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Remarks on the smoothing effect of the heat semigroup on $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$

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ABSTRACT

We consider the heat semigroup $e^{t\Delta} : \dot{B}^s_{p,\infty}(\mathbb{R}^n) \to \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $s \in \mathbb{R}$. We show that $e^{t\Delta} f \in BC((0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ for all $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$. In addition, we reveal a sufficient condition of $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ for ensuring the C_0-property $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$. We also give examples of $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ which have the property $e^{t\Delta} f \notin BUC((0,1); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$.

1. Introduction

In this paper, we consider the *heat semigroup* defined on *homogeneous Besov spaces* $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ for $n \ge 1$, $1 \le p \le \infty$, and $s \in \mathbb{R}$. We begin with recalling the definition of the homogeneous Besov spaces: Let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space. Then we define the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by setting

$$(\mathcal{F}\psi)(\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\psi(x)dx, \quad (\mathcal{F}^{-1}\psi)(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\psi(\xi)d\xi$$

for $\psi \in \mathscr{S}(\mathbb{R}^n)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ satisfy the following conditions:

$$\begin{cases} \operatorname{supp}\varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \le |\xi| \le 2\}, \\ \varphi(\xi) > 0 \text{ for all } 1/2 < |\xi| < 2, \\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{1.1}$$

We introduce the homogeneous dyadic blocks $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ by setting $\dot{\Delta}_j := \mathcal{F}^{-1}\varphi(2^{-j}\cdot)\mathcal{F}$ for $j \in \mathbb{Z}$. Then, for $1 \le p \le \infty$, $s \in \mathbb{R}$, and $1 \le q \le \infty$, define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by setting $\dot{B}^s_{p,q}(\mathbb{R}^n) := \{f \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) \mid \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty\}$ with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left\{\sum_{j\in\mathbb{Z}} (2^{sj}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)})^q\right\}^{1/q} & \text{if } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty, \end{cases}$$

where $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{P}(\mathbb{R}^n)$ denote the dual space of $\mathscr{S}(\mathbb{R}^n)$ and the set of all polynomials on $\mathbb{R}^n$, respectively. We also set $\mathscr{S}_0(\mathbb{R}^n) := \{f \in \mathscr{S}(\mathbb{R}^n) \mid 0 \notin \operatorname{supp}\mathcal{F}f\}$. Next we introduce the heat semigroup by setting $e^{t\Delta} := \mathcal{F}^{-1}e^{-t|\cdot|^2}\mathcal{F}$ for $0 < t < \infty$. Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ may be extended to the operators from $\mathscr{S}'(\mathbb{R}^n)$ onto itself, in what follows, we focus on the heat semigroup $e^{t\Delta} : \dot{B}^s_{p,\infty}(\mathbb{R}^n) \to \dot{B}^s_{p,\infty}(\mathbb{R}^n)$.

Our purpose of this paper is to show that

$$e^{t\Delta} f \in BC((0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n)) \tag{1.2}$$

for all $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$. In addition, if $\lim_{j\to\infty} 2^{sj}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} = 0$ holds, then we prove $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$. Next we give *examples* of $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ such that for any $0 < t < \infty$, the functions $e^{t\Delta} f$ *may not be approximated* in the topology of $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ by functions belonging to $\mathscr{S}_0(\mathbb{R}^n)$. We also reveal that $e^{t\Delta} f \notin BUC((0,1); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ for such examples.

To describe the details of our results, we shall recall the properties of the heat semigroup defined on homogeneous Besov spaces $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$. Kozono, Ogawa, and Taniuchi Ref. 1, Lemma 2.2 (ii) showed that for every $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$, it holds that $e^{t\Delta} f \in \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ for all $0 < t < \infty$ and $0 < \sigma < \infty$. This implies that $e^{t\Delta} f$ has *sufficient regularity*, so we may regard it as a *smoothing effect* of the heat semigroup. In addition, recently, the author Ref. 2, Theorems 3.1 and 3.3 improved the estimate given in Ref. 1, Lemma 2.2 (ii) and showed the *space–time analytic smoothing effect* of the heat semigroup. More precisely, for every $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$, the function $e^{t\Delta} f$ becomes *real-analytic* in $(0,\infty)\times\mathbb{R}^n$. Moreover, if $1 \le p < \infty$ and $1 \le q < \infty$, then the heat semigroup $e^{t\Delta} : \dot{B}^s_{p,q}(\mathbb{R}^n) \to \dot{B}^s_{p,q}(\mathbb{R}^n)$ is a bounded analytic C_0-semigroup Ref. 2, Theorem 3.6 (ii). In particular, it holds that $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,q}(\mathbb{R}^n))$ for all $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. Besides, Ozawa and the author Ref. 3, Theorem 1.1 refined the estimate given in Ref. 2, Lemma 5.1. As a by-product, we also refined the lower bound of the radius of convergence given in Ref. 2, Theorem 3.6 (i).

Next we focus on the case of $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$. In this case, it may *no longer* be expected that the heat semigroup is a C_0-semigroup. One of the typical ways to give some sense of the limit $\lim_{t\to+0} e^{t\Delta} f$ is to consider the *dual spaces*. In fact, we see that $e^{t\Delta} f \in BC_w([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ for all $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ and $1 < p \le \infty$ with the property

$$\lim_{t\to+0} \left\langle e^{t\Delta} f - f, \psi \right\rangle = 0 \quad \text{for all } \psi \in \dot{B}^{-s}_{p',1}(\mathbb{R}^n),$$

where $p' := p/(p-1)$. We refer to several results[4–10] for the construction of mild solutions to the parabolic PDEs with initial data in $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$. In

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these results, the dual spaces are used to verify that solutions converge to the initial data as $t \to +0$.

As mentioned above, it is not expected that $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ in general. It may be regarded that this claim is similar to the fact that $e^{t\Delta} f \notin BC([0,\infty); L^\infty(\mathbb{R}^n))$ for some $f \in L^\infty(\mathbb{R}^n)$. However, even so, we have $e^{t\Delta} f \in BC((0,\infty); L^\infty(\mathbb{R}^n))$ for all $f \in L^\infty(\mathbb{R}^n)$. The reason why such a property is obtained comes from the fact that $e^{t\Delta} f \in BUC(\mathbb{R}^n)$ for all $0 < t < \infty$ and $f \in L^\infty(\mathbb{R}^n)$. Therefore, even though the heat semigroup $e^{t\Delta} : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is *not* a C_0-semigroup, we see that $e^{t\Delta} f$ is *continuous* on $(0,\infty)$ in the topology of $L^\infty(\mathbb{R}^n)$. Hence, our results which state that (1.2) may be regarded as a *corresponding result* to the case of $L^\infty(\mathbb{R}^n)$. Here note that the author Ref. 2, Theorem 3.6 (iii) has already shown that the bounded linear operator $e^{t\Delta} : \dot{B}^s_{p,\infty}(\mathbb{R}^n) \to \dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ is real-analytic in $(0,\infty)$ for all $0 < \sigma < \infty$. However, in this case, the topology had to be replaced by that of the *different* spaces $\dot{B}^{s+\sigma}_{p,1}(\mathbb{R}^n)$ with that of the original one $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$.

In our results, we show that the property $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ is valid under the condition $\lim_{j\to\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} = 0$. In other words, we do *not* have to assume the condition

$$\lim_{j\to-\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} = 0, \tag{1.3}$$

which implies that there exists a function $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ such that $e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n))$ holds even though f may not be approximated in the topology of $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$ by functions belonging to $\mathscr{S}_0(\mathbb{R}^n)$. In addition, we also give examples of f such that $e^{t\Delta} f$ may not be approximated by functions belonging to $\mathscr{S}_0(\mathbb{R}^n)$. Such a phenomenon may be described by analyzing the smoothing effect of the heat semigroup precisely. From our results, we observe that

$$\lim_{j\to\infty} 2^{sj} \|\dot{\Delta}_j e^{t\Delta} f\|_{L^p(\mathbb{R}^n)} = 0 \tag{1.4}$$

for all $0 < t < \infty$ and $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$. However, we do not know whether *another limit* $j \to -\infty$ converges to 0. In fact, we may find examples of f such that

$$\liminf_{j\to-\infty} 2^{sj} \|\dot{\Delta}_j e^{t\Delta} f\|_{L^p(\mathbb{R}^n)} > 0$$

for all $0 < t < \infty$. So these results are given by the precise analysis of the smoothing effect that the heat semigroup has.

1.1. Main results

Here we shall state our main results. Recall that φ is the function satisfying (1.1). Let BC, BUC, and C^ω denote the space of the bounded continuous functions, the space of the bounded uniformly continuous functions, and the set of the real-analytic functions, respectively. In addition, we set the Sobolev spaces $H^m(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \mid \|f\|_{H^m(\mathbb{R}^n)} := \sum_{|\alpha|\le m} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} < \infty\}$ for $m \in \mathbb{N}$.

Theorem 1.1 (*Sufficient Condition for the C_0-Property*). *Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Then the following statements hold:*

(i) Suppose that $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$. Then it holds that

$$\begin{aligned} &2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p(\mathbb{R}^n)} \\ &\le C_n (1 - \exp(-2^{2j+4} t))^{1/2} \|\varphi\|_{H^n(\mathbb{R}^n)} \cdot 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} \end{aligned} \tag{1.5}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$, where $C_n > 0$ is a constant depending only on n.

(ii) Let $1 \le q \le \infty$ and suppose that $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. Then it holds that

$$\begin{cases} \displaystyle\lim_{t\to+0} \sum_{j=-\infty}^{j_0} (2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p(\mathbb{R}^n)})^q = 0 & \text{if } 1 \le q < \infty, \\ \displaystyle\lim_{t\to+0} \sup_{-\infty<j\le j_0} 2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p(\mathbb{R}^n)} = 0 & \text{if } q = \infty \end{cases} \tag{1.6}$$

for all $j_0 \in \mathbb{Z}$.

(iii) Suppose that $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$ satisfies

$$\lim_{j\to\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} = 0. \tag{1.7}$$

Then it holds that

$$e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{p,\infty}(\mathbb{R}^n)). \tag{1.8}$$

In addition, suppose that $1 \le q < \infty$. Then for every $f \in \dot{B}^s_{\infty,q}(\mathbb{R}^n)$, it holds that

$$e^{t\Delta} f \in BC([0,\infty); \dot{B}^s_{\infty,q}(\mathbb{R}^n)). \tag{1.9}$$

Remark 1.2. (i) It is known that the following relation holds for all $1 \le p < \infty$ and $s \in \mathbb{R}$:

$$\begin{aligned} \dot{B}^s_{p,\infty+}(\mathbb{R}^n) &:= \left\{ f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n) \;\middle|\; \begin{array}{l} \exists \{f_k\}_{k\in\mathbb{N}} \subset \mathscr{S}_0(\mathbb{R}^n), \\ \lim_{k\to\infty} \|f - f_k\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} = 0 \end{array} \right\} \\ &= \left\{ f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n) \;\middle|\; \lim_{j\to\pm\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)} = 0 \right\}. \end{aligned} \tag{1.10}$$

We will verify that (1.10) in the last part of this paper. If we assume that $f \in \dot{B}^s_{p,\infty+}(\mathbb{R}^n)$, then Ref. 2, Theorem 3.6 (ii') implies that the C_0-property (1.8) is valid. Compared with this result, we see by Theorem 1.1 (iii) that it is *sufficient* to assume that (1.7) to obtain the C_0-property (1.8). Hence, the relation (1.10) implies that we have *improved* the result of Ref. 2, Theorem 3.6 (ii') since we do *not* have to assume that (1.3). In particular, we do not have to take a function f which may be approximated by functions belonging to $\mathscr{S}_0(\mathbb{R}^n)$. We refer to Sawada Ref. 11, Definition 4 for comments related to the C_0-property of the heat semigroup on homogeneous and non-homogeneous Besov spaces.

(ii) The assertion such that we do not have to assume that (1.3) is *natural* in some sense: In our results, we focus on the heat semigroup on homogeneous Besov spaces $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$. Here let us consider the non-homogeneous Besov spaces $B^s_{p,\infty}(\mathbb{R}^n)$, namely, let us consider the following norm;

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \|\mathcal{F}^{-1}[\chi \mathcal{F} f]\|_{L^p(\mathbb{R}^n)} + \sup_{0\le j<\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^n)},$$

where $\chi := 1 - \sum_{0\le j<\infty} \varphi(2^{-j}\cdot)$. In this case, we may expect that the condition (1.3) is not necessary. Moreover, we may regard that the condition (1.7) corresponds to the assumption appearing in Ref. 7, Theorem 1.3, Ref. 4, Theorem 2.3, Ref. 12, Theorem 2 (ii), Ref. 5, Theorem 4 (2), and Ref. 10, Theorem 2.3. In these results, it is assumed that the *high-frequency part* of the initial data is small to ensure the existence of mild solutions. Similarly to our assumption, it is not assumed the smallness condition of the low-frequency part.

(iii) As a by-product of the proof of the C_0-property (1.8) under the condition (1.7), we also see that (1.9) for all $f \in \dot{B}^s_{\infty,q}(\mathbb{R}^n)$ although it is known that $e^{t\Delta} f \notin BC([0,\infty); L^\infty(\mathbb{R}^n))$ for some $f \in L^\infty(\mathbb{R}^n)$. This claim provides a clear difference between $L^\infty(\mathbb{R}^n)$ and $\dot{B}^s_{\infty,q}(\mathbb{R}^n)$, but we should mention that the assertion (1.9) is a known result. In fact, Iwabuchi Ref. 13, Theorem 1.2 has already shown that the C_0-property corresponding to (1.9) holds in a more general situation. We also note that the method of the proof of Theorem 1.1 is similar to that of Ref. 13, Theorem 1.1.

Theorem 1.3 (*Smoothing Effect of the Heat Semigroup on $\dot{B}^s_{p,\infty}(\mathbb{R}^n)$*). *Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. Suppose that $f \in \dot{B}^s_{p,\infty}(\mathbb{R}^n)$. Then the following statements hold:*

(i) It holds that

$$2^{sj} \|\dot{\Delta}_j e^{t\Delta} f\|_{L^p(\mathbb{R}^n)} \le C_n \exp(-2^{2j-5} t) \|\varphi\|_{H^n(\mathbb{R}^n)} \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^n)} \tag{1.11}$$

for all $0 < t < \infty$, where $C_n > 0$ is a constant depending only on n.

(ii) It holds that

$$e^{t\Delta}f \in BC((0,\infty);\dot B^s_{p,\infty}(\mathbb{R}^n)) \cap \bigcap_{m\in\mathbb{N}} BC^m((\varepsilon,\infty);\dot B^s_{p,\infty}(\mathbb{R}^n)) \cap C^\omega((0,\infty);\dot B^s_{p,\infty}(\mathbb{R}^n))$$

for all $0<\varepsilon<1$.

Remark 1.4. (i) Theorem 1.3 (i) states that (1.4) holds for any $f \in \dot B^s_{p,\infty}(\mathbb{R}^n)$. Namely, the function $e^{t\Delta}f$ satisfies the assumption for the C_0-property in Theorem 1.1 (iii). Therefore, we see that (1.2). As mentioned before, in the case of $f \in \dot B^s_{p,\infty}(\mathbb{R}^n)$, we consider the *dual space* to obtain the continuity of $e^{t\Delta}f$ in some sense. Indeed, we may obtain the weak-star continuity, so such a way is used to obtain the time-continuity of mild solutions to the parabolic PDEs in several papers.[4–9] However, Iwabuchi and Nakamura Ref. 14, Remark 1.3.2 have pointed out that we obtain (1.2) even though $f \in \dot B^s_{p,\infty}(\mathbb{R}^n)$. We give proof in detail of such a claim.

(ii) Concerning Theorem 1.3 (ii), the author Ref. 2, Theorem 3.6 (ii) has already shown that $e^{t\Delta} : \dot B^s_{p,q}(\mathbb{R}^n) \to \dot B^s_{p,q}(\mathbb{R}^n)$ is a bounded analytic C_0-semigroup under the conditions $1\le p<\infty$ and $1\le q<\infty$. What we have to do to show Theorem 1.3 (ii) is to verify that $e^{t\Delta}f \in C^\infty((0,\infty);\dot B^s_{p,\infty}(\mathbb{R}^n))$. Once we may show such a property, we have $e^{t\Delta}f \in C^\omega((0,\infty);\dot B^s_{p,\infty}(\mathbb{R}^n))$ in a similar manner to the proof of Ref. 2, Theorem 3.6 (ii). Moreover, by combining the method of the proof of Ref. 2, Theorem 3.6 (i) and the result of Ref. 3, Theorem 1.3, we also see that $e^{t\Delta} : \dot B^s_{p,\infty}(\mathbb{R}^n) \to \dot B^s_{p,\infty}(\mathbb{R}^n)$ may be extended analytically on

$$\Sigma := \left\{ t\in\mathbb{C}\setminus\{0\} \,\middle|\, |\arg t| < \sin^{-1}\left(\left(\frac{2e}{n}\right)^{n/2}\frac{\Gamma(n/2)}{2^{11}e}\right)\right\},$$

where $\Gamma(a) := \int_0^\infty \lambda^{a-1}e^{-\lambda}d\lambda$.

Theorem 1.5 (*Counterexamples*). *Let* $1\le p\le\infty$ *and* $0<\sigma<n$ *and let* $\delta \in \dot B^{-n+n/p}_{p,\infty}(\mathbb{R}^n)$ *denote the Dirac measure. Define* $f_0 := \delta$ *and* $f_1 := |\cdot|^{-\sigma} \in \dot B^{-\sigma+n/p}_{p,\infty}(\mathbb{R}^n)$. *Then it holds that*

$$2^{s_m j}\|\dot\Delta_j e^{t\Delta}f_m\|_{L^p(\mathbb{R}^n)} \ge C_{\sigma,n}\exp(-2^{2j+2}t)\|\varphi\|_{L^1(\mathbb{R}^n)}$$

for all $0<t<\infty$, $j\in\mathbb{Z}$, *and* $m=0,1$, *where* $s_0 := -n+n/p$ *and* $s_1 := -\sigma+n/p$ *and* $C_{\sigma,n}>0$ *is a constant depending only on* σ *and* n. *In particular, it holds that*

$$\liminf_{j\to-\infty} 2^{s_m j}\|\dot\Delta_j e^{t\Delta}f_m\|_{L^p(\mathbb{R}^n)} > 0. \tag{1.12}$$

Moreover, it holds that

$$e^{t\Delta}f_m \notin BUC((0,1);\dot B^{s_m}_{p,\infty}(\mathbb{R}^n)). \tag{1.13}$$

Remark 1.6. (i) From Theorem 1.3 (i), we see that the heat semigroup has a smoothing effect such that (1.4) holds for any $f\in\dot B^s_{p,\infty}(\mathbb{R}^n)$. However, we do *not* know whether

$$\lim_{j\to-\infty} 2^{sj}\|\dot\Delta_j e^{t\Delta}f\|_{L^p(\mathbb{R}^n)} = 0$$

holds or not. Concerning this problem, Theorem 1.5 gives *counterexamples* which satisfy (1.12). Therefore, the relation (1.10) implies that the function $e^{t\Delta}f_m$ may not be approximated by functions belonging to $\mathscr{S}_0(\mathbb{R}^n)$. In addition, the property (1.12) states that $e^{t\Delta}f_m \notin \dot B^{s_m}_{p,q}(\mathbb{R}^n)$ for any $1\le q<\infty$ and $0<t<\infty$. Since it holds by Ref. 1, Lemma 2.2 (ii) that $e^{t\Delta}f_m \in \dot B^{s_m+\sigma}_{p,1}(\mathbb{R}^n)$ for all $0<\sigma<\infty$, we also see that $e^{t\Delta}f_m \notin \dot B^{s_m-\sigma}_{p,\infty}(\mathbb{R}^n)$ for any $0<\sigma<\infty$ with the aid of Ref. 15, Proposition 2.22.

(ii) The property (1.13) implies that $e^{t\Delta}f_m \notin BC([0,1];\dot B^{s_m}_{p,\infty}(\mathbb{R}^n))$, so we may verify that $e^{t\Delta} : \dot B^{s_m}_{p,\infty}(\mathbb{R}^n) \to \dot B^{s_m}_{p,\infty}(\mathbb{R}^n)$ is *not* a C_0-semigroup certainly. However, we note that Theorem 1.3 (ii) yields $e^{t\Delta}f_m \in BC((0,\infty);\dot B^{s_m}_{p,\infty}(\mathbb{R}^n))$.

2. Proof of main results

In the following, we shall show our main results, Theorems 1.1, 1.3, and 1.5. The method of the proof of Theorems 1.1 and 1.3 is mainly based on analyzing the *symbol* $e^{-t|\cdot|^2}$ of the Fourier multiplier operator $e^{t\Delta} = \mathcal{F}^{-1}e^{-t|\cdot|^2}\mathcal{F}$. Since we consider only the case of the whole space $\mathbb{R}^n$, in what follows, we abbreviate like $L^p := L^p(\mathbb{R}^n)$, $H^m := H^m(\mathbb{R}^n)$, and $\dot B^s_{p,q} := \dot B^s_{p,q}(\mathbb{R}^n)$.

2.1. Preliminaries

Here we verify the basic properties of the function φ satisfying (1.1) and the homogeneous dyadic blocks $\{\dot\Delta_j\}_{j\in\mathbb{Z}}$. First we note that

$$\|\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)\psi]\|_{L^p} = 2^{(n-n/p)j}\|\mathcal{F}^{-1}[\varphi\psi(2^j\cdot)]\|_{L^p} \tag{2.1}$$

for all $j\in\mathbb{Z}$, $1\le p\le\infty$, and $\psi\in C^\infty(\mathbb{R}^n\setminus\{0\})$ from simple substitutions. In addition, noting that (1.1) yields $\operatorname{supp}\varphi(2^{-j}\cdot) = \{\xi\in\mathbb{R}^n \mid 2^{j-1}\le|\xi|\le 2^{j+1}\}$, we observe that

$$\begin{cases} \dot\Delta_j\dot\Delta_k = 0 \text{ for } |j-k|\ge 2, \\ \dot\Delta_j\psi = \displaystyle\sum_{k=j-1}^{j+1}\dot\Delta_j\dot\Delta_k\psi \text{ for } j\in\mathbb{Z} \text{ and } \psi\in\mathscr{S}'. \end{cases} \tag{2.2}$$

To show Theorems 1.1 and 1.3, let us verify Lemma 2.1, which provides the estimates of the symbol $e^{-t|\cdot|^2}$. Here we define the homogeneous Sobolev spaces $\dot H^m := \{f\in\mathscr{S}' \mid \|f\|_{\dot H^m} := \sum_{|\alpha|\le m}\|\partial_x^\alpha f\|_{L^2} < \infty\}$ for $m\in\mathbb{N}$.

Lemma 2.1. *(i) It holds that*

$$\max_{|\alpha|=j}|\partial_\xi^\alpha e^{-t|\xi|^2}| \le 2^{2j}t(1+t)^{j-1}(1+|\xi|)^j e^{-t|\xi|^2} \tag{2.3}$$

for all $0<t<\infty$, $\xi\in\mathbb{R}^n$, *and* $j\in\mathbb{N}$.

(ii) It holds that

$$\|\varphi(e^{-t|\cdot|^2}-\sigma)\|_{\dot H^m} \le C_{m,n}(e^{-t/8}+|\sigma|)\|\varphi\|_{H^m},$$

$$\|\varphi(t^{-1}(e^{-t|\cdot|^2}-1)+|\cdot|^2)\|_{\dot H^m} \le C_{m,n}\|\varphi\|_{H^m}$$

for all $0<t<\infty$, $\sigma\in\mathbb{R}$, *and* $m\in\mathbb{N}$, *where* $C_{m,n}>0$ *is a constant depending only on* m *and* n.

(iii) It holds that

$$\|\mathcal{F}^{-1}[\varphi\exp(-2^{2k}t|\cdot|^2)]\|_{L^1} \le C_n\exp(-2^{2k-3}t)\|\varphi\|_{H^n},$$

$$\|\mathcal{F}^{-1}[\varphi(\exp(-2^{2k}t|\cdot|^2)-1)]\|_{L^1} \le C_n(1-\exp(-2^{2k+2}t))^{1/2}\|\varphi\|_{H^n}$$

and

$$\left\|\mathcal{F}^{-1}\left[\varphi\left\{t^{-1}(\exp(-2^{2k}t|\cdot|^2)-1)+2^{2k}|\cdot|^2\right\}\right]\right\|_{L^1}$$
$$\le C_n 2^{2k+1}\left\{1-(2^{2k+2}t)^{-1}(1-\exp(-2^{2k+2}t))\right\}^{1/2}\|\varphi\|_{H^n}$$

for all $0<t<\infty$ *and* $k\in\mathbb{Z}$, *where* $C_n>0$ *is a constant depending only on* n.

Proof. (i) In case $j=1$, the estimate (2.3) trivially holds for all $0<t<\infty$ and $\xi\in\mathbb{R}^n$. Assume that (2.3) holds for $j=j_0$. Then, for any $\alpha\in(\mathbb{N}\cup\{0\})^n$ satisfying $|\alpha|=j_0$, it holds by (2.3) that

$$\begin{aligned}|\partial_{\xi_k}\partial_\xi^\alpha e^{-t|\xi|^2}| &= |\partial_\xi^\alpha(2t\xi_k e^{-t|\xi|^2})| \le 2t|\partial_\xi^\alpha\xi_k|e^{-t|\xi|^2} + 2t|\xi_k||\partial_\xi^\alpha e^{-t|\xi|^2}| \\ &\le 2t(1+|\xi|)e^{-t|\xi|^2} + 2t|\xi|\cdot 2^{2j_0}t(1+t)^{j_0-1}(1+|\xi|)^{j_0}e^{-t|\xi|^2} \\ &\le 2^{2j_0+2}t(1+t)^{j_0}(1+|\xi|)^{j_0+1}e^{-t|\xi|^2}\end{aligned}$$

for all $1\le k\le n$, $0<t<\infty$, and $\xi\in\mathbb{R}^n$. Thus we see that (2.3) holds for all $j\in\mathbb{N}$.

(ii) Since φ satisfies the condition (1.1), it suffices to consider the case of $1/2\le|\xi|\le 2$. Let $\alpha\in(\mathbb{N}\cup\{0\})^n$ satisfy $|\alpha|=m$. Then, by combining (2.3) and the Leibniz rule, we have

$$\begin{aligned}&|\partial_\xi^\alpha\{\varphi(\xi)(e^{-t|\xi|^2}-\sigma)\}| \\ &\le |\partial_\xi^\alpha\varphi(\xi)||e^{-t|\xi|^2}-\sigma| + C_{m,n}\sum_{\beta<\alpha}|\partial_\xi^\beta\varphi(\xi)||\partial_\xi^{\alpha-\beta}e^{-t|\xi|^2}| \\ &\le (e^{-t/4}+|\sigma|)|\partial_\xi^\alpha\varphi(\xi)| \\ &\quad + C_{m,n}\sum_{\beta<\alpha}|\partial_\xi^\beta\varphi(\xi)|\cdot 2^{2(|\alpha|-|\beta|)}t(1+t)^{|\alpha|-|\beta|-1}(1+|\xi|)^{|\alpha|-|\beta|}e^{-t|\xi|^2} \\ &\le (e^{-t/4}+|\sigma|)|\partial_\xi^\alpha\varphi(\xi)| + C_{m,n}e^{-t/4}t(1+t)^{m-1}\sum_{\beta<\alpha}|\partial_\xi^\beta\varphi(\xi)|\end{aligned}$$

for all $0 < t < \infty$ and $\sigma \in \mathbb{R}$. Noting that $\sup_{0<t<\infty} e^{-t/8} t(1+t)^{m-1} < \infty$, we observe that

$$|\partial_\xi^\alpha \{\varphi(\xi)(e^{-t|\xi|^2} - \sigma)\}| \le C_{m,n}(e^{-t/8} + |\sigma|) \sum_{\beta\le\alpha} |\partial_\xi^\beta \varphi(\xi)|,$$

which yields $\|\varphi(e^{-t|\cdot|^2} - \sigma)\|_{\dot{H}^m} \le C_{m,n}(e^{-t/8} + |\sigma|)\|\varphi\|_{H^m}$. Similarly, we also obtain

$$\begin{aligned}
&|\partial_\xi^\alpha \{\varphi(\xi)(t^{-1}(e^{-t|\xi|^2} - 1) + |\xi|^2)\}| \\
&\le |\partial_\xi^\alpha \varphi(\xi)| \left| t^{-1}(e^{-t|\xi|^2} - 1) + |\xi|^2 \right| \\
&\quad + C_{m,n} \sum_{\beta<\alpha} |\partial_\xi^\beta \varphi(\xi)| \left(t^{-1} |\partial_\xi^{\alpha-\beta} e^{-t|\xi|^2}| + |\partial_\xi^{\alpha-\beta} |\xi|^2| \right) \\
&\le |\partial_\xi^\alpha \varphi(\xi)| \left| |\xi|^2 - t^{-1}(1 - e^{-t|\xi|^2}) \right| \\
&\quad + C_{m,n} \sum_{\beta<\alpha} |\partial_\xi^\beta \varphi(\xi)| \left(2^{2(|\alpha|-|\beta|)}(1+t)^{|\alpha|-|\beta|-1}(1+|\xi|)^{|\alpha|-|\beta|} e^{-t|\xi|^2} + 2(1+|\xi|) \right) \\
&\le |\partial_\xi^\alpha \varphi(\xi)| \left| |\xi|^2 - t^{-1}(1 - e^{-t|\xi|^2}) \right| + C_{m,n}(e^{-t/4}(1+t)^{m-1} + 1) \sum_{\beta<\alpha} |\partial_\xi^\beta \varphi(\xi)|.
\end{aligned}$$

Here we set $\psi(t,r) := r - t^{-1}(1 - e^{-tr})$ for $t, r \in (0,\infty)$. Then we see by $\partial_t \psi(t,r) = t^{-2} e^{-tr}(tr + e^{tr} - 1) > 0$ and $\lim_{t\to+0} \psi(t,r) = 0$ that

$$0 < \psi(t_0, r) < \psi(t_1, r) \tag{2.4}$$

for all $0 < t_0 < t_1 < \infty$ and $0 < r < \infty$. In addition, since $\partial_r \psi(t,r) = 1 - e^{-tr} > 0$ yields $\sup_{1/4\le r\le 4} \psi(t,r) = \psi(t,4) = 4 - t^{-1}(1 - e^{-4t})$, we have

$$\sup_{1/2\le|\xi|\le 2} \left| |\xi|^2 - t^{-1}(1 - e^{-t|\xi|^2}) \right| \le 4 - t^{-1}(1 - e^{-4t}) \le 4 \tag{2.5}$$

for all $0 < t < \infty$. Noting that $\sup_{0<t<\infty} e^{-t/4}(1+t)^{m-1} < \infty$, we observe that

$$|\partial_\xi^\alpha \{\varphi(\xi)(t^{-1}(e^{-t|\xi|^2} - 1) + |\xi|^2)\}| \le C_{m,n} \sum_{\beta\le\alpha} |\partial_\xi^\beta \varphi(\xi)|,$$

which yields the desired estimate.

(iii) We see by Ref. 3, Proposition 3.2 that

$$\begin{aligned}
&\|\mathcal{F}^{-1}[\varphi \exp(-2^{2k} t|\cdot|^2)]\|_{L^1} \\
&\le C_n \|\varphi \exp(-2^{2k} t|\cdot|^2)\|_{L^2}^{1/2} \|\varphi \exp(-2^{2k} t|\cdot|^2)\|_{\dot{H}^n}^{1/2}
\end{aligned}$$

for all $0 < t < \infty$ and $k \in \mathbb{Z}$. Since it holds by the assertion of (ii) and (1.1) that

$$\|\varphi \exp(-2^{2k} t|\cdot|^2)\|_{L^2} \le \exp(-2^{2k-2} t)\|\varphi\|_{L^2},$$

$$\|\varphi \exp(-2^{2k} t|\cdot|^2)\|_{\dot{H}^n} \le C_n \exp(-2^{2k} t/8)\|\varphi\|_{H^n} = C_n \exp(-2^{2k-3} t)\|\varphi\|_{H^n},$$

we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}[\varphi \exp(-2^{2k} t|\cdot|^2)]\|_{L^1} \\
&\le C_n \cdot (\exp(-2^{2k-2} t)\|\varphi\|_{L^2})^{1/2} \cdot (\exp(-2^{2k-3} t)\|\varphi\|_{H^n})^{1/2} \\
&\le C_n \exp(-2^{2k-3} t)\|\varphi\|_{H^n}.
\end{aligned}$$

Similarly, by applying Ref. 3, Proposition 3.2 and the assertion of (ii), we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}[\varphi(\exp(-2^{2k} t|\cdot|^2) - 1)]\|_{L^1} \\
&\le C_n \|\varphi(\exp(-2^{2k} t|\cdot|^2) - 1)\|_{L^2}^{1/2} \|\varphi(\exp(-2^{2k} t|\cdot|^2) - 1)\|_{\dot{H}^n}^{1/2} \\
&\le C_n \cdot ((1 - \exp(-2^{2k+2} t))\|\varphi\|_{L^2})^{1/2} \cdot \|\varphi\|_{H^n}^{1/2} \\
&\le C_n (1 - \exp(-2^{2k+2} t))^{1/2} \|\varphi\|_{H^n}
\end{aligned}$$

for all $0 < t < \infty$ and $k \in \mathbb{Z}$. Moreover, we also obtain

$$\begin{aligned}
&\left\| \mathcal{F}^{-1} \left[\varphi \left\{ t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2 \right\} \right] \right\|_{L^1} \\
&\le C_n \|\varphi(t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2)\|_{L^2}^{1/2} \\
&\quad \times \|\varphi(t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2)\|_{\dot{H}^n}^{1/2}
\end{aligned}$$

for all $0 < t < \infty$ and $k \in \mathbb{Z}$. Here, by recalling (2.5), we observe that

$$\begin{aligned}
&\|\varphi(t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2)\|_{L^2} \\
&= 2^{2k} \|\varphi((2^{2k} t)^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + |\cdot|^2)\|_{L^2} \\
&\le 2^{2k} \cdot (4 - (2^{2k} t)^{-1}(1 - \exp(-4 \cdot 2^{2k} t)))\|\varphi\|_{L^2} \\
&= (2^{2k+2} - t^{-1}(1 - \exp(-2^{2k+2} t)))\|\varphi\|_{L^2}.
\end{aligned}$$

Since the assertion of (ii) yields

$$\begin{aligned}
&\|\varphi(t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2)\|_{\dot{H}^n} \\
&= 2^{2k} \|\varphi((2^{2k} t)^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + |\cdot|^2)\|_{\dot{H}^n} \le C_n 2^{2k} \|\varphi\|_{H^n}
\end{aligned}$$

for all $0 < t < \infty$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}
&\left\| \mathcal{F}^{-1} \left[\varphi \left\{ t^{-1}(\exp(-2^{2k} t|\cdot|^2) - 1) + 2^{2k}|\cdot|^2 \right\} \right] \right\|_{L^1} \\
&\le C_n \cdot \left\{ (2^{2k+2} - t^{-1}(1 - \exp(-2^{2k+2} t)))\|\varphi\|_{L^2} \right\}^{1/2} \cdot (2^{2k} \|\varphi\|_{H^n})^{1/2} \\
&\le C_n 2^{2k+1} \left\{ 1 - (2^{2k+2} t)^{-1}(1 - \exp(-2^{2k+2} t)) \right\}^{1/2} \|\varphi\|_{H^n},
\end{aligned}$$

which completes the proof of Lemma 2.1. □

2.2. *Smoothing effect of the heat semigroup: Proof of Theorems 1.1 and 1.3*

By using the estimates given in Lemma 2.1, we shall show our main results, Theorems 1.1 and 1.3.

Proof of Theorem 1.1. (i) Noting that (2.1) and (2.2), we see by the Young convolution inequality that

$$\begin{aligned}
&2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p} \\
&\le 2^{sj} \sum_{k=j-1}^{j+1} \|\dot{\Delta}_j \dot{\Delta}_k (e^{t\Delta} f - f)\|_{L^p} \\
&= 2^{sj} \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi(2^{-k}\cdot)(e^{-t|\cdot|^2} - 1)] * (\dot{\Delta}_j f)\|_{L^p} \\
&\le \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi(\exp(-2^{2k} t|\cdot|^2) - 1)]\|_{L^1} \cdot 2^{sj} \|\dot{\Delta}_j f\|_{L^p}
\end{aligned} \tag{2.6}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$. In addition, we see by Lemma 2.1 (iii) that

$$\begin{aligned}
&\sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi(\exp(-2^{2k} t|\cdot|^2) - 1)]\|_{L^1} \\
&\qquad \le C_n \sum_{k=j-1}^{j+1} (1 - \exp(-2^{2k+2} t))^{1/2} \|\varphi\|_{H^n} \\
&\qquad \le C_n (1 - \exp(-2^{2j+4} t))^{1/2} \|\varphi\|_{H^n}
\end{aligned}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$, which yields (1.5).

(ii) Let $j_0 \in \mathbb{Z}$. If $1 \le q < \infty$, then it holds by (1.5) that

$$\begin{aligned}
&\sum_{j=-\infty}^{j_0} (2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p})^q \\
&\le C_n (1 - \exp(-2^{2j_0+4} t))^{q/2} \|\varphi\|_{H^n}^q \sum_{j=-\infty}^{j_0} (2^{sj} \|\dot{\Delta}_j f\|_{L^p})^q \\
&\le C_n (1 - \exp(-2^{2j_0+4} t))^{q/2} \|\varphi\|_{H^n}^q \|f\|_{\dot{B}^s_{p,q}}^q.
\end{aligned}$$

If $q = \infty$, then we also see by (1.5) that

$$\begin{aligned}
&\sup_{-\infty<j\le j_0} 2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p} \\
&\le C_n (1 - \exp(-2^{2j_0+4} t))^{1/2} \|\varphi\|_{H^n} \sup_{-\infty<j\le j_0} 2^{sj} \|\dot{\Delta}_j f\|_{L^p} \\
&\le C_n (1 - \exp(-2^{2j_0+4} t))^{1/2} \|\varphi\|_{H^n} \|f\|_{\dot{B}^s_{p,\infty}}.
\end{aligned}$$

Therefore, by letting $t \to +0$, we have the desired result.

(iii) Let $f \in \dot{B}^s_{p,\infty}$. Then the estimate (1.5) implies that

$$\sup_{j_0\le j<\infty} 2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p} \le C_n \|\varphi\|_{H^n} \sup_{j_0\le j<\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}$$

for all $0 < t < \infty$ and $j_0 \in \mathbb{Z}$, which yields

$$\begin{aligned}
&\|e^{t\Delta} f - f\|_{\dot{B}^s_{p,\infty}} \\
&\le \sup_{-\infty<j\le j_0} 2^{sj} \|\dot{\Delta}_j (e^{t\Delta} f - f)\|_{L^p} + C_n \|\varphi\|_{H^n} \sup_{j_0\le j<\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}.
\end{aligned}$$

Therefore, by using (1.6), we observe that

$$\limsup_{t\to+0} \|e^{t\Delta} f - f\|_{\dot{B}^s_{p,\infty}} \le C_n \|\varphi\|_{H^n} \sup_{j_0\le j<\infty} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}$$

for all $j_0 \in \mathbb{Z}$. Since we have assumed that (1.7), by letting $j_0 \to \infty$, we obtain (1.8). Next we consider $f \in \dot{B}^s_{\infty,q}$, where $1 \le q < \infty$. In a similar manner to the above argument, by noting (1.5), we deduce that

$$\|e^{t\Delta}f - f\|_{\dot{B}^s_{\infty,q}} \le \left\{ \sum_{j=-\infty}^{j_0} (2^{sj}\|\dot{\Delta}_j(e^{t\Delta}f - f)\|_{L^\infty})^q \right\}^{1/q} + C_n\|\varphi\|_{H^n} \left\{ \sum_{j=j_0}^{\infty} (2^{sj}\|\dot{\Delta}_j f\|_{L^\infty})^q \right\}^{1/q}$$

for all $0 < t < \infty$ and $j_0 \in \mathbb{Z}$. Since $f \in \dot{B}^s_{\infty,q}$ and $1 \le q < \infty$, by letting $j_0 \to \infty$ after taking $t \to +0$ with the aid of (1.6), we obtain the desired result. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. (i) In a similar manner to the derivation of the estimate (2.6), we obtain

$$\begin{aligned} 2^{sj}\|\dot{\Delta}_j e^{t\Delta} f\|_{L^p} &\le \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi \exp(-2^{2k}t|\cdot|^2)]\|_{L^1} \cdot 2^{sj}\|\dot{\Delta}_j f\|_{L^p} \\ &\le \|f\|_{\dot{B}^s_{p,\infty}} \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi \exp(-2^{2k}t|\cdot|^2)]\|_{L^1} \end{aligned}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$. Here, Lemma 2.1 (iii) implies that

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi \exp(-2^{2k}t|\cdot|^2)]\|_{L^1} &\le C_n \sum_{k=j-1}^{j+1} \exp(-2^{2k-3}t)\|\varphi\|_{H^n} \\ &\le C_n \exp(-2^{2j-5}t)\|\varphi\|_{H^n} \end{aligned}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$, which yields (1.11).

(ii) Assume that $e^{t\Delta}f \in C^\infty((0,\infty); \dot{B}^s_{p,\infty})$ with the property

$$\partial_t^m e^{t\Delta} f = \Delta^m e^{t\Delta} f \quad \text{in } \dot{B}^s_{p,\infty} \tag{2.7}$$

for all $0 < t < \infty$ and $m \in \mathbb{N} \cup \{0\}$. Then we may show that $e^{t\Delta}f \in C^\omega((0,\infty); \dot{B}^s_{p,\infty})$ in the same way as in the proof of Ref. 2, Theorem 3.6 (iii). Moreover, by combining (2.7) and Ref. 1, Lemma 2.2 (ii), we observe that

$$\|\partial_t^m e^{t\Delta} f\|_{\dot{B}^s_{p,\infty}} \le C\|e^{t\Delta}f\|_{\dot{B}^{s+2m}_{p,\infty}} \le C\varepsilon^{-m}\|f\|_{\dot{B}^s_{p,\infty}}$$

for all $\varepsilon < t < \infty$ and $m \in \mathbb{N} \cup \{0\}$, which yields the desired result. Therefore, it is sufficient to show that (2.7). In case $m = 0$, the property (2.7) trivially holds. Assume that (2.7) holds for $m = m_0$. Since $-\Delta = \mathcal{F}^{-1}|\cdot|^2\mathcal{F}$, in a similar manner to the derivation of the estimate (2.6), we obtain

$$\begin{aligned} &2^{sj}\left\|\dot{\Delta}_j\left\{h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\}\right\|_{L^p} \\ &\le 2^{sj}\sum_{k=j-1}^{j+1}\left\|\dot{\Delta}_k\left\{h^{-1}(e^{h\Delta}(\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f) - (\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f)) - \Delta(\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f)\right\}\right\|_{L^p} \\ &= 2^{sj}\sum_{k=j-1}^{j+1}\left\|\mathcal{F}^{-1}\left[\varphi(2^{-k}\cdot)\left\{h^{-1}(e^{-h|\cdot|^2}-1) + |\cdot|^2\right\}\right] * (\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f)\right\|_{L^p} \\ &\le \sum_{k=j-1}^{j+1}\left\|\mathcal{F}^{-1}\left[\varphi\left\{h^{-1}(\exp(-2^{2k}h|\cdot|^2)-1) + 2^{2k}|\cdot|^2\right\}\right]\right\|_{L^1} \\ &\quad \times 2^{sj}\|\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f\|_{L^p} \end{aligned}$$

for all $0 < t < \infty$, $j \in \mathbb{Z}$, and $0 < h < 1$. Here, it holds by (2.4) and Lemma 2.1 (iii) that

$$\begin{aligned} &\sum_{k=j-1}^{j+1}\left\|\mathcal{F}^{-1}\left[\varphi\left\{h^{-1}(\exp(-2^{2k}h|\cdot|^2)-1) + 2^{2k}|\cdot|^2\right\}\right]\right\|_{L^1} \\ &\le C_n\sum_{k=j-1}^{j+1} 2^{2k+1}\left\{1 - (2^{2k+2}h)^{-1}(1-\exp(-2^{2k+2}h))\right\}^{1/2}\|\varphi\|_{H^n} \\ &\le C_n 2^{2j+3}\left\{1 - (2^{2j+4}h)^{-1}(1-\exp(-2^{2j+4}h))\right\}^{1/2}\|\varphi\|_{H^n} \end{aligned}$$

for all $j \in \mathbb{Z}$ and $0 < h < 1$. Therefore, we obtain

$$\begin{aligned} &2^{sj}\left\|\dot{\Delta}_j\left\{h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\}\right\|_{L^p} \\ &\le C_n 2^{2j+3}\left\{1 - (2^{2j+4}h)^{-1}(1-\exp(-2^{2j+4}h))\right\}^{1/2}\|\varphi\|_{H^n} \\ &\quad \times 2^{sj}\|\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f\|_{L^p} \end{aligned}$$

for all $0 < t < \infty$, $j \in \mathbb{Z}$, and $0 < h < 1$. In addition, if $m_0 = 0$, then (1.11) yields

$$2^{sj}\|\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f\|_{L^p} \le C_n\exp(-2^{2j-5}t)\|\varphi\|_{H^n}\|f\|_{\dot{B}^s_{p,\infty}}.$$

If $m_0 \ge 1$, then it holds by (1.11) and Ref. 1, Lemma 2.2 (ii) that

$$\begin{aligned} 2^{sj}\|\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f\|_{L^p} &\le M2^{sj}\|e^{(t/2)\Delta}\dot{\Delta}_j e^{(t/2)\Delta}f\|_{\dot{B}^{2m_0}_{p,1}} \\ &\le M2^{sj}(t/2)^{-m_0}\|\dot{\Delta}_j e^{(t/2)\Delta}f\|_{\dot{B}^0_{p,\infty}} \\ &\le Mt^{-m_0}\exp(-2^{2j-6}t)\|f\|_{\dot{B}^s_{p,\infty}}, \end{aligned}$$

where $M > 0$ is a constant independent of j, t, and f. Hence, we observe that

$$2^{sj}\|\dot{\Delta}_j\Delta^{m_0}e^{t\Delta}f\|_{L^p} \le M(1+t^{-m_0})\exp(-2^{2j-6}t)\|f\|_{\dot{B}^s_{p,\infty}}$$

for all $m_0 \in \mathbb{N} \cup \{0\}$. Now we fix $j_0 \in \mathbb{Z}$. Then we see by (2.4) that

$$\begin{aligned} &\sup_{-\infty<j\le j_0} 2^{sj}\left\|\dot{\Delta}_j\left\{h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\}\right\|_{L^p} \\ &\le M2^{2j_0+3}\left\{1 - (2^{2j_0+4}h)^{-1}(1-\exp(-2^{2j_0+4}h))\right\}^{1/2}(1+t^{-m_0})\|f\|_{\dot{B}^s_{p,\infty}}, \\ &\sup_{j_0\le j<\infty} 2^{sj}\left\|\dot{\Delta}_j\left\{h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\}\right\|_{L^p} \\ &\le M(1+t^{-m_0})\|f\|_{\dot{B}^s_{p,\infty}}\sup_{j_0\le j<\infty} 2^{2j+3}\exp(-2^{2j-6}t), \end{aligned}$$

which imply that

$$\begin{aligned} &\left\|h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\|_{\dot{B}^s_{p,\infty}} \\ &\le M2^{2j_0+3}\left\{1 - (2^{2j_0+4}h)^{-1}(1-\exp(-2^{2j_0+4}h))\right\}^{1/2}(1+t^{-m_0})\|f\|_{\dot{B}^s_{p,\infty}} \\ &\quad + M(1+t^{-m_0})\|f\|_{\dot{B}^s_{p,\infty}}\sup_{j_0\le j<\infty} 2^{2j+3}\exp(-2^{2j-6}t) \end{aligned}$$

for all $0 < t < \infty$, $j_0 \in \mathbb{Z}$, and $0 < h < 1$. Since

$$\begin{aligned} &\lim_{h\to+0}\left\{1 - (2^{2j_0+4}h)^{-1}(1-\exp(-2^{2j_0+4}h))\right\}^{1/2} \\ &= \lim_{h\to+0}\left\{1 + h^{-1}(e^{-h}-1)\right\}^{1/2} = 0, \end{aligned}$$

by letting $h \to +0$, we obtain

$$\begin{aligned} &\limsup_{h\to+0}\left\|h^{-1}(\Delta^{m_0}e^{(t+h)\Delta}f - \Delta^{m_0}e^{t\Delta}f) - \Delta^{m_0+1}e^{t\Delta}f\right\|_{\dot{B}^s_{p,\infty}} \\ &\le M(1+t^{-m_0})\|f\|_{\dot{B}^s_{p,\infty}}\sup_{j_0\le j<\infty} 2^{2j+3}\exp(-2^{2j-6}t). \end{aligned}$$

Hence, taking $j_0 \to \infty$ implies that $\partial_t(\Delta^{m_0}e^{t\Delta}f) = \Delta^{m_0+1}e^{t\Delta}f$ in $\dot{B}^s_{p,\infty}$. Since we have assumed that (2.7) holds for $m = m_0$, we obtain $\partial_t^{m_0+1}e^{t\Delta}f = \Delta^{m_0+1}e^{t\Delta}f$ in $\dot{B}^s_{p,\infty}$, so we have the desired result. This completes the proof of Theorem 1.3. □

2.3. *Counterexamples: Proof of Theorem 1.5*

Next we show Theorem 1.5. To give the lower bound of the L^p-norm for the function containing the Fourier transform, we choose examples f so that $(\mathcal{F}f)(\xi) > 0$ and $(\mathcal{F}f)(\lambda\xi) = a(\lambda)(\mathcal{F}f)(\xi)$ for all $\xi \in \mathbb{R}^n$ and $0 < \lambda < \infty$ with some function a of λ.

Proof of Theorem 1.5. We see by Ref. 16, p. 131 that $\mathcal{F}|\cdot|^{-\sigma} = c_0|\cdot|^{n-\sigma}$ and $\mathcal{F}\delta = 1$ with some constant $c_0 > 0$. Thus we have

$$(\mathcal{F}f_0)(2^j\cdot) = 1, \quad (\mathcal{F}f_1)(2^j\cdot) = c_0 2^{-(n-\sigma)j}|\cdot|^{n-\sigma} = 2^{-(n-\sigma)j}\mathcal{F}f_1$$

for all $j \in \mathbb{Z}$. Hence it holds by (2.1) that

$$\begin{aligned}&2^{s_m j}\|\dot{\Delta}_j e^{t\Delta} f_m\|_{L^p}\\&= 2^{s_m j}\|\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)e^{-t|\cdot|^2}\mathcal{F}f_m]\|_{L^p}\\&= 2^{(s_m+n-n/p)j}\|\mathcal{F}^{-1}[\varphi\exp(-2^{2j}t|\cdot|^2)(\mathcal{F}f_m)(2^j\cdot)]\|_{L^p}\\&= \|\mathcal{F}^{-1}[\varphi\exp(-2^{2j}t|\cdot|^2)\mathcal{F}f_m]\|_{L^p}\end{aligned} \tag{2.8}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$. Here, since φ satisfies the condition (1.1), by noting that $\mathcal{F}f_m > 0$ and $e^{i\xi\cdot x} = \cos(\xi\cdot x) + i\sin(\xi\cdot x)$, we observe that

$$\begin{aligned}&|\mathcal{F}^{-1}[\varphi\exp(-2^{2j}t|\cdot|^2)\mathcal{F}f_m](x)|^2\\&\quad= \frac{1}{(2\pi)^{2n}}\left|\int_{\mathbb{R}^n} e^{i\xi\cdot x}\varphi(\xi)\exp(-2^{2j}t|\xi|^2)(\mathcal{F}f_m)(\xi)d\xi\right|^2\\&\quad= \frac{1}{(2\pi)^{2n}}\left(\int_{\mathbb{R}^n}\cos(\xi\cdot x)\varphi(\xi)\exp(-2^{2j}t|\xi|^2)(\mathcal{F}f_m)(\xi)d\xi\right)^2\\&\qquad+ \frac{1}{(2\pi)^{2n}}\left(\int_{\mathbb{R}^n}\sin(\xi\cdot x)\varphi(\xi)\exp(-2^{2j}t|\xi|^2)(\mathcal{F}f_m)(\xi)d\xi\right)^2\\&\quad\geq \frac{1}{(2\pi)^{2n}}\left(\int_{\mathbb{R}^n}\cos(\xi\cdot x)\varphi(\xi)\exp(-2^{2j}t|\xi|^2)(\mathcal{F}f_m)(\xi)d\xi\right)^2\end{aligned}$$

for all $0 < t < \infty$, $x \in \mathbb{R}^n$, and $j \in \mathbb{Z}$. In addition, if $|x| \leq \pi/6$ and $|\xi| \leq 2$, then we have $|\xi\cdot x| \leq |\xi||x| \leq \pi/3$, which yields $1/2 \leq \cos(\xi\cdot x) \leq 1$. Therefore, noting that (1.1), we obtain

$$\begin{aligned}&\inf_{|x|\leq\pi/6}|\mathcal{F}^{-1}[\varphi\exp(-2^{2j}t|\cdot|^2)\mathcal{F}f_m](x)|^2\\&\qquad\geq C_{\sigma,n}\left(\exp(-2^{2j+2}t)\int_{\mathbb{R}^n}\varphi(\xi)d\xi\right)^2\end{aligned} \tag{2.9}$$

for all $0 < t < \infty$ and $j \in \mathbb{Z}$, which implies that

$$\begin{aligned}&2^{s_m j}\|\dot{\Delta}_j e^{t\Delta} f_m\|_{L^p}\\&\geq \inf_{|x|\leq\pi/6}|\mathcal{F}^{-1}[\varphi\exp(-2^{2j}t|\cdot|^2)\mathcal{F}f_m](x)|\left(\int_{|x|\leq\pi/6}dx\right)^{1/p}\\&\geq C_{\sigma,n}\exp(-2^{2j+2}t)\|\varphi\|_{L^1}.\end{aligned}$$

Next we show (1.13). We take t_0 and t_1 so that $0 < t_0 < t_1 < 1$. Then, in a similar manner to the derivation of the identity (2.8), we have

$$\begin{aligned}&2^{s_m j}\|\dot{\Delta}_j(e^{t_1\Delta}f_m - e^{t_0\Delta}f_m)\|_{L^p}\\&= \|\mathcal{F}^{-1}[\varphi(\exp(-2^{2j}t_1|\cdot|^2) - \exp(-2^{2j}t_0|\cdot|^2))\mathcal{F}f_m]\|_{L^p}\end{aligned}$$

for all $j \in \mathbb{Z}$. In addition, similarly to the estimate (2.9), we also obtain

$$\begin{aligned}&\inf_{|x|\leq\pi/6}|\mathcal{F}^{-1}[\varphi(\exp(-2^{2j}t_1|\cdot|^2) - \exp(-2^{2j}t_0|\cdot|^2))\mathcal{F}f_m](x)|^2\\&\geq C_{\sigma,n}\left(\int_{\mathbb{R}^n}\varphi(\xi)(\exp(-2^{2j}t_0|\xi|^2) - \exp(-2^{2j}t_1|\xi|^2))d\xi\right)^2.\end{aligned}$$

Here, since $t_0 < t_1$, the mean value theorem yields $e^{-t_0 r} - e^{-t_1 r} \geq re^{-t_1 r}(t_1 - t_0)$ for all $0 < r < \infty$. Thus we have

$$\begin{aligned}&\inf_{|x|\leq\pi/6}|\mathcal{F}^{-1}[\varphi(\exp(-2^{2j}t_1|\cdot|^2) - \exp(-2^{2j}t_0|\cdot|^2))\mathcal{F}f_m](x)|^2\\&\geq C_{\sigma,n}\left((t_1-t_0)\int_{\mathbb{R}^n}\varphi(\xi)\cdot 2^{2j}|\xi|^2\exp(-2^{2j}t_1|\xi|^2)d\xi\right)^2\\&\geq C_{\sigma,n}\left(2^{2j-2}(t_1-t_0)\exp(-2^{2j+2}t_1)\int_{\mathbb{R}^n}\varphi(\xi)d\xi\right)^2,\end{aligned}$$

which yields

$$\begin{aligned}&2^{s_m j}\|\dot{\Delta}_j(e^{t_1\Delta}f_m - e^{t_0\Delta}f_m)\|_{L^p}\\&\geq \inf_{|x|\leq\pi/6}|\mathcal{F}^{-1}[\varphi(\exp(-2^{2j}t_1|\cdot|^2) - \exp(-2^{2j}t_0|\cdot|^2))\mathcal{F}f_m](x)|\\&\quad\times\left(\int_{|x|\leq\pi/6}dx\right)^{1/p}\\&\geq C_{\sigma,n}2^{2j-2}(t_1-t_0)\exp(-2^{2j+2}t_1)\|\varphi\|_{L^1}\end{aligned}$$

for all $j \in \mathbb{Z}$. Let $0 < \eta < 1$. Then, by setting $t_0 = \eta/2$ and $t_1 = \eta$ in the above estimate, we observe that

$$\sup_{\substack{t_0,t_1\in(0,1),\\|t_1-t_0|<\eta}} 2^{s_m j}\|\dot{\Delta}_j(e^{t_1\Delta}f_m - e^{t_0\Delta}f_m)\|_{L^p} \geq C_{\sigma,n}2^{2j-3}\eta\exp(-2^{2j+2}\eta)\|\varphi\|_{L^1}$$

for all $j \in \mathbb{Z}$, which implies that

$$\begin{aligned}&\sup_{0\leq j<\infty}\sup_{\substack{t_0,t_1\in(0,1),\\|t_1-t_0|<\eta}} 2^{s_m j}\|\dot{\Delta}_j(e^{t_1\Delta}f_m - e^{t_0\Delta}f_m)\|_{L^p}\\&\qquad\geq C_{\sigma,n}\|\varphi\|_{L^1}\sup_{0\leq j<\infty}2^{2j-3}\eta\exp(-2^{2j+2}\eta).\end{aligned}$$

Here, since $0 < \eta < 1$, we may take $0 \leq j < \infty$ so that $2^{-2j} \leq \eta \leq 2^{-2j+2}$. Noting that $2^{2j-3}\eta\exp(-2^{2j+2}\eta) \geq 2^{-3}e^{-16}$, we observe that

$$\sup_{\substack{t_0,t_1\in(0,1),\\|t_1-t_0|<\eta}}\|e^{t_1\Delta}f_m - e^{t_0\Delta}f_m\|_{\dot{B}^{s_m}_{p,\infty}} \geq C_{\sigma,n}\|\varphi\|_{L^1}$$

for any $0 < \eta < 1$, which yields (1.13). This completes the proof of Theorem 1.5. □

3. Proof of the characterization (1.10) of the function spaces

Let us verify the characterization (1.10) of the spaces $\dot{B}^s_{p,\infty+}$. Such a fact is mentioned in Ref. 15, Remark 2.28, but for the convenience of the readers, we give the proof in detail here.

Proof of Eq. (1.10). Let $1 \leq p < \infty$ and $s \in \mathbb{R}$. Assume that $f \in \dot{B}^s_{p,\infty}$ and there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{S}_0$ of functions such that

$$\lim_{k\to\infty}\|f - f_k\|_{\dot{B}^s_{p,\infty}} = 0.$$

Then the Young convolution inequality and (2.1) imply that

$$\begin{aligned}2^{sj}\|\dot{\Delta}_j f_k\|_{L^p} &= 2^{sj}\|\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)|\cdot|^{-\sigma}|\cdot|^{\sigma}\mathcal{F}f_k]\|_{L^p}\\&= 2^{sj}\|\mathcal{F}^{-1}[\varphi(2^{-j}\cdot)|\cdot|^{-\sigma}] * \mathcal{F}^{-1}[|\cdot|^{\sigma}\mathcal{F}f_k]\|_{L^p}\\&\leq 2^{(s-\sigma)j}\|\mathcal{F}^{-1}[\varphi|\cdot|^{-\sigma}]\|_{L^1}\|\mathcal{F}^{-1}[|\cdot|^{\sigma}\mathcal{F}f_k]\|_{L^p}\end{aligned}$$

for all $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and $\sigma \in \mathbb{R}$. Here we note that $|\cdot|^{\sigma}\mathcal{F}f_k \in \mathscr{S}$ for any $\sigma \in \mathbb{R}$ from the condition $0 \notin \operatorname{supp}\mathcal{F}f_k$. Hence, by setting $\sigma = s+1$ and letting $j \to \infty$, we have $\lim_{j\to\infty}2^{sj}\|\dot{\Delta}_j f_k\|_{L^p} = 0$. In addition, by setting $\sigma = s-1$ and letting $j \to -\infty$, we also see that $\lim_{j\to-\infty}2^{sj}\|\dot{\Delta}_j f_k\|_{L^p} = 0$. Therefore, we obtain

$$\begin{aligned}\limsup_{j\to\pm\infty}2^{sj}\|\dot{\Delta}_j f\|_{L^p} &\leq \sup_{j\in\mathbb{Z}}2^{sj}\|\dot{\Delta}_j(f-f_k)\|_{L^p} + \limsup_{j\to\pm\infty}2^{sj}\|\dot{\Delta}_j f_k\|_{L^p}\\&= \|f - f_k\|_{\dot{B}^s_{p,\infty}}\end{aligned}$$

for any $k \in \mathbb{N}$. By letting $k \to \infty$, we have

$$\lim_{j\to\pm\infty}2^{sj}\|\dot{\Delta}_j f\|_{L^p} = 0. \tag{3.1}$$

Conversely, suppose that $f \in \dot{B}^s_{p,\infty}$ satisfies (3.1). Take $j_0 \in \mathbb{N}$ and set $f_{j_0} := \sum_{l=-j_0}^{j_0}\dot{\Delta}_l f$. Noting that (2.1) and (2.2), we see by the Young convolution inequality that

$$\begin{aligned}\|f_{j_0}\|_{\dot{B}^s_{p,1}} &= \sum_{j\in\mathbb{Z}}2^{sj}\left\|\dot{\Delta}_j\sum_{l=-j_0}^{j_0}\dot{\Delta}_l f\right\|_{L^p} \leq \sum_{j=-j_0-1}^{j_0+1}2^{sj}\sum_{l=j-1}^{j+1}\|\dot{\Delta}_j\dot{\Delta}_l f\|_{L^p}\\&= \sum_{j=-j_0-1}^{j_0+1}2^{sj}\sum_{l=j-1}^{j+1}\|\mathcal{F}^{-1}[\varphi(2^{-l}\cdot)] * (\dot{\Delta}_j f)\|_{L^p}\\&\leq \sum_{j=-j_0-1}^{j_0+1}\sum_{l=j-1}^{j+1}\|\mathcal{F}^{-1}\varphi\|_{L^1}\cdot 2^{sj}\|\dot{\Delta}_j f\|_{L^p}\\&\leq \|f\|_{\dot{B}^s_{p,\infty}}\sum_{j=-j_0-1}^{j_0+1}\sum_{l=j-1}^{j+1}\|\mathcal{F}^{-1}\varphi\|_{L^1}\\&\leq 3(2j_0+3)\|\mathcal{F}^{-1}\varphi\|_{L^1}\|f\|_{\dot{B}^s_{p,\infty}}.\end{aligned}$$

Hence, we obtain $f_{j_0} \in \dot{B}^s_{p,1}$, which implies that we may take a sequence $\{f_{j_0,k}\}_{k\in\mathbb{N}} \subset \mathscr{S}_0$ of functions satisfying

$$\lim_{k\to\infty}\|f_{j_0} - f_{j_0,k}\|_{\dot{B}^s_{p,1}} = 0$$

by virtue of Ref. 15, Proposition 2.27. Moreover, in a similar manner to the above computation, we have

$$\begin{aligned}\|f - f_{j_0}\|_{\dot{B}^s_{p,\infty}} &\le \sup_{j\in\mathbb{Z}} 2^{sj} \sum_{|l|\ge j_0+1,\,|l-j|\le 1} \|\dot{\Delta}_j \dot{\Delta}_l f\|_{L^p} \\ &\le \sup_{|j|\ge j_0} 2^{sj} \sum_{l=j-1}^{j+1} \|\mathcal{F}^{-1}[\varphi(2^{-l}\cdot)] * (\dot{\Delta}_j f)\|_{L^p} \\ &\le \sup_{|j|\ge j_0} \sum_{l=j-1}^{j+1} \|\mathcal{F}^{-1}\varphi\|_{L^1} \cdot 2^{sj} \|\dot{\Delta}_j f\|_{L^p} \\ &\le 3\|\mathcal{F}^{-1}\varphi\|_{L^1} \sup_{|j|\ge j_0} 2^{sj}\|\dot{\Delta}_j f\|_{L^p},\end{aligned}$$

which yields

$$\begin{aligned}\|f - f_{j_0,k}\|_{\dot{B}^s_{p,\infty}} &\le \|f - f_{j_0}\|_{\dot{B}^s_{p,\infty}} + \|f_{j_0} - f_{j_0,k}\|_{\dot{B}^s_{p,1}} \\ &\le 3\|\mathcal{F}^{-1}\varphi\|_{L^1} \sup_{|j|\ge j_0} 2^{sj}\|\dot{\Delta}_j f\|_{L^p} + \|f_{j_0} - f_{j_0,k}\|_{\dot{B}^s_{p,1}}\end{aligned}$$

for all $k \in \mathbb{N}$. Since letting $k \to \infty$ yields

$$\limsup_{k\to\infty} \|f - f_{j_0,k}\|_{\dot{B}^s_{p,\infty}} \le 3\|\mathcal{F}^{-1}\varphi\|_{L^1} \sup_{|j|\ge j_0} 2^{sj}\|\dot{\Delta}_j f\|_{L^p}$$

and since we have assumed that (3.1), by taking $j_0 \to \infty$, we have the desired result. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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