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MAXIMAL LORENTZ REGULARITY FOR THE KELLER-SEGEL SYSTEM OF PARABOLIC-ELLIPTIC TYPE

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Abstract. We construct local strong solutions of Keller-Segel system of parabolic-elliptic type for arbitrary initial data in the homogeneous Besov space which is scaling invariant. We also show that the solution exists globally in time for small initial data. The solutions belong to the Lorentz space in time direction since our method relies on the maximal Lorentz regularity of heat equations.

1. Introduction

We consider the Keller-Segel system of parabolic-elliptic type in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & t > 0,\ x \in \mathbb{R}^n, \\ 0 = \Delta v + u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ are the unknown functions standing for the density of amoebae and the concentration of the chemo-attractant, respectively, while $u_0 = u_0(x)$ is the given initial data. This system introduced by Keller-Segel [16] in 1970 is a famous mathematical model of chemotaxis.

In this article, we prove the existence and uniqueness of local strong solutions of (1.1) for initial data $u_0 \in \dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)$ for $1 \leq p < \infty$ and $1 \leq q \leq \infty$ (Notice that we can also construct solutions in the case of $q = \infty$ if high frequency part of the initial data is sufficiently small). Here, $\dot{B}^s_{p,q}(\mathbb{R}^n)$ denotes the homogeneous Besov space, which is defined by Definition 1.1. We also prove the existence and uniqueness of global strong solutions for small initial data $u_0 \in \dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)$. Now we assume that a pair of function (u,v) is the solution of (1.1). Then we see that the function $(u_\lambda(t,x), v_\lambda(t,x)) := (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x))$ is also the solution of (1.1) for all $\lambda > 0$. Notice that the relation

$$\|u_\lambda(0,\cdot)\|_{\dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)} = \|u_0\|_{\dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)}$$

holds for all $\lambda > 0$. Such a Banach space is called a *scaling invariant space*, and it is empirically known that this is a very crucial notion in the theory of PDEs. This argument is called Fujita-Kato principle since they first construct solutions of the Navier-Stokes system for initial data u_0 belonging to the homogeneous Sobolev space $\dot{H}^{1/2,2}(\mathbb{R}^n)$ as the scaling invariant space in their paper [10]. Since then, there have been a lot of researches on construction of solutions for the Navier-Stokes system in other scaling invariant spaces which are wider than $\dot{H}^{1/2,2}(\mathbb{R}^n)$ together with more precise analysis. For such previous works, see, e.g., Kato [15], Giga [11], Giga-Miyakawa [12], Taylor [26], Kozono-Yamazaki [21], Cannone [5] and Koch-Tataru [17].

As for the Keller-Segel system, there also exist various previous works. In particular, Jäger-Luckhaus [14] and Nagai [22] established fundamental works, and gave a several remarkable results. See, also Nagai-Senba [23]. Let us introduce some further contributions to (1.1). Biler [2] first

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constructed self-similar solutions of (1.1), and then Diaz-Nagai-Rakotoson [9] proved existence of local classical solutions of (1.1) for $u_0 \in L^1(\mathbb{R}^n) \cap H^{1,p}(\mathbb{R}^n)$ with $2 \le n < p$. In 2D-case, they also showed that the solution exists globally in time under the crucial hypothesis that the initial mass is less than 8π, i.e., $\int_{\mathbb{R}^2} u_0 dx < 8\pi$. Biler-Cannone-Guerra-Karch [3] constructed global solutions (u, v) of (1.1) in the class

$$u \in C_w([0,\infty); \mathcal{PM}^{n-2})$$

for small $u_0 \in \mathcal{PM}^{n-2}$ with $n \ge 4$, where $\mathcal{PM}^a$ denotes the pseudo measure defined by

$$\mathcal{PM}^a := \left\{ u \in \mathscr{S}' \,\middle|\, \mathcal{F}u \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \quad \|u\|_{\mathcal{PM}^a} := \||\cdot|^a \mathcal{F}u\|_{L^\infty(\mathbb{R}^n)} < \infty \right\}$$

for $a \ge 0$ with C_w denoting the set of weakly-star continuous functions. Corrias-Perthame-Zaag [8] constructed global weak solutions (u, v) of (1.1) satisfying

$$\|u(t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}, \quad \|u(t)\|_{L^p(\mathbb{R}^n)} \le \|u_0\|_{L^p(\mathbb{R}^n)}$$

for all $t > 0$ and $\max\{1, n/2-1\} \le p \le n/2$ if u_0 belongs to $L^1(\mathbb{R}^n) \cap L^{n/2}(\mathbb{R}^n)$ and $\|u_0\|_{L^{n/2}(\mathbb{R}^n)}$ is small with $n \ge 2$. Ogawa-Shimizu [24] treated the end-point case of $p = 1$ in the homogeneous Besov space $\dot{B}^0_{1,2}(\mathbb{R}^2)$ and constructed the local classical solution (u, v) of (1.1) in the class:

$$u \in C([0,T); \dot{B}^0_{1,2}(\mathbb{R}^2)) \cap C((0,T); \dot{B}^2_{1,2}(\mathbb{R}^2)) \cap C^1((0,T); \dot{B}^0_{1,2}(\mathbb{R}^2)) \cap L^2((0,T); \dot{B}^1_{1,2}(\mathbb{R}^2)).$$

Furthermore, if the initial data $u_0 \in \dot{B}^0_{1,2}(\mathbb{R}^2)$ is small, their solution is a global one, i.e., $T = \infty$. Based on the time-weighted homogeneous Sobolev space originally introduced by Fujita-Kato [10], Iwabuchi [13] considered the scaling invariant space equipped the norm given by

$$\begin{cases} \sup\limits_{t>0} \|u(t)\|_{\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)} + \sup\limits_{t>0} t^{1-\frac{n}{2q}} \|u(t)\|_{L^q(\mathbb{R}^n)} & \\ \qquad\qquad + \sup\limits_{t>0} t^{\frac{1}{2}-\frac{n}{2r}} \|(-\Delta)^{-\frac{1}{2}} u(t)\|_{L^r(\mathbb{R}^n)} < \infty & \text{if } p \le n, \\ \sup\limits_{t>0} \|u(t)\|_{\dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)} + \sup\limits_{t>0} t^{\frac{1}{2}-\frac{n}{2\rho}} \|u(t)\|_{\dot{B}^{-1}_{\rho,2}(\mathbb{R}^n)} < \infty & \text{if } n < p < \infty \end{cases}$$

with $1 < p < \infty$ and $n/2 \le p$, where q, r and ρ are arbitrary with some conditions, and constructed a global mild solution for small initial data $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$.

Our method is based on the maximal Lorentz regularity for the linear heat equation introduced by Kozono-Shimizu [20]. Indeed, we construct the global strong solution (u, v) of (1.1) in the class

$$\partial_t u, \Delta u \in L^{\alpha,q}((0,\infty); \dot{B}^s_{r,1}(\mathbb{R}^n))$$

with the initial data $u_0 \in \dot{B}^{-2+n/p}_{p,q}(\mathbb{R}^n)$ for $2/\alpha + n/r = 4 + s$ and $1 \le q \le \infty$. It should be noted that the third exponent q in the homogeneous Besov space coincides with the second exponent of the Lorentz space in the time direction. Hence our result covers the result based on the usual L^α maximal regularity. On the one hand, Ogawa-Shimizu [24] treated a marginal case $\dot{B}^0_{1,2}(\mathbb{R}^2)$ of the end-point in maximal regularity, on the other hand they are necessary restricted to $q = 2$. Our method makes it free to handle arbitrary exponent $1 \le q \le \infty$ including $q = \infty$. Notice that under some additional condition for the initial data $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}(\mathbb{R}^n)$, we are able to construct a local strong solution, unlike Iwabuchi [13] consider small initial data for $q = \infty$. In comparison with the iteration procedure for construction of mild solutions, the maximal regularity has an advantage for giving a straightforward method to that of strong solutions with higher regularity.

Before we state our main result, let us define some notations and function spaces. Let $\mathcal{F}$ denote the Fourier transform and let $(-\Delta)^{\frac{1}{2}s} := \mathcal{F}^{-1}|\cdot|^s \mathcal{F}$ for $s \in \mathbb{R}$. We also define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as follows.

Definition 1.1. We take a function $\varphi \in \mathscr{S}$ satisfying

$$\operatorname{supp}\varphi = \{\xi \in \mathbb{R}^n \,|\, 1/2 \le |\xi| \le 2\}, \quad \varphi(\xi) > 0 \quad \text{for } 1/2 < |\xi| < 2,$$
$$\sum_{j=-\infty}^{\infty} \varphi(\xi/2^j) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\mathscr{S}$ is the Schwartz spaces in $\mathbb{R}^n$. We define the sequence of function $\{\varphi_j\}_{j\in\mathbb{Z}}$ by

$$\varphi_j := \mathcal{F}^{-1}\varphi(\xi/2^j) \tag{1.2}$$

for $j \in \mathbb{Z}$. Then, for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ by

$$\dot{B}^s_{p,q}(\mathbb{R}^n) := \{f \in \mathscr{S}'/\mathscr{P} \,|\, \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left\{ \displaystyle\sum_{j=-\infty}^{\infty} \left(2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\right)^q \right\}^{\frac{1}{q}} & 1 \le q < \infty, \\ \displaystyle\sup_{j\in\mathbb{Z}} \left\{2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\right\} & q = \infty, \end{cases}$$

where $\mathscr{S}'$ and $\mathscr{P}$ are the sets of all tempered distributions and polynomials on $\mathbb{R}^n$, respectively.

We will abbreviate $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as $\dot{B}^s_{p,q}$. For $1 \le p, q \le \infty$, an interval $I \subset \mathbb{R}$ and a Banach space X, the X-valued Lebesgue space and Lorentz space on I are denoted by $L^p(I;X)$ and $L^{p,q}(I;X)$, respectively. We will write

$$\|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);X)} := \|\partial_t u\|_{L^{\alpha,q}((0,T);X)} + \|\Delta u\|_{L^{\alpha,q}((0,T);X)}$$

to simplify the notation.

In the following, we consider the single form;

$$\begin{cases} \partial_t u - \Delta u = -\nabla\cdot(u\nabla(-\Delta)^{-1}u) & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \qquad u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

by eliminating the function v from (1.1) formally. Our main result reads as follows.

Theorem 1.2. *(i) Let $1 \le p < \infty$ and $1 \le q < \infty$. For every $u_0 \in \dot{B}^{-2+n/p}_{p,q}$, there exist $0 < T \le \infty$ and a unique solution u on $(0,T)\times\mathbb{R}^n$ of (1.3) in the class*

$$\begin{gathered} \partial_t u, \Delta u \in L^{\alpha,q}((0,T);\dot{B}^s_{r,1}) \\ \text{for some } 1 < r < \infty,\ s \in \mathbb{R},\ 1 < \alpha < \infty \text{ such that } \frac{2}{\alpha} + \frac{n}{r} = 4 + s. \end{gathered} \tag{1.4}$$

In addition, it holds that

$$\begin{gathered} u \in L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1}) \\ \text{for some } r \le r^* < \infty,\ s^* \in \mathbb{R},\ \alpha < \alpha^* < \infty \text{ satisfying } \frac{2}{\alpha^*} + \frac{n}{r^*} = 2 + s^*. \end{gathered} \tag{1.5}$$

(ii) Let $n/2 < p < \infty$ and $\{\varphi_j\}_{j\in\mathbb{Z}}$ be defined by (1.2). There exists $\varepsilon = \varepsilon(n,p) > 0$ such that if $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}$ satisfies

$$\sup_{N\le j<\infty} 2^{\left(-2+\frac{n}{p}\right)j}\|\varphi_j * u_0\|_{L^p} \le \varepsilon \tag{1.6}$$

for some $N \in \mathbb{Z}$, then there exist $0 < T \le \infty$ and a solution u on $(0,T) \times \mathbb{R}^n$ of (1.3) satisfying (1.4) and (1.5) with q replaced by ∞.

Concerning the uniqueness, there exists a constant $\eta = \eta(n,p,r,s) > 0$ such that if the initial data u_0 and the solution u of (1.3) in the class (1.4) satisfy

$$\sup_{N \le j < \infty} 2^{\left(-2+\frac{n}{p}\right)j} \|\varphi_j * u_0\|_{L^p} + \limsup_{t\to\infty} \left\{ t\mu\left(\tau \in (0,T) \;\middle|\; \|\partial_t u(\tau), \Delta u(\tau)\|_{\dot{B}^s_{r,1}} > t\right)^{\frac{1}{\alpha}} \right\} \le \eta \tag{1.7}$$

for some $N \in \mathbb{Z}$, then u is unique.

(iii) Let $1 \le p < \infty$ and $1 \le q \le \infty$. There exists $\varepsilon_* = \varepsilon_*(n,p,q) > 0$ such that if $u_0 \in \dot{B}^{-2+n/p}_{p,q}$ satisfies

$$\|u_0\|_{\dot{B}^{-2+n/p}_{p,q}} \le \varepsilon_*, \tag{1.8}$$

there exists a solution u on $(0,\infty) \times \mathbb{R}^n$ of (1.3) satisfying (1.4) and (1.5) with $T = \infty$.

Concerning the uniqueness, in case $1 \le q < \infty$, u is unique in the class (1.4). In case $q = \infty$, u is unique in the class (1.4) under the condition (1.7) provided $n/2 < p < \infty$.

Remark 1.3. (i) The classes (1.4) and (1.5) are scaling invariant spaces to (1.1). In fact, we see that the following relations

$$\|\partial_t u_\lambda, \Delta u_\lambda\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} = \|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}, \quad \|u_\lambda\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} = \|u\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})}$$

are valid for all $\lambda > 0$, where $u_\lambda(t,x) := \lambda^2 u(\lambda^2 t, \lambda x)$.

(ii) By Theorem 1.2, we see that the effect of the third index q of the homogeneous Besov space $\dot{B}^{-2+n/p}_{p,q}$ of initial value appears in the solution class $L^{\alpha,q}((0,T);\dot{B}^s_{r,1})$ in time direction. Notice that the following elementary embedding in the homogeneous Besov spaces and Lorentz spaces

$$\dot{B}^{-2+n/p}_{p,q} \subset \dot{B}^{-2+n/p}_{p,q^*}, \quad L^{\alpha,q} \subset L^{\alpha,q^*} \quad \text{for } 1 \le q \le q^* \le \infty$$

holds. This may be regarded as a significant difference between Iwabuchi [13] treating $q = \infty$. Furthermore, as for regularity of solutions in the space variables, we may take $q = 1$ which may be regarded as gain of regularity of solutions compared with the initial data in $\dot{B}^{-2+n/p}_{p,q}$.

(iii) The argument of Theorem 1.2 (ii) is the result of (i) in case $q = \infty$. Notice that the condition $n/2 < p$ is required since it is essential that the derivative of homogeneous Besov space for initial data $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}$ should be negative. Notice that the condition (1.6) states that the high frequency part of $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}$ is sufficiently small. In particular, for $n/2 < p < \infty$, every $u_0 \in L^{n/2}(\mathbb{R}^n)$ necessary satisfies (1.6), since

$$\limsup_{j\to\infty} 2^{\left(-2+\frac{n}{p}\right)j} \|\varphi_j * u_0\|_{L^p} = 0.$$

(iv) Notice that the relation

$$\|\partial_t u, \Delta u\|_{L^{\alpha,\infty}((0,T);\dot{B}^s_{r,1})} = \sup_{t>0} \left\{ t\mu\left(\tau \in (0,T) \;\middle|\; \|\partial_t u(\tau), \Delta u(\tau)\|_{\dot{B}^s_{r,1}} > t\right)^{\frac{1}{\alpha}} \right\} \tag{1.9}$$

holds since the Lorentz space $L^{\alpha,\infty}$ is equivalent to the weak L^α space. Therefore, in case $q = \infty$, the conditions (1.7) states that the local norm of (1.9) is small enough. Notice that if $\partial_t u$ and Δu belong to $L^\alpha((0,T);\dot{B}^s_{r,1})$, it holds necessary that

$$\limsup_{t\to\infty} \left\{ t\mu\left(\tau \in (0,T) \;\middle|\; \|\partial_t u(\tau), \Delta u(\tau)\|_{\dot{B}^s_{r,1}} > t\right)^{\frac{1}{\alpha}} \right\} = 0.$$

(v) As is mentioned above, Theorem 1.2 is based on the maximal Lorentz regularity theorem by Kozono-Shimizu [20]. This theorem stems from the real interpolation theorem, so we have to exclude the case $\alpha = 1$ in Theorem 1.2.

Remark 1.4. Besides various existence and uniqueness of strong and mild solutions, there are several interesting results on blow-up phenomena of solutions to (1.1). See, for instance, Blanchet-Carrillo-Masmoudi [4], Cieślak-Winkler [6] and Conca-Espejo-Vilches [7].

2. Preliminary estimates for the nonlinear term

First, let us give the maximal Lorentz regularity theorem for heat equations introduced by Kozono-Shimizu [20]. Notice that some results in this section are shown in our previous paper. Now we define the function spaces $\mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$ by

$$\mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta}) := \{u : (0,T) \to \dot{B}^s_{r,\beta} \,|\, \partial_t u, \Delta u \in L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta}),\ u(0) = 0\},$$

where $1 < \alpha, r < \infty$, $1 \le \beta, q \le \infty$ and $s \in \mathbb{R}$. Here, $\mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta})} := \|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})} \quad \text{for } u \in \mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta}).$$

Proposition 2.1. *(i) Let $\beta \ge 0$, $\sigma < s + 2\beta$ and $1 \le p \le r \le \infty$. For every $a \in \dot{B}^\sigma_{p,\infty}$, it holds that $e^{t\Delta}a \in \dot{B}^{s+2\beta}_{r,1}$ with the estimate*

$$\|(-\Delta)^\beta e^{t\Delta} a\|_{\dot{B}^s_{r,1}} \le C t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{2}(s+2\beta-\sigma)} \|a\|_{\dot{B}^\sigma_{p,\infty}}, \quad 0 < t < \infty$$

where $C = C(n,p,r,\sigma,s,\beta) > 0$.

(ii) Let $\beta \ge 0$, $\sigma < s + 2\beta$, $1 \le p \le r \le \infty$, $1 \le q \le \infty$ and $1 < \alpha < \infty$ satisfy

$$\frac{1}{\alpha} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2}(s + 2\beta - \sigma).$$

For every $a \in \dot{B}^\sigma_{p,q}$, it holds that $e^{t\Delta}a \in L^{\alpha,q}((0,\infty);\dot{B}^{s+2\beta}_{r,1})$ with the estimate

$$\|(-\Delta)^\beta e^{t\Delta} a\|_{L^{\alpha,q}((0,\infty);\dot{B}^s_{r,1})} \le C\|a\|_{\dot{B}^\sigma_{p,q}},$$

where $C = C(n,p,r,q,\sigma,s,\beta) > 0$.

For the proof of (i), see Kozono-Ogawa-Taniuchi [18, Lemma 2.2]. For the proof of (ii), we refer to our previous paper [25, Proposition 3.2].

Theorem 2.2. *Let $1 < r < \infty$, $1 \le p \le r$, $s \in \mathbb{R}$, $1 \le q, \beta \le \infty$ and $1 < \alpha < \infty$ satisfy*

$$\frac{1}{p} < \frac{2}{n} + \frac{1}{r}, \quad -4 + \frac{n}{p} < s < -2 + \frac{n}{r}, \quad \frac{2}{\alpha} + \frac{n}{r} = 4 + s. \tag{2.1}$$

For every $u_0 \in \dot{B}^{-2+n/p}_{p,q}$ and $f \in L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$, there exists a unique solution u of

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T) \times \mathbb{R}^n, \\ u(0) = u_0 & \text{in } \mathbb{R}^n \end{cases}$$

in the class

$$\partial_t u, \Delta u \in L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$$

with the estimate

$$\|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})} \le C\left(\|u_0\|_{\dot{B}^{-2+n/p}_{p,q}} + \|f\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})}\right),$$

where $C = C(n,r,p,s,q,\beta) > 0$ is a constant independent of T.

Theorem 2.2 is the maximal Lorentz regularity theorem for heat equations. This theorem is shown by our previous paper [25, Theorem 3.1]. Next, we state a proposition on inequalities for the homogeneous Besov spaces.

Proposition 2.3. *(i) Let $1 \le p_0 \le p_1 \le \infty$, $1 \le q \le \infty$ and $s_0, s_1 \in \mathbb{R}$ satisfy*

$$\frac{1}{p_0} - \frac{s_0}{n} = \frac{1}{p_1} - \frac{s_1}{n}.$$

For every $f \in \dot{B}^{s_0}_{p_0,q}$, it holds that

$$\|f\|_{\dot{B}^{s_1}_{p_1,q}} \le C\|f\|_{\dot{B}^{s_0}_{p_0,q}},$$

where $C = C(n,p_0,p_1,s_0,s_1) > 0$.

(ii) Let $1 \le p, q \le \infty$, $s, \sigma, \mu > 0$ and $1 \le p_0, p_1, r_0, r_1 \le \infty$ satisfy

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{r_0} + \frac{1}{r_1}.$$

For every $f \in \dot{B}^{s+\sigma}_{p_0,q} \cap \dot{B}^{-\mu}_{r_0,\infty}$ and $g \in \dot{B}^{-\sigma}_{p_1,\infty} \cap \dot{B}^{s+\mu}_{r_1,q}$, it holds that $fg \in \dot{B}^s_{p,q}$ with the estimate

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\left(\|f\|_{\dot{B}^{s+\sigma}_{p_0,q}}\|g\|_{\dot{B}^{-\sigma}_{p_1,\infty}} + \|f\|_{\dot{B}^{-\mu}_{r_0,\infty}}\|g\|_{\dot{B}^{s+\mu}_{r_1,q}}\right),$$

where $C = C(n,p_0,p_1,r_0,r_1,s,\sigma,\mu) > 0$.

The first part of Proposition 2.3 is related to the basic theory on the Sobolev embedding. We refer to Bergh-Löfström [1, Theorem 6.5.1] for the proof in the case of the non-homogeneous Besov spaces. The second part of Proposition 2.3 is related to the Leibniz rule in the homogeneous Besov spaces. Kozono-Shimada [19, Lemma 2.1] showed the formula for the Triebel-Lizorkin spaces.

Lemma 2.4. *Let $1 < r < \infty$, $1 \le p \le r$, $s \in \mathbb{R}$, $1 \le q \le \infty$ and $1 < \alpha < \infty$ satisfy (2.1). We have $r \le r^* \le \infty$, $s^* \in \mathbb{R}$ and $\alpha < \alpha^* < \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{r} - \frac{s+4}{n} < \frac{1}{r^*}, \quad -2 + \frac{n}{p} < s^* < s + 2 - n\left(\frac{1}{r} - \frac{1}{r^*}\right), \quad \frac{2}{\alpha^*} + \frac{n}{r^*} = 2 + s^*. \tag{2.2}$$

For every function u on $(0,T) \times \mathbb{R}^n$ such that

$$\partial_t u, \Delta u \in L^{\alpha,q}((0,T);\dot{B}^s_{r,1}) \text{ and } u_0 := u(0) \in \dot{B}^{-2+n/p}_{p,q},$$

it holds that $e^{t\Delta}u_0 \in L^{\alpha^*,q}((0,\infty);\dot{B}^{s^*}_{r^*,1})$ and $u \in L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})$ with the estimates

$$\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,\infty);\dot{B}^{s^*}_{r^*,1})} \le C\|u_0\|_{\dot{B}^{-2+n/p}_{p,q}},$$

$$\|u\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} \le \|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + C\|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})},$$

where $C = C(n,r,p,s,q,r^*,s^*) > 0$ is a constant independent of T.

This lemma states that if both $\partial_t u$ and Δu belong to a Lorentz space, then u belongs to another Lorentz space. For the proof, see our previous paper [25, Lemma 4.3]. To give a condition on the solvability of (1.3), we prepare a basic proposition on exponents.

Proposition 2.5. *Let $1 < r < \infty$.*

(i) There exists $1 \le p \le r$ satisfying

$$\frac{1}{p} < \frac{1}{n} + \frac{1}{r}. \tag{2.3}$$

(ii) There exists $s \in \mathbb{R}$ satisfying

$$-4 + \frac{2n}{p} - \frac{n}{r} < s < -2 + \frac{n}{r}. \tag{2.4}$$

(iii) Let us take $\alpha \in \mathbb{R}$ satisfying

$$\frac{2}{\alpha} + \frac{n}{r} = 4 + s. \tag{2.5}$$

Then $\alpha \in \mathbb{R}$ also satisfies $1 < \alpha < \infty$.

(iv) If $1 < r < \infty$, $1 \le p \le r$, $s \in \mathbb{R}$ and $1 < \alpha < \infty$ satisfy (2.3), (2.4) and (2.5), there exists $s^* \in \mathbb{R}$ satisfying

$$\max\left\{-2 + \frac{n}{p}, \frac{1}{2}s, s+1\right\} < s^* \le \frac{1}{2}s + \frac{n}{2r}.$$

In addition, if we define α^* and r^* in such a way that

$$\alpha^* = 2\alpha, \quad \frac{1}{\alpha} + \frac{n}{r^*} = 2 + s^*, \tag{2.6}$$

it follows that

$$\begin{cases} r \le r^* < \infty, \quad \dfrac{1}{p} + \dfrac{1}{r} - \dfrac{s+4}{n} < \dfrac{1}{r^*}, \\ \max\left\{-2 + \dfrac{n}{p}, \dfrac{1}{2}s, s+1\right\} < s^* < s + 2 - n\left(\dfrac{1}{r} - \dfrac{1}{r^*}\right), \\ \alpha < \alpha^* < \infty, \quad \dfrac{2}{\alpha^*} + \dfrac{n}{r^*} = 2 + s^*. \end{cases} \tag{2.7}$$

Especially, the exponents (r, p, s, α) and (r^*, s^*, α^*) satisfy (2.1) and (2.2), respectively.

Since it is not difficult to show this proposition, we may omit it. Now we are in a position to estimate the nonlinear term $-\nabla \cdot (u\nabla(-\Delta)^{-1}u)$. In order to construct a solution of (1.3) by combining the maximal Lorentz regularity and the successive approximation method, we need to show that the nonlinear term belongs to the Lorentz space which $\partial_t u$ and Δu belong to.

Lemma 2.6. *Let $1 \le q \le \infty$, $1 < r < \infty$, $1 \le p \le r$, $s \in \mathbb{R}$ and $1 < \alpha < \infty$ satisfy (2.3), (2.4) and (2.5). Suppose that (u, w) is a pair of function on $(0,T) \times \mathbb{R}^n$ satisfying*

$$\partial_t u, \Delta u, \partial_t w, \Delta w \in L^{\alpha,q}((0,T); \dot{B}^s_{r,1}), \quad u_0 := u(0),\, w_0 := w(0) \in \dot{B}^{-2+n/p}_{p,q}.$$

Then it holds that $\nabla \cdot (u\nabla(-\Delta)^{-1}w) \in L^{\alpha,q}((0,T); \dot{B}^s_{r,1})$ with the estimate

$$\begin{aligned} \|\nabla \cdot (u\nabla(-\Delta)^{-1}w)\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} \le C &\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right) \\ &\times \left(\|e^{t\Delta}w_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t w, \Delta w\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right) \end{aligned}$$

for some $r \le r^* < \infty$, $s^* \in \mathbb{R}$ and $\alpha < \alpha^* < \infty$ satisfying

$$\frac{2}{\alpha^*} + \frac{n}{r^*} = s^*,$$

where $C = C(n, q, r, p, s, r^*, s^*) > 0$ is a constant independent of T.

Proof. Since $1 \le q \le \infty$, $1 < r < \infty$, $1 \le p \le r$, $s \in \mathbb{R}$ and $1 < \alpha < \infty$ satisfy (2.3), (2.4) and (2.5), by Proposition 2.5, there exists $r \le r^* < \infty$, $s^* \in \mathbb{R}$ and $\alpha < \alpha^* < \infty$ satisfying (2.7). By Lemma 2.4, we have

$$\begin{aligned}
\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,\infty);\dot{B}^{s^*}_{r^*,1})} &\le C\|u_0\|_{\dot{B}^{-2+n/p}_{p,q}}, \\
\|e^{t\Delta}w_0\|_{L^{\alpha^*,q}((0,\infty);\dot{B}^{s^*}_{r^*,1})} &\le C\|w_0\|_{\dot{B}^{-2+n/p}_{p,q}}, \\
\|u\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} &\le \|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + C\|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);\dot{B}^{s}_{r,1})} \\
\|w\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} &\le \|e^{t\Delta}w_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + C\|\partial_t w, \Delta w\|_{L^{\alpha,q}((0,T);\dot{B}^{s}_{r,1})}.
\end{aligned} \tag{2.8}$$

Since

$$\begin{aligned}
\nabla\cdot(u\nabla(-\Delta)^{-1}w) &= \nabla\cdot\begin{pmatrix} uD_1(-\Delta)^{-1}w \\ \vdots \\ uD_n(-\Delta)^{-1}w \end{pmatrix} \\
&= \sum_{i=1}^n D_i(uD_i(-\Delta)^{-1}w) \\
&= \sum_{i=1}^n \{(D_iu)D_i(-\Delta)^{-1}w + uD_i^2(-\Delta)^{-1}w\},
\end{aligned}$$

we see that

$$\|\nabla\cdot(u\nabla(-\Delta)^{-1}w)\|_{\dot{B}^s_{r,1}} \le \sum_{i=1}^n \left(\|(D_iu)D_i(-\Delta)^{-1}w\|_{\dot{B}^s_{r,1}} + \|uD_i^2(-\Delta)^{-1}w\|_{\dot{B}^s_{r,1}}\right).$$

By (2.7), we have $s/2 < s^*$, $s+1 < s^*$. Since we also have

$$-s^* - 1 < s^* - s - 1, \quad 0 < s^* - s - 1,$$

it is possible to choose $\sigma \in \mathbb{R}$ such that

$$\max\{0, -s^* - 1\} < \sigma < s^* - s - 1.$$

Let us define p_0 and p_1 as the relations

$$\frac{1}{p_0} = \frac{1}{r^*} - \frac{s^* - s - \sigma - 1}{n}, \quad \frac{1}{p_1} = \frac{1}{r^*} - \frac{s^* + \sigma + 1}{n}.$$

since $s^* - s - \sigma - 1$, $s^* + \sigma + 1 > 0$, we have

$$r^* < p_0, p_1.$$

Hence it holds by Proposition 2.3 that

$$\dot{B}^{s^*}_{r^*,1} \subset \dot{B}^{s+\sigma+1}_{p_0,1}, \quad \dot{B}^{s^*}_{r^*,1} \subset \dot{B}^{-\sigma-1}_{p_1,1}.$$

Since the relations (2.5) and (2.6) are valid, we observe

$$\begin{aligned}\frac{1}{p_0}+\frac{1}{p_1} &= \frac{1}{r^*}-\frac{s^*-s-\sigma-1}{n}+\frac{1}{r^*}-\frac{s^*+\sigma+1}{n}\\ &= \frac{1}{n}\left(\frac{2n}{r^*}-2s^*+s\right)\\ &= \frac{1}{n}\left(4-\frac{2}{\alpha}+s\right)\\ &= \frac{1}{r}.\end{aligned}$$

Hence we see from Proposition 2.3 that

$$\begin{aligned}\|(D_iu)D_i(-\Delta)^{-1}w\|_{\dot B^{s_1}_{r,1}} &\le C\left(\|D_iu\|_{\dot B^{s+\sigma}_{p_0,1}}\|D_i(-\Delta)^{-1}w\|_{\dot B^{-\sigma}_{p_1,\infty}}+\|D_iu\|_{\dot B^{-(\sigma+2)}_{p_1,\infty}}\|D_i(-\Delta)^{-1}w\|_{\dot B^{s+(\sigma+2)}_{p_0,1}}\right)\\ &\le C\left(\|u\|_{\dot B^{s+\sigma+1}_{p_0,1}}\|w\|_{\dot B^{-\sigma-1}_{p_1,1}}+\|u\|_{\dot B^{-\sigma-1}_{p_1,1}}\|w\|_{\dot B^{s+\sigma+1}_{p_0,1}}\right)\\ &\le C\left(\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}+\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}\right)\\ &\le C\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}\end{aligned}$$

and

$$\begin{aligned}\|uD_i^2(-\Delta)^{-1}w\|_{\dot B^{s_1}_{r,1}} &\le C\left(\|u\|_{\dot B^{s+(\sigma+1)}_{p_0,1}}\|D_i^2(-\Delta)^{-1}w\|_{\dot B^{-(\sigma+1)}_{p_1,\infty}}+\|u\|_{\dot B^{-(\sigma+1)}_{p_1,\infty}}\|D_i^2(-\Delta)^{-1}w\|_{\dot B^{s+(\sigma+1)}_{p_0,1}}\right)\\ &\le C\left(\|u\|_{\dot B^{s+\sigma+1}_{p_0,1}}\|w\|_{\dot B^{-\sigma-1}_{p_1,1}}+\|u\|_{\dot B^{-\sigma-1}_{p_1,1}}\|w\|_{\dot B^{s+\sigma+1}_{p_0,1}}\right)\\ &\le C\left(\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}+\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}\right)\\ &\le C\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}\end{aligned}$$

respectively, which implies that

$$\begin{aligned}\|\nabla\cdot(u\nabla(-\Delta)^{-1}w)\|_{\dot B^{s}_{r,1}} &\le \sum_{i=1}^{n}\left(\|(D_iu)D_i(-\Delta)^{-1}w\|_{\dot B^{s}_{r,1}}+\|uD_i^2(-\Delta)^{-1}w\|_{\dot B^{s}_{r,1}}\right)\\ &\le \sum_{i=1}^{n}\left(C\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}+C\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}\right)\\ &\le C\|u\|_{\dot B^{s^*}_{r^*,1}}\|w\|_{\dot B^{s^*}_{r^*,1}}.\end{aligned}$$

By (2.6), we have by the Hölder inequality and an elementary embedding in the Lorentz spaces that

$$\begin{aligned}
\|\nabla\cdot(u\nabla(-\Delta)^{-1}w)\|_{L^{\alpha,q}((0,T);\dot B^s_{r,1})} &= \left\| \|\nabla\cdot(u\nabla(-\Delta)^{-1}w)\|_{\dot B^s_{r,1}} \right\|_{L^{\alpha,q}(0,T)} \\
&\le \left\| C\|u\|_{\dot B^{s^*}_{r^*,1}} \|w\|_{\dot B^{s^*}_{r^*,1}} \right\|_{L^{\alpha,q}(0,T)} \\
&\le C \left\| \|u\|_{\dot B^{s^*}_{r^*,1}} \right\|_{L^{2\alpha,2q}(0,T)} \left\| \|w\|_{\dot B^{s^*}_{r^*,1}} \right\|_{L^{2\alpha,2q}(0,T)} \\
&\le C \left\| \|u\|_{\dot B^{s^*}_{r^*,1}} \right\|_{L^{\alpha^*,q}(0,T)} \left\| \|w\|_{\dot B^{s^*}_{r^*,1}} \right\|_{L^{\alpha^*,q}(0,T)} \\
&= C\|u\|_{L^{\alpha^*,q}((0,T);\dot B^{s^*}_{r^*,1})} \|w\|_{L^{\alpha^*,q}((0,T);\dot B^{s^*}_{r^*,1})}
\end{aligned}$$

By combining this estimate and (2.8), we obtain the desired estimate and we complete the proof. □

3. Construction of solutions

Let us state a proposition on the estimate for the recurrence inequality.

Proposition 3.1. *Let $C > 0$ and $0 < M < 1/(4C)$. Assume that a positive sequence $\{\alpha_j\}_{j=0}^{\infty}$ satisfies*

$$\alpha_0 \le CM^2, \quad \alpha_{j+1} \le C(\alpha_j^2 + 2M\alpha_j + M^2)$$

for all $j \ge 0$. Then it holds that

$$\sup_{j\ge 0} \alpha_j \le \frac{1 - 2CM - \sqrt{1-4CM}}{2C} < \infty.$$

Especially, it follows that

$$2C\left(M + \sup_{j\ge 0}\alpha_j\right) < 1.$$

This proposition is easily derived from induction, so we may omit the proof. Next, we show the proposition on the decomposition of functions. We will apply this proposition to prove the existence and uniqueness of solutions of (1.3) in the case of $q = \infty$.

Proposition 3.2. *(i) Let $n/2 < p < \infty$, $\varepsilon > 0$ and $\{\varphi_j\}_{j\in\mathbb{Z}}$ be defined by (1.2). If $u_0 \in \dot B^{-2+n/p}_{p,\infty}$ satisfies (1.6) for some $N \in \mathbb{Z}$, then it holds that*

$$u_0 = u_0^{(1)} + u_0^{(2)} \quad \textit{with} \quad u_0^{(1)} \in L^p, \quad \|u_0^{(2)}\|_{\dot B^{-2+n/p}_{p,\infty}} \le C\varepsilon \tag{3.1}$$

where $C = 3\|\mathcal{F}^{-1}\varphi\|_{L^1}$.

(ii) Let $1 < r < \infty$, $s \in \mathbb{R}$, $1 < \alpha < \infty$ and $\eta > 0$. If u in the class

$$\partial_t u, \Delta u \in L^{\alpha,\infty}((0,T);\dot B^s_{r,1})$$

satisfies

$$\limsup_{t\to\infty}\left\{ t\mu\left(\tau\in(0,T)\ \middle|\ \|\partial_t u(\tau),\Delta u(\tau)\|_{\dot B^s_{r,1}} > t\right)^{\frac{1}{\alpha}} \right\} \le \eta, \tag{3.2}$$

there exists $R_\eta > 0$ such that

$$\begin{gathered} N(t) := \|\partial_t u(t),\Delta u(t)\|_{\dot B^s_{r,1}} = N^{(1)}(t) + N^{(2)}(t), \quad 0 < t < T \\ \textit{with} \quad \|N^{(1)}\|_{L^\infty(0,T)} \le R_\eta, \quad \|N^{(2)}\|_{L^{\alpha,\infty}(0,T)} \le 2\eta. \end{gathered} \tag{3.3}$$

Proof. (i) Let us define

$$u_0^{(1)} := \sum_{j=-\infty}^{N} \varphi_j * u_0, \quad u_0^{(2)} := \sum_{j=N+1}^{\infty} \varphi_j * u_0.$$

Since $2 - n/p > 0$, it holds that

$$\begin{aligned}
\|u_0^{(1)}\|_{L^p} &\le \sum_{j=-\infty}^{N} \|\varphi_j * u_0\|_{L^p} \\
&\le \sup_{-\infty<j\le N} \left\{ 2^{\left(-2+\frac{n}{p}\right)j} \|\varphi_j * u_0\|_{L^p} \right\} \sum_{j=-\infty}^{N} 2^{\left(2-\frac{n}{p}\right)j} \\
&\le \frac{2^{\left(2-\frac{n}{p}\right)N}}{1-2^{\left(-2+\frac{n}{p}\right)}} \|u_0\|_{\dot{B}^{-2+n/p}_{p,\infty}}.
\end{aligned}$$

By the definition of the sequence of function $\{\varphi_j\}_{j\in\mathbb{Z}}$, we have

$$\varphi_j * \varphi_k = 0 \quad (|j-k| \ge 2), \quad \|\varphi_j\|_{L^1} = \|\mathcal{F}^{-1}\varphi\|_{L^1}$$

for all $j, k \in \mathbb{Z}$. Hence it holds that

$$\begin{aligned}
\|u_0^{(2)}\|_{\dot{B}^{-2+n/p}_{p,\infty}} &= \sup_{j\in\mathbb{Z}} \left\{ 2^{\left(-2+\frac{n}{p}\right)j} \|\varphi_j * u_0^{(2)}\|_{L^p} \right\} \\
&= \sup_{j\in\mathbb{Z}} \left\{ 2^{\left(-2+\frac{n}{p}\right)j} \left\| \varphi_j * \left(\sum_{k=N+1}^{\infty} \varphi_k * u_0 \right) \right\|_{L^p} \right\} \\
&= \sup_{N\le j<\infty} \left\{ 2^{\left(-2+\frac{n}{p}\right)j} \left\| \varphi_j * (\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * u_0 \right\|_{L^p} \right\} \\
&\le \sup_{j\in\mathbb{Z}} \left(\|\varphi_{j-1}\|_{L^1} + \|\varphi_j\|_{L^1} + \|\varphi_{j+1}\|_{L^1} \right) \sup_{N\le j<\infty} \left\{ 2^{\left(-2+\frac{n}{p}\right)j} \left\| \varphi_j * u_0 \right\|_{L^p} \right\} \\
&\le 3\|\mathcal{F}^{-1}\varphi\|_{L^1}\varepsilon.
\end{aligned}$$

This implies that the relation (3.1) is valid and we complete the proof.

(ii) Since the condition (3.2) holds, by taking $R_\eta > 0$ sufficiently large, we have that

$$\sup_{t\ge R_\eta} \left\{ t\mu \left(\tau \in (0,T) \;\middle|\; \|\partial_t u(\tau), \Delta u(\tau)\|_{\dot{B}^s_{r,1}} > t \right)^{\frac{1}{\alpha}} \right\} \le 2\eta.$$

We define

$$N(t) := \|\partial_t u(t), \Delta u(t)\|_{\dot{B}^s_{r,1}}, \quad 0 < t < T,$$

$$N^{(1)}(t) := \begin{cases} N(t) & \text{if } N(t) \le R_\eta, \\ 0 & \text{if } N(t) > R_\eta, \end{cases} \qquad N^{(2)}(t) := \begin{cases} 0 & \text{if } N(t) \le R_\eta, \\ N(t) & \text{if } N(t) > R_\eta. \end{cases}$$

Then we have $N = N^{(1)} + N^{(2)}$ and $\|N^{(1)}\|_{L^\infty(0,T)} \le R_\eta$. By the definition of $N^{(2)}$, we see that

$$
\begin{aligned}
&\|N^{(2)}\|_{L^{\alpha,\infty}(0,T)} \\
&= \max\left\{ \sup_{0<t\le R_\eta} \left\{ t\mu\left(\tau\in(0,T) \,\middle|\, N^{(2)}(\tau)>t\right)^{\frac{1}{\alpha}} \right\}, \sup_{t>R_\eta} \left\{ t\mu\left(\tau\in(0,T) \,\middle|\, N^{(2)}(\tau)>t\right)^{\frac{1}{\alpha}} \right\} \right\} \\
&= \max\left\{ R_\eta\mu\left(\tau\in(0,T) \,\middle|\, N^{(2)}(\tau)>R_\eta\right)^{\frac{1}{\alpha}}, \sup_{t>R_\eta} \left\{ t\mu\left(\tau\in(0,T) \,\middle|\, N^{(2)}(\tau)>t\right)^{\frac{1}{\alpha}} \right\} \right\} \\
&\le 2\eta.
\end{aligned}
$$

This completes the proof. □

Let us construct a solution of (1.3). By the successive approximation method, we regard (1.3) as a heat equation with the nonlinear term. The estimate of the nonlinear term yields a sequence of solutions to the heat equation by the maximal Lorentz regularity.

Proof of Theorem 1.2. (i) We first note that for every $1 \le p < \infty$, there exists $r \in \mathbb{R}$ such that

$$1 < r < \infty, \quad 1 \le p \le r, \quad \frac{1}{p} < \frac{1}{n} + \frac{1}{r}.$$

By Proposition 2.5, there also exists $s \in \mathbb{R}$ and $1 < \alpha < \infty$ satisfying (2.4) and (2.5). In the following, we abbreviate $\mathcal{M}_T := \mathcal{M}_{\alpha,q}((0,T); \dot{B}^s_{r,1})$. By Theorem 2.2, there exists a solution U^* of

$$
\begin{cases}
\partial_t U^* - \Delta U^* = 0 & \text{in } (0,\infty)\times\mathbb{R}^n, \\
U^*(0) = u_0 & \text{in } \mathbb{R}^n
\end{cases}
\tag{3.4}
$$

with the estimate

$$\|\partial_t U^*, \Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \le C\|u_0\|_{\dot{B}^{-2+n/p}_{p,q}}$$

for all $0 < T \le \infty$. We set $U_{-1} := 0$ and consider the sequence $\{U_j\}_{j=0}^\infty$ of function on $(0,T)\times\mathbb{R}^n$ satisfying the following equations

$$
\begin{cases}
\partial_t U_j - \Delta U_j = -\nabla\cdot((U^*+U_{j-1})\nabla(-\Delta)^{-1}(U^*+U_{j-1})) & \text{in } (0,T)\times\mathbb{R}^n, \\
U_j(0) = 0 & \text{in } \mathbb{R}^n
\end{cases}
\tag{3.5}
$$

for all $j \ge 0$. Let us define

$$M_T := \|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}.$$

Since

$$\partial_t U^*, \Delta U^* \in L^{\alpha,q}((0,T); \dot{B}^s_{r,1}), \quad u_0 = U^*(0) \in \dot{B}^{-2+n/p}_{p,q},$$

it follows from Lemma 2.6 that

$$\begin{aligned}
&\|\nabla\cdot(U^*\nabla(-\Delta)^{-1}U^*)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\\
&\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t U^*,\Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right)^2\le CM_T^2\\
&\|\nabla\cdot(U_j\nabla(-\Delta)^{-1}U^*)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\\
&\le C\|U_j\|_{\mathcal{M}_T}\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t U^*,\Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right)\le CM_T\|U_j\|_{\mathcal{M}_T}\\
&\|\nabla\cdot(U^*\nabla(-\Delta)^{-1}U_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\le CM_T\|U_j\|_{\mathcal{M}_T}\\
&\|\nabla\cdot(U_j\nabla(-\Delta)^{-1}U_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\le C\|U_j\|^2_{\mathcal{M}_T}
\end{aligned}$$

for some $r\le r^*<\infty$, $s^*\in\mathbb{R}$ and $\alpha<\alpha^*<\infty$ satisfying

$$\frac{2}{\alpha^*}+\frac{n}{r^*}=2+s^*.$$

Noting that $U_{-1}=0$, it holds by Theorem 2.2 that

$$\begin{aligned}
\|U_0\|_{\mathcal{M}_T}&\le C\|\nabla\cdot(U^*\nabla(-\Delta)^{-1}U^*)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\\
&\le CM_T^2,\\
\|U_{j+1}\|_{\mathcal{M}_T}&\le C\|\nabla\cdot((U^*+U_j)\nabla(-\Delta)^{-1}(U^*+U_j))\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\\
&\le CM_T^2+CM_T\|U_j\|_{\mathcal{M}_T}+CM_T\|U_j\|_{\mathcal{M}_T}+C\|U_j\|^2_{\mathcal{M}_T}\\
&=C\left(\|U_j\|^2_{\mathcal{M}_T}+2M_T\|U_j\|_{\mathcal{M}_T}+M_T^2\right).
\end{aligned}\tag{3.6}$$

Hence we have by induction that $U_j\in\mathcal{M}_T$ for all $j\ge 0$. We set $\overline{U}_j:=U_j-U_{j-1}$ for all $j\ge 0$. Notice that $\overline{U}_0=U_0$ and

$$\begin{aligned}
\overline{W}_j&:=-\nabla\cdot((U^*+U_j)\nabla(-\Delta)^{-1}(U^*+U_j)-(U^*+U_{j-1})\nabla(-\Delta)^{-1}(U^*+U_{j-1}))\\
&=-\nabla\cdot(U^*\nabla(-\Delta)^{-1}\overline{U}_j+\overline{U}_j\nabla(-\Delta)^{-1}U^*+\overline{U}_j\nabla(-\Delta)^{-1}U_j+U_{j-1}\nabla(-\Delta)^{-1}\overline{U}_j)
\end{aligned}$$

holds. Hence we have by (3.5) that

$$\begin{cases}\partial_t\overline{U}_{j+1}-\Delta\overline{U}_{j+1}=\overline{W}_j & \text{in } (0,T)\times\mathbb{R}^n,\\ \overline{U}_{j+1}(0)=0 & \text{in } \mathbb{R}^n\end{cases}$$

for all $j\ge 0$. Since $\overline{U}_j\in\mathcal{M}_T$ for all $j\ge 0$, it holds by Lemma 2.6 that

$$\begin{aligned}
&\|\nabla\cdot(U^*\nabla(-\Delta)^{-1}\overline{U}_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\\
&\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t U^*,\Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right)\|\overline{U}_j\|_{\mathcal{M}_T}=CM_T\|\overline{U}_j\|_{\mathcal{M}_T},\\
&\|\nabla\cdot(\overline{U}_j\nabla(-\Delta)^{-1}U^*)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\le CM_T\|\overline{U}_j\|_{\mathcal{M}_T},\\
&\|\nabla\cdot(\overline{U}_j\nabla(-\Delta)^{-1}U_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\le C\|\overline{U}_j\|_{\mathcal{M}_T}\|U_j\|_{\mathcal{M}_T},\\
&\|\nabla\cdot(U_{j-1}\nabla(-\Delta)^{-1}\overline{U}_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\le C\|U_{j-1}\|_{\mathcal{M}_T}\|\overline{U}_j\|_{\mathcal{M}_T}.
\end{aligned}$$

By Theorem 2.2, it follows that

$$\begin{aligned}
\|\overline{U}_{j+1}\|_{\mathcal{M}_T} &\le C\|\overline{W}_j\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \\
&\le C\left(M_T\|\overline{U}_j\|_{\mathcal{M}_T} + M_T\|\overline{U}_j\|_{\mathcal{M}_T} + \|\overline{U}_j\|_{\mathcal{M}_T}\|U_j\|_{\mathcal{M}_T} + \|U_{j-1}\|_{\mathcal{M}_T}\|\overline{U}_j\|_{\mathcal{M}_T}\right) \\
&= C\left(2M_T + \|U_j\|_{\mathcal{M}_T} + \|U_{j-1}\|_{\mathcal{M}_T}\right)\|\overline{U}_j\|_{\mathcal{M}_T} \\
&\le 2C\left(M_T + \sup_{j\ge 0}\|U_j\|_{\mathcal{M}_T}\right)\|\overline{U}_j\|_{\mathcal{M}_T}
\end{aligned}$$

and we have

$$\|\overline{U}_j\|_{\mathcal{M}_T} \le \left\{2C\left(M_T + \sup_{j\ge 0}\|U_j\|_{\mathcal{M}_T}\right)\right\}^j \|U_0\|_{\mathcal{M}_T}.$$

Noting that the relation (3.6), taking $0 < T \le \infty$ such that $M_T < 1/(4C)$, we have by Proposition 3.1 that

$$2C\left(M_T + \sup_{j\ge 0}\|U_j\|_{\mathcal{M}_T}\right) < 1.$$

Hence we see by

$$U_m = \sum_{j=0}^m \overline{U}_j, \quad \sum_{j=0}^\infty \left\{2C\left(M_T + \sup_{j\ge 0}\|U_j\|_{\mathcal{M}_T}\right)\right\}^j \|U_0\|_{\mathcal{M}_T} < \infty$$

that the limit of sequence

$$U := \lim_{m\to\infty} U_m = \sum_{j=0}^\infty \overline{U}_j \quad \text{in } \mathcal{M}_T$$

is convergence. Since $U \in \mathcal{M}_T$, we have $U(0) = 0$ and

$$\|\partial_t U_j - \Delta U_j - (\partial_t U - \Delta U)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \to 0 \quad (j\to\infty).$$

We set

$$\begin{aligned}
W_j &:= -\nabla\cdot\left((U^* + U_j)\nabla(-\Delta)^{-1}(U^*+U_j) - (U^*+U)\nabla(-\Delta)^{-1}(U^*+U)\right) \\
&= -\nabla\cdot\left(U^*\nabla(-\Delta)^{-1}(U_j - U) + (U_j - U)\nabla(-\Delta)^{-1}U^*\right. \\
&\quad \left. + (U_j - U)\nabla(-\Delta)^{-1}U_j + U\nabla(-\Delta)^{-1}(U_j - U)\right).
\end{aligned}$$

Since it holds by Lemma 2.6 that

$$\begin{aligned}
&\|\nabla\cdot(U^*\nabla(-\Delta)^{-1}(U_j - U))\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \\
&\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})}\right)\|U_j - U\|_{\mathcal{M}_T} \\
&\le CM_T\|U_j - U\|_{\mathcal{M}_T}, \\
&\|\nabla\cdot((U_j - U)\nabla(-\Delta)^{-1}U^*)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \le CM_T\|U_j - U\|_{\mathcal{M}_T}, \\
&\|\nabla\cdot((U_j - U)\nabla(-\Delta)^{-1}U_j)\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \le C\|U_j - U\|_{\mathcal{M}_T}\|U_j\|_{\mathcal{M}_T}, \\
&\|\nabla\cdot(U_{j-1}\nabla(-\Delta)^{-1}(U_j - U))\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} \le C\|U_{j-1}\|_{\mathcal{M}_T}\|U_j - U\|_{\mathcal{M}_T},
\end{aligned}$$

we see that

$$\begin{aligned}\|W_j\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,1})} &\le CM_T\|U_j-U\|_{\mathcal{M}_T}+CM_T\|U_j-U\|_{\mathcal{M}_T}\\&\quad+C\|U_j-U\|_{\mathcal{M}_T}\|U_j\|_{\mathcal{M}_T}+C\|U_{j-1}\|_{\mathcal{M}_T}\|U_j-U\|_{\mathcal{M}_T}\\&\to 0 \quad (j\to\infty).\end{aligned}$$

Therefore, by taking $j\to\infty$ in (3.5), we have that

$$\begin{cases}\partial_t U-\Delta U=-\nabla\cdot((U^*+U)\nabla(-\Delta)^{-1}(U^*+U)) & \text{in } (0,T)\times\mathbb{R}^n,\\ \qquad U(0)=0 & \text{in } \mathbb{R}^n.\end{cases}$$

Since U^* is a solution of (3.4), we see that $u:=U^*+U$ is a solution of (1.3). We also see by

$$\partial_t u,\Delta u\in L^{\alpha,q}((0,T);\dot{B}^s_{r,1}),\quad u_0=u(0)\in\dot{B}^{-2+n/p}_{p,q}$$

and Lemma 2.4 that $u\in L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})$.

Finally, we show that the solution of (1.3) is unique. Let us u_1 and u_2 be the solution of (1.3) in the class (1.4). We set $\overline{u}:=u_1-u_2$. Since

$$-\nabla\cdot(u_1\nabla(-\Delta)^{-1}u_1)+\nabla\cdot(u_2\nabla(-\Delta)^{-1}u_2)=-\nabla\cdot(\overline{u}\nabla(-\Delta)^{-1}u_1+u_2\nabla(-\Delta)^{-1}\overline{u})$$

holds, we have by (1.3) that

$$\begin{cases}\partial_t\overline{u}-\Delta\overline{u}=-\nabla\cdot(\overline{u}\nabla(-\Delta)^{-1}u_1+u_2\nabla(-\Delta)^{-1}\overline{u}) & \text{in } (0,T)\times\mathbb{R}^n,\\ \qquad \overline{u}(0)=0 & \text{in } \mathbb{R}^n.\end{cases}$$

Let us define

$$M'_\sigma:=\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,\sigma);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t u_1,\Delta u_1\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}+\|\partial_t u_2,\Delta u_2\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}.$$

It holds by Lemma 2.6 that

$$\begin{aligned}&\|\nabla\cdot(\overline{u}\nabla(-\Delta)^{-1}u_1)\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}\\&\le C\|\overline{u}\|_{\mathcal{M}_\sigma}\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,\sigma);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t u_1,\Delta u_1\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}\right)\le CM'_\sigma\|\overline{u}\|_{\mathcal{M}_\sigma},\\&\|\nabla\cdot(u_2\nabla(-\Delta)^{-1}\overline{u})\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}\\&\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,\sigma);\dot{B}^{s^*}_{r^*,1})}+\|\partial_t u_2,\Delta u_2\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}\right)\|\overline{u}\|_{\mathcal{M}_\sigma}\le CM'_\sigma\|\overline{u}\|_{\mathcal{M}_\sigma}\end{aligned}$$

for all $0<\sigma<T$. Hence we have by Theorem 2.2 that

$$\begin{aligned}\|\overline{u}\|_{\mathcal{M}_\sigma}&\le C\|\nabla\cdot(\overline{u}\nabla(-\Delta)^{-1}u_1+u_2\nabla(-\Delta)^{-1}\overline{u})\|_{L^{\alpha,q}((0,\sigma);\dot{B}^s_{r,1})}\\&\le CM'_\sigma\|\overline{u}\|_{\mathcal{M}_\sigma}+CM'_\sigma\|\overline{u}\|_{\mathcal{M}_\sigma}\\&=2CM'_\sigma\|\overline{u}\|_{\mathcal{M}_\sigma}.\end{aligned}$$

By taking $0<\sigma<T$ such that $M'_\sigma\le 1/(4C)$, it holds that

$$\|\overline{u}\|_{\mathcal{M}_\sigma}\le\frac{1}{2}\|\overline{u}\|_{\mathcal{M}_\sigma}$$

and we see that $\overline{u}(t)=0$ for all $0<t<\sigma$. By repeating this argument, we have that $\overline{u}(t)=0$ for all $0<t<T$ and we also see that $u_1\equiv u_2$.

(ii) Assume that $u_0 \in \dot{B}^{-2+n/p}_{p,\infty}$ satisfies (1.6). By Proposition 3.2, we obtain the decomposition satisfying (3.1). Notice that we need the relations $M_T < 1/(4C)$ and $M'_\sigma \le 1/(4C)$ to show the existence and uniqueness of the solution of (1.3). Since the exponents in (i) satisfy

$$\frac{2}{\alpha} + \frac{n}{r} = 4 + s, \quad \frac{2}{\alpha^*} + \frac{n}{r^*} = 2 + s^*, \tag{3.7}$$

we have by Proposition 2.1 that

$$\begin{aligned}
\|e^{t\Delta}u_0^{(1)}\|_{\dot{B}^{s^*}_{r^*,1}} &\le Ct^{-\frac{1}{2}n\left(\frac{1}{p}-\frac{1}{r^*}\right)-\frac{1}{2}s^*}\|u_0^{(1)}\|_{\dot{B}^0_{p,\infty}} \\
&= Ct^{-\frac{1}{\alpha^*}+\frac{1}{2}\left(2-\frac{n}{p}\right)}\|u_0^{(1)}\|_{\dot{B}^0_{p,\infty}} \\
&\le Ct^{-\frac{1}{\alpha^*}+\frac{1}{2}\left(2-\frac{n}{p}\right)}\|u_0^{(1)}\|_{L^p}
\end{aligned}$$

as well as

$$\begin{aligned}
\|\Delta e^{t\Delta}u_0^{(1)}\|_{\dot{B}^{s}_{r,1}} &\le Ct^{-\frac{1}{2}n\left(\frac{1}{p}-\frac{1}{r}\right)-\frac{1}{2}(s+2)}\|u_0^{(1)}\|_{\dot{B}^0_{p,\infty}} \\
&= Ct^{-\frac{1}{\alpha}+\frac{1}{2}\left(2-\frac{n}{p}\right)}\|u_0^{(1)}\|_{\dot{B}^0_{p,\infty}} \\
&\le Ct^{-\frac{1}{\alpha}+\frac{1}{2}\left(2-\frac{n}{p}\right)}\|u_0^{(1)}\|_{L^p}.
\end{aligned}$$

Noting that $2 - n/p > 0$, it holds that

$$\begin{aligned}
\|e^{t\Delta}u_0^{(1)}\|_{L^{\alpha^*,\infty}((0,T);\dot{B}^{s^*}_{r^*,1})} &\le \left(\int_0^T \|e^{t\Delta}u_0^{(1)}\|^{\alpha^*}_{\dot{B}^{s^*}_{r^*,1}}\,dt\right)^{\frac{1}{\alpha^*}} \\
&\le C\|u_0^{(1)}\|_{L^p}\left(\int_0^T t^{-1+\frac{1}{2}\alpha^*\left(2-\frac{n}{p}\right)}\,dt\right)^{\frac{1}{\alpha^*}} \\
&\le C\|u_0^{(1)}\|_{L^p}T^{\frac{1}{2}\left(2-\frac{n}{p}\right)}
\end{aligned} \tag{3.8}$$

as well as

$$\|\Delta e^{t\Delta}u_0^{(1)}\|_{L^{\alpha,\infty}((0,T);\dot{B}^{s}_{r,1})} \le C\|u_0^{(1)}\|_{L^p}T^{\frac{1}{2}\left(2-\frac{n}{p}\right)}. \tag{3.9}$$

Notice that we have by (3.7) that

$$\frac{1}{\alpha} = \frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{2}\left(s+4-\frac{n}{p}\right), \quad \frac{1}{\alpha^*} = \frac{n}{2}\left(\frac{1}{p}-\frac{1}{r^*}\right)+\frac{1}{2}\left(s^*+2-\frac{n}{p}\right),$$

it holds by Proposition 2.1 and (3.1) that

$$\begin{aligned}
\|e^{t\Delta}u_0^{(2)}\|_{L^{\alpha^*,\infty}((0,T);\dot{B}^{s^*}_{r^*,1})} &\le C\|u_0^{(2)}\|_{\dot{B}^{-2+n/p}_{p,\infty}} \le C\varepsilon, \\
\|\Delta e^{t\Delta}u_0^{(2)}\|_{L^{\alpha,\infty}((0,T);\dot{B}^{s}_{r,1})} &\le C\|u_0^{(2)}\|_{\dot{B}^{-2+n/p}_{p,\infty}} \le C\varepsilon.
\end{aligned} \tag{3.10}$$

Since U^* in (i) is a solution of (3.4), we have the representation $U^* = e^{t\Delta}u_0$ and the relation $\partial_t U^* = \Delta U^* = \Delta e^{t\Delta}u_0$. Therefore, taking $\varepsilon > 0$ and $T > 0$ such that

$$\varepsilon < \frac{1}{24C^2}, \quad T^{\frac{1}{2}\left(2-\frac{n}{p}\right)} < \frac{1}{24C^2\|u_0^{(1)}\|_{L^p}},$$

we have by (3.8), (3.9) and (3.10) that

$$\begin{aligned}
M_T &= \|e^{t\Delta}u_0\|_{L^{\alpha^*,\infty}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha,\infty}((0,T);\dot{B}^{s}_{r,1})} \\
&= \|e^{t\Delta}u_0\|_{L^{\alpha^*,\infty}((0,T);\dot{B}^{s^*}_{r^*,1})} + 2\|\Delta e^{t\Delta}u_0\|_{L^{\alpha,\infty}((0,T);\dot{B}^{s}_{r,1})} \\
&\le C\|u_0^{(1)}\|_{L^p} T^{\frac{1}{2}\left(2-\frac{n}{p}\right)} + C\varepsilon + 2C\|u_0^{(1)}\|_{L^p} T^{\frac{1}{2}\left(2-\frac{n}{p}\right)} + 2C\varepsilon \\
&< \frac{1}{4C}.
\end{aligned}$$

Hence we see that there exists a solution of (1.3).

Concerning the uniqueness, assume that u is a solution of (1.3) in the class (1.7). By Proposition 3.2, there exists $R_\eta > 0$ and it holds that (3.3). Then, we have that

$$\begin{aligned}
\|\partial_t u, \Delta u\|_{L^{\alpha,\infty}((0,\sigma);\dot{B}^{s}_{r,1})} &= \|N^{(1)} + N^{(2)}\|_{L^{\alpha,\infty}(0,\sigma)} \\
&\le \left\{\int_0^\sigma (N^{(1)}(t))^\alpha dt\right\}^{\frac{1}{\alpha}} + 2\eta \\
&\le R_\eta \sigma^{\frac{1}{\alpha}} + 2\eta.
\end{aligned}$$

It holds by (3.8) and (3.10) that

$$\|e^{t\Delta}u_0\|_{L^{\alpha^*,\infty}((0,\sigma);\dot{B}^{s^*}_{r^*,1})} \le C\|u_0^{(1)}\|_{L^p} \sigma^{\frac{1}{2}\left(2-\frac{n}{p}\right)} + C\eta.$$

Let us take $\eta > 0$ and $\sigma > 0$ such that

$$\eta < \frac{1}{8C(4+C)}, \quad \sigma^{\frac{1}{\alpha}} + \sigma^{\frac{1}{2}\left(2-\frac{n}{p}\right)} < \frac{1}{8C(C\|u_0^{(1)}\|_{L^p} + 2R_\eta)}.$$

Then, for the solutions u_1 and u_2 of (1.3) in the class (1.7), it holds that

$$\begin{aligned}
M'_\sigma &= \|e^{t\Delta}u_0\|_{L^{\alpha^*,\infty}((0,\sigma);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t u_1, \Delta u_1\|_{L^{\alpha,\infty}((0,\sigma);\dot{B}^{s}_{r,1})} + \|\partial_t u_2, \Delta u_2\|_{L^{\alpha,\infty}((0,\sigma);\dot{B}^{s}_{r,1})} \\
&\le C\|u_0^{(1)}\|_{L^p} \sigma^{\frac{1}{2}\left(2-\frac{n}{p}\right)} + C\eta + 2(R_\eta \sigma^{\frac{1}{\alpha}} + 2\eta) \\
&< \frac{1}{4C}.
\end{aligned}$$

Hence we see that the solution of (1.3) in the class (1.7) is unique.

(iii) Assume that $u_0 \in \dot{B}^{-2+n/p}_{p,q}$ satisfies (1.8). It is sufficient to show that $M_T < 1/(4C)$ holds for all $0 < T \le \infty$ to prove the existence a global solution of (1.3). It holds

$$\|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} \le C\varepsilon_*, \quad \|\Delta e^{t\Delta}u_0\|_{L^{\alpha,q}((0,T);\dot{B}^{s}_{r,1})} \le C\varepsilon_*$$

as well as (3.10). Therefore, taking $\varepsilon_* > 0$ such that

$$\varepsilon_* < \frac{1}{12C^2},$$

it holds that

$$\begin{aligned} M_T &= \|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha,q}((0,T);\dot{B}^{s}_{r,1})} \\ &= \|e^{t\Delta}u_0\|_{L^{\alpha^*,q}((0,T);\dot{B}^{s^*}_{r^*,1})} + 2\|\Delta e^{t\Delta}u_0\|_{L^{\alpha,q}((0,T);\dot{B}^{s}_{r,1})} \\ &< \frac{1}{4C} \end{aligned}$$

for all $0 < T \leq \infty$. This completes the proof of Theorem 1.2. □

References

[1] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[2] P. Biler, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, Studia Math. **114** (1995), no. 2, 181–205.

[3] P. Biler, M. Cannone, I. A. Guerra, and G. Karch, *Global regular and singular solutions for a model of gravitating particles*, Math. Ann. **330** (2004), no. 4, 693–708.

[4] A. Blanchet, J. A. Carrillo, and N. Masmoudi, *Infinite time aggregation for the critical Patlak-Keller-Segel model in* $\mathbb{R}^2$, Comm. Pure Appl. Math. **61** (2008), no. 10, 1449–1481.

[5] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Mat. Iberoamericana **13** (1997), no. 3, 515–541.

[6] T. Cieślak and M. Winkler, *Finite-time blow-up in a quasilinear system of chemotaxis*, Nonlinearity **21** (2008), no. 5, 1057–1076.

[7] C. Conca, E. Espejo, and K. Vilches, *Remarks on the blowup and global existence for a two species chemotactic Keller-Segel system in* $\mathbb{R}^2$, European J. Appl. Math. **22** (2011), no. 6, 553–580.

[8] L. Corrias, B. Perthame, and H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math. **72** (2004), 1–28.

[9] J. I. Diaz, T. Nagai, and J.-M. Rakotoson, *Symmetrization techniques on unbounded domains: application to a chemotaxis system on* $\mathbf{R}^N$, J. Differential Equations **145** (1998), no. 1, 156–183.

[10] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem. I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[11] Y. Giga, *Solutions for semilinear parabolic equations in* L^p *and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), no. 2, 186–212.

[12] Y. Giga and T. Miyakawa, *Navier-Stokes flow in* $\mathbf{R}^3$ *with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations **14** (1989), no. 5, 577–618.

[13] T. Iwabuchi, *Global well-posedness for Keller-Segel system in Besov type spaces*, J. Math. Anal. Appl. **379** (2011), no. 2, 930–948.

[14] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), no. 2, 819–824.

[15] T. Kato, *Strong* L^p*-solutions of the Navier-Stokes equation in* $\mathbf{R}^m$*, with applications to weak solutions*, Math. Z. **187** (1984), no. 4, 471–480.

[16] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. **26** (1970), no. 3, 399–415.

[17] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math. **157** (2001), no. 1, 22–35.

[18] H. Kozono, T. Ogawa, and Y. Taniuchi, *Navier-Stokes equations in the Besov space near* L^∞ *and BMO*, Kyushu J. Math. **57** (2003), no. 2, 303–324.

[19] H. Kozono and Y. Shimada, *Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations*, Math. Nachr. **276** (2004), 63–74.

[20] H. Kozono and S. Shimizu, *Strong solutions of the Navier-Stokes equations based on the maximal Lorentz regularity theorem in Besov spaces*, J. Funct. Anal. **276** (2019), no. 3, 896–931.

[21] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), no. 5-6, 959–1014.

[22] T. Nagai, *Global existence of solutions to a parabolic system for chemotaxis in two space dimensions*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 8 (Athens, 1996), 1997, pp. 5381–5388.

[23] T. Nagai and T. Senba, *Behavior of radially symmetric solutions of a system related to chemotaxis*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996), 1997, pp. 3837–3842.

[24] T. Ogawa and S. Shimizu, *End-point maximal regularity and its application to two-dimensional Keller-Segel system*, Math. Z. **264** (2010), no. 3, 601–628.

[25] T. Takeuchi, *The Keller-Segel system of parabolic-parabolic type in homogeneous Besov spaces framework* (submitted).

[26] M. E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations **17** (1992), no. 9-10, 1407–1456.

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