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Space–time analytic smoothing effect of the heat semigroup defined on homogeneous Besov spaces

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ABSTRACT

We refine the decay estimate of the heat semigroup $\{T(t)\}_{t \geq 0}$ defined on homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $p, q \in [1, \infty]$, which is obtained by Kozono et al. (2003). In particular, we give an explicit representation of a constant appeared in the decay estimate of $\{T(t)\}_{t \geq 0}$, which provides a space–time analytic smoothing effect of $\{T(t)\}_{t \geq 0}$. As a by-product, we obtain a radius of convergence of the Taylor expansion exactly. Furthermore, it is also showed that $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$, $p, q \in [1, \infty)$, where $\{T(t)\}_{t \geq 0}$ can be extended as an analytic function of t on the sector $\{t \in \mathbb{C} \setminus \{0\} \mid |\arg t| < \theta\}$ with an explicitly given constant θ .

1. Introduction

Let us consider the family of the operator $\{T(t)\}_{t \geq 0}$ in \mathbb{R}^n ($n \geq 1$) defined by $T(t) := \mathcal{F}^{-1} e^{-t|\xi|^2} \mathcal{F}$ for each $t \geq 0$, where \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and inverse Fourier transform, i.e.,

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(\xi) d\xi \quad (1.1)$$

for all f belonging to the Schwartz space \mathcal{S} in \mathbb{R}^n , respectively. Recalling the heat kernel G_t defined by

$$G_t(x) := \mathcal{F}^{-1}[e^{-t|\xi|^2}](x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \quad (1.2)$$

for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$, we see that $T(t) = G_t * \cdot$ holds for all $t > 0$. The family of such an operator is often denoted by $\{T(t)\}_{t \geq 0} = \{e^{t\Delta}\}_{t \geq 0}$ and called the heat semigroup.

In this article, we prove a *space–time analytic smoothing effect* of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and $1 \leq q \leq \infty$, together with a radius of convergence of the Taylor expansion. The homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ will be introduced in Definition 2.1. We also show that the heat semigroup $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ for $1 \leq p < \infty$, $s \in \mathbb{R}$ and $1 \leq q < \infty$, i.e., $T(t)$ can be extended to an $\mathcal{L}(\dot{B}_{p,q}^s(\mathbb{R}^n))$ -valued analytic function in the sector $\{t \in \mathbb{C} \setminus \{0\} \mid |\arg t| < \theta\}$ with an explicitly given constant θ . In addition, we reveal that even if the case of $q = \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on some closed subspace of $\dot{B}_{p,\infty}^s(\mathbb{R}^n)$.

For the proof of the space–time analytic smoothing effect of $\{T(t)\}_{t \geq 0}$, we shall show a *decay estimate* of the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$. Namely, we refine the inequalities given in Bergh-

Löfström¹, Chapter 6 and Kozono–Ogawa–Taniuchi², Lemma 2.2 and we obtain an increasing rate of the constant depending on the space–time derivative. Here, the derivations of those inequalities are mainly based on *Fourier multiplier theory*. Specifically, in order to improve the basic estimates on the homogeneous Besov spaces, we precisely analyze the estimates of the multipliers $|\xi|^\sigma$ and $(i\xi)^\alpha$ which are symbols of the fractional Laplacian $(-\Delta)^{\frac{\sigma}{2}}$ and differential operator ∂_x^α . Since the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ are continuously embedded in the space $L^\infty(\mathbb{R}^n)$ of bounded functions, we shall show a space–time analytic smoothing effect of the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$, by estimating the constant appeared in the decay estimate.

The paper is organized as follows: The next section introduces notations and function spaces used throughout this paper. In Section 3, we state our main results. More precisely, we first give a decay estimate of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$. Secondly, we state that $\{T(t)\}_{t \geq 0}$ defined on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ has a space–time analytic smoothing effect, which implies that $T(t)f$ becomes real analytic jointly in space and time even though f is singular data such as the Dirac measure δ that are unable to handle the standard Lebesgue space $L^p(\mathbb{R}^n)$ framework. Thirdly, we state that $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on $\dot{B}_{p,q}^s(\mathbb{R}^n)$. Fourthly, several properties of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ are given, which can be obtained by the above results. From Section 4, we show the several estimates on the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$, which is needed in the proof of our main results. Section 4 proves the estimates of the fractional Laplacian $(-\Delta)^{\frac{\sigma}{2}}$ and differential operator ∂_x^α on $\dot{B}_{p,q}^s(\mathbb{R}^n)$, where these estimates are based on a precise analysis of the multipliers $|\xi|^\sigma$ and $(i\xi)^\alpha$. In Section 5, we refine the decay estimate of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ given in Kozono–Ogawa–Taniuchi², Lemma 2.2 by combining the analysis of the

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multipliers and the estimate of the heat kernel G_t . Section 6 shows our main results. By applying the decay estimate of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ with the aid of higher derivative estimate of functions themselves on $\dot{B}_{p,q}^s(\mathbb{R}^n)$, we establish a precise decay estimate of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$, which provides a space–time analytic smoothing effect of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$. To prove the space–time analytic smoothing effect, the embedding $\dot{B}_{p,1}^{n/p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ and the Stirling formula play an important role. As a by-product, we obtain that $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on $\dot{B}_{p,q}^s(\mathbb{R}^n)$. Finally, in Section 7, we give the proof of Corollaries 3.9, 3.10, and 3.11 which are an immediate consequence of our theorems. In Appendix, we prove that $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is a bounded C_0 -semigroup which has differentiability on $t > 0$.

2. Preliminaries

In this section, we define notations and function spaces. The natural numbers are denoted by \mathbb{N} and the real numbers by \mathbb{R} . Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we set

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \dots + \alpha_n.$$

A binomial coefficient is written as $\binom{\alpha}{\beta}$. Let B_r be the open ball in \mathbb{R}^n centered at origin and radius $r > 0$. For Banach spaces X and Y , let $\mathcal{L}(X, Y)$ denote the Banach space of all bounded linear operators from X to Y . If $X = Y$, we write $\mathcal{L}(X) := \mathcal{L}(X, X)$. For an open set Ω , let $BC(\Omega)$ and $BUC(\Omega)$ denote the Banach spaces of all bounded continuous functions and bounded uniformly continuous functions on Ω , respectively. Define $BC^k(\Omega) := \{f \in C^k(\Omega) \mid \partial_x^\alpha f \in BC(\Omega), |\alpha| \leq k\}$ and $BUC^k(\Omega) := \{f \in C^k(\Omega) \mid \partial_x^\alpha f \in BUC(\Omega), |\alpha| \leq k\}$ for $k \in \mathbb{N}$. We write $C^\omega(\Omega)$ as the set of all analytic functions on Ω . Define $BUC^\infty(\Omega) := \bigcap_{k \in \mathbb{N}_0} BUC^k(\Omega)$ and $BUC^\omega(\Omega) := C^\omega(\Omega) \cap BUC^\infty(\Omega)$. For the case of X -valued function spaces, we write $BC^k(\Omega; X)$ and $BUC^k(\Omega; X)$. For $m \in \mathbb{N}$, we define the Sobolev spaces $H^m(\mathbb{R}^n)$ and $\dot{H}^m(\mathbb{R}^n)$ by

$$H^m(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' \mid \|f\|_{H^m(\mathbb{R}^n)} := \sum_{|\alpha| \leq m} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} < \infty \right\},$$

$$\dot{H}^m(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'/\mathcal{P} \mid \|f\|_{\dot{H}^m(\mathbb{R}^n)} := \|(-\Delta)^{\frac{m}{2}} f\|_{L^2(\mathbb{R}^n)} < \infty \right\},$$

where \mathcal{S}' and \mathcal{P} are the sets of all tempered distributions and polynomials on \mathbb{R}^n , respectively. Notice that the following relation

$$C_m^{-1} \|f\|_{\dot{H}^m(\mathbb{R}^n)} \leq \sum_{|\alpha|=m} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)} \leq C_m \|f\|_{H^m(\mathbb{R}^n)}$$

holds for all $f \in \dot{H}^m(\mathbb{R}^n)$, where $C_m > 0$ is a constant independent of f . In particular, for every $f \in H^m(\mathbb{R}^n)$, it holds that $C_m^{-1} \|f\|_{H^m(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} + \|f\|_{\dot{H}^m(\mathbb{R}^n)} \leq C_m \|f\|_{H^m(\mathbb{R}^n)}$.

It is well known that the Fourier transform \mathcal{F} and inverse Fourier transform \mathcal{F}^{-1} defined by (1.1) can be extended on \mathcal{S}' . We see that the relations

$$\mathcal{F}^{-1}[f\mathcal{F}g] = (\mathcal{F}^{-1}f) * g, \quad \mathcal{F}^{-1}[fg] = (\mathcal{F}^{-1}f) * \mathcal{F}^{-1}g$$

hold for all $f, g \in \mathcal{S}$. In addition, the Parseval identity $\|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)}$ holds for all $f \in L^2(\mathbb{R}^n)$. Since the elementary relations

$$\mathcal{F}[\partial_x^\alpha f] = (i\xi)^\alpha \mathcal{F}[f], \quad \mathcal{F}[(-\Delta)^N f] = |\xi|^{2N} \mathcal{F}[f]$$

hold for all $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}$, we can define the fractional Laplacian as a Fourier multiplier operator $(-\Delta)^{\frac{\sigma}{2}} := \mathcal{F}^{-1}|\xi|^\sigma \mathcal{F}$ for $\sigma \in \mathbb{R}$.

For $1 \leq p \leq \infty$, let us introduce the Banach spaces $M_p(\mathbb{R}^n)$ defined by

$$M_p(\mathbb{R}^n) := \left\{ \rho \in \mathcal{S}' \mid \|\rho\|_{M_p(\mathbb{R}^n)} := \sup_{f \in \mathcal{S} \setminus \{0\}} \frac{\|\mathcal{F}^{-1}[\rho\mathcal{F}f]\|_{L^p(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} < \infty \right\}. \tag{2.1}$$

It is known that the space $M_p(\mathbb{R}^n)$ has a scale invariance property. Namely, for every $\rho \in M_p(\mathbb{R}^n)$, the function $\rho_\lambda(\xi) := \rho(\lambda\xi)$ satisfies

$\|\rho_\lambda\|_{M_p(\mathbb{R}^n)} = \|\rho\|_{M_p(\mathbb{R}^n)}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. In addition, since $\overline{\mathcal{S}} = L^p(\mathbb{R}^n)$ holds for all $1 \leq p < \infty$, the Fourier multiplier operator $\mathcal{F}^{-1}\rho\mathcal{F}$ can be extended to the bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and we obtain $\|\rho\|_{M_p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}\rho\mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R}^n))}$. For the case of $p = \infty$, the above statement does not hold due to $\overline{\mathcal{S}} \neq L^\infty(\mathbb{R}^n)$. However, even if the case of $p = \infty$, the Hahn–Banach theorem gives that the Fourier multiplier operator $\mathcal{F}^{-1}\rho\mathcal{F}$ can be extended and we have $\|\rho\|_{M_\infty(\mathbb{R}^n)} = \|\mathcal{F}^{-1}\rho\mathcal{F}\|_{\mathcal{L}(L^\infty(\mathbb{R}^n))}$. For details, see Bergh–Löfström, Definition 6.1.1, Theorem 6.1.3. Therefore, since it holds by the Hausdorff–Young inequality that

$$\|\mathcal{F}^{-1}[\rho\mathcal{F}f]\|_{L^p(\mathbb{R}^n)} = \|(\mathcal{F}^{-1}\rho) * f\|_{L^p(\mathbb{R}^n)} \leq \|\mathcal{F}^{-1}\rho\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

we see that

$$\|\rho_\lambda\|_{M_p(\mathbb{R}^n)} = \|\rho\|_{M_p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}\rho\mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq \|\mathcal{F}^{-1}\rho\|_{L^1(\mathbb{R}^n)} \tag{2.2}$$

holds for all $1 \leq p \leq \infty$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ as follows.

Definition 2.1. Let us take a function $\varphi \in \mathcal{S}$ satisfying

$$\text{supp } \varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \leq |\xi| \leq 2\}, \quad \varphi(\xi) > 0 \quad \text{for } 1/2 < |\xi| < 2, \tag{2.3}$$

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

We also define the sequence $\{\varphi_j\}_{j \in \mathbb{Z}}$ of function by

$$\varphi_j := \mathcal{F}^{-1}[\varphi(2^{-j}\xi)] \tag{2.4}$$

for $j \in \mathbb{Z}$. Then, for $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and $1 \leq q \leq \infty$, we define the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ by

$$\dot{B}_{p,q}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'/\mathcal{P} \mid \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} := \begin{cases} \left\{ \sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)})^q \right\}^{\frac{1}{q}} & 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} \{2^{sj} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\} & q = \infty. \end{cases}$$

In case $q = \infty$, we introduce the closed subspace $\dot{B}_{p,\infty+}^s(\mathbb{R}^n)$ of $\dot{B}_{p,\infty}^s(\mathbb{R}^n)$ defined by

$$\dot{B}_{p,\infty+}^s(\mathbb{R}^n) := \left\{ f \in \dot{B}_{p,\infty}^s(\mathbb{R}^n) \mid \lim_{j \rightarrow \pm\infty} 2^{sj} \|\varphi_j * f\|_{L^p(\mathbb{R}^n)} = 0 \right\}. \tag{2.5}$$

We easily see that the sequence $\{\varphi_j\}_{j \in \mathbb{Z}}$ of function defined by (2.4) satisfies

$$\varphi_j * \varphi_k = 0 \quad (|j - k| \geq 2) \tag{2.6}$$

and the elementary embedding on the homogeneous Besov spaces

$$\dot{B}_{p,q_0}^s(\mathbb{R}^n) \subset \dot{B}_{p,q_1}^s(\mathbb{R}^n) \subset \dot{B}_{p,\infty+}^s(\mathbb{R}^n) \subset \dot{B}_{p,\infty}^s(\mathbb{R}^n) \quad (1 \leq q_0 \leq q_1 < \infty)$$

holds. Besides, we see that the embedding

$$\dot{B}_{p,1}^{n/p}(\mathbb{R}^n) \subset \dot{B}_{\infty,1}^0(\mathbb{R}^n) \subset BUC(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \quad (1 \leq p \leq \infty)$$

holds by Bahouri–Chemin–Danchin, Proposition 2.20 and Sawano, Theorem 3.21, Remark 3.22. Notice that we need to consider the space modulo polynomials \mathcal{P} in the definition of the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ in order to make it the Banach space. However, in case $s < n/p$ or $s = n/p$ with $q = 1$, we can obtain the Banach space without considering the space modulo polynomials \mathcal{P} by modification the definition of homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$. For details, see Sawano, Theorem 2.33, Definition 2.18. Then, we will write such Banach spaces as $\dot{B}_{p,q}^s(\mathbb{R}^n)$. In the following, for a space $X(\mathbb{R}^n)$ defined on \mathbb{R}^n , we will abbreviate $X := X(\mathbb{R}^n)$.

3. Main results

Here we state our first result on a decay estimate of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$.

Theorem 3.1. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}_{p,\infty}^s$, it holds that $\partial_x^\alpha \partial_t^j T(t)f \in \dot{B}_{p,1}^{s+\sigma}$ for all $\alpha \in \mathbb{N}_0^n, j \in \mathbb{N}_0, \sigma > 0$ and $t > 0$ with the estimate*

$$\|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} \leq K t^{-\frac{1}{2}(\sigma+|\alpha|+2j)} \|f\|_{\dot{B}_{p,\infty}^s},$$

where $K > 0$ is a constant defined by

$$K := C_{n,\sigma} \left(\frac{2^{12}n}{\pi} \right)^{\frac{|\alpha|}{2}+j} |\alpha|^{\frac{|\alpha|}{2}} j^j (|\alpha|+1)^n (j+1)^n$$

with $C_{n,\sigma} > 0$ independent of p, s, α, j, t and f . Here $0^0 := 1$.

Remark 3.2. Kozono–Ogawa–Taniuchi², Lemma 2.2 already showed the decay estimate of the heat semigroup defined on homogeneous Besov spaces. However, we further consider the space–time derivative and reveal an increasing rate of the constant depending on the space–time derivative in the inequality. Note that by setting $|\alpha| = j = 0$, our result coincides the result given by Ref. 2, Lemma 2.2.

By Theorem 3.1, we may show a space–time analytic smoothing effect of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$. Our second result reads as follows.

Theorem 3.3. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}_{p,\infty}^s$, it holds that $u := T(t)f \in C^\omega((0, \infty) \times \mathbb{R}^n)$. More precisely, for every $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, it holds that*

$$u(t, x) = \sum_{j,k=0}^{\infty} \sum_{|\alpha|=k} \frac{\partial_x^\alpha \partial_t^j u(t_0, x_0)}{\alpha! j!} (t-t_0)^j (x-x_0)^\alpha$$

for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$ satisfying $|t-t_0| < \pi t_0 / (2^{12}en)$.

Remark 3.4. (1) Theorem 3.3 states that the solution $u(t, x)$ of the heat equation with the initial data $f \in \dot{B}_{p,\infty}^s$ is necessarily analytic in space–time even though s is arbitrarily chosen in $(-\infty, \infty)$.

(2) By Theorem 3.3, we see that the solution $u(t, x)$ of the heat equation with the initial data $f \in \dot{B}_{p,\infty}^s$ is real analytic in space–time with radius of convergence for time variable at least $\pi t_0 / (2^{12}en)$ at the point $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$. In addition, a radius of convergence for spatial variables is infinity, which implies that it can be represented by the same Taylor expansion for every $x \in \mathbb{R}^n$ under the condition $t > 0$. The reason is due to the difference of the increasing rates $|\alpha|^{\frac{|\alpha|}{2}}$ and j^j , in Theorem 3.1. We also see by the Stirling formula that

$$\lim_{j \rightarrow \infty} \left\{ \frac{1}{j!} t^{-j} \left(\frac{2^{12}n}{\pi} \right)^j j^j (j+1)^n \right\}^{\frac{1}{j}} = \frac{2^{12}en}{\pi t}.$$

(3) As is mentioned before, in general, it is necessary to consider the space modulo polynomials in the definition of the homogeneous Besov spaces. However, it does not cause any troublesome since any polynomial is real analytic with respect to spatial variables.

Remark 3.5. We should refer^{6–8} for recent contributions to the space or time analyticity of the solutions of the heat equation. Furthermore, there are several results on the space or time analyticity of solutions of the Navier–Stokes system,^{9–11} Schrödinger equations,^{12,13} Burgers equation¹⁴ and primitive equations.¹⁵

As a by-product of Theorem 3.1, we see that the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$ is a bounded analytic C_0 -semigroup. Our third result reads as follows.

Theorem 3.6. *Let $1 \leq p \leq \infty, s \in \mathbb{R}$ and $1 \leq q \leq \infty$.*

(i) *In case $1 \leq p \leq \infty$ and $1 \leq q < \infty$. $T(t)$ can be extended to an $\mathcal{L}(\dot{B}_{p,q}^s)$ -valued analytic function in the sector Σ on the complex plane \mathbb{C} defined by*

$$\Sigma := \left\{ t \in \mathbb{C} \setminus \{0\} \mid |\arg t| < \sin^{-1} \left(\frac{\pi}{2^{12}en} \right) \right\},$$

i.e., $T(t) \in C^\omega(\Sigma; \mathcal{L}(\dot{B}_{p,q}^s))$. More precisely, for every $t_0 \in \Sigma$, it holds that

$$T(t) = \sum_{j=0}^{\infty} \frac{\partial_t^j T(t_0)}{j!} (t-t_0)^j \quad \text{in } \mathcal{L}(\dot{B}_{p,q}^s) \tag{3.1}$$

for all $t \in \mathbb{C} \setminus \{0\}$ satisfying

$$|t-t_0| < |t_0| \sin \left(\sin^{-1} \left(\frac{\pi}{2^{12}en} \right) - |\arg t_0| \right).$$

(i') *In case $1 \leq p \leq \infty$ with $q = \infty$. (3.1) holds with $\dot{B}_{p,\infty}^s$ replaced by $\dot{B}_{p,\infty}^{s+}$.*

(ii) *In case $1 \leq p < \infty$ and $1 \leq q < \infty$. The family of the operator $\{T(t)\}_{t \geq 0}$ is a bounded analytic C_0 -semigroup on $\dot{B}_{p,q}^s$. Namely, for an arbitrary small $\delta > 0$, it holds that*

$$\lim_{|h| \rightarrow 0, h \in \Sigma_\delta} \|T(h)f - f\|_{\dot{B}_{p,q}^s} = 0 \tag{3.2}$$

for all $f \in \dot{B}_{p,q}^s$, where

$$\Sigma_\delta := \left\{ t \in \mathbb{C} \setminus \{0\} \mid |\arg t| \leq \sin^{-1} \left(\frac{\pi}{2^{12}en} \right) - \delta \right\}.$$

(ii') *In case $1 \leq p < \infty$ with $q = \infty$. (3.2) holds with $\dot{B}_{p,\infty}^s$ replaced by $\dot{B}_{p,\infty}^{s+}$.*

(iii) *In case $1 \leq p \leq \infty$. It holds that $T(t) \in C^\omega((0, \infty); \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^{s+\sigma}))$ for all $\sigma > 0$. More precisely, for every $t_0 > 0$, it holds that*

$$T(t) = \sum_{j=0}^{\infty} \frac{\partial_t^j T(t_0)}{j!} (t-t_0)^j \quad \text{in } \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^{s+\sigma})$$

for all $t > 0$ satisfying $|t-t_0| < \pi t_0 / (2^{12}en)$.

Remark 3.7. (1) Theorem 3.6 states that the Taylor expansion of $T(t)$ like (3.1) is valid as an $\mathcal{L}(\dot{B}_{p,q}^s)$ -valued function for all $1 \leq p \leq \infty$ provided $1 \leq q < \infty$. However, in case $q = \infty$, neither C_0 -property nor analyticity is valid since the domain $D(-\Delta)$ is not dense in $\dot{B}_{p,\infty}^s$. To get around such difficulty, we may introduce the closed subspace $\dot{B}_{p,\infty}^{s+}$ as in (2.5), which yields both C_0 -property and analyticity of $T(t)$ as an $\mathcal{L}(\dot{B}_{p,\infty}^{s+})$ -valued function. On the other hand, since $(\dot{B}_{p,1}^{s+\sigma})' = \dot{B}_{p,\infty}^s$ holds for all $1 < p \leq \infty$ with $1/p + 1/p' = 1$, we see that $T(t)f$ is weakly-star continuous at $t = 0$ for all $f \in \dot{B}_{p,\infty}^s$, i.e.,

$$\lim_{h \rightarrow +0} \langle T(h)f - f, g \rangle = 0 \quad \text{for all } g \in \dot{B}_{p',1}^{-s}.$$

Furthermore, we may regard $T(t)$ as an analytic $\mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^{s+\sigma})$ -valued function for all $\sigma > 0$. This is due to the smoothing effect of $T(t)$.

(2) It is seen from general theory of semigroups that the heat semigroup on homogeneous Besov spaces is a bounded analytic C_0 -semigroup since it satisfies

$$\|\partial_t T(t)\|_{\mathcal{L}(\dot{B}_{p,q}^s)} = \|\Delta T(t)\|_{\mathcal{L}(\dot{B}_{p,q}^s)} = O(1/t).$$

For details, see Engel–Nagel.¹⁶, Theorem 4.6 Compared with this argument, in Theorem 3.6, we further reveal a sector Σ on which the heat semigroup is extended analytically. We may establish the same estimate of the $\mathcal{L}(\dot{B}_{p,q}^s)$ -valued function $T(t)$ in the similar manner as in Theorem 3.3, which implies that a radius of convergence is still at least $\pi t_0 / (2^{12}en)$ at the time $t_0 > 0$. Since this radius depends on time $t_0 > 0$, we may see that $T(t)$ can be extended analytically on the sector Σ by connecting the disk of convergence at each time $t_0 > 0$.

Remark 3.8. There are some results which show that the heat semigroup on non-homogeneous Besov spaces is an analytic semigroup. Bae¹⁷ showed that the Stokes operator $-P\Delta$ defined on non-homogeneous Besov spaces $B_{p,q}^s(\mathbb{R}_+^n)$ in the half space generates an analytic semigroup. In addition, Taniuchi¹⁸ treated in case $p = \infty$ under some additional conditions.

Finally, we state properties of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$, which are an immediate consequence of the above theorems.

Corollary 3.9. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. Then, it holds that $\partial_x^\alpha \partial_t^j T(t) \in \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s)$ for all $\alpha \in \mathbb{N}_0^n, j \in \mathbb{N}_0$ and $t > 0$ with $|\alpha| + j \geq 1$ having the estimate*

$$\|\partial_x^\alpha \partial_t^j T(t)\|_{\mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s)} = O\left(t^{-\frac{|\alpha|}{2}-j}\right) \quad \text{as } t \rightarrow +0 \text{ or } t \rightarrow \infty.$$

In addition, $\partial_x^\alpha \partial_t^j T(t) \in BUC((\varepsilon, \infty); \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s))$ holds for all $\varepsilon > 0$.

Corollary 3.10. *Let $1 \leq p < \infty, s \in \mathbb{R}$ and $1 \leq q \leq \infty$.*

(i) *In case $1 \leq p < \infty$ and $1 \leq q < \infty$. For every $f \in \dot{B}_{p,q}^s$, the function $u := T(t)f$ satisfies*

$$u \in BUC([0, \infty); \dot{B}_{p,q}^s) \cap BUC^\omega((\varepsilon, \infty); \dot{B}_{p,q}^s \cap \dot{B}_{p,1}^{s+\sigma}) \cap C^\omega((0, \infty) \times \mathbb{R}^n)$$

for all $\varepsilon, \sigma > 0$. In particular, u satisfies the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) = f & \text{in } \mathbb{R}^n \end{cases}$$

in the following sense:

$$\partial_t u(t, x) = \Delta u(t, x) \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad \lim_{t \rightarrow +0} \|u(t) - f\|_{\dot{B}_{p,q}^s} = 0.$$

(ii) *In case $1 \leq p < \infty$ with $q = \infty$. All of the above results hold with $\dot{B}_{p,\infty}^s$ replaced by $\dot{B}_{p,\infty}^{s+}$.*

Corollary 3.11. *Let $1 \leq p \leq \infty, s \in \mathbb{R}$ and $1 \leq q \leq \infty$. Assume that $s < n/p$ or $s = n/p$ with $q = 1$. Then, it holds that $\partial_x^\alpha \partial_t^j T(t) \in \mathcal{L}(\dot{B}_{p,q}^s, L^\infty)$ for all $\alpha \in \mathbb{N}_0^n, j \in \mathbb{N}_0$ and $t > 0$ with the estimate*

$$\|\partial_x^\alpha \partial_t^j T(t)\|_{\mathcal{L}(\dot{B}_{p,q}^s, L^\infty)} = O\left(t^{-\frac{1}{2}\left(\frac{n}{p} + |\alpha| + 2j - s\right)}\right) \quad \text{as } t \rightarrow +0 \text{ or } t \rightarrow \infty.$$

In particular, for every $f \in \dot{B}_{p,q}^s$, it holds that

$$\sup_{0 < t < \infty} t^{\frac{1}{2}\left(\frac{n}{p} + |\alpha| + 2j - s\right)} \|\partial_x^\alpha \partial_t^j T(t)f\|_{L^\infty} < \infty$$

and $T(t)f \in BUC^\omega((\varepsilon, \infty); BUC^\infty)$ holds for all $\varepsilon > 0$. Moreover, if $1 \leq p < \infty$, it holds that

$$\lim_{|x| \rightarrow \infty} |\partial_x^\alpha \partial_t^j T(t)f(x)| = 0.$$

4. Estimates of differential operators on the homogeneous Besov spaces

In this section, we will show the estimates of the fractional Laplacian $(-\Delta)^{\frac{\sigma}{2}}$ and differential operator ∂_x^α on the homogeneous Besov spaces. The method in this section is based on Fourier multiplier theory given in Bergh-Löfström,¹ Chapter 6

Lemma 4.1. *Let $1 \leq p \leq \infty, s \in \mathbb{R}$ and $1 \leq q \leq \infty$. For every $f \in \dot{B}_{p,q}^s$, it holds that $(-\Delta)^{\frac{\sigma}{2}} f \in \dot{B}_{p,q}^{s-\sigma}$ and $\partial_x^\alpha f \in \dot{B}_{p,q}^{s-|\alpha|}$ for all $\sigma \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$ with the estimates*

$$\begin{cases} \|(-\Delta)^{\frac{\sigma}{2}} f\|_{\dot{B}_{p,q}^{s-\sigma}} \leq 2^{2|\sigma|} (|\sigma| + 1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,q}^s}, \\ \|\partial_x^\alpha f\|_{\dot{B}_{p,q}^{s-|\alpha|}} \leq 2^{2|\alpha|} (|\alpha| + 1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,q}^s}, \end{cases}$$

where $C_n > 0$ is a constant independent of p, s, q, σ, α and f .

To prove Lemma 4.1, first we show the basic proposition on the higher derivative estimate of the function $|\xi|^\sigma$.

Proposition 4.2. *Let $R \geq 1$. It holds that*

$$|\partial_\xi^\beta |\xi|^\sigma| \leq C_\beta R^{|\beta|} (|\sigma| + 1)^{|\beta|} |\xi|^{\sigma-|\beta|} \tag{4.1}$$

for all $\sigma \in \mathbb{R}, \beta \in \mathbb{N}_0^n$ and $\xi \in B_R \setminus \{0\}$, where $C_\beta > 0$ is a constant independent of ξ, R and σ .

Proof. If $|\beta| = 0$ or $|\beta| = 1$, then (4.1) obviously holds for all $\sigma \in \mathbb{R}$. Assume that (4.1) holds for all $\sigma \in \mathbb{R}$ and for some $\beta \in \mathbb{N}_0^n$ with $|\beta| = k \in \mathbb{N}$. Then, we have

$$\begin{aligned} |\partial_\xi^\beta |\xi|^{\sigma-2}| &\leq C_\beta R^{k^2} (|\sigma - 2| + 1)^k |\xi|^{\sigma-2-k} \leq C_\beta R^{k^2} (|\sigma| + 3)^k |\xi|^{\sigma-2-k} \\ &\leq 3^k C_\beta R^{k^2} (|\sigma| + 1)^k |\xi|^{\sigma-2-k}. \end{aligned} \tag{4.2}$$

Since it holds by the Leibniz rule that

$$\begin{aligned} \partial_{\xi_j} \partial_\xi^\beta |\xi|^\sigma &= \partial_\xi^\beta (\sigma |\xi|^{\sigma-2} \xi_j) = \sigma \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} (\partial_\xi^\gamma |\xi|^{\sigma-2}) \partial_\xi^{\beta-\gamma} \xi_j \\ &= \sigma (\partial_\xi^\beta |\xi|^{\sigma-2}) \xi_j + \sigma \sum_{\gamma < \beta} \binom{\beta}{\gamma} (\partial_\xi^\gamma |\xi|^{\sigma-2}) \partial_\xi^{\beta-\gamma} \xi_j, \end{aligned}$$

it follows from (4.2) and $|\partial_\xi^{\beta-\gamma} \xi_j| \leq 1$ that

$$\begin{aligned} |\partial_{\xi_j} \partial_\xi^\beta |\xi|^\sigma| &\leq |\sigma| |\partial_\xi^\beta |\xi|^{\sigma-2}| |\xi_j| + |\sigma| \sum_{\gamma < \beta} \binom{\beta}{\gamma} |\partial_\xi^\gamma |\xi|^{\sigma-2}| |\partial_\xi^{\beta-\gamma} \xi_j| \\ &\leq |\sigma| \cdot 3^k C_\beta R^{k^2} (|\sigma| + 1)^k |\xi|^{\sigma-2-k} |\xi| \\ &\quad + |\sigma| \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma R^{|\gamma|^2} (|\sigma| + 1)^{|\gamma|} |\xi|^{\sigma-2-|\gamma|} \\ &\leq 3^k C_\beta R^{k^2} (|\sigma| + 1)^{k+1} |\xi|^{\sigma-(k+1)} + R^{k^2} (|\sigma| + 1)^{k+1} \\ &\quad \times \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma |\xi|^{\sigma-2-|\gamma|} \\ &= R^{k^2} (|\sigma| + 1)^{k+1} |\xi|^{\sigma-(k+1)} \\ &\quad \times \left\{ 3^k C_\beta + \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma |\xi|^{k-|\gamma|-1} \right\}. \end{aligned}$$

Notice that $|\xi|^{k-|\gamma|-1} \leq R^k$ holds since $\gamma < \beta$ is satisfied in the summation. Hence we have

$$\begin{aligned} R^{k^2} \left\{ 3^k C_\beta + \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma |\xi|^{k-|\gamma|-1} \right\} \\ \leq R^{k^2} \left\{ 3^k C_\beta + R^k \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma \right\} \\ \leq R^{(k+1)^2} \left\{ 3^k C_\beta + \sum_{\gamma < \beta} \binom{\beta}{\gamma} 3^{|\gamma|} C_\gamma \right\}, \end{aligned}$$

which implies that (4.1) holds for $\beta \in \mathbb{N}_0^n$ with $|\beta| = k + 1$. This proves Proposition 4.2. \square

Next, we prove the estimates of the multipliers $|\xi|^\sigma$ and $(i\xi)^\alpha$ which generate the Fourier multiplier operators $(-\Delta)^{\frac{\sigma}{2}} = \mathcal{F}^{-1} |\xi|^\sigma \mathcal{F}$ and $\partial_x^\alpha = \mathcal{F}^{-1} (i\xi)^\alpha \mathcal{F}$.

Proposition 4.3. *Let $\varphi \in \mathcal{S}$ be defined by (2.3). It holds that*

$$\|\rho(2^j \xi) \varphi\|_{L^2} \leq \begin{cases} 2^{\sigma j} 2^{|\sigma|} \|\varphi\|_{L^2} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha| j} 2^{|\alpha|} \|\varphi\|_{L^2} & \text{if } \rho(\xi) = (i\xi)^\alpha \end{cases}$$

and

$$\|\rho(2^j \xi) \varphi\|_{H^n} \leq \begin{cases} 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^n C_n \|\varphi\|_{H^n} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha| j} 2^{|\alpha|} (|\alpha| + 1)^n C_n \|\varphi\|_{H^n} & \text{if } \rho(\xi) = (i\xi)^\alpha \end{cases}$$

for all $\sigma \in \mathbb{R}, \alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{Z}$, where $C_n > 0$ is a constant independent of σ, α, j and φ .

Proof. By the definition (2.3) of the function $\varphi \in \mathcal{S}$, it is sufficient to consider on the annulus $1/2 \leq |\xi| \leq 2$. In the case of $\rho(\xi) = |\xi|^\sigma$, we have

$$|\rho(2^j \xi)| = |2^j \xi|^\sigma = 2^{\sigma j} |\xi|^\sigma \leq 2^{\sigma j} 2^{|\sigma|}.$$

In the case of $\rho(\xi) = (i\xi)^\alpha$, we also have

$$|\rho(2^j \xi)| = |(i2^j \xi)^\alpha| = 2^{|\alpha|j} |\xi|^\alpha \leq 2^{|\alpha|j} |\xi|^{|\alpha|} \leq 2^{|\alpha|j} 2^{|\alpha|}.$$

Hence it follows that

$$\begin{aligned} \|\rho(2^j \xi)\varphi\|_{L^2} &= \left(\int_{1/2 \leq |\xi| \leq 2} |\rho(2^j \xi)\varphi(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq \begin{cases} 2^{\sigma j} 2^{|\sigma|} \|\varphi\|_{L^2} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha|j} 2^{|\alpha|} \|\varphi\|_{L^2} & \text{if } \rho(\xi) = (i\xi)^\alpha. \end{cases} \end{aligned}$$

Next, we take a multi-index $\beta \in \mathbb{N}_0^n$ satisfying $|\beta| = n$. Then, in the case of $\rho(\xi) = |\xi|^\sigma$, it holds by the Leibniz rule and Proposition 4.2 that

$$\begin{aligned} |\partial_\xi^\beta (\rho(2^j \xi)\varphi(\xi))| &\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} 2^{\sigma j} |\partial_\xi^\gamma |\xi|^\sigma| |\partial_\xi^{\beta-\gamma} \varphi(\xi)| \\ &\leq 2^{\sigma j} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} C_\gamma (|\sigma| + 1)^{|\gamma|} |\xi|^{\sigma-|\gamma|} |\partial_\xi^{\beta-\gamma} \varphi(\xi)| \quad (4.3) \\ &\leq 2^{\sigma j} (|\sigma| + 1)^n C_n \sum_{\gamma \leq \beta} |\xi|^{\sigma-|\gamma|} |\partial_\xi^{\beta-\gamma} \varphi(\xi)|. \end{aligned}$$

Hence we have

$$\begin{aligned} \|\partial_\xi^\beta (\rho(2^j \xi)\varphi)\|_{L^2} &= \left(\int_{1/2 \leq |\xi| \leq 2} |\partial_\xi^\beta (\rho(2^j \xi)\varphi(\xi))|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq 2^{\sigma j} (|\sigma| + 1)^n C_n \\ &\quad \times \sum_{\gamma \leq \beta} \left(\int_{1/2 \leq |\xi| \leq 2} |\xi|^{2\sigma-2|\gamma|} |\partial_\xi^{\beta-\gamma} \varphi(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq 2^{\sigma j} (|\sigma| + 1)^n C_n \sum_{\gamma \leq \beta} 2^{|\sigma|+|\gamma|} \left(\int_{1/2 \leq |\xi| \leq 2} |\partial_\xi^{\beta-\gamma} \varphi(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^n C_n \sum_{\gamma \leq \beta} \|\partial_\xi^{\beta-\gamma} \varphi\|_{L^2}, \end{aligned}$$

which yields

$$\begin{aligned} \|\rho(2^j \xi)\varphi\|_{H^n} &\leq C_n \sum_{|\beta|=n} \|\partial_\xi^\beta (\rho(2^j \xi)\varphi)\|_{L^2} \\ &\leq 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^n C_n \sum_{|\beta|=n} \sum_{\gamma \leq \beta} \|\partial_\xi^{\beta-\gamma} \varphi\|_{L^2} \\ &\leq 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^n C_n \|\varphi\|_{H^n}. \end{aligned}$$

In the case of $\rho(\xi) = (i\xi)^\alpha$, we note that

$$\partial_\xi^\gamma \xi^\alpha = \begin{cases} \binom{\alpha}{\gamma} \xi^{\alpha-\gamma} & \text{if } \gamma \leq \alpha, \\ 0 & \text{otherwise} \end{cases}$$

and the estimate of binomial coefficients

$$\binom{k}{l} = \prod_{j=1}^l \frac{k-j+1}{j} \leq \prod_{j=1}^l \frac{k}{j} \leq (k+1)^l \quad \text{for } k, l \in \mathbb{N}_0 \text{ with } l \leq k$$

hold. Hence it follows that

$$\begin{aligned} |\partial_\xi^\gamma \xi^\alpha| &\leq \binom{\alpha}{\gamma} |\xi|^{\alpha-\gamma} \leq |\xi|^{|\alpha|-|\gamma|} \prod_{j=1}^n \binom{\alpha_j}{\gamma_j} \leq |\xi|^{|\alpha|-|\gamma|} \prod_{j=1}^n (\alpha_j + 1)^{\gamma_j} \\ &\leq |\xi|^{|\alpha|-|\gamma|} (|\alpha| + 1)^{|\gamma|} \end{aligned}$$

provided $\gamma \leq \alpha$. Since the estimate is also valid for all multi-index $\gamma \in \mathbb{N}_0^n$, we see that

$$|\partial_\xi^\gamma (\rho(2^j \xi))| = |\partial_\xi^\gamma (2^j i\xi)^\alpha| = |(2^j i)^\alpha \partial_\xi^\gamma \xi^\alpha| \leq 2^{|\alpha|j} (|\alpha| + 1)^{|\gamma|} |\xi|^{|\alpha|-|\gamma|}$$

holds for all $\alpha, \gamma \in \mathbb{N}_0^n$. Therefore, it holds by the Leibniz rule that

$$\begin{aligned} |\partial_\xi^\beta (\rho(2^j \xi)\varphi(\xi))| &\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\partial_\xi^\gamma (\rho(2^j \xi))| |\partial_\xi^{\beta-\gamma} \varphi(\xi)| \\ &\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} 2^{|\alpha|j} (|\alpha| + 1)^{|\gamma|} |\xi|^{|\alpha|-|\gamma|} |\partial_\xi^{\beta-\gamma} \varphi(\xi)| \\ &\leq 2^{|\alpha|j} (|\alpha| + 1)^n C_n \sum_{\gamma \leq \beta} |\xi|^{|\alpha|-|\gamma|} |\partial_\xi^{\beta-\gamma} \varphi(\xi)|. \end{aligned}$$

This inequality is similar compared with (4.3), so we can derive desired estimate in the similar manner as in the case of $\rho(\xi) = |\xi|^\sigma$. This completes the proof of Proposition 4.3. \square

By Proposition 4.3, we may prove the following estimate.

Proposition 4.4. Let $1 \leq p \leq \infty$ and $\varphi \in \mathcal{S}$ be defined by (2.3). It holds that

$$\|\rho\varphi(2^{-j}\xi)\|_{M_p} \leq \begin{cases} 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^{\frac{1}{2}n} C_n \|\varphi\|_{H^n} & \text{if } \rho(\xi) = |\xi|^\sigma, \\ 2^{|\alpha|j} 2^{|\alpha|} (|\alpha| + 1)^{\frac{1}{2}n} C_n \|\varphi\|_{H^n} & \text{if } \rho(\xi) = (i\xi)^\alpha \end{cases}$$

for all $\sigma \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{Z}$, where $C_n > 0$ is a constant independent of p, σ, α, j and φ .

Proof. By (2.2), we have $\|\rho\varphi(2^{-j}\xi)\|_{M_p} \leq \|\mathcal{F}^{-1}[\rho(2^j \xi)\varphi]\|_{L^1}$ for all $1 \leq p \leq \infty$. Let us define the constant $R_j > 0$ by

$$R_j := \left(\frac{\|\rho(2^j \xi)\varphi\|_{H^n}}{\|\rho(2^j \xi)\varphi\|_{L^2}} \right)^{\frac{1}{n}}. \quad (4.4)$$

In the L^2 framework, the Parseval identity $\|\mathcal{F}f\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}$ holds. Then, the Schwartz inequality and polar coordinates give that

$$\begin{aligned} \int_{|x| \leq R_j} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)| dx &\leq \left(\int_{|x| \leq R_j} dx \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{|x| \leq R_j} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq (|B_1| \int_0^{R_j} r^{n-1} dr)^{\frac{1}{2}} \|\mathcal{F}^{-1}[\rho(2^j \xi)\varphi]\|_{L^2} \\ &\leq C_n R_j^{\frac{n}{2}} \|\rho(2^j \xi)\varphi\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} \int_{|x| > R_j} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)| dx &\leq \left(\int_{|x| > R_j} |x|^{-2n} dx \right)^{\frac{1}{2}} \left(\int_{|x| > R_j} |x|^{2n} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq (|B_1| \int_{R_j}^\infty r^{-2n} \cdot r^{n-1} dr)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |x|^{2n} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C_n R_j^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} |x|^{2n} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

We set $P(\xi) := \rho(2^j \xi)\varphi(\xi)$ for simplicity. Since the relation $\mathcal{F}^{-1}[f](-x) = (2\pi)^{-n} \mathcal{F}[f](x)$ holds, the substitution $x = -\xi$ and the Parseval identity give that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |x|^{2n} |\mathcal{F}^{-1}[P](x)|^2 dx \right)^{\frac{1}{2}} &= \left(\int_{\mathbb{R}^n} |-\xi|^{2n} |\mathcal{F}^{-1}[P](-\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} |\xi|^{2n} |\mathcal{F}[P](\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2}} P(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= \|(-\Delta)^{\frac{n}{2}} P\|_{L^2} = \|P\|_{\dot{H}^n}. \end{aligned}$$

which implies that

$$\begin{aligned} \int_{|x|>R_j} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)| dx &\leq C_n R_j^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} |x|^{2n} |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C_n R_j^{-\frac{n}{2}} \|\rho(2^j \xi)\varphi\|_{H^n}. \end{aligned}$$

Combining these inequalities, it holds by (4.4) that

$$\begin{aligned} \|\mathcal{F}^{-1}[\rho(2^j \xi)\varphi]\|_{L^1} &= \left(\int_{|x|\leq R_j} + \int_{|x|>R_j} \right) |\mathcal{F}^{-1}[\rho(2^j \xi)\varphi](x)| dx \\ &\leq C_n \left(R_j^{\frac{n}{2}} \|\rho(2^j \xi)\varphi\|_{L^2} + R_j^{-\frac{n}{2}} \|\rho(2^j \xi)\varphi\|_{H^n} \right) \\ &= C_n \left(\|\rho(2^j \xi)\varphi\|_{L^2} \|\rho(2^j \xi)\varphi\|_{H^n} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, in the case of $\rho(\xi) = |\xi|^\sigma$, we have

$$\begin{aligned} \|\mathcal{F}^{-1}[\rho(2^j \xi)\varphi]\|_{L^1} &\leq C_n \left(\|\rho(2^j \xi)\varphi\|_{L^2} \|\rho(2^j \xi)\varphi\|_{H^n} \right)^{\frac{1}{2}} \\ &\leq C_n (2^{\sigma j} 2^{|\sigma|} \|\varphi\|_{L^2} \cdot 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^n C_n \|\varphi\|_{H^n})^{\frac{1}{2}} \\ &\leq 2^{\sigma j} 2^{|\sigma|} (|\sigma| + 1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n}. \end{aligned}$$

In the case of $\rho(\xi) = (i\xi)^\alpha$, we also have

$$\begin{aligned} \|\mathcal{F}^{-1}[\rho(2^j \xi)\varphi]\|_{L^1} &\leq C_n \left(\|\rho(2^j \xi)\varphi\|_{L^2} \|\rho(2^j \xi)\varphi\|_{H^n} \right)^{\frac{1}{2}} \\ &\leq C_n (2^{|\alpha|j} 2^{|\alpha|} \|\varphi\|_{L^2} \cdot 2^{|\alpha|j} 2^{|\alpha|} (|\alpha| + 1)^n C_n \|\varphi\|_{H^n})^{\frac{1}{2}} \\ &\leq 2^{|\alpha|j} 2^{|\alpha|} (|\alpha| + 1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n}, \end{aligned}$$

which completes the proof of Proposition 4.4. \square

Let us show Lemma 4.1 by applying Proposition 4.4.

Proof of Lemma 4.1. Let $\{\varphi_j\}_{j \in \mathbb{Z}}$ be defined by (2.4). By the definition (2.1) of the space M_p , (2.6) and the relation $\mathcal{F}[\varphi_k](\xi) = \varphi(2^{-k}\xi)$, we have

$$\begin{aligned} \|\varphi_j * \mathcal{F}^{-1}[\rho \mathcal{F} f]\|_{L^p} &= \sum_{k=j-1}^{j+1} \|\varphi_j * \varphi_k * \mathcal{F}^{-1}[\rho \mathcal{F} f]\|_{L^p} \\ &= \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[\rho \mathcal{F} \varphi_k] * (\varphi_j * f)\|_{L^p} \\ &= \sum_{k=j-1}^{j+1} \|\mathcal{F}^{-1}[(\rho \mathcal{F} \varphi_k) \mathcal{F}[\varphi_j * f]]\|_{L^p} \\ &\leq \|\varphi_j * f\|_{L^p} \sum_{k=j-1}^{j+1} \|\rho \mathcal{F} \varphi_k\|_{M_p} \\ &= \|\varphi_j * f\|_{L^p} \sum_{k=j-1}^{j+1} \|\rho \varphi(2^{-k}\xi)\|_{M_p}. \end{aligned}$$

Therefore, in the case of $\rho(\xi) = |\xi|^\sigma$, it holds by $\sum_{k=j-1}^{j+1} 2^{\sigma k} \leq 3 \cdot 2^{|\sigma|} 2^{\sigma j}$ and Proposition 4.4 that

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|\rho \varphi(2^{-k}\xi)\|_{M_p} &\leq \sum_{k=j-1}^{j+1} 2^{\sigma k} 2^{|\sigma|} (|\sigma| + 1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n} \\ &\leq 2^{\sigma j} 2^{2|\sigma|} (|\sigma| + 1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n}, \end{aligned}$$

which yields

$$2^{(s-\sigma)j} \|\varphi_j * (-\Delta)^{\frac{\sigma}{2}} f\|_{L^p} \leq 2^{2|\sigma|} (|\sigma| + 1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n} \cdot 2^{\sigma j} \|\varphi_j * f\|_{L^p}.$$

In the case of $\rho(\xi) = (i\xi)^\alpha$, we can show as well. This completes the proof of Lemma 4.1. \square

5. Decay estimate of the heat semigroup on homogeneous Besov spaces

The aim of this section is to obtain a decay estimate of the heat semigroup on homogeneous Besov spaces, i.e., to prove the following Lemma.

Lemma 5.1. Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}_{p,\infty}^s$, it holds that $T(t)f \in \dot{B}_{p,1}^{s+\sigma}$ for all $\sigma > 0$ and $t > 0$ with the estimate

$$\|T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} \leq \left(\frac{2^s n \sigma}{\pi t} \right)^{\frac{\sigma}{2}} (\sigma + 1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s},$$

where $C_n > 0$ is a constant independent of p, s, σ, t and f .

First we recall the elementary propositions. The following proposition is derived from theory of the real interpolation.

Proposition 5.2. (i) Let $1 \leq p, q, q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$. For $0 < \theta < 1$, it holds that $(\dot{B}_{p,q_0}^{s_0}, \dot{B}_{p,q_1}^{s_1})_{\theta,q} = \dot{B}_{p,q}^s$, where $s \in \mathbb{R}$ is defined by $s = (1 - \theta)s_0 + \theta s_1$.

(ii) Let X_0, X_1, Y_0 and Y_1 be Banach spaces. Then, if $T \in \mathcal{L}(X_0, X_1) \cap \mathcal{L}(Y_0, Y_1)$, it holds that $T \in \mathcal{L}((X_0, Y_0)_{\theta,q}, (X_1, Y_1)_{\theta,q})$ for all $1 \leq q \leq \infty$ and $0 < \theta < 1$ with the estimate

$$\|T\|_{\mathcal{L}((X_0, Y_0)_{\theta,q}, (X_1, Y_1)_{\theta,q})} \leq \|T\|_{\mathcal{L}(X_0, X_1)}^{1-\theta} \|T\|_{\mathcal{L}(Y_0, Y_1)}^\theta.$$

For the proof of (i), see Bergh–Löfström.¹, Theorem 6.4.5 For the proof of (ii), we refer to Lunardi.¹⁹, Theorem 1.6 Next proposition is easily derived.

Proposition 5.3. It holds that

$$(a + b)^{a+b} \leq 2^{a+b} a^a b^b$$

for all $a, b \geq 0$. Here $0^0 := 1$.

Proof. If $a = 0$ or $b = 0$, then the estimate clearly holds. So, we may assume that $a, b > 0$. Let us take the one-parameter function f defined by $f(x) := x \log x$ for $x > 0$. Then we see by $f''(x) = 1/x > 0$ that f is a convex function. Hence it holds that

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y),$$

i.e.,

$$(\theta x + (1 - \theta)y) \log(\theta x + (1 - \theta)y) \leq \theta x \log x + (1 - \theta)y \log y$$

for all $x, y > 0$ and $0 < \theta < 1$. Taking $x = 2a$, $y = 2b$ and $\theta = 1/2$, we see that

$$(a + b) \log(a + b) \leq a \log(2a) + b \log(2b)$$

holds. Hence it holds that

$$\log(a + b)^{a+b} \leq \log(2^{a+b} a^a b^b),$$

which yields the desired estimate. \square

We see the basic estimate of the heat kernel G_t defined by (1.2).

Proposition 5.4. It holds that

$$\|(-\Delta)^N G_t\|_{L^1} \leq \left(\frac{2nN}{\pi t} \right)^N$$

for all $N \in \mathbb{N}$ and $t > 0$.

Proof. For $t > 0$ and $1 \leq j \leq n$, it holds that

$$\begin{aligned} \|\partial_{x_j} G_t\|_{L^1} &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left| \frac{x_j}{2t} \right| e^{-\frac{|x|^2}{4t}} dx \\ &= \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left| \frac{x_j}{2t} \right| e^{-\frac{x_j^2}{4t}} dx_j \prod_{k \neq j} \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{x_k^2}{4t}} dx_k \\ &= \frac{2}{(4\pi t)^{\frac{1}{2}}} \int_0^{\infty} \frac{x_j}{2t} e^{-\frac{x_j^2}{4t}} dx_j = \left(\frac{1}{\pi t} \right)^{\frac{1}{2}}. \end{aligned}$$

Since the relation

$$\begin{aligned} G_{t+\tau} &= \mathcal{F}^{-1} e^{-(t+\tau)|\xi|^2} = \mathcal{F}^{-1} [e^{-t|\xi|^2} e^{-\tau|\xi|^2}] \\ &= (\mathcal{F}^{-1} e^{-t|\xi|^2}) * \mathcal{F}^{-1} e^{-\tau|\xi|^2} = G_t * G_\tau \end{aligned}$$

holds for all $t, \tau > 0$, for a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ satisfying $|\beta| = N$, we see by the Hausdorff–Young inequality that

$$\begin{aligned} \|\partial_x^\beta G_t\|_{L^1} &= \|\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} (G_{t/N} * \cdots * G_{t/N})\|_{L^1} \\ &= \|(\partial_{x_1} G_{t/N}) * \cdots * (\partial_{x_n} G_{t/N})\|_{L^1} \\ &\leq \prod_{j=1}^n \|\partial_{x_j} G_{t/N}\|_{L^1} \leq \prod_{j=1}^n \left(\frac{N}{\pi t}\right)^{\frac{\beta_j}{2}} = \left(\frac{N}{\pi t}\right)^{\frac{N}{2}}. \end{aligned}$$

Notice that it holds by the multinomial theorem that

$$(-\Delta)^N = (-\partial_{x_1}^2 - \cdots - \partial_{x_n}^2)^N = \sum_{|\beta|=N} \frac{N!}{\beta!} (-\partial_x^2)^\beta, \quad n^N = \sum_{|\beta|=N} \frac{N!}{\beta!}, \quad (5.1)$$

respectively. Hence we have

$$\begin{aligned} \|(-\Delta)^N G_t\|_{L^1} &= \left\| \sum_{|\beta|=N} \frac{N!}{\beta!} (-\partial_x^2)^\beta G_t \right\|_{L^1} \leq \sum_{|\beta|=N} \frac{N!}{\beta!} \|\partial_x^{2\beta} G_t\|_{L^1} \\ &\leq \sum_{|\beta|=N} \frac{N!}{\beta!} \left(\frac{2N}{\pi t}\right)^N = \left(\frac{2nN}{\pi t}\right)^N, \end{aligned}$$

which completes the proof of Proposition 5.4. \square

In the following, we show Lemma 5.1. Note that the proof relies on Kozono–Ogawa–Taniuchi. ², Lemma 2.2 Besides, since Proposition 5.4 is only valid in case N is a natural number, we need to apply real interpolation theory to extend the estimate for an arbitrary positive real number.

Proof of Lemma 5.1. Let $\{\varphi_j\}_{j \in \mathbb{Z}}$ be defined by (2.4). Then, for $N \in \mathbb{N}$, it holds by the Hausdorff–Young inequality and (2.6) that

$$\begin{aligned} \|\varphi_j * (T(t)f)\|_{L^p} &= \sum_{k=j-1}^{j+1} \|\varphi_j * \varphi_k * (G_t * f)\|_{L^p} \\ &= \sum_{k=j-1}^{j+1} \|((-\Delta)^{-N} \varphi_k) * ((-\Delta)^N G_t) * (\varphi_j * f)\|_{L^p} \\ &\leq \|\varphi_j * f\|_{L^p} \sum_{k=j-1}^{j+1} \|((-\Delta)^{-N} \varphi_k) * (-\Delta)^N G_t\|_{L^1}. \end{aligned}$$

Here, the definition (2.1) of the space M_p and the relation $\mathcal{F}[\varphi_k](\xi) = \varphi(2^{-k}\xi)$ gives

$$\begin{aligned} \|((-\Delta)^{-N} \varphi_k) * (-\Delta)^N G_t\|_{L^1} &= \|(\mathcal{F}^{-1}[|\xi|^{-2N} \mathcal{F}\varphi_k]) * (-\Delta)^N G_t\|_{L^1} \\ &= \|\mathcal{F}^{-1}[|\xi|^{-2N} \mathcal{F}\varphi_k] \mathcal{F}(-\Delta)^N G_t\|_{L^1} \\ &\leq \|\xi|^{-2N} \varphi(2^{-k}\xi)\|_{M_1} \|(-\Delta)^N G_t\|_{L^1}. \end{aligned}$$

Notice that it holds by $\sum_{k=j-1}^{j+1} 2^{-2Nk} \leq 3 \cdot 2^{2N} 2^{-2Nj}$ and Proposition 4.4 that

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|\xi|^{-2N} \varphi(2^{-k}\xi)\|_{M_1} &\leq \sum_{k=j-1}^{j+1} 2^{-2Nk} 2^{2N} (2N+1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n} \\ &\leq 2^{-2Nj} 2^{4N} (N+1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n}. \end{aligned}$$

Hence we see by Proposition 5.4 that

$$\begin{aligned} 2^{(s+2N)j} \|\varphi_j * (T(t)f)\|_{L^p} &\leq 2^{(s+2N)j} \|\varphi_j * f\|_{L^p} \cdot 2^{-2Nj} 2^{4N} (N+1)^{\frac{n}{2}} \\ &\quad \times C_n \|\varphi\|_{H^n} \cdot \left(\frac{2nN}{\pi t}\right)^N \\ &= \left(\frac{2^s n N}{\pi t}\right)^N (N+1)^{\frac{n}{2}} C_n \|\varphi\|_{H^n} \\ &\quad \cdot 2^{sj} \|\varphi_j * f\|_{L^p}, \end{aligned}$$

which implies that

$$\|T(t)f\|_{\dot{B}_{p,\infty}^{s+2N}} \leq \left(\frac{2^s n N}{\pi t}\right)^N (N+1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s} \quad (5.2)$$

holds for all $N \in \mathbb{N}$. Note that since $\|T(t)f\|_{\dot{B}_{p,\infty}^s} \leq \|f\|_{\dot{B}_{p,\infty}^s}$ holds, (5.2) is also valid $N = 0$ as $0^0 := 1$. For any $\sigma > 0$, there exists $N \in \mathbb{N}$ such

that $2(N-1) < \sigma < 2(N+1)$. Hence we may choose $0 < \theta < 1$ satisfying

$$\sigma = 2(1-\theta)(N-1) + 2\theta(N+1). \quad (5.3)$$

Since it holds by (5.2) that

$$\begin{cases} \|T(t)f\|_{\dot{B}_{p,\infty}^{s+2(N-1)}} \leq \left(\frac{2^s n(N-1)}{\pi t}\right)^{N-1} N^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s}, \\ \|T(t)f\|_{\dot{B}_{p,\infty}^{s+2(N+1)}} \leq \left(\frac{2^s n(N+1)}{\pi t}\right)^{N+1} (N+2)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s}, \end{cases}$$

we see that $T(t) \in \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,\infty}^{s+2(N-1)}) \cap \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,\infty}^{s+2(N+1)})$ holds. By Proposition 5.2, we have

$$(\dot{B}_{p,\infty}^{s+2(N-1)}, \dot{B}_{p,\infty}^{s+2(N+1)})_{\theta,1} = \dot{B}_{p,1}^{s+\sigma},$$

which yields $T(t) \in \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^{s+\sigma})$. Hence we see by Proposition 5.3 and (5.3) that

$$\begin{aligned} \|T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} &\leq \left\{ \left(\frac{2^s n(N-1)}{\pi t}\right)^{N-1} N^{\frac{n}{2}} \right\}^{1-\theta} \\ &\quad \times \left\{ \left(\frac{2^s n(N+1)}{\pi t}\right)^{N+1} (N+2)^{\frac{n}{2}} \right\}^{\theta} C_n \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq \left(\frac{2^s n}{\pi t}\right)^{\frac{\sigma}{2}} (N+1)^{\frac{\sigma}{2}} (N+2)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq \left(\frac{2^s n}{\pi t}\right)^{\frac{\sigma}{2}} \left(\frac{\sigma}{2} + 2\right)^{\frac{\sigma}{2}+2} \left(\frac{\sigma}{2} + 3\right)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq \left(\frac{2^s n}{\pi t}\right)^{\frac{\sigma}{2}} \cdot 2^{\frac{\sigma}{2}+2} \left(\frac{\sigma}{2}\right)^{\frac{\sigma}{2}} 2^2 \cdot 3^{\frac{n}{2}} (\sigma+1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq \left(\frac{2^s n \sigma}{\pi t}\right)^{\frac{\sigma}{2}} (\sigma+1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s}, \end{aligned}$$

which completes the proof of Lemma 5.1. \square

6. Space–time analytic smoothing effect of the heat semigroup on homogeneous Besov spaces

Before showing our main result on a space–time analytic smoothing effect, we shall show the properties of the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$. It is well known that the heat semigroup $\{T(t)\}_{t \geq 0}$ on L^p is a bounded analytic C_0 -semigroup for all $1 \leq p < \infty$. Hence, by theory of the L^p framework, we may also prove the properties of $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s$. In fact, we see that the following propositions hold.

Proposition 6.1. Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and $1 \leq q \leq \infty$.

(i) In case $1 \leq p < \infty$ and $1 \leq q < \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s$ is a bounded C_0 -semigroup which satisfies

$$\|T(t)f\|_{\dot{B}_{p,q}^s} \leq \|f\|_{\dot{B}_{p,q}^s}, \quad T(t+\tau)f = T(t)T(\tau)f, \quad (6.1)$$

$$\partial_t^j T(t)f = \Delta^j T(t)f \quad \text{in } \dot{B}_{p,q}^s$$

and

$$\lim_{h \rightarrow +0} \|T(h)f - f\|_{\dot{B}_{p,q}^s} = 0 \quad (6.2)$$

for all $t, \tau > 0$, $j \in \mathbb{N}$ and $f \in \dot{B}_{p,q}^s$.

(ii) In case $1 \leq p < \infty$ with $q = \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,\infty}^s$ is a bounded C_0 -semigroup which satisfies (6.1) and (6.2) with $\dot{B}_{p,\infty}^s$ replaced by $\dot{B}_{p,\infty+}^s$.

(iii) In case $p = \infty$ and $1 \leq q < \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{\infty,q}^s$ is a bounded semigroup which satisfies (6.1). In addition, for all $t > 0$ and $f \in \dot{B}_{\infty,q}^s$, it holds that

$$\lim_{h \rightarrow +0} \|T(t+h)f - T(t)f\|_{\dot{B}_{\infty,q}^s} = 0. \quad (6.3)$$

(iv) In case $p = q = \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{\infty, \infty+}^s$ is a bounded semigroup which satisfies (6.1) and (6.3) with $\dot{B}_{\infty, \infty}^s$ replaced by $\dot{B}_{\infty, \infty+}^s$.

Proposition 6.2. Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For every $f \in \dot{B}_{p, \infty}^s$, it holds that $T(t)f \in \dot{B}_{p,1}^{s+\sigma}$ and

$$\partial_t^j T(t)f = \Delta^j T(t)f \quad \text{in } \dot{B}_{p,1}^{s+\sigma}, \quad \lim_{h \rightarrow +0} \|T(t+h)f - T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} = 0$$

for all $\sigma > 0$ and $t > 0$.

The proof of Propositions 6.1 and 6.2 will be given in Appendix. In Proposition 6.1, we need to pay attention to the C_0 -property and differentiability on $t > 0$ for the case of $p = \infty$ or $q = \infty$. In case $1 \leq p < \infty$ with $q = \infty$, we need to consider the additional condition $\lim_{j \rightarrow \pm\infty} 2^{sj} \|\varphi_j * f\|_{L^p} = 0$ of (2.5) in order to obtain both C_0 -property and differentiability on $t > 0$. On the other hand, since $(\dot{B}_{p',1}^s)' = \dot{B}_{p,\infty}^s$ holds for all $1 < p \leq \infty$ with $1/p + 1/p' = 1$, we obtain the weakly-star continuity at $t = 0$ for all $f \in \dot{B}_{p,\infty}^s$, i.e.,

$$\lim_{h \rightarrow +0} \langle T(h)f - f, g \rangle = 0 \quad \text{for all } g \in \dot{B}_{p',1}^{-s}.$$

Furthermore, Proposition 6.2 states that regarding the function $T(t)f$ as a $\dot{B}_{p,1}^{s+\sigma}$ -valued function gives differentiability on $t > 0$ without considering the condition (2.5). In case $p = \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{\infty,q}^s$ no longer satisfies the C_0 -property. However, differentiability on $t > 0$ is still valid.

Now we are in a position to prove Theorem 3.1. The method of the estimate relies on Lemmas 4.1 and 5.1.

Proof of Theorem 3.1. For every $f \in \dot{B}_{p,\infty}^s$, it holds by Lemmas 4.1 and 5.1 that

$$\begin{aligned} \|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} &= \|\partial_x^\alpha \Delta^j T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} \\ &\leq 2^{2|\alpha|} (|\alpha| + 1)^{\frac{n}{2}} \cdot 2^{4j} (2j + 1)^{\frac{n}{2}} C_n \|T(t)f\|_{\dot{B}_{p,1}^{s+\sigma+|\alpha|+2j}} \\ &\leq 2^{2|\alpha|+4j} (|\alpha| + 1)^{\frac{n}{2}} (j + 1)^{\frac{n}{2}} C_n \|T(t)f\|_{\dot{B}_{p,1}^{s+\sigma+|\alpha|+2j}} \\ &\leq 2^{2|\alpha|+4j} (|\alpha| + 1)^{\frac{n}{2}} (j + 1)^{\frac{n}{2}} \\ &\quad \times \left(\frac{2^5 n (\sigma + |\alpha| + 2j)}{\pi t} \right)^{\frac{1}{2}(\sigma+|\alpha|+2j)} \\ &\quad \cdot (\sigma + |\alpha| + 2j + 1)^{\frac{n}{2}} C_n \|f\|_{\dot{B}_{p,\infty}^s}. \end{aligned}$$

Here, it holds that

$$(\sigma + |\alpha| + 2j + 1)^{\frac{n}{2}} \leq (\sigma + 1)^{\frac{n}{2}} (|\alpha| + 2j + 1)^{\frac{n}{2}} \leq C_{n,\sigma} (|\alpha| + 1)^{\frac{n}{2}} (j + 1)^{\frac{n}{2}}.$$

Besides, we see by Proposition 5.3 that

$$\begin{aligned} (\sigma + |\alpha| + 2j)^{\frac{1}{2}(\sigma+|\alpha|+2j)} &\leq \{2^{\sigma+|\alpha|+2j} \sigma^\sigma (|\alpha| + 2j)^{|\alpha|+2j}\}^{\frac{1}{2}} \\ &\leq C_\sigma \{2^{|\alpha|+2j} \cdot 2^{|\alpha|+2j} |\alpha|^{|\alpha|} (2j)^{2j}\}^{\frac{1}{2}} \\ &= C_\sigma 2^{|\alpha|+2j} |\alpha|^{\frac{|\alpha|}{2}} (2j)^j, \end{aligned}$$

which yields

$$\begin{aligned} &\left(\frac{2^5 n (\sigma + |\alpha| + 2j)}{\pi t} \right)^{\frac{1}{2}(\sigma+|\alpha|+2j)} \\ &= \left(\frac{2^5 n}{\pi t} \right)^{\frac{1}{2}(\sigma+|\alpha|+2j)} (\sigma + |\alpha| + 2j)^{\frac{1}{2}(\sigma+|\alpha|+2j)} \\ &\leq C_{n,\sigma} \left(\frac{2^5 n}{\pi} \right)^{\frac{|\alpha|}{2}+j} t^{-\frac{1}{2}(\sigma+|\alpha|+2j)} \cdot 2^{|\alpha|+2j} |\alpha|^{\frac{|\alpha|}{2}} (2j)^j \\ &= C_{n,\sigma} \left(\frac{2^7 n}{\pi} \right)^{\frac{|\alpha|}{2}+j} t^{-\frac{1}{2}(\sigma+|\alpha|+2j)} |\alpha|^{\frac{|\alpha|}{2}} (2j)^j. \end{aligned}$$

Hence we see that

$$\begin{aligned} \|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} &\leq 2^{2|\alpha|+4j} (|\alpha| + 1)^{\frac{n}{2}} (j + 1)^{\frac{n}{2}} \\ &\quad \times \left(\frac{2^7 n}{\pi} \right)^{\frac{|\alpha|}{2}+j} t^{-\frac{1}{2}(\sigma+|\alpha|+2j)} |\alpha|^{\frac{|\alpha|}{2}} (2j)^j \\ &\quad \cdot (|\alpha| + 1)^{\frac{n}{2}} (j + 1)^{\frac{n}{2}} C_{n,\sigma} \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq \left(\frac{2^{12} n}{\pi} \right)^{\frac{|\alpha|}{2}+j} |\alpha|^{\frac{|\alpha|}{2}} j^j (|\alpha| + 1)^n (j + 1)^n t^{-\frac{1}{2}(\sigma+|\alpha|+2j)} C_{n,\sigma} \|f\|_{\dot{B}_{p,\infty}^s}. \end{aligned}$$

This completes the proof of Theorem 3.1. \square

By Theorem 3.1, we may see a space–time analytic smoothing effect of the heat semigroup on homogeneous Besov spaces. Besides, estimating the constant appeared in the inequality by the Stirling formula provides a radius of convergence of the Taylor expansion.

Proof of Theorem 3.3. Let $f \in \dot{B}_{p,\infty}^s$ be arbitrary. Notice that the function f is an element of the quotient space \mathcal{S}'/\mathcal{P} by the definition of the homogeneous Besov spaces $\dot{B}_{p,\infty}^s$, which implies that the function f is represented by $f + P$ in the usual sense for some polynomial $P \in \mathcal{P}$. However, we may assume that $P \equiv 0$ holds. In fact, we have

$$T(t)(f + P) = T(t)f + T(t)P = T(t)f + F^{-1}[e^{-t|\xi|^2} \mathcal{F}P] = T(t)f + P$$

for all $t \geq 0$ since $\text{supp } \mathcal{F}P = \{0\}$ holds for any non-zero polynomial P . Furthermore, any polynomial P is real analytic with respect to spatial variables, and therefore it is sufficient to consider the case of $P \equiv 0$ when showing the space–time analyticity of the function $T(t)(f + P)$. Then, in the following, we show the space–time analyticity of the function $u := T(t)f$. For any $k_0, j_0 \in \mathbb{N}_0$ satisfying $s - n/p < k_0 + 2j_0$, we define

$$\sigma_0 := n/p + k_0 + 2j_0 - s > 0.$$

Then, we see by Theorem 3.1 that $\partial_x^\alpha \partial_t^j u = \partial_x^\alpha \partial_t^j T(t)f \in \dot{B}_{p,1}^{s+\sigma_0}$ holds for all $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}_0$. Now let us take a multi-index $\alpha_0 \in \mathbb{N}_0^n$ satisfying $|\alpha_0| = k_0$. By Lemma 4.1, we have

$$\begin{aligned} \|\partial_x^{\alpha+\alpha_0} \partial_t^{j+j_0} u\|_{\dot{B}_{p,1}^{n/p}} &= \|\partial_x^{\alpha_0} \Delta^{j_0} \partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{n/p}} \\ &\leq C_{n,k_0,j_0} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{n/p+k_0+2j_0}} \\ &= C_{n,k_0,j_0} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma_0}} \end{aligned}$$

for all $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}_0$. Since the embedding $\dot{B}_{p,1}^{n/p} \subset L^\infty$ holds, we see that $\partial_x^{\alpha+\alpha_0} \partial_t^{j+j_0} u \in L^\infty$ holds. Notice that in order to see the analyticity of the function u , it is sufficient to consider the function $\partial_x^{\alpha_0} \partial_t^{j_0} u$ instead of u . Given $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, the remainder term of the Taylor expansion is estimated by

$$\begin{aligned} &\left| \sum_{|\alpha|=k} \frac{\partial_x^{\alpha+\alpha_0} \partial_t^{j+j_0} u(t_*, x_*)}{\alpha! j!} (t - t_0)^j (x - x_0)^\alpha \right| \\ &\leq \sum_{|\alpha|=k} \frac{1}{\alpha! j!} \|\partial_x^{\alpha+\alpha_0} \partial_t^{j+j_0} u(t_*, \cdot)\|_{L^\infty} |t - t_0|^j |x - x_0|^k \\ &\leq \sum_{|\alpha|=k} \frac{1}{\alpha! j!} C_{n,p,k_0,j_0} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma_0}} |t - t_0|^j |x - x_0|^k \end{aligned}$$

for all $j, k \in \mathbb{N}_0$, $(t, x) \in (0, \infty) \times \mathbb{R}^n$ and for some $(t_*, x_*) \in (0, \infty) \times \mathbb{R}^n$ satisfying $\min\{t, t_0\} < t_* < \max\{t, t_0\}$. Since it holds by Theorem 3.1 that

$$\begin{aligned} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma_0}} &\leq C_{n,p,s,k_0,j_0} \left(\frac{2^{12} n}{\pi} \right)^{\frac{k}{2}+j} k^{\frac{k}{2}} j^j (k + 1)^n (j + 1)^n \\ &\quad \times t_*^{-\frac{1}{2}(\sigma_0+k+2j)} \|f\|_{\dot{B}_{p,\infty}^s} \\ &\leq j^j \left(\frac{2^{12} n}{\pi t_*} \right)^j (j + 1)^n \cdot k^{\frac{k}{2}} \left(\frac{2^{12} n}{\pi t_*} \right)^{\frac{k}{2}} (k + 1)^n \\ &\quad \cdot t_*^{-\frac{1}{2}\sigma_0} C_{n,p,s,k_0,j_0} \|f\|_{\dot{B}_{p,\infty}^s}, \end{aligned}$$

we see by (5.1) that

$$\begin{aligned} & \sum_{|\alpha|=k} \frac{1}{\alpha!j!} C_{n,p,k_0,j_0} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma_0}} |t-t_0|^j |x-x_0|^k \\ & \leq \frac{j^j}{j!} \left(\frac{2^{12}n}{\pi t_*}\right)^j (j+1)^n |t-t_0|^j \\ & \quad \times \sum_{|\alpha|=k} \frac{k!}{\alpha!} \cdot \frac{k^{\frac{k}{2}}}{k!} \left(\frac{2^{12}n}{\pi t_*}\right)^{\frac{k}{2}} (k+1)^n |x-x_0|^k \\ & \quad \cdot t_*^{-\frac{1}{2}\sigma_0} C_{n,p,s,k_0,j_0} \|f\|_{\dot{B}_{p,\infty}^s} \\ & = \frac{j^j}{j!} \left(\frac{2^{12}n}{\pi t_*}\right)^j (j+1)^n \\ & \quad \times k^{-\frac{k}{2}} \frac{k^k}{k!} \left(\frac{2^{12}n^3}{\pi t_*}\right)^{\frac{k}{2}} (k+1)^n \\ & \quad \cdot t_*^{-\frac{1}{2}\sigma_0} C_{n,p,s,k_0,j_0} \|f\|_{\dot{B}_{p,\infty}^s}. \end{aligned}$$

The following well known Stirling formula

$$\frac{N^N}{N!} \leq \frac{e^N}{\sqrt{2\pi N}} \tag{6.4}$$

gives that

$$\begin{aligned} & \sum_{|\alpha|=k} \frac{1}{\alpha!j!} C_{n,p,k_0,j_0} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma_0}} |t-t_0|^j |x-x_0|^k \\ & \leq \frac{e^j}{\sqrt{2\pi j}} \left(\frac{2^{12}n}{\pi t_*}\right)^j (j+1)^n \\ & \quad \times k^{-\frac{k}{2}} \frac{e^k}{\sqrt{2\pi k}} \left(\frac{2^{12}n^3}{\pi t_*}\right)^{\frac{k}{2}} (k+1)^n \\ & \quad \cdot t_*^{-\frac{1}{2}\sigma_0} C_{n,p,s,k_0,j_0} \|f\|_{\dot{B}_{p,\infty}^s} \\ & \leq \frac{(j+1)^n}{j^{\frac{1}{2}}} \left(\frac{2^{12}en}{\pi t_*}\right)^j \cdot \frac{(k+1)^n}{k^{\frac{k}{2}+\frac{1}{2}}} \left(\frac{2^{12}e^2n^3}{\pi t_*}\right)^{\frac{k}{2}} \\ & \quad \cdot t_*^{-\frac{1}{2}\sigma_0} C_{n,p,s,k_0,j_0} \|f\|_{\dot{B}_{p,\infty}^s}. \end{aligned}$$

Here, for an arbitrary $x \in \mathbb{R}^n$, we see that

$$\lim_{k \rightarrow \infty} \frac{(k+1)^n}{k^{\frac{k}{2}+\frac{1}{2}}} \left(\frac{2^{12}e^2n^3}{\pi t_*}\right)^{\frac{k}{2}} |x-x_0|^2 = 0$$

holds. Besides, taking $t > 0$ such that $|t-t_0|$ is sufficiently small, we also see that

$$\lim_{j \rightarrow \infty} \frac{(j+1)^n}{j^{\frac{1}{2}}} \left(\frac{2^{12}en}{\pi t_*}\right)^j |t-t_0|^j = 0$$

holds. Therefore, for such a $t > 0$, it follows that

$$\lim_{j+k \rightarrow \infty} \left| \sum_{|\alpha|=k} \frac{\partial_x^{\alpha+\alpha_0} \partial_t^{j+\alpha_0} u(t_*, x_*)}{\alpha!j!} (t-t_0)^j (x-x_0)^\alpha \right| = 0,$$

which implies that the following Taylor expansion

$$u(t, x) = \sum_{j,k=0}^{\infty} \sum_{|\alpha|=k} \frac{\partial_x^\alpha \partial_t^j u(t_0, x_0)}{\alpha!j!} (t-t_0)^j (x-x_0)^\alpha \tag{6.5}$$

is valid. Therefore, again by estimating the remainder term of (6.5), we see that (6.5) is absolutely convergence under the condition $|t-t_0| < \pi t_0 / (2^{12}en)$. This completes the proof of Theorem 3.3. \square

In the similar manner, we also see that the heat semigroup on homogeneous Besov spaces is a bounded analytic C_0 -semigroup.

Proof of Theorem 3.6. (i) Given $t_0 > 0$, the remainder term of the Taylor expansion is estimated by

$$\left\| \frac{\partial_t^{j+1} T(t_*)}{j!} (t-t_0)^j \right\|_{\mathcal{L}(\dot{B}_{p,q}^s)} \leq \frac{1}{j!} \|\partial_t^{j+1} T(t_*)\|_{\mathcal{L}(\dot{B}_{p,q}^s)} |t-t_0|^j$$

for all $j \in \mathbb{N}_0$, $t > 0$ and for some $t_* > 0$ satisfying $\min\{t, t_0\} < t_* < \max\{t, t_0\}$. Then, for every $f \in \dot{B}_{p,q}^s$, we see by Lemma 4.1 and Theorem 3.1 that

$$\begin{aligned} \|\partial_t^{j+1} T(t_*)f\|_{\dot{B}_{p,q}^s} & \leq \|\Delta \partial_t^j T(t_*)f\|_{\dot{B}_{p,1}^s} \leq C_n \|\partial_t^j T(t_*)f\|_{\dot{B}_{p,1}^{s+2}} \\ & \leq \left(\frac{2^{12}n}{\pi}\right)^j j^j (j+1)^n t_*^{-j} C_n \|f\|_{\dot{B}_{p,\infty}^s} \\ & \leq \left(\frac{2^{12}n}{\pi t_*}\right)^j j^j (j+1)^n C_n \|f\|_{\dot{B}_{p,q}^s}, \end{aligned}$$

which yields

$$\|\partial_t^{j+1} T(t_*)\|_{\mathcal{L}(\dot{B}_{p,q}^s)} \leq \left(\frac{2^{12}n}{\pi t_*}\right)^j j^j (j+1)^n C_n.$$

Hence we see by (6.4) that

$$\begin{aligned} \frac{1}{j!} \|\partial_t^{j+1} T(t_*)\|_{\mathcal{L}(\dot{B}_{p,q}^s)} |t-t_0|^j & \leq \frac{1}{j!} \cdot \left(\frac{2^{12}n}{\pi t_*}\right)^j j^j (j+1)^n C_n \cdot |t-t_0|^j \\ & = \frac{j^j}{j!} \left(\frac{2^{12}n}{\pi t_*}\right)^j (j+1)^n C_n \\ & \leq \frac{e^j}{\sqrt{2\pi j}} \left(\frac{2^{12}n}{\pi t_*}\right)^j (j+1)^n C_n \\ & \leq \frac{(j+1)^n}{j^{\frac{1}{2}}} \left(\frac{2^{12}en}{\pi t_*}\right)^j C_n. \end{aligned}$$

Taking $t > 0$ such that $|t-t_0|$ is sufficiently small, it holds that

$$\lim_{j \rightarrow \infty} \frac{(j+1)^n}{j^{\frac{1}{2}}} \left(\frac{2^{12}en}{\pi t_*}\right)^j |t-t_0|^j = 0,$$

which implies that the following Taylor expansion

$$T(t) = \sum_{j=0}^{\infty} \frac{\partial_t^j T(t_0)}{j!} (t-t_0)^j \quad \text{in } \mathcal{L}(\dot{B}_{p,q}^s) \tag{6.6}$$

is valid. Therefore, again by estimating the remainder term of (6.6), we see that (6.6) is absolutely convergence if $t \in \mathbb{C} \setminus \{0\}$ satisfies $|t-t_0| < \pi t_0 / (2^{12}en)$. Therefore, for every $t_0 > 0$, $T(t)$ can be extended to an analytic $\mathcal{L}(\dot{B}_{p,q}^s)$ -valued function in the following disk

$$t \in D(t_0) := \left\{ t \in \mathbb{C} \setminus \{0\} \mid |t-t_0| < \frac{\pi t_0}{2^{12}en} \right\}.$$

We see by proceeding above method for each $t_0 > 0$ that $T(t)$ is extended analytically on

$$\bigcup_{t_0>0} D(t_0) = \bigcup_{t_0>0} \left\{ t \in \mathbb{C} \setminus \{0\} \mid |t-t_0| < \frac{\pi t_0}{2^{12}en} \right\},$$

which coincides the sector

$$\Sigma := \left\{ t \in \mathbb{C} \setminus \{0\} \mid |\arg t| < \sin^{-1} \left(\frac{\pi}{2^{12}en} \right) \right\}.$$

Since the function $T(t)$ is extended analytically on Σ , we may consider the Taylor expansion (6.6) at the time $t_0 \in \Sigma$. The radius of convergence of (6.6) at the time $t_0 \in \Sigma$ is given the distance between t_0 and Σ , which implies that (6.6) is valid for all $t \in \mathbb{C} \setminus \{0\}$ satisfying

$$|t-t_0| < |t_0| \sin \left(\sin^{-1} \left(\frac{\pi}{2^{12}en} \right) - |\arg t_0| \right)$$

by the direct calculation.

(i') In case $1 \leq p \leq \infty$ with $q = \infty$, we can show as well by considering $\dot{B}_{p,\infty+}^s$ instead of $\dot{B}_{p,\infty}^s$.

(ii) Since the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s$ is a bounded C_0 -semigroup under the conditions $1 \leq p < \infty$ and $1 \leq q < \infty$, we see that the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s$ is a bounded analytic C_0 -semigroup.

(ii') In case $1 \leq p < \infty$ with $q = \infty$, notice that the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,\infty+}^s$ is a bounded C_0 -semigroup.

(iii) By [Theorem 3.1](#), for every $f \in \dot{B}_{p,\infty}^s$, it holds that

$$\|\partial_t^j T(t_*)f\|_{\dot{B}_{p,1}^{s+\sigma}} \leq \left(\frac{2^{12}n}{\pi t_*}\right)^j j^j(j+1)^n C_{n,\sigma} \|f\|_{\dot{B}_{p,\infty}^s}$$

for all $\sigma > 0$. In the same way as in (i), we see that for each $t_0 > 0$, the Taylor expansion

$$T(t) = \sum_{j=0}^{\infty} \frac{\partial_t^j T(t_0)}{j!} (t-t_0)^j \quad \text{in } \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^{s+\sigma})$$

holds for all $t > 0$ satisfying $|t-t_0| < \pi t_0/(2^{12}en)$. This completes the proof of [Theorem 3.6](#). \square

7. Properties of the heat semigroup on homogeneous Besov spaces

Finally, we show [Corollaries 3.9–3.11](#). Before showing them, we recall the basic proposition on the homogeneous Besov spaces.

Proposition 7.1. *Let $1 \leq p < \infty, s \in \mathbb{R}$ and $1 \leq q < \infty$. We set $\mathcal{S}_0 := \{f \in \mathcal{S} \mid 0 \notin \text{supp } Ff\}$. Then, it holds that $\overline{\mathcal{S}_0} = \dot{B}_{p,q}^s$. Namely, there exists a sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{S}_0$ of function satisfying*

$$\lim_{k \rightarrow \infty} \|f - f_k\|_{\dot{B}_{p,q}^s} = 0.$$

For the proof, we refer to Bahouri–Chemin–Danchin.³, Proposition 2.27

Proof of Corollary 3.9. In the following, we assume that $j \geq 1$ holds instead of the condition $|\alpha| + j \geq 1$. Hence we see by [Lemma 4.1](#) and [Theorem 3.1](#) that

$$\begin{aligned} \|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^s} &= \|\Delta \partial_x^\alpha \partial_t^{j-1} T(t)f\|_{\dot{B}_{p,1}^s} \leq C_n \|\partial_x^\alpha \partial_t^{j-1} T(t)f\|_{\dot{B}_{p,1}^{s+2}} \\ &\leq K t^{-\frac{1}{2}(2+|\alpha|+2j-2)} \|f\|_{\dot{B}_{p,\infty}^s} = K t^{-\frac{|\alpha|}{2}-j} \|f\|_{\dot{B}_{p,\infty}^s} \end{aligned} \quad (7.1)$$

for all $f \in \dot{B}_{p,\infty}^s$, which yields

$$\|\partial_x^\alpha \partial_t^j T(t)\|_{\mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s)} = O\left(t^{-\frac{|\alpha|}{2}-j}\right) \quad \text{as } t \rightarrow +0 \text{ or } t \rightarrow \infty.$$

In particular,

$$\|\partial_t T(t)f\|_{\dot{B}_{p,1}^s} \leq K t^{-1} \|f\|_{\dot{B}_{p,\infty}^s} \quad (7.2)$$

holds. Besides, for each $t > 0$, we take $h \in \mathbb{R}$ satisfying $|h| < t/4$. Then, it follows that

$$\begin{aligned} &(\partial_t^j T)(t+h)f - \partial_t^j T(t)f \\ &= \Delta^j T(t/4)T(t/2+h)T(t/4)f - \Delta^j T(t/4)T(t/2)T(t/4)f \\ &= \Delta^{j-1} T(t/4)T(t/2+h)\Delta T(t/4)f - \Delta^{j-1} T(t/4)T(t/2)\Delta T(t/4)f \\ &= (\partial_t^{j-1} T)(t/4)\{T(t/2+h)(\partial_t T)(t/4)f - T(t/2)(\partial_t T)(t/4)f\}. \end{aligned}$$

Since it holds by $(\partial_t T)(t/4)f \in \dot{B}_{p,1}^s$ that

$$\begin{aligned} &T(t/2+h)(\partial_t T)(t/4)f - T(t/2)(\partial_t T)(t/4)f \\ &= \int_{t/2}^{t/2+h} (\partial_t T)(\tau)(\partial_t T)(t/4)f \, d\tau \quad \text{in } \dot{B}_{p,1}^s, \end{aligned}$$

we see by (7.2) that

$$\begin{aligned} &\|T(t/2+h)(\partial_t T)(t/4)f - T(t/2)(\partial_t T)(t/4)f\|_{\dot{B}_{p,1}^s} \\ &\leq \int_{t/2}^{t/2+h} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^s)} \|(\partial_t T)(t/4)f\|_{\dot{B}_{p,1}^s} \, d\tau \\ &\leq \sup_{t/2-|h| \leq \tau \leq t/2+|h|} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^s)} \int_{t/2}^{t/2+h} K(t/4)^{-1} \|f\|_{\dot{B}_{p,\infty}^s} \, d\tau \\ &\leq K(t/4)^{-1} \sup_{t/4 \leq \tau \leq 3t/4} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^s)} \|f\|_{\dot{B}_{p,\infty}^s} h. \end{aligned}$$

Hence, by (7.1), we have

$$\begin{aligned} &\|\partial_x^\alpha (\partial_t^j T)(t+h)f - \partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^s} \\ &= \|\partial_x^\alpha (\partial_t^{j-1} T)(t/4)\{T(t/2+h)(\partial_t T)(t/4)f - T(t/2)(\partial_t T)(t/4)f\}\|_{\dot{B}_{p,1}^s} \\ &\leq K(t/4)^{-\frac{|\alpha|}{2}-j+1} \|T(t/2+h)(\partial_t T)(t/4)f - T(t/2)(\partial_t T)(t/4)f\|_{\dot{B}_{p,\infty}^s} \\ &\leq K(t/4)^{-\frac{|\alpha|}{2}-j} \sup_{t/4 \leq \tau \leq 3t/4} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^s)} \|f\|_{\dot{B}_{p,\infty}^s} h, \end{aligned}$$

which implies that

$$\lim_{h \rightarrow 0} \|\partial_x^\alpha (\partial_t^j T)(t+h) - \partial_x^\alpha \partial_t^j T(t)\|_{\mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s)} = 0$$

holds and we see that $\partial_x^\alpha \partial_t^j T(t) \in C((0, \infty); \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s))$ holds. By (7.1), We also have

$$\sup_{\varepsilon < t < \infty} \|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^s} \leq K \varepsilon^{-\frac{|\alpha|}{2}-j} \|f\|_{\dot{B}_{p,\infty}^s}$$

for all $\varepsilon > 0$. Hence we obtain $\partial_x^\alpha \partial_t^j T(t) \in BUC((\varepsilon, \infty); \mathcal{L}(\dot{B}_{p,\infty}^s, \dot{B}_{p,1}^s))$. In case $|\alpha| \geq 1$ with $j = 0$, we can show as well. This completes the proof of [Corollary 3.9](#). \square

Proof of Corollary 3.10. We set $u := T(t)f$. Since the heat semigroup $\{T(t)\}_{t \geq 0}$ on $\dot{B}_{p,q}^s$ is a bounded C_0 -semigroup under the conditions $1 \leq p < \infty$ and $1 \leq q < \infty$, we have $u \in BC([0, \infty); \dot{B}_{p,q}^s)$. If $j \geq 1$, we see by [Corollary 3.9](#) that

$$\partial_t^j u \in BUC((\varepsilon, \infty); \dot{B}_{p,1}^s) \subset BUC((\varepsilon, \infty); \dot{B}_{p,q}^s)$$

holds for all $\varepsilon > 0$, which yields

$$u \in BUC([0, \infty); \dot{B}_{p,q}^s) \cap BUC^\infty((\varepsilon, \infty); \dot{B}_{p,q}^s).$$

In the same way as in [Corollary 3.9](#), it holds by [Theorem 3.1](#) that

$$\begin{aligned} &\lim_{h \rightarrow 0} \|\partial_x^\alpha (\partial_t^j u)(t+h, \cdot) - \partial_x^\alpha \partial_t^j u(t, \cdot)\|_{\dot{B}_{p,1}^{s+\sigma}} \\ &= \lim_{h \rightarrow 0} \|\partial_x^\alpha (\partial_t^j T)(t+h)f - \partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{s+\sigma}} \\ &= \lim_{h \rightarrow 0} \|\partial_x^\alpha (\partial_t^j T)(t/2)\{T(t/2+h)f - T(t/2)f\}\|_{\dot{B}_{p,1}^{s+\sigma}} \\ &\leq \lim_{h \rightarrow 0} K(t/2)^{-\frac{1}{2}(\sigma+|\alpha|+2j)} \|T(t/2+h)f - T(t/2)f\|_{\dot{B}_{p,q}^s} = 0 \end{aligned}$$

and

$$\sup_{\varepsilon < t < \infty} \|\partial_x^\alpha \partial_t^j u\|_{\dot{B}_{p,1}^{s+\sigma}} \leq K \varepsilon^{-\frac{1}{2}(\sigma+|\alpha|+2j)} \|f\|_{\dot{B}_{p,q}^s}$$

holds for all $\varepsilon > 0$. Hence we see that $u \in BUC^\infty((\varepsilon, \infty); \dot{B}_{p,1}^{s+\sigma})$ holds. By combining the result, [Theorems 3.3](#) and [3.6](#), we obtain

$$u \in BUC([0, \infty); \dot{B}_{p,q}^s) \cap BUC^\omega((\varepsilon, \infty); \dot{B}_{p,q}^s \cap \dot{B}_{p,1}^{s+\sigma}) \cap C^\omega([0, \infty) \times \mathbb{R}^n).$$

This completes the proof of [Corollary 3.10](#). \square

Proof of Corollary 3.11. Assume that $f \in \dot{B}_{p,q}^s$ with $s < n/p$. Then, by [Theorem 3.1](#), we have

$$\|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{n/p}} = \|\partial_x^\alpha \partial_t^j T(t)f\|_{\dot{B}_{p,1}^{s+(n/p-s)}} \leq K t^{-\frac{1}{2}\left(\frac{n}{p}+|\alpha|+2j-s\right)} \|f\|_{\dot{B}_{p,q}^s} \quad (7.3)$$

for all $\alpha \in \mathbb{N}_0^n$ and $j \in \mathbb{N}_0$. Since $\dot{B}_{p,1}^{n/p} \subset BUC$ holds, it follows from (7.3) that

$$\sup_{0 < t < \infty} t^{\frac{1}{2}\left(\frac{n}{p}+|\alpha|+2j-s\right)} \|\partial_x^\alpha \partial_t^j T(t)f\|_{L^\infty} \leq K \|f\|_{\dot{B}_{p,q}^s}, \quad (7.4)$$

which implies that

$$\|\partial_x^\alpha \partial_t^j T(t)\|_{\mathcal{L}(\dot{B}_{p,q}^s, L^\infty)} = O\left(t^{-\frac{1}{2}\left(\frac{n}{p}+|\alpha|+2j-s\right)}\right) \quad \text{as } t \rightarrow +0 \text{ or } t \rightarrow \infty.$$

Hence we see that $\partial_x^\alpha T(t)f \in BUC$ holds for all $t > 0$, which yields $T(t)f \in BUC^\infty$. Furthermore, taking $s_0 \in \mathbb{R}$ such that $s < s_0 < n/p$, we see by [Theorem 3.1](#) that $T(t/4)f \in \dot{B}_{p,1}^{s_0}$ holds and $\partial_x^\alpha \partial_t^j T(t)g \in L^\infty$

holds for all $g \in \dot{B}_{p,1}^{s_0}$ with the estimates

$$\begin{cases} \|T(t/4)f\|_{\dot{B}_{p,1}^{s_0}} \leq K\|f\|_{\dot{B}_{p,q}^s}, \\ \|\partial_x^\alpha \partial_t^j T(t)g\|_{L^\infty} \leq t^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} K\|g\|_{\dot{B}_{p,1}^{s_0}}. \end{cases} \quad (7.5)$$

Hence, in the same way as in Corollary 3.9, it holds by (7.5) that

$$\begin{aligned} & \lim_{h \rightarrow 0} \sum_{|\alpha| \leq k} \|\partial_x^\alpha (\partial_t^j T)(t+h)f - \partial_x^\alpha \partial_t^j T(t)f\|_{L^\infty} \\ &= \lim_{h \rightarrow 0} \sum_{|\alpha| \leq k} \|\partial_x^\alpha (\partial_t^j T)(t/4)\{T(t/2+h)T(t/4)f - T(t/2)T(t/4)f\}\|_{L^\infty} \\ &\leq \lim_{h \rightarrow 0} \sum_{|\alpha| \leq k} K(t/4)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \\ &\quad \times \|T(t/2+h)T(t/4)f - T(t/2)T(t/4)f\|_{\dot{B}_{p,1}^{s_0}} \\ &\leq \lim_{h \rightarrow 0} \sum_{|\alpha| \leq k} K(t/4)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \\ &\quad \times \sup_{t/4 \leq \tau \leq 3t/4} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^{s_0})} \|T(t/4)f\|_{\dot{B}_{p,1}^{s_0}} h \\ &\leq \lim_{h \rightarrow 0} \sum_{|\alpha| \leq k} K(t/4)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \sup_{t/4 \leq \tau \leq 3t/4} \|\partial_t T(\tau)\|_{\mathcal{L}(\dot{B}_{p,1}^{s_0})} \|f\|_{\dot{B}_{p,q}^s} h = 0 \end{aligned}$$

and we see by (7.4) that

$$\sup_{\varepsilon < t < \infty} \sum_{|\alpha| \leq k} \|\partial_x^\alpha \partial_t^j T(t)f\|_{L^\infty} \leq \sum_{|\alpha| \leq k} K\varepsilon^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \|f\|_{\dot{B}_{p,q}^s} < \infty$$

holds for all $\varepsilon > 0$. Hence we have $T(t)f \in BUC^\omega((\varepsilon, \infty); BUC^\infty)$. By Theorem 3.6, the following relation

$$\partial_x^\alpha T(t)f \in C^\omega((0, \infty); \dot{B}_{p,1}^{n/p}) \subset C^\omega((0, \infty); BUC)$$

holds, which yields $T(t)f \in BUC^\omega((\varepsilon, \infty); BUC^\infty)$.

Next, assume that $1 \leq p < \infty$. First we note that $T(t)f \in BUC$ holds in the usual sense, under the condition $t > 0$. Hence the value $T(t)f(x) \in \mathbb{R}$ is well defined for each $(t, x) \in (0, \infty) \times \mathbb{R}^n$. Since we see by Proposition 7.1 that $\mathcal{S}_0 = \dot{B}_{p,1}^{s_0}$ holds, we may take a sequence $\{g_k\}_{k=1}^\infty \subset \mathcal{S}_0$ of function satisfying

$$\lim_{k \rightarrow \infty} \|T(t/2)f - g_k\|_{\dot{B}_{p,1}^{s_0}} = 0.$$

Notice that

$$\begin{aligned} \partial_x^\alpha \partial_t^j T(t)f(x) &= \partial_x^\alpha \Delta^j T(t)f(x) \\ &= \partial_x^\alpha \Delta^j T(t/2)\{T(t/2)f(x) - g_k(x)\} + \partial_x^\alpha \Delta^j T(t/2)g_k(x) \\ &= \partial_x^\alpha (\partial_t^j T)(t/2)\{T(t/2)f(x) - g_k(x)\} + \partial_x^\alpha (\partial_t^j T)(t/2)g_k(x) \end{aligned}$$

holds for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$. Since it holds by (7.5) that

$$\begin{aligned} \|\partial_x^\alpha (\partial_t^j T)(t/2)\{T(t/2)f - g_k\}\|_{L^\infty} &\leq K(t/2)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \\ &\quad \times \|T(t/2)f - g_k\|_{\dot{B}_{p,1}^{s_0}}, \end{aligned}$$

we have

$$\begin{aligned} |\partial_x^\alpha \partial_t^j T(t)f(x)| &\leq \|\partial_x^\alpha (\partial_t^j T)(t/2)\{T(t/2)f - g_k\}\|_{L^\infty} + |\partial_x^\alpha (\partial_t^j T)(t/2)g_k(x)| \\ &\leq K(t/2)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \|T(t/2)f - g_k\|_{\dot{B}_{p,1}^{s_0}} \\ &\quad + |\partial_x^\alpha (\partial_t^j T)(t/2)g_k(x)|. \end{aligned}$$

Hence we see by $g_k \in \mathcal{S}_0$ that

$$\begin{aligned} & \limsup_{|x| \rightarrow \infty} |\partial_x^\alpha \partial_t^j T(t)f(x)| \\ &\leq K(t/2)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \|T(t/2)f - g_k\|_{\dot{B}_{p,1}^{s_0}} + \limsup_{|x| \rightarrow \infty} |\partial_x^\alpha (\partial_t^j T)(t/2)g_k(x)| \\ &= K(t/2)^{-\frac{1}{2}(\frac{n}{p}+|\alpha|+2j-s_0)} \|T(t/2)f - g_k\|_{\dot{B}_{p,1}^{s_0}} \rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

For the case of $s = n/p$ with $q = 1$, since we see by Lemma 4.1 that $\partial_x^\alpha \partial_t^j T(t)f \in \dot{B}_{p,1}^{n/p-|\alpha|-2j}$ holds for all $f \in \dot{B}_{p,1}^{n/p}$, we can prove as in case $s < n/p$ under the condition $|\alpha| + j \geq 1$. If $|\alpha| = 0$ and $j = 0$,

note that $\|T(t)f\|_{\dot{B}_{p,1}^{n/p}} \leq \|f\|_{\dot{B}_{p,1}^{n/p}}$ and $\dot{B}_{p,1}^{n/p} \subset BUC$ hold. This proves Corollary 3.11. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

In this appendix, we give the proof of Propositions 6.1 and 6.2. Namely, we show that the heat semigroup $\{T(t)\}_{t \geq 0}$ on homogeneous Besov spaces $\dot{B}_{p,q}^s$ is a bounded C_0 -semi which has differentiability on $t > 0$, and we also show that the function $T(t)f$ has differentiability on $t > 0$ with respect to the norm on $\dot{B}_{p,1}^{s+\sigma}$ for all $f \in \dot{B}_{p,\infty}^s$.

Proof of Proposition 6.1. (i) First we note that for $1 \leq p < \infty$, the heat semigroup $\{T(t)\}_{t \geq 0}$ on L^p satisfies the following relations

$$\begin{cases} \|T(t)f\|_{L^p} \leq \|f\|_{L^p}, \quad T(t+\tau)f = T(t)T(\tau)f, \\ \lim_{h \rightarrow +0} \|T(h)f - f\|_{L^p} = 0, \quad \partial_t^j T(t)f = \Delta^j T(t)f \quad \text{in } L^p \end{cases} \quad (\text{A.1})$$

for all $t, \tau > 0, j \in \mathbb{N}$ and $f \in L^p$. Hence, for the C_0 -property, we see by (A.1) that

$$\begin{aligned} \lim_{h \rightarrow +0} \|T(h)f - f\|_{\dot{B}_{p,q}^s}^q &= \lim_{h \rightarrow +0} \left(\sum_{|j| \geq N+1} + \sum_{|j| \leq N} \right) \\ &\quad \times (2^{sj} \|T(h)(\varphi_j * f) - \varphi_j * f\|_{L^p})^q \\ &\leq \sum_{|j| \geq N+1} (2^{sj} \cdot 2 \|\varphi_j * f\|_{L^p})^q \rightarrow 0 \quad (N \rightarrow \infty) \end{aligned}$$

for all $f \in \dot{B}_{p,q}^s$. For differentiability of $T(t)$ on $t > 0$, first we note that $\Delta^j T(t)f \in \dot{B}_{p,1}^s$ holds for all $f \in \dot{B}_{p,q}^s, t > 0$ and $j \in \mathbb{N}$. In fact, it holds by Lemma 5.1 that $T(t)f \in \dot{B}_{p,1}^{s+2j}$ and we see by Lemma 5.1 that $\Delta^j T(t)f \in \dot{B}_{p,1}^s$. Besides, since the relation (A.1) holds, we have

$$\begin{aligned} T(t+h)g - T(t)g &= T(t/2+h)T(t/2)g - T(t/2)T(t/2)g \\ &= \int_{t/2}^{t/2+h} (\partial_t T)(\tau)T(t/2)g d\tau \quad \text{in } L^p \end{aligned}$$

for all $g \in L^p$, which yields

$$\begin{aligned} \|T(t+h)g - T(t)g\|_{L^p} &\leq \int_{t/2}^{t/2+h} \|\partial_t T(\tau)\|_{\mathcal{L}(L^p)} \|T(t/2)g\|_{L^p} d\tau \\ &\leq \sup_{t/2-|h| \leq \tau \leq t/2+|h|} \|\partial_t T(\tau)\|_{\mathcal{L}(L^p)} \int_{t/2}^{t/2+h} \|g\|_{L^p} d\tau \\ &\leq \sup_{t/4 \leq \tau \leq (3t)/4} \|\partial_t T(\tau)\|_{\mathcal{L}(L^p)} \|g\|_{L^p} h \end{aligned}$$

for $|h| \leq t/4$. By setting the constant $M_t := \sup_{t/4 \leq \tau \leq (3t)/4} \|\partial_t T(\tau)\|_{\mathcal{L}(L^p)} < \infty$ depending only on t and p , it holds that

$$\begin{aligned} \left\| \frac{T(t+h)g - T(t)g}{h} - \Delta T(t)g \right\|_{L^p} &\leq \left\| \frac{T(t+h)g - T(t)g}{h} \right\|_{L^p} + \|\Delta T(t)g\|_{L^p} \\ &\leq \sup_{t/4 \leq \tau \leq (3t)/4} \|\partial_t T(\tau)\|_{\mathcal{L}(L^p)} \|g\|_{L^p} \\ &\quad + \|\partial_t T(t)\|_{\mathcal{L}(L^p)} \|g\|_{L^p} \\ &\leq 2M_t \|g\|_{L^p}. \end{aligned}$$

Hence we see by (A.1) that

$$\begin{aligned} & \lim_{h \rightarrow +0} \left\| \frac{T(t+h)f - T(t)f}{h} - \Delta T(t)f \right\|_{\dot{B}_{p,q}^s}^q \\ &= \lim_{h \rightarrow +0} \left(\sum_{|j| \geq N+1} + \sum_{|j| \leq N} \right) \\ & \quad \times \left(2^{sj} \left\| \frac{T(t+h)(\varphi_j * f) - T(t)(\varphi_j * f)}{h} - \Delta T(t)(\varphi_j * f) \right\|_{L^p} \right)^q \\ & \leq \sum_{|j| \geq N+1} (2^{sj} \cdot 2M_t \|\varphi_j * f\|_{L^p})^q \rightarrow 0 \quad (N \rightarrow \infty), \end{aligned}$$

which implies that

$$\partial_t T(t)f = \Delta T(t)f \quad \text{in } \dot{B}_{p,q}^s$$

holds. Hence, we see by induction that (6.1) holds.

(ii) By the definition (2.5) of $\dot{B}_{p,\infty}^s$, it holds that

$$\lim_{j \rightarrow \pm\infty} 2^{sj} \|\varphi_j * f\|_{L^p} = 0$$

for all $f \in \dot{B}_{p,\infty}^s$. Hence, in the similar manner as in (i), we see that

$$\begin{aligned} \lim_{h \rightarrow +0} \|T(h)f - f\|_{\dot{B}_{p,\infty}^s} & \leq \lim_{h \rightarrow +0} \left(\sup_{|j| \geq N+1} + \sup_{|j| \leq N} \right) \\ & \quad \times \{2^{sj} \|T(h)(\varphi_j * f) - \varphi_j * f\|_{L^p}\} \\ & \leq \sup_{|j| \geq N+1} \{2^{sj} \cdot 2\|\varphi_j * f\|_{L^p}\} \rightarrow 0 \quad (N \rightarrow \infty) \end{aligned}$$

and

$$\begin{aligned} & \lim_{h \rightarrow +0} \left\| \frac{T(t+h)f - T(t)f}{h} - \Delta T(t)f \right\|_{\dot{B}_{p,\infty}^s} \\ & \leq \lim_{h \rightarrow +0} \left(\sup_{|j| \geq N+1} + \sup_{|j| \leq N} \right) \\ & \quad \times \left\{ 2^{sj} \left\| \frac{T(t+h)(\varphi_j * f) - T(t)(\varphi_j * f)}{h} - \Delta T(t)(\varphi_j * f) \right\|_{L^p} \right\} \\ & \leq \sup_{|j| \geq N+1} \{2^{sj} \cdot 2M_t \|\varphi_j * f\|_{L^p}\} \rightarrow 0 \quad (N \rightarrow \infty). \end{aligned}$$

(iii) Note that $T(t)f \in BUC$ holds for all $f \in L^\infty$ and $t > 0$. Since the heat semigroup $\{T(t)\}_{t \geq 0}$ on BUC is a bounded analytic C_0 -semigroup, we see that

$$\begin{cases} \|T(t)f\|_{L^\infty} \leq \|f\|_{L^\infty}, & T(t+\tau)f = T(t)T(\tau)f, \\ \lim_{h \rightarrow +0} \|T(t+h)f - T(t)f\|_{L^\infty} = 0, & \partial_t^j T(t)f = \Delta^j T(t)f \quad \text{in } L^\infty \end{cases}$$

for all $t, \tau > 0, j \in \mathbb{N}$ and $f \in L^\infty$. Hence we can show as in (i).

(iv) Combining the result of (ii) and (iii), it can be easily seen. This proves Proposition 6.1. \square

Proof of Proposition 6.2. We see by Lemma 5.1 that $T(t)f \in \dot{B}_{p,1}^{s+\sigma}$ holds for all $f \in \dot{B}_{p,\infty}^s$ and $\sigma > 0$. Hence, by the result of Proposition 6.1, we see that $T(t)f$ satisfies the desired properties as a $\dot{B}_{p,1}^{s+\sigma}$ -valued function. This proves Proposition 6.2. \square

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