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Abstract

In this paper, we will discuss asymptotic properties of two kernel type quantile estimators. The estimators are proposed from different point of views. Many researchers study quantile estimation because of its application to practical problems in the financial industry and risk assessment. Based on the kernel estimation method, two type estimators are proposed. One is a weighted combination of order statistics and the other is a direct inversion of the kernel distribution function estimator, which is equal to the quantile regression without covariates. In this paper we show that those estimators are asymptotically equivalent, and we compare asymptotic mean squared errors of them.

Key Words and Phrases: Kernel estimator, Nonparametric inference, Quantile estimation

1. Introduction

Many researchers study the estimation of the quantile because of its application to practical problems in financial industry and risk assessments. The risk measure VaR (Value at Risk) is a basic tool of financial engineering and tantamount quantile. For a continuous random variable X with a cumulative distribution function $F(x)$ and a density function $f(x)$. The p th quantile is defined by $Q(p) = \inf\{x : F(x) \geq q\}$. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed random variables with the distribution function $F(x)$. The simplest estimator of $Q(p)$ is the sample quantile estimator, that is

$$\widehat{Q}(p) = \inf\{x : F_n(x) \geq p\}$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ is the empirical distribution function. It is asymptotically normal with asymptotic variance $\sigma^2 = \frac{p(1-p)}{f^2(Q(p))}$.

An alternative estimator is the kernel quantile estimator

$$\widehat{Q}_{p,h} = \frac{1}{h}\int_0^1 F_n^{-1}(x)K\left(\frac{x-p}{h}\right)dx, \tag{1}$$

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where $F_n^{-1}(x)$ denotes the inverse of the empirical distribution function, and $K(\cdot)$ is a kernel function. Of course, the bandwidth should be chosen appropriately. Besides the obvious requirement $h \to 0$ $(n \to \infty)$, additional requirements must be placed to ensure consistency, asymptotic normality, asymptotic bias elimination, and higher order accuracy of the estimator $\widehat{Q}_{p,h}$.

1.1. Quantile estimators

Let us define order statistics $X_{1:n} \leq X_{2:n} \leq X_{n:n}$ based on the sample $X_1, X_2, \cdots, X_n$. Then the estimator $\widehat{Q}(p)$ is asymptotically equivalent to $X_{[np]+1:n}$ where $[a]$ denotes a maximum integer less than a. Further let us define order statistics $U_{1:n} \leq U_{2:n} \leq U_{n:n}$ based on a random sample $U_1, U_2, \cdots, U_n$ from the uniform distribution $U(0,1)$. Then $X_{[np]+1:n}$ and $F^{-1}(U_{[np]+1:n})$ are equivalent and we can discuss asymptotic expectation and variance of $\widehat{Q}(p)$. Since the distribution of $U_{[np]+1:n}$ is a beta distribution $Be([np]+1, n-[np])$, we can obtain the asymptotic mean squared error with residual term $O(n^{-2})$,which is given by

$$AMSE(\widehat{Q}(p)) = \frac{p(1-p)}{nf^2(Q(p))}. \tag{2}$$

The estimator in (1) has been studied by many researchers in the past (see Falk(1984, 1985), Sheather & Marron(1990), Xiang(1995) and Xiang & Vos(1997)). Clearly, $\widehat{Q}_{p,h}$ is an L-estimator since it can be written as a weighted sum of the order statistics $X_{i:n}, i = 1, 2, \ldots, n$:

$$\widehat{Q}_{p,h} = \sum_{i=1}^{n} v_{i,n} X_{i:n}, \qquad v_{i,n} = \frac{1}{h} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K\left(\frac{x-p}{h}\right) dx. \tag{3}$$

On the other hand, it is natural to get direct inversion of the kernel distribution function estimator. Let us define the kernel estimator of the distribution function as follows:

$$\widehat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} W\left(\frac{x-X_i}{h}\right) \tag{4}$$

where $W(t) = \int_{-\infty}^{t} K(u)du$. Then the natural quantile estimator is given by

$$Q^*_{p,h} = \inf\{x : \widehat{F}(x) \geq p\}. \tag{5}$$

The estimator $Q^*_{p,h}$ can be regarded as the special case of the quantile regression. Koenker & Bassett(1978) has proposed quantile regression which minimize the loss function

$$\min_{\alpha, \boldsymbol{\beta}} \sum_{i=1}^{n} \rho_p(X_i - \alpha - \boldsymbol{z}_i^t \boldsymbol{\beta}) \tag{6}$$

where $\boldsymbol{z}_i$ is a covariate vector and

$$\rho_p(u) = \left\{ \begin{array}{cc} pu, & u \geq 0 \\ (p-1)u, & u < 0 \end{array} \right. . \tag{7}$$

$\boldsymbol{\beta}$ is an unknown parameter vector. If $\boldsymbol{z}_i = \mathbf{0}$ $(i = 1, \ldots, n)$, the solution $\widehat{\alpha}$ of the equation (6) gives us a quantile estimator and $\widehat{\alpha} = Q^*_{p,h}$.

The purpose of this paper is to discuss asymptotic properties of the quantile estimators. We will obtain mean squared errors of them with residual term $o(n^{-1}h)$ and show that the estimators $\widehat{Q}_{p,h}$ and $Q^*_{p,h}$ are asymptotically equivalent.

1.2. Asymptotic properties of quantile estimators

For the kernel type estimator $\widehat{Q}_{p,h}$, Maesono & Penev (2011) have proved the following representation. Let us define $Y_i = F(X_i)$ and then $\{Y_i\}_{i=1,\cdots,n}$ are independent random variables uniformly distributed on $(0,1)$. Further let define

$$\bar{Q}(p) = \frac{1}{h}\int_0^1 F^{-1}(x)K(\frac{x-p}{h})dx,$$
$$\hat{I}_x(Y_1) = I(Y_1 \le p + hx) - (p + hx),$$
$$g_{1n}(Y_1) = -\int_{-1}^1 Q'(p+hx)K(x)\hat{I}_x(Y_1)dx,$$
$$g_{2n}(Y_1, Y_2) = -\int_{-1}^1 Q'(p+hx)K'(x)\hat{I}_x(Y_1)\hat{I}_x(Y_2)dx,$$
$$A_{1n} = \sum_{i=1}^n g_{1n}(Y_i), \qquad A_{2n} = \sum_{C_{n,2}} g_{2n}(Y_i, Y_j).$$

If the residual term R_n satisfies

$$E(R_n^2) = o(n^{-1}h),$$

we denote $R_n = o_p(n^{-1}h)$. Modifying the representation of Maesono & Penev (2011), we have the following lemma.

Lemma 1.1. *Assume $\int[F(x)(1-F(x))]^{1/5}dx < \infty$. Let $Q^{(4)}$ be uniformly bounded in a neighborhood of p $(0 < p < 1)$ and $f(Q(p)) > 0$. Let $K(x)$ be a symmetric kernel (i.e. $K(-x) = K(x)$) and $K^{(2)}(x) \in Lip(\alpha), \alpha > 0$. Further we assume that the support of the function $K(\cdot)$ is $[-1,\ 1]$. Denote $\delta = Q'(p)(\frac{1}{2} - p) + \frac{1}{2}Q^{(2)}(p)p(1-p)$. If the bandwidth h satisfies $n^{-1/3} \le h \le n^{-1/4}(\log n)^{-1}$, we have*

$$\widehat{Q}_{p,h} - \bar{Q}(p) = \frac{1}{n}A_{1n} + \frac{1}{n^2 h}A_{2n} + \frac{\delta}{n} + o_p(n^{-1}h). \tag{8}$$

Using this lemma, we can get asymptotic expectation and variance of $\widehat{Q}_{p,h}$.

Since the estimators $\widehat{Q}_{p,h}$ is an asymptotic U-statistic, we can show the asymptotic normality of $Q^*_{p,h}$ and $\widehat{Q}_{p,h}$.

Theorem 1.2. *Assume the conditions of Lemma 1.1. Then we have*

$$E(\widehat{Q}_{p,h}) = Q(p) + \frac{h^2\sigma_K^2 f'(Q(p))}{2f^3(Q(p))} + o(h^2),$$
$$V(\widehat{Q}_{p,h}) = \frac{p(1-p)}{nf^2(Q(p))} - \frac{2hk_1}{nf^2(Q(p))} + o(n^{-1}h)$$

and the asymptotic mean squared error is given by

$$AMSE(\widehat{Q}_{p,h}) = \frac{p(1-p)}{nf^2(Q(p))} - \frac{2hk_1}{nf^2(Q(p))} + \frac{h^4\sigma_K^4\{f'(Q(p))\}^2}{4f^6(Q(p))} \tag{9}$$

where $k_1 = \frac{1}{2}\{1 - \int_{-1}^{1} W^2(x)dx\}$.

Since $W(x)$ is a distribution function, $W^2(x) \le W(x)$ and then $k_1 > 0$.

Next we will obtain asymptotic representation of the $Q^*_{p,h}$. Note that $p = F(Q(p)) = \widehat{F}(Q^*_{p,h})$. Thus we have

$$\begin{aligned} 0 &= \widehat{F}(Q^*_{p,h}) - F(Q(p)) \\ &= \widehat{F}(Q^*_{p,h}) - \widehat{F}(Q(p)) + \widehat{F}(Q(p)) - F(Q(p)). \end{aligned}$$

Using the Taylor expansion, we get

$$\begin{aligned} & \widehat{F}(Q^*_{p,h}) - \widehat{F}(Q(p)) \qquad (10) \\ = \; & f(Q(p))\{Q^*_{p,h} - Q(p)\} + \{\widehat{f}(Q(p)) - f(Q(p))\}\{Q^*_{p,h} - Q(p)\} \\ & + \frac{\widehat{f}'(Q(p)) - f'(Q(p))}{2}\{Q^*_{p,h} - Q(p)\}^2 + \frac{f'(Q(p))}{2}\{Q^*_{p,h} - Q(p)\}^2 \\ & + \frac{\widehat{f}^{(2)}(Q(p)) - f^{(2)}(Q(p))}{6}\{Q^*_{p,h} - Q(p)\}^3 + \frac{f^{(2)}(Q(p))}{6}\{Q^*_{p,h} - Q(p)\}^3 \\ & + \frac{\widehat{f}^{(3)}(Q(p)) - f^{(3)}(Q(p))}{24}\{Q^*_{p,h} - Q(p)\}^4 + \frac{f^{(3)}(Q(p))}{24}\{Q^*_{p,h} - Q(p)\}^4 \\ & + \frac{\widehat{f}^{(4)}(\widetilde{Q})}{168}\{Q^*_{p,h} - Q(p)\}^5 \end{aligned}$$

where $\widetilde{Q}$ is a value between $Q(p)$ and $\widehat{Q}(p)$. As shown in the appendix, when we discuss an asymptotic expectation and variance of $Q^*_{p,h}$ until the order h^2 and $(nh)^{-1}$, we only need the asymptotic expectation and variance of the following approximation.

$$\begin{aligned} & Q^*_{p,h} - Q(p) \\ \approx \; & -\frac{1}{f(Q(p))}\left[\left\{\widehat{F}(Q(p)) - F(Q(p))\right\} + \{\widehat{f}(Q(p)) - f(Q(p))\}\{Q^*_{p,h} - Q(p)\}\right]. \end{aligned}$$

Using this representation, we can get the following theorem.

THEOREM 1.3. *Assume that the density function has bounded third differential* $f^{(3)}(x)$. *The kernel function* $K(\cdot)$ *is symmetric around the origin and has bounded derivative* $K'(\cdot)$. *If the bandwidth* h *satisfies* $n^{-1/3} \le h \le n^{-1/4}(\log n)^{-1}$, *we have*

$$\begin{aligned} E\left(Q^*_{p,h}\right) &= Q(p) - \frac{h^2\sigma_K^2 f'(Q(p))}{2f(Q(p))} + o(h^2), \\ V\left(Q^*_{p,h}\right) &= \frac{p(1-p)}{nf^2(Q(p))} - \frac{2hk_1}{nf(Q(p))} + o(n^{-1}h) \end{aligned}$$

Table 1: MSE ($p = 0.1$)

$n = 50$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.05489932	0.003280351	0.09482309
$\widehat{Q}_{p,h}$	**0.04602479**	0.002091292	**0.06389674**
$Q^*_{p,h}$	0.05378461	**0.001605083**	0.91770354
$n = 100$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.02916109	0.001331199	0.03869678
$\widehat{Q}_{p,h}$	**0.02525474**	0.0009585687	**0.03024975**
$Q^*_{p,h}$	0.03000956	**0.000905096**	0.91754283

where $k_1 = \int_{-\infty}^{\infty} uK(u)W(u)du$. *And then the asymptotic mean squared error is given by*

$$AMSE(Q^*_{p,h}) = \frac{p(1-p)}{nf^2(Q(p))} - \frac{2hk_1}{nf(Q(p))} + \frac{h^4\sigma_K^4\{f'(Q(p))\}^2}{4f^2(Q(p))} \tag{11}$$

Jones(1992) gave the mean squared error of the estimator $Q^*_{p,h}$. Comparing the equations (9) and (11), we have that if the support of $K(\cdot)$ is $[-1,\ 1]$, $k_1 = \frac{1}{2} - \frac{1}{2}\int_{-1}^{1} W^2(x)dx$. In many cases, the density function satisfies $|f(x)| \leq 1$. Then the square of the asymptotic bias of $Q^*_{p,h}$ is smaller than that of $\widehat{Q}_{p,h}$ and the asymptotic variance of $\widehat{Q}_{p,h}$ is smaller than that of $Q^*_{p,h}$.

We can show $Q^*_{p,h}$ and $\widehat{Q}_{p,h}$ are asymptotically equivalent in the sense of the first order asymptotic. Since main terms of $\widehat{Q}_{p,h}$ and $Q^*_{p,h}$ are sample means, they are asymptotically normal. We can obtain the following theorem.

Theorem 1.4. *Assume that the conditions of Theorem1.2 and Theorem1.3, we have*

$$\lim_{n\to\infty} E\left[\left\{\sqrt{n}\left(Q^*_{p,h} - \widehat{Q}_{p,h}\right)\right\}^2\right] = 0.$$

2. Simulation

Let us compare $\widehat{Q}(p)$, $\widehat{Q}_{p,h}$ and $Q^*_{p,h}$ by simulation. Here we consider when the underlying distribution F is standard normal (N(0,1)), exponential (Exp(1)) and gamma distribution ($\Gamma(2,2)$). Sample sizes are 50 and 100 and $p = 0.1, 0.25, 0.5, 0.75$. The repetition time is 1,000, the bandwidth is $h = n^{-1/4}(\log n)^{-1}$, and the kernel function is the Epanechnikov $K(u) = \frac{3}{4}(1-u^2)I(|u| \leq 1)$. In Table 1 $\sim$ 4, the bold numbers are smallest. The mean squared errors of both kernel type estimators are superior than the estimator $\widehat{Q}(p)$ which based on the empirical distribution. The kernel type estimators are comparable.

The kernel type estimators are comparable. Especially, $\widehat{Q}_{p,h}$ shows good performance. Since the $\widehat{Q}_{p,h}$ does not need inversion, the estimator has small variances.

Table 2: MSE ($p = 0.25$)

$n = 50$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.03690784	0.007444699	0.1179751
$\widehat{Q}_{p,h}$	**0.03274878**	0.006419052	**0.1055921**
$Q^*_{p,h}$	0.0348852	**0.00571473**	2.6710586
$n = 100$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.01913422	0.00361322	0.05961159
$\widehat{Q}_{p,h}$	**0.01799609**	**0.003143485**	**0.05189728**
$Q^*_{p,h}$	0.01980991	0.003218674	2.67672001

Table 3: MSE ($p = 0.5$)

$n = 50$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.03104016	0.022871	0.2264228
$\widehat{Q}_{p,h}$	**0.02765495**	**0.01922293**	**0.1982005**
$Q^*_{p,h}$	0.02954645	0.01922293	7.1302426
$n = 100$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.01678377	0.010381502	0.10393733
$\widehat{Q}_{p,h}$	**0.01585592**	**0.009513172**	**0.09748764**
$Q^*_{p,h}$	0.01630611	0.010234096	7.11420621

Table 4: MSE ($p = 0.75$)

$n = 50$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.03626702	0.06958815	0.4741157
$\widehat{Q}_{p,h}$	**0.03189623**	**0.06313137**	**0.4397589**
$Q^*_{p,h}$	0.03371842	0.06454984	15.9881487
$n = 100$	$N(0,1)$	$Exp(1)$	$\Gamma(2,2)$
$\widehat{Q}(p)$	0.0190663	0.03067551	0.2707586
$\widehat{Q}_{p,h}$	**0.01740836**	**0.02747103**	**0.2420617**
$Q^*_{p,h}$	0.01779845	0.02842799	16.0689921

Whereas, since we use the Newton- Raphson approximation to get the approximation of the inversion, the estimator $Q^*_{p,h}$ has the big mean squared errors. Then we recommend the quantile estimator $\widehat{Q}_{p,h}$.

3. Appendices

Proof of Theorem1.2

Since $\widehat{Q}_{p,h} - \overline{Q}(p)$ is an asymptotic U-statistic, we have that

$$\begin{aligned} E[\widehat{Q}_{p,h} - \overline{Q}(p)] &= \frac{\delta}{n} + o(n^{-1}), \\ V(\widehat{Q}_{p,h}) &= \frac{1}{n}E[g_{1n}^2(Y_1)] + \frac{1}{2n^2h^2}E[g_{2n}^2(Y_1)] + o(n^{-2}). \end{aligned}$$

It follows from the definition of g_{1n} that

$$\begin{aligned} & E[g_{1n}^2(Y_1)] \\ = \; & E\left(\int_{-1}^{1} Q'(p+hx)K(x)\hat{I}_x(Y_1)dx \int_{-1}^{1} Q'(p+hy)K(y)\hat{I}_y(Y_1)dy\right) \\ = \; & E\left(\int_{-1}^{1}\int_{-1}^{1} Q'(p+hx)Q'(p+hy)K(x)K(y)\hat{I}_x(Y_1)\hat{I}_y(Y_1)dxdy\right) \\ = \; & 2\iint_{x<y} Q'(p+hx)Q'(p+hy)K(x)K(y)\{(p+hx)-(p+hx)(p+hy)\}dxdy \\ = \; & 2\iint_{x<y} \{Q'(p)+hxQ''(p)\}\{Q'(p)+hyQ''(p)\}K(x)K(y) \\ & \times\{(p+hx)-(p+hx)(p+hy)\}dxdy + O(h^2) \\ = \; & 2\iint_{x<y} \{Q'(p)^2 + h(x+y)Q'(p)Q''(p)\}K(x)K(y) \\ & \times\{p(1-p)+h((1-p)x-py)\}dxdy + O(h^2) \\ = \; & 2Q'(p)^2p(1-p)\iint_{x<y} K(x)K(y)dxdy \\ & +2hQ'(p)Q''(p)p(1-p)\iint_{x<y}(x+y)K(x)K(y)dxdy \\ & -2hQ'(p)^2p\iint_{x<y}(x+y)K(x)K(y)dxdy \\ & +2hQ'(p)^2\iint_{x<y} xK(x)K(y)dxdy + O(h^2). \end{aligned}$$

Since the function $K(\cdot)$ is symmetric around the origin, we have

$$\iint_{x<y} K(x)K(y)dxdy = \frac{1}{2}, \qquad \iint_{x<y}(x+y)K(x)K(y)dxdy = 0$$

and

$$\iint_{x<y} xK(x)K(y)dxdy = \frac{1}{2}\left\{\int_{-1}^{1} W^2(x)dx - 1\right\}.$$

Thus we get

$$E[g_{1n}^2(Y_1)]$$

$$\begin{aligned}
&= Q'(p)^2 p(1-p) + 2hQ'(p)p\{(1-p)Q''(p) - Q'(p)\} \iint_{x<y} (x+y)K(x)K(y)dxdy \\
&\quad +2hQ'(p)^2 \iint_{x<y} xK(x)K(y)dxdy + O(h^2) \\
&= Q'(p)^2 p(1-p) + hQ'(p)^2 \left\{ \int_{-1}^{1} W^2(x)dx - 1 \right\} + O(h^2).
\end{aligned}$$

Similarly, we can show that $E[g_{2n}^2(Y_1, Y_2)] = O(h)$, and then we have the variance of $\widehat{Q}_{p,h}$.

It is easy to get the asymptotic bias. Since $E[A_{1n}] = E[A_{2n}] = 0$, the asymptotic bias is $\overline{Q} - Q(p) + \frac{\delta}{n}$. Using the transformation $u = (x-p)/h$, for sufficiently large n, we have

$$\begin{aligned}
&\frac{1}{h}\int_{-1}^{1} F^{-1}(x)K\left(\frac{x-p}{h}\right)dx - Q(p) \\
&= \int_{-p/h}^{(1-p)/h} Q(p+hu)K(u)du - Q(p) \\
&= \int_{-1}^{1} \{Q(p) + Q'(p)hu + \frac{h^2u^2}{2}Q''(p) + O(h^3)u^3\}K(u)du - Q(p) \\
&= \frac{h^2\sigma_K^2}{2}Q''(p) + O(h^3).
\end{aligned}$$

Thus we have the desired result. □

Proof of Theorem1.3

From the equation (10), the main term of the approximation $Q^*_{p,h} - Q(p)$ is given by

$$-\frac{1}{f(Q(p))}\{\widehat{F}(Q(p)) - F(Q(p))\} = -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{f(Q(p))}\left\{W\left(\frac{Q(p)-X_i}{h}\right) - F(Q(p))\right\}.$$

Here we only consider the expectation. The asymptotic variance is similarly obtained. It follows from an algebraic inequality that

$$|\widehat{F}(Q(p)) - F(Q(p))|^\ell \le C\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\left[W\left(\frac{Q(p)-X_i}{h}\right) - E(W)\right]\right|^\ell + |E(W) - Q(p)|^\ell\right\},$$

where $E(W) = E\left[W\left(\frac{Q(p)-X_1}{h}\right)\right]$. Since

$$\widehat{f}^{(4)}(x) = \frac{1}{h^5}\sum_{i=1}^{n}K^{(4)}\left(\frac{\widetilde{Q}-X_i}{h}\right),$$

we have $\widehat{f}^{(4)}(x) = O(h^{-5})$. Using the moment evaluation of *i.i.d.* sum, and the bias of $\widehat{F}(\cdot)$ we can get

$$E|\widehat{F}(Q(p)) - F(Q(p))|^\ell = O(n^{-\ell/2} + h^{2\ell})$$

and then

$$E|Q^*_{p,h} - Q(p)|^\ell = O(n^{-\ell/2}h^{-\ell} + h^{2\ell}).$$

Therefore, we get

$$E\left|\frac{\widehat{f}^{(4)}(\widetilde{Q})}{168}\{Q^*_{p,h} - Q(p)\}^5\right| = O(h^{-5}\{n^{-5/2} + h^{10}\}) = O(n^{-5/6}) = o(h^2).$$

For the second term of the equation (10), it follows from the asymptotic mean squared error of the kernel type density estimator that

$$\begin{aligned} & E|\{\widehat{f}(Q(p)) - f(Q(p))\}\{Q^*_{p,h} - Q(p)\}| \\ \le\ & \left\{E[\widehat{f}(Q(p)) - f(Q(p))]^2 E[Q^*_{p,h} - Q(p)]^2\right\}^{1/2} \\ =\ & \left\{O((nh)^{-1} + h^4)O(n^{-1} + h^4)\right\}^{1/2} = o(h^2). \end{aligned}$$

Similarly we can show that the expectation of the rest terms of the equation (10) are $o(h^2)$. Using the mean squared error of the kernel type distribution function (see Shirahata & Chu (1992)), we get the Theorem1.3. □

Proof of Theorem1.4
From the definition, we get

$$\begin{aligned} & E\left[\left\{\sqrt{n}\left(Q^*_{p,h} - \widehat{Q}_{p,h}\right)\right\}^2\right] \\ =\ & E\left[n\left(Q^*_{p,h} - Q(p)\right)^2\right] - E\left[2n\left(Q^*_{p,h} - Q(p)\right)\left(\widehat{Q}_{p,h} - Q(p)\right)\right] \\ & +E\left[n\left(\widehat{Q}_{p,h} - Q(p)\right)^2\right] \\ =\ & nE\left[\left(Q^*_{p,h} - Q(p)\right)^2\right] - 2nE\left[\left(Q^*_{p,h} - Q(p)\right)\left(\widehat{Q}_{p,h} - Q(p)\right)\right] \\ & +nE\left[\left(\widehat{Q}_{p,h} - Q(p)\right)^2\right]. \end{aligned}$$

It follows from Theorem1.2 and Theorem1.3 that

$$nE\left[\left(Q^*_{p,h} - Q(p)\right)^2\right] = \frac{p(1-p)}{f^2(Q(p))} + o(1)$$

and

$$nE\left[\left(\widehat{Q}_{p,h} - Q(p)\right)^2\right] = \frac{p(1-p)}{f^2(Q(p))} + o(1).$$

Let us evaluate $nE\left[\left(Q^*_{p,h} - Q(p)\right)\left(\widehat{Q}_{p,h} - Q(p)\right)\right]$. Since the biases of $Q^*_{p,h}$ and $\widehat{Q}_{p,h}$ are both $o(n^{-1/2})$, we have

$$\begin{aligned} & nE\left[\left(Q^*_{p,h} - Q(p)\right)\left(\widehat{Q}_{p,h} - Q(p)\right)\right] \\ =\ & Q'(p)Cov\left[-W\left(\frac{Q(p) - X_1}{h}\right), g_{1n}(F(X_1))\right] + o(1). \end{aligned}$$

Further, using the fact that the distributions X_1 and $Q(Y_1)$ are same, we get

$$\begin{aligned}
& Cov\left[-W\left(\frac{Q(p)-X_1}{h}\right), g_{1n}(F(X_1))\right] \\
= \ & Cov\left[-W\left(\frac{Q(p)-Q(Y_1)}{h}\right), g_{1n}(Y_1)\right] \\
= \ & E\left[-W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\widehat{I}_x(Y_1)dx\right] \\
& -E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\right]\cdot E\left[g_{1n}(Y_1)\right] \\
= \ & E\left[-W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\widehat{I}_x(Y_1)dx\right].
\end{aligned}$$

It follows from the definition that

$$\begin{aligned}
& E\left[\widehat{I}_x(Y_1)-\widehat{I}_0(Y_1)\right]^2 \\
= \ & E\left[I(Y_1 \le p+hx) - I(Y_1 \le p) - hx\right]^2 \\
= \ & E\left[I^2(Y_1 \le p+hx) + I^2(Y_1 \le p) + h^2x^2 - 2I(Y_1 \le p+hx)I(Y_1 \le p)\right. \\
& \left. -2hxI(Y_1 \le p+hx) + 2hxI(Y_1 \le p)\right] \\
= \ & p+hx+p+h^2x^2-2\min\{p+hx,\ p\}-2hx(p+hx)+2hxp \\
= \ & 2p-2\min\{p+hx,\ p\}+hx-h^2x^2 \\
\le \ & h|x|+h^2x^2
\end{aligned}$$

Since $W(x)$ is bounded $|W\left(\frac{Q(p)-Q(y)}{h}\right)| \le C$, we have that

$$\begin{aligned}
& E\left[-W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\widehat{I}_x(Y_1)dx\right] \\
= \ & E\left[-W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\widehat{I}_0(Y_1)dx\right] + o(h^{1/2}).
\end{aligned}$$

Using Taylor expansion, we get

$$\begin{aligned}
& E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)I(Y_1 \le p+hx)dx\right] \\
= \ & \int_{-\infty}^{\infty} W\left(\frac{Q(p)-Q(y)}{h}\right) f(y)\int_{-1}^{1} Q'(p+hx)K(x)I(y \le p+hx)dxdy \\
= \ & \int_{0}^{1} W\left(\frac{Q(p)-Q(y)}{h}\right)\frac{1}{1-0}\int_{-1}^{1} Q'(p+hx)K(x)I(y \le p+hx)dxdy \\
= \ & \int_{0}^{1} W\left(\frac{Q(p)-Q(y)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)I(y \le p+hx)dxdy,
\end{aligned}$$

and then we get

$$\left|E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\left\{I(Y_1 \le p+hx)-I(Y_1 \le p)\right\}dx\right]\right|$$

$$
\begin{aligned}
&= \left| \int_{-1}^{1} Q'(p+hx)K(x)E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\{I(Y_1 \le p+hx) - I(Y_1 \le p)\}\right] dx \right| \\
&\le C\left| \int_{-1}^{1} Q'(p+hx)K(x)\cdot CE\left[\{I(Y_1 \le p+hx) - I(Y_1 \le p)\}\right] dx \right| \\
&= \left| \int_{-1}^{1} Q'(p+hx)K(x)E\left[I(Y_1 \le p+hx) - I(Y_1 \le p)\right] dx \right| \\
&\le \left(\int_{-1}^{1} \{Q'(p+hx)K(x)\}^2 dx \right)^{\frac{1}{2}} \left(\int_{-1}^{1} E\left[\{I(Y_1 \le p+hx) - I(Y_1 \le p)\}^2\right] dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

Here we have

$$
\begin{aligned}
& E\left[\{I(Y_1 \le p+hx) - I(Y_1 \le p)\}^2\right] \\
&= E\left[\{I(Y_1 \le p+hx)\}^2 - 2I(Y_1 \le p+hx)I(Y_1 \le p) + \{I(Y_1 \le p)\}^2\right] \\
&= E\left[\{I(Y_1 \le p+hx)\}^2\right] - 2E\left[I(Y_1 \le p+hx)I(Y_1 \le p)\right] + E\left[\{I(Y_1 \le p)\}^2\right] \\
&= \int_0^1 I(y \le p+hx)dy - 2\int_0^1 I(y \le p+hx)I(y \le p)dy + \int_0^1 I(y \le p)dy \\
&= \int_0^{p+hx} dy - 2\int_0^{\min\{p+hx,p\}} dy + \int_0^p dy \\
&= p + hx - 2\min\{p+hx, p\} + p \\
&= \begin{cases} hx & (x \ge 0) \\ -hx & (x < 0). \end{cases}
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
& \left| E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)\{I(Y_1 \le p+hx) - I(Y_1 \le p)\} dx\right] \right| \\
&\le C\left(\int_{-1}^{1} \{Q'(p+hx)K(x)\}^2 dx \right)^{\frac{1}{2}} \left(\int_{-1}^{1} h|x|dx \right)^{\frac{1}{2}} \\
&= O(h^{\frac{1}{2}})\left(\int_{-1}^{1} \{Q'(p+hx)K(x)\}^2 dx \right)^{\frac{1}{2}} \\
&= O(h^{\frac{1}{2}})
\end{aligned}
$$

Further, using the Taylor expansion of $Q'(p+hx)$, we can show that

$$
\begin{aligned}
& E\left[W\left(\frac{Q(p)-Q(Y_1)}{h}\right)\int_{-1}^{1} Q'(p+hx)K(x)I(Y_1 \le p)dx\right] \\
&= Q'(p)\int_0^1 \left\{W\left(\frac{Q(p)-Q(y)}{h}\right)I(y \le p)\int_{-1}^{1} K(x)dx\right\} dy + O(h^2) \\
&= Q'(p)\int_0^p W\left(\frac{Q(p)-Q(y)}{h}\right) dy + O(h^2).
\end{aligned}
$$

Using the integration by parts and the transformation $u = \{Q(p) - Q(y)\}/h$, we have $y = F(Q(p) - hu)$ and $du = -\frac{Q'(y)}{h}dy$. Since $F(Q(p)) = p$, we get

$$\begin{aligned}
& Q'(p) \int_0^p W\left(\frac{Q(p) - Q(y)}{h}\right) dy \\
= \quad & Q'(p) \left(\left[W\left(\frac{Q(p) - Q(y)}{h}\right)\right]_0^p - \int_0^p K\left(\frac{Q(p) - Q(y)}{h}\right)\left\{-\frac{Q'(y)}{h}\right\} ydy \right) \\
= \quad & Q'(p) \left(W(0)p - \int_\infty^0 K(u) F(Q(p) - hu) du \right) \\
= \quad & \frac{pQ'(p)}{2} + Q'(p) \int_0^\infty K(u) F(Q(p)) du + O(h) \\
= \quad & pQ'(p) + O(h).
\end{aligned}$$

Thus we have the desired result. □

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