n/(2p_0)} \cdot (t/2)^{n/(2p_0)-1/(\kappa-1)}[\Phi u]_{X_t} + Ct^{-1/(\kappa-1)}[u]_{X_t}^{\kappa-1} \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0},$$

which yields

$$\sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|(\Phi u)(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0} \leq C\left([e^{t\Delta}u_0]_{X_t} + [u]_{X_t}^\kappa + [u]_{X_t}^{\kappa-1} \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau,\cdot)\|_{\dot{B}_{\infty,1}^0}\right)$$

from (3.15). By Proposition 2.2 (ii) and the definition (3.8) of X, we obtain

$$\lim_{t\to+0}\left([\Phi u]_{X_t} + t^{1/(\kappa-1)}\|(\Phi u)(t,\cdot)\|_{\dot{B}_{\infty,1}^0}\right) = 0.$$

Since it holds by (3.15) and Proposition 2.2 (i) that

(3.16) $\qquad\qquad\qquad \|\Phi u\|_X \leq C\|u_0\|_{L^{(\kappa-1)n/2}} + C\|u\|_X^\kappa,$

where

(3.17) $\qquad\qquad \|u\|_X := \|u\|_{L^\infty(L^{(\kappa-1)n/2})} + [u]_{X_\infty} + \sup_{0<t<\infty} t^{1/(\kappa-1)}\|u(t,\cdot)\|_{\dot{B}_{\infty,1}^0},$

we have $\Phi u \in X$. In addition, noting that

$$(\Phi u)(t, \cdot) - (\Phi v)(t, \cdot) = \lambda \mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot)$$

for all $0 < t < \infty$ and $u, v \in X$, we also see by (3.12) that

(3.18) $\qquad \|(\Phi u)(t, \cdot) - (\Phi v)(t, \cdot)\|_{L^{(\kappa-1)n/2}} + [\Phi u - \Phi v]_{X_t} \le C([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})[u - v]_{X_t}.$

Moreover, it holds by

$$(\Phi u)(t, \cdot) - (\Phi v)(t, \cdot)$$
$$= e^{(t/2)\Delta}((\Phi u)(t/2, \cdot) - (\Phi v)(t/2, \cdot)) + \lambda \int_{t/2}^t e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)d\tau$$

and (3.13) that

$$\|(\Phi u)(t, \cdot) - (\Phi v)(t, \cdot)\|_{\dot{B}_{\infty,1}^0}$$
$$\le C(t/2)^{-n/(2p_0)}\|(\Phi u)(t/2, \cdot) - (\Phi v)(t/2, \cdot)\|_{L^{p_0}}$$
$$+ Ct^{-1/(\kappa-1)}([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1}) \sup_{0<\tau<t} \tau^{1/(\kappa-1)}\|u(\tau, \cdot) - v(\tau, \cdot)\|_{\dot{B}_{\infty,1}^0}$$
$$\le Ct^{-n/(2p_0)} \cdot (t/2)^{n/(2p_0)-1/(\kappa-1)}[\Phi u - \Phi v]_{X_t} + Ct^{-1/(\kappa-1)}([u]_{X_t}^{\kappa-1} + [v]_{X_t}^{\kappa-1})\|u - v\|_X$$

with the aid of Proposition 2.2 (i). Therefore, we see by (3.18) that

(3.19) $\qquad \|\Phi u - \Phi v\|_X \le C(\|u\|_X^{\kappa-1} + \|v\|_X^{\kappa-1})\|u - v\|_X.$

Here we assume that $u_0 \in L^{(\kappa-1)n/2}$ satisfies

(3.20) $\qquad \|u_0\|_{L^{(\kappa-1)n/2}} \le 3(4C)^{-\kappa/(\kappa-1)}.$

Note that the condition (3.20) may be fulfilled by the assumption (1.4). We also assume that $u, v \in X$ satisfy

(3.21) $\qquad \max\{\|u\|_X, \|v\|_X\} \le (4C)^{-1/(\kappa-1)}.$

Then we see by (3.16), (3.19), (3.20), and (3.21) that

$$\|\Phi u\|_X \le C \cdot 3(4C)^{-\kappa/(\kappa-1)} + C \cdot (4C)^{-\kappa/(\kappa-1)} \quad = (4C)^{-1/(\kappa-1)},$$
$$\|\Phi u - \Phi v\|_X \le C((4C)^{-1} + (4C)^{-1})\|u - v\|_X \quad = (1/2)\|u - v\|_X.$$

Therefore, the Banach fixed point theorem ensures the existence of a unique $u \in X$ such that $\Phi u = u$. In addition, since it holds by (3.16) and (3.21) that

$$\|u\|_X \le C\|u_0\|_{L^{(\kappa-1)n/2}} + C \cdot (4C)^{-1}\|u\|_X = C\|u_0\|_{L^{(\kappa-1)n/2}} + (1/4)\|u\|_X,$$

we have (3.5). Noting that

$$u(t, \cdot) - u_0 = e^{t\Delta}u_0 - u_0 + \lambda \mathcal{I}_\kappa^{(l)}(u, 0)(t, \cdot)$$

for all $0 < t < \infty$, by using (3.12), the definition (3.8) of X, and the strong continuity of the heat semigroup $\{e^{t\Delta}\}_{t>0}$, we observe that

$$\limsup_{t \to +0} \|u(t, \cdot) - u_0\|_{L^{(\kappa-1)n/2}} \le \limsup_{t \to +0} \|e^{t\Delta}u_0 - u_0\|_{L^{(\kappa-1)n/2}} + C\limsup_{t \to +0}[u]_{X_t}^\kappa = 0.$$

Hence we have (3.4).

(ii) By the statement of (i), we see that $u, v \in X$ satisfy (3.21) and

$$u(t, \cdot) = e^{t\Delta}u_0 + \lambda \mathcal{I}_\kappa^{(l)}(u, 0)(t, \cdot), \quad v(t, \cdot) = e^{t\Delta}v_0 + \lambda \mathcal{I}_\kappa^{(l)}(v, 0)(t, \cdot)$$

for all $0 < t < \infty$. Since

$$u(t, \cdot) - v(t, \cdot) = e^{t\Delta}(u_0 - v_0) + \lambda \mathcal{I}_\kappa^{(l)}(u, v)(t, \cdot)$$
$$= e^{(t/2)\Delta}(u(t/2, \cdot) - v(t/2, \cdot)) + \lambda \int_{t/2}^t e^{(t-\tau)\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)d\tau,$$

it holds by (3.12) and Proposition 2.2 (i) that

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^{(\kappa-1)n/2}} + [u - v]_{X_t} \leq C\|u_0 - v_0\|_{L^{(\kappa-1)n/2}} + C(\|u\|_X^{\kappa-1} + \|v\|_X^{\kappa-1})\|u - v\|_X$$

and it holds by (3.13) and Proposition 2.2 (i) that

$$\|u(t, \cdot) - v(t, \cdot)\|_{\dot{B}_{\infty,1}^0}$$
$$\leq C(t/2)^{-n/(2p_0)}\|u(t/2, \cdot) - v(t/2, \cdot)\|_{L^{p_0}} + Ct^{-1/(\kappa-1)}(\|u\|_X^{\kappa-1} + \|v\|_X^{\kappa-1})\|u - v\|_X$$
$$\leq Ct^{-n/(2p_0)} \cdot (t/2)^{n/(2p_0)-1/(\kappa-1)}[u - v]_{X_t} + Ct^{-1/(\kappa-1)}(\|u\|_X^{\kappa-1} + \|v\|_X^{\kappa-1})\|u - v\|_X.$$

Therefore, the estimate (3.21) implies that

$$\|u - v\|_X \leq C\|u_0 - v_0\|_{L^{(\kappa-1)n/2}} + C(\|u\|_X^{\kappa-1} + \|v\|_X^{\kappa-1})\|u - v\|_X$$
$$\leq C\|u_0 - v_0\|_{L^{(\kappa-1)n/2}} + C((4C)^{-1} + (4C)^{-1})\|u - v\|_X$$
$$= C\|u_0 - v_0\|_{L^{(\kappa-1)n/2}} + (1/2)\|u - v\|_X,$$

which yields (3.6). This completes the proof of Proposition 3.1. $\qquad\square$

4. Smoothing properties of solutions

This section is devoted to the proof of the smoothing effects of the solutions u to (1.3), namely, Theorem 1.3.

4.1. **Basic properties.** To obtain the regularities of the solutions, we combine the parabolic smoothing effects and the estimates of the terms $|u|^{\kappa-l}u^l$ with the power nonlinearity. Here we state the smoothing properties of mild solutions to the heat equation.

Proposition 4.1. *Let $1 \leq r_0 < \infty$ and suppose that $f \in L_{\text{loc}}^1((0, \infty) \times \mathbb{R}^n)$ satisfies*

$$\sup_{0<t<\infty} \int_0^t \|e^{(t-\tau)\Delta}f(\tau, \cdot)\|_{L^{r_0}} d\tau < \infty.$$

Define $\mathcal{I}f$ by setting

$$(4.1) \qquad (\mathcal{I}f)(t, \cdot) := \int_0^t e^{(t-\tau)\Delta}f(\tau, \cdot)d\tau, \quad 0 < t < \infty.$$

Then the following statements hold:

(i) Let $r_0 \leq r_1 < \infty$. Suppose that $f \in L_{\text{loc}}^\infty((0, \infty); L^{r_1})$. Then it holds that

$$\mathcal{I}f \in \bigcap_{0<\theta<1} C_{\text{loc}}^{1-\theta}((0, \infty); B_{r_1,1}^{2\theta}).$$

(ii) Let $r_0 \leq r_1 \leq \infty$ and $0 < s < \infty$. Suppose that $f \in L_{\text{loc}}^\infty((0, \infty); B_{r_1,\infty}^s)$. Then it holds that

$$\mathcal{I}f \in \bigcap_{0<\theta<1} C_{\text{loc}}^{1-\theta}((0, \infty); B_{r_1,1}^{s+2\theta}).$$

(iii) Let $k = 0, 1$, $m \in \mathbb{N} \cup \{0\}$, and $0 < \gamma < 1$. Suppose that $f \in C_{\text{loc}}^{k+\gamma}((0, \infty); BUC^m)$. Then it holds that

$$\mathcal{I}f \in C_{\text{loc}}^{1+k+\gamma}((0, \infty); BUC^m) \cap C_{\text{loc}}^{k+\gamma}((0, \infty); BUC^{m+2})$$

with the identity $\partial_t \mathcal{I} f - \Delta \mathcal{I} f = f$ in $(0, \infty) \times \mathbb{R}^n$.

The proof of Proposition 4.1 is given in Appendix. Concerning the estimates of $|u|^{\kappa - l} u^l$, we begin with giving the simple estimates. Here we use the abbreviations like $\|\cdot\|_{L_\varepsilon^\infty(L^\infty)} := \|\cdot\|_{L^\infty((\varepsilon, \varepsilon^{-1}); L^\infty)}$.

Proposition 4.2. *Let $1 < \kappa < \infty$ and $l = 0, 1$. Then the following assertions hold:*
(i) Let $0 < \sigma < 1$. For every $\varphi \in C^\sigma$, it holds that $|\varphi|^{\kappa - l} \varphi^l \in C^\sigma$ with the estimate

$$\||\varphi|^{\kappa - l} \varphi^l\|_{C^\sigma} \le (\max\{2\kappa, 2^{\kappa-1}\kappa\} + 1)\|\varphi\|_{L^\infty}^{\kappa-1}\|\varphi\|_{C^\sigma}.$$

(ii) Let $0 < \gamma < 1$. For every $f \in C_{loc}^\gamma((0, \infty); BUC)$, it holds that $|f|^{\kappa-l} f^l \in C_{loc}^\gamma((0, \infty); BUC)$ with the estimate

$$\||f|^{\kappa-l} f^l\|_{C_\varepsilon^\gamma(L^\infty)} \le (\max\{2\kappa, 2^{\kappa-1}\kappa\} + 1)\|f\|_{L_\varepsilon^\infty(L^\infty)}^\kappa \|f\|_{C_\varepsilon^\gamma(L^\infty)}$$

for all $0 < \varepsilon < 1$.

Proof. (i) Since $\||\varphi|^{\kappa-l}\varphi^l\|_{L^\infty} \le \|\varphi\|_{L^\infty}^\kappa$ and since Proposition 2.3 (ii) implies that

$$|(|\varphi|^{\kappa-l}\varphi^l)(x_1) - (|\varphi|^{\kappa-l}\varphi^l)(x_0)| \le \max\{\kappa, 2^{\kappa-2}\kappa\}(|\varphi(x_1)|^{\kappa-1} + |\varphi(x_0)|^{\kappa-1})|\varphi(x_1) - \varphi(x_0)|$$

$$\le \max\{\kappa, 2^{\kappa-2}\kappa\} \cdot 2\|\varphi\|_{L^\infty}^{\kappa-1} \cdot \|\varphi\|_{C^\sigma}|x_1 - x_0|^\sigma$$

for all $x_0, x_1 \in \mathbb{R}^n$, we have the desired result.

(ii) Since $f(t, \cdot) \in BUC$ holds for each $0 < t < \infty$, we see by Proposition 2.3 (ii) that $(|f|^{\kappa-l} f^l)(t, \cdot) \in BUC$. In addition, it holds that $\||f|^{\kappa-l} f^l\|_{L_\varepsilon^\infty(L^\infty)} \le \|f\|_{L_\varepsilon^\infty(L^\infty)}^\kappa$. By applying Proposition 2.3 (ii) again, we also obtain

$$|(|f|^{\kappa-l} f^l)(t_1, x) - (|f|^{\kappa-l} f^l)(t_0, x)|$$

$$\le \max\{\kappa, 2^{\kappa-2}\kappa\}(|f(t_1, x)|^{\kappa-1} + |f(t_0, x)|^{\kappa-1})|f(t_1, x) - f(t_0, x)|$$

$$\le \max\{\kappa, 2^{\kappa-2}\kappa\} \cdot 2\|f\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-1} \cdot \|f\|_{C_\varepsilon^\gamma(L^\infty)}|t_1 - t_0|^\gamma$$

for all $0 < \varepsilon < 1$, $t_0, t_1 \in (\varepsilon, \varepsilon^{-1})$, and $x \in \mathbb{R}^n$. Thus we have the desired result. This completes the proof of Proposition 4.2. $\square$

4.2. **Auxiliary regularities.** By combining Propositions 4.1 and 4.2, we may show the following regularities of the solutions u to (1.3):

Lemma 4.3. *Let $1 + 2/n < \kappa < \infty$, $l = 0, 1$, and $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $u_0 \in L^{(\kappa-1)n/2}$. Then the solution u to (1.3) obtained in Proposition 3.1 (i) has the following regularities*

$$(4.2) \qquad u \in \bigcap_{0 < \theta < 1} C_{loc}^{1-\theta}((0, \infty); B_{\infty,1}^{2\theta} \cap B_{(\kappa-1)n/2,1}^{2\theta}),$$

$$(4.3) \qquad u \in C_{loc}^{3/2}((0, \infty); BUC) \cap C_{loc}^{1/2}((0, \infty); BUC^2).$$

Proof. First we notice that

$$(4.4) \qquad e^{t\Delta} u_0 \in \bigcap_{j \in \mathbb{N}} C^\infty((0, \infty); BUC^j \cap W^{j,(\kappa-1)n/2})$$

from $u_0 \in L^{(\kappa-1)n/2}$. In the same way as the derivation of the estimate (3.11), we see that

$$(4.5) \qquad \sup_{0 < t < \infty} \int_0^t \|e^{(t-\tau)\Delta}(|u|^{\kappa-l} u^l)(\tau, \cdot)\|_{L^{(\kappa-1)n/2}} d\tau \le C\|u\|_X^\kappa,$$

where $\|\cdot\|_X$ is given by (3.17). Moreover, since $\dot{B}_{r,1}^0 \subset L^r$ for all $1 \le r \le \infty$, the condition (3.3) and Proposition 2.3 (iii) imply that

$$\|(|u|^{\kappa-l}u^l)(t,\cdot)\|_{L^{p_0/(\kappa-1)}} \le C\|u(t,\cdot)\|_{L^{p_0}}^{\kappa-1}\|u(t,\cdot)\|_{L^\infty} \qquad \le Ct^{-1+(\kappa-1)n/(2p_0)-1/(\kappa-1)}\|u\|_X^\kappa,$$

$$\|(|u|^{\kappa-l}u^l)(t,\cdot)\|_{L^{(\kappa-1)n/2}} \le C\|u(t,\cdot)\|_{L^\infty}^{\kappa-1}\|u(t,\cdot)\|_{L^{(\kappa-1)n/2}} \quad \le Ct^{-1}\|u\|_X^\kappa$$

for all $0 < t < \infty$, which yield

$$(4.6) \qquad\qquad |u|^{\kappa-l}u^l \in L_{\mathrm{loc}}^\infty((0,\infty); L^{p_0/(\kappa-1)} \cap L^{(\kappa-1)n/2}).$$

Hence it holds by Proposition 4.1 (i) that

$$(4.7) \qquad\qquad \mathcal{I}_\kappa^{(l)}(u,0) \in \bigcap_{0<\theta<1} C_{\mathrm{loc}}^{1-\theta}((0,\infty); B_{p_0/(\kappa-1),1}^{2\theta} \cap B_{(\kappa-1)n/2,1}^{2\theta}),$$

where $\mathcal{I}_\kappa^{(l)}(u,0)$ is defined by (3.9). Here we note that (3.7) yields $0 < (\kappa-1)n/(2p_0) < 1$, so we may take θ_0 such that

$$(\kappa-1)n/(2p_0) < \theta_0 < 1.$$

Then, since Proposition 2.1 (iv) implies that $B_{p_0/(\kappa-1),1}^{2\theta_0} \subset B_{\infty,1}^{2\theta_0-(\kappa-1)n/p_0}$, it holds by (4.7) that

$$\mathcal{I}_\kappa^{(l)}(u,0) \in C_{\mathrm{loc}}^{1-\theta_0}((0,\infty); B_{\infty,1}^{2\theta_0-(\kappa-1)n/p_0} \cap B_{(\kappa-1)n/2,1}^{2\theta_0}).$$

Thus we see by (4.4) that

$$u \in C_{\mathrm{loc}}^\sigma((0,\infty); B_{\infty,1}^\sigma \cap B_{(\kappa-1)n/2,1}^\sigma)$$

for sufficiently small $0 < \sigma < 1$. Noting that $B_{\infty,1}^\sigma \subset C^\sigma$, we see by (4.6) and Proposition 4.2 (i) that

$$|u|^{\kappa-l}u^l \in L_{\mathrm{loc}}^\infty((0,\infty); C^\sigma \cap L^{(\kappa-1)n/2}).$$

Therefore, by combining (4.4), (4.5), and Proposition 4.1 (i) and (ii), we have (4.2). Moreover, since $u \in C_{\mathrm{loc}}^{1/2}((0,\infty); BUC)$ from (4.2), Proposition 4.2 (ii) yields

$$|u|^{\kappa-l}u^l \in C_{\mathrm{loc}}^{1/2}((0,\infty); BUC).$$

Thus we have (4.3) by virtue of (4.4), (4.5), and Proposition 4.1 (iii). This completes the proof of Lemma 4.3. $\qquad\square$

4.3. Estimates of higher order derivatives of the nonlinear term.
In the previous subsection, we showed that the solutions u have the regularities (4.2) and (4.3). Here, the proof is based only on the simple estimates given by Proposition 4.2. Since we do not consider the derivatives of $|u|^{\kappa-l}u^l$ yet, we next focus on the estimates of the derivatives of $|u|^{\kappa-l}u^l$. To this end, we will apply the key estimates given in Lemma 2.4. As well as the previous subsection, we use the abbreviations like $\|\cdot\|_{BC_\varepsilon^1(L^\infty)} := \|\cdot\|_{BC^1((\varepsilon,\varepsilon^{-1});L^\infty)}$.

Lemma 4.4. *Let $1 \le \kappa < \infty$ and $l = 0, 1$. Suppose that $m \in \mathbb{Z}$ satisfies $0 \le m < \kappa$. Then the following assertions hold:*
(i) For every $\varphi \in BUC^m$, it holds that $|\varphi|^{\kappa-l}\varphi^l \in BUC^m$ with the estimate

$$\||\varphi|^{\kappa-l}\varphi^l\|_{BC^m} \le C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1}\|\varphi\|_{BC^m},$$

where $C > 0$ is a constant independent of φ.
(ii) Let $1 \le r < \infty$. For every $\varphi \in BC^m \cap W^{m,r}$, it holds that $|\varphi|^{\kappa-l}\varphi^l \in W^{m,r}$ with the estimate

$$\||\varphi|^{\kappa-l}\varphi^l\|_{W^{m,r}} \le C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1}\|\varphi\|_{W^{m,r}}.$$

Lemma 4.5. *Let $l = 0, 1$. Then the following assertions hold:*

(i) Suppose that $1 < \kappa < \infty$ satisfies $\kappa \notin \mathbb{N}$. For every $\varphi \in C^\kappa$, it holds that $|\varphi|^{\kappa-l}\varphi^l \in C^\kappa$ with the estimate

$$\||\varphi|^{\kappa-l}\varphi^l\|_{C^\kappa} \leq C(1 + \|\varphi\|_{BC^{[\kappa]}})^{\kappa+[\kappa]-1}\|\varphi\|_{C^\kappa},$$

where $C > 0$ is a constant independent of φ.

(ii) Suppose that $1 + 2/n < \kappa < \infty$ satisfies $\kappa \notin \mathbb{N}$. Set $\rho := (\kappa - [\kappa])^{-1}(\kappa - 1)n/2$. For every $\varphi \in BC^{[\kappa]} \cap W^{1,(\kappa-1)n/2} \cap B^\kappa_{\rho,\rho}$, it holds that $|\varphi|^{\kappa-l}\varphi^l \in B^{\kappa-\sigma}_{\rho,\rho}$ with the estimate

$$\||\varphi|^{\kappa-l}\varphi^l\|_{B^{\kappa-\sigma}_{\rho,\rho}} \leq C_\sigma(1 + \|\varphi\|_{BC^{[\kappa]}})^{\kappa+[\kappa]-1}(\|\varphi\|_{W^{1,(\kappa-1)n/2}}^{\kappa-[\kappa]} + \|\varphi\|_{B^\kappa_{\rho,\rho}})$$

for all $0 < \sigma < \kappa - [\kappa]$, where $C_\sigma > 0$ is a constant independent of φ.

Lemma 4.6. *Let $j = 0, 1$ and $l = 0, 1$. Suppose that $1 < \kappa < \infty$ satisfies $\kappa \notin \mathbb{N}$. Then, for every*

$$f \in C^1((0, \infty); BC) \cap C^{j+\kappa-[\kappa]}_{\mathrm{loc}}((0, \infty); C^{\kappa-j}),$$

it holds that $|f|^{\kappa-l}f^l \in C^{j+\kappa-[\kappa]}_{\mathrm{loc}}((0, \infty); BUC^{[\kappa]-j})$ with the estimate

$$\||f|^{\kappa-l}f^l\|_{C^{j+\kappa-[\kappa]}_\varepsilon(BC^{[\kappa]-j})} \leq C(1 + \|f\|_{BC^j_\varepsilon(C^{\kappa-j})})^{\kappa+[\kappa]-1}(\|f\|_{BC^1_\varepsilon(L^\infty)}^{\kappa-[\kappa]} + \|f\|_{C^{j+\kappa-[\kappa]}_\varepsilon(C^{\kappa-j})})$$

for all $0 < \varepsilon < 1$, where $C > 0$ is a constant independent of ε and f.

Proof of Lemma 4.4. (i) In the case of $m = 0$, the desired result may be obtained by Proposition 2.3 (i) or (ii). Hence, in the following, we assume that $1 < \kappa < \infty$ and $1 \leq m < \kappa$. Then it holds by Lemma 2.4 (ii) that

(4.8)
$$\sum_{|\alpha|=m_*} |\partial^\alpha_x(|\varphi|^{\kappa-l}\varphi^l)(x)|$$
$$\leq C|(|\varphi|^{\kappa-2m_*+l}\varphi^{m_*-l})(x)| \sum_{|\alpha|=1} |\partial^\alpha_x\varphi(x)|^{m_*} + C(1 + \|\varphi\|_{BC^{m_*}})^{\kappa+m_*-1} \sum_{|\alpha|\leq m_*} |\partial^\alpha_x\varphi(x)|$$
$$\leq C\|\varphi\|_{L^\infty}^{\kappa-m_*}\|\varphi\|_{BC^1}^{m_*-1} \sum_{|\alpha|=1} |\partial^\alpha_x\varphi(x)| + C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1} \sum_{|\alpha|\leq m} |\partial^\alpha_x\varphi(x)|$$
$$\leq C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1} \sum_{|\alpha|\leq m} |\partial^\alpha_x\varphi(x)|$$

for all $1 \leq m_* \leq m$. Thus we see by $\||\varphi|^{\kappa-l}\varphi^l\|_{L^\infty} \leq \|\varphi\|_{L^\infty}^\kappa$ that

$$\sum_{|\alpha|\leq m} \|\partial^\alpha_x(|\varphi|^{\kappa-l}\varphi^l)\|_{L^\infty} \leq C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1}\|\varphi\|_{BC^m}.$$

In addition, by applying Lemma 2.4 (ii) again, we observe that

(4.9)
$$\sum_{|\alpha|=m} |\partial^\alpha_x\{(|\varphi|^{\kappa-l}\varphi^l)(x_1) - (|\varphi|^{\kappa-l}\varphi^l)(x_0)\}|$$
$$\leq C|(|\varphi|^{\kappa-2m+l}\varphi^{m-l})(x_1) - (|\varphi|^{\kappa-2m+l}\varphi^{m-l})(x_0)| \sum_{|\alpha|=1} |\partial^\alpha_x\varphi(x_1)|^m$$
$$+ C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1} \sum_{|\alpha|\leq m} |\partial^\alpha_x\{\varphi(x_1) - \varphi(x_0)\}|$$

for all $x_0, x_1 \in \mathbb{R}^n$. Here we note that $|\varphi|^{\kappa-2m+l}\varphi^{m-l} = |\varphi|^{(\kappa-m)-(m-l)}\varphi^{m-l}$ and $\kappa - m > 0$. Since $\varphi \in BUC^m$ and since (4.9) holds, Proposition 2.3 (i) or (ii) implies that $|\varphi|^{\kappa-l}\varphi^l \in BUC^m$, which yields the desired result.

(ii) Since $\||\varphi|^{\kappa-l}\varphi^l\|_{L^r} \le \|\varphi\|_{L^\infty}^{\kappa-1}\|\varphi\|_{L^r}$, we may assume that $1 \le m < \kappa$. Then (4.8) is valid for all $1 \le m_* \le m$. Thus we have the desired estimate, which completes the proof of Lemma 4.4. $\square$

Proof of Lemma 4.5. (i) We set $m := [\kappa]$. Since $0 < \kappa - m < 1$ holds from $\kappa \notin \mathbb{N}$ and since $|\varphi|^{\kappa-2m+l}\varphi^{m-l} = |\varphi|^{\kappa-m-(m-l)}\varphi^{m-l}$, by combining (4.9), Proposition 2.3 (i), and the mean value theorem, we obtain

$$\sum_{|\alpha|=m} |\partial_x^\alpha\{(|\varphi|^{\kappa-l}\varphi^l)(x_1) - (|\varphi|^{\kappa-l}\varphi^l)(x_0)\}|$$

$$\le C|\varphi(x_1) - \varphi(x_0)|^{\kappa-m} \cdot \|\varphi\|_{BC^1}^m + C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1} \sum_{|\alpha|\le m} |\partial_x^\alpha\{\varphi(x_1) - \varphi(x_0)\}|$$

$$\le C(\|\varphi\|_{BC^1}|x_1 - x_0|)^{\kappa-m} \cdot \|\varphi\|_{BC^1}^m + C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1} \cdot \|\varphi\|_{C^\kappa}|x_1 - x_0|^{\kappa-m}$$

$$\le C(1 + \|\varphi\|_{BC^m})^{\kappa+m-1}\|\varphi\|_{C^\kappa}|x_1 - x_0|^{\kappa-m}$$

for all $x_0, x_1 \in \mathbb{R}^n$. Therefore, we have the desired result with the aid of Lemma 4.4 (i).

(ii) Since $\kappa \notin \mathbb{N}$, we have $0 < \kappa - [\kappa] < 1$. By setting $m := [\kappa]$, we see that $(\kappa - m)\rho = (\kappa-1)n/2$. Here, by (4.9) and Proposition 2.3 (i), we deduce that

$$\sum_{|\alpha|=m} |\partial_x^\alpha\{(|\varphi|^{\kappa-l}\varphi^l)(x_1) - (|\varphi|^{\kappa-l}\varphi^l)(x_0)\}|^\rho$$

$$\le C(|\varphi(x_1) - \varphi(x_0)|^{\kappa-m})^\rho\|\varphi\|_{BC^1}^{m\rho} + C(1 + \|\varphi\|_{BC^m})^{(\kappa+m-1)\rho} \sum_{|\alpha|\le m} |\partial_x^\alpha\{\varphi(x_1) - \varphi(x_0)\}|^\rho$$

for all $x_0, x_1 \in \mathbb{R}^n$. Hence it holds by Proposition 2.1 (iii) that

$$\sum_{|\alpha|=m} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\partial_x^\alpha\{(|\varphi|^{\kappa-l}\varphi^l)(x_1) - (|\varphi|^{\kappa-l}\varphi^l)(x_0)\}|^\rho}{|x_1 - x_0|^{(\kappa-m-\sigma)\rho+n}} dx_0 dx_1$$

$$\le C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\varphi(x_1) - \varphi(x_0)|^{(\kappa-1)n/2}}{|x_1 - x_0|^{(\kappa-m-\sigma)\rho+n}} dx_0 dx_1 \cdot \|\varphi\|_{BC^1}^{m\rho}$$

$$+ C(1 + \|\varphi\|_{BC^m})^{(\kappa+m-1)\rho} \sum_{|\alpha|\le m} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\partial_x^\alpha\{\varphi(x_1) - \varphi(x_0)\}|^\rho}{|x_1 - x_0|^{(\kappa-m-\sigma)\rho+n}} dx_0 dx_1$$

$$\le C(1 + \|\varphi\|_{BC^m})^{(\kappa+m-1)\rho} \left(\|\varphi\|_{B_{(\kappa-1)n/2,(\kappa-1)n/2}^{1-\sigma/(\kappa-m)}}^{(\kappa-m)\rho} + \|\varphi\|_{B_{\rho,\rho}^{\kappa-\sigma}}^\rho\right)$$

$$\le C_\sigma(1 + \|\varphi\|_{BC^m})^{(\kappa+m-1)\rho}(\|\varphi\|_{W^{1,(\kappa-1)n/2}}^{\kappa-m} + \|\varphi\|_{B_{\rho,\rho}^\kappa})^\rho.$$

Therefore, we have the desired result with the aid of Proposition 2.1 (iii) and Lemma 4.4 (ii). This completes the proof of Lemma 4.5. $\square$

Proof of Lemma 4.6. We set $m := [\kappa]$. Then we have $0 < \kappa - m < 1$ from $\kappa \notin \mathbb{N}$. First we consider the case of $j = 0$. Since $f \in C((0,\infty); C^\kappa)$, Lemma 4.5 (i) yields $|f|^{\kappa-l}f^l \in L_{\text{loc}}^\infty((0,\infty); C^\kappa)$ with the estimate

(4.10) $$\||f|^{\kappa-l}f^l\|_{L_\varepsilon^\infty(C^\kappa)} \le C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1}\|f\|_{L_\varepsilon^\infty(C^\kappa)}.$$

In addition, since $|f|^{\kappa-2m+l}f^{m-l} = |f|^{\kappa-m-(m-l)}f^{m-l}$ and since $0 < \kappa - m < 1$, by combining Proposition 2.3 (i) and Lemma 2.4 (ii), we observe that

$$\sum_{|\alpha|=m} |\partial_x^\alpha\{(|f|^{\kappa-l}f^l)(t_1,x) - (|f|^{\kappa-l}f^l)(t_0,x)\}|$$

$$\leq C|(|f|^{\kappa-2m+l}f^{m-l})(t_1,x) - (|f|^{\kappa-2m+l}f^{m-l})(t_0,x)| \sum_{|\alpha|=1} |\partial_x^\alpha f(t_1,x)|^m$$

$$+ C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1} \sum_{|\alpha|\leq m} |\partial_x^\alpha\{f(t_1,x) - f(t_0,x)\}|$$

$$\leq C|f(t_1,x) - f(t_0,x)|^{\kappa-m} \cdot \|f\|_{L_\varepsilon^\infty(BC^1)}^m + C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1}\|f(t_1,\cdot) - f(t_0,\cdot)\|_{BC^m}$$

for all $0 < \varepsilon < 1$, $t_0, t_1 \in (\varepsilon, \varepsilon^{-1})$, and $x \in \mathbb{R}^n$. Here, the mean value theorem yields

$$\sum_{|\alpha|=m} |\partial_x^\alpha\{(|f|^{\kappa-l}f^l)(t_1,x) - (|f|^{\kappa-l}f^l)(t_0,x)\}|$$

$$\leq C(\|f\|_{BC_\varepsilon^1(L^\infty)}|t_1 - t_0|)^{\kappa-m} \cdot \|f\|_{L_\varepsilon^\infty(BC^1)}^m$$

$$+ C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1} \cdot \|f\|_{C_\varepsilon^{\kappa-m}(BC^m)}|t_1 - t_0|^{\kappa-m}$$

$$\leq C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1}(\|f\|_{BC_\varepsilon^1(L^\infty)}^{\kappa-m} + \|f\|_{C_\varepsilon^{\kappa-m}(BC^m)})|t_1 - t_0|^{\kappa-m}.$$

Noting that $|f|^{\kappa-l}f^l \in L_{\text{loc}}^\infty((0,\infty); C^\kappa)$, we have

$$[|f|^{\kappa-l}f^l]_{C_\varepsilon^{\kappa-m}(BC^m)} \leq C(1 + \|f\|_{L_\varepsilon^\infty(BC^m)})^{\kappa+m-1}(\|f\|_{BC_\varepsilon^1(L^\infty)}^{\kappa-m} + \|f\|_{C_\varepsilon^{\kappa-m}(BC^m)}).$$

Therefore, by combining the above estimate and (4.10), we have the desired result.

Next we consider the case of $j = 1$. Now we assume that

$$(4.11) \qquad |f|^{\kappa-l}f^l, \partial_t(|f|^{\kappa-l}f^l) \in L_{\text{loc}}^\infty((0,\infty); BUC^{m-1})$$

with the estimate

$$(4.12) \quad \||f|^{\kappa-l}f^l\|_{L_\varepsilon^\infty(BC^{m-1})} + \|\partial_t(|f|^{\kappa-l}f^l)\|_{L_\varepsilon^\infty(BC^{m-1})} \leq C(1 + \|f\|_{BC_\varepsilon^1(C^{\kappa-1})})^{\kappa+m-1}\|f\|_{BC_\varepsilon^1(C^{\kappa-1})}.$$

Then, by combining Proposition 2.3 (i) and Lemma 2.4 (iii), we observe that

$$\sum_{|\alpha|=m-1} |\partial_t\partial_x^\alpha\{(|f|^{\kappa-l}f^l)(t_1,x) - (|f|^{\kappa-l}f^l)(t_0,x)\}|$$

$$\leq C|(|f|^{\kappa-2m+l}f^{m-l})(t_1,x) - (|f|^{\kappa-2m+l}f^{m-l})(t_0,x)||\partial_t f(t_1,x)| \sum_{|\alpha|=1} |\partial_x^\alpha f(t_1,x)|^{m-1}$$

$$+ C(1 + \|f\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1} \sum_{|\alpha|\leq m-1} \left(|\partial_x^\alpha\{f(t_1,x) - f(t_0,x)\}| + |\partial_t\partial_x^\alpha\{f(t_1,x) - f(t_0,x)\}| \right)$$

$$\leq C|f(t_1,x) - f(t_0,x)|^{\kappa-m} \cdot \|f\|_{BC_\varepsilon^1(L^\infty)}\|f\|_{L_\varepsilon^\infty(BC^1)}^{m-1}$$

$$+ C(1 + \|f\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1}\left(\|f(t_1,\cdot) - f(t_0,\cdot)\|_{BC^{m-1}} + \|\partial_t\{f(t_1,\cdot) - f(t_0,\cdot)\}\|_{BC^{m-1}} \right)$$

for all $0 < \varepsilon < 1$, $t_0, t_1 \in (\varepsilon, \varepsilon^{-1})$, and $x \in \mathbb{R}^n$. Here, the mean value theorem yields

$$\sum_{|\alpha|=m-1} |\partial_t \partial_x^\alpha \{(|f|^{\kappa-l} f^l)(t_1, x) - (|f|^{\kappa-l} f^l)(t_0, x)\}|$$

$$\leq C(\|f\|_{BC_\varepsilon^1(L^\infty)} |t_1 - t_0|)^{\kappa-m} \cdot \|f\|_{BC_\varepsilon^1(BC^{m-1})}^m$$

$$+ C(1 + \|f\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1} \cdot \|f\|_{C_\varepsilon^{1+\kappa-m}(BC^{m-1})} |t_1 - t_0|^{\kappa-m}$$

$$\leq C(1 + \|f\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1}(\|f\|_{BC_\varepsilon^1(L^\infty)}^{\kappa-m} + \|f\|_{C_\varepsilon^{1+\kappa-m}(BC^{m-1})}) |t_1 - t_0|^{\kappa-m}.$$

Noting that (4.11), we have

$$[|f|^{\kappa-l} f^l]_{C_\varepsilon^{1+\kappa-m}(BC^{m-1})} \leq C(1 + \|f\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1}(\|f\|_{BC_\varepsilon^1(L^\infty)}^{\kappa-m} + \|f\|_{C_\varepsilon^{1+\kappa-m}(BC^{m-1})}).$$

Therefore, by combining the above estimate and (4.12), we have the desired result. Hence, it remains to verify that (4.11) with the estimate (4.12).

In case $\kappa > 2$, note that $|f|^{\kappa-l} f^l = |f|^{\kappa-1-(1-l)} f^{1-l} \cdot f$ and $\partial_t(|f|^{\kappa-l} f^l) = \kappa |f|^{\kappa-1-(1-l)} f^{1-l} \partial_t f$. Since $f \in C^1((0,\infty); C^{\kappa-1})$, it holds by Lemma 4.5 (i) that the following estimates

$$\||f|^{\kappa-l} f^l\|_{L_\varepsilon^\infty(C^{\kappa-1})} \leq C\||f|^{\kappa-1-(1-l)} f^{1-l}\|_{L_\varepsilon^\infty(C^{\kappa-1})} \|f\|_{L_\varepsilon^\infty(C^{\kappa-1})}$$

$$\leq C(1 + \|f\|_{L_\varepsilon^\infty(BC^{m-1})})^{\kappa+m-3} \|f\|_{L_\varepsilon^\infty(C^{\kappa-1})}^2,$$

$$\|\partial_t(|f|^{\kappa-l} f^l)\|_{L_\varepsilon^\infty(C^{\kappa-1})} \leq C\||f|^{\kappa-1-(1-l)} f^{1-l}\|_{L_\varepsilon^\infty(C^{\kappa-1})} \|f\|_{BC_\varepsilon^1(C^{\kappa-1})}$$

$$\leq C(1 + \|f\|_{BC_\varepsilon^1(C^{\kappa-1})})^{\kappa+m-2} \|f\|_{L_\varepsilon^\infty(C^{\kappa-1})}$$

hold. This implies that (4.11) and (4.12) are valid.

In case $1 < \kappa < 2$, note that $0 < \kappa - 1 < 1$. Thus we see by $f \in C^1((0,\infty); C^{\kappa-1})$ and Proposition 4.2 (i) that the estimate

$$\||f|^{\kappa-l} f^l\|_{L_\varepsilon^\infty(C^{\kappa-1})} \leq C\|f\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-1} \|f\|_{L_\varepsilon^\infty(C^{\kappa-1})}$$

holds. In addition, it holds by $\partial_t(|f|^{\kappa-l} f^l) = \kappa |f|^{\kappa-2+l} f^{1-l} \partial_t f$ that $\partial_t(|f|^{\kappa-l} f^l) \in L_{\text{loc}}^\infty((0,\infty); L^\infty)$ with the estimate

$$\|\partial_t(|f|^{\kappa-l} f^l)\|_{L_\varepsilon^\infty(L^\infty)} \leq \kappa \|f\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-1} \|f\|_{BC_\varepsilon^1(L^\infty)}.$$

Noting that $|f|^{\kappa-2+l} f^{1-l} = |f|^{(\kappa-1)-(1-l)} f^{1-l}$, by applying Proposition 2.3 (i), we also obtain

$$|\partial_t\{(|f|^{\kappa-l} f^l)(t, x_1) - (|f|^{\kappa-l} f^l)(t, x_0)\}|$$

$$\leq \kappa \left(|(|f|^{\kappa-2+l} f^{1-l})(t, x_1) - (|f|^{\kappa-2+l} f^{1-l})(t, x_0)||\partial_t f(t, x_1)| \right.$$

$$+ |(|f|^{\kappa-2+l} f^{1-l})(t, x_0)||\partial_t\{f(t, x_1) - f(t, x_0)\}| \Big)$$

$$\leq C\left(|f(t, x_1) - f(t, x_0)|^{\kappa-1} \|f\|_{BC_\varepsilon^1(L^\infty)} + \|f\|_{L_\varepsilon^\infty(L^\infty)} |\partial_t\{f(t, x_1) - f(t, x_0)\}| \right),$$

which yields $\partial_t(|f|^{\kappa-l} f^l) \in L_{\text{loc}}^\infty((0,\infty); BUC)$ from $f \in C_{\text{loc}}^\kappa((0,\infty); C^{\kappa-1})$. This implies that (4.11) and (4.12) are valid as well. This completes the proof of Lemma 4.6. $\qquad\square$

4.4. **Proof of Theorem 1.3.** Once we obtain the estimates of higher order derivatives of the nonlinear term $|u|^{\kappa-l} u^l$, we may show the smoothing effects of the solutions u, i.e., Theorem 1.3. The remaining step is just combining Proposition 4.1 and the estimates in the previous subsection.

Proof of Theorem 1.3. Since

$$u \in C^{1/2}_{\text{loc}}((0,\infty); B^1_{\infty,1} \cap B^1_{(\kappa-1)n/2,1}) \subset C^{1/2}_{\text{loc}}((0,\infty); BUC^1 \cap W^{1,(\kappa-1)n/2})$$

from (4.2), we see by Lemma 4.4 (ii) that $|u|^{\kappa-l}u^l \in L^\infty_{\text{loc}}((0,\infty); W^{1,(\kappa-1)n/2})$. Hence, the embedding $W^{1,(\kappa-1)n/2} \subset B^1_{(\kappa-1)n/2,\infty}$ and Proposition 4.1 (ii) imply that

$$\mathcal{I}^{(l)}_\kappa(u,0) \in \bigcap_{0<\theta<1} C^{1-\theta}_{\text{loc}}((0,\infty); B^{1+2\theta}_{(\kappa-1)n/2,1})$$

with the aid of (4.5), where $\mathcal{I}^{(l)}_\kappa(u,0)$ is defined by (3.9). Thus we obtain

$$(4.13) \qquad u \in \bigcap_{0<\theta<1} C^{1-\theta}_{\text{loc}}((0,\infty); B^{2\theta}_{\infty,1} \cap B^{1+2\theta}_{(\kappa-1)n/2,1})$$

from (4.2) and (4.4).

Here we consider the case of $1 + 2/n < \kappa < 2$. Since the regularity (4.13) implies that

$$(4.14) \qquad u \in L^\infty_{\text{loc}}((0,\infty); C^\kappa \cap B^\kappa_{(\kappa-1)n/2,1}),$$

it holds by Lemma 4.5 that

$$(4.15) \qquad |u|^{\kappa-l}u^l \in L^\infty_{\text{loc}}((0,\infty); C^\kappa \cap B^{\kappa-\sigma}_{\rho,\rho})$$

for all $0 < \sigma < \kappa - [\kappa]$. Therefore, by combining (4.5) and Proposition 4.1 (ii) again, we obtain

$$(4.16) \qquad \mathcal{I}^{(l)}_\kappa(u,0) \in \bigcap_{\theta,\sigma\in(0,1)} C^{1-\theta}_{\text{loc}}((0,\infty); B^{\kappa+2\theta}_{\infty,1} \cap B^{\kappa-\sigma+2\theta}_{\rho,1}),$$

which yields (1.5) with the aid of (4.4). Moreover, we see by (1.5) and (4.3) that

$$u \in C^1((0,\infty); BUC) \cap C^{\kappa-1}_{\text{loc}}((0,\infty); C^\kappa).$$

Hence it follows from Lemma 4.6 that $|u|^{\kappa-l}u^l \in C^{\kappa-1}_{\text{loc}}((0,\infty); BUC^1)$. Here, (4.5) and Proposition 4.1 (iii) imply that

$$\mathcal{I}^{(l)}_\kappa(u,0) \in C^\kappa_{\text{loc}}((0,\infty); BUC^1) \cap C^{\kappa-1}_{\text{loc}}((0,\infty); BUC^3),$$

which yields

$$(4.17) \qquad u \in C^\kappa_{\text{loc}}((0,\infty); BUC^1) \cap C^{\kappa-1}_{\text{loc}}((0,\infty); BUC^3)$$

with the aid of (4.4). Since the regularity (4.17) yields

$$u \in C^1((0,\infty); BUC) \cap C^\kappa_{\text{loc}}((0,\infty); C^{\kappa-1}),$$

by applying Lemma 4.6 again, we have $|u|^{\kappa-l}u^l \in C^\kappa_{\text{loc}}((0,\infty); BUC)$. Proposition 4.1 (iii) and (4.4) imply that

$$u \in C^{1+\kappa}_{\text{loc}}((0,\infty); BUC) \cap C^\kappa_{\text{loc}}((0,\infty); BUC^2).$$

By combining the above regularity and (4.17), we have the desired result.

Next we consider the case of $2 < \kappa < \infty$ and $\kappa \notin \mathbb{N}$. Since (4.13) yields

$$u \in C^{1/2}_{\text{loc}}((0,\infty); B^1_{\infty,1} \cap B^2_{(\kappa-1)n/2,1}) \subset C^{1/2}_{\text{loc}}((0,\infty); BUC^1 \cap W^{2,(\kappa-1)n/2}),$$

we see by Lemma 4.4 that

$$|u|^{\kappa-l}u^l \in L^\infty_{\text{loc}}((0,\infty); BUC^1 \cap W^{2,(\kappa-1)n/2}).$$

By combining (4.5) and Proposition 4.1 (ii), we deduce that

$$\mathcal{I}_\kappa^{(l)}(u,0) \in \bigcap_{0<\theta<1} C_{\mathrm{loc}}^{1-\theta}((0,\infty); B_{\infty,1}^{1+2\theta} \cap B_{(\kappa-1)n/2,1}^{2+2\theta}),$$

which yields

$$u \in \bigcap_{0<\theta<1} C_{\mathrm{loc}}^{1-\theta}((0,\infty); B_{\infty,1}^{1+2\theta} \cap B_{(\kappa-1)n/2,1}^{2+2\theta})$$

with the aid of (4.4). Moreover, by repeating the above argument, we have

(4.18) $$u \in \bigcap_{0<\theta<1} C_{\mathrm{loc}}^{1-\theta}((0,\infty); B_{\infty,1}^{[\kappa]-1+2\theta} \cap B_{(\kappa-1)n/2,1}^{[\kappa]+2\theta}).$$

Since $\kappa \notin \mathbb{N}$, it holds by (4.18) that (4.14). Hence, Lemma 4.5 implies that (4.15). Since (4.5) and Proposition 4.1 (ii) yield (4.16) and since (4.18), we obtain (1.5). Moreover, from (1.5) and (4.3), we have

$$u \in C^1((0,\infty); BUC) \cap C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); C^\kappa).$$

Therefore, it holds by Lemma 4.6 that $|u|^{\kappa-l}u^l \in C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]})$, which yields

$$\mathcal{I}_\kappa^{(l)}(u,0) \in C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]}) \cap C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]+2})$$

by virtue of (4.5) and Proposition 4.1 (iii). Thus we have

(4.19) $$u \in C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]}) \cap C_{\mathrm{loc}}^{\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]+2})$$

with the aid of (4.4). Since the regularity (4.19) implies that

$$u \in C^1((0,\infty); BUC) \cap C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); C^{\kappa-1}),$$

by applying Lemma 4.6 again, we obtain $|u|^{\kappa-l}u^l \in C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]-1})$. Hence we see by Proposition 4.1 (iii) and (4.4) that

$$u \in C_{\mathrm{loc}}^{2+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]-1}) \cap C_{\mathrm{loc}}^{1+\kappa-[\kappa]}((0,\infty); BUC^{[\kappa]+1}).$$

By combining the above regularity and (4.19), we have the desired result. This completes the proof of Theorem 1.3. $\qquad\square$

5. Breakdown of C^∞-smoothing effects

In this section, we show that the solutions u to (1.1) do not belong to C^∞-class in space by taking special initial data u_0 under the condition $\kappa \notin \mathbb{N}$. To show such a property, we choose sign-changing initial data u_0. In addition, it is necessary to use the fact that the solutions u have the properties $u(t_0, x_0) = 0$ and $\nabla u(t_0, x_0) \neq 0$ for some $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^n$. First we give the following lemma, which ensures that the solutions u are close to u_0 in the sense of BC^1 if $t > 0$ is small:

Lemma 5.1. *In the statement of Proposition 3.1 (i), assume that the initial data $u_0 \in L^{(\kappa-1)n/2} \cap BUC^1$. There exists a constant $0 < \delta_* < 1$ independently of u_0 such that if $\|u_0\|_{L^{(\kappa-1)n/2} \cap BC^1} \leq \delta_*$, then the corresponding solution u to (1.1) also satisfies $u \in BC([0,1); BUC^1)$ with the properties*

$$\|u\|_{L^\infty((0,1); BC^1)} \leq C\|u_0\|_{L^{(\kappa-1)n/2} \cap BC^1}, \quad \lim_{t \to +0} \|u(t,\cdot) - u_0\|_{BC^1} = 0,$$

where $C > 0$ is a constant independent of δ_*, u_0, and u.

Proof. Define the function space Y by setting $Y := X \cap BC([0,1); BUC^1)$ with the norm $\|u\|_Y := \|u\|_X + \|u\|_{L^\infty((0,1); BC^1)}$, where X is defined by (3.8) and (3.17). We consider the mapping Φ given by (3.14) again. Here, since Propositions 2.2 (i) and 2.3 (iii) imply that

(5.1)
$$\|\nabla^j e^{t\Delta}(|u|^{\kappa-l}u^l - |v|^{\kappa-l}v^l)(\tau, \cdot)\|_{L^\infty}$$
$$\leq Ct^{-j/2}\|(|u|^{\kappa-l}u^l)(\tau, \cdot) - (|v|^{\kappa-l}v^l)(\tau, \cdot)\|_{L^\infty}$$
$$\leq Ct^{-j/2}(\|u(\tau, \cdot)\|_{L^\infty}^{\kappa-1} + \|v(\tau, \cdot)\|_{L^\infty}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{L^\infty}$$

for $j = 0, 1$ and for all $t, \tau \in (0,1)$ and $u, v \in Y$, we observe that

$$\|\nabla^j(\Phi u)(t, \cdot)\|_{L^\infty} \leq \|\nabla^j u_0\|_{L^\infty} + C\int_0^t (t-\tau)^{-j/2}\|u(\tau, \cdot)\|_{L^\infty}^\kappa d\tau \leq \|\nabla^j u_0\|_{L^\infty} + C\|u\|_Y^\kappa$$

and

$$\|\nabla^j(\Phi u)(t, \cdot) - \nabla^j(\Phi v)(t, \cdot)\|_{L^\infty}$$
$$\leq C\int_0^t (t-\tau)^{-j/2}(\|u(\tau, \cdot)\|_{L^\infty}^{\kappa-1} + \|v(\tau, \cdot)\|_{L^\infty}^{\kappa-1})\|u(\tau, \cdot) - v(\tau, \cdot)\|_{L^\infty}d\tau$$
$$\leq C(\|u\|_Y^{\kappa-1} + \|v\|_Y^{\kappa-1})\|u - v\|_Y.$$

Hence we have

$$\|\Phi u\|_{L^\infty((0,1); BC^1)} \leq C\|u_0\|_{BC^1} + C\|u\|_Y^\kappa,$$
$$\|\Phi u - \Phi v\|_{L^\infty((0,1); BC^1)} \leq C(\|u\|_Y^{\kappa-1} + \|v\|_Y^{\kappa-1})\|u - v\|_Y$$

for all $u, v \in Y$. By the estimates (3.16) and (3.19), we obtain

$$\|\Phi u\|_Y \leq C\|u_0\|_{L^{(\kappa-1)n/2}\cap BC^1} + C\|u\|_Y^\kappa, \quad \|\Phi u - \Phi v\|_Y \leq C(\|u\|_Y^{\kappa-1} + \|v\|_Y^{\kappa-1})\|u - v\|_Y.$$

Therefore, by assuming that $u_0 \in L^{(\kappa-1)n/2} \cap BUC^1$ satisfies $\|u_0\|_{L^{(\kappa-1)n/2}\cap BC^1} \leq 3(4C)^{-\kappa/(\kappa-1)}$ and $u, v \in Y$ satisfy $\max\{\|u\|_Y, \|v\|_Y\} \leq (4C)^{-1/(\kappa-1)}$, we obtain a unique global solution $u \in Y$ in a similar manner to the proof of Proposition 3.1 (i). In addition, it holds by (5.1) that

$$\|\nabla^j u(t, \cdot) - \nabla^j u_0\|_{L^\infty} \leq \|\nabla^j e^{t\Delta}u_0 - \nabla^j u_0\|_{L^\infty} + C\int_0^t (t-\tau)^{-j/2}\|u(\tau, \cdot)\|_{L^\infty}^\kappa d\tau$$
$$\leq \|\nabla^j e^{t\Delta}u_0 - \nabla^j u_0\|_{L^\infty} + Ct^{1-j/2}\|u\|_Y^\kappa$$

for $j = 0, 1$ and for all $0 < t < 1$, which yields

$$\|u(t, \cdot) - u_0\|_{BC^1} \leq C\|e^{t\Delta}u_0 - u_0\|_{BC^1} + C(t + t^{1/2})\|u\|_Y.$$

Therefore, by letting $t \to +0$, we have the desired result. This proves Lemma 5.1. $\qquad \square$

In what follows, we show Theorem 1.5. We will make use of the key estimates in Lemma 2.4 again.

Proof of Theorem 1.5. First we take a sufficiently small constant $0 < \eta < 1$ and an initial data $u_0 \in C_0^\infty$ such that

$$u_0(x) = \eta\cos(\pi|x|^2) \text{ for } |x| \leq 1, \quad \|u_0\|_{L^{(\kappa-1)n/2}\cap BC^1} \leq \delta_*.$$

Then we see that

$$u_0(x) \geq \eta/2 \text{ if } |x| \leq \sqrt{1/3}, \quad u_0(x) \leq -\eta/2 \text{ if } \sqrt{2/3} \leq |x| \leq 1$$

by direct computations. In addition, by Lemma 5.1, we may take $0 < T < 1$ so that the corresponding solution u to (1.1) satisfies

(5.2)
$$\|u(t, \cdot) - u_0\|_{L^\infty} + \|\nabla u(t, \cdot) - \nabla u_0\|_{L^\infty} \leq \eta/4$$

for all $0 < t < T$. Now we fix $0 < t_0 < T$. If $|x| \leq \sqrt{1/3}$, we see by (5.2) that

$$u(t_0, x) = u_0(x) - u_0(x) + u(t_0, x) \geq u_0(x) - \|u(t_0, \cdot) - u_0\|_{L^\infty} \geq \eta/4.$$

Similarly, if $\sqrt{2/3} \leq |x| \leq 1$, we obtain

$$u(t_0, x) = u_0(x) - u_0(x) + u(t_0, x) \leq u_0(x) + \|u(t_0, \cdot) - u_0\|_{L^\infty} \leq -\eta/4.$$

Hence, by applying the intermediate value theorem with the aid of $u(t_0, \cdot) \in BUC^1$, we may take $x_0 \in \mathbb{R}^n$ satisfying $\sqrt{1/3} \leq |x_0| \leq \sqrt{2/3}$ and

$$(5.3) \qquad\qquad\qquad\qquad u(t_0, x_0) = 0.$$

In addition, noting that $\nabla u_0(x) = -2\pi\eta \sin(\pi|x|^2)x$ for $|x| \leq 1$, we have

$$|\nabla u_0(x)| \geq 2\pi\eta \left|\sin(\pi|x|^2)\right| |x| \geq 2\pi\eta \cdot 1/\sqrt{2} \cdot 1/2 = \sqrt{2}\pi\eta/2$$

for $1/2 \leq |x| \leq \sqrt{3}/2$. By using (5.2) again, we also obtain

$$|\nabla u(t_0, x)| = |\nabla u_0(x) - \nabla u_0(x) + \nabla u(t_0, x)|$$
$$\geq |\nabla u_0(x)| - \|\nabla u(t_0, \cdot) - \nabla u_0\|_{L^\infty} \geq (\sqrt{2}\pi/2 - 1/4)\eta.$$

Here, the above estimate implies the existence of $1 \leq j \leq n$ such that

$$(5.4) \qquad\qquad\qquad |\partial_j u(t_0, x)| \geq b := (\sqrt{2}\pi/2 - 1/4)(\eta/n) > 0$$

for $1/2 \leq |x| \leq \sqrt{3}/2$. Since u satisfies (1.1), it holds that

$$\partial_j^m \partial_t u(t_0, x) - \partial_j^m \Delta u(t_0, x) = \lambda \partial_j^m (|u|^{\kappa-l} u^l)(t_0, x)$$

for all $x \in \mathbb{R}^n$, where $m := [\kappa]$. Therefore, we see that

$$(5.5) \quad \begin{aligned} &- (\partial_j^m \Delta u(t_0, x_1) - \partial_j^m \Delta u(t_0, x_0)) \\ &= \lambda(\partial_j^m(|u|^{\kappa-l} u^l)(t_0, x_1) - \partial_j^m(|u|^{\kappa-l} u^l)(t_0, x_0)) - (\partial_j^m \partial_t u(t_0, x_1) - \partial_j^m \partial_t u(t_0, x_0)) \end{aligned}$$

for all $x_1 \in \mathbb{R}^n$. Here we recall that $x_0 \in \mathbb{R}^n$ satisfies (5.3). In what follows, let $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ denote the multi-index satisfying $\alpha_j = m$ and $\alpha_i = 0$ for $i \neq j$. We note that $\kappa \notin \mathbb{N}$, i.e., $0 < \kappa - m < 1$. Hence, by using the definition (2.2) of $\mathcal{R}_\alpha^{(l)}$, we may obtain the following representation

$$\partial_j^m(|u|^{\kappa-l} u^l)(t_0, x) = \left\{\prod_{i=1}^m (\kappa + 1 - i)\right\} (|u|^{\kappa-2m+l} u^{m-l})(t_0, x)(\partial_j u(t_0, x))^m + (\mathcal{R}_\alpha^{(l)} u)(t_0, x)$$

for all $x \in \mathbb{R}^n$. Then, from the relation (5.3), we have $\partial_j^m(|u|^{\kappa-l} u^l)(t_0, x_0) = (\mathcal{R}_\alpha^{(l)} u)(t_0, x_0)$. Therefore, it holds by Lemma 2.4 (i) that

$$(5.6) \quad \begin{aligned} &|\partial_j^m(|u|^{\kappa-l} u^l)(t_0, x_1) - \partial_j^m(|u|^{\kappa-l} u^l)(t_0, x_0)| \\ &\geq C|(|u|^{\kappa-2m+l} u^{m-l})(t_0, x_1)(\partial_j u(t_0, x_1))^m| - |(\mathcal{R}_\alpha^{(l)} u)(t_0, x_1) - (\mathcal{R}_\alpha^{(l)} u)(t_0, x_0)| \\ &\geq C|u(t_0, x_1)|^{\kappa-m}|\partial_j u(t_0, x_1)|^m - CM \sum_{|\beta| \leq m} |\partial_x^\beta u(t_0, x_1) - \partial_x^\beta u(t_0, x_0)|, \end{aligned}$$

where $M := (1 + \|u\|_{L^\infty(BC^m)})^{\kappa+m-1}$. Noting that $u \in C^1((0, \infty); BUC^{[\kappa]+1})$ from (1.6), by combining (5.5) and (5.6), we observe that

$$
\begin{aligned}
&|\partial_j^m \Delta u(t_0, x_1) - \partial_j^m \Delta u(t_0, x_0)| \\
&\geq C|u(t_0, x_1)|^{\kappa-m}|\partial_j u(t_0, x_1)|^m - CM \sum_{|\beta| \leq m} |\partial_x^\beta u(t_0, x_1) - \partial_x^\beta u(t_0, x_0)| \\
(5.7) \quad &\quad - |\partial_j^m \partial_t u(t_0, x_1) - \partial_j^m \partial_t u(t_0, x_0)| \\
&\geq C|u(t_0, x_1)|^{\kappa-m}|\partial_j u(t_0, x_1)|^m - CM\|u(t_0, \cdot)\|_{C^{\kappa+\sigma}} \cdot |x_1 - x_0|^{\kappa-m+\sigma} \\
&\quad - C\|\partial_t u(t_0, \cdot)\|_{C^{\kappa+\sigma}} \cdot |x_1 - x_0|^{\kappa-m+\sigma}
\end{aligned}
$$

for all $0 < \sigma < 1 - (\kappa - m)$. Since $1/2 < \sqrt{1/3} \leq |x_0| \leq \sqrt{2/3} < \sqrt{3}/2$, we may take a sufficiently small $0 < h_0 < 1$ so that $1/2 \leq |x_0 + he_j| \leq \sqrt{3}/2$ and

$$
\left| \frac{u(t_0, x_0 + he_j) - u(t_0, x_0)}{h} - \partial_j u(t_0, x_0) \right| \leq \frac{b}{2}
$$

for all $0 < h < h_0$, where $e_j \in \mathbb{R}^n$ denotes the j-th standard basis vector. Then, noting that (5.4) is valid for $1/2 \leq |x| \leq \sqrt{3}/2$, we see by (5.3) and (5.4) that

$$
\begin{aligned}
&|u(t_0, x_0 + he_j)|^{\kappa-m}|\partial_j u(t_0, x_0 + he_j)|^m \\
&= \frac{|\partial_j u(t_0, x_0 + he_j)|^m}{|u(t_0, x_0 + he_j)|^\sigma} \left(\frac{|u(t_0, x_0 + he_j) - u(t_0, x_0)|}{h} \right)^{\kappa-m+\sigma} h^{\kappa-m+\sigma} \\
&\geq \frac{|\partial_j u(t_0, x_0 + he_j)|^m}{|u(t_0, x_0 + he_j)|^\sigma} \left(|\partial_j u(t_0, x_0)| - \frac{b}{2} \right)^{\kappa-m+\sigma} h^{\kappa-m+\sigma} \\
&\geq \frac{b^m}{|u(t_0, x_0 + he_j)|^\sigma} \left(\frac{b}{2} \right)^{\kappa-m+\sigma} h^{\kappa-m+\sigma}
\end{aligned}
$$

for all $0 < h < h_0$. Therefore, by letting $x_1 = x_0 + he_j$ in the estimate (5.7), we observe that

$$
\begin{aligned}
|\partial_j^m \Delta u(t_0, x_0 + he_j) - \partial_j^m \Delta u(t_0, x_0)| &\geq \frac{Cb^m}{|u(t_0, x_0 + he_j)|^\sigma} \left(\frac{b}{2} \right)^{\kappa-m+\sigma} h^{\kappa-m+\sigma} \\
&\quad - CM(\|u(t_0, \cdot)\|_{C^{\kappa+\sigma}} + \|\partial_t u(t_0, \cdot)\|_{C^{\kappa+\sigma}})h^{\kappa-m+\sigma}.
\end{aligned}
$$

Consequently, since (5.3) holds, we have

$$
\liminf_{h \to +0} \frac{1}{h^{\kappa-m+\sigma}} |\partial_j^m \Delta u(t_0, x_0 + he_j) - \partial_j^m \Delta u(t_0, x_0)| = \infty,
$$

which yields $u(t_0, \cdot) \notin C_{\text{loc}}^{\kappa+2+\sigma}$ for all $0 < \sigma < 1 - (\kappa - m)$. This completes the proof of Theorem 1.5. $\qquad\square$

ACKNOWLEDGMENT

APPENDIX A. KEY ESTIMATES FOR THE TERMS WITH THE POWER NONLINEARITY

In this appendix, we give the proof of the key estimates, i.e., Lemma 2.4. To this end, we focus on the representation of higher order derivatives of $|\varphi|^{\kappa-l}\varphi^l$. The proof is based only on induction, but we have to deal with a lot of complicated calculations.

Lemma A.1. *Let $2 < \kappa < \infty$ and $l = 0, 1$. Suppose that $m \in \mathbb{N}$ satisfies $2 \leq m < \kappa$. Then the following statements hold:*

(i) Suppose that $\varphi \in BC^m$. Then it holds that

$$\partial_x^\alpha(|\varphi|^{\kappa-l}\varphi^l)(x) = M_{\kappa,m}(|\varphi|^{\kappa-2m+l}\varphi^{m-l})(x)\prod_{j=1}^n(\partial_j\varphi(x))^{\alpha_j}$$

(A.1)
$$+ \sum_{j=1}^m\left\{(|\varphi|^{\kappa-2m+l}\varphi^{m-l+j})(x)(\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)(x)\right\}$$

for all $x \in \mathbb{R}^n$ and $\alpha \in (\mathbb{N}\cup\{0\})^n$ satisfying $|\alpha| = m$, where

$$M_{\kappa,m} := \prod_{j=1}^m(\kappa+1-j), \quad \mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi := \prod_{h=1}^m\sum_{|\beta|\leq m}c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta\varphi$$

and $c_{\kappa,\alpha,j,h,\beta}^{(l)} \in \mathbb{R}$ are some constants independent of x and φ.

(ii) Suppose that $f \in C^1((0,\infty); BC^{m-1})$. Then it holds that

$$\partial_t\partial_x^\alpha(|f|^{\kappa-l}f^l)(t,x) = M_{\kappa,m}(|f|^{\kappa-2m+l}f^{m-l})(t,x)\partial_t f(t,x)\prod_{j=1}^n(\partial_j f(t,x))^{\alpha_j}$$

(A.2)
$$+ \sum_{j=1}^m\left\{(|f|^{\kappa-2m+l}f^{m-l+j})(t,x)(\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f)(t,x)\right\}$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and $\alpha \in (\mathbb{N}\cup\{0\})^n$ satisfying $|\alpha| = m-1$, where

$$\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f := \left(\prod_{h=1}^{m-1}\sum_{|\beta|\leq m-1}c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta f\right)\sum_{|\beta|\leq m-1}c_{\kappa,\alpha,j,m,\beta}^{(l)}\partial_t\partial_x^\beta f$$

and $c_{\kappa,\alpha,j,h,\beta}^{(l)} \in \mathbb{R}$ are some constants independent of t, x, and f.

Proof. (i) First we consider the case of $m = 2$. By a direct computation, we have

$$\partial_{i_1}\partial_{i_0}(|\varphi|^{\kappa-l}\varphi^l) = \partial_{i_1}(\kappa|\varphi|^{\kappa-2+l}\varphi^{1-l}\partial_{i_0}\varphi)$$
$$= \kappa(\kappa-1)|\varphi|^{\kappa-4+l}\varphi^{2-l}(\partial_{i_1}\varphi)\partial_{i_0}\varphi + \kappa|\varphi|^{\kappa-2+l}\varphi^{1-l}\partial_{i_1}\partial_{i_0}\varphi$$

for all $i_0, i_1 \in [1,n]$. Thus we have the desired result by taking suitable constants $c_{\kappa,\alpha,j,h,\beta}^{(l)} \in \mathbb{R}$. Since the condition $2 < \kappa \leq 3$ yields $m = 2$, in the following, we assume that $3 < \kappa < \infty$. Then we may take $m \in \mathbb{N}$ so that $2 \leq m < \kappa - 1$. Here we also assume that (A.1) holds for all $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n$ satisfying $|\alpha| = m$. By computing the derivative of (A.1), we observe that

$$\partial_i\partial_x^\alpha(|\varphi|^{\kappa-l}\varphi^l) = M_{\kappa,m}\{\partial_i(|\varphi|^{\kappa-2m+l}\varphi^{m-l})\}\prod_{j=1}^n(\partial_j\varphi)^{\alpha_j} + M_{\kappa,m}|\varphi|^{\kappa-2m+l}\varphi^{m-l}\partial_i\prod_{j=1}^n(\partial_j\varphi)^{\alpha_j}$$

(A.3)
$$+ \partial_i\sum_{j=1}^m(|\varphi|^{\kappa-2m+l}\varphi^{m-l+j}\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)$$

for all $1 \leq i \leq n$. Here it holds that

$$
\begin{aligned}
\partial_i(&|\varphi|^{\kappa-2m+l}\varphi^{m-l+j}) \\
&= (\kappa - 2m + l)|\varphi|^{\kappa-2m+l-2}\varphi\partial_i\varphi \cdot \varphi^{m-l+j} + |\varphi|^{\kappa-2m+l} \cdot (m-l+j)\varphi^{m-l+j-1}\partial_i\varphi \\
&= (\kappa - m + j)|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}\partial_i\varphi
\end{aligned}
$$
(A.4)

for all $1 \leq i \leq n$ and $j \in \mathbb{N} \cup \{0\}$. In addition, we also obtain

$$
\partial_i \prod_{j=1}^{n} (\partial_j\varphi)^{\alpha_j} = \sum_{j=1}^{n} \left\{ \alpha_j(\partial_j\varphi)^{\alpha_j-1}\partial_i\partial_j\varphi \prod_{j_*\neq j}(\partial_{j_*}\varphi)^{\alpha_{j*}} \right\}.
$$
(A.5)

By using (A.4), we observe that

$$
\begin{aligned}
\partial_i \sum_{j=1}^{m}(|\varphi|^{\kappa-2m+l}\varphi^{m-l+j}\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi) = &\sum_{j=1}^{m} \left\{ (\kappa - m + j)|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}(\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)\partial_i\varphi \right\} \\
&+ \sum_{j=2}^{m+1}(|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}\partial_i\mathcal{P}_{\kappa,\alpha,j-1}^{(l)}\varphi).
\end{aligned}
$$
(A.6)

Now we set $\alpha' := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n)$. Then, by combining (A.3), (A.4), (A.5), and (A.6), we have

$$
\begin{aligned}
\partial_x^{\alpha'}(|\varphi|^{\kappa-l}\varphi^l) = &M_{\kappa,m+1}|\varphi|^{\kappa-2m-2+l}\varphi^{m-l+1}\partial_i\varphi \prod_{j=1}^{n}(\partial_j\varphi)^{\alpha_j} \\
&+ M_{\kappa,m}|\varphi|^{\kappa-2m+l}\varphi^{m-l}\sum_{j=1}^{n}\left\{ \alpha_j(\partial_j\varphi)^{\alpha_j-1}\partial_i\partial_j\varphi \prod_{j_*\neq j}(\partial_{j_*}\varphi)^{\alpha_{j*}} \right\} \\
&+ \sum_{j=1}^{m}\left\{ (\kappa-m+j)|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}(\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)\partial_i\varphi \right\} \\
&+ \sum_{j=2}^{m+1}(|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}\partial_i\mathcal{P}_{\kappa,\alpha,j-1}^{(l)}\varphi) \\
= &M_{\kappa,m+1}|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l}\prod_{j=1}^{n}(\partial_j\varphi)^{\alpha'_j} + \sum_{j=1}^{m+1}(|\varphi|^{\kappa-2m-2+l}\varphi^{m+1-l+j}\widetilde{\mathcal{P}}_{\kappa,\alpha',j}^{(l)}\varphi),
\end{aligned}
$$

where we set

$$
\widetilde{\mathcal{P}}_{\kappa,\alpha',1}^{(l)}\varphi := M_{\kappa,m}\sum_{h=1}^{n}\left\{ \alpha_h(\partial_h\varphi)^{\alpha_h-1}\partial_i\partial_h\varphi \prod_{h_*\neq h}(\partial_{h_*}\varphi)^{\alpha_{h*}} \right\} + (\kappa-m+1)(\mathcal{P}_{\kappa,\alpha,1}^{(l)}\varphi)\partial_i\varphi,
$$

$$
\widetilde{\mathcal{P}}_{\kappa,\alpha',j}^{(l)}\varphi := (\kappa-m+j)(\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)\partial_i\varphi + \partial_i\mathcal{P}_{\kappa,\alpha,j-1}^{(l)}\varphi, \quad (2 \leq j \leq m),
$$

$$
\widetilde{\mathcal{P}}_{\kappa,\alpha',m+1}^{(l)}\varphi := \partial_i\mathcal{P}_{\kappa,\alpha,m}^{(l)}\varphi.
$$

By taking suitable constants $c^{(l)}_{\kappa,\alpha',1,h,\beta} \in \mathbb{R}$, we observe that

$$M_{\kappa,m} \sum_{h=1}^{n} \left\{ \alpha_h (\partial_h \varphi)^{\alpha_h - 1} \partial_i \partial_h \varphi \prod_{h_* \neq h} (\partial_{h_*} \varphi)^{\alpha_{h*}} \right\} + (\kappa - m + 1)(\mathcal{P}^{(l)}_{\kappa,\alpha,1} \varphi) \partial_i \varphi$$

$$= \prod_{h=1}^{m+1} \sum_{|\beta| \leq m} c^{(l)}_{\kappa,\alpha',1,h,\beta} \partial_x^\beta \varphi.$$

Moreover, since we may take constants $\widetilde{c}^{(l)}_{\kappa,\alpha',j,h,\beta} \in \mathbb{R}$ so that

$$\partial_i \mathcal{P}^{(l)}_{\kappa,\alpha,j} \varphi = \sum_{h=1}^{m} \left\{ \left(\sum_{|\beta| \leq m} c^{(l)}_{\kappa,\alpha,j,h,\beta} \partial_i \partial_x^\beta \varphi \right) \prod_{h_* \neq h} \sum_{|\beta| \leq m} c^{(l)}_{\kappa,\alpha,j,h_*,\beta} \partial_x^\beta \varphi \right\}$$

$$= \prod_{h=1}^{m} \sum_{|\beta| \leq m+1} \widetilde{c}^{(l)}_{\kappa,\alpha',j,h,\beta} \partial_x^\beta \varphi,$$

we also obtain the representations

$$(\kappa - m + j)(\mathcal{P}^{(l)}_{\kappa,\alpha,j} \varphi) \partial_i \varphi + \partial_i \mathcal{P}^{(l)}_{\kappa,\alpha,j-1} \varphi = \prod_{h=1}^{m+1} \sum_{|\beta| \leq m+1} c^{(l)}_{\kappa,\alpha',j,h,\beta} \partial_x^\beta \varphi,$$

$$\partial_i \mathcal{P}^{(l)}_{\kappa,\alpha,m} \varphi = \prod_{h=1}^{m} \sum_{|\beta| \leq m+1} c^{(l)}_{\kappa,\alpha',m,h,\beta} \partial_x^\beta \varphi.$$

Hence we deduce that

$$\partial_x^{\alpha'} (|\varphi|^{\kappa-l} \varphi^l) = M_{\kappa,m+1} |\varphi|^{\kappa-2m-2+l} \varphi^{m+1-l} \prod_{j=1}^{n} (\partial_j \varphi)^{\alpha'_j} + \sum_{j=1}^{m+1} (|\varphi|^{\kappa-2m-2+l} \varphi^{m+1-l+j} \mathcal{P}^{(l)}_{\kappa,\alpha',j} \varphi)$$

with

$$\mathcal{P}^{(l)}_{\kappa,\alpha',j} \varphi := \widetilde{\mathcal{P}}^{(l)}_{\kappa,\alpha',j} \varphi = \prod_{h=1}^{m+1} \sum_{|\beta| \leq m+1} c^{(l)}_{\kappa,\alpha',j,h,\beta} \partial_x^\beta \varphi.$$

This implies that (A.1) is also valid for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = m + 1$. Hence we have the desired result.

(ii) Since $f(t, \cdot) \in BC^{m-1}$ for each $0 < t < \infty$, the representation (A.1) implies that

$$\partial_x^\alpha (|f|^{\kappa-l} f^l) = M_{\kappa,m-1} |f|^{\kappa-2m+2+l} f^{m-1-l} \prod_{j=1}^{n} (\partial_j f)^{\alpha_j} + \sum_{j=1}^{m-1} (|f|^{\kappa-2m+2+l} f^{m-1-l+j} \mathcal{P}^{(l)}_{\kappa,\alpha,j} f)$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = m - 1$. Noting that

$$\partial_t (|f|^{\kappa-2m+2+l} f^{m-1-l+j}) = (\kappa - m + 1 + j) |f|^{\kappa-2m+l} f^{m-l+j} \partial_t f,$$

$$\partial_t \prod_{j=1}^{n} (\partial_j f)^{\alpha_j} = \sum_{j=1}^{n} \left\{ \alpha_j (\partial_j f)^{\alpha_j - 1} \partial_t \partial_j f \prod_{j_* \neq j} (\partial_{j_*} f)^{\alpha_{j*}} \right\}$$

for all $j \in \mathbb{N} \cup \{0\}$ in a similar manner to the calculations of (A.4) and (A.5), we have

$$\partial_t \partial_x^\alpha (|f|^{\kappa-l} f^l) = M_{\kappa,m} |f|^{\kappa-2m+l} f^{m-l} \partial_t f \prod_{j=1}^n (\partial_j f)^{\alpha_j}$$

$$+ M_{\kappa,m-1} |f|^{\kappa-2m+2+l} f^{m-1-l} \sum_{j=1}^n \left\{ \alpha_j (\partial_j f)^{\alpha_j-1} \partial_t \partial_j f \prod_{j_* \neq j} (\partial_{j_*} f)^{\alpha_{j_*}} \right\}$$

$$+ \sum_{j=1}^{m-1} \left\{ (\kappa-m+1+j)|f|^{\kappa-2m+l} f^{m-l+j} (\mathcal{P}_{\kappa,\alpha,j}^{(l)} f) \partial_t f \right\}$$

$$+ \sum_{j=2}^m (|f|^{\kappa-2m+l} f^{m-l+j} \partial_t \mathcal{P}_{\kappa,\alpha,j-1}^{(l)} f)$$

$$= M_{\kappa,m} |f|^{\kappa-2m+l} f^{m-l} \partial_t f \prod_{j=1}^n (\partial_j f)^{\alpha_j} + \sum_{j=1}^m (|f|^{\kappa-2m+l} f^{m-l+j} \widetilde{\mathcal{Q}}_{\kappa,\alpha,j}^{(l)} f),$$

where we set

$$\widetilde{\mathcal{Q}}_{\kappa,\alpha,1}^{(l)} f := M_{\kappa,m-1} \sum_{h=1}^n \left\{ \alpha_h (\partial_h f)^{\alpha_h-1} \partial_t \partial_h f \prod_{h_* \neq h} (\partial_{h_*} f)^{\alpha_{h_*}} \right\} + (\kappa-m+2)(\mathcal{P}_{\kappa,\alpha,1}^{(l)} f) \partial_t f,$$

$$\widetilde{\mathcal{Q}}_{\kappa,\alpha,j}^{(l)} f := (\kappa-m+1+j)(\mathcal{P}_{\kappa,\alpha,j}^{(l)} f) \partial_t f + \partial_t \mathcal{P}_{\kappa,\alpha,j-1}^{(l)} f, \quad (2 \leq j \leq m),$$

$$\widetilde{\mathcal{Q}}_{\kappa,\alpha,m+1}^{(l)} f := \partial_t \mathcal{P}_{\kappa,\alpha,m}^{(l)} f.$$

By taking suitable constants $c_{\kappa,\alpha',1,h,\beta}^{(l)} \in \mathbb{R}$, we observe that

$$M_{\kappa,m-1} \sum_{h=1}^n \left\{ \alpha_h (\partial_h f)^{\alpha_h-1} \partial_t \partial_h f \prod_{h_* \neq h} (\partial_{h_*} f)^{\alpha_{h_*}} \right\} + (\kappa-m+2)(\mathcal{P}_{\kappa,\alpha,1}^{(l)} f) \partial_t f$$

$$= \left(\prod_{h=1}^{m-1} \sum_{|\beta| \leq m-1} c_{\kappa,\alpha,j,h,\beta}^{(l)} \partial_x^\beta f \right) \sum_{|\beta| \leq m-1} c_{\kappa,\alpha,j,m,\beta}^{(l)} \partial_t \partial_x^\beta f.$$

Moreover, since we may take constants $\widetilde{c}_{\kappa,\alpha',j,h,\beta}^{(l)} \in \mathbb{R}$ so that

$$\partial_t \mathcal{P}_{\kappa,\alpha,j}^{(l)} f = \sum_{h=1}^{m-1} \left\{ \left(\sum_{|\beta| \leq m-1} c_{\kappa,\alpha,j,h,\beta}^{(l)} \partial_t \partial_x^\beta f \right) \prod_{h_* \neq h} \sum_{|\beta| \leq m-1} c_{\kappa,\alpha,j,h_*,\beta}^{(l)} \partial_x^\beta f \right\}$$

$$= \left(\prod_{h=1}^{m-1} \sum_{|\beta| \leq m-1} \widetilde{c}_{\kappa,\alpha,j,h,\beta}^{(l)} \partial_x^\beta f \right) \sum_{|\beta| \leq m-1} \widetilde{c}_{\kappa,\alpha,j,m,\beta}^{(l)} \partial_t \partial_x^\beta f,$$

we also obtain the representations

$$(\kappa - m + 1 + j)(\mathcal{P}_{\kappa,\alpha,j}^{(l)}f)\partial_t f + \partial_t \mathcal{P}_{\kappa,\alpha,j-1}^{(l)}f = \left(\prod_{h=1}^{m-1} \sum_{|\beta|\le m-1} c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta f\right)\sum_{|\beta|\le m-1} c_{\kappa,\alpha,j,m,\beta}^{(l)}\partial_t\partial_x^\beta f,$$

$$\partial_t \mathcal{P}_{\kappa,\alpha,m}^{(l)}f = \left(\prod_{h=1}^{m-1} \sum_{|\beta|\le m-1} c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta f\right)\sum_{|\beta|\le m-1} c_{\kappa,\alpha,j,m,\beta}^{(l)}\partial_t\partial_x^\beta f.$$

Therefore, by setting $\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f := \widetilde{\mathcal{Q}}_{\kappa,\alpha,j}^{(l)}f$, we obtain the representation (A.2). This completes the proof of Lemma A.1. □

Remark A.2. The statement of Lemma A.1 may be extended to the case of the general domain $\Omega \subset \mathbb{R}^n$ as well.

By applying Lemma A.1, we may show the key estimates.

Proof of Lemma 2.4. (i) First we consider the case of $m = 1$. Then, since it holds that $\partial_i(|\varphi|^{\kappa-l}\varphi^l) = \kappa|\varphi|^{\kappa-2+l}\varphi^{1-l}\partial_i\varphi$ for $1 \le i \le n$ and for all $\varphi \in BC^1$, we see that $\mathcal{R}_\alpha^{(l)}\varphi \equiv 0$ for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = 1$. Thus we have the desired result. Since the condition $1 < \kappa \le 2$ yields $m = 1$, in the following, we assume that $2 < \kappa < \infty$ and $2 \le m < \kappa$. In this case, we see by Lemma A.1 that the representation (A.1) holds. Therefore, by taking $m_* \in \mathbb{N}$ such that $2 \le m_* \le m$, we observe that

$$\mathcal{R}_\alpha^{(l)}\varphi = \sum_{j=1}^{m_*}(|\varphi|^{\kappa-2m_*+l}\varphi^{m_*-l+j}\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi)$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = m_*$, which implies that

$$|\mathcal{R}_\alpha^{(l)}\varphi_1 - \mathcal{R}_\alpha^{(l)}\varphi_0| \le \sum_{j=1}^{m_*}\left(||\varphi_1|^{\kappa-2m_*+l}\varphi_1^{m_*-l+j} - |\varphi_0|^{\kappa-2m_*+l}\varphi_0^{m_*-l+j}||\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi_1|\right)$$

$$+ \sum_{j=1}^{m_*}\left(|\varphi_0|^{\kappa-m_*+j}|\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi_1 - \mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi_0|\right).$$

Here, since $|\varphi|^{\kappa-2m_*+l}\varphi^{m_*-l+j} = |\varphi|^{\kappa-m_*+j-(m_*-l+j)}\varphi^{m_*-l+j}$ and since $\kappa - m_* + j > 1$ for $j \ge 1$, Proposition 2.3 (ii) yields

$$(A.7) \quad ||\varphi_1|^{\kappa-2m_*+l}\varphi_1^{m_*-l+j} - |\varphi_0|^{\kappa-2m_*+l}\varphi_0^{m_*-l+j}| \le C(\|\varphi_1\|_{L^\infty}^{\kappa-m_*+j-1} + \|\varphi_0\|_{L^\infty}^{\kappa-m_*+j-1})|\varphi_1 - \varphi_0|.$$

In addition, noting that

$$\mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi_1 - \mathcal{P}_{\kappa,\alpha,j}^{(l)}\varphi_0 = \left\{\sum_{|\beta|\le m_*} c_{\kappa,\alpha,j,1,\beta}^{(l)}\partial_x^\beta(\varphi_1 - \varphi_0)\right\}\prod_{h=2}^{m_*}\sum_{|\beta|\le m_*} c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta\varphi_1 + \cdots$$

$$+ \left(\prod_{h=1}^{m_*-1}\sum_{|\beta|\le m_*} c_{\kappa,\alpha,j,h,\beta}^{(l)}\partial_x^\beta\varphi_0\right)\sum_{|\beta|\le m_*} c_{\kappa,\alpha,j,m_*,\beta}^{(l)}\partial_x^\beta(\varphi_1 - \varphi_0),$$

we observe that

$$
\begin{aligned}
&|\mathcal{P}^{(l)}_{\kappa,\alpha,j}\varphi_1 - \mathcal{P}^{(l)}_{\kappa,\alpha,j}\varphi_0| \\
&\leq C \sum_{|\beta|\leq m_*} \left(|\partial_x^\beta(\varphi_1 - \varphi_0)| \|\varphi_1\|_{BC^{m_*}}^{m_*-1} + \cdots + \|\varphi_0\|_{BC^{m_*}}^{m_*-1} |\partial_x^\beta(\varphi_1 - \varphi_0)| \right) \\
&\leq C(\|\varphi_1\|_{BC^{m_*}} + \|\varphi_0\|_{BC^{m_*}})^{m_*-1} \sum_{|\beta|\leq m_*} |\partial_x^\beta(\varphi_1 - \varphi_0)|.
\end{aligned}
$$
(A.8)

Therefore, by combining (A.7) and (A.8), we obtain

$$
\begin{aligned}
|\mathcal{R}^{(l)}_\alpha \varphi_1 - \mathcal{R}^{(l)}_\alpha \varphi_0| &\leq C \sum_{j=1}^{m_*} (\|\varphi_1\|_{L^\infty}^{\kappa-m_*+j-1} + \|\varphi_0\|_{L^\infty}^{\kappa-m_*+j-1}) |\varphi_1 - \varphi_0| \cdot \|\varphi_1\|_{BC^{m_*}}^{m_*-1} \sum_{|\beta|\leq m_*} |\partial_x^\beta \varphi_1| \\
&\quad + C \sum_{j=1}^{m_*} \|\varphi_0\|_{L^\infty}^{\kappa-m_*+j} \cdot (\|\varphi_1\|_{BC^{m_*}} + \|\varphi_0\|_{BC^{m_*}})^{m_*-1} \sum_{|\beta|\leq m_*} |\partial_x^\beta(\varphi_1 - \varphi_0)| \\
&\leq C(1 + \|\varphi_1\|_{BC^{m_*}} + \|\varphi_0\|_{BC^{m_*}})^{\kappa+m_*-1} \sum_{|\beta|\leq m_*} |\partial_x^\beta(\varphi_1 - \varphi_0)|
\end{aligned}
$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = m_*$. Since the above estimate is valid for all $2 \leq m_* \leq m$ and since $\mathcal{R}^{(l)}_\alpha \varphi \equiv 0$ for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = 1$, we obtain the desired result.

(ii) By the definition (2.2) of $\mathcal{R}^{(l)}_\alpha \varphi$, we have

$$
\partial_x^\alpha(|\varphi|^{\kappa-l}\varphi^l) = \left\{ \prod_{j=1}^m (\kappa + 1 - j) \right\} |\varphi|^{\kappa-2m+l}\varphi^{m-l} \prod_{j=1}^n (\partial_j\varphi)^{\alpha_j} + \mathcal{R}^{(l)}_\alpha \varphi
$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^n$ satisfying $|\alpha| = m$. Thus we deduce that

$$
\begin{aligned}
|\partial_x^\alpha(|\varphi_1|^{\kappa-l}\varphi_1^l - |\varphi_0|^{\kappa-l}\varphi_0^l)| &\leq C||\varphi_1|^{\kappa-2m+l}\varphi_1^{m-l} - |\varphi_0|^{\kappa-2m+l}\varphi_0^{m-l}| \prod_{j=1}^n |\partial_j\varphi_1|^{\alpha_j} \\
&\quad + C|\varphi_0|^{\kappa-m} \left| \prod_{j=1}^n (\partial_j\varphi_1)^{\alpha_j} - \prod_{j=1}^n (\partial_j\varphi_0)^{\alpha_j} \right| + |\mathcal{R}^{(l)}_\alpha \varphi_1 - \mathcal{R}^{(l)}_\alpha \varphi_0|.
\end{aligned}
$$

Here, since it holds that

$$
\left| \prod_{j=1}^n (\partial_j\varphi_1)^{\alpha_j} - \prod_{j=1}^n (\partial_j\varphi_0)^{\alpha_j} \right| \leq C(\|\varphi_1\|_{BC^1}^{m-1} + \|\varphi_0\|_{BC^1}^{m-1}) \sum_{|\beta|=1} |\partial_x^\beta(\varphi_1 - \varphi_0)|,
$$

by applying the assertion of (i), we observe that

$$|\partial_x^\alpha(|\varphi_1|^{\kappa-l}\varphi_1^l - |\varphi_0|^{\kappa-l}\varphi_0^l)| \leq C||\varphi_1|^{\kappa-2m-l}\varphi_1^{m-l} - |\varphi_0|^{\kappa-2m-l}\varphi_0^{m-l}| \sum_{|\beta|=1} |\partial_x^\beta\varphi_1|^m$$

$$+ C\|\varphi_1\|_{L^\infty}^{\kappa-m}(\|\varphi_1\|_{BC^1}^{m-1} + \|\varphi_0\|_{BC^1}^{m-1}) \sum_{|\beta|=1} |\partial_x^\beta(\varphi_1 - \varphi_0)|$$

$$+ C(1 + \|\varphi_1\|_{BC^m} + \|\varphi_0\|_{BC^m})^{\kappa+m-1} \sum_{|\beta|\leq m} |\partial_x^\beta(\varphi_1 - \varphi_0)|$$

$$\leq C||\varphi_1|^{\kappa-2m-l}\varphi_1^{m-l} - |\varphi_0|^{\kappa-2m-l}\varphi_0^{m-l}| \sum_{|\beta|=1} |\partial_x^\beta\varphi_1|^m$$

$$+ C(1 + \|\varphi_1\|_{BC^m} + \|\varphi_0\|_{BC^m})^{\kappa+m-1} \sum_{|\beta|\leq m} |\partial_x^\beta(\varphi_1 - \varphi_0)|.$$

(iii) First we consider the case of $m = 1$. Then, since $\partial_t(|f|^{\kappa-l}f^l) = \kappa|f|^{\kappa-2+l}f^{1-l}\partial_t f$, we see that

$$|\partial_t(|f_1|^{\kappa-l}f_1^l - |f_0|^{\kappa-l}f_0^l)| \leq \kappa\left(||f_1|^{\kappa-2+l}f_1^{1-l} - |f_0|^{\kappa-2+l}f_0^{1-l}||\partial_t f_1| + \|f_0\|_{L_\varepsilon^\infty(L^\infty)}|\partial_t(f_1 - f_0)|\right).$$

Thus we have the desired result. Since the condition $1 < \kappa \leq 2$ yields $m = 1$, in the following, we assume that $2 < \kappa < \infty$ and $2 \leq m < \kappa$. In this case, by using the representation (A.2), we have

$$|\partial_t\partial_x^\alpha(|f_1|^{\kappa-l}f_1^l - |f_0|^{\kappa-l}f_0^l)|$$

$$\leq C||f_1|^{\kappa-2m+l}f_1^{m-l} - |f_0|^{\kappa-2m+l}f_0^{m-l}||\partial_t f_1| \prod_{j=1}^n |\partial_j f_1|^{\alpha_j}$$

$$+ C|f_0|^{\kappa-m}\left|\partial_t f_1 \prod_{j=1}^n(\partial_j f_1)^{\alpha_j} - \partial_t f_0 \prod_{j=1}^n(\partial_j f_0)^{\alpha_j}\right|$$

$$+ \sum_{j=1}^m\left(||f_1|^{\kappa-2m+l}f_1^{m-l+j} - |f_0|^{\kappa-2m+l}f_0^{m-l+j}||\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_1|\right)$$

$$+ \sum_{j=1}^m\left(|f_0|^{\kappa-m+j}|\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_1 - \mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_0|\right)$$

$$\leq C||f_1|^{\kappa-2m+l}f_1^{m-l} - |f_0|^{\kappa-2m+l}f_0^{m-l}||\partial_t f_1| \sum_{|\beta|=1} |\partial_x^\beta f_1|^{m-1} + CS_0 + S_1 + S_2,$$

where

$$S_0 := |f_0|^{\kappa-m}\left|\partial_t f_1 \prod_{j=1}^n(\partial_j f_1)^{\alpha_j} - \partial_t f_0 \prod_{j=1}^n(\partial_j f_0)^{\alpha_j}\right|,$$

$$S_1 := \sum_{j=1}^m\left(||f_1|^{\kappa-2m+l}f_1^{m-l+j} - |f_0|^{\kappa-2m+l}f_0^{m-l+j}||\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_1|\right),$$

$$S_2 := \sum_{j=1}^m\left(|f_0|^{\kappa-m+j}|\mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_1 - \mathcal{Q}_{\kappa,\alpha,j}^{(l)}f_0|\right).$$

Here we see that

$$S_0 \le \|f_0\|_{L_\varepsilon^\infty(L^\infty)} \left\{ |\partial_t(f_1 - f_0)| \prod_{j=1}^n |\partial_j f_1|^{\alpha_j} + |\partial_t f_0| \left| \prod_{j=1}^n (\partial_j f_1)^{\alpha_j} - \prod_{j=1}^n (\partial_j f_0)^{\alpha_j} \right| \right\}$$

$$\le C\|f_0\|_{L_\varepsilon^\infty(L^\infty)} \left\{ |\partial_t(f_1 - f_0)| \|f_1\|_{L_\varepsilon^\infty(BC^1)}^{m-1} \right.$$

(A.9)

$$\left. + \|f_0\|_{BC_\varepsilon^1(L^\infty)} (\|f_1\|_{L_\varepsilon^\infty(BC^1)}^{m-2} + \|f_0\|_{L_\varepsilon^\infty(BC^1)}^{m-2}) \sum_{|\beta|=1} |\partial_x^\beta(f_1 - f_0)| \right\}$$

$$\le C(1 + \|f_1\|_{BC_\varepsilon^1(BC^1)} + \|f_0\|_{BC_\varepsilon^1(BC^1)})^m \left(\sum_{|\beta|=1} |\partial_x^\beta(f_1 - f_0)| + |\partial_t(f_1 - f_0)| \right).$$

Moreover, in a similar manner to the derivation of the estimate (A.8), we deduce that

$$|\mathcal{Q}_{\kappa,\alpha,j}^{(l)} f_1 - \mathcal{Q}_{\kappa,\alpha,j}^{(l)} f_0|$$

$$\le C \sum_{|\beta| \le m-1} \left(|\partial_x^\beta(f_1 - f_0)| \|f_1\|_{L_\varepsilon^\infty(BC^{m-1})}^{m-2} \|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \cdots \right.$$

$$\left. + \|f_0\|_{L_\varepsilon^\infty(BC^{m-1})}^{m-2} |\partial_x^\beta(f_1 - f_0)| \|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{L_\varepsilon^\infty(BC^{m-1})}^{m-1} |\partial_t \partial_x^\beta(f_1 - f_0)| \right)$$

$$\le C(\|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{BC_\varepsilon^1(BC^{m-1})})^{m-1} \sum_{|\beta| \le m-1} \left(|\partial_x^\beta(f_1 - f_0)| + |\partial_t \partial_x^\beta(f_1 - f_0)| \right).$$

Thus we see by (A.7) that

(A.10)

$$\begin{cases} S_1 \le C \sum_{j=1}^m (\|f_1\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-m+j-1} + \|f_0\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-m+j-1})|f_1 - f_0| \\ \qquad \times \|f_1\|_{BC_\varepsilon^1(BC^{m-1})}^{m-1} \sum_{|\beta| \le m-1} (|\partial_x^\beta f_1| + |\partial_t \partial_x^\beta f_1|) \\ \qquad \le C(1 + \|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1}|f_1 - f_0|, \\ S_2 \le C \sum_{j=1}^m \|f_0\|_{L_\varepsilon^\infty(L^\infty)}^{\kappa-m+j} (\|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{BC_\varepsilon^1(BC^{m-1})})^{m-1} \\ \qquad \times \sum_{|\beta| \le m-1} \left(|\partial_x^\beta(f_1 - f_0)| + |\partial_t \partial_x^\beta(f_1 - f_0)| \right) \\ \qquad \le C(1 + \|f_1\|_{BC_\varepsilon^1(BC^{m-1})} + \|f_0\|_{BC_\varepsilon^1(BC^{m-1})})^{\kappa+m-1} \\ \qquad \times \sum_{|\beta| \le m-1} \left(|\partial_x^\beta(f_1 - f_0)| + |\partial_t \partial_x^\beta(f_1 - f_0)| \right). \end{cases}$$

Therefore, by combining (A.9) and (A.10), we have the desired result. This completes the proof of Lemma 2.4. □

Appendix B. Smoothing properties of mild solutions to the heat equation

In this appendix, we give the proof of the smoothing properties of mild solutions to the heat equation, i.e., Proposition 4.1. Such a kind of parabolic smoothing effect may be well-known, but

we have to pay attention to some viewpoints. The proof of Proposition 4.1 is strongly based on the argument due to Lunardi [29, Proposition 4.2.1 and Theorem 4.3.1]. Here we note that it is not so obvious that the heat semigroup $e^{t\Delta}: B_{r,\infty}^s \to B_{r,\infty}^s$ is an analytic semigroup. So we do not know whether we may use abstract results like [29, Proposition 4.2.1] directly. Moreover, the assumptions in Proposition 4.1 are slightly weaker than those of [29, Proposition 4.2.1 and Theorem 4.3.1]. For these reasons, we show the smoothing effects by direct computations.

Proposition B.1. *(i) Let $1 \le r_0 \le \infty$, $0 < s < \infty$, and $1 < T < \infty$. Then, for every $\varphi \in B_{r_0,\infty}^s$, it holds that $\partial_t^j e^{t\Delta}\varphi \in B_{r_1,1}^{s+\sigma}$ with the estimate*

$$\|\partial_t^j e^{t\Delta}\varphi\|_{B_{r_1,1}^{s+\sigma}} \le CT^{\sigma/2}t^{-\sigma/2-j-(n/2)(1/r_0-1/r_1)}\|\varphi\|_{B_{r_0,\infty}^s}$$

for all $r_0 \le r_1 \le \infty$, $0 < \sigma < \infty$, $j \in \mathbb{N} \cup \{0\}$, and $0 < t < T$, where $C > 0$ is a constant independent of T, t, and φ.

(ii) Let $1 \le r_1 \le \infty$ and $0 < s < \infty$. Suppose that $f \in L_{\mathrm{loc}}^\infty([0,\infty); B_{r_1,\infty}^s)$. Then it holds that

$$\mathcal{I}f \in \bigcap_{0 < \theta < 1} C_{\mathrm{loc}}^{1-\theta}((0,\infty); B_{r_1,1}^{s+2\theta}),$$

where $\mathcal{I}f$ is defined by (4.1).

Proof. (i) Note that $\partial_t^j e^{t\Delta}\varphi = \Delta^j e^{t\Delta}\varphi$ for all $j \in \mathbb{N} \cup \{0\}$. The usual estimate of the heat semigroup yields

$$\|\Delta^j e^{t\Delta}\varphi\|_{L^{r_1}} \le Ct^{-j-(n/2)(1/r_0-1/r_1)}\|\varphi\|_{L^{r_0}}$$

for all $j \in \mathbb{N} \cup \{0\}$ and $0 < t < \infty$. In addition, since $\dot{B}_{r_0,1}^{s+\sigma+n(1/r_0-1/r_1)} \subset \dot{B}_{r_1,1}^{s+\sigma}$ from Proposition 2.1 (iv), we see by [24, Lemma 2.2 (ii)] and [43, Theorem 2.28] that

$$\|\Delta^j e^{t\Delta}\varphi\|_{\dot{B}_{r_1,1}^{s+\sigma}} \le C\|\Delta^j e^{t\Delta}\varphi\|_{\dot{B}_{r_0,1}^{s+\sigma+n(1/r_0-1/r_1)}} \le C\|e^{t\Delta}\varphi\|_{\dot{B}_{r_0,1}^{s+\sigma+2j+n(1/r_0-1/r_1)}}$$

$$\le Ct^{-\sigma/2-j-(n/2)(1/r_0-1/r_1)}\|\varphi\|_{\dot{B}_{r_0,\infty}^s}$$

for all $0 < \sigma < \infty$, $j \in \mathbb{N} \cup \{0\}$, and $0 < t < \infty$. Therefore, noting that $B_{r,q}^s = L^r \cap \dot{B}_{r,q}^s$ holds for $0 < s < \infty$, $1 \le r \le \infty$, and $1 \le q \le \infty$ from [4, Theorem 6.3.2], we observe that

$$\|\Delta^j e^{t\Delta}\varphi\|_{B_{r_1,1}^{s+\sigma}} \le C(\|\Delta^j e^{t\Delta}\varphi\|_{L^{r_1}} + \|\Delta^j e^{t\Delta}\varphi\|_{\dot{B}_{r_1,1}^{s+\sigma}})$$

$$\le Ct^{-\sigma/2-j-(n/2)(1/r_0-1/r_1)}(t^{\sigma/2}\|\varphi\|_{L^{r_0}} + \|\varphi\|_{\dot{B}_{r_0,\infty}^s})$$

$$\le CT^{\sigma/2}t^{-\sigma/2-j-(n/2)(1/r_0-1/r_1)}\|\varphi\|_{B_{r_0,\infty}^s}$$

for all $0 < t < T$.

(ii) Let $1 < T < \infty$. By using the estimate in (i), we have

$$\|(\mathcal{I}f)(t,\cdot)\|_{B_{r_1,1}^{s+2\theta}} \le CT^\theta \int_0^t (t-\tau)^{-\theta}\|f(\tau,\cdot)\|_{B_{r_1,\infty}^s}\,d\tau$$

$$\le CT^\theta\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)}\left[-(t-\tau)^{1-\theta}\right]_{\tau=0}^{\tau=t}$$

$$\le CT\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)}$$

for all $0 < t < T$ and $0 < \theta < 1$. Thus we obtain

$$\|\mathcal{I}f\|_{L^\infty((0,T);B_{r_1,1}^{s+2\theta})} \le CT\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)}.$$

In addition, since it holds that

$$(\mathcal{I}f)(t_1,\cdot) - (\mathcal{I}f)(t_0,\cdot) = \int_0^{t_0} (e^{(t_1-\tau)\Delta} - e^{(t_0-\tau)\Delta})f(\tau,\cdot)d\tau + \int_{t_0}^{t_1} e^{(t_1-\tau)\Delta}f(\tau,\cdot)d\tau$$

$$= \int_0^{t_0} \left(\int_{t_0-\tau}^{t_1-\tau} \partial_\eta e^{\eta\Delta}f(\tau,\cdot)d\eta\right)d\tau + \int_{t_0}^{t_1} e^{(t_1-\tau)\Delta}f(\tau,\cdot)d\tau$$

for all $0 < t_0 < t_1 < T$, by using the estimate in (i) again, we observe that

$$\|(\mathcal{I}f)(t_1,\cdot) - (\mathcal{I}f)(t_0,\cdot)\|_{B_{r_1,1}^{s+2\theta}}$$

$$\leq CT^{\theta+1} \int_0^{t_0} \int_{t_0-\tau}^{t_1-\tau} \eta^{-\theta-1}\|f(\tau,\cdot)\|_{B_{r_1,\infty}^s}d\eta d\tau + CT^\theta \int_{t_0}^{t_1} (t_1-\tau)^{-\theta}\|f(\tau,\cdot)\|_{B_{r_1,\infty}^s}d\tau$$

$$\leq C(T^{\theta+1} + T^\theta)\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)} \left(\int_0^{t_0} \left[-\eta^{-\theta}\right]_{\eta=t_0-\tau}^{\eta=t_1-\tau}d\tau + \left[-(t_1-\tau)^{1-\theta}\right]_{\tau=t_0}^{\tau=t_1}\right)$$

$$\leq C(T^{\theta+1} + T^\theta)\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)} \left(\left[(t_1-\tau)^{1-\theta} - (t_0-\tau)^{1-\theta}\right]_{\tau=0}^{\tau=t_0} + (t_1-t_0)^{1-\theta}\right)$$

$$\leq C(T^{\theta+1} + T^\theta)\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)} \left\{(t_1-t_0)^{1-\theta} - t_1^{1-\theta} + t_0^{1-\theta}\right\}$$

for all $0 < \theta < 1$. Since $t_0 < t_1$, we have

$$\|(\mathcal{I}f)(t_1,\cdot) - (\mathcal{I}f)(t_0,\cdot)\|_{B_{r_1,1}^{s+2\theta}} \leq C(T^{\theta+1} + T^\theta)\|f\|_{L^\infty((0,T);B_{r_1,\infty}^s)}(t_1-t_0)^{1-\theta},$$

which implies the desired result. This proves Proposition B.1. $\qquad\qquad\square$

Proof of Proposition 4.1. (i) Let $0 < \varepsilon < 1$ and let $\chi_\varepsilon \in C^\infty(0,\infty)$ satisfy

$$0 \leq \chi_\varepsilon \leq 1, \quad \chi_\varepsilon(t) = \begin{cases} 0 & \text{for } t \in (0, \varepsilon/8), \\ 1 & \text{for } t \in (\varepsilon/4, \infty). \end{cases}$$

Noting that $f = (1 - \chi_\varepsilon)f + \chi_\varepsilon f$, we have

(B.1) $\quad (\mathcal{I}f)(t,\cdot) = e^{(t/2)\Delta}\int_0^{\varepsilon/4} e^{(t/2-\tau)\Delta}((1-\chi_\varepsilon(\tau))f(\tau,\cdot))d\tau + \int_0^t e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau)f(\tau,\cdot))d\tau$

for all $\varepsilon < t < \infty$. Concerning the first term of the right-hand side in (B.1), we see by the condition $\sup_{0<t<\infty}\int_0^t \|e^{(t-\tau)\Delta}f(\tau,\cdot)\|_{L^{r_0}}d\tau < \infty$, the embedding $B_{r_0,\infty}^{\sigma/2} \subset L^{r_0}$, and Proposition B.1 (i) that

$$\left\|\partial_t^j e^{(t/2)\Delta}\int_0^{\varepsilon/4} e^{(t/2-\tau)\Delta}((1-\chi_\varepsilon(\tau))f(\tau,\cdot))d\tau\right\|_{B_{r_1,1}^\sigma}$$

$$\leq CT^{\sigma/4}(t/2)^{-\sigma/4-j-(n/2)(1/r_0-1/r_1)} \left\|\int_0^{\varepsilon/4} (1-\chi_\varepsilon(\tau))e^{(t/2-\tau)\Delta}f(\tau,\cdot)d\tau\right\|_{B_{r_0,\infty}^{\sigma/2}}$$

$$\leq CT^{\sigma/4}\varepsilon^{-\sigma/4-j-(n/2)(1/r_0-1/r_1)} \int_0^{t/2} \|e^{(t/2-\tau)\Delta}f(\tau,\cdot)\|_{L^{r_0}}d\tau < \infty$$

for all $1 < T < \infty$, $\varepsilon < t < T$, $j \in \mathbb{N} \cup \{0\}$, and $0 < \sigma < \infty$. Therefore, we have

(B.2) $\qquad\qquad \int_0^{\varepsilon/4} e^{(t/2-\tau)\Delta}((1-\chi_\varepsilon(\tau))f(\tau,\cdot))d\tau \in \bigcap_{j\in\mathbb{N}} C^\infty((\varepsilon,\infty); W^{j,r_1}).$

In addition, since $\chi_\varepsilon = 0$ on $(0, \varepsilon/8)$ and since $f \in L^\infty_{\mathrm{loc}}((0, \infty); L^{r_1})$, it holds that

$$\chi_\varepsilon f \in L^\infty_{\mathrm{loc}}([0, \infty); L^{r_1}).$$

Noting that the heat semigroup $e^{t\Delta} : L^{r_1} \to L^{r_1}$ is an analytic semigroup, by applying [29, Proposition 4.2.1], we observe that

$$\int_0^t e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau) f(\tau, \cdot)) d\tau \in \bigcap_{0 < \theta < 1} C^{1-\theta}_{\mathrm{loc}}([0, \infty); (L^{r_1}, W^{2, r_1})_{\theta, 1}).$$

Since $(L^{r_1}, W^{2, r_1})_{\theta, 1} = B^{2\theta}_{r_1, 1}$ from [48, Remark 4, p.186], we have the desired result.

(ii) We consider the decomposition (B.1) again. Then, since $\chi_\varepsilon = 0$ on $(0, \varepsilon/8)$ and since $f \in L^\infty_{\mathrm{loc}}((0, \infty); B^s_{r_1, \infty})$, we have $\chi_\varepsilon f \in L^\infty_{\mathrm{loc}}([0, \infty); B^s_{r_1, \infty})$. Therefore, Proposition B.1 (ii) implies that

$$\int_0^t e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau) f(\tau, \cdot)) d\tau \in \bigcap_{0 < \theta < 1} C^{1-\theta}_{\mathrm{loc}}([0, \infty); B^{s+2\theta}_{r_1, 1}).$$

By combining the above regularity and (B.2), we have the desired result.

(iii) First we consider the case of $k = 0$. Take $0 < \varepsilon < 1$ sufficiently small. Since $\chi_\varepsilon = 0$ on $(0, \varepsilon/8)$ and since $f \in C^\gamma_{\mathrm{loc}}((0, \infty); BUC^m)$, it holds that $\chi_\varepsilon f \in C^\gamma([0, T]; BUC^m)$ for all $0 < T < \infty$. Therefore, since the heat semigroup $e^{t\Delta} : BUC^m \to BUC^m$ is an analytic semigroup, we see by [29, Theorem 4.3.1] that

$$\int_0^t e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau) f(\tau, \cdot)) d\tau \in C^{1+\gamma}_{\mathrm{loc}}((0, T]; BUC^m) \cap C^\gamma_{\mathrm{loc}}((0, T]; BUC^{m+2}).$$

Noting that (B.1) and (B.2), we deduce that

(B.3) $$\mathcal{I}f \in C^{1+\gamma}_{\mathrm{loc}}((0, \infty); BUC^m) \cap C^\gamma_{\mathrm{loc}}((0, \infty); BUC^{m+2}).$$

Next we consider the case of $k = 1$. Then it holds that $\partial_t f \in C^\gamma_{\mathrm{loc}}((0, \infty); BUC^m)$. Here we set

$$(\mathcal{J}_\varepsilon f)(t, \cdot) := \int_0^t e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau) \partial_\tau f(\tau, \cdot)) d\tau$$

for $0 < t < \infty$. Since $\chi_\varepsilon = 0$ on $(0, \varepsilon/8)$, we see that $\chi_\varepsilon \partial_t f \in C^\gamma([0, T]; BUC^m)$ for all $0 < T < \infty$. By the same argument as in the case of $k = 0$, we observe that

$$\mathcal{J}_\varepsilon f \in C^{1+\gamma}_{\mathrm{loc}}((0, T]; BUC^m) \cap C^\gamma_{\mathrm{loc}}((0, T]; BUC^{m+2})$$

as well. Moreover, we have

$$\partial_\tau(e^{(t-\tau)\Delta} f(\tau, \cdot)) = -\Delta e^{(t-\tau)\Delta} f(\tau, \cdot) + e^{(t-\tau)\Delta} \partial_\tau f(\tau, \cdot)$$

for all $0 < \tau < t < \infty$. Hence, by integrating from $\varepsilon/2$ to t, we obtain

$$f(t, \cdot) - e^{(t-\varepsilon/2)\Delta} f(\varepsilon/2, \cdot) = -\int_{\varepsilon/2}^t \Delta e^{(t-\tau)\Delta} f(\tau, \cdot) d\tau + \int_{\varepsilon/2}^t e^{(t-\tau)\Delta} \partial_\tau f(\tau, \cdot) d\tau$$

for all $\varepsilon < t < \infty$. Thus we deduce that

$$\Delta(\mathcal{I}f)(t, \cdot) = \left(\int_0^{\varepsilon/2} + \int_{\varepsilon/2}^t \right) \Delta e^{(t-\tau)\Delta} f(\tau, \cdot) d\tau$$

$$= \int_0^{\varepsilon/2} \Delta e^{(t-\tau)\Delta} f(\tau, \cdot) d\tau - f(t, \cdot) + e^{(t-\varepsilon/2)\Delta} f(\varepsilon/2, \cdot) + \int_{\varepsilon/2}^t e^{(t-\tau)\Delta} \partial_\tau f(\tau, \cdot) d\tau$$

with the aid of (B.3). Notice that

$$e^{(t-\varepsilon/2)\Delta}f(\varepsilon/2,\cdot) \in \bigcap_{j\in\mathbb{N}} C^{\infty}((\varepsilon,\infty); W^{j,r_1}).$$

In a similar manner to the proof of (B.2), we also see that

$$\int_0^{\varepsilon/2} \Delta e^{(t-\tau)\Delta} f(\tau,\cdot)d\tau \in \bigcap_{j\in\mathbb{N}} C^{\infty}((\varepsilon,\infty); W^{j,r_1}).$$

Moreover, since $\chi_\varepsilon = 0$ on $(0,\varepsilon/8)$ and $\chi_\varepsilon = 1$ on $(\varepsilon/2,\infty)$, we have

$$\int_{\varepsilon/2}^{t} e^{(t-\tau)\Delta}\partial_\tau f(\tau,\cdot)d\tau = (\mathcal{J}_\varepsilon f)(t,\cdot) - \int_{\varepsilon/8}^{\varepsilon/2} e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau)\partial_\tau f(\tau,\cdot))d\tau$$

for all $\varepsilon < t < \infty$. Therefore, since $\partial_t f \in C^{\gamma}_{\text{loc}}((0,\infty); BUC^m)$, it holds by Proposition B.1 (i) that

$$\int_{\varepsilon/8}^{\varepsilon/2} \|\partial_t^j e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau)\partial_\tau f(\tau,\cdot))\|_{B^{m+\sigma}_{\infty,1}}d\tau \leq CT^{\sigma/2}\int_{\varepsilon/8}^{\varepsilon/2}(t-\tau)^{-\sigma/2-j}\|\chi_\varepsilon(\tau)\partial_\tau f(\tau)\|_{B^m_{\infty,\infty}}d\tau$$

$$\leq CT^{\sigma/2}\varepsilon^{1-\sigma/2-j}\|\partial_t f\|_{L^{\infty}((\varepsilon/8,\varepsilon/2);BC^m)} < \infty$$

for all $1 < T < \infty$, $\varepsilon < t < T$, $0 < \sigma < \infty$, and $j \in \mathbb{N}\cup\{0\}$, which yields

$$\int_{\varepsilon/8}^{\varepsilon/2} e^{(t-\tau)\Delta}(\chi_\varepsilon(\tau)\partial_\tau f(\tau,\cdot))d\tau \in \bigcap_{j\in\mathbb{N}} C^{\infty}((\varepsilon,\infty); BUC^j).$$

Hence, by combining the above relations, we obtain $\Delta\mathcal{I}f \in C^{1+\gamma}_{\text{loc}}((0,\infty); BUC^m)$. In addition, by using (B.3), we also obtain $\mathcal{I}f \in C^{1+\gamma}_{\text{loc}}((0,\infty); BUC^{m+2})$. Noting that $\partial_t\mathcal{I}f = \Delta\mathcal{I}f + f$ and $f \in C^{1+\gamma}_{\text{loc}}((0,\infty); BUC^m)$, we have the desired result. This completes the proof of Proposition 4.1. $\qquad\square$

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