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バージョン：

THE KELLER-SEGEL SYSTEM OF PARABOLIC-PARABOLIC TYPE IN HOMOGENEOUS BESOV SPACES FRAMEWORK

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Abstract. We show the existence and uniqueness of local strong solutions of Keller-Segel system of parabolic-parabolic type for arbitrary initial data in the homogeneous Besov space which is scaling invariant. We also construct global strong solutions for small initial data, where the solutions belong to the Lorentz space in time direction. The proof is based on the maximal Lorentz regularity theorem of heat equations.

1. Introduction

Consider the Keller-Segel system of parabolic-parabolic type in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \quad\partial_t u = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n, \ \ t > 0, \\ \quad\partial_t v = \Delta v + u, & x \in \mathbb{R}^n, \ \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ stand for the density of amoebae and the concentration of the chemo-attractant, respectively, while $u_0 = u_0(x)$ and $v_0 = v_0(x)$ are the given initial data.

The aim of this article is to show the existence and uniqueness of local strong solutions of (1.1) for arbitrary initial data $(u_0, v_0) \in \dot{B}^{-2+n/p_1}_{p_1,q}(\mathbb{R}^n) \times \dot{B}^{n/p_2}_{p_2,q}(\mathbb{R}^n)$ for some $1 \leq p_1, p_2 < \infty$ and $1 \leq q < \infty$. Here, $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the homogeneous Besov space. We also prove the existence and uniqueness of global strong solutions for small $(u_0, v_0) \in \dot{B}^{-2+n/p_1}_{p_1,q}(\mathbb{R}^n) \times \dot{B}^{n/p_2}_{p_2,q}(\mathbb{R}^n)$, including the case of $q = \infty$. If a pair of function (u, v) solves (1.1), then $(u_\lambda(t,x), v_\lambda(t,x)) := (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x))$ becomes also the solution of (1.1) for all $\lambda > 0$. In addition, the Banach space $X \times Y$ with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ is said to be *scaling invariant* to (1.1) if the conditions $\|u_\lambda(0,\cdot)\|_X = \|u_0\|_X$ and $\|v_\lambda(0,\cdot)\|_Y = \|v_0\|_Y$ are satisfied for all $\lambda > 0$. Notice that $\dot{B}^{-2+n/p_1}_{p_1,q}(\mathbb{R}^n) \times \dot{B}^{n/p_2}_{p_2,q}(\mathbb{R}^n)$ is one of scaling invariant spaces to (1.1). The notion of scaling invariant spaces is traced back to the Fujita-Kato principle for the Navier-Stokes system:

$$\begin{cases} \quad\partial_t u = \Delta u - (u \cdot \nabla) u - \nabla \pi, & x \in \mathbb{R}^n, \ \ t > 0, \\ \ \nabla \cdot u = 0, & x \in \mathbb{R}^n, \ \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Fujita and Kato [11] chose $\dot{H}^{1/2,2}(\mathbb{R}^3)$ as the scaling invariant space to (1.2). After their work, Kato [15] and Giga [12] showed the existence of the strong solution of (1.2) for $L^n(\mathbb{R}^n)$ and Cannone [8] also showed for $\dot{B}^{-1+3/p}_{3,\infty}(\mathbb{R}^3)$ with $3 < p \leq 6$. For further results in other function spaces, see, e.g., Taylor [27], Giga-Miyakawa [13], Kozono-Yamazaki [22] and Koch-Tataru [16]. For the Keller-Segel

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system, Biler [2] constructed self-similar solutions in $\mathbb{R}^n$, and then Kozono-Sugiyama [20,21] showed the existence of strong solutions (u,v) of (1.1) in the class

$$\begin{aligned}
&u \in C([0,\infty); H^{-2+n/p,p}(\mathbb{R}^n)) \cap C((0,\infty); H^{2,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n)),\\
&v \in C([0,\infty); H^{n/p,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n))
\end{aligned}$$

for small $(u_0,v_0) \in H^{-2+n/p,p}(\mathbb{R}^n) \times H^{n/p,p}(\mathbb{R}^n)$ with $\max\{1,n/4\} < p < n/2$ and

$$\begin{aligned}
&u \in BC([0,\infty); L^1(\mathbb{R}^n)) \cap C((0,\infty); H^{2,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n)),\\
&v \in C([0,\infty); L^1(\mathbb{R}^n)) \cap C((0,\infty); H^{n/p,p}(\mathbb{R}^n)) \cap C^1((0,\infty); L^p(\mathbb{R}^n)) \quad (n/3 < \exists p < n/2)
\end{aligned}$$

for small $(u_0,v_0) \in L^{n/2,\infty}(\mathbb{R}^n) \times \mathrm{BMO}$ with additional conditions, respectively. Notice that they obtained the strong solutions by showing the regularity of *mild solutions*.

Compared with previous results, we shall construct a strong solution (u,v) in the class

$$\partial_t u, \Delta u \in L^{\alpha_1,q}((0,T); \dot{B}^{s_1}_{r,1}(\mathbb{R}^n)), \quad \partial_t v, \Delta v \in L^{\alpha_2,q}((0,T); \dot{B}^{s_2}_{r,1}(\mathbb{R}^n))$$

with $2/\alpha_1 + n/r = 4 + s_1$, $2/\alpha_2 + n/r = 2 + s_2$ and $1 \le q < \infty$ for sufficiently small $0 < T < \infty$. To this end, we follow the approach by Kozono-Shimizu [19], i.e., our method relies on the maximal Lorentz regularity theorem for the Keller-Segel system in the homogeneous Besov spaces framework. The crucial point of the approach by Kozono-Shimizu [19] stems from the analysis of the heat equation:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T) \times \mathbb{R}^n,\\ u(0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

The heat equation (1.3) is divided into two parts, namely

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T) \times \mathbb{R}^n,\\ u(0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.4}$$

and

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T) \times \mathbb{R}^n,\\ u(0) = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.5}$$

For (1.4), by the estimate of the heat semigroup $\{e^{t\Delta}\}_{t\ge 0}$ (cf. Kozono-Ogawa-Taniuchi [17, Lemma 2.2]) and the theory of real interpolation, they obtain

$$\|\Delta e^{t\Delta} u_0\|_{L^{\alpha,q}((0,\infty); \dot{B}^s_{r,1})} \le C\|u_0\|_{\dot{B}^\sigma_{p,q}}, \quad \text{for } \frac{1}{\alpha} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2}(s + 2 - \sigma). \tag{1.6}$$

On the other hand, for (1.5), by the maximal L^p regularity for the Laplacian (cf. Hieber [14, Corollary 1.6.10]) and real interpolation, they also prove

$$\|\partial_t u\|_{L^{\alpha,q}((0,T); \dot{B}^s_{r,\beta})} + \|\Delta u\|_{L^{\alpha,q}((0,T); \dot{B}^s_{r,\beta})} \le C\|f\|_{L^{\alpha,q}((0,T); \dot{B}^s_{r,\beta})}. \tag{1.7}$$

Combining the estimates (1.6) and (1.7), they derive the maximal Lorentz regularity of (1.3), which is employed to prove the existence of the strong solution of (1.2) for initial data $u_0 \in \dot{B}^{-1+n/p}_{p,q}(\mathbb{R}^n)$. Here, notice that, the class of solutions to (1.2) is determined from the estimate (1.6). In our situation, however, the Keller-Segel system (1.1) consists of two parabolic-type PDEs so that the suitable choice of function spaces for the initial data $\dot{B}^{-2+n/p_1}_{p_1,q}(\mathbb{R}^n) \times \dot{B}^{n/p_2}_{p_2,q}(\mathbb{R}^n)$ becomes more delicate due to the nonlinear term $-\nabla \cdot (u\nabla v)$. Particularly, in the case of $q = \infty$, the initial data v_0 belong to $\dot{B}^{n/p_2}_{p_2,\infty}(\mathbb{R}^n)$ with $n/p_2 > 0$, and hence we cannot directly apply the argument of [19] to show the existence of the local strong solution to (1.1) because the number n/p_2 expects to be negative if we follow the approach due to [19].

Let us give a few comments on our results. First, our results are based on the maximal Lorentz regularity estimate for the Laplacian, which is established by Kozono and Shimizu [19]. As far as the author knows, this paper is the first article, which succeeds to apply the maximal Lorentz regularity of the heat equation to a nonlinear parabolic-type PDE excluded the Navier-Stokes system. Namely, arguments used in this paper will work not only for the Keller-Segel system but for other parabolic-type PDEs, especially, for coupling systems of parabolic-type PDEs. The advantage of the maximal Lorentz regularity approach is that we can obtain strong solutions *directly*, which means that it is not necessary to construct mild solutions — the existence proof of strong solutions becomes more elegant and short.

The main result of this paper reads as follows.

Theorem 1.1. *(i) Let* $1 \le p_1, p_2 < \infty$ *and* $1 \le q < \infty$. *Assume that*

$$-\frac{1}{n} < \frac{1}{p_1} - \frac{1}{p_2} < \frac{2}{n}. \tag{1.8}$$

For every $u_0 \in \dot{B}^{-2+n/p_1}_{p_1,q}$ *and* $v_0 \in \dot{B}^{n/p_2}_{p_2,q}$, *there exist* $0 < T < \infty$ *and a unique solution* (u, v) *on* $(0,T) \times \mathbb{R}^n$ *of* (1.1) *in the class*

$$\partial_t u, \Delta u \in L^{\alpha_1,q}((0,T); \dot{B}^{s_1}_{r,1}), \quad \partial_t v, \Delta v \in L^{\alpha_2,q}((0,T); \dot{B}^{s_2}_{r,1}) \tag{1.9}$$

for some $1 < r < \infty$, $s_1, s_2 \in \mathbb{R}$ *and* $1 < \alpha_1 < \alpha_2 < \infty$ *such that*

$$\frac{2}{\alpha_1} + \frac{n}{r} = 4 + s_1, \quad \frac{2}{\alpha_2} + \frac{n}{r} = 2 + s_2.$$

In addition, it holds that

$$u \in L^{\alpha_2,q}((0,T); \dot{B}^{s_2}_{r,1}), \quad v \in L^{\alpha_2^*,q}((0,T); \dot{B}^{s_2^*}_{r_2^*,1}) \tag{1.10}$$

for some $r \le r_2^* \le \infty$, $s_2^* \in \mathbb{R}$ *and* $\alpha_2 < \alpha_2^* < \infty$ *satisfying*

$$\frac{2}{\alpha_2^*} + \frac{n}{r_2^*} = s_2^*.$$

(ii) Let $1 \le p_1, p_2 < \infty$ *satisfy* (1.8). *Suppose that* $1 \le q \le \infty$. *There exists* $\varepsilon = \varepsilon(n, p_1, p_2, q) > 0$ *such that if* $u_0 \in \dot{B}^{-2+n/p_1}_{p_1,q}$ *and* $v_0 \in \dot{B}^{n/p_2}_{p_2,q}$ *satisfy*

$$\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}} + \|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}} \le \varepsilon,$$

then we may take $T = \infty$ *in* (1.9) *and* (1.10). *Concerning the uniqueness for* $q = \infty$, *there exists a constant* $\eta = \eta(n, p_1, p_2, r, s_1, s_2) > 0$ *such that if* (u_1, v_1) *and* (u_2, v_2) *are the solution of* (1.1) *in the class* (1.9) *with*

$$\begin{aligned}\sum_{i=1}^{2} \Bigg[&\limsup_{t\to\infty} \left\{ t\mu\left(\tau \in (0,\infty) \;\middle|\; \|\partial_t u_i(\tau)\|_{\dot{B}^{s_1}_{r,1}} + \|\Delta u_i(\tau)\|_{\dot{B}^{s_1}_{r,1}} > t\right)^{\frac{1}{\alpha_1}} \right\} \\ &+ \limsup_{t\to\infty} \left\{ t\mu\left(\tau \in (0,\infty) \;\middle|\; \|\partial_t v_i(\tau)\|_{\dot{B}^{s_2}_{r,1}} + \|\Delta v_i(\tau)\|_{\dot{B}^{s_2}_{r,1}} > t\right)^{\frac{1}{\alpha_2}} \right\} \Bigg] \le \eta,\end{aligned} \tag{1.11}$$

then it holds that $u_1 \equiv u_2$ *and* $v_1 \equiv v_2$.

Remark 1.2. (i) We have not yet obtained a local strong solution for arbitrary large initial data in the case $q = \infty$ due to the reasons mentioned above.

(ii) Compared with the previous studies due to [20,21], the initial data belong to the homogeneous Besov spaces. In particular, the initial data u_0 of the density of amoebae can be taken as a singular data, e.g., the Dirac measure $\varepsilon\delta(x)$ with small coefficient $\varepsilon > 0$ in 2D case, since $\delta \in \dot{B}^{-n+n/p}_{p,\infty}(\mathbb{R}^n)$.

(iii) In 2D case, it is well-known that there exist global solutions of (1.1) if $u_0 \in L^1(\mathbb{R}^2)$ satisfies the condition $\int_{\mathbb{R}^2} u_0 dx < 8\pi$, without taking v_0 small. We should refer to [7,25] for the result. Their method is mainly based on the *a priori estimate* of (1.1) by skillful technique. On the other hand, we need smallness assumption on both $u_0 \in \dot{B}^{-2+2/p_1}_{p_1,q}(\mathbb{R}^2)$ and $v_0 \in \dot{B}^{2/p_2}_{p_2,q}(\mathbb{R}^2)$. Although our result does not yield the threshold initial mass of u_0 like 8π, we are able to obtain more general class $\dot{B}^{-2+2/p_1}_{p_1,q}(\mathbb{R}^2)$ of u_0 which ensures the existence of global solutions. For instance, by taking $p_1 = 1$ and $q = \infty$, we see that the class $\dot{B}^{0}_{1,\infty}(\mathbb{R}^2)$ of u_0 with smallness assumption is larger than that in $L^1(\mathbb{R}^2)$, i.e., $L^1(\mathbb{R}^2) \subset \dot{B}^{0}_{1,\infty}(\mathbb{R}^2)$.

Remark 1.3. (i) We should notice that it is also known results on the existence of global solution of the Keller-Segel system for Neumann problems in bounded domains Ω. Establishment a pioneer work to deal with such a problem might be in the paper [9] by Cao, who obtained global classical solutions of the Neumann problems under the condition that $\|u_0\|_{L^{n/2}(\Omega)}$ and $\|\nabla v_0\|_{L^n(\Omega)}$ are small. Winkler [36] recently treated 1D case and showed the existence of global classical solutions, even if u_0 is Radon measures on $\overline{\Omega}$, whenever $v_0 \in L^2(\Omega)$ holds. Besides, the global existence and blow-up phenomena of solutions for Neumann problems have been fully in the series of papers of Winkler [29,30,32,33].

(ii) Moreover, we can expect that one obtain farther-reaching results by considering additional structures of the Keller-Segel system, e.g., with considering logistic sources. For contributions to the asymptotic behavior, blow-up phenomena and instantaneous regularization of the solution of the logistic Keller-Segel system, we refer to [23,24,31,34,35,37]. We also note that Biler et al. [3–6] obtained various results for other settings in the Keller-Segel system.

The rest of this paper is organized as follows. In Section 2, we will recall the notations of functional spaces and the basic propositions. Section 3 shows the maximal Lorentz regularity theorem for the heat equations based on the argument due to Kozono-Shimizu [19]. Section 4 deals with estimating the nonlinear term $-\nabla \cdot (u\nabla v)$. Finally, combining the results obtained in Sections 3 and 4, we show our main result, Theorem 1.1, in Section 5.

2. Preliminaries

In the following, we define the notation and the function spaces. We write $\mathcal{F}$ the Fourier transform and we set $(-\Delta)^{\frac{1}{2}s} := \mathcal{F}^{-1}|\cdot|^s\mathcal{F}$ for $s \in \mathbb{R}$. We also define the homogeneous Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^n)$ and the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as follows.

Definition 2.1. Let us take a function $\varphi \in \mathscr{S}$ satisfying

$$\operatorname{supp}\varphi = \{\xi \in \mathbb{R}^n \mid 1/2 \leq |\xi| \leq 2\}, \quad \varphi(\xi) > 0 \quad \text{for } 1/2 < |\xi| < 2,$$
$$\sum_{j=-\infty}^{\infty} \varphi(\xi/2^j) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\mathscr{S}$ is the Schwartz spaces in $\mathbb{R}^n$. We set $\varphi_j := \mathcal{F}^{-1}\varphi(\xi/2^j)$ for $j \in \mathbb{Z}$. Then, for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, we define the homogeneous Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^n)$ and the homogeneous Besov spaces

$\dot{B}^s_{p,q}(\mathbb{R}^n)$ by

$$\begin{aligned}\dot{H}^{s,p}(\mathbb{R}^n) &:= \{f \in \mathscr{S}' \mid \|f\|_{\dot{H}^{s,p}(\mathbb{R}^n)} < \infty\},\\ \dot{B}^s_{p,q}(\mathbb{R}^n) &:= \{f \in \mathscr{S}'/\mathscr{P} \mid \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty\}\end{aligned}$$

respectively, with the norms

$$\|f\|_{\dot{H}^{s,p}(\mathbb{R}^n)} := \|(-\Delta)^{\frac{1}{2}s} f\|_{L^p(\mathbb{R}^n)},$$

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left\{ \displaystyle\sum_{j=-\infty}^{\infty} \left(2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\right)^q \right\}^{\frac{1}{q}} & 1 \le q < \infty,\\ \displaystyle\sup_{j\in\mathbb{Z}} \left\{2^{sj}\|\varphi_j * f\|_{L^p(\mathbb{R}^n)}\right\} & q = \infty, \end{cases}$$

where $\mathscr{S}'$ and $\mathscr{P}$ are the sets of all tempered distributions and polynomials on $\mathbb{R}^n$, respectively.

In the following, we will abbreviate $\dot{H}^{s,p} := \dot{H}^{s,p}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q} := \dot{B}^s_{p,q}(\mathbb{R}^n)$. For $1 \le p, q \le \infty$, an interval $I \subset \mathbb{R}$ and a Banach space X, we write $L^p(I; X)$ as the X-valued Bochner-Lebesgue space on I. We also define the Lorentz space $L^{p,q}(I; X)$ as the quasi-Banach space of all X-valued locally integrable functions on I such that

$$\|f\|_{L^{p,q}(I;X)} := \begin{cases} \{\int_0^\infty (t^{1/p} f^*(t))^q dt/t\}^{1/q} < \infty & 1 \le q < \infty,\\ \sup_{t>0}\{t^{1/p} f^*(t)\} < \infty & q = \infty. \end{cases}$$

Here, f^* is the decreasing rearrangement of f defined by $f^*(t) := \inf\{\lambda > 0 \mid \mu_f(\lambda) \le t\}$, where $\mu_f(\lambda) := \mu(t \in I \mid \|f(t)\|_X > \lambda)$. Note that the relation $\|f\|_{L^{p,q}(I;X)} = \|\|f(\cdot)\|_X\|_{L^{p,q}(I;\mathbb{R})}$ holds by a straightforward computation. For more details on the Lorentz space, see Castillo-Humberto [10].

Next we introduce the basic propositions about the estimate of heat semigroup and real interpolation that we will use later.

Proposition 2.2. *Let* $\beta \ge 0$, $s_0 < s_1 + 2\beta$ *and* $1 \le p \le q \le \infty$. *For every* $a \in \dot{B}^{s_0}_{p,\infty}$, *it holds that* $e^{t\Delta}a \in \dot{B}^{s_1+2\beta}_{q,1}$ *with the estimate*

$$\|(-\Delta)^\beta e^{t\Delta} a\|_{\dot{B}^{s_1}_{q,1}} \le Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}(s_1+2\beta-s_0)}\|a\|_{\dot{B}^{s_0}_{p,\infty}}, \quad 0 < t < \infty$$

where $C = C(n, p, q, s_0, s_1, \beta) > 0$.

Proposition 2.3. *(i) Let* $1 \le q, q_0, q_1, p_0, p_1 \le \infty$ *with* $p_0 \ne p_1$. *For* $0 < \theta < 1$, *it holds that*

$$(L^{p_0,q_0}(I; X), L^{p_1,q_1}(I; X))_{\theta,q} = L^{p,q}(I; X),$$

where $1 < p < \infty$ *defined by* $p^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$. *Particularly, it holds that*

$$(L^{p_0}(I; X), L^{p_1}(I; X))_{\theta,q} = L^{p,q}(I; X).$$

(ii) Let $1 \le p, q, q_0, q_1 \le \infty$ *and* $s_0, s_1 \in \mathbb{R}$ *with* $s_0 \ne s_1$. *For* $0 < \theta < 1$, *it holds that*

$$(\dot{H}^{s_0,p}, \dot{H}^{s_1,p})_{\theta,q} = \dot{B}^s_{p,q}, \quad (\dot{B}^{s_0}_{p,q_0}, \dot{B}^{s_1}_{p,q_1})_{\theta,q} = \dot{B}^s_{p,q},$$

where $s \in \mathbb{R}$ *defined by* $s = (1-\theta)s_0 + \theta s_1$.

For the proof of Proposition 2.2, see Kozono-Ogawa-Taniuchi [17, Lemma 2.2]. For the proof of Proposition 2.3, we refer to Triebel [28, p.134] and Bergh-Löfström [1, Theorem 6.4.5]. Notice that an elementary embedding in real interpolation spaces [1, Theorem 3.4.1] and Proposition 2.3 (i) imply that

$$L^{p,q_0}(I; X) \subset L^{p,q_1}(I; X) \quad \text{for} \quad 1 \le q_0 \le q_1 \le \infty \tag{2.1}$$

holds.

3. MAXIMAL LORENTZ REGULARITY THEOREM

Consider the following heat equations

$$
\begin{aligned}
&\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T)\times\mathbb{R}^n, \\ \quad u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}\\
&\begin{cases} \partial_t v - \Delta v = g & \text{in } (0,T)\times\mathbb{R}^n, \\ \quad v(0) = v_0 & \text{in } \mathbb{R}^n, \end{cases}
\end{aligned}
\tag{3.1}
$$

where $0 < T \le \infty$. The aim of this section is to prove the maximal Lorentz regularity theorem for (3.1). In the following, we will write

$$\|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);X)} := \|\partial_t u\|_{L^{\alpha,q}((0,T);X)} + \|\Delta u\|_{L^{\alpha,q}((0,T);X)}$$

to simplify the notation.

Theorem 3.1. *Let* $1 < r_1, r_2 < \infty,\ 1 \le p_1 \le r_1,\ 1 \le p_2 \le r_2,\ s_1, s_2 \in \mathbb{R},\ 1 \le q_1, q_2, \beta_1, \beta_2 \le \infty$ *and* $1 < \alpha_1, \alpha_2 < \infty$ *satisfy*

$$
\begin{cases}
\dfrac{1}{p_1} < \dfrac{2}{n} + \dfrac{1}{r_1}, & -4 + \dfrac{n}{p_1} < s_1 < -2 + \dfrac{n}{r_1}, \qquad \dfrac{2}{\alpha_1} + \dfrac{n}{r_1} = 4 + s_1, \\
\dfrac{1}{p_2} < \dfrac{2}{n} + \dfrac{1}{r_2}, & -2 + \dfrac{n}{p_2} < s_2 < \dfrac{n}{r_2}, \qquad \dfrac{2}{\alpha_2} + \dfrac{n}{r_2} = 2 + s_2.
\end{cases}
\tag{3.2}
$$

For every $u_0 \in \dot{B}^{-2+n/p_1}_{p_1,q_1}$, $v_0 \in \dot{B}^{n/p_2}_{p_2,q_2}$, $f \in L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})$ *and* $g \in L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})$, *there exists a unique solution* (u,v) *of* (3.1), *given by* $u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta} f(\tau)d\tau$ *and* $v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-\tau)\Delta} g(\tau)d\tau$, *in the class*

$$\partial_t u, \Delta u \in L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1}), \quad \partial_t v, \Delta v \in L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})$$

with the estimates

$$
\begin{aligned}
\|\partial_t u, \Delta u\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} &\le C\left(\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}} + \|f\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})}\right),\\
\|\partial_t v, \Delta v\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} &\le C\left(\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}} + \|g\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})}\right),
\end{aligned}
$$

where $C = C(n, r_1, r_2, p_1, p_2, s_1, s_2, q_1, q_2, \beta_1, \beta_2) > 0$ *is independent of* T.

To show Theorem 3.1, we first introduce the next proposition.

Proposition 3.2. *Let* $\beta \ge 0,\ \sigma < s + 2\beta,\ 1 \le p \le r \le \infty,\ 1 \le q \le \infty$ *and* $1 < \alpha < \infty$ *satisfy*

$$\frac{1}{\alpha} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2}(s + 2\beta - \sigma).$$

For every $a \in \dot{B}^{\sigma}_{p,q}$, *it holds that* $e^{t\Delta}a \in L^{\alpha,q}((0,\infty);\dot{B}^{s+2\beta}_{r,1})$ *with the estimate*

$$\|(-\Delta)^{\beta} e^{t\Delta} a\|_{L^{\alpha,q}((0,\infty);\dot{B}^{s}_{r,1})} \le C\|a\|_{\dot{B}^{\sigma}_{p,q}},$$

where $C = C(n, p, r, q, \sigma, s, \beta) > 0$.

Proof. Let $i = 0, 1$. Since $\sigma < s + 2\beta$, we choose $\sigma_i \in \mathbb{R}$ sufficiently close to σ such that $\sigma_0 < \sigma < \sigma_1 < s + 2\beta$. Let us take $\alpha_i \in \mathbb{R}$ so that

$$\frac{1}{\alpha_i} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right) + \frac{1}{2}(s + 2\beta - \sigma_i).$$

Notice that it holds $1 < \alpha_i < \infty$ due to $1 < \alpha < \infty$. From Proposition 2.2, we have

$$\|(-\Delta)^\beta e^{t\Delta} a_i\|_{\dot{B}^s_{r,1}} \leq C t^{-\frac{1}{\alpha_i}} \|a_i\|_{\dot{B}^{\sigma_i}_{p,\infty}}$$

for all $a_i \in \dot{B}^{\sigma_i}_{p,\infty}$, which implies that

$$\left\| \|(-\Delta)^\beta e^{t\Delta} a_i\|_{\dot{B}^s_{r,1}} \right\|_{L^{\alpha_i,\infty}(0,\infty)} \leq \sup_{\lambda > 0} \left\{ \lambda \mu \left(t \in (0,\infty) \;\middle|\; C t^{-\frac{1}{\alpha_i}} \|a_i\|_{\dot{B}^{\sigma_i}_{p,\infty}} > \lambda \right)^{\frac{1}{\alpha_i}} \right\} = C \|a_i\|_{\dot{B}^{\sigma_i}_{p,\infty}}.$$

Since $\sigma_0 < \sigma < \sigma_1$, there exists $0 < \theta < 1$ such that $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$. Hence, by the definition of α_i, we observe $\alpha^{-1} = (1-\theta)\alpha_0^{-1} + \theta\alpha_1^{-1}$. Thus, Proposition 2.3 yields

$$\|(-\Delta)^\beta e^{t\Delta} a\|_{L^{\alpha,q}((0,\infty);\dot{B}^s_{r,1})} = \left\| \|(-\Delta)^\beta e^{t\Delta} a\|_{\dot{B}^s_{r,1}} \right\|_{L^{\alpha,q}(0,\infty)} \leq C \|a\|_{\dot{B}^\sigma_{p,q}}$$

for all $a \in \dot{B}^\sigma_{p,q}$. □

Corollary 3.3. *Let* $1 \leq p_1 \leq r_1 \leq \infty$, $1 \leq p_2 \leq r_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$, $1 \leq q_1, q_2 \leq \infty$ *and* $1 < \alpha_1, \alpha_2 < \infty$ *satisfy*

$$-4 + \frac{n}{p_1} < s_1, \quad \frac{2}{\alpha_1} + \frac{n}{r_1} = 4 + s_1, \quad -2 + \frac{n}{p_2} < s_2, \quad \frac{2}{\alpha_2} + \frac{n}{r_2} = 2 + s_2.$$

Then, for every $u_0 \in \dot{B}^{-2+n/p_1}_{p_1,q_1}$ *and* $v_0 \in \dot{B}^{n/p_2}_{p_2,q_2}$, *it holds that* $e^{t\Delta}u_0 \in L^{\alpha_1,q_1}((0,\infty);\dot{B}^{s_1+2}_{r_1,1})$ *and* $e^{t\Delta}v_0 \in L^{\alpha_2,q_2}((0,\infty);\dot{B}^{s_2+2}_{r_2,1})$ *with the estimates*

$$\|\Delta e^{t\Delta} u_0\|_{L^{\alpha_1,q_1}((0,\infty);\dot{B}^{s_1}_{r_1,1})} \leq C \|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}}, \quad \|\Delta e^{t\Delta} v_0\|_{L^{\alpha_2,q_2}((0,\infty);\dot{B}^{s_2}_{r_2,1})} \leq C \|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}},$$

where $C = C(n, r_1, r_2, p_1, p_2, s_1, s_2, q_1, q_2) > 0$.

For a Banach space X, we define

$$\mathcal{M}_{\alpha,q}((0,T);X) := \{u : (0,T) \to X \;|\; \partial_t u, \Delta u \in L^{\alpha,q}((0,T);X), \;\; u(0) = 0\},$$

where $1 < \alpha < \infty$ and $1 \leq q \leq \infty$. If X is a UMD space, $\mathcal{M}_{\alpha,q}((0,T);X)$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{M}_{\alpha,q}((0,T);X)} := \|\partial_t u, \Delta u\|_{L^{\alpha,q}((0,T);X)} \quad \text{for } u \in \mathcal{M}_{\alpha,q}((0,T);X).$$

To simplify the notation, we will write $\mathcal{M}_\alpha((0,T);X) := \mathcal{M}_{\alpha,\alpha}((0,T);X)$ for $1 < \alpha < \infty$. If X is homogeneous Besov spaces, we have the following proposition.

Proposition 3.4. *Let* $1 < \alpha, r < \infty$, $1 \leq \beta, q \leq \infty$ *and* $s \in \mathbb{R}$. *For every* $f \in L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$, *there exists a unique solution* $u \in \mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta})$ *of*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T) \times \mathbb{R}^n, \\ u(0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{3.3}$$

given by $u(t) = \int_0^t e^{(t-\tau)\Delta} f(\tau)\,d\tau$, *with the estimate*

$$\|u\|_{\mathcal{M}_{\alpha,q}((0,T);\dot{B}^s_{r,\beta})} \leq C \|f\|_{L^{\alpha,q}((0,T);\dot{B}^s_{r,\beta})},$$

where $C = C(n, \alpha, r, \beta, q, s) > 0$ *is independent of* T.

Proof. Let $i = 0, 1$. We take $s_i \in \mathbb{R}$ such that $s_0 < s < s_1$. For $1 < \alpha_i < \infty$ and $f \in L^{\alpha_i}((0,T); \dot{H}^{s_i,r})$, there exists a unique solution $u \in \mathcal{M}_{\alpha_i}((0,T); \dot{H}^{s_i,r})$ of (3.3), given by $u(t) = \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau$, with the estimate

$$\|u\|_{\mathcal{M}_{\alpha_i}((0,T);\dot{H}^{s_i,r})} \leq C\|f\|_{L^{\alpha_i}((0,T);\dot{H}^{s_i,r})}.$$

For the details, see Hieber [14, Corollary 1.6.10]. We define the solution operator S that maps from $L^{\alpha_i}((0,T); \dot{H}^{s_i,r})$ to $\mathcal{M}_{\alpha_i}((0,T); \dot{H}^{s_i,r})$. Since $s_0 < s < s_1$ and since Proposition 2.3, there exists $0 < \theta < 1$ such that $(\dot{H}^{s_0,r}, \dot{H}^{s_1,r})_{\theta,\beta} = \dot{B}^s_{r,\beta}$ for all $1 \leq \beta \leq \infty$. Then, by the real interpolation theorem, we obtain

$$S \in \mathcal{L}(L^{\alpha_i}((0,T); \dot{B}^s_{r,\beta}), \mathcal{M}_{\alpha_i}((0,T); \dot{B}^s_{r,\beta})).$$

Taking $0 < \eta < 1$ such that $\alpha^{-1} = (1-\eta)\alpha_0^{-1} + \eta\alpha_1^{-1}$ is valid, by Proposition 2.3, we observe

$$(L^{\alpha_0}((0,T); \dot{B}^s_{r,\beta}), L^{\alpha_1}((0,T); \dot{B}^s_{r,\beta}))_{\eta,q} = L^{\alpha,q}((0,T); \dot{B}^s_{r,\beta}).$$

Hence, the real interpolation theorem shows that S is a bounded operator in the sense that

$$S : L^{\alpha,q}((0,T); \dot{B}^s_{r,\beta}) \to \mathcal{M}_{\alpha,q}((0,T); \dot{B}^s_{r,\beta}).$$

Here, it is clear that the operator norm of S is independent of T. This completes the proof of Proposition 3.4. □

Now, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We see that $(u_1, v_1) = (e^{t\Delta}u_0, e^{t\Delta}v_0)$ solves the following equations

$$\begin{cases} \partial_t u_1 - \Delta u_1 = 0 & \text{in } (0,T) \times \mathbb{R}^n, \\ u_1(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \partial_t v_1 - \Delta v_1 = 0 & \text{in } (0,T) \times \mathbb{R}^n, \\ v_1(0) = v_0 & \text{in } \mathbb{R}^n. \end{cases}$$

Noting that $\partial_t u_1 = \Delta u_1 = \Delta e^{t\Delta}u_0$ and $\partial_t v_1 = \Delta v_1 = \Delta e^{t\Delta}v_0$, it follows from Corollary 3.3 that

$$\begin{aligned} \|\partial_t u_1, \Delta u_1\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} &= 2\|\Delta e^{t\Delta}u_0\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} \\ &\leq 2\|\Delta e^{t\Delta}u_0\|_{L^{\alpha_1,q_1}((0,\infty);\dot{B}^{s_1}_{r_1,1})} \\ &\leq C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}} \end{aligned}$$

as well as

$$\|\partial_t v_1, \Delta v_1\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} \leq C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}}.$$

For $f \in L^{\alpha_1,q_1}((0,T); \dot{B}^{s_1}_{r_1,\beta_1})$ and $g \in L^{\alpha_2,q_2}((0,T); \dot{B}^{s_2}_{r_2,\beta_2})$, by Proposition 3.4, there exists a unique solution $(u_2, v_2) \in \mathcal{M}_{\alpha_1,q_1}((0,T); \dot{B}^{s_1}_{r_1,\beta_1}) \times \mathcal{M}_{\alpha_2,q_2}((0,T); \dot{B}^{s_2}_{r_2,\beta_2})$ of

$$\begin{cases} \partial_t u_2 - \Delta u_2 = f & \text{in } (0,T) \times \mathbb{R}^n, \\ u_2(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \partial_t v_2 - \Delta v_2 = g & \text{in } (0,T) \times \mathbb{R}^n, \\ v_2(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

given by $u_2(t) = \int_0^t e^{(t-\tau)\Delta} f(\tau)d\tau$ and $v_2(t) = \int_0^t e^{(t-\tau)\Delta} g(\tau)d\tau$, with the estimates

$$\|u_2\|_{\mathcal{M}_{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} \le C\|f\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})}, \quad \|v_2\|_{\mathcal{M}_{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} \le C\|g\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})}.$$

Then we see that $(u, v) := (u_1 + u_2, v_1 + v_2)$ solves (3.1) with the estimates

$$\begin{aligned}
\|\partial_t u, \Delta u\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} &\le \|\partial_t u_1, \Delta u_1\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} + \|u_2\|_{\mathcal{M}_{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})} \\
&\le C\Big(\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}} + \|f\|_{L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,\beta_1})}\Big), \\
\|\partial_t v, \Delta v\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} &\le \|\partial_t v_1, \Delta v_1\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} + \|v_2\|_{\mathcal{M}_{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})} \\
&\le C\Big(\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}} + \|g\|_{L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,\beta_2})}\Big).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. PRELIMINARY ESTIMATES FOR THE NONLINEAR TERM

In this section, we will estimate the nonlinear term $-\nabla \cdot (u\nabla v)$ to obtain a solution of (1.1) by applying the maximal Lorentz regularity theorem. For that purpose, we first introduce the following well-known the Hardy-Littlewood-Sobolev inequality.

Proposition 4.1. *Let $0 < \theta < 1$, $1 < \alpha < 1/\theta$ and $1 \le q \le \infty$. For $f \in L^{\alpha,q}(0,\infty)$, we define*

$$(I_\theta f)(t) := \int_0^\infty |t-\tau|^{\theta-1} f(\tau)d\tau.$$

Then it holds that $I_\theta f \in L^{\alpha^*,q}(0,\infty)$ with the estimate

$$\|I_\theta f\|_{L^{\alpha^*,q}(0,\infty)} \le C\|f\|_{L^{\alpha,q}(0,\infty)},$$

with $C = C(\alpha,\theta) > 0$, where $\alpha < \alpha^* < \infty$ defined by

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \theta.$$

This inequality is easily derived from Stein [26, V, Theorem 1] and Proposition 2.3. Next, we give the propositions of embedding and paraproduct formula for the homogeneous Besov spaces.

Proposition 4.2. *(i) Let $1 \le p_0 \le p_1 \le \infty$, $1 \le q \le \infty$ and $s_0, s_1 \in \mathbb{R}$ satisfy*

$$\frac{1}{p_0} - \frac{s_0}{n} = \frac{1}{p_1} - \frac{s_1}{n}.$$

For every $f \in \dot{B}^{s_0}_{p_0,q}$, it holds that

$$\|f\|_{\dot{B}^{s_1}_{p_1,q}} \le C\|f\|_{\dot{B}^{s_0}_{p_0,q}},$$

where $C = C(n, p_0, p_1, s_0, s_1) > 0$.

(ii) Let $1 \le p, q \le \infty$, $s, \sigma, \mu > 0$ and $1 \le p_0, p_1, r_0, r_1 \le \infty$ satisfy

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{r_0} + \frac{1}{r_1}.$$

For every $f \in \dot{B}^{s+\sigma}_{p_0,q} \cap \dot{B}^{-\mu}_{r_0,\infty}$ and $g \in \dot{B}^{-\sigma}_{p_1,\infty} \cap \dot{B}^{s+\mu}_{r_1,q}$, it holds that $fg \in \dot{B}^s_{p,q}$ with the estimate

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\left(\|f\|_{\dot{B}^{s+\sigma}_{p_0,q}}\|g\|_{\dot{B}^{-\sigma}_{p_1,\infty}} + \|f\|_{\dot{B}^{-\mu}_{r_0,\infty}}\|g\|_{\dot{B}^{s+\mu}_{r_1,q}}\right),$$

where $C = C(n, p_0, p_1, r_0, r_1, s, \sigma, \mu) > 0$.

For the proof of (i), the case of the non-homogeneous Besov spaces is shown by Bergh-Löfström [1, Theorem 6.5.1]. For (ii), the case of the Triebel-Lizorkin spaces is shown by Kozono-Shimada [18, Lemma 2.1]. By applying them, we have the following Lemma. This shows that if $\partial_t u$ and Δu belong to the same Lorentz space, then u also belongs to the Lorentz space of a different class.

Lemma 4.3. *Let* $1 < r_1, r_2 < \infty,\ 1 \le p_1 \le r_1,\ 1 \le p_2 \le r_2,\ s_1, s_2 \in \mathbb{R},\ 1 \le q_1, q_2 \le \infty$ *and* $1 < \alpha_1, \alpha_2 < \infty$ *satisfy* (3.2). *We have* $r_1 \le r_1^* \le \infty,\ r_2 \le r_2^* \le \infty,\ s_1^*, s_2^* \in \mathbb{R},\ \alpha_1 < \alpha_1^* < \infty$ *and* $\alpha_2 < \alpha_2^* < \infty$ *satisfy*

$$\begin{cases} \dfrac{1}{p_1} + \dfrac{1}{r_1} - \dfrac{s_1+4}{n} < \dfrac{1}{r_1^*}, & -2 + \dfrac{n}{p_1} < s_1^* < s_1 + 2 - n\left(\dfrac{1}{r_1} - \dfrac{1}{r_1^*}\right), \qquad \dfrac{2}{\alpha_1^*} + \dfrac{n}{r_1^*} = 2 + s_1^*, \\ \dfrac{1}{p_2} + \dfrac{1}{r_2} - \dfrac{s_2+2}{n} < \dfrac{1}{r_2^*}, & \dfrac{n}{p_2} < s_2^* < s_2 + 2 - n\left(\dfrac{1}{r_2} - \dfrac{1}{r_2^*}\right), \qquad \dfrac{2}{\alpha_2^*} + \dfrac{n}{r_2^*} = s_2^*. \end{cases} \tag{4.1}$$

For every functions u *and* v *on* $(0,T) \times \mathbb{R}^n$ *such that*

$$\partial_t u, \Delta u \in L^{\alpha_1, q_1}((0,T); \dot{B}^{s_1}_{r_1,1}), \quad \partial_t v, \Delta v \in L^{\alpha_2, q_2}((0,T); \dot{B}^{s_2}_{r_2,1})$$

with $u_0 := u(0) \in \dot{B}^{-2+n/p_1}_{p_1,q_1}$ *and* $v_0 := v(0) \in \dot{B}^{n/p_2}_{p_2,q_2}$*, it holds that* $e^{t\Delta}u_0 \in L^{\alpha_1^*, q_1}((0,\infty); \dot{B}^{s_1^*}_{r_1^*,1})$, $e^{t\Delta}v_0 \in L^{\alpha_2^*, q_2}((0,\infty); \dot{B}^{s_2^*}_{r_2^*,1})$, $u \in L^{\alpha_1^*, q_1}((0,T); \dot{B}^{s_1^*}_{r_1^*,1})$ *and* $v \in L^{\alpha_2^*, q_2}((0,T); \dot{B}^{s_2^*}_{r_2^*,1})$ *with the estimates*

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^{\alpha_1^*, q_1}((0,\infty); \dot{B}^{s_1^*}_{r_1^*,1})} &\le C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}}, \\ \|e^{t\Delta}v_0\|_{L^{\alpha_2^*, q_2}((0,\infty); \dot{B}^{s_2^*}_{r_2^*,1})} &\le C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}}, \\ \|u\|_{L^{\alpha_1^*, q_1}((0,T); \dot{B}^{s_1^*}_{r_1^*,1})} &\le \|e^{t\Delta}u_0\|_{L^{\alpha_1^*, q_1}((0,T); \dot{B}^{s_1^*}_{r_1^*,1})} + C\|\partial_t u, \Delta u\|_{L^{\alpha_1, q_1}((0,T); \dot{B}^{s_1}_{r_1,1})}, \\ \|v\|_{L^{\alpha_2^*, q_2}((0,T); \dot{B}^{s_2^*}_{r_2^*,1})} &\le \|e^{t\Delta}v_0\|_{L^{\alpha_2^*, q_2}((0,T); \dot{B}^{s_2^*}_{r_2^*,1})} + C\|\partial_t v, \Delta v\|_{L^{\alpha_2, q_2}((0,T); \dot{B}^{s_2}_{r_2,1})}, \end{aligned}$$

where $C = C(n, r_1, r_2, p_1, p_2, s_1, s_2, q_1, q_2, r_1^*, r_2^*, s_1^*, s_2^*) > 0$ *is independent of* T.

Proof. Let us define $f := \partial_t u - \Delta u$ and $g := \partial_t v - \Delta v$. Then it holds $f \in L^{\alpha_1, q_1}((0,T); \dot{B}^{s_1}_{r_1,1})$ and $g \in L^{\alpha_2, q_2}((0,T); \dot{B}^{s_2}_{r_2,1})$ with

$$\begin{aligned} \|f\|_{L^{\alpha_1, q_1}((0,T); \dot{B}^{s_1}_{r_1,1})} &\le \|\partial_t u, \Delta u\|_{L^{\alpha_1, q_1}((0,T); \dot{B}^{s_1}_{r_1,1})}, \\ \|g\|_{L^{\alpha_2, q_2}((0,T); \dot{B}^{s_2}_{r_2,1})} &\le \|\partial_t v, \Delta v\|_{L^{\alpha_2, q_2}((0,T); \dot{B}^{s_2}_{r_2,1})}, \end{aligned} \tag{4.2}$$

with the identities

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \partial_t v - \Delta v = g & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad v(0) = v_0 & \text{in } \mathbb{R}^n. \end{cases}$$

Hence we see by Theorem 3.1 that u and v have the following representations

$$\begin{aligned} u(t) &= e^{t\Delta}u_0 + F(t), \quad F(t) := \int_0^t e^{(t-\tau)\Delta} f(\tau)\, d\tau, \\ v(t) &= e^{t\Delta}v_0 + G(t), \quad G(t) := \int_0^t e^{(t-\tau)\Delta} g(\tau)\, d\tau. \end{aligned} \tag{4.3}$$

It holds by (4.1) and Proposition 3.2 that

$$\|e^{t\Delta}u_0\|_{L^{\alpha_1^*,q_1}((0,\infty);\dot{B}^{s_1^*}_{r_1^*,1})} \le C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q_1}}, \quad \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q_2}((0,\infty);\dot{B}^{s_2^*}_{r_2^*,1})} \le C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q_2}}.$$

It holds by Proposition 2.2 that

$$\|e^{(t-\tau)\Delta}f(\tau)\|_{\dot{B}^{s_1^*}_{r_1^*,1}} \le C(t-\tau)^{-\frac{n}{2}\left(\frac{1}{r_1}-\frac{1}{r_1^*}\right)-\frac{1}{2}(s_1^*-s_1)}\|f(\tau)\|_{\dot{B}^{s_1}_{r_1,1}},$$

$$\|e^{(t-\tau)\Delta}g(\tau)\|_{\dot{B}^{s_2^*}_{r_2^*,1}} \le C(t-\tau)^{-\frac{n}{2}\left(\frac{1}{r_2}-\frac{1}{r_2^*}\right)-\frac{1}{2}(s_2^*-s_2)}\|g(\tau)\|_{\dot{B}^{s_2}_{r_2,1}}$$

for all $0<\tau<t$, since conditions (3.2) and (4.1) are valid. We define

$$\theta_1 := 1-\frac{n}{2}\left(\frac{1}{r_1}-\frac{1}{r_1^*}\right)-\frac{1}{2}(s_1^*-s_1), \quad \theta_2 := 1-\frac{n}{2}\left(\frac{1}{r_2}-\frac{1}{r_2^*}\right)-\frac{1}{2}(s_2^*-s_2).$$

We find that

$$\begin{aligned}
\|F(t)\|_{\dot{B}^{s_1^*}_{r_1^*,1}} &\le \int_0^t \|e^{(t-\tau)\Delta}f(\tau)\|_{\dot{B}^{s_1^*}_{r_1^*,1}}\,d\tau \\
&\le \int_0^t C(t-\tau)^{-\frac{n}{2}\left(\frac{1}{r_1}-\frac{1}{r_1^*}\right)-\frac{1}{2}(s_1^*-s_1)}\|f(\tau)\|_{\dot{B}^{s_1}_{r_1,1}}\,d\tau \\
&= C\int_0^t (t-\tau)^{\theta_1-1}\|f(\tau)\|_{\dot{B}^{s_1}_{r_1,1}}\,d\tau \\
&\le C\int_0^T |t-\tau|^{\theta_1-1}\|f(\tau)\|_{\dot{B}^{s_1}_{r_1,1}}\,d\tau
\end{aligned}$$

as well as

$$\|G(t)\|_{\dot{B}^{s_2^*}_{r_2^*,1}} \le C\int_0^T |t-\tau|^{\theta_2-1}\|g(\tau)\|_{\dot{B}^{s_2}_{r_2,1}}\,d\tau.$$

Since it holds that $1<\alpha_1<\alpha_1^*$ and $1<\alpha_2<\alpha_2^*$ with

$$0<\frac{1}{\alpha_1}-\frac{1}{\alpha_1^*}<\frac{1}{\alpha_1}<1, \quad 0<\frac{1}{\alpha_2}-\frac{1}{\alpha_2^*}<\frac{1}{\alpha_2}<1$$

and since

$$\frac{1}{\alpha_1}-\frac{1}{\alpha_1^*}=\theta_1, \quad \frac{1}{\alpha_2}-\frac{1}{\alpha_2^*}=\theta_2,$$

we see that $\theta_1,\theta_2\in\mathbb{R}$ satisfy $0<\theta_1,\theta_2<1$, $1<\alpha_1<1/\theta_1$ and $1<\alpha_2<1/\theta_2$. Let us define

$$N_f(\tau) := \begin{cases} \|f(\tau)\|_{\dot{B}^{s_1}_{r_1,1}} & 0<\tau\le T, \\ 0 & T\le\tau, \end{cases} \qquad N_g(\tau) := \begin{cases} \|g(\tau)\|_{\dot{B}^{s_2}_{r_2,1}} & 0<\tau\le T, \\ 0 & T\le\tau. \end{cases}$$

Since $f\in L^{\alpha_1,q_1}((0,T);\dot{B}^{s_1}_{r_1,1})$ and $g\in L^{\alpha_2,q_2}((0,T);\dot{B}^{s_2}_{r_2,1})$, we have that $N_f\in L^{\alpha_1,q_1}(0,\infty)$ and $N_g\in L^{\alpha_2,q_2}(0,\infty)$. Since

$$\frac{1}{\alpha_1^*}=\frac{1}{\alpha_1}-\theta_1, \quad \frac{1}{\alpha_2^*}=\frac{1}{\alpha_2}-\theta_2,$$

we deduce from Proposition 4.1 that

$$\|I_{\theta_1}N_f\|_{L^{\alpha_1^*,q_1}(0,\infty)} \le C\|N_f\|_{L^{\alpha_1,q_1}(0,\infty)} = C\left\| \|f(\tau)\|_{\dot B^{s_1}_{r_1,1}}\right\|_{L^{\alpha_1,q_1}(0,T)} = C\|f\|_{L^{\alpha_1,q_1}((0,T);\dot B^{s_1}_{r_1,1})}.$$

Here, it should be noticed that

$$(I_{\theta_1}N_f)(t) := \int_0^\infty |t-\tau|^{\theta_1-1}N_f(\tau)d\tau = \int_0^T |t-\tau|^{\theta_1-1}\|f(\tau)\|_{\dot B^{s_1}_{r_1,1}}d\tau.$$

Similarly, we have that

$$\|I_{\theta_2}N_g\|_{L^{\alpha_2^*,q_2}(0,\infty)} \le C\|g\|_{L^{\alpha_2,q_2}((0,T);\dot B^{s_2}_{r_2,1})}.$$

Since

$$\|F(t)\|_{\dot B^{s_1^*}_{r_1^*,1}} \le C(I_{\theta_1}N_f)(t), \quad \|G(t)\|_{\dot B^{s_2^*}_{r_2^*,1}} \le C(I_{\theta_2}N_g)(t),$$

it follows that

$$\|F\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})} = \left\| \|F(\cdot)\|_{\dot B^{s_1^*}_{r_1^*,1}}\right\|_{L^{\alpha_1^*,q_1}(0,T)} \le \|CI_{\theta_1}N_f\|_{L^{\alpha_1^*,q_1}(0,T)} \le C\|f\|_{L^{\alpha_1,q_1}((0,T);\dot B^{s_1}_{r_1,1})}$$

as well as

$$\|G\|_{L^{\alpha_2^*,q_2}((0,T);\dot B^{s_2^*}_{r_2^*,1})} \le C\|g\|_{L^{\alpha_2,q_2}((0,T);\dot B^{s_2}_{r_2,1})}.$$

Hence, from (4.2) and (4.3), we have that

$$\begin{aligned}\|u\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})} &\le \|e^{t\Delta}u_0\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})} + \|F\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})}\\ &\le \|e^{t\Delta}u_0\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})} + C\|f\|_{L^{\alpha_1,q_1}((0,T);\dot B^{s_1}_{r_1,1})}\\ &\le \|e^{t\Delta}u_0\|_{L^{\alpha_1^*,q_1}((0,T);\dot B^{s_1^*}_{r_1^*,1})} + C\|\partial_t u, \Delta u\|_{L^{\alpha_1,q_1}((0,T);\dot B^{s_1}_{r_1,1})}\end{aligned}$$

as well as

$$\|v\|_{L^{\alpha_2^*,q_2}((0,T);\dot B^{s_2^*}_{r_2^*,1})} \le \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q_2}((0,T);\dot B^{s_2^*}_{r_2^*,1})} + C\|\partial_t v, \Delta v\|_{L^{\alpha_2,q_2}((0,T);\dot B^{s_2}_{r_2,1})}.$$

This proves Lemma 4.3. □

Next, we state the basic proposition about the exponents. This proposition gives a condition on the solvability of (1.1).

Proposition 4.4. *Let $1 < r < \infty$.*

(i) There exist $1 \le p_1, p_2 \le r$ satisfying

$$\frac{1}{p_1} < \frac{2}{n} + \frac{1}{r}, \quad \max\left\{\frac{1}{p_2}, \frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\right\} < \frac{1}{n} + \frac{1}{r}. \tag{4.4}$$

(ii) There exist $s_1, s_2 \in \mathbb{R}$ satisfying

$$\begin{cases}\max\left\{-4+n\left(\frac{2}{p_2}-\frac{1}{r}\right), -4+n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{r}\right)\right\} < s_1 < -2+\frac{n}{r},\\ \max\left\{-2+\frac{n}{p_2}, -2+\frac{n}{p_1}, s_1+1, \frac{1}{2}s_1+\frac{n}{2r}\right\} < s_2 < s_1+2-n\left(\frac{1}{p_2}-\frac{1}{r}\right).\end{cases} \tag{4.5}$$

Particularly, it holds that

$$-4 + \frac{n}{p_1} < s_1 < -2 + \frac{n}{r}, \quad -2 + \frac{n}{p_2} < s_2 < \frac{n}{r}.$$

(iii) Let us take $\alpha_1, \alpha_2 \in \mathbb{R}$ *satisfying*

$$\frac{2}{\alpha_1} + \frac{n}{r} = 4 + s_1, \quad \frac{2}{\alpha_2} + \frac{n}{r} = 2 + s_2. \tag{4.6}$$

Then $\alpha_1, \alpha_2 \in \mathbb{R}$ *also satisfy* $1 < \alpha_1 < \alpha_2$.

(iv) If $1 < r < \infty,\ 1 \le p_1, p_2 \le r,\ s_1, s_2 \in \mathbb{R}$ *and* $1 < \alpha_1 < \alpha_2$ *satisfy* (4.4), (4.5) *and* (4.6), *there exist* $r \le r_2^* \le \infty,\ s_2^* \in \mathbb{R}$ *and* $\alpha_2 < \alpha_2^* < \infty$ *satisfying*

$$\begin{cases} \dfrac{1}{p_2} + \dfrac{1}{r} - \dfrac{s_2 + 2}{n} < \dfrac{1}{r_2^*}, \\ \dfrac{n}{p_2} < s_2^* < s_2 + 2 - n\left(\dfrac{1}{r} - \dfrac{1}{r_2^*}\right), \\ \dfrac{2}{\alpha_2^*} + \dfrac{n}{r_2^*} = s_2^*, \\ s_1 - s_2 - s_2^* + \dfrac{n}{r_2^*} + 2 = 0. \end{cases} \tag{4.7}$$

(v) If $1 < r < \infty,\ 1 \le p_1, p_2 \le r,\ s_1, s_2 \in \mathbb{R}$ *and* $1 < \alpha_1 < \alpha_2$ *satisfy* (4.4), (4.5) *and* (4.6), *it follows that*

$$\begin{cases} r \le r_1^* \le \infty, \quad \dfrac{1}{p_1} + \dfrac{1}{r} - \dfrac{s_1 + 4}{n} < \dfrac{1}{r_1^*}, \\ -2 + \dfrac{n}{p_1} < s_1^* < s_1 + 2 - n\left(\dfrac{1}{r} - \dfrac{1}{r_1^*}\right), \\ \alpha_1 < \alpha_1^* < \infty, \quad \dfrac{2}{\alpha_1^*} + \dfrac{n}{r_1^*} = 2 + s_1^*. \end{cases} \tag{4.8}$$

where $s_1^* := s_2,\ r_1^* := r,\ \alpha_1^* := \alpha_2$.

The proof is done by a straightforward calculation. So, we may omit it. By the previous proposition, we will estimate the nonlinear term $-\nabla \cdot (u \nabla v)$. If the nonlinear term belongs to the same space as the Lorentz space to which $\partial_t u$ and Δu belong, we can construct the solution by applying the maximal Lorentz regularity and the successive approximation method.

Lemma 4.5. *Let* $1 \le q \le \infty,\ 1 < r < \infty,\ 1 \le p_1, p_2 \le r,\ s_1, s_2 \in \mathbb{R}$ *and* $1 < \alpha_1 < \alpha_2$ *satisfy* (4.4), (4.5) *and* (4.6). *Suppose that* (u, v) *is a pair of function on* $(0, T) \times \mathbb{R}^n$ *satisfying*

$$\partial_t u, \Delta u \in L^{\alpha_1, q}((0, T); \dot{B}^{s_1}_{r,1}), \quad \partial_t v, \Delta v \in L^{\alpha_2, q}((0, T); \dot{B}^{s_2}_{r,1})$$

with $u_0 := u(0) \in \dot{B}^{-2+n/p_1}_{p_1,q}$ *and* $v_0 := v(0) \in \dot{B}^{n/p_2}_{p_2,q}$. *Then it holds that* $\nabla \cdot (u \nabla v) \in L^{\alpha_1, q}((0, T); \dot{B}^{s_1}_{r,1})$ *with the estimate*

$$\begin{aligned} \|\nabla \cdot (u \nabla v)\|_{L^{\alpha_1, q}((0,T); \dot{B}^{s_1}_{r,1})} \le C &\left(\|e^{t\Delta} u_0\|_{L^{\alpha_2, q}((0,T); \dot{B}^{s_2}_{r,1})} + \|\partial_t u, \Delta u\|_{L^{\alpha_1, q}((0,T); \dot{B}^{s_1}_{r,1})} \right) \\ &\times \left(\|e^{t\Delta} v_0\|_{L^{\alpha_2^*, q}((0,T); \dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t v, \Delta v\|_{L^{\alpha_2, q}((0,T); \dot{B}^{s_2}_{r,1})} \right), \end{aligned}$$

for some $r \le r_2^* \le \infty,\ \ s_2^* \in \mathbb{R}$ *and* $\alpha_2 < \alpha_2^* < \infty$ *satisfying*

$$\frac{2}{\alpha_2^*} + \frac{n}{r_2^*} = s_2^*,$$

where $C = C(n, q, r, p_1, p_2, s_1, s_2, r_2^*, s_2^*) > 0$ *is independent of* T.

Proof. Since $1 \le q \le \infty,\ 1 < r < \infty,\ 1 \le p_1, p_2 \le r,\ s_1, s_2 \in \mathbb{R}$ and $1 < \alpha_1 < \alpha_2$ satisfy (4.4), (4.5) and (4.6), by Proposition 4.4, there exist $r \le r_2^* \le \infty,\ s_2^* \in \mathbb{R}$ and $\alpha_2 < \alpha_2^* < \infty$ satisfying (4.7). By (4.8), applying Lemma 4.3 to $r_1^* = r,\ s_1^* = s_2$ and $\alpha_1^* = \alpha_2$, we have

$$\begin{aligned}
\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,\infty);\dot{B}^{s_2}_{r,1})} &\le C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}}, \\
\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,\infty);\dot{B}^{s_2^*}_{r_2^*,1})} &\le C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}}, \\
\|u\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} &\le \|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + C\|\partial_t u, \Delta u\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}, \\
\|v\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} &\le \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} + C\|\partial_t v, \Delta v\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}.
\end{aligned} \tag{4.9}$$

Since

$$\nabla \cdot (u\nabla v) = \nabla \cdot \begin{pmatrix} uD_1 v \\ \vdots \\ uD_n v \end{pmatrix} = \sum_{i=1}^{n} D_i(uD_i v) = \sum_{i=1}^{n} \{(D_i u)D_i v + uD_i^2 v\},$$

we have

$$\|\nabla \cdot (u\nabla v)\|_{\dot{B}^{s_1}_{r,1}} \le \sum_{i=1}^{n} \left(\|(D_i u)D_i v\|_{\dot{B}^{s_1}_{r,1}} + \|uD_i^2 v\|_{\dot{B}^{s_1}_{r,1}} \right).$$

Since $s_1 - s_2 - s_2^* + n/r_2^* + 2 = 0$, we have that

$$\frac{1}{\alpha_2} + \frac{1}{\alpha_2^*} = \frac{1}{\alpha_1}. \tag{4.10}$$

Since

$$s_1 + 2 = s_2^* + s_2 - \frac{n}{r_2^*} < s_2^* + s_2, \quad s_1 + 1 < s_2, \quad s_1 + 2 < s_2^*,$$

it follows that

$$1 - s_2^* < s_2 - s_1 - 1, \quad -s_2 < s_2^* - s_1 - 2, \quad 0 < s_2 - s_1 - 1, \quad 0 < s_2^* - s_1 - 2.$$

Hence it is possible to choose $\sigma, \mu \in \mathbb{R}$ such that

$$\max\{0, 1 - s_2^*\} < \sigma < s_2 - s_1 - 1, \quad \max\{0, -s_2\} < \mu < s_2^* - s_1 - 2.$$

Let us define r_0^1, r_0^2, p_0^1 and p_0^2 in such a way that

$$\begin{aligned}
\frac{1}{r_0^1} &= \frac{1}{r} - \frac{s_2 - s_1 - \sigma - 1}{n}, &\qquad \frac{1}{r_0^2} &= \frac{1}{r_2^*} - \frac{s_2^* + \sigma - 1}{n}, \\
\frac{1}{p_0^1} &= \frac{1}{r} - \frac{s_2 + \mu}{n}, &\qquad \frac{1}{p_0^2} &= \frac{1}{r_2^*} - \frac{s_2^* - s_1 - \mu - 2}{n}.
\end{aligned}$$

since $s_2 - s_1 - \sigma - 1,\ s_2^* + \sigma - 1,\ s_2 + \mu,\ s_2^* - s_1 - \mu - 2 > 0$, it holds that

$$r < r_0^1, p_0^1, \quad r_2^* < r_0^2, p_0^2.$$

Hence we have by Proposition 4.2 that

$$\dot B^{s_2}_{r,1} \subset \dot B^{s_1+\sigma+1}_{r_0^1,1}, \quad \dot B^{s_2^*}_{r_2^*,1} \subset \dot B^{-\sigma+1}_{r_0^2,1}, \quad \dot B^{s_2}_{r,1} \subset \dot B^{-\mu}_{p_0^1,1}, \quad \dot B^{s_2^*}_{r_2^*,1} \subset \dot B^{s_1+\mu+2}_{p_0^2,1}.$$

Since $s_1 - s_2 - s_2^* + n/r_2^* + 2 = 0$, we observe

$$\begin{aligned}
\frac{1}{r_0^1} + \frac{1}{r_0^2} &= \frac{1}{r} - \frac{s_2 - s_1 - \sigma - 1}{n} + \frac{1}{r_2^*} - \frac{s_2^* + \sigma - 1}{n} \\
&= \frac{1}{r} + \frac{1}{n}\left(-s_2 + s_1 + \sigma + 1 + \frac{n}{r_2^*} - s_2^* - \sigma + 1 \right) \\
&= \frac{1}{r} + \frac{1}{n}\left(s_1 - s_2 - s_2^* + \frac{n}{r_2^*} + 2 \right) \\
&= \frac{1}{r}
\end{aligned}$$

as well as

$$\frac{1}{p_0^1} + \frac{1}{p_0^2} = \frac{1}{r}.$$

Hence it follows from Proposition 4.2 that

$$\begin{aligned}
\|(D_i u) D_i v\|_{\dot B^{s_1}_{r,1}} &\le C \left(\|D_i u\|_{\dot B^{s_1+\sigma}_{r_0^1,1}} \|D_i v\|_{\dot B^{-\sigma}_{r_0^2,\infty}} + \|D_i u\|_{\dot B^{-(\mu+1)}_{p_0^1,\infty}} \|D_i v\|_{\dot B^{s_1+(\mu+1)}_{p_0^2,1}} \right) \\
&\le C \left(\|u\|_{\dot B^{s_1+\sigma+1}_{r_0^1,1}} \|v\|_{\dot B^{-\sigma+1}_{r_0^2,1}} + \|u\|_{\dot B^{-\mu}_{p_0^1,1}} \|v\|_{\dot B^{s_1+\mu+2}_{p_0^2,1}} \right) \\
&\le C \left(\|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}} + \|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}} \right) \\
&\le C \|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}}
\end{aligned}$$

and

$$\begin{aligned}
\|u D_i^2 v\|_{\dot B^{s_1}_{r,1}} &\le C \left(\|u\|_{\dot B^{s_1+(\sigma+1)}_{r_0^1,1}} \|D_i^2 v\|_{\dot B^{-(\sigma+1)}_{r_0^2,\infty}} + \|u\|_{\dot B^{-\mu}_{p_0^1,\infty}} \|D_i^2 v\|_{\dot B^{s_1+\mu}_{p_0^2,1}} \right) \\
&\le C \left(\|u\|_{\dot B^{s_1+(\sigma+1)}_{r_0^1,1}} \|v\|_{\dot B^{-\sigma+1}_{r_0^2,1}} + \|u\|_{\dot B^{-\mu}_{p_0^1,1}} \|v\|_{\dot B^{s_1+\mu+2}_{p_0^2,1}} \right) \\
&\le C \left(\|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}} + \|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}} \right) \\
&\le C \|u\|_{\dot B^{s_2}_{r,1}} \|v\|_{\dot B^{s_2^*}_{r_2^*,1}},
\end{aligned}$$

which yields that

$$
\begin{aligned}
\|\nabla\cdot(u\nabla v)\|_{\dot{B}^{s_1}_{r,1}} &\le \sum_{i=1}^{n}\left(\|(D_iu)D_iv\|_{\dot{B}^{s_1}_{r,1}} + \|uD_i^2v\|_{\dot{B}^{s_1}_{r,1}}\right)\\
&\le \sum_{i=1}^{n}\left(C\|u\|_{\dot{B}^{s_2}_{r,1}}\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}} + C\|u\|_{\dot{B}^{s_2}_{r,1}}\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}}\right)\\
&\le C\|u\|_{\dot{B}^{s_2}_{r,1}}\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}}.
\end{aligned}
$$

Since (4.10) is valid, we have by the Hölder inequality and (2.1) that

$$
\begin{aligned}
\|\nabla\cdot(u\nabla v)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &= \left\|\|\nabla\cdot(u\nabla v)\|_{\dot{B}^{s_1}_{r,1}}\right\|_{L^{\alpha_1,q}(0,T)}\\
&\le \left\|C\|u\|_{\dot{B}^{s_2}_{r,1}}\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}}\right\|_{L^{\alpha_1,q}(0,T)}\\
&\le C\left\|\|u\|_{\dot{B}^{s_2}_{r,1}}\right\|_{L^{\alpha_2,\frac{\alpha_2}{\alpha_1}q}(0,T)}\left\|\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}}\right\|_{L^{\alpha_2^*,\frac{\alpha_2^*}{\alpha_1}q}(0,T)}\\
&\le C\left\|\|u\|_{\dot{B}^{s_2}_{r,1}}\right\|_{L^{\alpha_2,q}(0,T)}\left\|\|v\|_{\dot{B}^{s_2^*}_{r_2^*,1}}\right\|_{L^{\alpha_2^*,q}(0,T)}\\
&= C\|u\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\|v\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})}.
\end{aligned}
$$

This implies the desired estimate and we complete the proof. □

5. CONSTRUCTION OF SOLUTIONS

Let us first introduce the following fundamental proposition.

Proposition 5.1. *(i) Let* $C \ge 1$ *and* $0 < M \le 1/(8C^2)$. *Assume that the positive sequences* $\{\alpha_j\}_{j=0}^{\infty}$ *and* $\{\beta_j\}_{j=0}^{\infty}$ *satisfy*

$$\alpha_0 \le CM^2, \quad \beta_0 \le CM, \quad \alpha_{j+1} \le CM^2 + CM\alpha_j + CM\beta_j + C\alpha_j\beta_j, \quad \beta_{j+1} \le CM + C\alpha_j$$

for all $j \ge 0$. *Then it holds that*

$$\alpha_j \le \frac{1}{8C^2}, \quad \beta_j \le \frac{1}{4C}$$

for all $j \ge 0$.

(ii) Let $C \ge 1$. *Assume that the positive sequences* $\{\overline{\alpha}_j\}_{j=0}^{\infty}$ *and* $\{\overline{\beta}_j\}_{j=0}^{\infty}$ *satisfy*

$$\overline{\alpha}_0 \le \frac{1}{8C^2}, \quad \overline{\beta}_0 \le \frac{1}{4C}, \quad \overline{\alpha}_{j+1} \le \frac{3}{8}\overline{\alpha}_j + \frac{1}{4C}\overline{\beta}_j, \quad \overline{\beta}_{j+1} \le C\overline{\alpha}_j$$

for all $j \ge 0$. *Then it holds that*

$$\overline{\alpha}_j \le \frac{1}{2C}\left(\frac{3}{4}\right)^{j+1}, \quad \overline{\beta}_j \le \frac{1}{2}\left(\frac{3}{4}\right)^{j}$$

for all $j \ge 0$.

It is easy to check that this argument by induction and the proof may be omitted. Now, we will construct a solution.

Proof of Theorem 1.1. (i) We first show that if $1 \le p_1, p_2 < \infty$ and $1 \le q < \infty$ satisfy (1.8), then there exist $1 < r < \infty,\ s_1, s_2 \in \mathbb{R}$ and $1 < \alpha_1 < \alpha_2 < \infty$ satisfying (4.4), (4.5) and (4.6). By Proposition 4.4, it is sufficient to show the existence $r \in \mathbb{R}$ satisfying

$$1 < r < \infty, \quad 1 \le p_1, p_2 \le r, \quad \frac{1}{p_1} < \frac{2}{n} + \frac{1}{r}, \quad \max\left\{\frac{1}{p_2}, \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right\} < \frac{1}{n} + \frac{1}{r}. \tag{5.1}$$

It is easily seen that the following relation

$$\max\left\{0, \frac{1}{p_1} - \frac{2}{n}, \frac{1}{p_2} - \frac{1}{n}, \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - \frac{1}{n}\right\} < \min\left\{\frac{1}{p_1}, \frac{1}{p_2}\right\} \le 1$$

is valid by (1.8). Hence it is possible to choose $r \in \mathbb{R}$ such that

$$\max\left\{0, \frac{1}{p_1} - \frac{2}{n}, \frac{1}{p_2} - \frac{1}{n}, \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - \frac{1}{n}\right\} < \frac{1}{r} < \min\left\{\frac{1}{p_1}, \frac{1}{p_2}\right\}$$

and it is clear that such an $r \in \mathbb{R}$ satisfies (5.1). In the following, we construct a solution of (1.1). Since $1 < r < \infty,\ 1 \le p_1, p_2 \le r,\ s_1, s_2 \in \mathbb{R}$ and $1 < \alpha_1 < \alpha_2$ satisfy (4.4), (4.5) and (4.6), by Theorem 3.1, there exists a solution (U^*, V^*) of

$$\begin{aligned}
&\begin{cases} \partial_t U^* - \Delta U^* = 0 & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \qquad U^*(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \\
&\begin{cases} \partial_t V^* - \Delta V^* = 0 & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \qquad V^*(0) = v_0 & \text{in } \mathbb{R}^n, \end{cases}
\end{aligned} \tag{5.2}$$

with the estimates

$$\begin{aligned}
&\|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} \le C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}}, \\
&\|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} \le C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}}
\end{aligned} \tag{5.3}$$

for all $0 < T \le \infty$. Let us consider the sequences $\{U_j\}_{j=0}^\infty, \{V_j\}_{j=0}^\infty$ of function on $(0,T) \times \mathbb{R}^n$ satisfying the following equations

$$\begin{aligned}
&\begin{cases} \partial_t U_0 - \Delta U_0 = -\nabla \cdot (U^* \nabla V^*) & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad U_0(0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \\
&\begin{cases} \partial_t V_0 - \Delta V_0 = U^* & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad V_0(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&\begin{cases} \partial_t U_{j+1} - \Delta U_{j+1} = -\nabla \cdot ((U^* + U_j)\nabla(V^* + V_j)) & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad U_{j+1}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \\
&\begin{cases} \partial_t V_{j+1} - \Delta V_{j+1} = U^* + U_j & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad V_{j+1}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}
\end{aligned} \tag{5.5}$$

for all $j \ge 0$. Let us define

$$\begin{aligned}
M_T := {}& \|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} \\
&+ \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}.
\end{aligned}$$

In the following, we abbreviate $\mathcal{M}_T^1 := \mathcal{M}_{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})$ and $\mathcal{M}_T^2 := \mathcal{M}_{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})$. Since

$$\partial_t U^*, \Delta U^* \in L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1}), \quad \partial_t V^*, \Delta V^* \in L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})$$

with $u_0 = U^*(0) \in \dot{B}^{-2+n/p_1}_{p_1,q}$ and $v_0 = V^*(0) \in \dot{B}^{n/p_2}_{p_2,q}$, it follows from Lemma 4.5 and (4.9) that

$$\begin{aligned}
\|U^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} &\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\right) \\
&\le CM_T, \\
\|\nabla\cdot(U^*\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\right) \\
&\quad \times \left(\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\right) \\
&\le CM_T^2
\end{aligned} \tag{5.6}$$

for some $r \le r_2^* \le \infty$, $s_2^* \in \mathbb{R}$ and $\alpha_2 < \alpha_2^* < \infty$ satisfying

$$\frac{2}{\alpha_2^*} + \frac{n}{r_2^*} = s_2^*.$$

By Proposition 3.4, there exists a unique solution $(U_0, V_0) \in \mathcal{M}_T^1 \times \mathcal{M}_T^2$ of (5.4) with the estimates

$$\begin{aligned}
\|U_0\|_{\mathcal{M}_T^1} &\le C\|\nabla\cdot(U^*\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}, \\
\|V_0\|_{\mathcal{M}_T^2} &\le C\|U^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}.
\end{aligned} \tag{5.7}$$

Then it holds by (5.6) and (5.7) that

$$\|U_0\|_{\mathcal{M}_T^1} \le CM_T^2, \quad \|V_0\|_{\mathcal{M}_T^2} \le CM_T. \tag{5.8}$$

Assume that $(U_j, V_j) \in \mathcal{M}_T^1 \times \mathcal{M}_T^2$. Noting that

$$-\nabla\cdot((U^* + U_j)\nabla(V^* + V_j)) = -\nabla\cdot(U^*\nabla V^* + U_j\nabla V^* + U^*\nabla V_j + U_j\nabla V_j)$$

and $U_j(0) = V_j(0) = 0$, it follows from Lemma 4.5 and (4.9) that

$$\begin{aligned}
\|U^*\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} &\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})}\right)\\
&\le CM_T,\\
\|U_j\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} &\le C\|U_j\|_{\mathcal{M}^1_T},\\
\|\nabla\cdot(U^*\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})} &\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})}\right)\\
&\quad\times\left(\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot B^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})}\right)\\
&\le CM_T^2,\\
\|\nabla\cdot(U_j\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})} &\le C\|U_j\|_{\mathcal{M}^1_T}\left(\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot B^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})}\right)\\
&\le CM_T\|U_j\|_{\mathcal{M}^1_T},\\
\|\nabla\cdot(U^*\nabla V_j)\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})} &\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})}\right)\|V_j\|_{\mathcal{M}^2_T}\\
&\le CM_T\|V_j\|_{\mathcal{M}^2_T},\\
\|\nabla\cdot(U_j\nabla V_j)\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})} &\le C\|U_j\|_{\mathcal{M}^1_T}\|V_j\|_{\mathcal{M}^2_T}.
\end{aligned}$$

By Proposition 3.4, there exists a unique solution $(U_{j+1}, V_{j+1}) \in \mathcal{M}^1_T \times \mathcal{M}^2_T$ of (5.5) with the estimates

$$\begin{aligned}
\|U_{j+1}\|_{\mathcal{M}^1_T} &\le C\|\nabla\cdot((U^*+U_j)\nabla(V^*+V_j))\|_{L^{\alpha_1,q}((0,T);\dot B^{s_1}_{r,1})}\\
&\le CM_T^2 + CM_T\|U_j\|_{\mathcal{M}^1_T} + CM_T\|V_j\|_{\mathcal{M}^2_T} + C\|U_j\|_{\mathcal{M}^1_T}\|V_j\|_{\mathcal{M}^2_T},\\
\|V_{j+1}\|_{\mathcal{M}^2_T} &\le C\|U^*+U_j\|_{L^{\alpha_2,q}((0,T);\dot B^{s_2}_{r,1})} \le CM_T + C\|U_j\|_{\mathcal{M}^1_T}.
\end{aligned} \tag{5.9}$$

Hence we have by induction that $(U_j, V_j) \in \mathcal{M}^1_T \times \mathcal{M}^2_T$ for all $j \ge 0$. By (5.8) and (5.9), taking $0 < T \le \infty$ such that $M_T \le 1/(8C^2)$, we have by Proposition 5.1 that

$$\|U_j\|_{\mathcal{M}^1_T} \le \frac{1}{8C^2}, \quad \|V_j\|_{\mathcal{M}^2_T} \le \frac{1}{4C}.$$

We set $U_{-1} := 0,\ V_{-1} := 0,\ \overline{U}_j := U_j - U_{j-1}$ and $\overline{V}_j := V_j - V_{j-1}$ for $j \ge 0$. Noting that $\overline{U}_0 = U_0,\ \overline{V}_0 = V_0$ and

$$\begin{aligned}
\overline{W}_j &:= -\nabla\cdot((U^*+U_j)\nabla(V^*+V_j)) + \nabla\cdot((U^*+U_{j-1})\nabla(V^*+V_{j-1}))\\
&= -\nabla\cdot(\overline{U}_j\nabla V^* + U^*\nabla\overline{V}_j + \overline{U}_j\nabla V_j + U_{j-1}\nabla\overline{V}_j),
\end{aligned}$$

we have by (5.5) that

$$\begin{cases} \partial_t\overline{U}_{j+1} - \Delta\overline{U}_{j+1} = \overline{W}_j & \text{in } (0,T)\times\mathbb{R}^n,\\ \overline{U}_{j+1}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$
$$\begin{cases} \partial_t\overline{V}_{j+1} - \Delta\overline{V}_{j+1} = \overline{U}_j & \text{in } (0,T)\times\mathbb{R}^n,\\ \overline{V}_{j+1}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{5.12}$$

for $j \geq 1$. Notice that (5.12) is valid for $j = 0$. Since $(\overline{U}_j, \overline{V}_j) \in \mathcal{M}^1_T \times \mathcal{M}^2_T$, it holds by Lemma 4.5 and (4.9) that

$$\begin{aligned}
\|\overline{U}_j\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} &\leq C\|\overline{U}_j\|_{\mathcal{M}^1_T},\\
\|\nabla\cdot(\overline{U}_j\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &\leq C\|\overline{U}_j\|_{\mathcal{M}^1_T}\left(\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\right)\\
&\leq CM_T\|\overline{U}_j\|_{\mathcal{M}^1_T},\\
\|\nabla\cdot(U^*\nabla\overline{V}_j)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &\leq C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\right)\|\overline{V}_j\|_{\mathcal{M}^2_T}\\
&\leq CM_T\|\overline{V}_j\|_{\mathcal{M}^2_T},\\
\|\nabla\cdot(\overline{U}_j\nabla V_j)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &\leq C\|\overline{U}_j\|_{\mathcal{M}^1_T}\|V_j\|_{\mathcal{M}^2_T},\\
\|\nabla\cdot(U_{j-1}\nabla\overline{V}_j)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} &\leq C\|U_{j-1}\|_{\mathcal{M}^1_T}\|\overline{V}_j\|_{\mathcal{M}^2_T}.
\end{aligned}$$

Hence it follows from Proposition 3.4 that

$$\begin{aligned}
\|\overline{U}_{j+1}\|_{\mathcal{M}^1_T} &\leq C\|\overline{W}_j\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\\
&\leq CM_T\|\overline{U}_j\|_{\mathcal{M}^1_T} + CM_T\|\overline{V}_j\|_{\mathcal{M}^2_T} + C\|\overline{U}_j\|_{\mathcal{M}^1_T}\|V_j\|_{\mathcal{M}^2_T} + C\|U_{j-1}\|_{\mathcal{M}^1_T}\|\overline{V}_j\|_{\mathcal{M}^2_T}
\end{aligned}$$

and

$$\|\overline{V}_{j+1}\|_{\mathcal{M}^2_T} \leq C\|\overline{U}_j\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} \leq C\|\overline{U}_j\|_{\mathcal{M}^1_T}.$$

Since

$$M_T \leq \frac{1}{8C^2}, \quad \|U_j\|_{\mathcal{M}^1_T} \leq \frac{1}{8C^2}, \quad \|V_j\|_{\mathcal{M}^2_T} \leq \frac{1}{4C}$$

for all $j \geq 0$, we have that

$$\|\overline{U}_{j+1}\|_{\mathcal{M}^1_T} \leq \frac{3}{8}\|\overline{U}_j\|_{\mathcal{M}^1_T} + \frac{1}{4C}\|\overline{V}_j\|_{\mathcal{M}^2_T}.$$

Since

$$\|\overline{U}_0\|_{\mathcal{M}^1_T} = \|U_0\|_{\mathcal{M}^1_T} \leq \frac{1}{8C^2}, \quad \|\overline{V}_0\|_{\mathcal{M}^2_T} = \|V_0\|_{\mathcal{M}^2_T} \leq \frac{1}{4C},$$

it follows from Proposition 5.1 that

$$\|\overline{U}_j\|_{\mathcal{M}^1_T} \leq \frac{1}{2C}\left(\frac{3}{4}\right)^{j+1}, \quad \|\overline{V}_j\|_{\mathcal{M}^2_T} \leq \frac{1}{2}\left(\frac{3}{4}\right)^{j}.$$

Notice that the following relations

$$U_N = \sum_{j=0}^{N}\overline{U}_j, \quad V_N = \sum_{j=0}^{N}\overline{V}_j, \quad \sum_{j=0}^{\infty}\frac{1}{2C}\left(\frac{3}{4}\right)^{j+1} = \frac{3}{2C} < \infty, \quad \sum_{j=0}^{\infty}\frac{1}{2}\left(\frac{3}{4}\right)^{j} = 2 < \infty$$

are valid. Therefore,

$$U := \lim_{N\to\infty} U_N = \sum_{j=0}^{\infty}\overline{U}_j \quad \text{in } \mathcal{M}^1_T, \quad V := \lim_{N\to\infty} V_N = \sum_{j=0}^{\infty}\overline{V}_j \quad \text{in } \mathcal{M}^2_T$$

are convergence. It holds by $(U,V)\in\mathcal{M}_T^1\times\mathcal{M}_T^2$ that $U(0)=V(0)=0$ and

$$\begin{aligned}
&\|\partial_t U_{j+1}-\Delta U_{j+1}-(\partial_t U-\Delta U)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\to 0 \quad (j\to\infty),\\
&\|\partial_t V_{j+1}-\Delta V_{j+1}-(\partial_t V-\Delta V)\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\to 0 \quad (j\to\infty).
\end{aligned}$$

We set

$$\begin{aligned}
W_j &:= -\nabla\cdot((U^*+U_j)\nabla(V^*+V_j))+\nabla\cdot((U^*+U)\nabla(V^*+V))\\
&= -\nabla\cdot((U_j-U)\nabla V^*+U^*\nabla(V_j-V)+(U_j-U)\nabla V_j+U\nabla(V_j-V)).
\end{aligned}$$

Since it holds by Lemma 4.5 and (4.9) that

$$\begin{aligned}
&\|U-U_j\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\le C\|U-U_j\|_{\mathcal{M}_T^1},\\
&\|\nabla\cdot((U_j-U)\nabla V^*)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\\
&\le C\|U_j-U\|_{\mathcal{M}_T^1}\left(\|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})}+\|\partial_t V^*,\Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\right)\\
&\le CM_T\|U_j-U\|_{\mathcal{M}_T^1},\\
&\|\nabla\cdot(U^*\nabla(V_j-V))\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\\
&\le C\left(\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}+\|\partial_t U^*,\Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\right)\|V_j-V\|_{\mathcal{M}_T^2}\\
&\le CM_T\|V_j-V\|_{\mathcal{M}_T^2},\\
&\|\nabla\cdot((U_j-U)\nabla V_j)\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\le C\|U_j-U\|_{\mathcal{M}_T^1}\|V_j\|_{\mathcal{M}_T^2},\\
&\|\nabla\cdot(U_{j-1}\nabla(V_j-V))\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})}\le C\|U_{j-1}\|_{\mathcal{M}_T^1}\|V_j-V\|_{\mathcal{M}_T^2},
\end{aligned}$$

we have that

$$\|U-U_j\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}\to 0 \quad (j\to\infty)$$

and

$$\begin{aligned}
\|W_j\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})}&\le CM_T\|U_j-U\|_{\mathcal{M}_T^1}+CM_T\|V_j-V\|_{\mathcal{M}_T^2}\\
&\quad+C\|U_j-U\|_{\mathcal{M}_T^1}\|V_j\|_{\mathcal{M}_T^2}+C\|U_{j-1}\|_{\mathcal{M}_T^1}\|V_j-V\|_{\mathcal{M}_T^2}\\
&\to 0 \quad (j\to\infty).
\end{aligned}$$

Therefore, by taking $j\to\infty$ in (5.5), we have that

$$\begin{cases}\partial_t U-\Delta U=-\nabla\cdot((U^*+U)\nabla(V^*+V)) & \text{in } (0,T)\times\mathbb{R}^n,\\ \qquad U(0)=0 & \text{in } \mathbb{R}^n,\end{cases}$$

$$\begin{cases}\partial_t V-\Delta V=U^*+U & \text{in } (0,T)\times\mathbb{R}^n,\\ \qquad V(0)=0 & \text{in } \mathbb{R}^n.\end{cases}$$

Since (U^*,V^*) is a solution of (5.2), we see that $(u,v):=(U^*+U,V^*+V)$ is a solution of (1.1). Since

$$\partial_t u,\Delta u\in L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1}),\quad \partial_t v,\Delta v\in L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})$$

with $u_0=u(0)\in\dot{B}^{-2+n/p_1}_{p_1,q}$ and $v_0=v(0)\in\dot{B}^{n/p_2}_{p_2,q}$, we have by (4.9) that

$$u\in L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1}),\quad v\in L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1}).$$

Finally, we see that the solution of (1.1) is unique. Let us (u_1, v_1) and (u_2, v_2) be the solution of (1.1) in the class (1.9). We set $\overline{u} := u_1 - u_2,\ \ \overline{v} := v_1 - v_2$. Noting that

$$-\nabla \cdot (u_1 \nabla v_1) + \nabla \cdot (u_2 \nabla v_2) = -\nabla \cdot (\overline{u} \nabla v_1 + u_2 \nabla \overline{v}),$$

we have by (1.1) that

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} = -\nabla \cdot (\overline{u} \nabla v_1 + u_2 \nabla \overline{v}) & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad \overline{u}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

$$\begin{cases} \partial_t \overline{v} - \Delta \overline{v} = \overline{u} & \text{in } (0,T) \times \mathbb{R}^n, \\ \qquad \overline{v}(0) = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Let us define

$$\begin{aligned} M'_\sigma &:= \|e^{t\Delta} u_0\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})} + \|\partial_t u_2, \Delta u_2\|_{L^{\alpha_1,q}((0,\sigma);\dot{B}^{s_1}_{r,1})} \\ &\quad + \|e^{t\Delta} v_0\|_{L^{\alpha_2^*,q}((0,\sigma);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t v_1, \Delta v_1\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})}. \end{aligned}$$

It holds by Lemma 4.5 and (4.9) that

$$\begin{aligned} \|\overline{u}\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})} &\le C\|\overline{u}\|_{\mathcal{M}^1_\sigma}, \\ \|\nabla \cdot (\overline{u} \nabla v_1)\|_{L^{\alpha_1,q}((0,\sigma);\dot{B}^{s_1}_{r,1})} &\le C\|\overline{u}\|_{\mathcal{M}^1_\sigma} \left(\|e^{t\Delta} v_0\|_{L^{\alpha_2^*,q}((0,\sigma);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t v_1, \Delta v_1\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})} \right) \\ &\le CM'_\sigma \|\overline{u}\|_{\mathcal{M}^1_\sigma}, \\ \|\nabla \cdot (u_2 \nabla \overline{v})\|_{L^{\alpha_1,q}((0,\sigma);\dot{B}^{s_1}_{r,1})} &\le C \left(\|e^{t\Delta} u_0\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})} + \|\partial_t u_2, \Delta u_2\|_{L^{\alpha_1,q}((0,\sigma);\dot{B}^{s_1}_{r,1})} \right) \|\overline{v}\|_{\mathcal{M}^2_\sigma} \\ &\le CM'_\sigma \|\overline{v}\|_{\mathcal{M}^2_\sigma} \end{aligned}$$

for all $0 < \sigma < T$. Therefore, by Proposition 3.4, it holds that

$$\|\overline{u}\|_{\mathcal{M}^1_\sigma} \le C\|\nabla \cdot (\overline{u} \nabla v_1 + u_2 \nabla \overline{v})\|_{L^{\alpha_1,q}((0,\sigma);\dot{B}^{s_1}_{r,1})} \le CM'_\sigma \|\overline{u}\|_{\mathcal{M}^1_\sigma} + CM'_\sigma \|\overline{v}\|_{\mathcal{M}^2_\sigma}$$

and

$$\|\overline{v}\|_{\mathcal{M}^2_\sigma} \le C\|\overline{u}\|_{L^{\alpha_2,q}((0,\sigma);\dot{B}^{s_2}_{r,1})} \le C\|\overline{u}\|_{\mathcal{M}^1_\sigma}.$$

Hence it holds by above inequalities that

$$\|\overline{u}\|_{\mathcal{M}^1_\sigma} \le CM'_\sigma \|\overline{u}\|_{\mathcal{M}^1_\sigma} + C^2 M'_\sigma \|\overline{u}\|_{\mathcal{M}^1_\sigma} \le 2C^2 M'_\sigma \|\overline{u}\|_{\mathcal{M}^1_\sigma}.$$

By taking $0 < \sigma < T$ such that $M'_\sigma \le 1/(4C^2)$, we observe $\overline{u}(t) = \overline{v}(t) = 0$ for all $0 < t < \sigma$ since it holds that

$$\|\overline{u}\|_{\mathcal{M}^1_\sigma} \le \frac{1}{2} \|\overline{u}\|_{\mathcal{M}^1_\sigma}.$$

By repeating this argument, it holds that $\overline{u}(t) = \overline{v}(t) = 0$ for all $0 < t < T$. Thus we have that $u_1 \equiv u_2$ and $v_1 \equiv v_2$.

(ii) Assume that $u_0 \in \dot{B}^{-2+n/p_1}_{p_1,q}$ and $v_0 \in \dot{B}^{n/p_2}_{p_2,q}$ satisfy

$$\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}} + \|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}} \le \varepsilon$$

for some $\varepsilon > 0$. Notice that we have to take $0 < T \leq \infty$ such that $M_T \leq 1/(8C^2)$ to obtain the solution of (1.1). However, since we have by (4.9) that

$$\|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,\infty);\dot{B}^{s_2}_{r,1})} \leq C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}}, \quad \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,\infty);\dot{B}^{s_2^*}_{r_2^*,1})} \leq C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}}$$

and since the relation (5.3) holds, by taking $\varepsilon \leq 1/(16C^3)$, it holds that

$$\begin{aligned}
M_T &= \|e^{t\Delta}u_0\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} + \|\partial_t U^*, \Delta U^*\|_{L^{\alpha_1,q}((0,T);\dot{B}^{s_1}_{r,1})} \\
&\quad + \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,q}((0,T);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t V^*, \Delta V^*\|_{L^{\alpha_2,q}((0,T);\dot{B}^{s_2}_{r,1})} \\
&\leq 2C\left(\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,q}} + \|v_0\|_{\dot{B}^{n/p_2}_{p_2,q}}\right) \\
&\leq 2C\varepsilon \\
&\leq \frac{1}{8C^2}
\end{aligned}$$

for all $0 < T \leq \infty$. Therefore, we may take $T = \infty$.

Concerning the uniqueness for $q = \infty$, it is sufficient to show the existence $0 < \sigma < \infty$ satisfying $M_\sigma \leq 1/(4C^2)$ if (u_1, v_1) and (u_2, v_2) satisfy (1.11). We set

$$N_{u_i}(t) := \|\partial_t u_i(t), \Delta u_i(t)\|_{\dot{B}^{s_1}_{r,1}}, \quad N_{v_i}(t) := \|\partial_t v_i(t), \Delta v_i(t)\|_{\dot{B}^{s_2}_{r,1}}, \quad 0 < t < \infty, \quad i = 1, 2.$$

Since the condition (1.11) holds, by taking $R_\eta > 0$ sufficiently large, we have that

$$\begin{aligned}
&\sup_{t \geq R_\eta} \left\{ t\mu\left(\tau \in (0,\infty) \ \middle|\ N_{u_i}(\tau) > t\right)^{\frac{1}{\alpha_1}} \right\} \leq 2\eta, \\
&\sup_{t \geq R_\eta} \left\{ t\mu\left(\tau \in (0,\infty) \ \middle|\ N_{v_i}(\tau) > t\right)^{\frac{1}{\alpha_2}} \right\} \leq 2\eta.
\end{aligned}$$

We define $N^{(1)}_{u_i}(t)$ and $N^{(2)}_{u_i}(t)$ by

$$N^{(1)}_{u_i}(t) := \begin{cases} N_{u_i}(t) & \text{if } N_{u_i}(t) \leq R_\eta, \\ 0 & \text{if } N_{u_i}(t) > R_\eta, \end{cases} \qquad N^{(2)}_{u_i}(t) := \begin{cases} 0 & \text{if } N_{u_i}(t) \leq R_\eta, \\ N_{u_i}(t) & \text{if } N_{u_i}(t) > R_\eta, \end{cases} \qquad 0 < t < \infty$$

respectively. Then, we see that $N_{u_i} = N^{(1)}_{u_i} + N^{(2)}_{u_i}$ and $N^{(1)}_{u_i} \in L^\infty(0,\infty)$ with $\|N^{(1)}_{u_i}\|_{L^\infty(0,\infty)} \leq R_\eta$. By the definition of $N^{(2)}_{u_i}$, we have that

$$\begin{aligned}
&\|N^{(2)}_{u_i}\|_{L^{\alpha_1,\infty}(0,\infty)} \\
&= \max\left\{ \sup_{0 < t \leq R_\eta} \left\{ t\mu\left(\tau \in (0,\infty) \ \middle|\ N_{u_i}(\tau) > t\right)^{\frac{1}{\alpha_1}} \right\}, \sup_{t > R_\eta} \left\{ t\mu\left(\tau \in (0,\infty) \ \middle|\ N_{u_i}(\tau) > t\right)^{\frac{1}{\alpha_1}} \right\} \right\} \\
&= \max\left\{ R_\eta \mu\left(\tau \in (0,\infty) \ \middle|\ N_{u_i}(\tau) > R_\eta\right)^{\frac{1}{\alpha_1}}, \sup_{t > R_\eta} \left\{ t\mu\left(\tau \in (0,\infty) \ \middle|\ N_{u_i}(\tau) > t\right)^{\frac{1}{\alpha_1}} \right\} \right\} \\
&\leq 2\eta.
\end{aligned}$$

Hence we observe

$$\begin{aligned}\|\partial_t u_i, \Delta u_i\|_{L^{\alpha_1,\infty}((0,\sigma);\dot{B}^{s_1}_{r,1})} &\le 2\|N_{u_i}\|_{L^{\alpha_1,\infty}(0,\sigma)}\\ &\le C\left(\|N^{(1)}_{u_i}\|_{L^{\alpha_1,\infty}(0,\sigma)} + \|N^{(2)}_{u_i}\|_{L^{\alpha_1,\infty}(0,\sigma)}\right)\\ &\le C\left\{C\left(\int_0^\sigma (N^{(1)}_{u_i}(t))^{\alpha_1}dt\right)^{\frac{1}{\alpha_1}} + 2\eta\right\}\\ &\le CR_\eta\sigma^{\frac{1}{\alpha_1}} + C\eta.\end{aligned}$$

Similarly, it holds that

$$\|\partial_t v_i, \Delta v_i\|_{L^{\alpha_2,\infty}((0,\sigma);\dot{B}^{s_2}_{r,1})} \le CR_\eta\sigma^{\frac{1}{\alpha_2}} + C\eta.$$

Since we have by (4.9) that

$$\|e^{t\Delta}u_0\|_{L^{\alpha_2,\infty}((0,\infty);\dot{B}^{s_2}_{r,1})} \le C\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,\infty}}, \quad \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,\infty}((0,\infty);\dot{B}^{s_2^*}_{r_2^*,1})} \le C\|v_0\|_{\dot{B}^{n/p_2}_{p_2,\infty}},$$

by taking

$$\varepsilon \le \frac{1}{12C^3}, \quad \eta \le \frac{1}{24C^3}, \quad \sigma^{\frac{1}{\alpha_1}} + \sigma^{\frac{1}{\alpha_2}} \le \frac{1}{12C^3R_\eta},$$

it holds that

$$\begin{aligned}M'_\sigma &= \|e^{t\Delta}u_0\|_{L^{\alpha_2,\infty}((0,\sigma);\dot{B}^{s_2}_{r,1})} + \|\partial_t u_2, \Delta u_2\|_{L^{\alpha_1,\infty}((0,\sigma);\dot{B}^{s_1}_{r,1})}\\ &\quad + \|e^{t\Delta}v_0\|_{L^{\alpha_2^*,\infty}((0,\sigma);\dot{B}^{s_2^*}_{r_2^*,1})} + \|\partial_t v_1, \Delta v_1\|_{L^{\alpha_2,\infty}((0,\sigma);\dot{B}^{s_2}_{r,1})}\\ &\le C\left(\|u_0\|_{\dot{B}^{-2+n/p_1}_{p_1,\infty}} + \|v_0\|_{\dot{B}^{n/p_2}_{p_2,\infty}}\right) + CR_\eta(\sigma^{\frac{1}{\alpha_1}} + \sigma^{\frac{1}{\alpha_2}}) + 2C\eta\\ &\le C\varepsilon + \frac{1}{6C^2}\\ &\le \frac{1}{4C^2}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

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