Various regularity estimates for the Keller-Segel-Navier-Stokes system in Besov spaces

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VARIOUS REGULARITY ESTIMATES FOR THE KELLER-SEGEL-NAVIER-STOKES SYSTEM IN BESOV SPACES

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Abstract. We show the local well-posedness for the Keller-Segel-Navier-Stokes system with initial data in the scaling invariant Besov spaces, where the solution exists globally in time if the initial data is sufficiently small. We also reveal that the solution belongs to the Lorentz spaces in time direction, while the solution is smooth in space and time. Moreover, we obtain the maximal regularity estimates of solutions under the certain conditions. We further show that the solution has the additional regularities if the initial data has higher regularities. This result implies that global solutions decay as the limit $t \to \infty$ in the same norm of the space of the initial data. Our results on the Lorentz regularity estimates are based on the strategy by Kozono-Shimizu (J. Funct. Anal. **276** (2019), no. 3, 896–931).

1. Introduction

In this paper, we consider the initial value problem for the Keller-Segel-Navier-Stokes system of parabolic-elliptic type in $\mathbb{R}^N$, $N \geq 2$;

$$(1.1) \qquad \begin{cases} \partial_t n - d\Delta n = -\nabla \cdot (n\nabla c) - \boldsymbol{u} \cdot \nabla n, & t > 0,\ x \in \mathbb{R}^N, \\ \qquad -\Delta c = n, & t > 0,\ x \in \mathbb{R}^N, \\ \partial_t \boldsymbol{u} - \nu\Delta \boldsymbol{u} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p + n\nabla c, & t > 0,\ x \in \mathbb{R}^N, \\ \quad \nabla \cdot \boldsymbol{u} = 0, & t > 0,\ x \in \mathbb{R}^N, \\ \quad n(0,x) = a(x), \quad \boldsymbol{u}(0,x) = \boldsymbol{b}(x), & \qquad x \in \mathbb{R}^N, \end{cases}$$

where $n = n(t,x)$, $c = c(t,x)$, $\boldsymbol{u} = \boldsymbol{u}(t,x)$, and $p = p(t,x)$ are the unknown functions standing for the density of the cell, the concentration of the chemo-attractant, the velocity of the fluid, and the pressure, respectively. In addition, $(a, \boldsymbol{b}) = (a(x), \boldsymbol{b}(x))$ is the given initial data and $0 < d, \nu < \infty$ are the given constant.

Our purpose in this paper is to show the *local well-posedness* for (1.1) with initial data $(a, \boldsymbol{b})$ in the *scaling invariant Besov spaces*, i.e., $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N) \times \dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)$ with suitable conditions. Here we also treat the case of $\rho = \infty$. Moreover, we reveal that the solution belongs to the *Lorentz spaces* in time direction, while the solution is *smooth*, i.e., in C^∞ class in space and time. We also show that the solution exists *globally in time* if $(a, \boldsymbol{b})$ is sufficiently small. In fact, the global solution decays as the limit $t \to \infty$ in the *same* norm of the space of the initial data. In addition, we obtain the *maximal regularity estimates* of solutions under the certain conditions. Finally, we prove the additional regularities of solutions if the initial data has higher regularities.

The system (1.1) is regarded as a mathematical model of chemotaxis taking into account the effect of the viscous fluid flow. The original model of chemotaxis without flow effect is well-known

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as the Keller-Segel system [21] given by

$$\begin{cases} \partial_t n - d\Delta n = -\nabla \cdot (n\nabla c), & t>0,\ x\in\mathbb{R}^N,\\ -\Delta c = n, & t>0,\ x\in\mathbb{R}^N,\\ n(0,x)=a(x), & x\in\mathbb{R}^N. \end{cases} \tag{1.2}$$

Here the system (1.2) is called parabolic-elliptic type. On the other hand, if the second equation in (1.2) is $\partial_t c - \Delta c = n$ instead of $-\Delta c = n$, then the system is called parabolic-parabolic type. Both models are well-established to analyze the effect of chemotaxis, so they have been researched mathematically from various aspects [2, 16, 30, 36, 39, 41]. As for the system (1.1), it is expected that the more complicated phenomenon occurs on account of the effect of the viscous fluid flow. Let us mention on some previous works dealing with the Keller-Segel-Navier-Stokes system briefly here. In the case of the whole space $\mathbb{R}^N$, Duan-Lorz-Markowich [11] achieved to construct global classical solutions for $N = 3$ provided that the initial data have sufficient regularities in the Sobolev spaces with a smallness condition. Kozono-Miura-Sugiyama [22] obtained global mild solutions for $N \geq 2$ in the scaling invariant spaces with a smallness condition. Yang-Fu-Sun [50] enlarged the spaces of the initial data compared with the case of [22]. Kang-Lee-Winkler [18] recently showed the existence of global weak solutions for $N = 3$ without any smallness assumption of the initial data. Finally, for fairly recent contribution, let us refer to Yomgne [10]. He introduced a new function space which may be regarded as an extension of $BMO^{-1}(\mathbb{R}^N)$ and showed the existence of mild solutions. This result might give the largest space of the initial data ensuring the well-posedness. In the case of a bounded convex domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary, Winkler [40, 43] constructed global classical solutions for $N = 2$ and global weak solutions for $N = 3$ even if the initial data are arbitrarily large. In addition, the smoothing effects and the stability of global solutions are also discussed in [42, 44]. We also notice that damping terms of logistic type might affect the original system in a positive way, so there are several results on construction of global solutions of the Keller-Segel-Navier-Stokes system with logistic terms. We should refer to Tao-Winkler [37] and Winkler [46, 48] for such results. Here, although a similar system to (1.1), i.e., the case of parabolic-parabolic type has been fully studied in the above literatures, the corresponding results to those of such a system (1.1) has not been obtained yet. The system (1.1) might be initially proposed by Gong-He [14], who considered (1.1) with $d = \nu = 1$ in 2D case and showed that the solution $(n, \boldsymbol{u})$ exists globally in time for arbitrary initial velocities $\boldsymbol{b}$ provided $\|a\|_{L^1(\mathbb{R}^2)} < 8\pi$. This result may be regarded as that of the 2D original Keller-Segel system dealing with the critical mass [3, 4, 9, 13, 15, 29]. On the other hand, our motivation is to reveal the properties of solutions of (1.1) in higher dimensional case, including the well-posedness, regularity estimates, smoothing effects, and time-decay properties of global solutions.

Concerning the well-posedness, we notice that

$$\|n_\lambda(0,\cdot)\|_{\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N)} = \|n(0,\cdot)\|_{\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N)}, \quad \|\boldsymbol{u}_\lambda(0,\cdot)\|_{\dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)} = \|\boldsymbol{u}(0,\cdot)\|_{\dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)} \tag{1.3}$$

provided that $\lambda = 2^j$ for $j \in \mathbb{Z}$, where $(n_\lambda(t,x), \boldsymbol{u}_\lambda(t,x)) := (\lambda^2 n(\lambda^2 t, \lambda x), \lambda \boldsymbol{u}(\lambda^2 t, \lambda x))$. This implies that the space $\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N) \times \dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)$ is one of *scaling invariant spaces* to (1.1). Our results on the local and global well-posedness for (1.1) with initial data in $\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N) \times \dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)$ are based on the standard method by Fujita-Kato [12, 20] and Kato [19], namely, introducing the time weighted spaces and construction of mild solutions of (1.1). In fact, we further show that the solution belongs to the *Lorentz spaces* in time direction with the exponent ρ appearing in the space $\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N) \times \dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)$ of the initial data. Our result may be regarded as an improved version compared with [12, 19, 20]. Hence, this is an advantage of considering the initial data in $\dot{B}^{-2+N/r}_{r,\rho}(\mathbb{R}^N) \times \dot{B}^{-1+N/q}_{q,\rho}(\mathbb{R}^N)$ for *general* $1 \le \rho \le \infty$. In addition, we show the *smoothing effects* of solutions, namely, the solution belongs to C^∞ class in space and time. Although it may be expected that solutions of parabolic type PDEs are smooth, we construct solutions with the initial data in the space with the *homogeneous* norm. Since the inclusions on the derivative indices fail to hold, i.e., $\dot{B}^{s_1}_{r,\rho}(\mathbb{R}^N) \not\subset \dot{B}^{s_0}_{r,\rho}(\mathbb{R}^N)$ even if $s_0 < s_1$, we need to pay attention to the proof of the smoothing effects of solutions for such spaces. Regarding this problem, we give the *simple procedure* via bootstrap argument by showing the regularity properties of mild solutions of the linear heat equation. We may expect that our method to obtain the smoothing effects is still valid for any semilinear parabolic type PDEs in the homogeneous Besov spaces framework. Moreover, we show the *maximal regularity estimates* of solutions under the certain conditions. According to the result by Kozono-Shimizu [26, Theorem 2] who showed the maximal Lorentz regularity theorem for the Navier-Stokes system, the solution belongs to another scaling invariant space obtained from the structure of the Stokes system [26, Lamma 3.1]. Thus we introduce *another scaling invariant space* as well. Once we establish the nonlinear estimates for such a space, we may show the maximal regularity estimates immediately.

Before considering the system (1.1), we shall simplify (1.1) by eliminating the unknown functions ∇c and ∇p. Since the Poisson equation $-\Delta c = n$ in $\mathbb{R}^N$ has the well-known solution formula

$$c = \begin{cases} -(2\pi)^{-1}(\log|\cdot|) * n & \text{if } N = 2, \\ ((N-2)\omega_{N-1})^{-1}|\cdot|^{2-N} * n & \text{if } N \ge 3, \end{cases}$$

by setting the function

$$\boldsymbol{K}(x) := -\frac{x}{\omega_{N-1}|x|^N} \tag{1.4}$$

for $x \in \mathbb{R}^N \setminus \{0\}$, we have $\nabla c = \boldsymbol{K} * n$. Here ω_{N-1} denotes the surface area of a unit ball in $\mathbb{R}^N$. Therefore, by operating the Helmholtz projection $P := I + \nabla(-\Delta)^{-1}\nabla\cdot$ to both sides in the third equation of (1.1), we may obtain

$$\begin{cases} \partial_t n - d\Delta n = -\nabla\cdot(n(\boldsymbol{K} * n)) - \boldsymbol{u}\cdot\nabla n & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + P(n(\boldsymbol{K} * n)) & \text{in } (0,\infty)\times\mathbb{R}^N, \\ n(0) = a, \quad \boldsymbol{u}(0) = \boldsymbol{b} & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

This paper is organized as follows: We state our main results in the next section. More precisely, we give the local and global well-posedness results for (1.5) with initial data in the scaling invariant Besov spaces. In addition, we also give the maximal regularity estimates and additional regularities of solutions under the certain conditions. In Section 3, we recall the definitions of some function spaces and fundamental properties. Section 4 is devoted to the proof of the local well-posedness results. In Section 5, we show the maximal regularity estimates and additional regularities of solutions under the certain conditions. We further show the global well-posedness with the time-decay properties of global solutions. In Appendix, we prove the regularity properties of mild solutions of the linear heat equation.

2. Main results

In this section, we shall state our main results. In the following, let $\dot{B}^s_{r,\rho} := \dot{B}^s_{r,\rho}(\mathbb{R}^N)$ and $L^{\alpha,\rho}$ denote the homogeneous Besov spaces and the Lorentz spaces, respectively. We also abbreviate $\|\cdot\|_{L^\infty_T(X)} := \|\cdot\|_{L^\infty((0,T);X)}$ and $\|\cdot\|_{L^{\alpha,\rho}_T(X)} := \|\cdot\|_{L^{\alpha,\rho}((0,T);X)}$ for simplicity. We will introduce more details of notations and function spaces in Section 3.

2.1. **Local and global well-posedness in the scaling invariant Besov spaces.** In this subsection, we state our main results on the local well-posedness for (1.5) with initial data in the scaling invariant Besov spaces, where the solution exists globally in time if the initial data is sufficiently small. We also give the time-decay properties of global solutions.

Theorem 2.1. *Let $1 \le \rho < \infty$. Suppose that $1 \le r, q < N$ satisfy*

$$-1/N < 1/r - 1/q < 2/N, \quad 1/N < 3/r - 2/q, \quad 2/r - 3/q < 1/N \tag{2.1}$$

and $0 < s, \alpha, \beta < 1$ satisfy

$$\begin{cases} (1/2)\max\{3 - N/r, 3 - N/q\} < s, \alpha < \min\{1 + N(1/r - 1/q), 2 - N(1/r - 1/q)\}, \\ \max\{3 - N/r - \alpha, (1/2)(2 - N/q)\} < \beta < 2\alpha. \end{cases} \tag{2.2}$$

In addition, suppose that the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. Then the following statements hold:

(i) There exist $0 < T < \infty$ and a solution $(n, \boldsymbol{u})$ on $(0, T) \times \mathbb{R}^N$ of (1.5) satisfying

$$\begin{cases} n \in BC([0,T); \dot{B}^{-2+N/r}_{r,\rho}) \cap \bigcap_{0<\gamma<\infty} C^\infty((0,T); \dot{B}^{-2+N/r}_{r,\rho} \cap \dot{B}^{\gamma-2+N/r}_{r,1}), \\ \partial_t n \in C^\infty((0,T); \dot{B}^{2s-4+N/r}_{r,1}), \\ t^{\alpha/2} n \in BC([0,T); \dot{B}^{\alpha-2+N/r}_{r,1}), \quad t^{s/2} n \in BC([0,T); \dot{B}^{s-2+N/r}_{r,1}), \\ n \in L^{2/\alpha,\rho}((0,T); \dot{B}^{\alpha-2+N/r}_{r,1}) \cap L^{2/s,\rho}((0,T); \dot{B}^{s-2+N/r}_{r,1}), \\ \boldsymbol{u} \in BC([0,T); P(\dot{B}^{-1+N/q}_{q,\rho})^N) \cap \bigcap_{0<\gamma<\infty} C^\infty((0,T); P(\dot{B}^{-1+N/q}_{q,\rho} \cap \dot{B}^{\gamma-1+N/q}_{q,1})^N), \\ \partial_t \boldsymbol{u} \in C^\infty((0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N), \\ t^{\beta/2} \boldsymbol{u} \in BC([0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N), \quad t^{s/2} \boldsymbol{u} \in BC([0,T); P(\dot{B}^{s-2+N/r}_{r,1})^N), \\ \boldsymbol{u} \in L^{2/\beta,\rho}((0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N) \cap L^{2/s,\rho}((0,T); P(\dot{B}^{s-1+N/q}_{q,1})^N) \end{cases} \tag{2.3}$$

with

$$\begin{cases} \lim_{t\to+0} \|n(t) - a\|_{\dot{B}^{-2+N/r}_{r,\rho}} = 0, \quad \lim_{t\to+0} t^{\alpha/2} \|n(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,1}} = 0, \quad \lim_{t\to+0} t^{s/2} \|n(t)\|_{\dot{B}^{s-2+N/r}_{r,1}} = 0, \\ \lim_{t\to+0} \|\boldsymbol{u}(t) - \boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} = 0, \quad \lim_{t\to+0} t^{\beta/2} \|\boldsymbol{u}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,1}} = 0, \quad \lim_{t\to+0} t^{s/2} \|\boldsymbol{u}(t)\|_{\dot{B}^{s-1+N/q}_{q,1}} = 0. \end{cases} \tag{2.4}$$

Moreover, the following estimates

$$\begin{cases} \|n\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,\rho})} \\ \le C\left((1 + d^{-\alpha/2})\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1 + \nu^{-\beta/2})\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right), \\ \|t^{\alpha/2} n\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|n\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|t^{\beta/2}\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,1})} + \|\boldsymbol{u}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \\ \le C\left(d^{-\alpha/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) \end{cases} \tag{2.5}$$

hold with some constant $C = C(N, r, q, \rho, s, \alpha, \beta) > 0$ independent of $d, \nu, T, a, \boldsymbol{b}, n,$ and $\boldsymbol{u}$. Likewise, the estimates (2.5) hold with α and β replaced by s.

(ii) There exists a constant $0 < \kappa < 1$ independent of the initial data $(a, \boldsymbol{b})$ such that a mild solution $(n, \boldsymbol{u})$ on $(0,T) \times \mathbb{R}^N$ of (1.5) satisfying

$$\begin{cases} n \in L^{2/\alpha,\infty}((0,T); \dot{B}^{\alpha-2+N/r}_{r,\infty}), \quad \boldsymbol{u} \in L^{2/\beta,\infty}((0,T); P(\dot{B}^{\beta-1+N/q}_{q,\infty})^N), \\ \limsup_{\lambda\to\infty} \Big\{ \lambda\mu(t \in (0,T) \mid \|n(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}} > \lambda)^{\alpha/2} \\ \quad + \lambda\mu(t \in (0,T) \mid \|\boldsymbol{u}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}} > \lambda)^{\beta/2} \Big\} \le \kappa \end{cases} \tag{2.6}$$

is unique, where μ denotes the usual Lebesgue measure on $(0,T)$.

(iii) Suppose that $(n, \boldsymbol{u})$ is the solution on $(0,T) \times \mathbb{R}^N$ of (1.5) with the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ obtained in (i). Likewise, suppose that $(n_*, \boldsymbol{u}_*)$ is a solution of (1.5) with an initial data $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. There is a constant $0 < \delta < 1$ such that if $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ satisfies

$$\|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}} < \delta, \tag{2.7}$$

then it holds that

$$\begin{cases} \|n - n_*\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u} - \boldsymbol{u}_*\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,\rho})} \\ \le C \left((1 + d^{-\alpha/2}) \|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1 + \nu^{-\beta/2}) \|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right), \\ \|t^{\alpha/2}(n - n_*)\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|n - n_*\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} \\ + \|t^{\beta/2}(\boldsymbol{u} - \boldsymbol{u}_*)\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,1})} + \|\boldsymbol{u} - \boldsymbol{u}_*\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \\ \le C \left(d^{-\alpha/2} \|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2} \|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right) \end{cases} \tag{2.8}$$

with some constant $C = C(N, r, q, \rho, s, \alpha, \beta) > 0$ independent of $\delta, d, \nu, T, a, \boldsymbol{b}, a_*, \boldsymbol{b}_*, n, \boldsymbol{u}, n_*,$ and $\boldsymbol{u}_*$. Likewise, it holds that (2.8) with α and β replaced by s.

Remark 2.2. (i) The left-hand sides of the estimates (2.5) are invariant under the change of scaling $(n_\lambda(t,x), \boldsymbol{u}_\lambda(t,x)) := (\lambda^2 n(\lambda^2 t, \lambda x), \lambda \boldsymbol{u}(\lambda^2 t, \lambda x))$, where $\lambda = 2^j$ for $j \in \mathbb{Z}$. This property may be regarded as a scaling invariance corresponding to that of the initial data (1.3).

(ii) The method of construction of solutions relies on introducing the time weighted spaces, i.e., the approach by Fujita-Kato [12, 20] and Kato [19]. In fact, we may obtain the regularity of solutions in space compared with the initial data since $\dot{B}^s_{r,1} \subset \dot{B}^s_{r,\rho}$ holds for all $1 \le r \le \infty$, $s \in \mathbb{R}$, and $1 \le \rho \le \infty$. This result is obtained from the smoothing estimates of the heat semigroup in $\dot{B}^s_{r,\rho}$ given by Kozono-Ogawa-Taniuchi [23].

(iii) Moreover, we show that the solution also belongs to the Lorentz spaces in time direction. Here it should be noticed that the interpolation exponent ρ of the initial data appears in the regularity of the solution in the Lorentz spaces. This is an advantage of considering the initial data in $\dot{B}^s_{r,\rho}$ for general $1 \le \rho \le \infty$. We also note that this idea stems from Kozono-Shimizu [26].

(iv) We reveal that the solution is smooth, i.e., in C^∞ class in space and time even if the initial data belongs to the space with the homogeneous norm. In addition, we may expect that our method to obtain the smoothing effects is still valid for any semilinear parabolic type PDEs in the homogeneous Besov spaces framework. Here note that it seems to be difficult to show $n \in C^\infty((0,T); \dot{B}^{-2+N/r}_{r,1})$ and $\boldsymbol{u} \in C^\infty((0,T); P(\dot{B}^{-1+N/q}_{q,1})^N)$, i.e., smoothing effects for the interpolation exponent, but we see that $\partial_t n$ and $\partial_t \boldsymbol{u}$ have the desired regularity. These smoothing effects correspond to that of the heat semigroup. In fact, the author [35] recently showed the space-time analytic smoothing effects of the heat semigroup in $\dot{B}^s_{r,\rho}$.

(v) Theorem 2.1 (ii) states the uniqueness of mild solutions of (1.5). Notice that all of mild solutions $(n, \boldsymbol{u})$ satisfying (2.3) necessarily fulfill the condition (2.6). For details, see Proposition 3.7. Hence, our result may be regarded as an improved version compared with the usual uniqueness obtained by the Banach fixed point theorem.

Theorem 2.3 (In case $\rho = \infty$)**.** *Suppose that $1 < r, q < N$ satisfy (2.1) and $0 < s, \alpha, \beta < 1$ satisfy (2.2). There exists a constant $0 < \varepsilon_0 < 1$ such that if the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$ satisfies*

$$\limsup_{j\to\infty} \left(2^{(-2+N/r)j} \|\dot{\Delta}_j a\|_{L^r} + 2^{(-1+N/q)j} \|\dot{\Delta}_j \boldsymbol{b}\|_{L^q} \right) < \varepsilon_0, \tag{2.9}$$

Then the following statements hold, where $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ denotes the dyadic decomposition:

(i) There exist $0 < T < \infty$ and a solution $(n, \boldsymbol{u})$ on $(0,T) \times \mathbb{R}^N$ of (1.5) satisfying

$$\begin{cases} n \in \bigcap_{0<\gamma<\infty} \left(BC_w([0,T); \dot{B}^{-2+N/r}_{r,\infty}) \cap C^\infty((0,T); \dot{B}^{\gamma-2+N/r}_{r,1}) \right), \\ \partial_t n \in C^\infty((0,T); \dot{B}^{2s-4+N/r}_{r,1}), \\ t^{\alpha/2} n \in BC((0,T); \dot{B}^{\alpha-2+N/r}_{r,1}), \quad t^{s/2} n \in BC((0,T); \dot{B}^{s-2+N/r}_{r,1}), \\ \boldsymbol{u} \in \bigcap_{0<\gamma<\infty} \left(BC_w([0,T); P(\dot{B}^{-1+N/q}_{q,\infty})^N) \cap C^\infty((0,T); P(\dot{B}^{\gamma-1+N/q}_{q,1})^N) \right), \\ \partial_t \boldsymbol{u} \in C^\infty((0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N), \\ t^{\beta/2} \boldsymbol{u} \in BC((0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N), \quad t^{s/2} \boldsymbol{u} \in BC((0,T); P(\dot{B}^{s-2+N/r}_{r,1})^N) \end{cases} \tag{2.10}$$

with

$$\begin{cases} \lim_{t\to+0} \langle n(t) - a, \varphi_0 \rangle = 0, \quad \lim_{t\to+0} t^{\alpha/2} \langle n(t), \varphi_\alpha \rangle = 0, \quad \lim_{t\to+0} t^{s/2} \langle n(t), \varphi_s \rangle = 0, \\ \lim_{t\to+0} \langle \boldsymbol{u}(t) - \boldsymbol{b}, \boldsymbol{f}_0 \rangle = 0, \quad \lim_{t\to+0} t^{\beta/2} \langle \boldsymbol{u}(t), \boldsymbol{f}_\beta \rangle = 0, \quad \lim_{t\to+0} t^{s/2} \langle \boldsymbol{u}(t), \boldsymbol{f}_s \rangle = 0 \end{cases} \tag{2.11}$$

for all $\varphi_\theta \in \dot{B}^{-\theta+2-N/r}_{r/(r-1),1}$ and $\boldsymbol{f}_\theta \in (\dot{B}^{-\theta+1-N/q}_{q/(q-1),1})^N$, where $\theta = 0, \alpha, \beta, s$. Here BC_w denotes the space of all bounded weakly-star continuous functions and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Moreover, the following estimates

$$\begin{cases} \|n\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,\infty})} + \|\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,\infty})} \\ \le C \left((1 + d^{-\alpha/2}) \|a\|_{\dot{B}^{-2+N/r}_{r,\infty}} + (1 + \nu^{-\beta/2}) \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\infty}} \right), \\ \|t^{\alpha/2} n\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|t^{\beta/2} \boldsymbol{u}\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,1})} \\ \le C \left(d^{-\alpha/2} \|a\|_{\dot{B}^{-2+N/r}_{r,\infty}} + \nu^{-\beta/2} \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\infty}} \right) \end{cases} \tag{2.12}$$

hold with some constant $C = C(N, r, q, s, \alpha, \beta) > 0$ independent of $d, \nu, \varepsilon_0, T, a, \boldsymbol{b}, n,$ and $\boldsymbol{u}$. Likewise, the estimates (2.12) hold with α and β replaced by s. The uniqueness assertion remains true in the same way as in Theorem 2.1 (ii).

(ii) Suppose that $(n, \boldsymbol{u})$ is the solution on $(0,T) \times \mathbb{R}^N$ of (1.5) with the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$ satisfying (2.9) obtained in (i). Likewise, suppose that $(n_*, \boldsymbol{u}_*)$ is a

solution of (1.5) *with an initial data* $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$. *There is a constant* $0 < \delta < 1$ *such that if* $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$ *satisfies* (2.7) *with* ρ *replaced by* ∞, *then it holds that*

$$(2.13) \qquad \begin{cases} \|n - n_*\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,\infty})} + \|\boldsymbol{u} - \boldsymbol{u}_*\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,\infty})} \\ \leq C\left((1 + d^{-\alpha/2})\|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\infty}} + (1 + \nu^{-\beta/2})\|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\infty}}\right), \\ \|t^{\alpha/2}(n - n_*)\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|t^{\beta/2}(\boldsymbol{u} - \boldsymbol{u}_*)\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,1})} \\ \leq C\left(d^{-\alpha/2}\|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\infty}} + \nu^{-\beta/2}\|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\infty}}\right) \end{cases}$$

with some constant $C = C(N, r, q, s, \alpha, \beta) > 0$ *independent of* $\delta, d, \nu, \varepsilon_0, T, a, \boldsymbol{b}, a_*, \boldsymbol{b}_*, n, \boldsymbol{u}, n_*$, *and* $\boldsymbol{u}_*$. *Likewise, it holds that* (2.13) *with* α *and* β *replaced by* s.

Remark 2.4. (i) In case $\rho = \infty$, since the Schwartz space $\mathscr{S} \cap \dot{B}^s_{r,\infty}$ is not dense in $\dot{B}^s_{r,\infty}$, it is necessary to assume the smallness condition (2.9) for the high frequency part of the initial data. Here notice that the function φ in the closure of $\mathscr{S} \cap \dot{B}^s_{r,\infty}$ for the norm $\|\cdot\|_{\dot{B}^s_{r,\infty}}$ satisfies $\lim_{j\to\pm\infty} 2^{sj}\|\dot{\Delta}_j\varphi\|_{L^r} = 0$.

(ii) Although it is unknown whether the continuity like (2.4) is valid due to the lack of the density, we may show the weak-star continuity by the duality argument. For this reason, the case of $r = 1$ or $q = 1$ is excluded.

(iii) In case $\rho = \infty$, we do not have to consider the Lorentz regularity of solutions since it holds that $n \in L^{2/\alpha,\infty}((0,T); \dot{B}^{\alpha-2+N/r}_{r,1})$ and $\boldsymbol{u} \in L^{2/\beta,\infty}((0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N)$ for all $(n, \boldsymbol{u})$ satisfying (2.10). For details, see Proposition 3.7.

Theorem 2.5 (Global existence and time-decay properties)**.** *In Theorems 2.1 and 2.3, there exists a constant* $\varepsilon = \varepsilon(d, \nu, N, r, q, \rho, s, \alpha, \beta) > 0$ *such that if* $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ *satisfies*

$$(2.14) \qquad \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} < \varepsilon,$$

then $T = \infty$ *holds. In particular, the decay estimates of the global solution are given by*

$$(2.15) \qquad \|n(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,1}} = O(t^{-\alpha/2}), \quad \|\boldsymbol{u}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,1}} = O(t^{-\beta/2})$$

as $t \to \infty$. *Likewise, it holds that* (2.15) *with* α *and* β *replaced by* s. *Moreover, if* $1 \leq \rho < \infty$, *it holds that*

$$(2.16) \qquad \lim_{t\to\infty} \|n(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} = 0, \quad \lim_{t\to\infty} \|\boldsymbol{u}(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}} = 0.$$

If $\rho = \infty$, *it holds that*

$$(2.17) \qquad \lim_{t\to\infty} \langle n(t), \varphi\rangle = 0, \quad \lim_{t\to\infty} \langle \boldsymbol{u}(t), \boldsymbol{f}\rangle = 0$$

for all $\varphi \in \dot{B}^{2-N/r}_{r/(r-1),1}$ *and* $\boldsymbol{f} \in (\dot{B}^{1-N/q}_{q/(q-1),1})^N$.

Remark 2.6. (i) The decay rates (2.15) of the global solutions coincide with the rates of the solutions of the linear heat equation. In fact, we see by Kozono-Ogawa-Taniuchi [23, Lemma 2.2] that $\|e^{t\Delta}\varphi\|_{\dot{B}^{s+\beta}_{r,1}} \leq Ct^{-\beta/2}\|\varphi\|_{\dot{B}^s_{r,\infty}}$. We may expect that these results are obtained since the method of construction of global solutions relies on the linear analysis and perturbation theory.

(ii) Chae-Kang-Lee [6–8] obtained the decay rates of global solutions of the Keller-Segel-Navier-Stokes system in $\mathbb{R}^N$, $N = 2, 3$. Compared with our results, they [6] showed the decays as the limit $t \to \infty$ in the sense of L^∞. Although it is assumed sufficient regularities of the initial data, they achieved to relax the smallness assumptions. The method in [6] is based on the a priori estimates, so our method is entirely different from that of [6]. On the other hand, they [7] also considered a similar model and showed that its solutions behave like the heat kernel asymptotically. Unlike the case of [6], the method in [7] has some similarities to ours since the method relies on introducing the time-weighted spaces and estimating the integral systems.

(iii) The properties (2.16) imply that the global solution $(n, \boldsymbol{u})$ decays as the limit $t \to \infty$ in the same norm of the space of the initial data. This may be regarded as a corresponding result to that of Kato [19, Note] who considered the Navier-Stokes system with initial data in $P(L^N)^N$. Moreover, Kozono-Okada-Shimizu [24, Theorem 1] also showed the time-decay properties with initial data in $P(\dot{B}^{-1+N/q}_{q,\rho})^N$ provided $N < q < \infty$ and $1 \le \rho < \infty$. Here it should be noticed that we further obtain the time-decay properties (2.17) in the sense of the weak-star topology even if $\rho = \infty$.

2.2. **Maximal regularity estimates.** In this subsection, we state our main results on the maximal regularity estimates of solutions obtained in Theorem 2.1 under the certain conditions.

Theorem 2.7. *In Theorem 2.1, suppose that* $1 \le r, q < 3N/5$ *satisfy* (2.1)*. If* s *satisfies* $0 < s < 2/3$ *along with* (2.2)*, then the solution* $(n, \boldsymbol{u})$ *on* $(0,T) \times \mathbb{R}^N$ *of* (1.5) *has the following properties*

$$
\begin{cases} t^s \partial_t n, t^s \Delta n \in BC([0,T); \dot{B}^{2s-4+N/r}_{r,1}), & \partial_t n, \Delta n \in L^{1/s,\rho}((0,T); \dot{B}^{2s-4+N/r}_{r,1}), \\ t^s \partial_t \boldsymbol{u}, t^s \Delta \boldsymbol{u} \in BC([0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N), & \partial_t \boldsymbol{u}, \Delta \boldsymbol{u} \in L^{1/s,\rho}((0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N) \end{cases} \tag{2.18}
$$

with

$$
\lim_{t \to +0} t^s \|n(t)\|_{\dot{B}^{2s-2+N/r}_{r,1}} = 0, \quad \lim_{t \to +0} t^s \|\boldsymbol{u}(t)\|_{\dot{B}^{2s-1+N/q}_{q,1}} = 0 \tag{2.19}
$$

having the estimates

$$
\begin{aligned} &\|t^s n\|_{L^\infty_T(\dot{B}^{2s-2+N/r}_{r,1})} + \|n\|_{L^{1/s,\rho}_T(\dot{B}^{2s-2+N/r}_{r,1})} + \|t^s \boldsymbol{u}\|_{L^\infty_T(\dot{B}^{2s-1+N/q}_{q,1})} + \|\boldsymbol{u}\|_{L^{1/s,\rho}_T(\dot{B}^{2s-1+N/q}_{q,1})} \\ &\le C \left(d^{-s} \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-s} \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right) \end{aligned} \tag{2.20}
$$

and

$$
\begin{cases} \|t^s \partial_t n\|_{L^\infty_T(\dot{B}^{2s-4+N/r}_{r,1})} + \|\partial_t n\|_{L^{1/s,\rho}_T(\dot{B}^{2s-4+N/r}_{r,1})} \\ \le C \left(d^{-s}(d + \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}}) \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-s}(d + \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}) \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right), \\ \|t^s \partial_t \boldsymbol{u}\|_{L^\infty_T(\dot{B}^{2s-3+N/q}_{q,1})} + \|\partial_t \boldsymbol{u}\|_{L^{1/s,\rho}_T(\dot{B}^{2s-3+N/q}_{q,1})} \\ \le C \left(d^{-s}(\nu + \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}}) \|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-s}(\nu + \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}) \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right), \end{cases} \tag{2.21}
$$

where $C = C(N, r, q, \rho, s) > 0$ *is a constant independent of* $d, \nu, T, a, \boldsymbol{b}, n,$ *and* $\boldsymbol{u}$.

Remark 2.8. (i) To show Theorem 2.7, we suppose the stronger conditions $1 \le r, q < 3N/5$ so that we may take $0 < s < 2/3$. In this case, the derivative indices of the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ are greater than $-1/3$ and $2/3$, respectively.

(ii) Since it holds that (2.18), we may conceive of Theorem 2.7 as a maximal regularity theorem for the system (1.5). Here, by the real interpolation, we have

$$
(\dot{B}^{2s-4+N/r}_{r,1}, \dot{B}^{2s-2+N/r}_{r,1})_{1-s,\rho} = \dot{B}^{-2+N/r}_{r,\rho}, \quad (\dot{B}^{2s-3+N/q}_{q,1}, \dot{B}^{2s-1+N/q}_{q,1})_{1-s,\rho} = \dot{B}^{-1+N/q}_{q,\rho}. \tag{2.22}
$$

Notice that $(X, D(A))_{1-1/\alpha,\alpha}$ is known as a space of initial data in theory of the maximal L^α-regularity [31, Definition 3.5.1]. Hence, the relations (2.22) may be regarded as a corresponding space to $(X, D(A))_{1-1/\alpha,\alpha}$. Moreover, in the case of $t^{1-\mu}$-weighted L^α space, the space of initial data is given by $(X, D(A))_{\mu-1/\alpha,\alpha}$ [31, Theorem 3.5.5]. Hence, by letting $\mu = 1 - s$ and $\alpha = \infty$ formally, we have similar relations to (2.22) as well.

2.3. **Additional regularities.** In this subsection, we state our main results on the additional regularities of solutions obtained in Theorem 2.1 if the initial data has higher regularities.

Theorem 2.9. *Let $1 \le r, q < N$, $1 \le \rho < \infty$, and $0 < s < 1$ be as in Theorem 2.1 and let $N/(N-1+s) \le \theta < N$ and $0 < \sigma < s$. Suppose that $(n, \boldsymbol{u})$ is the solution on $(0,T) \times \mathbb{R}^N$ of (1.5) with the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ obtained in Theorem 2.1. If the initial data has the additional regularity*

$$(a, \boldsymbol{b}) \in (\dot{B}^{-2+N/r}_{r,\rho} \cap L^\theta) \times P(\dot{B}^{-1+N/q}_{q,\rho} \cap L^{N\theta/(N-\theta)})^N,$$

then the solution also satisfies

$$\begin{cases} n \in BC([0,T); L^\theta), & t^{\sigma/2} n \in BC([0,T); \dot{B}^\sigma_{\theta,1}), \\ \boldsymbol{u} \in BC([0,T); P(L^{N\theta/(N-\theta)})^N), & t^{\sigma/2}\boldsymbol{u} \in BC([0,T); P(\dot{B}^\sigma_{N\theta/(N-\theta),1})^N) \end{cases} \tag{2.23}$$

with

$$\begin{cases} \lim_{t\to+0} \|n(t) - a\|_{L^\theta} = 0, & \lim_{t\to+0} t^{\sigma/2}\|n(t)\|_{\dot{B}^\sigma_{\theta,1}} = 0, \\ \lim_{t\to+0} \|\boldsymbol{u}(t) - \boldsymbol{b}\|_{L^{N\theta/(N-\theta)}} = 0, & \lim_{t\to+0} t^{\sigma/2}\|\boldsymbol{u}(t)\|_{\dot{B}^\sigma_{N\theta/(N-\theta),1}} = 0. \end{cases} \tag{2.24}$$

Moreover, the following estimates

$$\begin{cases} \|n\|_{L^\infty_{T_*}(L^\theta)} + \|\boldsymbol{u}\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})} \le 2(\|a\|_{L^\theta} + \|\boldsymbol{b}\|_{L^{N\theta/(N-\theta)}}), \\ \|t^{\sigma/2} n\|_{L^\infty_{T_*}(\dot{B}^\sigma_{\theta,1})} + \|t^{\sigma/2}\boldsymbol{u}\|_{L^\infty_{T_*}(\dot{B}^\sigma_{N\theta/(N-\theta),1})} \le C\left(d^{-\sigma/2}\|a\|_{L^\theta} + \nu^{-\sigma/2}\|\boldsymbol{b}\|_{L^{N\theta/(N-\theta)}}\right) \end{cases} \tag{2.25}$$

hold with some $0 < T_* < T$ and constant $C = C(N, r, q, \rho, s, \alpha, \beta, \theta, \sigma) > 0$ independent of $d, \nu, T, T_*, a, \boldsymbol{b}, n$, and $\boldsymbol{u}$. In addition, there exists a constant $\varepsilon_* = \varepsilon_*(d, \nu, N, r, q, \rho, s, \alpha, \beta, \theta, \sigma) > 0$ such that if $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ satisfies (2.14) with ε replaced by ε_*, then $T = T_* = \infty$ holds.

Remark 2.10. Theorem 2.9 plays a key role in the proof of the time-decay properties of global solutions, i.e., Theorem 2.5. It should be noticed that we do not have to assume the smallness condition for the norm of $L^\theta \times P(L^{N\theta/(N-\theta)})^N$ to obtain the estimates (2.25) with $T_* = \infty$. The proof relies on considering the linearized problem by using the solution obtained in Theorem 2.1. We refer to [24, Lemma 3.2] for this strategy.

Remark 2.11. (i) As mentioned before, it has been also considered damping terms of logistic type [37, 46, 48]. Since these results yield the global existence of solutions with the aid of logistic terms, it might be expected that our results are improved by considering such structures as well.

(ii) Concerning the nonlinear term $\nabla\cdot(n\nabla c)$ in (1.1), which stands for the effect of the chemotactic cross-diffusion, it should be noticed that a slight modification of the term is also interesting in some applications. Here, Xue-Othmer [49] have been proposed that the term $\nabla \cdot (nS(x, n, c) \cdot \nabla c)$ is used instead of $\nabla \cdot (n\nabla c)$ to describe the more exact physical model from the experimental observation. However, since $S(x, n, c)$ is an $\mathbb{R}^{N\times N}$-valued function, studying such a model is not so easy in contrast to the usual model. In spite of the fact, by assuming that $N = 2$ and the domain $\Omega \subset \mathbb{R}^2$ is bounded, Winkler [45, 47] showed the global existence of classical solutions without the smallness assumption of the initial data. Here, since results in the higher dimensional case are not enough yet, it is also interesting to extend our results to the corresponding system.

3. Preliminaries

3.1. **Notations and function spaces.** In the following, let us introduce notations and function spaces used throughout this paper. Let $\mathscr{S}(\mathbb{R}^N)$ and $\mathscr{S}'(\mathbb{R}^N)$ denote the Schwartz space and its dual space, respectively. We also set $\mathscr{S}_0(\mathbb{R}^N) := \{\varphi \in \mathscr{S}(\mathbb{R}^N) \,|\, 0 \notin \operatorname{supp} \mathcal{F}\varphi\}$, where $\mathcal{F}$ denotes the Fourier transform. The homogeneous Besov spaces are defined by

$$\dot{B}^s_{r,\rho}(\mathbb{R}^N) := \{\varphi \in \mathscr{S}'(\mathbb{R}^N) \,|\, \|\varphi\|_{\dot{B}^s_{r,\rho}(\mathbb{R}^N)} := \|\{2^{js}\|\dot{\Delta}_j\varphi\|_{L^r(\mathbb{R}^N)}\}_{j\in\mathbb{Z}}\|_{l^\rho(\mathbb{Z})} < \infty\}$$

for $1 \le r \le \infty$, $s \in \mathbb{R}$, and $1 \le \rho \le \infty$, where $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ denotes the dyadic decomposition defined by Bahouri-Chemin-Danchin [1, Proposition 2.10 and Definition 2.15]. By the definition, we see immediately that $\dot{B}^s_{r,\rho_0}(\mathbb{R}^N) \subset \dot{B}^s_{r,\rho_1}(\mathbb{R}^N)$ for $1 \le \rho_0 \le \rho_1 \le \infty$. Here we note that the space $\dot{B}^s_{r,\rho}(\mathbb{R}^N)$ is suitable to consider the scaling invariant spaces, but $\dot{B}^s_{r,\rho}(\mathbb{R}^N)$ defined as above is not Banach space. In fact, any function $\varphi \in \mathscr{S}'(\mathbb{R}^N)$ satisfying $\|\varphi\|_{\dot{B}^s_{r,\rho}(\mathbb{R}^N)} = 0$ is not identically zero but polynomials. However, if $s < N/r$ or $s = N/r$ with $\rho = 1$, then the spaces $\dot{B}^s_{r,\rho}(\mathbb{R}^N)$ become Banach spaces [32, Theorem 3.20].

Let X be a Banach space. Then, the Banach space of all bounded continuous X-valued functions on an interval $I \subset \mathbb{R}$ is denoted by $BC(I;X)$. Let $L^\alpha(I;X)$ denote the Bochner-Lebesgue spaces on I and we write $\varphi \in L^\alpha_{\mathrm{loc}}(I;X)$ if $\varphi \in L^\alpha(K;X)$ holds for arbitrary compact subintervals $K \subset I$. The Lorentz spaces on I are defined by $L^{\alpha,\rho}(I;X) := \{\varphi \in L^1_{\mathrm{loc}}(I;X) \,|\, \|\varphi\|_{L^{\alpha,\rho}(I;X)} < \infty\}$ for $1 < \alpha < \infty$ and $1 \le \rho \le \infty$ with the norm

$$\|\varphi\|_{L^{\alpha,\rho}(I;X)} := \begin{cases} \{\int_0^\infty (\tau^{1/\alpha}\varphi^*(\tau))^\rho d\tau/\tau\}^{1/\rho} & \text{if } 1 \le \rho < \infty, \\ \sup_{0<\tau<\infty} \tau^{1/\alpha}\varphi^*(\tau) & \text{if } \rho = \infty, \end{cases}$$

where φ^* denotes the rearrangement of φ given by Castillo-Rafeiro [5, Definitions 4.4 and 6.1]. Notice that the Lorentz space is a quasi-Banach space with the properties $L^{\alpha,\alpha}(I;X) = L^\alpha(I;X)$ and $L^{\alpha,\rho_0}(I;X) \subset L^{\alpha,\rho_1}(I;X)$ for $1 \le \rho_0 \le \rho_1 \le \infty$ [5, Definition 6.1 and Theorem 6.3]. Moreover, in case $\rho = \infty$, the Lorentz space $L^{\alpha,\infty}(I;X)$ coincides with the weak Lebesgue space. In fact, it holds that

$$\|\varphi\|_{L^{\alpha,\infty}(I;X)} = \sup_{0<\lambda<\infty} \lambda\mu(t \in I \,|\, \|\varphi(t)\|_X > \lambda)^{1/\alpha}$$

for $1 < \alpha < \infty$, where μ denotes the usual Lebesgue measure on I [5, Theorem 6.6].

Let $\{e^{t\Delta}\}_{0<t<\infty}$ denote the heat semigroup on $\mathbb{R}^N$. For the Helmholtz projection P on $\mathbb{R}^N$, we write $PX := \{P\varphi \,|\, \varphi \in X\}$. It should be noticed that P is a bounded operator from $\dot{B}^s_{r,\rho}(\mathbb{R}^N)$ onto itself for any $1 \le r \le \infty$, $s \in \mathbb{R}$, and $1 \le \rho \le \infty$ [25, Proposition 2.1]. In the following, we will abbreviate $X := X(\mathbb{R}^N)$ for a function space $X(\mathbb{R}^N)$ defined on $\mathbb{R}^N$. Also, we write $\|\cdot\|_{L^\infty_T(X)} := \|\cdot\|_{L^\infty((0,T);X)}$, $\|\cdot\|_{L^{\alpha,\rho}_T(X)} := \|\cdot\|_{L^{\alpha,\rho}((0,T);X)}$, and $\|\varphi,\psi\|_X := \|\varphi\|_X + \|\psi\|_X$ for simplicity.

3.2. **Fundamental properties of the homogeneous Besov and Lorentz spaces.** In this subsection, we shall recall the fundamental properties on the spaces defined in the previous subsection. In what follows, let X and Y denote Banach spaces. The following proposition plays a crucial role as the Sobolev embeddings in the homogeneous Besov spaces:

Proposition 3.1. *(i) For $1 \le r \le q \le \infty$, $s \in \mathbb{R}$, and $1 \le \rho \le \infty$, the following continuous embedding $\dot{B}^s_{r,\rho} \subset \dot{B}^{s-N(1/r-1/q)}_{q,\rho}$ holds.*

(ii) For $1 \le r \le \infty$, the continuous embedding $\dot{B}^{N/r}_{r,1} \subset BC \subset L^\infty$ holds.

For the proof of (i), see Bahouri-Chemin-Danchin [1, Proposition 2.20]. On the other hand, we may show (ii) by Sawano [32, Theorem 3.21] with the aid of (i), i.e., $\dot{B}^{N/r}_{r,1} \subset \dot{B}^0_{\infty,1} \subset BC$. The density and duality properties for the homogeneous Besov spaces are given as follows:

Proposition 3.2. *Let $1 \le r < \infty$, $s \in \mathbb{R}$, and $1 \le \rho < \infty$. Then the set $\mathscr{S}_0 \subset \dot{B}^s_{r,\rho}$ is dense in $\dot{B}^s_{r,\rho}$. Moreover, it holds that $(\dot{B}^s_{r,\rho})^* = \dot{B}^{-s}_{r/(r-1),\rho/(\rho-1)}$.*

Proof. The first assertion is given by Bahouri-Chemin-Danchin [1, Proposition 2.27]. See also Sawano [32, Theorem 3.15]. The duality properties may be shown by Bahouri-Chemin-Danchin [1, Proposition 2.29] and the fact that $\mathscr{S} \cap \dot{B}^s_{r,\rho}$ is dense in $\dot{B}^s_{r,\rho}$ provided $1 \le r, \rho < \infty$. □

Next we give the fractional Leibniz rule in the homogeneous Besov spaces.

Proposition 3.3. *(i) Let $1 \le r \le \infty$, $0 < s < \infty$, $1 \le \rho \le \infty$, and $0 < \lambda_0, \lambda_1 < \infty$. Assume that $1 \le q_0, q_1, r_0, r_1 \le \infty$ satisfy $1/r = 1/q_0 + 1/q_1 = 1/r_0 + 1/r_1$. Then, for every $\varphi \in \dot{B}^{s+\lambda_0}_{q_0,\rho} \cap \dot{B}^{-\lambda_1}_{r_1,\infty}$ and $\psi \in \dot{B}^{s+\lambda_1}_{r_0,\rho} \cap \dot{B}^{-\lambda_0}_{q_1,\infty}$, it holds that $\varphi\psi \in \dot{B}^s_{r,\rho}$ with the estimate*

$$\|\varphi\psi\|_{\dot{B}^s_{r,\rho}} \le C(\|\varphi\|_{\dot{B}^{s+\lambda_0}_{q_0,\rho}}\|\psi\|_{\dot{B}^{-\lambda_0}_{q_1,\infty}} + \|\varphi\|_{\dot{B}^{-\lambda_1}_{r_1,\infty}}\|\psi\|_{\dot{B}^{s+\lambda_1}_{r_0,\rho}}),$$

where $C = C(N, q_0, q_1, r_0, r_1, s, \rho, \lambda_0, \lambda_1) > 0$ is a constant independent of φ and ψ.

(ii) Let $1 \le r \le \infty$, $0 < s < \infty$, and $1 \le \rho \le \infty$. Assume that $1 \le q_0, q_1, r_0, r_1 \le \infty$ satisfy $1/r = 1/q_0 + 1/q_1 = 1/r_0 + 1/r_1$. Then, for every $\varphi \in \dot{B}^s_{q_0,\rho} \cap L^{r_1}$ and $\psi \in \dot{B}^s_{r_0,\rho} \cap L^{q_1}$, it holds that $\varphi\psi \in \dot{B}^s_{r,\rho}$ with the estimate

$$\|\varphi\psi\|_{\dot{B}^s_{r,\rho}} \le C(\|\varphi\|_{\dot{B}^s_{q_0,\rho}}\|\psi\|_{L^{q_1}} + \|\varphi\|_{L^{r_1}}\|\psi\|_{\dot{B}^s_{r_0,\rho}}),$$

where $C = C(N, q_0, q_1, r_0, r_1, s, \rho) > 0$ is a constant independent of φ and ψ.

The proof of Proposition 3.3 is given by Kaneko-Kozono-Shimizu [17, Proposition 2.2]. Concerning the estimates of the function $\boldsymbol{K}$ defined by (1.4), we may show the following proposition by the Hardy-Littlewood-Sobolev inequality:

Proposition 3.4. *Let $1 < r < N$, $s \in \mathbb{R}$, and $1 \le \rho \le \infty$. Then the following estimates*

$$\|\boldsymbol{K} * \varphi\|_{L^{r_0}} \le C\|\varphi\|_{L^r}, \quad \|\boldsymbol{K} * \varphi\|_{\dot{B}^s_{r_0,\rho}} \le C\|\varphi\|_{\dot{B}^s_{r,\rho}}$$

hold for all $\varphi \in \mathscr{S}_0$, where $r_0 := (1/r - 1/N)^{-1}$ and $C = C(N, r, s, \rho) > 0$ is a constant independent of φ.

Proof. It is sufficient to show the first estimate. Since $|\boldsymbol{K}(x)| \le \omega_{N-1}^{-1}|x|^{N-1}$ for all $x \in \mathbb{R}^N$ from the definition (1.4) of $\boldsymbol{K}$, we have $\|\boldsymbol{K} * \varphi\|_{L^{r_0}} \le \omega_{N-1}^{-1}\||\cdot|^{N-1} * |\varphi|\|_{L^{r_0}}$. Thus we obtain the desired estimate by virtue of the Hardy-Littlewood-Sobolev inequality [33, V, Theorem 1]. □

Let us verify the Hölder inequality for the Lorentz spaces.

Proposition 3.5. *Let $1 < \alpha < \infty$, $1 \le \rho \le \infty$, and $I \subset \mathbb{R}$. Suppose that $1 < \alpha_0, \alpha_1 < \infty$ and $1 \le \rho_0, \rho_1 \le \infty$ satisfy $1/\alpha = 1/\alpha_0 + 1/\alpha_1$ and $1/\rho = 1/\rho_0 + 1/\rho_1$, respectively. Then, for every $\varphi \in L^{\alpha_0,\rho_0}(I;\mathbb{R})$ and $\psi \in L^{\alpha_1,\rho_1}(I;\mathbb{R})$, it holds that $\varphi\psi \in L^{\alpha,\rho}(I;\mathbb{R})$ with the Hölder inequality*

$$\|\varphi\psi\|_{L^{\alpha,\rho}(I;\mathbb{R})} \le 2^{1/\alpha}\|\varphi\|_{L^{\alpha_0,\rho_0}(I;\mathbb{R})}\|\psi\|_{L^{\alpha_1,\rho_1}(I;\mathbb{R})}.$$

In particular, it holds that $\|\varphi\psi\|_{L^{\alpha,\rho}(I;\mathbb{R})} \le C\|\varphi\|_{L^{\alpha_0,\rho}(I;\mathbb{R})}\|\psi\|_{L^{\alpha_1,\rho}(I;\mathbb{R})}$ for all $\varphi \in L^{\alpha_0,\rho}(I;\mathbb{R})$ and $\psi \in L^{\alpha_1,\rho}(I;\mathbb{R})$ with some constant $C = C(\alpha_0, \alpha_1, \rho) > 0$ independent of I, φ, and ψ.

Proof. Notice that it holds by Castillo-Rafeiro [5, Theorem 4.11] that $(\varphi\psi)^*(\tau) \le \varphi^*(\tau/2)\psi^*(\tau/2)$. Hence, in case $1 \le \rho, \rho_0, \rho_1 < \infty$, the usual Hölder inequality yields

$$\begin{aligned}
&\left\{\int_0^\infty (\tau^{1/\alpha}(\varphi\psi)^*(\tau))^\rho \frac{d\tau}{\tau}\right\}^{1/\rho} \\
&\le 2^{1/\alpha}\left\{\int_0^\infty (\lambda^{1/\alpha_0-1/\rho_0}\varphi^*(\lambda)\lambda^{1/\alpha_1-1/\rho_1}\psi^*(\lambda))^\rho d\lambda\right\}^{1/\rho} \\
&\le 2^{1/\alpha}\left\{\int_0^\infty (\lambda^{1/\alpha_0-1/\rho_0}\varphi^*(\lambda))^{\rho_0} d\lambda\right\}^{1/\rho_0}\left\{\int_0^\infty (\lambda^{1/\alpha_1-1/\rho_1}\psi^*(\lambda))^{\rho_1} d\lambda\right\}^{1/\rho_1} \\
&= 2^{1/\alpha}\|\varphi\|_{L^{\alpha_0,\rho_0}(I;\mathbb{R})}\|\psi\|_{L^{\alpha_1,\rho_1}(I;\mathbb{R})}.
\end{aligned}$$

The remaining cases may be shown in a similar manner. This completes the proof of Proposition 3.5. □

By combining the Hölder inequality and the Hardy-Littlewood-Sobolev inequality for the Lorentz spaces, we may show the estimates of the bilinear singular integral operators in the Lorentz spaces.

Proposition 3.6. *Let $1 < \alpha < \infty$, $1 \le \rho \le \infty$, $0 < \lambda < 1 - 1/\alpha$, and $0 < T \le \infty$. Suppose that $\varphi \in L^{\alpha_0,\rho}((0,T);X)$ and $\psi \in L^{\alpha_1,\rho}((0,T);Y)$, where $1 < \alpha_0, \alpha_1 < \infty$ satisfy $1/\alpha + \lambda = 1/\alpha_0 + 1/\alpha_1$. Then, for the function $\mathcal{I}_\lambda(\varphi,\psi)$ defined by*

$$\mathcal{I}_\lambda(\varphi,\psi)(t) := \int_0^t (t-\tau)^{\lambda-1}\|\varphi(\tau)\|_X\|\psi(\tau)\|_Y d\tau, \quad 0 < t < T,$$

it holds that $\mathcal{I}_\lambda(\varphi,\psi) \in L^{\alpha,\rho}((0,T);\mathbb{R})$ with the estimate

$$\|\mathcal{I}_\lambda(\varphi,\psi)\|_{L_T^{\alpha,\rho}(\mathbb{R})} \le C\|\varphi\|_{L_T^{\alpha_0,\rho}(X)}\|\psi\|_{L_T^{\alpha_1,\rho}(Y)},$$

where $C = C(\alpha_0,\alpha_1,\rho,\lambda) > 0$ is a constant inpedendent of T, φ, and ψ.

Proof. We define $\overline{\varphi}$ and $\overline{\psi}$ by setting

$$\overline{\varphi}(t) := \begin{cases} \|\varphi(t)\|_X & t \in (0,T), \\ 0 & t \in \mathbb{R}\setminus(0,T), \end{cases} \qquad \overline{\psi}(t) := \begin{cases} \|\psi(t)\|_Y & t \in (0,T), \\ 0 & t \in \mathbb{R}\setminus(0,T). \end{cases}$$

Then it holds that

$$|\mathcal{I}_\lambda(\varphi,\psi)(t)| \le \int_{-\infty}^\infty |t-\tau|^{\lambda-1}|\overline{\varphi}(\tau)||\overline{\psi}(\tau)|d\tau$$

for all $t \in \mathbb{R}$. Thus we have

$$\|\mathcal{I}_\lambda(\varphi,\psi)\|_{L^{\alpha,\rho}((0,T);\mathbb{R})} \le C\|\overline{\varphi}\overline{\psi}\|_{L^{(1/\alpha+\lambda)^{-1},\rho}(\mathbb{R};\mathbb{R})} \tag{3.1}$$

since the Hardy-Littlewood-Sobolev inequality [33, V, Theorem 1] is still valid for Lorentz spaces by virtue of real interpolation theory [38, p.134]. Here we note that $0 < 1/\alpha + \lambda < 1$. On the other hand, Proposition 3.5 gives that $\overline{\varphi}\overline{\psi} \in L^{(1/\alpha+\lambda)^{-1},\rho}(\mathbb{R};\mathbb{R})$ holds with the estimate

$$\|\overline{\varphi}\overline{\psi}\|_{L^{(1/\alpha+\lambda)^{-1},\rho}(\mathbb{R};\mathbb{R})} \le C\|\overline{\varphi}\|_{L^{\alpha_0,\rho}(\mathbb{R};\mathbb{R})}\|\overline{\psi}\|_{L^{\alpha_1,\rho}(\mathbb{R};\mathbb{R})} = C\|\varphi\|_{L^{\alpha_0,\rho}((0,T);X)}\|\psi\|_{L^{\alpha_1,\rho}((0,T);Y)}. \tag{3.2}$$

Hence, we may show the desired estimate by (3.1) and (3.2). This completes the proof of Proposition 3.6. □

The following proposition gives the relation between the time weighted spaces and the weak Lebesgue spaces:

Proposition 3.7. *Let $0 < \beta < 1$ and $0 < T \le \infty$. Then, for every $\varphi \in L^\infty_{\mathrm{loc}}((0,T);X)$ satisfying $t^\beta \varphi \in L^\infty((0,T);X)$, it holds that $\varphi \in L^{1/\beta,\infty}((0,T);X)$ with the estimate*

$$\|\varphi\|_{L_T^{1/\beta,\infty}(X)} \le \|t^\beta \varphi\|_{L_T^\infty(X)}.$$

Moreover, if $\lim_{t\to+0} t^\beta \|\varphi(t)\|_X = 0$, then it holds that $\lim_{\lambda\to\infty} \lambda \mu(t \in (0,T) \,|\, \|\varphi(t)\|_X > \lambda)^\beta = 0$.

Proof. By the assumption, we have $\|\varphi(t)\|_X \le M t^{-\beta}$ for all $0 < t < T$, where $M := \|t^\beta \varphi\|_{L_T^\infty(X)}$. Thus we see that

$$\begin{aligned}\mu(t \in (0,T) \,|\, \|\varphi(t)\|_X > \lambda)^\beta &\le \mu(t \in (0,T) \,|\, M t^{-\beta} > \lambda)^\beta \\ &\le \mu(t \in (0,\infty) \,|\, (M/\lambda)^{1/\beta} > t)^\beta = M/\lambda,\end{aligned}$$

which yields $\|\varphi\|_{L_T^{1/\beta,\infty}(X)} \le M$. Next we assume that $\lim_{t\to+0} t^\beta \|\varphi(t)\|_X = 0$. Then, for arbitrarily small $0 < \varepsilon < T$, there exists $0 < T_\varepsilon < T$ such that $\|\varphi(t)\|_X \le \varepsilon t^{-\beta}$ for all $0 < t < T_\varepsilon$. On the other hand, we have $\|\varphi(t)\|_X \le M T_\varepsilon^{-\beta/2} t^{-\beta/2}$ for all $T_\varepsilon < t < T$. Hence we observe that

$$\begin{aligned}&\mu(t \in (0,T) \,|\, \|\varphi(t)\|_X > \lambda)^\beta \\ &\le \mu(t \in (0,T_\varepsilon) \,|\, \varepsilon t^{-\beta} > \lambda)^\beta + \mu(t \in (T_\varepsilon,T) \,|\, M T_\varepsilon^{-\beta/2} t^{-\beta/2} > \lambda)^\beta \\ &\le \mu(t \in (0,\infty) \,|\, (\varepsilon/\lambda)^{1/\beta} > t)^\beta + \mu(t \in (0,\infty) \,|\, T_\varepsilon^{-1} (M/\lambda)^{2/\beta} > t)^\beta = \varepsilon/\lambda + T_\varepsilon^{-\beta} (M/\lambda)^2,\end{aligned}$$

which implies $\limsup_{\lambda\to\infty} \lambda \mu(t \in (0,T) \,|\, \|\varphi(t)\|_X > \lambda)^\beta \le \varepsilon$. Therefore, letting $\varepsilon \to +0$ yields the desired result. This completes the proof of Proposition 3.7. □

3.3. **Linear theory of the heat semigroup.** In this subsection, we shall give some properties of the heat semigroup. First we recall the smoothing estimates of the heat semigroup in the homogeneous Besov spaces.

Proposition 3.8. *Let $1 \le r \le q \le \infty$, $s \in \mathbb{R}$, $1 \le \rho \le \infty$, and $0 < \beta < \infty$. Then it holds that*

$$\begin{cases} \|e^{t\Delta}\varphi\|_{\dot B^s_{r,\rho}} \le \|\varphi\|_{\dot B^s_{r,\rho}}, \\ \|e^{t\Delta}\varphi\|_{\dot B^s_{q,1}} \le C t^{-(N/2)(1/r-1/q)} \|\varphi\|_{\dot B^s_{r,\infty}} \quad \text{if } r < q, \\ \|e^{t\Delta}\varphi\|_{\dot B^{s+\beta}_{q,1}} \le C t^{-(N/2)(1/r-1/q)-\beta/2} \|\varphi\|_{\dot B^s_{r,\infty}} \end{cases}$$

for all $0 < t < \infty$ and $\varphi \in \mathscr{S}_0$, where $C = C(N,r,q,s,\rho,\beta) > 0$ is a constant independent of t and φ.

Proposition 3.8 may be shown by Kozono-Ogawa-Taniuchi [23, Lemma 2.2] with the aid of Proposition 3.1. We also recall the space-time estimates of the heat semigroup.

Proposition 3.9. *Let $1 \le r \le q \le \infty$, $s \in \mathbb{R}$, $1 \le \rho \le \infty$, $0 < \beta < \infty$, and $0 < \nu < \infty$. Assume that $1 < \alpha < \infty$ satisfies $1/\alpha = (N/2)(1/r - 1/q) + \beta/2$. Then, for every $\varphi \in \dot B^s_{r,\rho}$, it holds that $\Phi \in L^{\alpha,\rho}((0,\infty); \dot B^{s+\beta}_{q,1})$ with the estimate*

$$\|\Phi\|_{L^{\alpha,\rho}_\infty(\dot B^{s+\beta}_{q,1})} \le C \nu^{-1/\alpha} \|\varphi\|_{\dot B^s_{r,\rho}},$$

where $\Phi(t) := e^{\nu t \Delta}\varphi$ for $0 < t < \infty$ and $C = C(N,r,q,s,\rho,\beta) > 0$ is a constant independent of ν and φ.

The proof of Proposition 3.9 is given by the author [34, Proposition 3.2]. In what follows, when there is no danger of confusion, we will use the notations $e^{\nu t\Delta}\varphi \in L^{\alpha,\rho}((0,\infty);\dot{B}^{s+\beta}_{q,1})$ and $\|e^{\nu t\Delta}\varphi\|_{L^{\alpha,\rho}_\infty(\dot{B}^{s+\beta}_{q,1})} \le C\nu^{-1/\alpha}\|\varphi\|_{\dot{B}^s_{r,\rho}}$, where t denotes an integration variable. Namely, it does not make particular sense to use the time variable t in these notations. We should refer to Kozono-Shimizu [26], who established the maximal Lorentz regularity theorem for the Stokes system by using the above estimates. In the last of this subsection, we verify the vanishing properties of the heat semigroup.

Proposition 3.10. *(i) Let $1 \le r < \infty$, $s \in \mathbb{R}$, $1 \le \rho < \infty$, and $0 < \beta < \infty$. Then, for every $\varphi \in \dot{B}^s_{r,\rho}$, it holds that $\lim_{t\to+0} t^{\beta/2}\|e^{t\Delta}\varphi\|_{\dot{B}^{s+\beta}_{r,1}} = 0$ and $\lim_{t\to\infty}\|e^{t\Delta}\varphi\|_{\dot{B}^s_{r,\rho}} = 0$.*

(ii) Let $1 < r \le \infty$, $s \in \mathbb{R}$, and $0 < \beta < \infty$. Then, for every $\varphi \in \dot{B}^s_{r,\infty}$, it holds that $\lim_{t\to+0}\left\langle e^{t\Delta}\varphi - \varphi, \psi\right\rangle = 0$ for all $\psi \in \dot{B}^{-s}_{r/(r-1),1}$. In addition, it holds that $\lim_{t\to+0} t^{\beta/2}\left\langle e^{t\Delta}\varphi, \psi\right\rangle = 0$ for all $\psi \in \dot{B}^{-s-\beta}_{r/(r-1),1}$ and $\lim_{t\to\infty}\left\langle e^{t\Delta}\varphi, \psi\right\rangle = 0$ for all $\psi \in \dot{B}^{-s}_{r/(r-1),1}$.

Proof. (i) Let $\varphi \in \dot{B}^s_{r,\rho}$ be arbitrary. By Proposition 3.2, we may take a sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{S}_0$ of functions satisfying $\lim_{j\to\infty}\|\varphi - \varphi_j\|_{\dot{B}^s_{r,\rho}} = 0$. Then we see by Proposition 3.8 that

$$\begin{aligned}
\limsup_{t\to+0} t^{\beta/2}\|e^{t\Delta}\varphi\|_{\dot{B}^{s+\beta}_{r,1}} &\le \limsup_{t\to+0} t^{\beta/2}\|e^{t\Delta}(\varphi-\varphi_j)\|_{\dot{B}^{s+\beta}_{r,1}} + \limsup_{t\to+0} t^{\beta/2}\|e^{t\Delta}\varphi_j\|_{\dot{B}^{s+\beta}_{r,1}}\\
&\le C\|\varphi-\varphi_j\|_{\dot{B}^s_{r,\rho}} + \limsup_{t\to+0} t^{\beta/2}\|\varphi_j\|_{\dot{B}^{s+\beta}_{r,1}} = C\|\varphi-\varphi_j\|_{\dot{B}^s_{r,\rho}},\\
\limsup_{t\to\infty}\|e^{t\Delta}\varphi\|_{\dot{B}^s_{r,\rho}} &\le \limsup_{t\to\infty}\|e^{t\Delta}(\varphi-\varphi_j)\|_{\dot{B}^s_{r,\rho}} + \limsup_{t\to\infty}\|e^{t\Delta}\varphi_j\|_{\dot{B}^s_{r,\rho}}\\
&\le \|\varphi-\varphi_j\|_{\dot{B}^s_{r,\rho}} + C\limsup_{t\to\infty} t^{-1}\|\varphi_j\|_{\dot{B}^{s-2}_{r,\rho}} = \|\varphi-\varphi_j\|_{\dot{B}^s_{r,\rho}},
\end{aligned}$$

which yield the desired results by letting $j \to \infty$.

(ii) The first assertion is a simple consequence of the strong continuity of the heat semigroup $\{e^{t\Delta}\}_{0<t<\infty}$ in $\dot{B}^{-s}_{r/(r-1),1}$. In fact, it holds that

$$\limsup_{t\to+0}\left\langle e^{t\Delta}\varphi-\varphi,\psi\right\rangle = \limsup_{t\to+0}\left\langle \varphi, e^{t\Delta}\psi-\psi\right\rangle \le \limsup_{t\to+0}\|\varphi\|_{\dot{B}^s_{r,\infty}}\|e^{t\Delta}\psi-\psi\|_{\dot{B}^{-s}_{r/(r-1),1}} = 0$$

for all $\psi \in \dot{B}^{-s}_{r/(r-1),1}$ since $(\dot{B}^{-s}_{r/(r-1),1})^* = \dot{B}^s_{r,\infty}$ from Proposition 3.2. Next, let $\varphi \in \dot{B}^s_{r,\infty}$ and $\psi \in \dot{B}^{-s-\beta}_{r/(r-1),1}$ be arbitrary. By Proposition 3.2, we may take a sequence $\{\psi_j\}_{j=1}^\infty \subset \mathscr{S}_0$ of functions satisfying $\lim_{j\to\infty}\|\psi-\psi_j\|_{\dot{B}^{-s-\beta}_{r/(r-1),1}} = 0$. Noting that $(\dot{B}^{-s-\beta}_{r/(r-1),1})^* = \dot{B}^{s+\beta}_{r,\infty}$ and $(\dot{B}^{-s}_{r/(r-1),1})^* = \dot{B}^s_{r,\infty}$, we see by Proposition 3.8 that

$$\begin{aligned}
\left|t^{\beta/2}\left\langle e^{t\Delta}\varphi,\psi\right\rangle\right| &\le t^{\beta/2}\left|\left\langle e^{t\Delta}\varphi,\psi-\psi_j\right\rangle\right| + t^{\beta/2}\left|\left\langle e^{t\Delta}\varphi,\psi_j\right\rangle\right|\\
&\le t^{\beta/2}\|e^{t\Delta}\varphi\|_{\dot{B}^{s+\beta}_{r,\infty}}\|\psi-\psi_j\|_{\dot{B}^{-s-\beta}_{r/(r-1),1}} + t^{\beta/2}\|\varphi\|_{\dot{B}^s_{r,\infty}}\|e^{t\Delta}\psi_j\|_{\dot{B}^{-s}_{r/(r-1),1}}\\
&\le C\|\varphi\|_{\dot{B}^s_{r,\infty}}\|\psi-\psi_j\|_{\dot{B}^{-s-\beta}_{r/(r-1),1}} + t^{\beta/2}\|\varphi\|_{\dot{B}^s_{r,\infty}}\|\psi_j\|_{\dot{B}^{-s}_{r/(r-1),1}},
\end{aligned}$$

which implies $\limsup_{t\to+0}\left|t^{\beta/2}\left\langle e^{t\Delta}\varphi,\psi\right\rangle\right| \le C\|\varphi\|_{\dot{B}^s_{r,\infty}}\|\psi-\psi_j\|_{\dot{B}^{-s-\beta}_{r/(r-1),1}}$. Hence, letting $j\to\infty$ gives that $\lim_{t\to+0} t^{\beta/2}\left\langle e^{t\Delta}\varphi,\psi\right\rangle = 0$. The rest statement may be shown in a similar manner. This completes the proof of Proposition 3.10. □

Remark 3.11. The reason why we exclude the case of $r=\infty$ or $\rho=\infty$ in (i) is due to the lack of the density. Indeed, the set $\mathscr{S}_0$ is not dense in $\dot{B}^s_{r,\rho}$ if $r=\infty$ or $\rho=\infty$. Likewise, since the proof of (ii) is based on the duality argument, we also exclude the case of $r=1$ in (ii).

4. Local well-posedness in the scaling invariant Besov spaces

In this section, we shall show the local well-posedness for (1.5) with initial data in the scaling invariant Besov spaces. To this end, we construct mild solutions of (1.5) by the Banach fixed point theorem.

4.1. **Nonlinear estimates.** First we establish the nonlinear estimates in the homogeneous Besov spaces.

Lemma 4.1. *Let $1\le r,q<N$, $1\le\rho\le\infty$, $0<\alpha,\beta<1$, and $0\le\eta<\infty$. Then the following statements hold for all $\varphi,\psi\in\mathscr{S}$ and $\boldsymbol{f},\boldsymbol{g}\in\mathscr{S}^N$ with some constant $C=C(N,r,q,\rho,\alpha,\beta,\eta)>0$ independent of $\varphi,\psi,\boldsymbol{f}$, and $\boldsymbol{g}$:*

(i) If $\alpha>(1/2)(3-N/r)$, it holds that

$$\|\nabla\cdot(\varphi(\boldsymbol{K}*\psi))\|_{\dot{B}^{2\alpha-4+N/r+\eta}_{r,\rho}}\le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}+\|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}).$$

(ii) If $1/r-1/q<2/N$, $\alpha<2-N(1/r-1/q)$, $\alpha+\beta>3-N/r$, and $\nabla\cdot\boldsymbol{f}=0$, it holds that

$$\|\boldsymbol{f}\cdot\nabla\varphi\|_{\dot{B}^{\alpha+\beta-4+N/r+\eta}_{r,\rho}}\le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}\|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}+\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}}\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}).$$

(iii) If $\beta>(1/2)(2-N/q)$ and $\nabla\cdot\boldsymbol{f}=\nabla\cdot\boldsymbol{g}=0$, it holds that

$$\|(\boldsymbol{f}\cdot\nabla)\boldsymbol{g}\|_{\dot{B}^{2\beta-3+N/q+\eta}_{q,\rho}}\le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}\|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}}+\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}}\|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}).$$

(iv) If $1/r-1/q>-1/N$, $2/r-1/q>1/N$, and $(1/2)(3-N/q)<\alpha<1+N(1/r-1/q)$, it holds that

$$\|\varphi(\boldsymbol{K}*\psi)\|_{\dot{B}^{2\alpha-3+N/q+\eta}_{q,\rho}}\le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}+\|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}).$$

Proof. (i) Since $1\le r<N$ and $0<\alpha<1$, we may take r_0,r_1, and λ such that

$$0<1/r_1<\min\{(1-\alpha)/N,1/r-1/N\},\quad 1/r_0=1/r-1/r_1,\quad \lambda=1-\alpha-N/r_1.$$

Then we have $r<r_0<N$, $r<(1/r_0-1/N)^{-1},(1/r_1+1/N)^{-1}<\infty$, and $\lambda>0$. Therefore, it holds by Proposition 3.1 that

$$\begin{cases}\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}\subset\dot{B}^{\alpha-2+N/r_0+\eta}_{r_0,\rho}=\dot{B}^{2\alpha-3+N/r+\eta+\lambda}_{r_0,\rho},\\ \dot{B}^{\alpha-2+N/r}_{r,\rho}\subset\dot{B}^{\alpha-1+N/r_1}_{(1/r_1+1/N)^{-1},\infty}=\dot{B}^{-\lambda}_{(1/r_1+1/N)^{-1},\infty}.\end{cases}$$

Noting that $2\alpha-3+N/r>0$, we see by Propositions 3.3 and 3.4 that

$$\begin{aligned}&\|\nabla\cdot(\varphi(\boldsymbol{K}*\psi))\|_{\dot{B}^{2\alpha-4+N/r+\eta}_{r,\rho}}\le C\|\varphi(\boldsymbol{K}*\psi)\|_{\dot{B}^{2\alpha-3+N/r+\eta}_{r,\rho}}\\&\le C(\|\varphi\|_{\dot{B}^{2\alpha-3+N/r+\eta+\lambda}_{r_0,\rho}}\|\boldsymbol{K}*\psi\|_{\dot{B}^{-\lambda}_{r_1,\infty}}+\|\varphi\|_{\dot{B}^{-\lambda}_{(1/r_1+1/N)^{-1},\infty}}\|\boldsymbol{K}*\psi\|_{\dot{B}^{2\alpha-3+N/r+\eta+\lambda}_{(1/r_0-1/N)^{-1},\rho}})\\&\le C(\|\varphi\|_{\dot{B}^{2\alpha-3+N/r+\eta+\lambda}_{r_0,\rho}}\|\psi\|_{\dot{B}^{-\lambda}_{(1/r_1+1/N)^{-1},\infty}}+\|\varphi\|_{\dot{B}^{-\lambda}_{(1/r_1+1/N)^{-1},\infty}}\|\psi\|_{\dot{B}^{2\alpha-3+N/r+\eta+\lambda}_{r_0,\rho}})\\&\le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}+\|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}}\|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}).\end{aligned}$$

(ii) Since $1/r - 1/q < 2/N$, $0 < \alpha < \min\{1, 2 - N(1/r - 1/q)\}$, and $0 < \beta < 1$, we may take $q_0, q_1, r_0, r_1, \lambda_\alpha$, and λ_β such that

$$\begin{cases} \max\{0, 1/r - 1/q\} < 1/q_1 < \min\{1/r, (2-\alpha)/N\}, & 1/q_0 = 1/r - 1/q_1, \quad \lambda_\alpha = 2 - \alpha - N/q_1, \\ 0 < 1/r_1 < \min\{1/r, 1/q, (1-\beta)/N\}, & 1/r_0 = 1/r - 1/r_1, \quad \lambda_\beta = 1 - \beta - N/r_1. \end{cases}$$

Then we have $q < q_0, r_1 < \infty$, $r < q_1, r_0 < \infty$, and $\lambda_\alpha, \lambda_\beta > 0$. Therefore, it holds by Proposition 3.1 that

$$\begin{cases} \dot{B}^{\beta-1+N/q+\eta}_{q,\rho} \subset \dot{B}^{\beta-1+N/q_0+\eta}_{q_0,\rho} = \dot{B}^{\alpha+\beta-3+N/r+\eta+\lambda_\alpha}_{q_0,\rho}, \quad \dot{B}^{\alpha-2+N/r}_{r,\rho} \subset \dot{B}^{\alpha-2+N/q_1}_{q_1,\infty} = \dot{B}^{-\lambda_\alpha}_{q_1,\infty}, \\ \dot{B}^{\alpha-2+N/r+\eta}_{r,\rho} \subset \dot{B}^{\alpha-2+N/r_0+\eta}_{r_0,\rho} = \dot{B}^{\alpha+\beta-3+N/r+\eta+\lambda_\beta}_{r_0,\rho}, \quad \dot{B}^{\beta-1+N/q}_{q,\rho} \subset \dot{B}^{\beta-1+N/r_1}_{r_1,\infty} = \dot{B}^{-\lambda_\beta}_{r_1,\infty}. \end{cases}$$

Noting that $\alpha + \beta - 3 + N/r > 0$ and $\boldsymbol{f} \cdot \nabla\varphi = \nabla \cdot (\boldsymbol{f}\varphi)$, we see by Proposition 3.3 that

$$\begin{aligned} \|\boldsymbol{f} \cdot \nabla\varphi\|_{\dot{B}^{\alpha+\beta-4+N/r+\eta}_{r,\rho}} &\le C\|\boldsymbol{f}\varphi\|_{\dot{B}^{\alpha+\beta-3+N/r+\eta}_{r,\rho}} \\ &\le C(\|\boldsymbol{f}\|_{\dot{B}^{\alpha+\beta-3+N/r+\eta+\lambda_\alpha}_{q_0,\rho}}\|\varphi\|_{\dot{B}^{-\lambda_\alpha}_{q_1,\infty}} + \|\boldsymbol{f}\|_{\dot{B}^{-\lambda_\beta}_{r_1,\infty}}\|\varphi\|_{\dot{B}^{\alpha+\beta-3+N/r+\eta+\lambda_\beta}_{r_0,\rho}}) \\ &\le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}\|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} + \|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}}\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}). \end{aligned}$$

(iii) Since $0 < \beta < 1$, we may take q_0, q_1, and λ such that

$$0 < 1/q_1 < \min\{1/q, (1-\beta)/N\}, \quad 1/q_0 = 1/q - 1/q_1, \quad \lambda = 1 - \beta - N/q_1.$$

Then we have $q < q_0, q_1 < \infty$ and $\lambda > 0$. Therefore, it holds by Proposition 3.1 that

$$\begin{cases} \dot{B}^{\beta-1+N/q+\eta}_{q,\rho} \subset \dot{B}^{\beta-1+N/q_0+\eta}_{q_0,\rho} = \dot{B}^{2\beta-2+N/q+\eta+\lambda}_{q_0,\rho}, \\ \dot{B}^{\beta-1+N/q}_{q,\rho} \subset \dot{B}^{\beta-1+N/q_1}_{q_1,\infty} = \dot{B}^{-\lambda}_{q_1,\infty}. \end{cases}$$

Noting that $2\beta - 2 + N/q > 0$ and $(\boldsymbol{f} \cdot \nabla)\boldsymbol{g} = \nabla \cdot (\boldsymbol{f} \otimes \boldsymbol{g})$, we see by Proposition 3.3 that

$$\begin{aligned} \|(\boldsymbol{f} \cdot \nabla)\boldsymbol{g}\|_{\dot{B}^{2\beta-3+N/q+\eta}_{q,\rho}} &\le C\|\boldsymbol{f} \otimes \boldsymbol{g}\|_{\dot{B}^{2\beta-2+N/q+\eta}_{q,\rho}} \\ &\le C(\|\boldsymbol{f}\|_{\dot{B}^{2\beta-2+N/q+\eta+\lambda}_{q_0,\rho}}\|\boldsymbol{g}\|_{\dot{B}^{-\lambda}_{q_1,\infty}} + \|\boldsymbol{f}\|_{\dot{B}^{-\lambda}_{q_1,\infty}}\|\boldsymbol{g}\|_{\dot{B}^{2\beta-2+N/q+\eta+\lambda}_{q_0,\rho}}) \\ &\le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}\|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}} + \|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}}\|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}). \end{aligned}$$

(iv) Since $1/r - 1/q > -1/N$, $2/r - 1/q > 1/N$, and $0 < \alpha < \min\{1, 1 + N(1/r - 1/q)\}$, we may take q_0, q_1, and λ such that

$$\begin{cases} \max\{0, -(1/r - 1/q), \} < 1/q_1 < \min\{1/r - 1/N, 1/q - 1/N, (1-\alpha)/N\}, \\ 1/q_0 = 1/q - 1/q_1, \quad \lambda = 1 - \alpha - N/q_1. \end{cases}$$

Thus we have $r < q_0 < N$, $q < (1/q_0 - 1/N)^{-1} < \infty$, $r < (1/q_1 + 1/N)^{-1} < \infty$, and $\lambda > 0$. Therefore, it holds by Proposition 3.1 that

$$\begin{cases} \dot{B}^{\alpha-2+N/r+\eta}_{r,\rho} \subset \dot{B}^{\alpha-2+N/q_0+\eta}_{q_0,\rho} = \dot{B}^{2\alpha-3+N/q+\eta+\lambda}_{q_0,\rho}, \\ \dot{B}^{\alpha-2+N/r}_{r,\rho} \subset \dot{B}^{\alpha-1+N/q_1}_{(1/q_1+1/N)^{-1},\infty} = \dot{B}^{-\lambda}_{(1/q_1+1/N)^{-1},\infty}. \end{cases}$$

Noting that $2\alpha - 3 + N/q > 0$, we see by Propositions 3.3 and 3.4 that

$$
\begin{aligned}
&\|\varphi(\boldsymbol{K} * \psi)\|_{\dot{B}^{2\alpha-3+N/q+\eta}_{q,\rho}} \\
&\le C(\|\varphi\|_{\dot{B}^{2\alpha-3+N/q+\eta+\lambda}_{q_0,\rho}} \|\boldsymbol{K} * \psi\|_{\dot{B}^{-\lambda}_{q_1,\infty}} + \|\varphi\|_{\dot{B}^{-\lambda}_{(1/q_1+1/N)^{-1},\infty}} \|\boldsymbol{K} * \psi\|_{\dot{B}^{2\alpha-3+N/q+\eta+\lambda}_{(1/q_0-1/N)^{-1},\rho}}) \\
&\le C(\|\varphi\|_{\dot{B}^{2\alpha-3+N/q+\eta+\lambda}_{q_0,\rho}} \|\psi\|_{\dot{B}^{-\lambda}_{(1/q_1+1/N)^{-1},\infty}} + \|\varphi\|_{\dot{B}^{-\lambda}_{(1/q_1+1/N)^{-1},\infty}} \|\psi\|_{\dot{B}^{2\alpha-3+N/q+\eta+\lambda}_{q_0,\rho}}) \\
&\le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} + \|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}).
\end{aligned}
$$

This completes the proof of Lemma 4.1. □

Lemma 4.1 determines the condition of the space of initial data. Namely, we consider the following condition:

Proposition 4.2. *Let $1 \le \rho \le \infty$. Suppose that $1 \le r, q < N$ satisfy* (2.1). *Then there exist $0 < s, \alpha, \beta < 1$ satisfying* (2.2). *In particular, for the nonlinear estimates in Lemma 4.1, it holds that*

$$
\begin{cases}
\|\nabla \cdot (\varphi(\boldsymbol{K} * \psi))\|_{\dot{B}^{2\alpha-4+N/r+\eta}_{r,\rho}} \\
\qquad \le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} + \|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}), \\
\|\boldsymbol{f} \cdot \nabla\varphi\|_{\dot{B}^{\alpha+\beta-4+N/r+\eta}_{r,\rho}} \\
\qquad \le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}} \|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} + \|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}} \|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}}), \\
\|(\boldsymbol{f} \cdot \nabla)\boldsymbol{g}\|_{\dot{B}^{2\beta-3+N/q+\eta}_{q,\rho}} \\
\qquad \le C(\|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}} \|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}} + \|\boldsymbol{f}\|_{\dot{B}^{\beta-1+N/q}_{q,\rho}} \|\boldsymbol{g}\|_{\dot{B}^{\beta-1+N/q+\eta}_{q,\rho}}), \\
\|\varphi(\boldsymbol{K} * \psi)\|_{\dot{B}^{2\alpha-3+N/q+\eta}_{q,\rho}} \\
\qquad \le C(\|\varphi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} + \|\varphi\|_{\dot{B}^{\alpha-2+N/r}_{r,\rho}} \|\psi\|_{\dot{B}^{\alpha-2+N/r+\eta}_{r,\rho}})
\end{cases}
\tag{4.1}
$$

for all $0 \le \eta < \infty$. Likewise, it holds that (4.1) *with α and β replaced by s.*

4.2. **Construction of mild solutions.** We define the following function spaces

$$
X^{\alpha,\beta}_T := \left\{ (n, \boldsymbol{u}) \;\middle|\; \begin{array}{c} n \in BC([0,T); \dot{B}^{-2+N/r}_{r,\rho}), \quad \boldsymbol{u} \in BC([0,T); P(\dot{B}^{-1+N/q}_{q,\rho})^N), \\ t^{\alpha/2} n \in BC([0,T); \dot{B}^{\alpha-2+N/r}_{r,1}), \quad t^{\beta/2}\boldsymbol{u} \in BC([0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N), \\ \lim_{t \to +0} \|\tau^{\alpha/2} n\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,1})} = 0, \quad \lim_{t \to +0} \|\tau^{\beta/2}\boldsymbol{u}\|_{L^\infty_t(\dot{B}^{\beta-1+N/q}_{q,1})} = 0 \end{array} \right\}
\tag{4.2}
$$

with the norm

$$
\|n, \boldsymbol{u}\|_{X^{\alpha,\beta}_T} := \|n\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,\rho})} + [n, \boldsymbol{u}]_{X^{\alpha,\beta}_T},
$$

where

$$
[n, \boldsymbol{u}]_{X^{\alpha,\beta}_T} := \|t^{\alpha/2} n\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|t^{\beta/2}\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,1})}.
$$

To construct of solutions of (1.5), we begin with considering the integral forms of (1.5), i.e.,

$$
\begin{cases}
n = e^{dt\Delta} a - I_1(n, n) - I_2(\boldsymbol{u}, n) & \text{in } (0,\infty) \times \mathbb{R}^N, \\
\boldsymbol{u} = e^{\nu t\Delta} \boldsymbol{b} - J_1(\boldsymbol{u}, \boldsymbol{u}) + J_2(n, n) & \text{in } (0,\infty) \times \mathbb{R}^N,
\end{cases}
\tag{4.3}
$$

where the nonlinear terms I_1, I_2, J_1, and J_2 are defined by

$$
\begin{cases}
I_1(n,m)(t) := \displaystyle\int_0^t e^{d(t-\tau)\Delta}\nabla\cdot(n(\tau)(\boldsymbol{K}*m)(\tau))d\tau, \\
I_2(\boldsymbol{u},m)(t) := \displaystyle\int_0^t e^{d(t-\tau)\Delta}(\boldsymbol{u}(\tau)\cdot\nabla m(\tau))d\tau, \\
J_1(\boldsymbol{u},\boldsymbol{v})(t) := \displaystyle\int_0^t e^{\nu(t-\tau)\Delta}P(\boldsymbol{u}(\tau)\cdot\nabla)\boldsymbol{v}(\tau)d\tau, \\
J_2(n,m)(t) := \displaystyle\int_0^t e^{\nu(t-\tau)\Delta}P(n(\tau)(\boldsymbol{K}*m)(\tau))d\tau
\end{cases} \tag{4.4}
$$

for all $0 < t < \infty$. The aim of this subsection is to construct of solutions of (4.3). Here, in case the initial data $(a,\boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$, it is more complicated to show the corresponding result. Hence, first we consider in case the interpolation exponent is finite.

Theorem 4.3. *Let $1 \le \rho < \infty$. Suppose that $1 \le r,q < N$ satisfy (2.1) and $0 < s,\alpha,\beta < 1$ satisfy (2.2). In addition, suppose that the initial data $(a,\boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. Then the following statements hold:*

(i) There exist $0 < T < \infty$ and a unique solution $(n,\boldsymbol{u}) \in X^{\alpha,\beta}_T \cap X^{s,s}_T$ on $(0,T)\times\mathbb{R}^N$ of (4.3) satisfying

$$
\lim_{t\to+0}\|n(t)-a\|_{\dot{B}^{-2+N/r}_{r,\rho}} = 0, \quad \lim_{t\to+0}\|\boldsymbol{u}(t)-\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} = 0. \tag{4.5}
$$

Moreover, the following estimates

$$
\begin{cases}
\|n,\boldsymbol{u}\|_{X^{\alpha,\beta}_T} \le C\left((1+d^{-\alpha/2})\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1+\nu^{-\beta/2})\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right), \\
[n,\boldsymbol{u}]_{X^{\alpha,\beta}_T} \le C\left(d^{-\alpha/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right)
\end{cases} \tag{4.6}
$$

hold, where $C = C(N,r,q,\rho,s,\alpha,\beta) > 0$ is a constant independent of $d,\nu,T,a,\boldsymbol{b},n$, and $\boldsymbol{u}$. Likewise, the estimates (4.6) hold with α and β replaced by s.

(ii) Suppose that $(n,\boldsymbol{u}) \in X^{\alpha,\beta}_T \cap X^{s,s}_T$ is the solution on $(0,T)\times\mathbb{R}^N$ of (4.3) with the initial data $(a,\boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ obtained in (i). Likewise, suppose that $(n_*,\boldsymbol{u}_*)$ is a solution of (4.3) with an initial data $(a_*,\boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. There is a constant $0 < \delta < 1$ such that if $(a_*,\boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ satisfies (2.7), then it holds that

$$
\begin{cases}
\|n-n_*,\boldsymbol{u}-\boldsymbol{u}_*\|_{X^{\alpha,\beta}_T} \le C\left((1+d^{-\alpha/2})\|a-a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1+\nu^{-\beta/2})\|\boldsymbol{b}-\boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right), \\
[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X^{\alpha,\beta}_T} \le C\left(d^{-\alpha/2}\|a-a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}-\boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right),
\end{cases} \tag{4.7}
$$

where $C = C(N,r,q,\rho,s,\alpha,\beta) > 0$ is a constant independent of $d,\nu,T,a,\boldsymbol{b},a_*,\boldsymbol{b}_*,n,\boldsymbol{u},n_*$, and $\boldsymbol{u}_*$. Likewise, it holds that (4.7) with α and β replaced by s.

(iii) In the statements of (i) and (ii), there exists a constant $\varepsilon = \varepsilon(d,\nu,N,r,q,\rho,s,\alpha,\beta) > 0$ such that if $(a,\boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ satisfies (2.14), then $T = \infty$ holds.

Since we obtained the nonlinear estimates (4.1), we may construct solutions of (4.3) by the Banach fixed point theorem, i.e., we may prove Theorem 4.3.

Lemma 4.4. *Let $1 \le \rho \le \infty$ and $0 < T \le \infty$. Suppose that $1 \le r,q < N$ satisfy (2.1) and $0 < s,\alpha,\beta < 1$ satisfy (2.2). Then, for the nonlinear terms I_1, I_2, J_1, and J_2 defined by (4.4), it* *holds that*

$$
(4.8)\qquad
\begin{cases}
\|I_1(n,m)(t)\|_{\dot{B}^{\alpha j-2+N/r}_{r,1}} \le C_{d,\nu} t^{-(\alpha/2)j} \|\tau^{\alpha/2} n\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \|\tau^{\alpha/2} m\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,\infty})},\\
\|I_2(\boldsymbol{u},m)(t)\|_{\dot{B}^{\alpha j-2+N/r}_{r,1}} \le C_{d,\nu} t^{-(\alpha/2)j} \|\tau^{\beta/2} \boldsymbol{u}\|_{L^\infty_t(\dot{B}^{\beta-1+N/q}_{q,\infty})} \|\tau^{\alpha/2} m\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,\infty})},\\
\|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot{B}^{\beta j-1+N/q}_{q,1}} \le C_{d,\nu} t^{-(\beta/2)j} \|\tau^{\beta/2} \boldsymbol{u}\|_{L^\infty_t(\dot{B}^{\beta-1+N/q}_{q,\infty})} \|\tau^{\beta/2} \boldsymbol{v}\|_{L^\infty_t(\dot{B}^{\beta-1+N/q}_{q,\infty})},\\
\|J_2(n,m)(t)\|_{\dot{B}^{\beta j-1+N/q}_{q,1}} \le C_{d,\nu} t^{-(\beta/2)j} \|\tau^{\alpha/2} n\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \|\tau^{\alpha/2} m\|_{L^\infty_t(\dot{B}^{\alpha-2+N/r}_{r,\infty})}
\end{cases}
$$

for all $0<t<T$, $0<d,\nu<\infty$, $j=0,1$, *and* $(n,\boldsymbol{u}),(m,\boldsymbol{v})\in X^{\alpha,\beta}_T$, *where* $C_{d,\nu}=C(d,\nu,N,r,q,\rho,s,\alpha,\beta)>0$ *is a constant independent of* $t,T,n,m,\boldsymbol{u}$, *and* $\boldsymbol{v}$. *Moreover, the following estimates*

$$
(4.9)\qquad
\begin{cases}
\|I_1(n,m)\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} \le C_{d,\nu} \|n\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \|m\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})},\\
\|I_2(\boldsymbol{u},m)\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} \le C_{d,\nu} \|\boldsymbol{u}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,\infty})} \|m\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})},\\
\|J_1(\boldsymbol{u},\boldsymbol{v})\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \le C_{d,\nu} \|\boldsymbol{u}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,\infty})} \|\boldsymbol{v}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,\infty})},\\
\|J_2(n,m)\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \le C_{d,\nu} \|n\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \|m\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})}
\end{cases}
$$

hold for all $(n,\boldsymbol{u}),(m,\boldsymbol{v})\in L^{2/\alpha,\rho}((0,T);\dot{B}^{\alpha-2+N/r}_{r,\infty})\times L^{2/\beta,\rho}((0,T);P(\dot{B}^{\beta-1+N/q}_{q,\infty})^N)$. *Likewise, it holds that* (4.8) *and* (4.9) *with* α *and* β *replaced by* s.

Proof. First, we see by (4.1) that

$$
(4.10)\qquad
\begin{cases}
\|\nabla\cdot(n(t)(\boldsymbol{K}*m)(t))\|_{\dot{B}^{2\alpha-4+N/r}_{r,\infty}} \le C\|n(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}} \|m(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}},\\
\|\boldsymbol{u}(t)\cdot\nabla m(t)\|_{\dot{B}^{\alpha+\beta-4+N/r}_{r,\infty}} \le C\|\boldsymbol{u}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}} \|m(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}},\\
\|(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{v}(t)\|_{\dot{B}^{2\beta-3+N/q}_{q,\infty}} \le C\|\boldsymbol{u}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}} \|\boldsymbol{v}(t)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}},\\
\|n(t)(\boldsymbol{K}*m)(t)\|_{\dot{B}^{2\alpha-3+N/q}_{q,\infty}} \le C\|n(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}} \|m(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}}
\end{cases}
$$

hold for all $0<t<T$. Hence, it holds by Proposition 3.8 and (4.10) that

$$
\begin{cases}
\|I_1(n,m)(t)\|_{\dot{B}^{\alpha j-2+N/r}_{r,1}} \le C_{d,\nu}\int_0^t (t-\tau)^{-(\alpha j+2-2\alpha)/2} \|\nabla\cdot(n(\tau)(\boldsymbol{K}*m)(\tau))\|_{\dot{B}^{2\alpha-4+N/r}_{r,\infty}}d\tau\\
\qquad\le C_{d,\nu}\int_0^t (t-\tau)^{\alpha-(\alpha/2)j-1} \|n(\tau)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}} \|m(\tau)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}}d\tau,\\
\|I_2(\boldsymbol{u},m)(t)\|_{\dot{B}^{\alpha j-2+N/r}_{r,1}} \le C_{d,\nu}\int_0^t (t-\tau)^{-(\alpha j+2-\alpha-\beta)/2} \|\boldsymbol{u}(\tau)\cdot\nabla m(\tau)\|_{\dot{B}^{\alpha+\beta-4+N/r}_{r,\infty}}d\tau\\
\qquad\le C_{d,\nu}\int_0^t (t-\tau)^{(\alpha+\beta)/2-(\alpha/2)j-1} \|\boldsymbol{u}(\tau)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}} \|m(\tau)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}}d\tau,\\
\|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot{B}^{\beta j-1+N/q}_{q,1}} \le C_{d,\nu}\int_0^t (t-\tau)^{-(\beta j+2-2\beta)/2} \|(\boldsymbol{u}(\tau)\cdot\nabla)\boldsymbol{v}(\tau)\|_{\dot{B}^{2\beta-3+N/q}_{q,\infty}}d\tau\\
\qquad\le C_{d,\nu}\int_0^t (t-\tau)^{\beta-(\beta/2)j-1} \|\boldsymbol{u}(\tau)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}} \|\boldsymbol{v}(\tau)\|_{\dot{B}^{\beta-1+N/q}_{q,\infty}}d\tau,\\
\|J_2(n,m)(t)\|_{\dot{B}^{\beta j-1+N/q}_{q,1}} \le C_{d,\nu}\int_0^t (t-\tau)^{-(\beta j+2-2\alpha)/2} \|n(\tau)(\boldsymbol{K}*m)(\tau)\|_{\dot{B}^{2\alpha-3+N/q}_{q,\infty}}d\tau\\
\qquad\le C_{d,\nu}\int_0^t (t-\tau)^{\alpha-(\beta/2)j-1} \|n(\tau)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}} \|m(\tau)\|_{\dot{B}^{\alpha-2+N/r}_{r,\infty}}d\tau
\end{cases}
$$

for $j = 0, 1$. Here we note that the relation $\int_0^t (t-\tau)^{\lambda-1}\tau^{\mu-1}d\tau = B(\lambda,\mu)t^{\lambda+\mu-1}$ holds for $0 < t < \infty$ and $0 < \lambda, \mu < \infty$, where $B(\lambda,\mu)$ denotes the beta function defined by $B(\lambda,\mu) := \int_0^1 (1-\tau)^{\lambda-1}\tau^{\mu-1}d\tau < \infty$. Since we see by (2.2) that $0 < \alpha, \beta < 1$ satisfy $\beta < 2\alpha$, we obtain (4.8). On the other hand, we may show (4.9) with the aid of Proposition 3.6. This completes the proof. $\square$

Proof of Theorem 4.3. (i) Let $0 < T < \infty$ be arbitrary. Define the mappings Φ_1 and Φ_2 by setting

$$\begin{cases} \Phi_1(n,\boldsymbol{u})(t) := e^{dt\Delta}a - I_1(n,n)(t) - I_2(\boldsymbol{u},n)(t), & 0 < t < T, \\ \Phi_2(n,\boldsymbol{u})(t) := e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u},\boldsymbol{u})(t) + J_2(n,n)(t), & 0 < t < T \end{cases} \tag{4.11}$$

for $(n,\boldsymbol{u}) \in X_T^{\alpha,\beta} \cap X_T^{s,s}$. Since it holds by (4.8) that

$$\begin{cases} \|\Phi_1(n,\boldsymbol{u})(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} \le \|e^{dt\Delta}a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + 2C_{d,\nu}[n,\boldsymbol{u}]^2_{X_t^{\alpha,\beta}}, \\ \|\Phi_1(n,\boldsymbol{u})(t)\|_{\dot{B}^{\alpha-2+N/r}_{r,1}} \le \|e^{dt\Delta}a\|_{\dot{B}^{\alpha-2+N/r}_{r,1}} + 2C_{d,\nu}t^{-\alpha/2}[n,\boldsymbol{u}]^2_{X_t^{\alpha,\beta}}, \\ \|\Phi_2(n,\boldsymbol{u})(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}} \le \|e^{\nu t\Delta}\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} + 2C_{d,\nu}[n,\boldsymbol{u}]^2_{X_t^{\alpha,\beta}}, \\ \|\Phi_2(n,\boldsymbol{u})(t)\|_{\dot{B}^{\beta-1+N/q}_{q,1}} \le \|e^{\nu t\Delta}\boldsymbol{b}\|_{\dot{B}^{\beta-1+N/q}_{q,1}} + 2C_{d,\nu}t^{-\beta/2}[n,\boldsymbol{u}]^2_{X_t^{\alpha,\beta}} \end{cases}$$

for all $0 < t < T$, we have

$$[\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})]_{X_t^{\alpha,\beta}} \le t^{\alpha/2}\|e^{dt\Delta}a\|_{\dot{B}^{\alpha-2+N/r}_{r,1}} + t^{\beta/2}\|e^{\nu t\Delta}\boldsymbol{b}\|_{\dot{B}^{\beta-1+N/q}_{q,1}} + 4C_{d,\nu}[n,\boldsymbol{u}]^2_{X_t^{\alpha,\beta}}. \tag{4.12}$$

Thus we obtain $\lim_{t\to+0}[\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})]_{X_t^{\alpha,\beta}} = 0$ from Proposition 3.10. In addition, we also see by Proposition 3.8 that

$$\begin{aligned} &\|\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})\|_{X_T^{\alpha,\beta}} \\ &\le C\left((1+d^{-\alpha/2})\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1+\nu^{-\beta/2})\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) + 8C_{d,\nu}[n,\boldsymbol{u}]^2_{X_T^{\alpha,\beta}}, \end{aligned} \tag{4.13}$$

which yields $(\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})) \in X_T^{\alpha,\beta}$. In a similar manner, it holds that

$$\begin{aligned} &\|\Phi_1(n,\boldsymbol{u}) - \Phi_1(m,\boldsymbol{v}), \Phi_2(n,\boldsymbol{u}) - \Phi_2(m,\boldsymbol{v})\|_{X_T^{\alpha,\beta}} \\ &\qquad\qquad \le 16C_{d,\nu}([n,\boldsymbol{u}]_{X_T^{\alpha,\beta}} + [m,\boldsymbol{v}]_{X_T^{\alpha,\beta}})\|n-m,\boldsymbol{u}-\boldsymbol{v}\|_{X_T^{\alpha,\beta}}. \end{aligned} \tag{4.14}$$

Notice that the same method implies that (4.12), (4.13), and (4.14) hold with α and β replaced by s. Now we take $0 < T < \infty$ satisfying

$$[e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta}\cap X_T^{s,s}} < 1/(2^7 C_{d,\nu}) \tag{4.15}$$

and consider the condition

$$[n,\boldsymbol{u}]_{X_T^{\alpha,\beta}\cap X_T^{s,s}} \le 1/(2^6 C_{d,\nu}). \tag{4.16}$$

Here notice that we may take such a T by virtue of Proposition 3.10. Then, for all $(n,\boldsymbol{u}) \in X_T^{\alpha,\beta} \cap X_T^{s,s}$ satisfying (4.16), it holds by (4.12), (4.15), and (4.16) that

$$[\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})]_{X_T^{\alpha,\beta}\cap X_T^{s,s}} \le 1/(2^6 C_{d,\nu}).$$

Besides, since it holds by (4.14) and (4.16) that

$$\|\Phi_1(n,\boldsymbol{u}) - \Phi_1(m,\boldsymbol{v}), \Phi_2(n,\boldsymbol{u}) - \Phi_2(m,\boldsymbol{v})\|_{X_T^{\alpha,\beta}\cap X_T^{s,s}} \le (1/2)\|n-m,\boldsymbol{u}-\boldsymbol{v}\|_{X_T^{\alpha,\beta}\cap X_T^{s,s}},$$

we may apply the Banach fixed point theorem to the mapping (Φ_1, Φ_2). Therefore, there exists a unique solution $(n, \boldsymbol{u}) \in X_T^{\alpha,\beta} \cap X_T^{s,s}$ of (4.3) with the estimate (4.16). Note that it holds by Proposition 3.8 that

$$[e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta}} \leq C\left(d^{-\alpha/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right). \tag{4.17}$$

Thus we obtain

$$[n, \boldsymbol{u}]_{X_T^{\alpha,\beta}} \leq C\left(d^{-\alpha/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) + (1/16)[n, \boldsymbol{u}]_{X_T^{\alpha,\beta}}$$

from (4.12), which yields

$$[n, \boldsymbol{u}]_{X_T^{\alpha,\beta}} \leq 2C\left(d^{-\alpha/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right).$$

Likewise, it holds by (4.13) that

$$\|n, \boldsymbol{u}\|_{X_T^{\alpha,\beta}} \leq 2C\left((1 + d^{-\alpha/2})\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (1 + \nu^{-\beta/2})\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right).$$

In addition, we see by the system $n = e^{dt\Delta}a - I_1(n, n) - I_2(\boldsymbol{u}, n)$ and $\boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u}, \boldsymbol{u}) + J_2(n, n)$ that

$$\begin{cases} \|n(t) - a\|_{\dot{B}^{-2+N/r}_{r,\rho}} \leq \|e^{dt\Delta}a - a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + 2C_{d,\nu}[n, \boldsymbol{u}]^2_{X_t^{\alpha,\beta}}, \\ \|\boldsymbol{u}(t) - \boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} \leq \|e^{\nu t\Delta}\boldsymbol{b} - \boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} + 2C_{d,\nu}[n, \boldsymbol{u}]^2_{X_t^{\alpha,\beta}} \end{cases}$$

by virtue of (4.8). Hence, the strong continuity of the heat semigroup yields (4.5).

(ii) Take an arbitrary initial data $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. Notice that by the condition (4.15), there exists a constant $0 < \lambda < 1$ satisfying

$$[e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} < 1/(2^7 C_{d,\nu}) - \lambda.$$

Hence, we see by Proposition 3.8 that

$$\begin{aligned} &[e^{dt\Delta}a_*, e^{\nu t\Delta}\boldsymbol{b}_*]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} \\ &\leq [e^{dt\Delta}(a_* - a), e^{\nu t\Delta}(\boldsymbol{b}_* - \boldsymbol{b})]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} + [e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} \\ &< C\left((d^{-\alpha/2} + d^{-s/2})\|a_* - a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + (\nu^{-\beta/2} + \nu^{-s/2})\|\boldsymbol{b}_* - \boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) + 1/(2^7 C_{d,\nu}) - \lambda \end{aligned}$$

hold. By taking $\delta := \lambda/(C(d^{-\alpha/2} + d^{-s/2} + \nu^{-\beta/2} + \nu^{-s/2}))$, we obtain

$$[e^{dt\Delta}a_*, e^{\nu t\Delta}\boldsymbol{b}_*]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} < 1/(2^7 C_{d,\nu})$$

provided that $(a_*, \boldsymbol{b}_*)$ satisfies (2.7). Thus we see that the existence time interval of $(n_*, \boldsymbol{u}_*)$ may be chosen the same time interval T as the original one. Moreover, we have

$$[n, \boldsymbol{u}]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} + [n_*, \boldsymbol{u}_*]_{X_T^{\alpha,\beta} \cap X_T^{s,s}} \leq 1/(2^5 C_{d,\nu}) \tag{4.18}$$

by virtue of (4.16). Since $(n, \boldsymbol{u})$ and $(n_*, \boldsymbol{u}_*)$ satisfy the system

$$\begin{cases} n = e^{dt\Delta}a - I_1(n, n) - I_2(\boldsymbol{u}, n), & \boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u}, \boldsymbol{u}) + J_2(n, n), \\ n_* = e^{dt\Delta}a_* - I_1(n_*, n_*) - I_2(\boldsymbol{u}_*, n_*), & \boldsymbol{u}_* = e^{\nu t\Delta}\boldsymbol{b}_* - J_1(\boldsymbol{u}_*, \boldsymbol{u}_*) + J_2(n_*, n_*), \end{cases}$$

we observe that

$$\begin{cases} \|n(t)-n_*(t)\|_{\dot B^{-2+N/r}_{r,\rho}} \le \|e^{dt\Delta}(a-a_*)\|_{\dot B^{-2+N/r}_{r,\rho}} + (1/8)[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}}, \\ \|n(t)-n_*(t)\|_{\dot B^{\alpha-2+N/r}_{r,1}} \le \|e^{dt\Delta}(a-a_*)\|_{\dot B^{\alpha-2+N/r}_{r,1}} + (1/8)t^{-\alpha/2}[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}}, \\ \|\boldsymbol{u}(t)-\boldsymbol{u}_*(t)\|_{\dot B^{-1+N/q}_{q,\rho}} \le \|e^{\nu t\Delta}(\boldsymbol{b}-\boldsymbol{b}_*)\|_{\dot B^{-1+N/q}_{q,\rho}} + (1/8)[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}}, \\ \|\boldsymbol{u}(t)-\boldsymbol{u}_*(t)\|_{\dot B^{\beta-1+N/q}_{q,1}} \le \|e^{\nu t\Delta}(\boldsymbol{b}-\boldsymbol{b}_*)\|_{\dot B^{\beta-1+N/q}_{q,1}} + (1/8)t^{-\beta/2}[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}} \end{cases}$$

with the aid of (4.8) and (4.18). Therefore, it holds by Proposition 3.8 that

$$\begin{cases} \|n-n_*,\boldsymbol{u}-\boldsymbol{u}_*\|_{X_T^{\alpha,\beta}} \le C\left((1+d^{-\alpha/2})\|a-a_*\|_{\dot B^{-2+N/r}_{r,\rho}} + (1+\nu^{-\beta/2})\|\boldsymbol{b}-\boldsymbol{b}_*\|_{\dot B^{-1+N/q}_{q,\rho}}\right) \\ \qquad\qquad + (1/2)[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}}, \\ [n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}} \le C\left(d^{-\alpha/2}\|a-a_*\|_{\dot B^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}-\boldsymbol{b}_*\|_{\dot B^{-1+N/q}_{q,\rho}}\right) \\ \qquad\qquad + (1/4)[n-n_*,\boldsymbol{u}-\boldsymbol{u}_*]_{X_T^{\alpha,\beta}}, \end{cases}$$

which yield (4.7).

(iii) In the proof of (i), we note that the existence time interval T is determined by (4.15). Therefore, if $(a,\boldsymbol{b}) \in \dot B^{-2+N/r}_{r,\rho} \times P(\dot B^{-1+N/q}_{q,\rho})^N$ satisfies

$$\|a\|_{\dot B^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}} < \frac{1}{2^7 C_{d,\nu} C(d^{-\alpha/2}+d^{-s/2}+\nu^{-\beta/2}+\nu^{-s/2})},$$

we see by (4.17) that (4.15) is valid for any $0<T<\infty$. Thus we may take $T=\infty$. This completes the proof of Theorem 4.3. □

Remark 4.5. It should be noticed that Theorem 4.3 yields the existence of solutions of (4.3), but we do not know whether the density of the cell n is non-negative in $(0,T)$ even if the initial density a is so. This is the most fundamental question to see that the system (4.3) describes the physical model exactly. Unfortunately, since we consider the case where the initial density a might be chosen in the distribution class, it is not so easy to show the non-negativity. Regarding this problem, by imposing the additional assumption such that a belongs to the space of bounded continuous functions, we may expect to obtain the non-negativity. We also refer to Kozono-Sugiyama [27, Theorem 1.3] for the result on the non-negativity of solutions of the Keller-Segel system in the scaling invariant spaces framework.

4.3. **Lorentz regularity in time direction, uniqueness assertion, and smoothing effects.** In this subsection, we shall show the following lemmas, which yield the proof of Theorem 2.1:

Lemma 4.6. *(i) In Theorem 4.3, by taking* $0<T<\infty$ *small as necessary, the solution* $(n,\boldsymbol{u}) \in X_T^{\alpha,\beta} \cap X_T^{s,s}$ *of* (4.3) *also satisfies*

$$(4.19)\qquad \begin{cases} n \in L^{2/\alpha,\rho}((0,T);\dot B^{\alpha-2+N/r}_{r,1}) \cap L^{2/s,\rho}((0,T);\dot B^{s-2+N/r}_{r,1}), \\ \boldsymbol{u} \in L^{2/\beta,\rho}((0,T);P(\dot B^{\beta-1+N/q}_{q,1})^N) \cap L^{2/s,\rho}((0,T);P(\dot B^{s-1+N/q}_{q,1})^N) \end{cases}$$

with the estimate

$$(4.20)\qquad \|n\|_{L^{2/\alpha,\rho}_T(\dot B^{\alpha-2+N/r}_{r,1})} + \|\boldsymbol{u}\|_{L^{2/\beta,\rho}_T(\dot B^{\beta-1+N/q}_{q,1})} \le C\left(d^{-\alpha/2}\|a\|_{\dot B^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}\right),$$

where $C=C(N,r,q,\rho,s,\alpha,\beta)>0$ *is a constant independent of* $d,\nu,T,a,\boldsymbol{b},n,$ *and* $\boldsymbol{u}$*. Likewise, the estimate* (4.20) *holds with* α *and* β *replaced by* s*.*

(ii) Suppose that $(n, \boldsymbol{u})$ *is the solution on* $(0,T) \times \mathbb{R}^N$ *of* (4.3) *with the initial data* $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ *obtained in (i). Likewise, suppose that* $(n_*, \boldsymbol{u}_*)$ *is a solution of* (4.3) *with an initial data* $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$*. There is a constant* $0 < \delta < 1$ *such that if* $(a_*, \boldsymbol{b}_*) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ *satisfies* (2.7)*, then it holds that*

$$
\begin{aligned}
&\|n - n_*\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|\boldsymbol{u} - \boldsymbol{u}_*\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \\
&\qquad\qquad \le C\left(d^{-\alpha/2}\|a - a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\beta/2}\|\boldsymbol{b} - \boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right),
\end{aligned}
\tag{4.21}
$$

where $C = C(N, r, q, \rho, s, \alpha, \beta) > 0$ *is a constant independent of* $d, \nu, T, a, \boldsymbol{b}, a_*, \boldsymbol{b}_*, n, \boldsymbol{u}, n_*,$ *and* $\boldsymbol{u}_*$*. Likewise, it holds that* (4.21) *with* α *and* β *replaced by* s*.*

(iii) In the statements of (i) and (ii), there exists a constant $\varepsilon = \varepsilon(d, \nu, N, r, q, \rho, s, \alpha, \beta) > 0$ *such that if* $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$ *satisfies* (2.14)*, then* $T = \infty$ *holds in* (4.19)*,* (4.20)*, and* (4.21)*.*

Lemma 4.7. *There exists a constant* $0 < \kappa < 1$ *independent of the initial data* $(a, \boldsymbol{b})$ *such that a solution* $(n, \boldsymbol{u})$ *of* (4.3) *satisfying* (2.6) *is unique.*

Lemma 4.8. *In Theorem 4.3, the solution* $(n, \boldsymbol{u}) \in X^{\alpha,\beta}_T \cap X^{s,s}_T$ *of* (4.3) *is a solution of* (1.5) *in a classical sense. In fact, it holds that*

$$
\begin{cases}
n \in \displaystyle\bigcap_{0<\gamma<\infty} C^\infty((0,T); \dot{B}^{-2+N/r}_{r,\rho} \cap \dot{B}^{\gamma-2+N/r}_{r,1}), & \partial_t n \in C^\infty((0,T); \dot{B}^{2s-4+N/r}_{r,1}), \\
\boldsymbol{u} \in \displaystyle\bigcap_{0<\gamma<\infty} C^\infty((0,T); P(\dot{B}^{-1+N/q}_{q,\rho} \cap \dot{B}^{\gamma-1+N/q}_{q,1})^N), & \partial_t \boldsymbol{u} \in C^\infty((0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N).
\end{cases}
\tag{4.22}
$$

Note that the method of the proof of Lemma 4.6 is based on Kozono-Shimizu [26]. Lemma 4.6 shows that the solution also belongs to the Lorentz spaces in time direction, where the interpolation exponent ρ in the homogeneous Besov spaces affects the Lorentz regularity. Lemma 4.7 is the uniqueness assertion of solutions of (4.3). Since the solution $(n, \boldsymbol{u})$ obtained in Theorem 4.3 necessarily fulfills the condition (4.7) by virtue of Proposition 3.7, we may regard Lemma 4.7 as an improved result compared with the usual uniqueness given by the Banach fixed point theorem. Lemma 4.8 implies the smoothing effects of the solution of (4.3). Here it should be noticed that we consider the initial data $(a, \boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\rho} \times P(\dot{B}^{-1+N/q}_{q,\rho})^N$. Since the inclusions on the derivative indices fail to hold, i.e., $\dot{B}^{s_1}_{r,\rho} \not\subset \dot{B}^{s_0}_{r,\rho}$ even if $s_0 < s_1$, we need to pay attention to the proof of Lemma 4.8.

Proof of Lemma 4.6. (i) We introduce the following function space

$$
Y^{\alpha,\beta}_T := L^{2/\alpha,\rho}((0,T); \dot{B}^{\alpha-2+N/r}_{r,1}) \times L^{2/\beta,\rho}((0,T); P(\dot{B}^{\beta-1+N/q}_{q,1})^N)
\tag{4.23}
$$

with the norm $\|n, \boldsymbol{u}\|_{Y^{\alpha,\beta}_T} := \|n\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + \|\boldsymbol{u}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})}$. Then the mappings Φ_1 and Φ_2 defined by (4.11) satisfy the following estimates

$$
\begin{cases}
\|\Phi_1(n, \boldsymbol{u})\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} \le \|e^{dt\Delta}a\|_{L^{2/\alpha,\rho}_T(\dot{B}^{\alpha-2+N/r}_{r,1})} + 2C_{d,\nu}\|n, \boldsymbol{u}\|^2_{Y^{\alpha,\beta}_T}, \\
\|\Phi_2(n, \boldsymbol{u})\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} \le \|e^{\nu t\Delta}\boldsymbol{b}\|_{L^{2/\beta,\rho}_T(\dot{B}^{\beta-1+N/q}_{q,1})} + 2C_{d,\nu}\|n, \boldsymbol{u}\|^2_{Y^{\alpha,\beta}_T}
\end{cases}
$$

by virtue of (4.9). Thus we have

$$
\|\Phi_1(n, \boldsymbol{u}), \Phi_2(n, \boldsymbol{u})\|_{Y^{\alpha,\beta}_T} \le \|e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}\|_{Y^{\alpha,\beta}_T} + 4C_{d,\nu}\|n, \boldsymbol{u}\|^2_{Y^{\alpha,\beta}_T}.
\tag{4.24}
$$

In a similar manner, it holds that

$$\begin{aligned}(4.25)\quad &\|\Phi_1(n,\boldsymbol{u})-\Phi_1(m,\boldsymbol{v}),\Phi_2(n,\boldsymbol{u})-\Phi_2(m,\boldsymbol{v})\|_{Y_T^{\alpha,\beta}}\\ &\qquad\le 8C_{d,\nu}(\|n,\boldsymbol{u}\|_{Y_T^{\alpha,\beta}}+\|m,\boldsymbol{v}\|_{Y_T^{\alpha,\beta}})\|n-m,\boldsymbol{u}-\boldsymbol{v}\|_{Y_T^{\alpha,\beta}}.\end{aligned}$$

Now we take $0<T<\infty$ satisfying

$$(4.26)\qquad \|e^{dt\Delta}a,e^{\nu t\Delta}\boldsymbol{b}\|_{Y_T^{\alpha,\beta}\cap Y_T^{s,s}}+[e^{dt\Delta}a,e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta}\cap X_T^{s,s}}<1/(2^7C_{d,\nu})$$

and consider the condition

$$(4.27)\qquad \|n,\boldsymbol{u}\|_{Y_T^{\alpha,\beta}\cap Y_T^{s,s}}+[n,\boldsymbol{u}]_{X_T^{\alpha,\beta}\cap X_T^{s,s}}\le 1/(2^6C_{d,\nu}).$$

Then, for all $(n,\boldsymbol{u})\in X_T^{\alpha,\beta}\cap X_T^{s,s}\cap Y_T^{\alpha,\beta}\cap Y_T^{s,s}$ satisfying (4.27), it holds by (4.12), (4.24), (4.26), and (4.27) that

$$\|\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})\|_{Y_T^{\alpha,\beta}\cap Y_T^{s,s}}+[\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u})]_{X_T^{\alpha,\beta}\cap X_T^{s,s}}\le 1/(2^6C_{d,\nu}).$$

Here we notice that $(\Phi_1(n,\boldsymbol{u}),\Phi_2(n,\boldsymbol{u}))\in X_T^{\alpha,\beta}\cap X_T^{s,s}\cap Y_T^{\alpha,\beta}\cap Y_T^{s,s}$ holds. Besides, since it holds by (4.14), (4.25), and (4.27) that

$$\begin{aligned}&\|\Phi_1(n,\boldsymbol{u})-\Phi_1(m,\boldsymbol{v}),\Phi_2(n,\boldsymbol{u})-\Phi_2(m,\boldsymbol{v})\|_{X_T^{\alpha,\beta}\cap X_T^{s,s}\cap Y_T^{\alpha,\beta}\cap Y_T^{s,s}}\\ &\qquad\le(1/2)\|n-m,\boldsymbol{u}-\boldsymbol{v}\|_{X_T^{\alpha,\beta}\cap X_T^{s,s}\cap Y_T^{\alpha,\beta}\cap Y_T^{s,s}},\end{aligned}$$

we may apply the Banach fixed point theorem to the mapping (Φ_1,Φ_2). Therefore, there exists a unique solution $(n,\boldsymbol{u})\in X_T^{\alpha,\beta}\cap X_T^{s,s}\cap Y_T^{\alpha,\beta}\cap Y_T^{s,s}$ of (4.3) with the estimate (4.27), where we use the fact that the solution $(n,\boldsymbol{u})\in X_T^{\alpha,\beta}\cap X_T^{s,s}$ of (4.3) is unique. Note that it holds by Proposition 3.9 that

$$(4.28)\qquad \|e^{dt\Delta}a,e^{\nu t\Delta}\boldsymbol{b}\|_{Y_T^{\alpha,\beta}}\le C\left(d^{-\alpha/2}\|a\|_{\dot B^{-2+N/r}_{r,\rho}}+\nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}\right).$$

Thus we obtain

$$\|n,\boldsymbol{u}\|_{Y_T^{\alpha,\beta}}\le C\left(d^{-\alpha/2}\|a\|_{\dot B^{-2+N/r}_{r,\rho}}+\nu^{-\beta/2}\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}\right)+(1/16)\|n,\boldsymbol{u}\|_{Y_T^{\alpha,\beta}}$$

from (4.24). Hence, we have (4.20).

(ii) We may show in the same way as in the proof of Theorem 4.3 (ii).

(iii) In the proof of (i), we note that the existence time interval T is determined by (4.26). Therefore, if $(a,\boldsymbol{b})\in\dot B^{-2+N/r}_{r,\rho}\times P(\dot B^{-1+N/q}_{q,\rho})^N$ satisfies

$$\|a\|_{\dot B^{-2+N/r}_{r,\rho}}+\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}<\frac{1}{2^7C_{d,\nu}C(d^{-\alpha/2}+d^{-s/2}+\nu^{-\beta/2}+\nu^{-s/2})},$$

we see by (4.17) and (4.28) that (4.26) is valid for any $0<T<\infty$. Thus we may take $T=\infty$. This completes the proof of Lemma 4.6. □

Before showing Lemma 4.7, we give the following proposition, which ensures that the pair of functions $(n,\boldsymbol{u})$ satisfying (2.6) are small by taking a time interval t sufficiently small:

Proposition 4.9. *Let $1<\alpha<\infty$, $0<T\le\infty$, and X be a Banach space. Suppose that $\varphi\in L^{\alpha,\infty}((0,T);X)$ satisfies*

$$\limsup_{\lambda\to\infty}\lambda\mu(t\in(0,T)\mid\|\varphi(t)\|_X>\lambda)^{1/\alpha}\le\kappa$$

with some constant $0 < \kappa < \infty$. Then there exist a constant $R_\kappa > 0$ and functions $\varphi_0 \in L^\infty((0,T);X)$ and $\varphi_1 \in L^{\alpha,\infty}((0,T);X)$ such that

$$\|\varphi_0\|_{L^\infty_T(X)} \le R_\kappa, \quad \|\varphi_1\|_{L^{\alpha,\infty}_T(X)} \le 2\kappa, \quad \varphi = \varphi_0 + \varphi_1.$$

Proof. The assumption allows us to take $R_\kappa > 0$ so that

$$\sup_{R_\kappa \le \lambda < \infty} \lambda \mu(t \in (0,T) \,|\, \|\varphi(t)\|_X > \lambda)^{1/\alpha} \le 2\kappa$$

holds. Now we define

$$\varphi_0(t) := \begin{cases} \varphi(t) & \text{if } \|\varphi(t)\|_X \le R_\kappa, \\ 0 & \text{if } \|\varphi(t)\|_X > R_\kappa, \end{cases} \qquad \varphi_1(t) := \begin{cases} 0 & \text{if } \|\varphi(t)\|_X \le R_\kappa, \\ \varphi(t) & \text{if } \|\varphi(t)\|_X > R_\kappa. \end{cases}$$

Then we have $\varphi = \varphi_0 + \varphi_1$ and $\varphi_0 \in L^\infty((0,T);X)$ with $\|\varphi_0\|_{L^\infty_T(X)} \le R_\kappa$. In addition, we obtain

$$\begin{aligned} &\|\varphi_1\|_{L^{\alpha,\infty}_T(X)} \\ &= \max\left\{ \sup_{0<\lambda\le R_\kappa} \lambda\mu(t\in(0,T)\,|\,\|\varphi_1(t)\|_X > \lambda)^{1/\alpha}, \sup_{R_\kappa<\lambda<\infty} \lambda\mu(t\in(0,T)\,|\,\|\varphi_1(t)\|_X > \lambda)^{1/\alpha} \right\} \\ &\le \max\left\{ R_\kappa \mu(t\in(0,T)\,|\,\|\varphi(t)\|_X > R_\kappa)^{1/\alpha}, \sup_{R_\kappa<\lambda<\infty} \lambda\mu(t\in(0,T)\,|\,\|\varphi(t)\|_X > \lambda)^{1/\alpha} \right\} \\ &\le 2\kappa, \end{aligned}$$

which completes the proof of Proposition 4.9. □

Proof of Lemma 4.7. First we shall show that if $(n, \boldsymbol{u})$ satisfies (2.6), then it holds that

$$\|n\|_{L_h^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}\|_{L_h^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})} \le C\kappa \tag{4.29}$$

for sufficiently small $0 < h < T$, where $C > 0$ is a constant independent of h and κ. In fact, by Proposition 4.9, there exist a constant $R_\kappa > 0$ such that the following decompositions

$$\begin{cases} \|n_0\|_{L^\infty_T(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \le R_\kappa, \quad \|n_1\|_{L_T^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \le 2\kappa, \quad n = n_0 + n_1, \\ \|\boldsymbol{u}_0\|_{L^\infty_T(\dot{B}^{\beta-1+N/q}_{q,\infty})} \le R_\kappa, \quad \|\boldsymbol{u}_1\|_{L_T^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})} \le 2\kappa, \quad \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1 \end{cases}$$

hold. Thus we obtain

$$\begin{cases} \|n\|_{L_t^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} \le C(\|n_0\|_{L_t^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} + \|n_1\|_{L_t^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})}) \\ \qquad\qquad \le C(\|n_0\|_{L_t^{2/\alpha}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} + 2\kappa) \le C(R_\kappa t^{\alpha/2} + 2\kappa), \\ \|\boldsymbol{u}\|_{L_t^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})} \le C(\|\boldsymbol{u}_0\|_{L_t^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})} + \|\boldsymbol{u}_1\|_{L_t^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})}) \\ \qquad\qquad \le C(\|\boldsymbol{u}_0\|_{L_t^{2/\beta}(\dot{B}^{\beta-1+N/q}_{q,\infty})} + 2\kappa) \le C(R_\kappa t^{\beta/2} + 2\kappa), \end{cases}$$

which yield $\|n\|_{L_t^{2/\alpha,\infty}(\dot{B}^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}\|_{L_t^{2/\beta,\infty}(\dot{B}^{\beta-1+N/q}_{q,\infty})} \le CR_\kappa(t^{\alpha/2} + t^{\beta/2}) + 4C\kappa$. Hence, by taking $0 < h < T$ sufficiently small, we have (4.29). Now we assume that $(n, \boldsymbol{u})$ and $(m, \boldsymbol{v})$ are solutions of (4.3) satisfying (2.6). Since $(n, \boldsymbol{u})$ and $(m, \boldsymbol{v})$ satisfy

$$\begin{cases} n = e^{dt\Delta}a - I_1(n,n) - I_2(\boldsymbol{u}, n), \quad \boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u},\boldsymbol{u}) + J_2(n,n), \\ m = e^{dt\Delta}a - I_1(m,m) - I_2(\boldsymbol{v}, m), \quad \boldsymbol{v} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{v},\boldsymbol{v}) + J_2(m,m), \end{cases}$$

we observe that

$$\begin{cases} \|n-m\|_{L_h^{2/\alpha,\infty}(\dot B^{\alpha-2+N/r}_{r,\infty})} \le 4C_{d,\nu}C\kappa\left(\|n-m\|_{L_h^{2/\alpha,\infty}(\dot B^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}-\boldsymbol{v}\|_{L_h^{2/\beta,\infty}(\dot B^{\beta-1+N/q}_{q,\infty})}\right), \\ \|\boldsymbol{u}-\boldsymbol{v}\|_{L_h^{2/\beta,\infty}(\dot B^{\beta-1+N/q}_{q,\infty})} \le 4C_{d,\nu}C\kappa\left(\|n-m\|_{L_h^{2/\alpha,\infty}(\dot B^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}-\boldsymbol{v}\|_{L_h^{2/\beta,\infty}(\dot B^{\beta-1+N/q}_{q,\infty})}\right) \end{cases}$$

from (4.9) and (4.29). Hence, by taking $\kappa \le 1/(2^4 C_{d,\nu}C)$, we have

$$\|n-m\|_{L_h^{2/\alpha,\infty}(\dot B^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}-\boldsymbol{v}\|_{L_h^{2/\beta,\infty}(\dot B^{\beta-1+N/q}_{q,\infty})} = 0,$$

which yields $n=m$ and $\boldsymbol{u}=\boldsymbol{v}$ in $(0,h)$. Since we have

$$\|n\|_{L^{2/\alpha,\infty}((h,2h);\dot B^{\alpha-2+N/r}_{r,\infty})} + \|\boldsymbol{u}\|_{L^{2/\beta,\infty}((h,2h);\dot B^{\beta-1+N/q}_{q,\infty})} \le C\kappa$$

like (4.29), we may show that $n=m$ and $\boldsymbol{u}=\boldsymbol{v}$ in $(0,2h)$ in the same way. Thus we obtain $n=m$ and $\boldsymbol{u}=\boldsymbol{v}$ in $(0,T)$, which complete the proof of Lemma 4.7. □

Finally, we show Lemma 4.8. To this end, we give the following propositions:

Proposition 4.10. *Let $1 \le r < \infty$, $s \in \mathbb{R}$, $1 \le \rho < \infty$, and $0 < \nu < \infty$. Then, for every $\varphi \in \dot B^s_{r,\rho}$, it holds that*

$$e^{\nu t\Delta}\varphi \in BC([0,\infty);\dot B^s_{r,\rho}) \cap \bigcap_{0<\gamma<\infty} C^\infty((0,\infty);\dot B^s_{r,\rho}\cap \dot B^{\gamma+s}_{r,1}).$$

In addition, for every $\varphi \in \dot B^s_{r,\infty}$, it holds that

$$e^{\nu t\Delta}\varphi \in \bigcap_{0<\gamma<\infty}\left(BC_w([0,\infty);\dot B^s_{r,\infty}) \cap C^\infty((0,\infty);\dot B^{\gamma+s}_{r,1})\right)$$

provided $1 < r < \infty$.

Proposition 4.11. *Let $1 \le r < \infty$, $s \in \mathbb{R}$, $0 < \nu < \infty$, and $0 < T < \infty$. Then the following statements hold:*

(i) Let $0 \le \alpha < 2$ and assume that $\varphi \in L^\infty_{\mathrm{loc}}((0,T);\dot B^{s-\alpha}_{r,\infty})$ satisfies $\|\Phi\|_{L^\infty_T(\dot B^s_{r,\infty})} < \infty$, where the function Φ is defined by

$$\Phi(t) := \int_0^t e^{\nu(t-\tau)\Delta}\varphi(\tau)d\tau, \quad 0<t<T. \tag{4.30}$$

Then it holds that

$$\Phi \in \bigcap_{0<\beta<2-\alpha} C^{1-(\alpha+\beta)/2}_{\mathrm{loc}}((0,T);\dot B^{s+\beta}_{r,1})$$

with the estimate

$$\begin{aligned} &\|\Phi\|_{C^{1-(\alpha+\beta)/2}((\varepsilon,T);\dot B^{s+\beta}_{r,1})} \\ &\le C\nu^{-\beta/2}(\varepsilon^{-\beta/2}+T^{\alpha/2}\varepsilon^{-1})\|\Phi\|_{L^\infty_T(\dot B^s_{r,\infty})} + C\nu^{-(\alpha+\beta)/2}(1+T^{1-(\alpha+\beta)/2})\|\varphi\|_{L^\infty((\varepsilon/2,T);\dot B^{s-\alpha}_{r,\infty})} \end{aligned}$$

for all $0<\varepsilon<T$ and $0<\beta<2-\alpha$, where $C = C(N,r,s,\alpha,\beta) > 0$ is a constant independent of ν, T, ε, and φ. Here C^γ_{loc} denotes the space of all locally bounded γ-Hölder continuous functions.

(ii) Let $1 \le \rho < \infty$, $0 < \gamma < 1$, and $0 \le \eta < \infty$ and assume that $\varphi \in C^\gamma_{\mathrm{loc}}((0,T);\dot B^{s+\eta}_{r,\rho})$ satisfies $\|\Phi\|_{L^\infty_T(\dot B^s_{r,\infty})} < \infty$, where the function Φ is defined by (4.30). Then it holds that $\partial_t\Phi, \Delta\Phi \in C^\gamma_{\mathrm{loc}}((0,T);\dot B^{s+\eta}_{r,\rho})$ with the identity $\partial_t\Phi - \nu\Delta\Phi = \varphi$ having the estimate

$$\begin{aligned} &\|\partial_t\Phi, \nu\Delta\Phi\|_{C^\gamma((\varepsilon,T);\dot B^{s+\eta}_{r,\rho})} \\ &\le C\varepsilon^{-1}(\nu\varepsilon)^{-\eta/2}(1+\varepsilon^{-\gamma})\|\Phi\|_{L^\infty_T(\dot B^s_{r,\infty})} + C(1+T^\gamma+\varepsilon^{-\gamma})\|\varphi\|_{C^\gamma((\varepsilon/2,T);\dot B^{s+\eta}_{r,\rho})} \end{aligned}$$

for all $0 < \varepsilon < T$*, where* $C = C(N, r, s, \rho, \gamma, \eta) > 0$ *is a constant independent of* ν, T, ε*, and* φ*.*

The proof of Proposition 4.10 is given by the author [35, Corollary 3.10]. In fact, it is shown that the heat semigroup in the homogeneous Besov spaces has space-time analytic smoothing effects in [35]. For Proposition 4.11, we will give the proof in Appendix. Here, Proposition 4.11 is variant properties of Lunardi [28, Proposition 4.2.1 and Theorem 4.3.4] by considering the space with the homogeneous norm. In addition, note that the assumption on the behavior of φ near $t = 0$ is slightly weaker than that of [28, Proposition 4.2.1 and Theorem 4.3.4].

Proof of Lemma 4.8. By the estimates (4.1), we obtain

$$\begin{aligned}&\|\nabla \cdot (n(t)(\boldsymbol{K} * n)(t)), \boldsymbol{u}(t) \cdot \nabla n(t)\|_{\dot{B}^{2s-4+N/r}_{r,1}} \\ &+ \|(\boldsymbol{u}(t) \cdot \nabla)\boldsymbol{u}(t), n(t)(\boldsymbol{K} * n)(t)\|_{\dot{B}^{2s-3+N/q}_{q,1}} \le Ct^{-s}[n, \boldsymbol{u}]^2_{X^{s,s}_T}.\end{aligned} \tag{4.31}$$

Thus we see that the solution $(n, \boldsymbol{u}) \in X^{s,s}_T$ of (4.3) satisfies

$$\begin{cases}\nabla \cdot (n(\boldsymbol{K} * n)), \boldsymbol{u} \cdot \nabla n \in L^\infty_{\mathrm{loc}}((0, T); \dot{B}^{-2+N/r-2(1-s)}_{r,1}), \\ \quad (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, n(\boldsymbol{K} * n) \in L^\infty_{\mathrm{loc}}((0, T); (\dot{B}^{-1+N/q-2(1-s)}_{q,1})^N).\end{cases} \tag{4.32}$$

Moreover, it holds that

$$\|I_1(n, n), I_2(\boldsymbol{u}, n)\|_{L^\infty_T(\dot{B}^{-2+N/r}_{r,1})} + \|J_1(\boldsymbol{u}, \boldsymbol{u}), J_2(n, n)\|_{L^\infty_T(\dot{B}^{-1+N/q}_{q,1})} \le 4C_{d,\nu}[n, \boldsymbol{u}]^2_{X^{s,s}_T} \tag{4.33}$$

from (4.8). Hence we may apply Proposition 4.11 (i) by virtue of (4.32) and (4.33). This implies that

$$\begin{cases} I_1(n, n), I_2(\boldsymbol{u}, n) \in \bigcap_{0<s_0<2s} C^{s-s_0/2}_{\mathrm{loc}}((0, T); \dot{B}^{s_0-2+N/r}_{r,1}), \\ J_1(\boldsymbol{u}, \boldsymbol{u}), J_2(n, n) \in \bigcap_{0<s_0<2s} C^{s-s_0/2}_{\mathrm{loc}}((0, T); P(\dot{B}^{s_0-1+N/q}_{q,1})^N),\end{cases}$$

which yield $n \in C^{s/2}_{\mathrm{loc}}((0, T); \dot{B}^{s-2+N/r}_{r,1})$ and $\boldsymbol{u} \in C^{s/2}_{\mathrm{loc}}((0, T); P(\dot{B}^{s-1+N/q}_{q,1})^N)$ from the system $n = e^{dt\Delta}a - I_1(n, n) - I_2(\boldsymbol{u}, n)$ and $\boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u}, \boldsymbol{u}) + J_2(n, n)$ combined with Proposition 4.10. In addition, since it holds that (4.1) for all $0 \le \eta < \infty$, letting $\eta = 0$ in (4.1) yields

$$\begin{cases}\nabla \cdot (n(\boldsymbol{K} * n)), \boldsymbol{u} \cdot \nabla n \in C^{s/2}_{\mathrm{loc}}((0, T); \dot{B}^{2s-4+N/r}_{r,1}), \\ \quad (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, n(\boldsymbol{K} * n) \in C^{s/2}_{\mathrm{loc}}((0, T); (\dot{B}^{2s-3+N/q}_{q,1})^N).\end{cases} \tag{4.34}$$

Here we see that

$$\|I_1(n, n), I_2(\boldsymbol{u}, n)\|_{L^\infty_T(\dot{B}^{2s-4+N/r}_{r,1})} + \|J_1(\boldsymbol{u}, \boldsymbol{u}), J_2(n, n)\|_{L^\infty_T(\dot{B}^{2s-3+N/q}_{q,1})} \le CT^{1-s}[n, \boldsymbol{u}]^2_{X^{s,s}_T} \tag{4.35}$$

by the direct computation with the aid of (4.31). Hence we may apply Proposition 4.11 (ii) by virtue of (4.34) and (4.35). This implies that

$$\begin{cases}\partial_t I_1(n, n), \Delta I_1(n, n), \partial_t I_2(\boldsymbol{u}, n), \Delta I_2(\boldsymbol{u}, n) \in C^{s/2}_{\mathrm{loc}}((0, T); \dot{B}^{2s-4+N/r}_{r,1}), \\ \partial_t J_1(\boldsymbol{u}, \boldsymbol{u}), \Delta J_1(\boldsymbol{u}, \boldsymbol{u}), \partial_t J_2(n, n), \Delta J_2(n, n) \in C^{s/2}_{\mathrm{loc}}((0, T); P(\dot{B}^{2s-3+N/q}_{q,1})^N)\end{cases}$$

with the identities

$$\begin{cases}\partial_t I_1(n, n) - d\Delta I_1(n, n) = \nabla \cdot (n(\boldsymbol{K} * n)), & \partial_t I_2(\boldsymbol{u}, n) - d\Delta I_2(\boldsymbol{u}, n) = \boldsymbol{u} \cdot \nabla n, \\ \partial_t J_1(\boldsymbol{u}, \boldsymbol{u}) - \nu\Delta J_1(\boldsymbol{u}, \boldsymbol{u}) = P(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, & \partial_t J_2(n, n) - \nu\Delta J_2(n, n) = P(n(\boldsymbol{K} * n)).\end{cases}$$

Therefore, we see by Proposition 4.10 and the system $n = e^{dt\Delta}a - I_1(n,n) - I_2(\boldsymbol{u},n)$ and $\boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u},\boldsymbol{u}) + J_2(n,n)$ that $(n,\boldsymbol{u})$ satisfies

$$\begin{cases} n \in BC([0,T); \dot{B}^{-2+N/r}_{r,\rho}) \cap C^{s/2}_{\mathrm{loc}}((0,T); \dot{B}^{-2+N/r}_{r,\rho} \cap \dot{B}^{2s-2+N/r}_{r,1}), \\ \partial_t n \in C^{s/2}_{\mathrm{loc}}((0,T); \dot{B}^{2s-4+N/r}_{r,1}), \\ \boldsymbol{u} \in BC([0,T); P(\dot{B}^{-1+N/q}_{q,\rho})^N) \cap C^{s/2}_{\mathrm{loc}}((0,T); P(\dot{B}^{-1+N/q}_{q,\rho} \cap \dot{B}^{2s-1+N/q}_{q,1})^N), \\ \partial_t \boldsymbol{u} \in C^{s/2}_{\mathrm{loc}}((0,T); P(\dot{B}^{2s-3+N/q}_{q,1})^N). \end{cases}$$

Note that letting $\eta = 2s$ in (4.1) implies that

$$\begin{cases} \nabla \cdot (n(\boldsymbol{K} * n)), \boldsymbol{u} \cdot \nabla n \in C^{s/2}_{\mathrm{loc}}((0,T); \dot{B}^{4s-4+N/r}_{r,1}), \\ (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, n(\boldsymbol{K} * n) \in C^{s/2}_{\mathrm{loc}}((0,T); (\dot{B}^{4s-3+N/q}_{q,1})^N). \end{cases} \tag{4.36}$$

Hence we may apply Proposition 4.11 (ii) again by virtue of (4.35) and (4.36). This yields the gain of regularity of $(n,\boldsymbol{u})$ in space. Repeating this argument implies that

$$\begin{cases} n \in \bigcap_{0<\gamma<\infty} C^{s/2}_{\mathrm{loc}}((0,T); \dot{B}^{-2+N/r}_{r,\rho} \cap \dot{B}^{\gamma-2+N/r}_{r,1}), \\ \partial_t n \in \bigcap_{0<\gamma<\infty} C^{s/2}_{\mathrm{loc}}((0,T); \dot{B}^{2s-4+N/r}_{r,1} \cap \dot{B}^{\gamma-2+N/r}_{r,1}), \\ \boldsymbol{u} \in \bigcap_{0<\gamma<\infty} C^{s/2}_{\mathrm{loc}}((0,T); P(\dot{B}^{-1+N/q}_{q,\rho} \cap \dot{B}^{\gamma-1+N/q}_{q,1})^N), \\ \partial_t \boldsymbol{u} \in \bigcap_{0<\gamma<\infty} C^{s/2}_{\mathrm{loc}}((0,T); P(\dot{B}^{2s-3+N/q}_{q,1} \cap \dot{B}^{\gamma-1+N/q}_{q,1})^N) \end{cases}$$

hold. Thus we see that $(n,\boldsymbol{u})$ is a solution of (1.5) in a classical sense. In addition, taking η arbitrarily large in (4.1) gives

$$\begin{cases} \nabla \cdot (n(\boldsymbol{K} * n)), \boldsymbol{u} \cdot \nabla n \in \bigcap_{0<\gamma<\infty} C^{1+s/2}_{\mathrm{loc}}((0,T); \dot{B}^{-2+N/r}_{r,\rho} \cap \dot{B}^{\gamma-2+N/r}_{r,1}), \\ (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, n(\boldsymbol{K} * n) \in \bigcap_{0<\gamma<\infty} C^{1+s/2}_{\mathrm{loc}}((0,T); (\dot{B}^{-1+N/q}_{q,\rho} \cap \dot{B}^{\gamma-1+N/q}_{q,1})^N). \end{cases}$$

This yields the gain of regularity of $(n,\boldsymbol{u})$ in time from the system $\partial_t n = d\Delta n - \nabla\cdot(n(\boldsymbol{K}*n)) - \boldsymbol{u}\cdot\nabla n$ and $\partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + P(n(\boldsymbol{K}*n))$. Therefore, repeating this argument gives that (4.22) holds. This completes the proof of Lemma 4.8. □

Proof of Theorem 2.1. (i) By Theorem 4.3 (i), we obtain the solution $(n,\boldsymbol{u}) \in X^{\alpha,\beta}_T \cap X^{s,s}_T$ of (4.3) with (2.4) having the estimates (4.6). Moreover, Lemmas 4.6 (i) and 4.8 imply that the solution $(n,\boldsymbol{u})$ of (4.3) is a solution of (1.5) in a classical sense satisfying (4.19) and (4.22) with the estimates (4.20). Thus we see that $(n,\boldsymbol{u})$ satisfies (2.5).

(ii) It is shown from Lemma 4.7.

(iii) By combining Theorem 4.3 (ii) and Lemma 4.6 (ii), we obtain (2.8). This completes the proof of Theorem 2.1. □

4.4. **In case the interpolation exponent is infinity.** This subsection is devoted to consideration in case the initial data $(a,\boldsymbol{b}) \in \dot{B}^{-2+N/r}_{r,\infty} \times P(\dot{B}^{-1+N/q}_{q,\infty})^N$. In this case, since $\mathscr{S} \cap \dot{B}^s_{r,\infty}$ is not dense in $\dot{B}^s_{r,\infty}$, the heat semigroup $\{e^{t\Delta}\}_{0<t<\infty}$ does not have the strong continuity. To get over this problem, we shall decompose the initial data into the smooth part and the remeinder part. Besides, although it is unknown whether (2.4) hold, we may expect the weak-star continuity by the duality argument.

Proposition 4.12. *Let $1 \le r \le \infty$ and $s \in \mathbb{R}$. Suppose that $\varphi \in \dot{B}^s_{r,\infty}$. Then, for the functions φ_0 and φ_1 defined by $\varphi_0 := \sum_{j=-\infty}^{j_*} \dot{\Delta}_j \varphi$ and $\varphi_1 := \sum_{j=j_*+1}^{\infty} \dot{\Delta}_j \varphi$ with $j_* \in \mathbb{Z}$, it holds that*

$$\varphi_0 \in \bigcap_{0<\gamma<\infty} (\dot{B}^s_{r,\infty} \cap \dot{B}^{s+\gamma}_{r,1}), \quad \varphi_1 \in \bigcap_{0<\gamma<\infty} (\dot{B}^s_{r,\infty} \cap \dot{B}^{s-\gamma}_{r,1}), \quad \varphi = \varphi_0 + \varphi_1$$

with the estimates

$$\begin{cases} \|\varphi_0\|_{\dot{B}^{s+\gamma}_{r,1}} \le \dfrac{2^{\gamma(j_*+1)}C}{1-2^{-\gamma}} \|\varphi\|_{\dot{B}^s_{r,\infty}}, & \|\varphi_0\|_{\dot{B}^s_{r,\infty}} \le C \displaystyle\sup_{-\infty<k\le j_*+1} 2^{sk} \|\dot{\Delta}_k \varphi\|_{L^r}, \\ \|\varphi_1\|_{\dot{B}^{s-\gamma}_{r,1}} \le \dfrac{2^{-\gamma j_*}C}{1-2^{-\gamma}} \|\varphi\|_{\dot{B}^s_{r,\infty}}, & \|\varphi_1\|_{\dot{B}^s_{r,\infty}} \le C \displaystyle\sup_{j_*\le k<\infty} 2^{sk} \|\dot{\Delta}_k \varphi\|_{L^r} \end{cases}$$

for all $0 < \gamma < \infty$, where $C = C(N, r, s) > 0$ is a constant independent of γ, j_*, and φ. Here $\{\dot{\Delta}_j\}_{j\in\mathbb{Z}}$ denotes the dyadic decomposition.

Proof. Since $\dot{\Delta}_j \dot{\Delta}_k = 0$ holds for $|j-k| \ge 2$ by Bahouri-Chemin-Danchin [1, Proposition 2.10], we have

$$\begin{cases} \|\varphi_0\|_{\dot{B}^{s+\gamma}_{r,1}} = \displaystyle\sum_{k\in\mathbb{Z}} 2^{(s+\gamma)k} \|\dot{\Delta}_k \varphi_0\|_{L^r} \le C \|\varphi\|_{\dot{B}^s_{r,\infty}} \sum_{k=-\infty}^{j_*+1} 2^{\gamma k} = \dfrac{2^{\gamma(j_*+1)}C}{1-2^{-\gamma}} \|\varphi\|_{\dot{B}^s_{r,\infty}}, \\ \|\varphi_0\|_{\dot{B}^s_{r,\infty}} = \displaystyle\sup_{k\in\mathbb{Z}} 2^{sk} \|\dot{\Delta}_k \varphi_0\|_{L^r} \le C \sup_{-\infty<k\le j_*+1} 2^{sk} \|\dot{\Delta}_k \varphi\|_{L^r}. \end{cases}$$

The remaining estimates are shown in the same way. This completes the proof. □

To show that solutions are weakly-star continuous at $t = 0$, we verify the following lemma:

Lemma 4.13. *Let $1 < r \le \infty$, $s, s_0 \in \mathbb{R}$, $0 < \alpha < 2$, $0 \le \beta < \infty$, $0 < \nu < \infty$, and $0 < T \le \infty$. Suppose that $F \in L^\infty_{\mathrm{loc}}((0,T); \dot{B}^{s_0}_{r,\infty})$ satisfies*

$$M := \sup_{0<t<T} t^{\alpha/2} \|F(t)\|_{\dot{B}^{s_0}_{r,\infty}} + \sup_{0<t<T} t^{\beta/2} \left\| \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau \right\|_{\dot{B}^{s+\beta}_{r,\infty}} < \infty.$$

Then it holds that

$$\lim_{t\to+0} t^{\beta/2} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau, \varphi \right\rangle = 0$$

for all $\varphi \in \dot{B}^{-s-\beta}_{r/(r-1),1}$.

Proof. Let $\varphi \in \dot{B}^{-s-\beta}_{r/(r-1),1}$ be arbitrary. By Proposition 3.2, we may take a sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{S}_0$ of functions satisfying $\lim_{j\to\infty} \|\varphi - \varphi_j\|_{\dot{B}^{-s-\beta}_{r/(r-1),1}} = 0$. Here, noting that $(\dot{B}^{-s_0}_{r/(r-1),1})^* = \dot{B}^{s_0}_{r,\infty}$ from

Proposition 3.2, we see by Proposition 3.8 that

$$\begin{aligned}\left|\left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi_j \right\rangle\right| &\le \int_0^t \left|\left\langle F(\tau), e^{\nu(t-\tau)\Delta}\varphi_j \right\rangle\right| d\tau \\ &\le \int_0^t \|F(\tau)\|_{\dot B^{s_0}_{r,\infty}} \|e^{\nu(t-\tau)\Delta}\varphi_j\|_{\dot B^{-s_0}_{r/(r-1),1}} d\tau \\ &\le \int_0^t M\tau^{-\alpha/2} C(\nu(t-\tau))^{-(1/2-\alpha/4)} \|\varphi_j\|_{\dot B^{-s_0-(1-\alpha/2)}_{r/(r-1),1}} d\tau \\ &\le MCB(1/2+\alpha/4, 1-\alpha/2)\nu^{-1/2+\alpha/4} t^{1/2-\alpha/4} \|\varphi_j\|_{\dot B^{-s_0-1+\alpha/2}_{r/(r-1),1}},\end{aligned}$$

where B denotes the beta function. Thus we have

$$\lim_{t\to+0} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi_j \right\rangle = 0$$

since $0 < \alpha < 2$. Noting that $(\dot B^{-s-\beta}_{r/(r-1),1})^* = \dot B^{s+\beta}_{r,\infty}$, we observe that

$$\begin{aligned}&\left| t^{\beta/2} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi \right\rangle\right| \\ &\le t^{\beta/2}\left|\left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi-\varphi_j \right\rangle\right| + t^{\beta/2}\left|\left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi_j \right\rangle\right| \\ &\le M\|\varphi-\varphi_j\|_{\dot B^{-s-\beta}_{r/(r-1),1}} + t^{\beta/2}\left|\left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi_j \right\rangle\right|,\end{aligned}$$

which implies

$$\limsup_{t\to+0} \left| t^{\beta/2} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau)d\tau, \varphi \right\rangle\right| \le M\|\varphi-\varphi_j\|_{\dot B^{-s-\beta}_{r/(r-1),1}}.$$

Therefore, letting $j \to \infty$ yields the desired result. This completes the proof of Lemma 4.13. □

Proof of Theorem 2.3. In case $\rho = \infty$, we introduce the space

$$\widetilde X^{\alpha,\beta}_T := \left\{ (n, \boldsymbol{u}) \,\middle|\, \begin{array}{ll} n \in BC_w([0,T); \dot B^{-2+N/r}_{r,\infty}), & \boldsymbol{u} \in BC_w([0,T); P(\dot B^{-1+N/q}_{q,\infty})^N), \\ t^{\alpha/2} n \in BC((0,T); \dot B^{\alpha-2+N/r}_{r,1}), & t^{\beta/2}\boldsymbol{u} \in BC((0,T); P(\dot B^{\beta-1+N/q}_{q,1})^N) \end{array} \right\}$$

instead of the space $X^{\alpha,\beta}_T$ defined by (4.2). Then, in the same way as in the proof of Theorem 4.3 (i), we may show that

$$\begin{cases} [\Phi_1(n,\boldsymbol{u}), \Phi_2(n,\boldsymbol{u})]_{X^{\alpha,\beta}_t} \le t^{\alpha/2}\|e^{dt\Delta}a\|_{\dot B^{\alpha-2+N/r}_{r,1}} + t^{\beta/2}\|e^{\nu t\Delta}\boldsymbol{b}\|_{\dot B^{\beta-1+N/q}_{q,1}} + 4C_{d,\nu}[n,\boldsymbol{u}]^2_{X^{\alpha,\beta}_t}, \\ \|\Phi_1(n,\boldsymbol{u}) - \Phi_1(m,\boldsymbol{v}), \Phi_2(n,\boldsymbol{u}) - \Phi_2(m,\boldsymbol{v})\|_{X^{\alpha,\beta}_T} \\ \qquad\qquad \le 16C_{d,\nu}([n,\boldsymbol{u}]_{X^{\alpha,\beta}_T} + [m,\boldsymbol{v}]_{X^{\alpha,\beta}_T})\|n-m, \boldsymbol{u}-\boldsymbol{v}\|_{X^{\alpha,\beta}_T} \end{cases}$$

like the estimates (4.12) and (4.14), where Φ_1 and Φ_2 are defined by (4.11). Here, Proposition 4.12 implies that the initial data $(a, \boldsymbol{b})$ can be decomposed into $a = a_0 + a_1$ and $\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_1$ with the properties

$$\begin{cases} a_0 \in \displaystyle\bigcap_{0<\gamma<\infty} (\dot B^{-2+N/r}_{r,\infty} \cap \dot B^{\gamma-2+N/r}_{r,1}), & \|a_1\|_{\dot B^{-2+N/r}_{r,\infty}} \le C \displaystyle\sup_{j_*\le k<\infty} 2^{(-2+N/r)k}\|\dot\Delta_k a\|_{L^r}, \\ \boldsymbol{b}_0 \in \displaystyle\bigcap_{0<\gamma<\infty} P(\dot B^{-1+N/q}_{q,\infty} \cap \dot B^{\gamma-1+N/q}_{q,1})^N, & \|\boldsymbol{b}_1\|_{\dot B^{-1+N/q}_{q,\infty}} \le C \displaystyle\sup_{j_*\le k<\infty} 2^{(-1+N/q)k}\|\dot\Delta_k \boldsymbol{b}\|_{L^q}. \end{cases}$$

Hence we see by Proposition 3.8 that

$$\begin{aligned}[e^{dt\Delta}a, e^{\nu t\Delta}\boldsymbol{b}]_{X_T^{\alpha,\beta}} &\le [e^{dt\Delta}a_0, e^{\nu t\Delta}\boldsymbol{b}_0]_{X_T^{\alpha,\beta}} + [e^{dt\Delta}a_1, e^{\nu t\Delta}\boldsymbol{b}_1]_{X_T^{\alpha,\beta}} \\ &\le [e^{dt\Delta}a_0, e^{\nu t\Delta}\boldsymbol{b}_0]_{X_T^{\alpha,\beta}} \\ &+ Cd^{-\alpha/2} \sup_{j_*\le k<\infty} 2^{(-2+N/r)k}\|\dot{\Delta}_k a\|_{L^r} + C\nu^{-\beta/2} \sup_{j_*\le k<\infty} 2^{(-1+N/q)k}\|\dot{\Delta}_k \boldsymbol{b}\|_{L^q}.\end{aligned}$$

On the one hand, it holds by Proposition 3.8 that

$$\begin{aligned}[e^{dt\Delta}a_0, e^{\nu t\Delta}\boldsymbol{b}_0]_{X_T^{\alpha,\beta}\cap X_T^{s,s}} \le C \Big(& d^{-\alpha/4}T^{\alpha/4}\|a_0\|_{\dot{B}^{\alpha/2-2+N/r}_{r,\infty}} + d^{-s/4}T^{s/4}\|a_0\|_{\dot{B}^{s/2-2+N/r}_{r,\infty}} \\ & + \nu^{-\beta/4}T^{\beta/4}\|\boldsymbol{b}_0\|_{\dot{B}^{\beta/2-1+N/q}_{q,\infty}} + \nu^{-s/4}T^{s/4}\|\boldsymbol{b}_0\|_{\dot{B}^{s/2-1+N/q}_{q,\infty}} \Big)\end{aligned}$$

since $a_0 \in \dot{B}^{\alpha/2-2+N/r}_{r,1} \cap \dot{B}^{s/2-2+N/r}_{r,1}$ and $\boldsymbol{b}_0 \in P(\dot{B}^{\beta/2-1+N/q}_{q,1} \cap \dot{B}^{s/2-1+N/q}_{q,1})^N$, on the other hand the assumption (2.9) allows us to suppose that

$$\sup_{j_*\le k<\infty} \Big(2^{(-2+N/r)k}\|\dot{\Delta}_k a\|_{L^r} + 2^{(-1+N/q)k}\|\dot{\Delta}_k \boldsymbol{b}\|_{L^q} \Big) < \frac{1}{2^8 C_{d,\nu} C (d^{-\alpha/2} + d^{-s/2} + \nu^{-\beta/2} + \nu^{-s/2})}.$$

Thus we observe that (4.15) holds by taking $0 < T < \infty$ sufficiently small. This gives that we may apply the Banach fixed point theorem in the same way as in the proof of Theorem 4.3 (i). As a by-product, we obtain (2.12). In addition, we may show (2.10) in the same way as in the proof of Lemma 4.8. Next, let us verify the weak-star continuity (2.11). By the system $n = e^{dt\Delta}a - I_1(n,n) - I_2(\boldsymbol{u}, n)$ and $\boldsymbol{u} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\boldsymbol{u},\boldsymbol{u}) + J_2(n,n)$, we have

$$\begin{cases} \langle n(t) - a, \varphi\rangle = \Big\langle e^{dt\Delta}a - a, \varphi \Big\rangle - \langle I_1(n,n), \varphi\rangle - \langle I_1(\boldsymbol{u}, n), \varphi\rangle, \\ \langle \boldsymbol{u}(t) - \boldsymbol{b}, \boldsymbol{f}\rangle = \big\langle e^{\nu t\Delta}\boldsymbol{b} - \boldsymbol{b}, \boldsymbol{f}\big\rangle - \langle J_1(\boldsymbol{u},\boldsymbol{u}), \boldsymbol{f}\rangle + \langle J_2(n,n), \boldsymbol{f}\rangle \end{cases}$$

for all $\varphi \in \dot{B}^{2-N/r}_{r/(r-1),1}$ and $\boldsymbol{f} \in (\dot{B}^{1-N/q}_{q/(q-1),1})^N$. Hence we observe that $\lim_{t\to+0}\langle n(t) - a, \varphi\rangle = 0$ and $\lim_{t\to+0}\langle \boldsymbol{u}(t) - \boldsymbol{b}, \boldsymbol{f}\rangle = 0$ with the aid of Proposition 3.10 and Lemma 4.13. In a similar manner, we have (2.11). For the statement (ii), we may show in a similar manner to the proof of Theorem 4.3 (ii). This completes the proof of Theorem 2.3. □

5. Additional properties of solutions under the certain conditions

In this section, we shall show the additional properties of solutions of (1.5) obtained in Theorem 2.1 under the certain conditions. We also show that global solutions of (1.5) decay as the limit $t \to \infty$ in the same norm of the space of the initial data. Continuing from the previous section, we will use the function space $X_T^{s,s}$ defined by (4.2).

5.1. Maximal regularity estimates.

Let us show the maximal regularity estimates of solutions, i.e., Theorem 2.7, where the idea of the proof relies on Kozono-Shimizu [26, Theorem 2]. To obtain the maximal regularity estimates, we introduce another scaling invariant space obtained from the structure of the bilinear estimates. Once we establish the nonlinear estimates for such a space, we may show the desired estimates immediately.

Lemma 5.1. *Let $1 \le \rho \le \infty$ and $0 < T \le \infty$. Suppose that $1 \le r, q < N$ satisfy (2.1) and $0 < s < 1$ satisfies (2.2). Then it holds that*

$$\begin{cases} \|t^s \nabla \cdot (n(\boldsymbol{K} * m))\|_{L^\infty_T(\dot{B}^{2s-4+N/r}_{r,1})} \le C\|t^{s/2} n\|_{L^\infty_T(\dot{B}^{s-2+N/r}_{r,1})} \|t^{s/2} m\|_{L^\infty_T(\dot{B}^{s-2+N/r}_{r,1})}, \\ \|t^s(\boldsymbol{u}\cdot\nabla m)\|_{L^\infty_T(\dot{B}^{2s-4+N/r}_{r,1})} \le C\|t^{s/2}\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{s-1+N/q}_{q,1})} \|t^{s/2} m\|_{L^\infty_T(\dot{B}^{s-2+N/r}_{r,1})}, \\ \|t^s(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{L^\infty_T(\dot{B}^{2s-3+N/q}_{q,1})} \le C\|t^{s/2}\boldsymbol{u}\|_{L^\infty_T(\dot{B}^{s-1+N/q}_{q,1})} \|t^{s/2} \boldsymbol{v}\|_{L^\infty_T(\dot{B}^{s-1+N/q}_{q,1})}, \\ \|t^s n(\boldsymbol{K} * m)\|_{L^\infty_T(\dot{B}^{2s-3+N/q}_{q,1})} \le C\|t^{s/2} n\|_{L^\infty_T(\dot{B}^{s-2+N/r}_{r,1})} \|t^{s/2} m\|_{L^\infty_T(\dot{B}^{s-2+N/r}_{r,1})} \end{cases} \tag{5.1}$$

for all $(n, \boldsymbol{u}), (m, \boldsymbol{v}) \in X^{s,s}_T$, where $C = C(N, r, q, \rho, s) > 0$ is a constant independent of $t, T, n, m, \boldsymbol{u}$, and $\boldsymbol{v}$. Moreover, the following estimates

$$\begin{cases} \|\nabla \cdot (n(\boldsymbol{K} * m))\|_{L^{1/s,\rho}_T(\dot{B}^{2s-4+N/r}_{r,1})} \le C\|n\|_{L^{2/s,\rho}_T(\dot{B}^{s-2+N/r}_{r,1})} \|m\|_{L^{2/s,\rho}_T(\dot{B}^{s-2+N/r}_{r,1})}, \\ \|\boldsymbol{u}\cdot\nabla m\|_{L^{1/s,\rho}_T(\dot{B}^{2s-4+N/r}_{r,1})} \le C\|\boldsymbol{u}\|_{L^{2/s,\rho}_T(\dot{B}^{s-1+N/q}_{q,1})} \|m\|_{L^{2/s,\rho}_T(\dot{B}^{s-2+N/r}_{r,1})}, \\ \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{L^{1/s,\rho}_T(\dot{B}^{2s-3+N/q}_{q,1})} \le C\|\boldsymbol{u}\|_{L^{2/s,\rho}_T(\dot{B}^{s-1+N/q}_{q,1})} \|\boldsymbol{v}\|_{L^{2/s,\rho}_T(\dot{B}^{s-1+N/q}_{q,1})}, \\ \|n(\boldsymbol{K} * m)\|_{L^{1/s,\rho}_T(\dot{B}^{2s-3+N/q}_{q,1})} \le C\|n\|_{L^{2/s,\rho}_T(\dot{B}^{s-2+N/r}_{r,1})} \|m\|_{L^{2/s,\rho}_T(\dot{B}^{s-2+N/r}_{r,1})} \end{cases} \tag{5.2}$$

hold for all $(n, \boldsymbol{u}), (m, \boldsymbol{v}) \in L^{2/s,\rho}((0,T); \dot{B}^{s-2+N/r}_{r,1}) \times L^{2/s,\rho}((0,T); P(\dot{B}^{s-1+N/q}_{q,1})^N)$.

Proof. Since we see by (4.1) that

$$\begin{cases} \|\nabla \cdot (n(t)(\boldsymbol{K} * m)(t))\|_{\dot{B}^{2s-4+N/r}_{r,1}} \le C\|n(t)\|_{\dot{B}^{s-2+N/r}_{r,1}} \|m(t)\|_{\dot{B}^{s-2+N/r}_{r,1}}, \\ \|\boldsymbol{u}(t)\cdot\nabla m(t)\|_{\dot{B}^{2s-4+N/r}_{r,1}} \le C\|\boldsymbol{u}(t)\|_{\dot{B}^{s-1+N/q}_{q,1}} \|m(t)\|_{\dot{B}^{s-2+N/r}_{r,1}}, \\ \|(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{v}(t)\|_{\dot{B}^{2s-3+N/q}_{q,1}} \le C\|\boldsymbol{u}(t)\|_{\dot{B}^{s-1+N/q}_{q,1}} \|\boldsymbol{v}(t)\|_{\dot{B}^{s-1+N/q}_{q,1}}, \\ \|n(t)(\boldsymbol{K} * m)(t)\|_{\dot{B}^{2s-3+N/q}_{q,1}} \le C\|n(t)\|_{\dot{B}^{s-2+N/r}_{r,1}} \|m(t)\|_{\dot{B}^{s-2+N/r}_{r,1}} \end{cases}$$

hold for all $0 < t < T$, we obtain (5.1) and (5.2) with the aid of Proposition 3.5. This completes the proof. □

Proof of Theorem 2.7. We introduce the function spaces $X^{2s,2s}_T$ and $Y^{2s,2s}_T$ defined by (4.2) and (4.23), respectively. Then, by setting $\eta = s$ in (4.1), we may show the following nonlinear estimates

$$\begin{cases} \|I_1(n,m), I_2(\boldsymbol{u},m)\|_{L^\infty_T(\dot{B}^{2s-2+N/r}_{r,1})} + \|J_1(\boldsymbol{u},\boldsymbol{v}), J_2(n,m)\|_{L^\infty_T(\dot{B}^{2s-1+N/q}_{q,1})} \\ \le C_{d,\nu}\|n,\boldsymbol{u}\|_{X^{s,s}_T \cap X^{2s,2s}_T} \|m,\boldsymbol{v}\|_{X^{s,s}_T \cap X^{2s,2s}_T}, \\ \|I_1(n,m), I_2(\boldsymbol{u},m)\|_{L^{1/s,\rho}_T(\dot{B}^{2s-2+N/r}_{r,1})} + \|J_1(\boldsymbol{u},\boldsymbol{v}), J_2(n,m)\|_{L^{1/s,\rho}_T(\dot{B}^{2s-1+N/q}_{q,1})} \\ \le C_{d,\nu}\|n,\boldsymbol{u}\|_{Y^{s,s}_T \cap Y^{2s,2s}_T} \|m,\boldsymbol{v}\|_{Y^{s,s}_T \cap Y^{2s,2s}_T} \end{cases}$$

in a similar manner to the proof of (4.8) and (4.9). Here we note that these estimates are valid provided that s satisfies $0 < s < 2/3$ along with (2.2). Therefore, in the same way as in the proof of Theorem 4.3 (i) and Lemma 4.6, we see that the solution $(n, \boldsymbol{u})$ also satisfies

$$\begin{cases} t^s n \in BC([0,T); \dot{B}^{2s-2+N/r}_{r,1}), & n \in L^{1/s,\rho}((0,T); \dot{B}^{2s-2+N/r}_{r,1}), \\ t^s \boldsymbol{u} \in BC([0,T); P(\dot{B}^{2s-1+N/q}_{q,1})^N), & \boldsymbol{u} \in L^{1/s,\rho}((0,T); P(\dot{B}^{2s-1+N/q}_{q,1})^N) \end{cases}$$

with (2.19) having the estimates (2.20). On the other hand, we see by the system $\partial_t n = d\Delta n - \nabla \cdot (n(\boldsymbol{K} * n)) - \boldsymbol{u}\cdot\nabla n$ and $\partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + P(n(\boldsymbol{K}*n))$ that

$$
\begin{cases} \|\partial_t n(t)\|_{\dot B^{2s-4+N/r}_{r,1}} \le Cd\|n(t)\|_{\dot B^{2s-2+N/r}_{r,1}} + \|\nabla\cdot(n(t)(\boldsymbol{K}*n)(t)), \boldsymbol{u}(t)\cdot\nabla n(t)\|_{\dot B^{2s-4+N/r}_{r,1}}, \\ \|\partial_t \boldsymbol{u}(t)\|_{\dot B^{2s-3+N/q}_{q,1}} \le C\nu\|\boldsymbol{u}(t)\|_{\dot B^{2s-1+N/q}_{q,1}} + \|P(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t), P(n(t)(\boldsymbol{K}*n)(t))\|_{\dot B^{2s-3+N/q}_{q,1}} \end{cases} \tag{5.3}
$$

hold. Since

$$
\begin{aligned} &\|t^{s/2}n\|_{L^\infty_T(\dot B^{s-2+N/r}_{r,1})} + \|n\|_{L^{2/s,\rho}_T(\dot B^{s-2+N/r}_{r,1})} + \|t^{s/2}\boldsymbol{u}\|_{L^\infty_T(\dot B^{s-1+N/q}_{q,1})} + \|\boldsymbol{u}\|_{L^{2/s,\rho}_T(\dot B^{s-1+N/q}_{q,1})} \\ &\le C\left(d^{-s/2}\|a\|_{\dot B^{-2+N/r}_{r,\rho}} + \nu^{-s/2}\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}\right) \end{aligned}
$$

hold from (2.5), we see by (5.1) and (5.2) that

$$
\begin{aligned} &\|t^s\nabla\cdot(n(\boldsymbol{K}*n)), t^s(\boldsymbol{u}\cdot\nabla n)\|_{L^\infty_T(\dot B^{2s-4+N/r}_{r,1})} + \|t^s(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, t^s n(\boldsymbol{K}*n)\|_{L^\infty_T(\dot B^{2s-3+N/q}_{q,1})} \\ &+ \|\nabla\cdot(n(\boldsymbol{K}*n)), \boldsymbol{u}\cdot\nabla n\|_{L^{1/s,\rho}_T(\dot B^{2s-4+N/r}_{r,1})} + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, n(\boldsymbol{K}*n)\|_{L^{1/s,\rho}_T(\dot B^{2s-3+N/q}_{q,1})} \\ &\le C\left(d^{-s/2}\|a\|_{\dot B^{-2+N/r}_{r,\rho}} + \nu^{-s/2}\|\boldsymbol{b}\|_{\dot B^{-1+N/q}_{q,\rho}}\right)^2. \end{aligned}
$$

Hence, it holds by (5.3) that (2.21), which complete the proof of Theorem 2.7. □

5.2. **Additional regularities.** We shall show that if the initial data has higher regularities, then the solution also has the additional regularities. To this end, we consider the linearized problem by using the solution obtained in Theorem 2.1.

Lemma 5.2. *Let $1 \le \rho \le \infty$ and $0 < T \le \infty$. Suppose that $1 \le r, q < N$ satisfy (2.1) and $0 < s < 1$ satisfies (2.2). In addition, suppose that $N/(N-1+s) \le \theta < N$ and $0 < \sigma < s$. Then, for the nonlinear terms I_1, I_2, J_1, and J_2 defined by (4.4), it holds that*

$$
\begin{cases} \|I_1(n,m)(t)\|_{\dot B^{\sigma j}_{\theta,1}} \le C_{d,\nu} t^{-(\sigma/2)j}\|n\|_{L^\infty_t(L^\theta)}\|\tau^{s/2}m\|_{L^\infty_t(\dot B^{s-2+N/r}_{r,1})}, \\ \|I_2(\boldsymbol{u},m)(t)\|_{\dot B^{\sigma j}_{\theta,1}} \le C_{d,\nu} t^{-(\sigma/2)j}\|m\|_{L^\infty_t(L^\theta)}\|\tau^{s/2}\boldsymbol{u}\|_{L^\infty_t(\dot B^{s-1+N/q}_{q,1})}, \\ \|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot B^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu} t^{-(\sigma/2)j}\|\boldsymbol{u}\|_{L^\infty_t(L^{N\theta/(N-\theta)})}\|\tau^{s/2}\boldsymbol{v}\|_{L^\infty_t(\dot B^{s-1+N/q}_{q,1})}, \\ \|J_2(n,m)(t)\|_{\dot B^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu} t^{-(\sigma/2)j}\|n\|_{L^\infty_t(L^\theta)}\|\tau^{s/2}m\|_{L^\infty_t(\dot B^{s-2+N/r}_{r,1})} \end{cases} \tag{5.4}
$$

for all $0 < t < T$, $0 < d, \nu < \infty$, $j = 0, 1$, and $(n,\boldsymbol{u}), (m,\boldsymbol{v}) \in X^{s,s}_T \cap BC([0,T); L^\theta \times P(L^{N\theta/(N-\theta)})^N)$, where $C_{d,\nu} = C(d,\nu,N,r,q,\rho,s,\theta,\sigma) > 0$ is a constant independent of $t, T, n, m, \boldsymbol{u}$, and $\boldsymbol{v}$.

Proof. First we note that $(2-s)/N < 1/\theta + (1-s)/N \le 1$. Hence, by taking θ_s such that $1/\theta_s = 1/\theta + (1-s)/N$, we have $1 \le \theta_s < N$. Since it holds by Proposition 3.1 that $\dot B^{s-2+N/r}_{r,1} \subset L^{N/(2-s)}$ and $\dot B^{s-1+N/q}_{q,1} \subset L^{N/(1-s)}$, we obtain

$$
\begin{cases} \|\varphi(\boldsymbol{K}*\psi)\|_{L^{\theta_s}} \le C\|\varphi\|_{L^\theta}\|\psi\|_{L^{N/(2-s)}} \le C\|\varphi\|_{L^\theta}\|\varphi\|_{\dot B^{s-2+N/r}_{r,1}}, \\ \|\varphi\psi\|_{L^{\theta_s}} \le C\|\varphi\|_{L^\theta}\|\psi\|_{L^{N/(1-s)}} \le C\|\varphi\|_{L^\theta}\|\psi\|_{\dot B^{s-1+N/q}_{q,1}}, \\ \|\varphi\psi\|_{L^{(1/\theta_s-1/N)^{-1}}} \le C\|\varphi\|_{L^{N\theta/(N-\theta)}}\|\psi\|_{L^{N/(1-s)}} \le C\|\varphi\|_{L^{N\theta/(N-\theta)}}\|\psi\|_{\dot B^{s-1+N/q}_{q,1}} \end{cases}
$$

for all $\varphi, \psi \in \mathscr{S}$ by virtue of Proposition 3.4. In addition, we see by Proposition 3.8 that

$$\begin{cases} \|e^{dt\Delta}\nabla\cdot \boldsymbol{f}\|_{\dot{B}^{\sigma j}_{\theta,1}} \le C_{d,\nu} t^{-(1-s)/2-1/2-(\sigma/2)j}\|\boldsymbol{f}\|_{L^{\theta_s}}, \\ \|e^{\nu t\Delta}\nabla\cdot \boldsymbol{f}\|_{\dot{B}^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu} t^{-(1-s)/2-1/2-(\sigma/2)j}\|\boldsymbol{f}\|_{L^{(1/\theta_s-1/N)^{-1}}}, \\ \|e^{\nu t\Delta}\boldsymbol{f}\|_{\dot{B}^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu} t^{-(2-s)/2-(\sigma/2)j}\|\boldsymbol{f}\|_{L^{\theta_s}} \end{cases}$$

for all $\boldsymbol{f} \in \mathscr{S}^N$. Since the condition $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{v} = 0$ implies that $\boldsymbol{u}\cdot\nabla m = \nabla\cdot(\boldsymbol{u}m)$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{v} = \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{v})$, we observe that

$$\begin{cases} \|I_1(n,m)(t)\|_{\dot{B}^{\sigma j}_{\theta,1}} \le C_{d,\nu}\displaystyle\int_0^t (t-\tau)^{s/2-(\sigma/2)j-1}\|n(\tau)(\boldsymbol{K}*m)(\tau)\|_{L^{\theta_s}}\,d\tau, \\ \|I_2(\boldsymbol{u},m)(t)\|_{\dot{B}^{\sigma j}_{\theta,1}} \le C_{d,\nu}\displaystyle\int_0^t (t-\tau)^{s/2-(\sigma/2)j-1}\|\boldsymbol{u}(\tau)m(\tau)\|_{L^{\theta_s}}\,d\tau, \\ \|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot{B}^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu}\displaystyle\int_0^t (t-\tau)^{s/2-(\sigma/2)j-1}\|\boldsymbol{u}(\tau)\otimes\boldsymbol{v}(\tau)\|_{L^{(1/\theta_s-1/N)^{-1}}}\,d\tau, \\ \|J_2(n,m)(t)\|_{\dot{B}^{\sigma j}_{N\theta/(N-\theta),1}} \le C_{d,\nu}\displaystyle\int_0^t (t-\tau)^{s/2-(\sigma/2)j-1}\|n(\tau)(\boldsymbol{K}*m)(\tau)\|_{L^{\theta_s}}\,d\tau, \end{cases}$$

which yield (5.4) combined with the above estimates. This completes the proof. □

Proof of Theorem 2.9. We begin with considering the following linearized problem;

$$(5.5)\qquad \begin{cases} \overline{n} = e^{dt\Delta}a - I_1(\overline{n},n) - I_2(\boldsymbol{u},\overline{n}) & \text{in } (0,T)\times\mathbb{R}^N, \\ \overline{\boldsymbol{u}} = e^{\nu t\Delta}\boldsymbol{b} - J_1(\overline{\boldsymbol{u}},\boldsymbol{u}) + J_2(\overline{n},n) & \text{in } (0,T)\times\mathbb{R}^N, \end{cases}$$

where $(n,\boldsymbol{u}) \in X^{s,s}_T$ is the solution on $(0,T)\times\mathbb{R}^N$ of (1.5) and $(\overline{n},\overline{\boldsymbol{u}})$ is the unknown function. Let $0 < T_* < T$ be arbitrary. Define the mappings Ψ_1 and Ψ_2 by setting

$$\begin{cases} \Psi_1(\overline{n},\overline{\boldsymbol{u}})(t) = e^{dt\Delta}a - I_1(\overline{n},n)(t) - I_2(\boldsymbol{u},\overline{n})(t), & 0<t<T_*, \\ \Psi_2(\overline{n},\overline{\boldsymbol{u}})(t) = e^{\nu t\Delta}\boldsymbol{b} - J_1(\overline{\boldsymbol{u}},\boldsymbol{u})(t) + J_2(\overline{n},n)(t), & 0<t<T_* \end{cases}$$

for $(\overline{n},\overline{\boldsymbol{u}}) \in X^{s,s}_{T_*} \cap BC([0,T_*); L^\theta \times P(L^{N\theta/(N-\theta)})^N)$. Then, we see by (4.8) and (5.4) that

$$(5.6)\qquad \begin{cases} \|\Psi_1(\overline{n},\overline{\boldsymbol{u}}),\Psi_2(\overline{n},\overline{\boldsymbol{u}})\|_{X^{s,s}_{T_*}} \\ \le C\left(d^{-s/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-s/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) + C_{d,\nu}[n,\boldsymbol{u}]_{X^{s,s}_{T_*}}[\overline{n},\overline{\boldsymbol{u}}]_{X^{s,s}_{T_*}}, \\ \|\Psi_1(\overline{n},\overline{\boldsymbol{u}})\|_{L^\infty_{T_*}(L^\theta)} + \|\Psi_2(\overline{n},\overline{\boldsymbol{u}})\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})} \\ \le \|a\|_{L^\theta} + \|\boldsymbol{b}\|_{L^{N\theta/(N-\theta)}} + C_{d,\nu}[n,\boldsymbol{u}]_{X^{s,s}_{T_*}}\left(\|\overline{n}\|_{L^\infty_{T_*}(L^\theta)} + \|\overline{\boldsymbol{u}}\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})}\right) \end{cases}$$

with the aid of Proposition 3.8. Thus we have $(\Psi_1(\overline{n},\overline{\boldsymbol{u}}),\Psi_2(\overline{n},\overline{\boldsymbol{u}})) \in X^{s,s}_{T_*} \cap BC([0,T_*); L^\theta \times P(L^{N\theta/(N-\theta)})^N)$. In a similar manner, we also have

$$\begin{cases} \|\Psi_1(\overline{n},\overline{\boldsymbol{u}}) - \Psi_1(\overline{m},\overline{\boldsymbol{v}}), \Psi_2(\overline{n},\overline{\boldsymbol{u}}) - \Psi_2(\overline{m},\overline{\boldsymbol{v}})\|_{X^{s,s}_{T_*}} \le C_{d,\nu}[n,\boldsymbol{u}]_{X^{s,s}_{T_*}}\|\overline{n}-\overline{m},\overline{\boldsymbol{u}}-\overline{\boldsymbol{v}}\|_{X^{s,s}_{T_*}}, \\ \|\Psi_1(\overline{n},\overline{\boldsymbol{u}}) - \Psi_1(\overline{m},\overline{\boldsymbol{v}})\|_{L^\infty_{T_*}(L^\theta)} + \|\Psi_2(\overline{n},\overline{\boldsymbol{u}}) - \Psi_2(\overline{m},\overline{\boldsymbol{v}})\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})} \\ \qquad \le C_{d,\nu}[n,\boldsymbol{u}]_{X^{s,s}_{T_*}}\left(\|\overline{n}-\overline{m}\|_{L^\infty_{T_*}(L^\theta)} + \|\overline{\boldsymbol{u}}-\overline{\boldsymbol{v}}\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})}\right). \end{cases}$$

Hence, by taking $0 < T_* < T$ such that

$$(5.7)\qquad [n,\boldsymbol{u}]_{X^{s,s}_{T_*}} < 1/(2C_{d,\nu}),$$

we may apply the Banach fixed point theorem to the mapping (Ψ_1, Ψ_2). Therefore, there exists a unique pair

$$(\overline{n}, \overline{\boldsymbol{u}}) \in X^{s,s}_{T_*} \cap BC([0,T_*); L^\theta \times P(L^{N\theta/(N-\theta)})^N) \tag{5.8}$$

of functions satisfying (5.5). In fact, we see that $(\overline{n}, \overline{\boldsymbol{u}})$ coincides with $(n, \boldsymbol{u})$ since $(n, \boldsymbol{u}) \in X^{s,s}_T$ satisfies (4.3) and since the solution $(\overline{n}, \overline{\boldsymbol{u}})$ of (5.5) satisfying (5.8) is unique. Thus we obtain

$$\|n\|_{L^\infty_{T_*}(L^\theta)} + \|\boldsymbol{u}\|_{L^\infty_{T_*}(L^{N\theta/(N-\theta)})} \le 2(\|a\|_{L^\theta} + \|\boldsymbol{b}\|_{L^{N\theta/(N-\theta)}})$$

by combining the estimates (5.6) and (5.7). Likewise, we obtain

$$\|t^{\sigma/2} n\|_{L^\infty_{T_*}(\dot{B}^\sigma_{\theta,1})} + \|t^{\sigma/2}\boldsymbol{u}\|_{L^\infty_{T_*}(\dot{B}^\sigma_{N\theta/(N-\theta),1})} \le C\left(d^{-\sigma/2}\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \nu^{-\sigma/2}\|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right)$$

from (5.4) and (5.7). For the properties (2.24), we may show by the estimates

$$\begin{cases} \|n(t) - a\|_{L^\theta} \le \|e^{dt\Delta}a - a\|_{L^\theta} + MC_{d,\nu}[n, \boldsymbol{u}]_{X^{s,s}_t}, \\ \|\boldsymbol{u}(t) - \boldsymbol{b}\|_{L^{N\theta/(N-\theta)}} \le \|e^{\nu t\Delta}\boldsymbol{b} - \boldsymbol{b}\|_{L^{N\theta/(N-\theta)}} + MC_{d,\nu}[n, \boldsymbol{u}]_{X^{s,s}_t}, \\ \|n(t)\|_{\dot{B}^\sigma_{\theta,1}} \le \|e^{dt\Delta}a\|_{\dot{B}^\sigma_{\theta,1}} + MC_{d,\nu}t^{-\sigma/2}[n, \boldsymbol{u}]_{X^{s,s}_t}, \\ \|\boldsymbol{u}(t)\|_{\dot{B}^\sigma_{N\theta/(N-\theta),1}} \le \|e^{\nu t\Delta}\boldsymbol{b}\|_{\dot{B}^\sigma_{N\theta/(N-\theta),1}} + MC_{d,\nu}t^{-\sigma/2}[n, \boldsymbol{u}]_{X^{s,s}_t} \end{cases}$$

with the aid of Proposition 3.10, where $M := \|n\|_{L^\infty_T(L^\theta)} + \|\boldsymbol{u}\|_{L^\infty_T(L^{N\theta/(N-\theta)})}$. Moreover, since the condition (5.7) of T_* is independent of the initial data $(a, \boldsymbol{b})$, we can extend the existence time interval T_* to T by repeating this argument with replacing initial data within many steps. Thus we obtain (2.23), which complete the proof of Theorem 2.9. □

5.3. **Time-decay properties of global solutions.** Finally, we consider the time-decay properties of global solutions. Since it is shown that the solution exists globally in time from Theorem 4.3 (iii), it is sufficient to show that global solutions decay as the limit $t \to \infty$. To show the time-decay in the same norm of the space of the initial data, we use the density argument.

Lemma 5.3. *Let $1 \le \rho \le \infty$ and $0 < T \le \infty$. Suppose that $1 \le r, q < N$ satisfy (2.1) and $0 < s < 1$ satisfies (2.2). In addition, suppose that θ satisfies*

$$\max\{1/N, 1/N - (1/r - 1/q), s/N + 1/r - 1/q\} < 1/\theta < (s+2)/N. \tag{5.9}$$

Then, for the nonlinear terms $I_1, I_2, J_1,$ and J_2 defined by (4.4), it holds that

$$\begin{cases} \|I_1(n,m)(t)\|_{\dot{B}^{-2+N/r}_{r,1}} \le C_{d,\nu}t^{1-N/(2\theta)}\|n,m\|_{L^\infty_t(L^\theta)}\|\tau^{s/2}n, \tau^{s/2}m\|_{L^\infty_t(\dot{B}^{s-2+N/r}_{r,1})}, \\ \|I_2(\boldsymbol{u},m)(t)\|_{\dot{B}^{-2+N/r}_{r,1}} \le C_{d,\nu}t^{1-N/(2\theta)}\left(\|m\|_{L^\infty_t(L^\theta)} + \|\boldsymbol{u}\|_{L^\infty_t(L^{N\theta/(N-\theta)})}\right) \\ \qquad\qquad \times\left(\|\tau^{s/2}m\|_{L^\infty_t(\dot{B}^{s-2+N/r}_{r,1})} + \|\tau^{s/2}\boldsymbol{u}\|_{L^\infty_t(\dot{B}^{s-1+N/q}_{q,1})}\right), \\ \|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot{B}^{-1+N/q}_{q,1}} \le C_{d,\nu}t^{1-N/(2\theta)}\|\boldsymbol{u},\boldsymbol{v}\|_{L^\infty_t(L^{N\theta/(N-\theta)})}\|\tau^{s/2}\boldsymbol{u}, \tau^{s/2}\boldsymbol{v}\|_{L^\infty_t(\dot{B}^{s-1+N/q}_{q,1})}, \\ \|J_2(n,m)(t)\|_{\dot{B}^{-1+N/q}_{q,1}} \le C_{d,\nu}t^{1-N/(2\theta)}\|n,m\|_{L^\infty_t(L^\theta)}\|\tau^{s/2}n, \tau^{s/2}m\|_{L^\infty_t(\dot{B}^{s-2+N/r}_{r,1})} \end{cases} \tag{5.10}$$

for all $0 < t < T$, $0 < d, \nu < \infty$, and $(n, \boldsymbol{u}), (m, \boldsymbol{v}) \in X^{s,s}_T \cap BC([0,T); L^\theta \times P(L^{N\theta/(N-\theta)})^N)$, where $C_{d,\nu} = C(d, \nu, N, r, q, \rho, s, \theta, \sigma) > 0$ is a constant independent of $t, T, n, m, \boldsymbol{u},$ and $\boldsymbol{v}$.

Proof. First notice that we may take θ satisfying (5.9) by (2.1). Since $2-N/r<(1/2)(3-N/r)<s$ holds from $1\le r<N$ and (2.2), there exist $r<r_*<N/(2-s)$ and $q<q_*<N/(1-s)$ such that

$$\begin{cases}\max\{1/r+1/N-1/\theta,1/q+1/N-1/\theta\}\le 1/r_*\le\min\{1+1/N-1/\theta,(1-s)/N+1/q\},\\ 1/q+1/N-1/\theta\le 1/q_*\le 1+1/N-1/\theta.\end{cases}$$

Then it holds by Proposition 3.1 that

$$\dot{B}^{s-2+N/r}_{r,1}\subset\dot{B}^{s-2+N/r_*}_{r_*,1},\quad \dot{B}^{s-1+N/q}_{q,1}\subset\dot{B}^{s-1+N/q_*}_{q_*,1},\quad \dot{B}^{s-1+N/q}_{q,1}\subset\dot{B}^{s-2+N/r_*}_{(1/r_*-1/N)^{-1},1}.$$

Hence, for $1\le r_\theta\le r$ and $1\le q_\theta\le q$ satisfying

$$1/r_\theta=1/r_*+1/\theta-1/N,\quad 1/q_\theta=1/q_*+1/\theta-1/N,$$

we obtain

$$\begin{cases}\|\varphi(\boldsymbol{K}*\psi)\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}\le C(\|\varphi\|_{\dot{B}^{s-2+N/r_*}_{r_*,1}}\|\boldsymbol{K}*\psi\|_{L^{(1/\theta-1/N)^{-1}}}+\|\varphi\|_{L^\theta}\|\boldsymbol{K}*\psi\|_{\dot{B}^{s-2+N/r_*}_{(1/r_*-1/N)^{-1},1}})\\ \qquad\le C(\|\varphi\|_{\dot{B}^{s-2+N/r}_{r,1}}\|\psi\|_{L^\theta}+\|\varphi\|_{L^\theta}\|\psi\|_{\dot{B}^{s-2+N/r}_{r,1}}),\\ \|\boldsymbol{f}\varphi\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}\le C(\|\boldsymbol{f}\|_{\dot{B}^{s-2+N/r_*}_{(1/r_*-1/N)^{-1},1}}\|\varphi\|_{L^\theta}+\|\boldsymbol{f}\|_{L^{N\theta/(N-\theta)}}\|\varphi\|_{\dot{B}^{s-2+N/r_*}_{r_*,1}})\\ \qquad\le C(\|\boldsymbol{f}\|_{\dot{B}^{s-1+N/q}_{q,1}}\|\varphi\|_{L^\theta}+\|\boldsymbol{f}\|_{L^{N\theta/(N-\theta)}}\|\varphi\|_{\dot{B}^{s-2+N/r}_{r,1}}),\\ \|\boldsymbol{f}\otimes\boldsymbol{g}\|_{\dot{B}^{s-1+N/q_*}_{q_\theta,1}}\le C(\|\boldsymbol{f}\|_{\dot{B}^{s-1+N/q_*}_{q_*,1}}\|\boldsymbol{g}\|_{L^{N\theta/(N-\theta)}}+\|\boldsymbol{f}\|_{L^{N\theta/(N-\theta)}}\|\boldsymbol{g}\|_{\dot{B}^{s-1+N/q_*}_{q_*,1}})\\ \qquad\le C(\|\boldsymbol{f}\|_{\dot{B}^{s-1+N/q}_{q,1}}\|\boldsymbol{g}\|_{L^{N\theta/(N-\theta)}}+\|\boldsymbol{f}\|_{L^{N\theta/(N-\theta)}}\|\boldsymbol{g}\|_{\dot{B}^{s-1+N/q}_{q,1}})\end{cases}$$

for all $\varphi,\psi\in\mathscr{S}$ and $\boldsymbol{f},\boldsymbol{g}\in\mathscr{S}^N$ by virtue of Propositions 3.3 and 3.4. In addition, we see by Proposition 3.8 that

$$\begin{cases}\|e^{dt\Delta}\nabla\cdot\boldsymbol{f}\|_{\dot{B}^{-2+N/r}_{r,1}}\le C_{d,\nu}t^{-(N/2)(1/r_\theta-1/r)-(1/2)(1-s+N/r-N/r_*)}\|\boldsymbol{f}\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}},\\ \|e^{\nu t\Delta}\nabla\cdot\boldsymbol{f}\|_{\dot{B}^{-1+N/q}_{q,1}}\le C_{d,\nu}t^{-(N/2)(1/q_\theta-1/q)-(1/2)(1-s+N/q-N/q_*)}\|\boldsymbol{f}\|_{\dot{B}^{s-1+N/q_*}_{q_\theta,1}},\\ \|e^{\nu t\Delta}\boldsymbol{f}\|_{\dot{B}^{-1+N/q}_{q,1}}\le C_{d,\nu}t^{-(N/2)(1/r_\theta-1/q)-(1/2)(1-s+N/q-N/r_*)}\|\boldsymbol{f}\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}\end{cases}$$

for all $\boldsymbol{f}\in\mathscr{S}^N$. Since the condition $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{v}=0$ implies that $\boldsymbol{u}\cdot\nabla m=\nabla\cdot(\boldsymbol{u}m)$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}=\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{v})$, we observe that

$$\begin{cases}\|I_1(n,m)(t)\|_{\dot{B}^{-2+N/r}_{r,1}}\le C_{d,\nu}\int_0^t(t-\tau)^{s/2-N/(2\theta)}\|n(\tau)(\boldsymbol{K}*m)(\tau)\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}d\tau,\\ \|I_2(\boldsymbol{u},m)(t)\|_{\dot{B}^{-2+N/r}_{r,1}}\le C_{d,\nu}\int_0^t(t-\tau)^{s/2-N/(2\theta)}\|\boldsymbol{u}(\tau)m(\tau)\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}d\tau,\\ \|J_1(\boldsymbol{u},\boldsymbol{v})(t)\|_{\dot{B}^{-1+N/q}_{q,1}}\le C_{d,\nu}\int_0^t(t-\tau)^{s/2-N/(2\theta)}\|\boldsymbol{u}(\tau)\otimes\boldsymbol{v}(\tau)\|_{\dot{B}^{s-1+N/q_*}_{q_\theta,1}}d\tau,\\ \|J_2(n,m)(t)\|_{\dot{B}^{-1+N/q}_{q,1}}\le C_{d,\nu}\int_0^t(t-\tau)^{s/2-N/(2\theta)}\|n(\tau)(\boldsymbol{K}*m)(\tau)\|_{\dot{B}^{s-2+N/r_*}_{r_\theta,1}}d\tau,\end{cases}$$

which yield (5.10) combined with the above estimates. This completes the proof. □

In case the interpolation exponent $\rho=\infty$, it is unknown whether global solutions decay in the same norm of the space of the initial data. However, we may obtain the time-decay properties in the sense of the weak-star topology.

Lemma 5.4. *Let* $1 < r \leq \infty$, $s, s_0 \in \mathbb{R}$, $0 < \alpha < 2$, *and* $0 < \nu < \infty$. *Suppose that* $F \in L^\infty_{\mathrm{loc}}((0,\infty); \dot{B}^{s_0}_{r,\infty})$ *satisfies*

$$M := \sup_{0<t<\infty} t^{\alpha/2} \|F(t)\|_{\dot{B}^{s_0}_{r,\infty}} + \sup_{0<t<\infty} \left\| \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau \right\|_{\dot{B}^{s}_{r,\infty}} < \infty.$$

Then it holds that

$$\lim_{t\to\infty} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau, \varphi \right\rangle = 0$$

for all $\varphi \in \dot{B}^{-s}_{r/(r-1),1}$.

Proof. Let $\varphi \in \dot{B}^{-s}_{r/(r-1),1}$ be arbitrary. By Proposition 3.2, we may take a sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathscr{S}_0$ of functions satisfying $\lim_{j\to\infty} \|\varphi - \varphi_j\|_{\dot{B}^{-s}_{r/(r-1),1}} = 0$. Hence, in a similar manner to the proof of Lemma 4.13, we have

$$\begin{aligned} \left| \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau, \varphi_j \right\rangle \right| &\leq \int_0^t M\tau^{-\alpha/2} C(\nu(t-\tau))^{-(1-\alpha/4)} \|\varphi_j\|_{\dot{B}^{-s_0-(2-\alpha/2)}_{r/(r-1),1}} d\tau \\ &\leq MCB(\alpha/4, 1-\alpha/2)\nu^{-1+\alpha/4} t^{-\alpha/4} \|\varphi_j\|_{\dot{B}^{-s_0-2+\alpha/2}_{r/(r-1),1}}, \end{aligned}$$

which yields

$$\lim_{t\to\infty} \left\langle \int_0^t e^{\nu(t-\tau)\Delta} F(\tau) d\tau, \varphi_j \right\rangle = 0.$$

Thus we may show the desired property in the same way as in the proof of Lemma 4.13. □

Proof of Theorem 2.5. First, notice that we may take $T = \infty$, i.e., obtain global solutions $(n, \boldsymbol{u})$ by virtue of Theorem 4.3 (iii). Hence, it remains to prove (2.16) and (2.17). Suppose that $1 \leq \rho < \infty$. Since $1 \leq r, q < N$ satisfy (2.1), we may take θ so that

$$\max\{2/N, s/N + 1/r - 1/q\} < 1/\theta < (s+2)/N$$

holds. Since θ also satisfies $N/(N-1+s) \leq \theta < N$, Theorem 2.9 implies that if we choose an initial data

$$(a_*, \boldsymbol{b}_*) \in (\dot{B}^{-2+N/r}_{r,\rho} \cap L^\theta) \times P(\dot{B}^{-1+N/q}_{q,\rho} \cap L^{N\theta/(N-\theta)})^N$$

satisfying

$$\|a_*\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b}_*\|_{\dot{B}^{-1+N/q}_{q,\rho}} < \varepsilon_* \tag{5.11}$$

with sufficiently small $0 < \varepsilon_* < 1$, then the corresponding solution has the additional regularity $(n_*, \boldsymbol{u}_*) \in BC([0,\infty); L^\theta \times P(L^{N\theta/(N-\theta)})^N)$. Here, by the assumption (2.14), we may assume that the initial data $(a, \boldsymbol{b})$ satisfies $\|a\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b}\|_{\dot{B}^{-1+N/q}_{q,\rho}} < \varepsilon_*/2$. In addition, we may take a sequence $\{(a_j, \boldsymbol{b}_j)\}_{j=1}^\infty \subset \mathscr{S}_0 \times (\mathscr{S}_0)^N$ of functions satisfying

$$\lim_{j\to\infty} \left(\|a - a_j\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{b} - \boldsymbol{b}_j\|_{\dot{B}^{-1+N/q}_{q,\rho}} \right) = 0$$

by virtue of Proposition 3.2. Thus we see that $\{(a_j, \boldsymbol{b}_j)\}_{j=j_*}^\infty$ fulfills the corresponding condition (5.11) by taking j_* sufficiently large. On the one hand, Theorem 2.1 (iii) yields

$$\lim_{j\to\infty} \left(\|n - n_j\|_{L^\infty_\infty(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u} - \boldsymbol{u}_j\|_{L^\infty_\infty(\dot{B}^{-1+N/q}_{q,\rho})} \right) = 0,$$

where $\{(n_j,\boldsymbol{u}_j)\}_{j=j_*}^{\infty}$ denotes the corresponding solution with the initial data $\{(a_j,\boldsymbol{b}_j)\}_{j=j_*}^{\infty}$. On the other hand, since $\{(n_j,\boldsymbol{u}_j)\}_{j=j_*}^{\infty}$ satisfies (4.3) and since θ satisfies (5.9), we see by (5.10) that

$$\begin{cases} \|n_j(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} \\ \le \|e^{dt\Delta}a_j\|_{\dot{B}^{-2+N/r}_{r,\rho}} + C_{d,\nu}t^{1-N/(2\theta)}\left(\|n_j\|_{L^\infty_t(L^\theta)} + \|\boldsymbol{u}_j\|_{L^\infty_t(L^{N\theta/(N-\theta)})}\right)[n_j,\boldsymbol{u}_j]_{X^{s,s}_t}, \\ \|\boldsymbol{u}_j(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}} \\ \le \|e^{\nu t\Delta}\boldsymbol{b}_j\|_{\dot{B}^{-1+N/q}_{q,\rho}} + C_{d,\nu}t^{1-N/(2\theta)}\left(\|n_j\|_{L^\infty_t(L^\theta)} + \|\boldsymbol{u}_j\|_{L^\infty_t(L^{N\theta/(N-\theta)})}\right)[n_j,\boldsymbol{u}_j]_{X^{s,s}_t}. \end{cases}$$

Thus we have

$$\lim_{t\to\infty}\|n_j(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} = 0, \quad \lim_{t\to\infty}\|\boldsymbol{u}_j(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}} = 0$$

for each $j_* \le j$ from $1-N/(2\theta) < 0$ and Proposition 3.10. Therefore, we observe that

$$\begin{aligned} &\limsup_{t\to\infty}\left(\|n(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{u}(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) \\ &\le \limsup_{t\to\infty}\left(\|n_j(t)\|_{\dot{B}^{-2+N/r}_{r,\rho}} + \|\boldsymbol{u}_j(t)\|_{\dot{B}^{-1+N/q}_{q,\rho}}\right) + \|n-n_j\|_{L^\infty_\infty(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u}-\boldsymbol{u}_j\|_{L^\infty_\infty(\dot{B}^{-1+N/q}_{q,\rho})} \\ &= \|n-n_j\|_{L^\infty_\infty(\dot{B}^{-2+N/r}_{r,\rho})} + \|\boldsymbol{u}-\boldsymbol{u}_j\|_{L^\infty_\infty(\dot{B}^{-1+N/q}_{q,\rho})}, \end{aligned}$$

which yields (2.16) by letting $j\to\infty$. Finally, we shall verify (2.17). In case $\rho=\infty$, since $(n,\boldsymbol{u})$ satisfies (4.3), we have

$$\begin{cases} \langle n(t),\varphi\rangle = \left\langle e^{dt\Delta}a,\varphi\right\rangle - \langle I_1(n,n),\varphi\rangle - \langle I_1(\boldsymbol{u},n),\varphi\rangle, \\ \langle \boldsymbol{u}(t),\boldsymbol{f}\rangle = \left\langle e^{\nu t\Delta}\boldsymbol{b},\boldsymbol{f}\right\rangle - \langle J_1(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{f}\rangle + \langle J_2(n,n),\boldsymbol{f}\rangle \end{cases}$$

for all $\varphi \in \dot{B}^{2-N/r}_{r/(r-1),1}$ and $\boldsymbol{f} \in (\dot{B}^{1-N/q}_{q/(q-1),1})^N$. Therefore, by combining Proposition 3.10 and Lemma 5.4, we conclude that (2.17) hold. This completes the proof of Theorem 2.5. □

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Appendix A. Regularity properties of the heat semigroup

In this appendix, we give the proof of Proposition 4.11, i.e., the regularity properties of mild solutions of the linear heat equation. We note that the proof of Proposition 4.11 is mainly based on Lunardi [28, Proposition 4.2.1 and Theorem 4.3.4]. Here, let us recall the definition of the γ-Hölder spaces, namely, we define

$$\|\varphi\|_{C^\gamma((\varepsilon,T);X)} := \|\varphi\|_{L^\infty((\varepsilon,T);X)} + \sup_{\varepsilon<t,\tau<T,\, t\neq\tau}\frac{\|\varphi(t)-\varphi(\tau)\|_X}{|t-\tau|^\gamma}$$

for $0<\varepsilon<T<\infty$, $0<\gamma<1$, and a Banach space X.

Proof of Proposition 4.11. (i) Take an arbitrary $0<\varepsilon<T$ and fix it. Then the following decomposition

$$\Phi(t) = e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2) + \int_{\varepsilon/2}^{t} e^{\nu(t-\tau)\Delta}\varphi(\tau)d\tau \tag{A.1}$$

holds for all $\varepsilon < t < T$. Thus we see by Proposition 3.8 that

$$\begin{aligned}\|\Phi(t)\|_{\dot{B}^{s+\beta}_{r,1}} &\le C(\nu(t-\varepsilon/2))^{-\beta/2}\|\Phi(\varepsilon/2)\|_{\dot{B}^{s}_{r,\infty}} + C\int_{\varepsilon/2}^{t}(\nu(t-\tau))^{-(\alpha+\beta)/2}\|\varphi(\tau)\|_{\dot{B}^{s-\alpha}_{r,\infty}}d\tau \\ &\le C(\nu\varepsilon)^{-\beta/2}\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CS_\varepsilon(\nu T)^{-(\alpha+\beta)/2}T\end{aligned}$$

holds for all $0 < \beta < 2-\alpha$ and $\varepsilon < t < T$, where $S_\varepsilon := \|\varphi\|_{L^\infty((\varepsilon/2,T);\dot{B}^{s-\alpha}_{r,\infty})}$. Noting that

$$(e^{\nu h\Delta}-I)e^{\nu(t-\tau)\Delta} = \int_{t-\tau}^{t+h-\tau}\partial_\lambda e^{\nu\lambda\Delta}d\lambda = \nu\int_{t-\tau}^{t+h-\tau}\Delta e^{\nu\lambda\Delta}d\lambda \tag{A.2}$$

holds for all $\varepsilon < t < T$, $0 \le \tau \le t$, and $0 < h < T-t$, we see by (A.1) that

$$\begin{aligned}&\Phi(t+h)-\Phi(t) \\ &= (e^{\nu h\Delta}-I)e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2) + \int_{\varepsilon/2}^{t}(e^{\nu h\Delta}-I)e^{\nu(t-\tau)\Delta}\varphi(\tau)d\tau + \int_{t}^{t+h}e^{\nu(t+h-\tau)\Delta}\varphi(\tau)d\tau \\ &= \nu\int_0^h \Delta e^{\nu\lambda\Delta}e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2)d\lambda + \nu\int_{\varepsilon/2}^{t}\int_{t-\tau}^{t+h-\tau}\Delta e^{\nu\lambda\Delta}\varphi(\tau)d\lambda d\tau + \int_{t}^{t+h}e^{\nu(t+h-\tau)\Delta}\varphi(\tau)d\tau\end{aligned}$$

holds. Hence, it holds by Proposition 3.8 that

$$\begin{aligned}&\|\Phi(t+h)-\Phi(t)\|_{\dot{B}^{s+\beta}_{r,1}} \\ &\le C\nu\int_0^h(\nu\lambda)^{-\beta/2}d\lambda\|\Delta e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2)\|_{\dot{B}^{s}_{r,\infty}} \\ &+ CS_\varepsilon\nu\int_{\varepsilon/2}^{t}\int_{t-\tau}^{t+h-\tau}(\nu\lambda)^{-1-(\alpha+\beta)/2}d\lambda d\tau + CS_\varepsilon\int_{t}^{t+h}(\nu(t+h-\tau))^{-(\alpha+\beta)/2}d\tau \\ &\le C(\nu h)^{1-\beta/2}(\nu(t-\varepsilon/2))^{-1}\|\Phi(\varepsilon/2)\|_{\dot{B}^{s}_{r,\infty}} \\ &+ CS_\varepsilon\nu^{-(\alpha+\beta)/2}\int_{\varepsilon/2}^{t}((t-\tau)^{-(\alpha+\beta)/2}-(t+h-\tau)^{-(\alpha+\beta)/2})d\tau + CS_\varepsilon\nu^{-(\alpha+\beta)/2}h^{1-(\alpha+\beta)/2} \\ &\le C\nu^{-\beta/2}\varepsilon^{-1}h^{1-\beta/2}\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CS_\varepsilon\nu^{-(\alpha+\beta)/2}h^{1-(\alpha+\beta)/2},\end{aligned}$$

which yields

$$\frac{1}{h^{1-(\alpha+\beta)/2}}\|\Phi(t+h)-\Phi(t)\|_{\dot{B}^{s+\beta}_{r,1}} \le C\nu^{-\beta/2}T^{\alpha/2}\varepsilon^{-1}\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CS_\varepsilon\nu^{-(\alpha+\beta)/2}.$$

(ii) Take an arbitrary $0 < \varepsilon < T$ and fix it. Then it holds that

$$\int_{\varepsilon/2}^{t}\Delta e^{\nu(t-\tau)\Delta}d\tau = -\nu^{-1}\int_{\varepsilon/2}^{t}\partial_\tau e^{\nu(t-\tau)\Delta}d\tau = -\nu^{-1}(I-e^{\nu(t-\varepsilon/2)\Delta}) \tag{A.3}$$

for all $\varepsilon < t < T$. Hence, it holds by (A.1) that

$$\Delta\Phi(t) = \Delta e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2) + \int_{\varepsilon/2}^{t}\Delta e^{\nu(t-\tau)\Delta}(\varphi(\tau)-\varphi(t))d\tau - \nu^{-1}(I-e^{\nu(t-\varepsilon/2)\Delta})\varphi(t). \tag{A.4}$$

By the definition of the Hölder spaces, we have

$$\|\varphi(\tau)-\varphi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}} \le K_\varepsilon(t-\tau)^\gamma$$

for all $\varepsilon/2 < \tau < t$, where $K_\varepsilon := \|\varphi\|_{C^\gamma((\varepsilon/2,T);\dot{B}^{s+\eta}_{r,\rho})}$. Thus we see by Proposition 3.8 that

$$
\begin{aligned}
&\|\Delta\Phi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}}\\
&\le C(\nu(t-\varepsilon/2))^{-1-\eta/2}\|\Phi(\varepsilon/2)\|_{\dot{B}^{s}_{r,\infty}} + C\int_{\varepsilon/2}^{t}(\nu(t-\tau))^{-1}\|\varphi(\tau)-\varphi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}}d\tau + 2K_\varepsilon\nu^{-1}\\
&\le C(\nu\varepsilon)^{-1-\eta/2}\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CK_\varepsilon\nu^{-1}(1+T^\gamma),
\end{aligned}
$$

Besides, we see by (A.2), (A.3), and (A.4) that the following decomposition

$$
\begin{aligned}
&\Delta\Phi(t+h)-\Delta\Phi(t)\\
&=\Delta(e^{\nu h\Delta}-I)e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2) + \int_{\varepsilon/2}^{t}\Delta(e^{\nu(t+h-\tau)\Delta}-e^{\nu(t-\tau)\Delta})(\varphi(\tau)-\varphi(t))d\tau\\
&+\int_{\varepsilon/2}^{t}\Delta e^{\nu(t+h-\tau)\Delta}(\varphi(t)-\varphi(t+h))d\tau + \int_{t}^{t+h}\Delta e^{\nu(t+h-\tau)\Delta}(\varphi(\tau)-\varphi(t+h))d\tau\\
&-\nu^{-1}(I-e^{\nu(t+h-\varepsilon/2)\Delta})\varphi(t+h) + \nu^{-1}(I-e^{\nu(t-\varepsilon/2)\Delta})\varphi(t)\\
&=\nu\int_0^h(-\Delta)^2e^{\nu\lambda\Delta}e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2)d\lambda + \nu\int_{\varepsilon/2}^{t}\int_{t-\tau}^{t+h-\tau}(-\Delta)^2e^{\nu\lambda\Delta}(\varphi(\tau)-\varphi(t))d\lambda d\tau\\
&-\nu^{-1}(e^{\nu h\Delta}-e^{\nu(t+h-\varepsilon/2)\Delta})(\varphi(t)-\varphi(t+h)) + \int_t^{t+h}\Delta e^{\nu(t+h-\tau)\Delta}(\varphi(\tau)-\varphi(t+h))d\tau\\
&-\nu^{-1}(I-e^{\nu(t+h-\varepsilon/2)\Delta})(\varphi(t+h)-\varphi(t)) + \int_0^h\Delta e^{\nu\lambda\Delta}e^{\nu(t-\varepsilon/2)\Delta}\varphi(t)d\lambda
\end{aligned}
$$

holds for all $\varepsilon < t < T$ and $0 < h < T-t$. Therefore, noting that the following calcuration

$$
\int_{\varepsilon/2}^{t}(t-\tau)^\gamma\int_{t-\tau}^{t+h-\tau}\lambda^{-2}d\lambda d\tau = \int_{\varepsilon/2}^{t}h(t-\tau)^{\gamma-1}(t+h-\tau)^{-1}d\tau \le h^\gamma\int_0^\infty\eta^{\gamma-1}(\eta+1)^{-1}d\eta \le Ch^\gamma
$$

from a substitution $\eta = (t-\tau)/h$, we see by Proposition 3.8 that

$$
\begin{aligned}
&\|\Delta\Phi(t+h)-\Delta\Phi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}}\\
&\le C\nu\int_0^h(\nu\lambda)^{\gamma-1}d\lambda\|(-\Delta)^{1+\gamma}e^{\nu(t-\varepsilon/2)\Delta}\Phi(\varepsilon/2)\|_{\dot{B}^{s+\eta}_{r,\infty}} + C\nu\int_{\varepsilon/2}^{t}\int_{t-\tau}^{t+h-\tau}(\nu\lambda)^{-2}\|\varphi(\tau)-\varphi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}}d\lambda d\tau\\
&+2\nu^{-1}\|\varphi(t)-\varphi(t+h)\|_{\dot{B}^{s+\eta}_{r,\rho}} + C\int_t^{t+h}(\nu(t+h-\tau))^{-1}\|\varphi(\tau)-\varphi(t+h)\|_{\dot{B}^{s+\eta}_{r,\rho}}d\tau\\
&+2\nu^{-1}\|\varphi(t+h)-\varphi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}} + C\int_0^h(\nu\lambda)^{\gamma-1}d\lambda\|(-\Delta)^\gamma e^{\nu(t-\varepsilon/2)\Delta}\varphi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}}\\
&\le C(\nu h)^\gamma(\nu(t-\varepsilon/2))^{-1-\gamma-\eta/2}\|\Phi(\varepsilon/2)\|_{\dot{B}^{s}_{r,\infty}} + CK_\varepsilon\nu^{-1}\int_{\varepsilon/2}^{t}(t-\tau)^\gamma\int_{t-\tau}^{t+h-\tau}\lambda^{-2}d\lambda d\tau\\
&+2K_\varepsilon\nu^{-1}h^\gamma + CK_\varepsilon\nu^{-1}\int_t^{t+h}(t+h-\tau)^{\gamma-1}d\tau + 2K_\varepsilon\nu^{-1}h^\gamma + C\nu^{\gamma-1}h^\gamma K_\varepsilon(\nu(t-\varepsilon/2))^{-\gamma}\\
&\le C(\nu\varepsilon)^{-1-\eta/2}\varepsilon^{-\gamma}h^\gamma\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CK_\varepsilon\nu^{-1}(1+\varepsilon^{-\gamma})h^\gamma,
\end{aligned}
$$

which yields

$$
\frac{1}{h^\gamma}\|\Delta\Phi(t+h)-\Delta\Phi(t)\|_{\dot{B}^{s+\eta}_{r,\rho}} \le C(\nu\varepsilon)^{-1-\eta/2}\varepsilon^{-\gamma}\|\Phi\|_{L^\infty_T(\dot{B}^{s}_{r,\infty})} + CK_\varepsilon\nu^{-1}(1+\varepsilon^{-\gamma}).
$$

Thus we see that $\Delta\Phi \in C^{\gamma}_{\mathrm{loc}}((0,T);\dot{B}^{s+\eta}_{r,\rho})$ holds with the estimate

$$\|\nu\Delta\Phi\|_{C^{\gamma}((\varepsilon,T);\dot{B}^{s+\eta}_{r,\rho})} \le C\varepsilon^{-1}(\nu\varepsilon)^{-\eta/2}(1+\varepsilon^{-\gamma})\|\Phi\|_{L^{\infty}_{T}(\dot{B}^{s}_{r,\infty})} + CK_{\varepsilon}(1+T^{\gamma}+\varepsilon^{-\gamma}).$$

Hence, we also have $\partial_t\Phi = \nu\Delta\Phi + \varphi \in C^{\gamma}_{\mathrm{loc}}((0,T);\dot{B}^{s+\eta}_{r,\rho})$. This completes the proof of Proposition 4.11. □

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011.

[2] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. **25** (2015), no. 9, 1663–1763.

[3] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), no. 2, 715–743.

[4] A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations (2006), No. 44, 32.

[5] R. E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, [Cham], 2016.

[6] M. Chae, K. Kang, and J. Lee, *Global existence and temporal decay in Keller-Segel models coupled to fluid equations*, Comm. Partial Differential Equations **39** (2014), no. 7, 1205–1235.

[7] ———, *Asymptotic behaviors of solutions for an aerotaxis model coupled to fluid equations*, J. Korean Math. Soc. **53** (2016), no. 1, 127–146.

[8] ———, *Global well-posedness and long time behaviors of chemotaxis-fluid system modeling coral fertilization*, Discrete Contin. Dyn. Syst. **40** (2020), no. 4, 2135–2163.

[9] J. I. Diaz, T. Nagai, and J.-M. Rakotoson, *Symmetrization techniques on unbounded domains: application to a chemotaxis system on* $\mathbf{R}^N$, J. Differential Equations **145** (1998), no. 1, 156–183.

[10] G. Diebou Yomgne, *Well-posedness for chemotaxis-fluid models in arbitrary dimensions*, arXiv:2111.04792.

[11] R. Duan, A. Lorz, and P. Markowich, *Global solutions to the coupled chemotaxis-fluid equations*, Comm. Partial Differential Equations **35** (2010), no. 9, 1635–1673.

[12] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[13] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. **195** (1998), 77–114.

[14] Y. Gong and S. He, *On the* 8π*-critical-mass threshold of a Patlak-Keller-Segel-Navier-Stokes system*, SIAM J. Math. Anal. **53** (2021), no. 3, 2925–2956.

[15] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), no. 3, 103–165.

[16] S. Ishida, K. Seki, and T. Yokota, *Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains*, J. Differential Equations **256** (2014), no. 8, 2993–3010.

[17] K. Kaneko, H. Kozono, and S. Shimizu, *Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity*, Indiana Univ. Math. J. **68** (2019), no. 3, 857–880.

[18] K. Kang, J. Lee, and M. Winkler, *Global weak solutions to a chemotaxis-Navier-Stokes system in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. **42** (2022), no. 11, 5201–.

[19] T. Kato, *Strong* L^p*-solutions of the Navier-Stokes equation in* $\mathbf{R}^m$*, with applications to weak solutions*, Math. Z. **187** (1984), no. 4, 471–480.

[20] T. Kato and H. Fujita, *On the nonstationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[21] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. **26** (1970), no. 3, 399–415.

[22] H. Kozono, M. Miura, and Y. Sugiyama, *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid*, J. Funct. Anal. **270** (2016), no. 5, 1663–1683.

[23] H. Kozono, T. Ogawa, and Y. Taniuchi, *Navier-Stokes equations in the Besov space near* L^∞ *and BMO*, Kyushu J. Math. **57** (2003), no. 2, 303–324.

[24] H. Kozono, A. Okada, and S. Shimizu, *Characterization of initial data in the homogeneous Besov space for solutions in the Serrin class of the Navier-Stokes equations*, J. Funct. Anal. **278** (2020), no. 5, 108390, 47.

[25] H. Kozono and S. Shimizu, *Navier-Stokes equations with external forces in time-weighted Besov spaces*, Math. Nachr. **291** (2018), no. 11-12, 1781–1800.

[26] ———, *Strong solutions of the Navier-Stokes equations based on the maximal Lorentz regularity theorem in Besov spaces*, J. Funct. Anal. **276** (2019), no. 3, 896–931.

[27] H. Kozono and Y. Sugiyama, *The Keller-Segel system of parabolic-parabolic type with initial data in weak $L^{n/2}(\mathbb{R}^n)$ and its application to self-similar solutions*, Indiana Univ. Math. J. **57** (2008), no. 4, 1467–1500.

[28] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 16, Birkhäuser Verlag, Basel, 1995.

[29] T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac. **40** (1997), no. 3, 411–433.

[30] K. Osaki and A. Yagi, *Finite dimensional attractor for one-dimensional Keller-Segel equations*, Funkcial. Ekvac. **44** (2001), no. 3, 441–469.

[31] J. Prüss and G. Simonett, *Moving interfaces and quasilinear parabolic evolution equations*, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, [Cham], 2016.

[32] Y. Sawano, *Homogeneous Besov spaces*, Kyoto J. Math. **60** (2020), no. 1, 1–43.

[33] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[34] T. Takeuchi, *The Keller-Segel system of parabolic-parabolic type in homogeneous Besov spaces framework*, J. Differential Equations **298** (2021), 609–640.

[35] ———, *Space-time analytic smoothing effect of the heat semigroup defined on homogeneous Besov spaces*, Partial Differential Equations in Applied Mathematics **4** (2021), 100174.

[36] Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations **252** (2012), no. 1, 692–715.

[37] ———, *Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system*, Z. Angew. Math. Phys. **67** (2016), no. 6, Art. 138, 23.

[38] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam-New York, 1978.

[39] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations **248** (2010), no. 12, 2889–2905.

[40] ———, *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops*, Comm. Partial Differential Equations **37** (2012), no. 2, 319–351.

[41] ———, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl. (9) **100** (2013), no. 5, 748–767.

[42] ———, *Stabilization in a two-dimensional chemotaxis-Navier-Stokes system*, Arch. Ration. Mech. Anal. **211** (2014), no. 2, 455–487.

[43] ———, *Global weak solutions in a three-dimensional chemotaxis–Navier-Stokes system*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **33** (2016), no. 5, 1329–1352.

[44] ———, *How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3067–3125.

[45] ———, *Global mass-preserving solutions in a two-dimensional chemotaxis-Stokes system with rotational flux components*, J. Evol. Equ. **18** (2018), no. 3, 1267–1289.

[46] ———, *A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: global weak solutions and asymptotic stabilization*, J. Funct. Anal. **276** (2019), no. 5, 1339–1401.

[47] ———, *Can rotational fluxes impede the tendency toward spatial homogeneity in nutrient Taxis(-Stokes) systems?*, Int. Math. Res. Not. IMRN **11** (2021), 8106–8152.

[48] ———, *Reaction-driven relaxation in three-dimensional Keller-Segel-Navier-Stokes interaction*, Comm. Math. Phys. **389** (2022), no. 1, 439–489.

[49] C. Xue and H. G. Othmer, *Multiscale models of taxis-driven patterning in bacterial populations*, SIAM J. Appl. Math. **70** (2009), no. 1, 133–167.

[50] M. Yang, Z. Fu, and J. Sun, *Existence and large time behavior to coupled chemotaxis-fluid equations in Besov-Morrey spaces*, J. Differential Equations **266** (2019), no. 9, 5867–5894.

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