Homotpy invariants for finite spaces

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Homotopy invariants for finite spaces

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Introduction

Digital topology treats two or three dimensional digital images that is understood and analyzed in terms of topological invariants such that path-connectivity, fundamental groups or characteristic numbers of digital objects, and has a lot of applications in a wide range of fields including medicine and informatics (for example, [**Udu92, US96, Kov89, SSB15**]).

In [**KKR90**], Khalimsky et al. introduced a digital model by giving a topological structure to lattice points, and they proposed a method of image compression using a circle represented by finitely many points which are now called a Khalimsky circle. For instance, the finite pixels and voxels in the digital image correspond to a "finite space" in Khalimsky's digital model. We remark that a finite space is nothing but a topological space with finitely many points. For example, the world of RPG known as Dragon Quest is quite similar to a Khalimsky torus, the product of two Khalimsky circles.

One important thing in topology is a motion of an object which is considered as a path in the configuration space of the object. A numerical invariant representing a complexity to give a continuous planning of such motion was introduced by M. Farber in [**Far03**] which is known as the topological complexity and is denoted by TC. Topological complexity is studied by many authors in some 20 years, but it looks very difficult to determine even for polyhedra in principle.

On the other hand, if we restrict our attention to finite spaces, the situation is a little bit different. In principle, there exists an algorithm with finitely many steps to determine the topological complexity by verifying the conditions in the definition of topological complexity. However, if we try to do that in reality, the required computer resources will easily become inaccessibly large.

Another important thing is that: a finite space is shown to be equipped with a polyhedron as its topological counterpart with a weak homotopy equivalence from the polyhedron to the finite space. For example, the Khalimsky circle denoted by S^1_n, which is consisting of $2n$ points, is equipped with a circle S^1 as its counterpart which satisfies $\mathrm{TC}(S^1) = 1$. One of Tanaka's two conjectures in [Tan18] tells us that the topological complexity of a Khalimsky circle would be different from that of S^1 the topological counter part of a Khalimsky circle. If it is the case, we must accept that the topological complexity of a finite space X can not approximate the topological complexity of $\mathcal{K}(X)$ the topological counter part of X.

In Chapter 1, we answer this conjecture by studying properties of the topological invariants lying deeply in digital topology. For example, homology is often used in topological data analysis, but in general, homology alone is not enough to distinguish two different figures. In fact, our result implies there are two digital images that cannot be distinguished by homology but can be distinguished by deeper invariants. In other words, the topological complexity is not a weak homotopy invariant, but a homotopy invariant. From this perspective, we hope that the computation of various invariants in digital topology will open-up wider applications.

As an analogue of topological complexity, D. Fernández-Ternero et al. introduced in the discrete topological complexity for a simplicial complex [**FTMVMV18**], which is related to the simplicial LS category for a simplicial complex which is introduced in [**FTMVV15**]. The contiguity distance for a simplicial map was introduced by Macías-Virgós and Mosquera-Lois in [MVML18] as a more general notion for the above two invariants. In Chapter 2, we determine the discrete topological complexity for a simplicial complex homotopy equivalent to a wedge of two circles. This gives a counterexample to triangle inequality for contiguity distance.

In Chapter 3, we introduce the fundamental monoid of a based finite space as an analogy for the fundamental group. The fundamental monoid is defined by using Khalimsky circles instead of a usual circle. Moreover, we shows that there is a simply-connected finite space with non-trivial fundamental monoid. This result implies that the fundamental monoid is not a weak homotopy invariant, thus fundamental monoid a stronger invariant of a fundamental group for a finite space.

Contents

CHAPTER 1

Topological complexity of Khalimsky circles

The topological complexity was first introduced by M. Farber in [**Far03**] as a numerical homotopy invariant in a relation with the robot motion planning problem. It attracts many authors such as M. Grant, J. González, D. Davis, L. Vandembroucq, A. Dranishnikov, N. Iwase and etc.

In these 10 years, a number of authors defined versions of topological complexity for a discrete or a finite space. For example, combinatorial complexity for finite spaces is introduced by K. Tanaka [**Tan18**], discrete topological complexity for simplicial complexes is introduced by D. Fernández-Ternero, E. Macías-Virgós, E. Minuz, and J. A. Vilches [**FTMVMV18**] and simplicial complexity for simplicial complexes is introduced by J. González [**Gon18**] and digital topological complexity for digital images by İ. Karaca and M. İs [**Kİ18**].

Topological complexity of a space X is defined as follows: for a space X, let PX be the path space on X equipped with the compact-open topology, or more precisely, all continuous maps from the unit interval $[0,1]$ to X, which is equipped with a projection $\pi : PX \to X\times X$ given by $\pi(\gamma) = (\gamma(0), \gamma(1))$. The topological complexity $\mathrm{TC}(X)$ of a space X is the minimal number $m \geqslant 0$ such that $X\times X$ is covered by $m+1$ open subsets each on which there is a continuous section to π, i.e., *section-categorical* (cf. James [**Jam95**]).

For a space X, let LX be a subspace of PX consisting of all paths γ in X with $\gamma(0) = *$ the base point of X, which is equipped with a projection $\pi_0 : LX \to X$ given by the restriction of π to LX, where we regard $\{*\}\times X = X$. Then we obtain a similar but a more classical invariant called L-S category $\operatorname{cat} X$, which is the minimal number $m \geqslant 0$ such that X is covered by $m+1$ open subsets on which there is a section to π_0. Since LX is contractible, each open set of the covering is contractible in X, in other words, categorical. Then the following inequality is well known:

$$\mathrm{cat}(X) \leqslant \mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X) \leqslant (\mathrm{cat}(X)+1)^2 - 1.$$

In this chapter, we consider the topological complexity and the L-S category of a finite space, though it is completely determined by the combinatorial structures. Note that Tanaka calls topological complexity for finite space X combinatorially defined as combinatorial complexity and use the notation $\mathrm{CC}(X)$ [**Tan18**], but we will not distinguish it from topological complexity.

Let us denote by $\mathbb{S}^1_n$ the *Khalimsky circle* of $2n$-points, for $n \geq 3$:

$$\mathbb{S}^1_n = \{a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}\},$$

whose topology is determined by the open sub-basis $\{a_i, b_i, a_{i+1}\}$ where i runs over $\mathbb{Z}/n$. Exceptionally, for $n = 2$, the Khalimsky circle $\mathbb{S}^1_2 = \{a_0, a_1; b_0, b_1\}$ is topologized by the the open basis $\{\{a_0\}, \{a_1\}, \{a_0, b_0, a_1\}, \{a_0, b_1, a_1\}\}$.

Then order complex $\mathcal{K}(\mathbb{S}^1_n)$ is a circle complex with n zero-cells and n one-cells. We remark that $\mathbb{S}^1_n$ is weakly homotopy equivalent with the topological circle S^1, more precisely, there is a weak homotopy equivalence from $S^1 = |\mathcal{K}(\mathbb{S}^1_n)|$ to $\mathbb{S}^1_n$. So, Khalimsky circles are good example to observe the difference between the topological complexity of the realization and that of a finite space.

Recently, Tanaka showed $\mathrm{TC}(\mathbb{S}^1_2) = \mathrm{cat}(\mathbb{S}^1_2 \times \mathbb{S}^1_2) = 3$ and $\mathrm{TC}(\mathbb{S}^1_3) = \mathrm{cat}(\mathbb{S}^1_3 \times \mathbb{S}^1_3) = 2$ in [**Tan18**], while we know $\mathrm{TC}(S^1) = 1$ and $\mathrm{cat}(S^1 \times S^1) = 2$ for a topological circle S^1. More recently, S. Kandola showed $\mathrm{cat}(\mathbb{S}^1_n \times \mathbb{S}^1_n) = 2$ for all $n \geqslant 3$ in [**Kan19**]. In [**Tan18**], K. Tanaka raised two conjectures on the topological complexity of finite spaces.

Conjecture. *(1)* [**Tan18**, Conjecture 4.16] $\mathrm{TC}(\mathbb{S}^1_{2k})) = 2$ *for any* $k \geq 2$.

(2) [**Tan18**, Conjecture 3.11] *There is a finite space* X *satisfying* $\mathrm{TC}(X) < \mathrm{cat}(X \times X)$.

The purpose of this chapter is to answer the above conjectures. More precisely, we answer in the negative to (1) and positive to (2) by giving the following result.

Theorem. *The topological complexity of Khalimsky circle* $\mathbb{S}^1_n$ *is given as follows:*

$$\mathrm{TC}(\mathbb{S}^1_n) = \begin{cases} 2, & n = 4, \\ 1, & n \geqslant 5. \end{cases}$$

This chapter is organized as follows: In Section 1.1, we explain some basics of finite T_0-spaces following J. A. Barmak [**Bar11**]. In Sections 1.2 and 1.3, we determine the topological complexity of Khalimsky circles. Finally, in Appendix, we give a precise proof of the homotopy classification of maps between Khalimsky circles, which is used in the previous two sections.

1.1. Finite T_0-spaces

We recall the basic important results of finite spaces.

A finite space has a finite T_0-subspace as its strong deformation retract (see J. A. Barmak [**Bar11**, Remark 1.3.2] for example). So from a homotopy-theoretical view-point, there are no differences between a finite space and a finite T_0-space. From now on, we work in the category of finite T_0-spaces and continuous maps between them, unless otherwise stated.

Firstly, P. Alexandroff pointed out a one-to-one correspondence between a finite T_0-space and a finite poset in [**Ale41**]: for a given finite T_0-space X with a family of open sets $\mathcal{O}$ as its topology, we define a relation $\leq_{\mathcal{O}}$ as follows: for $x, y \in X$,

$$x \leq_{\mathcal{O}} y \overset{\text{def}}{\iff} \text{for any } U \in \mathcal{O},\ y \in U \text{ implies } x \in U.$$

Then $\leq_{\mathcal{O}}$ gives a poset structure on X, since $\mathcal{O}$ is a T_0-topology. Conversely, for a finite poset P with poset structure $\leq$, we define a topology $\mathcal{O}_{\leq}$ on P: for $V \subset P$,

$$V \in \mathcal{O}_{\leq} \overset{\text{def}}{\iff} x \in V \text{ iff } x \leq y \text{ for some } y \in V.$$

Then $\mathcal{O}_{\leq}$ is a T_0-topology on P. So a finite T_0-space is a finite poset, and vice versa. This implies that a homotopy-theoretical properties of finite spaces are described combinatorially.

Secondly, in [**McC66**], M.C. McCord showed a tight connection between a finite poset and a finite abstract simplicial complex: for a finite poset P, we obtain an order complex $\mathcal{K}(P)$ such that $\sigma \in \mathcal{K}(P) \iff \sigma$ is a linearly ordered subset of P. Conversely, for a finite simplicial complex K, we obtain a finite poset $\chi(K) = (K, \leq)$ such that $\tau \leq \sigma \iff \tau \subset \sigma$. We remark that $\mathcal{K}(\chi(K))$ is nothing but the barycentric subdivision of K. Furthermore, he obtained that, for a finite T_0-space X which is in fact a poset, there is a weak homotopy equivalence from $|\mathcal{K}(X)|$ to X, and also that, for a finite simplicial complex K, there is a weak homotopy equivalence $|K| \to \chi(K)$. But unfortunately, it does not imply $\mathrm{TC}(|\mathcal{K}(X)|) = \mathrm{TC}(X)$ nor $\mathrm{cat}(|\mathcal{K}(X)|) = \mathrm{cat}(X)$. For instance, we have $\mathrm{TC}(S^1) = 1 \neq 3 = \mathrm{TC}(\mathbb{S}^1_2)$ and $\mathrm{cat}(S^1 \times S^1) = 2 \neq 3 = \mathrm{cat}(\mathbb{S}^1_2 \times \mathbb{S}^1_2)$.

Thirdly, R. E. Stong discovered about combinatorial description of homotopy between finite spaces in [**Sto66**]. Even for maps $f, g : X \to Y$ between finite T_0-spaces, we adopt the usual notation of homotopy, i.e. $f \simeq g$ if and only if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In that case, we say f and g are homotopic as usual.

Theorem 1.1.1 ([**Sto66**]). *Let $f, g : X \to Y$ be two maps between finite T_0-spaces. Then, $f \simeq g$ if and only if there exists a sequence of maps $f_0, f_1, \ldots, f_n$ such that*

$$f = f_0 \leq f_1 \geq f_2 \leq \cdots \geq (\leq) f_n = g.$$

, where $f_i \leq f_{i+1}$ if and only if $f_i(x) \leq f_{i+1}(x)$ for any $x \in X$.

Finally, we recall the following important theorem.

Theorem 1.1.2 ([**Bar11**], for example). *Let $f, g : X \to Y$ be two homotopic maps between finite T_0-spaces. Then, two maps $|\mathcal{K}(f)|, |\mathcal{K}(g)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ are homotopic.*

For a given map $f : X \to Y$ such that $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ are two compact oriented manifolds of the same dimension, we define the *degree* of f as the degree of $|\mathcal{K}(f)|$, and denote by $\deg f$.

Let X be a finite T_0-space which is naturally a poset. Let us denote by

$$[a, b]_X = \{x \in X | a \leq x \leq b\}, \quad [a, -]_X = \{x \in X | a \leq x\} \text{ and } [-, b]_X = \{x \in X | x \leq b\}.$$

We abbreviate as $[a, b] = [a, b]_X$, $[a, -] = [a, -]_X$ and $[-, b] = [-, b]_X$, if it causes no confusion.

Proposition 1.1.3. *Let U be an open set in $X \times X$. Then, U is section-categorical if and only if* $\mathrm{Pr}_1|_U \simeq \mathrm{Pr}_2|_U$*, where* $\mathrm{Pr}_i : X \times X \to X$ *is the i-th projection for $i = 1, 2$.*

The above proposition holds for any topological space because the interval I is locally compact Hausdorff and the exponential law holds (see the proof of [**MVML18**, Proposition 2.7]).

For an open cover $\mathcal{U}$ of a finite space X, we say that $\mathcal{U}$ is *principal* if for each open set U in $\mathcal{U}$, there exist some maximal elements $m_0, m_1, \ldots, m_l$ in X such that

$$U = \bigcup_i [-, m_i]_X.$$

Theorem 1.1.4. *Let X be a finite space of* $\mathrm{TC}(X) = k$*. Then there exists a principal open covering of $X \times X$ consisting of $k+1$ section-categorical open subsets.*

Proof. For a family of subsets $\mathcal{V}$ of a finite topological space Y, we define

$$\mathcal{V}_M = \{V_M \mid V \in \mathcal{V}\},$$

where $V_M = \bigcup_{m \in M(Y) \cap V} [-, m]$ and $M(Y)$ is the set of all maximal elements of Y. We remark that V_M is a subset of V if V is open. Then, for any open covering $\mathcal{O}$ of Y, $\mathcal{O}_M$ is also an open covering of Y, and is a refinement of $\mathcal{O}$. In our case, we have an open covering $\mathcal{V} = \{V_0, V_1 \ldots, V_k\}$ of $X \times X$, since $\mathrm{TC}(X) = k$. Thus $\mathcal{U} = \mathcal{V}_M$ gives our desired open covering of $X \times X$ for a finite space X. $\square$

1.2. The proof of $\mathrm{TC}(\mathbb{S}^1_k) = 1$, $k \geqslant 5$

The following theorem is the key to prove that $\mathrm{TC}(\mathbb{S}^1_k) = 1$, $k \geqslant 5$.

Theorem 1.2.1. *For given two distinct maps $f, g : \mathbb{S}^1_m \to \mathbb{S}^1_n$, f and g are homotopic if and only if* $\deg f = \deg g = d$ *for some d with $|d| < m/n$.*

If f, g are homotopic, then $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)| : S^1 \to S^1$ by Theorem 1.1.2, so we have $\deg f := \deg |\mathcal{K}(f)| = \deg |\mathcal{K}(g)| =: \deg g$. Theorem 1.2.1 shall be completely proved in the section 1.4.

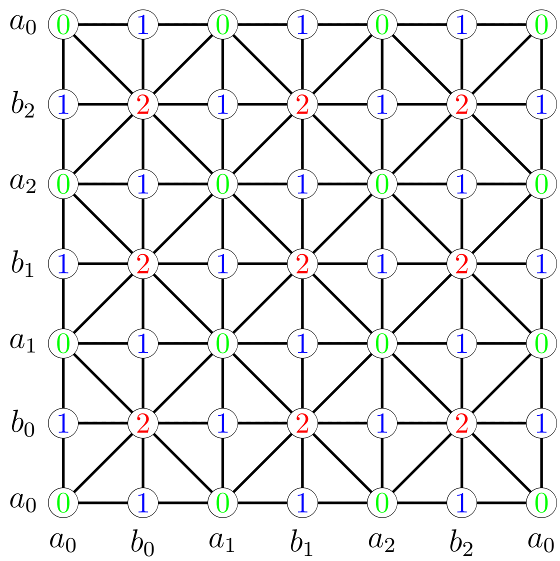


FIGURE 1. The poset struture of $\mathbb{S}^1_k \times \mathbb{S}^1_k$

This theorem implies the following main theorem.

THEOREM 1.2.2. $\mathrm{TC}(\mathbb{S}^1_k) = 1$, $k \geqslant 5$.

PROOF. Let $k \geqslant 5$ and $m = 2\lfloor k/2 \rfloor + 1$, that is $m = k$ if k is odd and $m = k+1$ if k is even. It suffices to show that $\mathrm{TC}(\mathbb{S}^1_k) \leqslant 1$, since we know $1 = \mathrm{TC}(|\mathbb{S}^1_k|) \leqslant \mathrm{TC}(\mathbb{S}^1_k)$. From now on, let us denote the elements a_i and b_i in $\mathbb{S}^1_k$ by just integers $2i+1$ and $2i+2$, respectively. In this proof, we use "$\leqslant$" as the order of numbers, i.e., the usual meaning of "greater than or equal to" and "$\leq$" as the order of posets. Using the above notation, we define an open cover of $\mathbb{S}^1_k \times \mathbb{S}^1_k$ of section-categorical open subsets.

Let U, V be open subsets covering $\mathbb{S}^1_k \times \mathbb{S}^1_k$ defined as follows:

(1) $U = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$,

(2) V is the smallest open neighborhood of $\mathbb{S}^1_k \times \mathbb{S}^1_k \setminus U$,

where A_i's are defined as follows.

$$A_0 = \{1, 2, \ldots, 2k{-}1\}\times\{1, 2, 3\}$$

$$A_1 = \{m, m{+}1, m{+}2\}\times\{3, 4, \ldots, 2k{-}1\}$$

$$A_2 = \{m{+}2, m{+}3, \ldots, 2k, 1\}\times\{2k{-}3, 2k{-}2, 2k{-}1\}$$

$$A_3 = \{1, 2, \ldots, m{-}2\}\times\{5, 6, \ldots, 2k{-}1\}$$

$$A_4 = \{1, 2, 3\}\times\{2k{-}1, 2k, 1\}.$$

Then, we give a proof of the fact that U is section-categorical, because a similar argument works also for V.

Firstly, for each non-negative integer $i \leqslant 2k{-}m{-}3$, we define a continuous map $f_i : U \to U$ by

$$f_i(x, y) = \begin{cases} (k_i, y), & (x, y) \in A_0 \cap \{(x, y) \mid k_i \leqslant x\} \subset U, \\ (l_i, y), & (x, y) \in A_3 \cap \{(x, y) \mid \max\{3, l_i\} \leqslant x\} \subset U, \\ (x, y), & \text{otherwise.} \end{cases}$$

where $k_i = 2k{-}1{-}i$, $l_i = m{-}2{-}i$ so that $(k_i, y) \in A_0 \subset U$ if $(x, y) \in A_0$ and $(l_i, y) \in A_3 \subset U$ if $(x, y) \in A_3$. Since A_0 and A_3 are disjoint, so f_i is well-defined.

Then we can easily see that $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$, and hence $f_i \simeq f_{i+1}$. Thus we have $1_U = f_0 \simeq f_{2k-m-3}$ and hence we obtain that $C_1 = f_{2k-m-3}(U)$ is a deformation retract of U. Since $k \geqslant 5$, we have $m{+}2 \leqslant k{+}3 \leqslant 2k{-}1$ and $3 \leqslant k \leqslant m$, and we can easily see the following equation.

$$C_1 = A'_0 \cup A_1 \cup A_2 \cup A'_3 \cup A_4$$

where $A'_0 \subset A_0$ and $A'_3 \subset A_3$ are given as follows.

$$A'_0 = \{1, 2, \ldots, m{+}2\}\times\{1, 2, 3\} \subset A_0$$

$$A'_3 = \{1, 2, 3\}\times\{5, 6, \ldots, 2k{-}1\} \subset A_3.$$

Secondly, for each non-negative integer $0 \leqslant i \leqslant 2k{-}8$, we define a continuous map $g_i : C_1 \to C_1$ by

$$g_i(x, y) = \begin{cases} (x, 5{+}i), & (x, y) \in A'_3 \cap \{(x, y) \mid y \leq 5{+}i\} \subset C_1, \\ (x, y), & \text{otherwise.} \end{cases}$$

Since $(x, 5+i) \in A_3' \subset C_1$ provided that $(x, y) \in A_3'$, g_i is well-defined.

Then we can easily see that $g_i \simeq g_{i+1}$. Thus we have $1_{C_1} = g_0 \simeq g_{2k-8}$, and hence we obtain that $C_2 = g_{2k-8}(C_1)$ is a deformation retract of C_1.

Since $k \geqslant 5$, we have $5 \leqslant 2k-3$, and we can easily see the following equation.

$$C_2 = A_0' \cup A_1 \cup A_2 \cup A_3'' \cup A_4$$

where $A_3'' \subset A_3'$ are given as follows.

$$A_3'' = \{1, 2, 3\} \times \{2k-3, 2k-2, 2k-1\}$$

Thirdly, we define a continuous map $h_0 : C_2 \to C_2$ by

$$h_0(x, y) = \begin{cases} (x+1, y), & (x, y) \in A^{\rightarrow}, \\ (x-1, y), & (x, y) \in A^{\leftarrow}, \\ (x, y+1), & (x, y) \in A^{\uparrow}, \\ (x, y-1), & (x, y) \in A^{\downarrow}, \\ (x-1, y+1), & (x, y) \in A^{\nwarrow}, \\ (x+1, y-1), & (x, y) \in A^{\searrow}, \\ (x, y), & \text{otherwise}, \end{cases}$$

where the sets $A^{\rightarrow}$, $A^{\leftarrow}$, $A^{\uparrow}$, $A^{\downarrow}$, $A^{\nwarrow}$, $A^{\searrow}$ are defined as follows:

$$A^{\rightarrow} := (\{1\} \times \{1, 2, 2k\}) \cup (\{m\} \times \{4, 5, \ldots, 2k-2\}),$$

$$A^{\leftarrow} := (\{3\} \times \{2k-2, 2k-1, 2k\}) \cup (\{m+2\} \times \{2, 3, \ldots, 2k-4\}),$$

$$A^{\uparrow} := (\{4, 5, \ldots, m+1\} \times \{1\}) \cup (\{1, 2\} \times \{2k-3\}) \cup (\{m+3, m+4, \ldots, 2k\} \times \{2k-3\}),$$

$$A^{\downarrow} := (\{2, 3, \ldots, m-1\} \times \{3\}) \cup (\{m+1, m+2, \ldots, 2k\} \times \{2k-1\}),$$

$$A^{\nwarrow} := \{(m+2, 1), (3, 2k-3)\},$$

$$A^{\searrow} := \{(1, 3), (m, 2k-1)\}.$$

Then we can easily see that $1_{C_2} \simeq h_0$, and hence we obtain that $C_3 = h_0(C_2)$ is a deformation retract of C_2.

Finally, we define a continuous map $h_1 : C_3 \to C_3$ by

$$h_1(x,y) = \begin{cases} (1, 2k{-}1), & (x,y) = (1, 2k{-}2), (2, 2k{-}2), (2, 2k{-}1), \\ (3,1), & (x,y) = (2,1), (2,2), (3,2), \\ (m,3), & (x,y) = (m,2), (m{+}1, 2), (m{+}1, 3), \\ (m{+}2, 2k{-}3), & (x,y) = (m{+}1, 2k{-}3), (m{+}1, 2k{-}2), (m{+}2, 2k{-}2), \\ (x,y) & \text{otherwise.} \end{cases}$$

Then we can easily see that $1_{C_3} \simeq h_1$, and hence we obtain that $C = h_1(C_3)$ is a deformation retract of C_3, where C can be described as follows.

$$\begin{aligned} C = &\{(a, a{-}2) \mid a = 3, 4\} \cup \{(b, 2) \mid b = 4, \ldots, m{-}1\} \\ &\cup \{(c, c{+}3{-}m) \mid c = m{-}1, m, m{+}1\} \cup \{(m{+}1, d) \mid d = 4, \ldots, 2k{-}4\} \\ &\cup \{(e, e{+}2k{-}m{-}5) \mid e = m{+}1, m{+}2, m{+}3\} \cup \{(f, 2k{-}2) \mid f = m{+}3, \ldots, 2k\} \\ &\cup \{(g, g{+}2k{-}2) \mid g = 1, 2\}. \end{aligned}$$

Then the arguments given above show that C is a deformation retract of U.

Since $\deg(\mathrm{Pr}_1|_C) = \deg(\mathrm{Pr}_2|_C) = 1$ and $C \approx \mathbb{S}^1_{2k-4}$, we can proceed to obtain that $\mathrm{Pr}_1|_C \simeq \mathrm{Pr}_2|_C$ by Theorem 1.2.1. Thus U is section-categorical and so is V, and hence we obtain $\mathrm{TC}(\mathbb{S}^1_k) = 1$. □

Remark 1.2.3. *The Figure 4 below shows how the open set U in the first figure is deformed into the core C as a retract in the last figure, in the case when $k = 6$.*

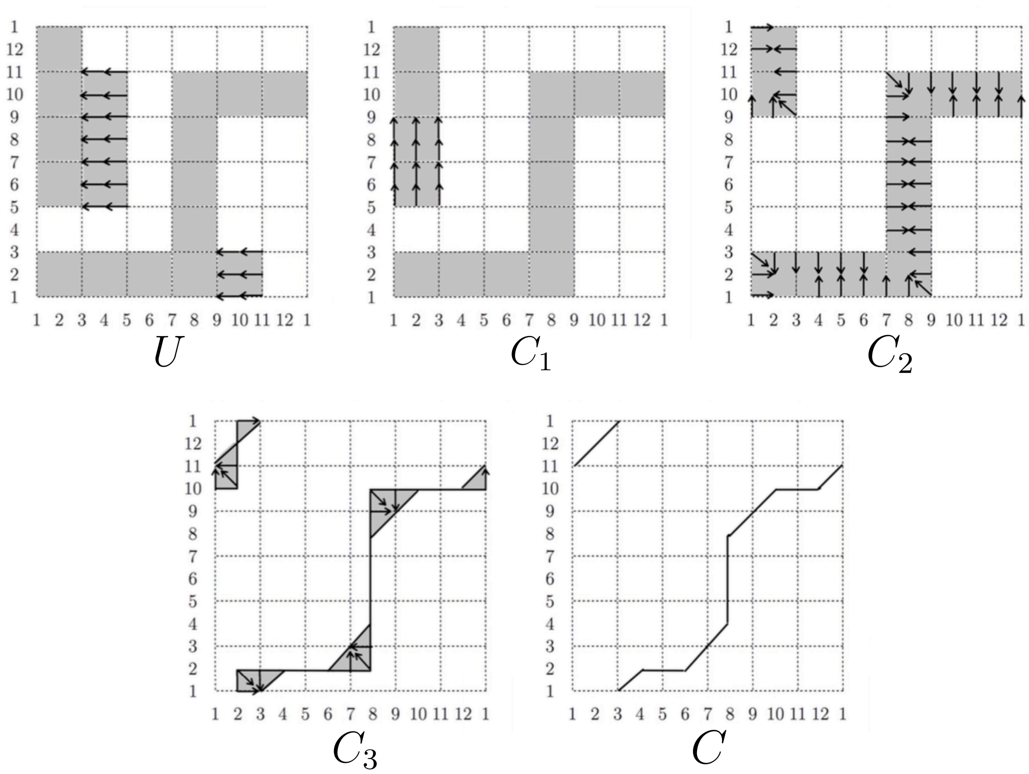


FIGURE 2.

1.3. The proof of $\mathrm{TC}(\mathbb{S}^1_4) = 2$

To prove $\mathrm{TC}(\mathbb{S}^1_4) = 2$, we use a square-shaped figure of a Khalimsky torus $\mathbb{S}^1_4 \times \mathbb{S}^1_4$, where $|\mathcal{K}(\mathbb{S}^1_4 \times \mathbb{S}^1_4)| \approx S^1 \times S^1$, decomposed as a collection of small squares.

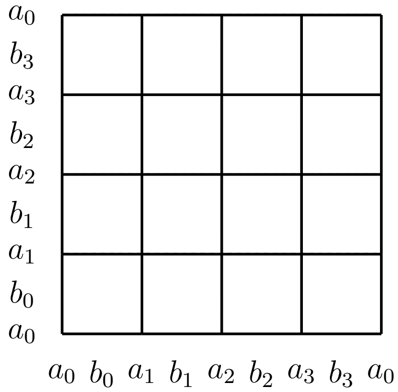


FIGURE 3. The square decomposition

In the square-shaped figure, each small square represents the following open set in $\mathbb{S}^1_4 \times \mathbb{S}^1_4$:

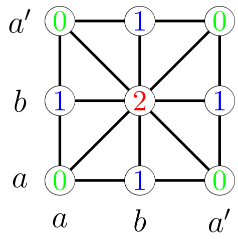


FIGURE 4.

Then a principal open cover of $\mathbb{S}^1_4 \times \mathbb{S}^1_4$ by n-open subsets is in one-to-one correspondence with an n-coloring on C the set of all small squares in Figure 3. So we must consider a coloring on C, where an n-coloring on C can be identified with a map $\chi : C \to \mathbb{Z}/n = \{0, 1, \ldots, n-1\}$.

PROPOSITION 1.3.1. *If $|\chi^{-1}(i)|$ contains a whole horizontal or a whole vertical line, then $\chi^{-1}(i)$ is not section-categorical.*

PROOF. In the case when $|\chi^{-1}(i)|$ contains a horizontal line, we have the subspace V of $\chi^{-1}(i)$ which corresponds to the line, and V is represented as the following set for some element $a \in \mathbb{S}^1_4$:

$$V = \{(x, a) \mid x \in \mathbb{S}^1_4\}.$$

Hence $V \approx \mathbb{S}^1_4$, $\deg \mathrm{Pr}_1|_V = 1$ and $\deg \mathrm{Pr}_2|_V = 0$. If we assume that $\chi^{-1}(i)$ is section-categorical, then $\mathrm{Pr}_1|_{\chi^{-1}(i)} \simeq \mathrm{Pr}_2|_{\chi^{-1}(i)}$. Since $V \subset \chi^{-1}(i)$, we obtain $\mathrm{Pr}_1|_V \simeq \mathrm{Pr}_2|_V$, and hence $\deg \mathrm{Pr}_1|_V = \deg \mathrm{Pr}_2|_V$ by Theorem 1.2.1, which is a contradiction. In the case when $|\chi^{-1}(i)|$ contains a vertical line, a similar argument lead us to a contradiction, too. $\square$

So, let us call a coloring $\chi : C \to \mathbb{Z}/n$ a simple coloring if $|\chi^{-1}(i)|$ does not contain a horizontal line nor a vertical line for all $i \in \mathbb{Z}/n$.

PROPOSITION 1.3.2. *There are only two simple colorings up to isomorphism.*

1	0	0	1
0	0	1	1
0	1	1	0
1	1	0	0

(I)

1	0	1	1
0	0	1	0
1	1	1	0
1	0	0	0

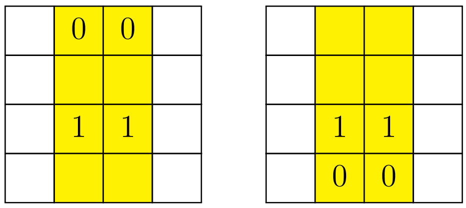
(II)

Proof. By the assumption on the coloring χ, we have

(1) In the continued two rows of squares $\begin{array}{|c|c|c|c|}\hline \, & \, & \, & \, \\\hline \, & \, & \, & \, \\\hline\end{array}$, we must find out $\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}$ and $\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$.

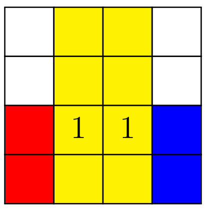
(2) In the continued two columns of squares $\begin{array}{|c|c|}\hline \, & \, \\\hline \, & \, \\\hline \, & \, \\\hline \, & \, \\\hline\end{array}$, we must find out $\begin{array}{|c|c|}\hline 0 & 0\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}$.

In (2) above, there are only two cases. $\begin{array}{|c|c|}\hline 0 & 0\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array}$ are connected or not. Focusing on the middle two columns, we may consider the following two patterns without loss of generality.

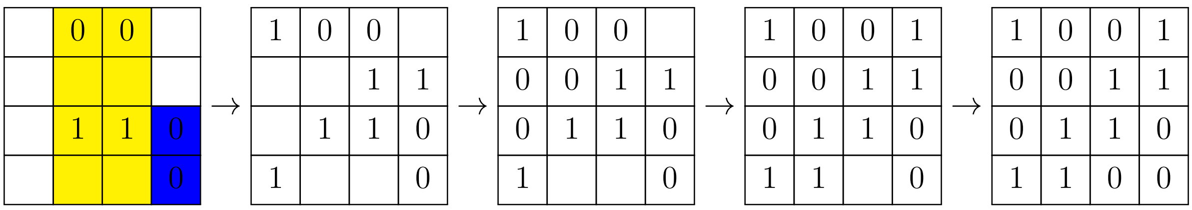


(I) (II)

Ignoring the upper part, either red or blue part must be $\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}$ in both the cases.



Without loss of generality, we assume that the blue part is $\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}$. Using the conditions (1) and (2), the remaining part can be filled as follows in cases (I) and (II).



(I)

$$
\underset{\text{(II)}}{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & 1 & 1 & 0 \\ \hline & 0 & 0 & 0 \\ \hline \end{array}}
\to
\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & & \\ \hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}
\to
\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline 0 & 0 & & 0 \\ \hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}
\to
\begin{array}{|c|c|c|c|}\hline 1 & & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}
\to
\begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}
$$

It completes the proof of the proposition. □

THEOREM 1.3.3. $\mathrm{TC}(\mathbb{S}^1_4) = 2$.

PROOF. Since $1 = \mathrm{TC}(S^1) \leqslant \mathrm{TC}(\mathbb{S}^1_4) \leqslant \mathrm{cat}(\mathbb{S}^1_4 \times \mathbb{S}^1_4) = 2$, it is sufficient to show that $1 < \mathrm{TC}(\mathbb{S}^1_4)$ where we know $1 \leqslant \mathrm{TC}(\mathbb{S}^1_4) \leqslant 2$. By assuming $\mathrm{TC}(\mathbb{S}^1_4) = 1$, we are lead to a contradiction. If $\mathrm{TC}(\mathbb{S}^1_4) = 1$, then there is an open covering $\{U_1, U_2\}$ of $\mathbb{S}^1_4 \times \mathbb{S}^1_4$ such that $\mathrm{Pr}_1|_{U_i} \simeq \mathrm{Pr}_2|_{U_i}$, where U_i's should not include a whole horizontal or a whole vertical line by Proposition 1.3.1. Thus they must be isomorphic to (I) or (II) in Proposition 1.3.2.

$$
\underset{\text{(I)}}{\begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}}
\qquad
\underset{\text{(II)}}{\begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}}
$$

In either case, U_0 or U_1 must contain at least one non-trivial Khalimsky circle $\mathbb{S} \approx \mathbb{S}^1_4$ on the blue part or on the red part, which is not on the diagonal line.

Then by the definition of topological complexity, we have $\mathrm{Pr}_1|_{\mathbb{S}} \simeq \mathrm{Pr}_2|_{\mathbb{S}}$. If they were distinct, then since they have same degree and $1 = |\deg \mathrm{Pr}_1| \geqslant 4/4$, they would not be homotopic by Theorem 1.2.1 , which is contradicting that $\mathbb{S}$ is not on the diagonal line. Thus $1 < \mathrm{TC}(\mathbb{S}^1_4)$ and $\mathrm{TC}(\mathbb{S}^1_4) = 2$. □

1.4. The homotopy classification of maps between Khalimsky circles

E. Khalimsky, R. Kopperman, and P. R. Meyer. defined a digital plane as finite spaces in [**KKR90**]. The circle that comes up in considering the Jordan curve on the digital plane it was called the Khalimsky circle (for example, [**Kis00**]).

In this section, we keep in mind that the relation "$\leqslant$" on $\mathbb{Z}$ is used in the usual meaning of "greater than or equal to".

We begin this chapter with introducing a Khalimsky line denoted simply by $\mathbb{Z}$. The Khalimsky line $\mathbb{Z}$ is the set of integers $\mathbb{Z}$ with the topology generated by an open sub-basis $\{\{2k,\ 2k{+}1,\ 2k{+}2\} \mid k \in \mathbb{Z}\}$.

Khalimsky line is known to be an Alexandroff space, and an Alexandroff space has a natural ordering and, in fact, it is in one-to-one correspondence with a poset, like a finite topological space. Therefore, the topological properties of an Alexandroff space can be described in terms of posets.

In particular, we use a relation $\leq$ on Khalimsky line as follows : for $a, b \in \mathbb{Z}$ satisfying $|a{-}b| \leqslant 1$,

$$a \leq b \overset{\text{def}}{\Longleftrightarrow} a = b \text{ or } a \text{ is even.}$$

We denote by $[k, l]_{\mathbb{Z}}$ the subspace $\{z \in \mathbb{Z} \mid k \leqslant z \leqslant l\}$ of $\mathbb{Z}$.

PROPOSITION 1.4.1. *A function $f : [k, l]_{\mathbb{Z}} \to \mathbb{Z}$ is continuous if and only if it is an order-preserving map $f : ([k, l]_{\mathbb{Z}}, \leq) \to (\mathbb{Z}, \leq)$.*

PROOF. That is because $f| : [k, l]_{\mathbb{Z}} \to f([k, l]_{\mathbb{Z}})$ is continuous iff $f|$ is order-preserving. $\square$

In this section, for any two continuous functions $f, g; [k, l]_{\mathbb{Z}} \to \mathbb{Z}$ with $f(k) = g(k)$ and $f(l) = g(l)$, we denote by $f \sim g$, if they are homotopic relative to the both ends k and l. We denote by $\varepsilon_a : A \to \mathbb{Z}$ the constant function at a for an Alexandroff space A and $a \in \mathbb{Z}$. From now on, $f : [k, l]_{\mathbb{Z}} \to \mathbb{Z}$ denotes a continuous function.

PROPOSITION 1.4.2. *If $f : [k, l]_{\mathbb{Z}} \to \mathbb{Z}$ satisfies $f(k) = f(l) = a$, then $f \sim \varepsilon_a$.*

PROOF. Because $f([k, l]_{\mathbb{Z}})$ is contractible in $\mathbb{Z}$ to $f(k) = f(l)$, we have $f \sim \varepsilon_{f(k)}$. $\square$

From the above proposition we obtain the following corollary using an induction on the number in $[k, l]_{\mathbb{Z}}$.

Corollary 1.4.3. *For a function $f : [k,l]_{\mathbb{Z}} \to \mathbb{Z}$, there is a monotone function g in the normal integer order such that $f \sim g$.*

Definition 1.4.4. *For a continuous function $g : [k,l]_{\mathbb{Z}} \to \mathbb{Z}$ satisfying $g(k) \leqslant g(l)$ and* $\mathrm{height}(k) = \mathrm{height}(g(k))$, *we can define a continuous function $h_g : [k,l]_{\mathbb{Z}} \to \mathbb{Z}$ by the following formula:*

$$h_g(z) = \begin{cases} z + g(k) - k, & z + g(k) - k \leqslant g(l), \\ g(l), & z + g(k) - k \geqslant g(l). \end{cases}$$

We can easily show that h_g is continuous by Theorem 1.4.1.

Proposition 1.4.5. *If $f : [k,l]_{\mathbb{Z}} \to \mathbb{Z}$ satisfies $f(k) \leqslant f(l)$ and* $\mathrm{height}(k) = \mathrm{height}(f(k))$, *then we obtain $f \sim h_f$.*

Proof. Without loss of generality, we may assume that $k = 0$ and f is monotone increasing satisfying $f(0) = 0$ or $f(0) = 1$ by Corollary 1.4.3. In the case when $f(0) = 1$, since f is monotone and f is continuous, we have $f(0) = f(1) = 1$. Since f is homotopic to the map f' defined by $f'(0) = 0$ and $f'(z) = f(z)$ if $z \neq 0$, we consentrate in the case when $f(0) = 0$. Let $f_0 = f$. Then by definition, f_0 is a continuous monotone increasing function, and hence we have $f_0(z+1) \in \{f_0(z), f_0(z)+1\}$. Let $k_0 := \min\{z \in [k,l]_{\mathbb{Z}} \mid f_0(z+1) = z\}$ and $l_0 := \max\{z \in [k,l]_{\mathbb{Z}} \mid f_0(z) = k_0\}$. Then we have $f_0|_{[0,k_0]} = h_f|_{[0,k_0]}$ and $k_0 \leqslant f_0(l)$. In case when $k_0 = f_0(l)$, $f_0 = h_f$ and we have done. So we assume that $k_0 < f_0(l)$. Then we have that $f_0(l_0+1) = k_0+1$. Let $f_1 : [0,l]_{\mathbb{Z}} \to \mathbb{Z}$ as follows:

$$f_1(z) = \begin{cases} k_0, & z = k_0, \\ k_0 + 1, & z \in [k_0 + 1, l_0 + 1], \\ f_0(z), & \text{otherwise}, \end{cases}$$

We then have $f_0 \sim f_1$. Let $k_1 := \min\{z \in [k,l]_{\mathbb{Z}} \mid f_1(z+1) = z\}$ and $l_1 := \max\{z \in [k,l]_{\mathbb{Z}} \mid f_1(z) = k_1\}$. Then we obtain $k_1 = k_0 + 1$. If $k_1 = f(l)$, then $f_0 \sim f_1 = h_f$, and we have done. If $k_1 < f(l)$, we can continue the above process to obtain a sequence of functions $f_1, f_2, \ldots, f_n$ for a sufficiently large $n \leqslant l-k$, such that $k_n = f(l)$ and $f = f_0 \sim f_1 \sim \cdots \sim f_n = h_f$. □

For $m \geqslant 2$, the quotient space $\mathbb{Z}/2m = \{\overline{0}, \overline{1}, \ldots, \overline{2m-1}\}$ is homeomorphic with the Khalimsky circle $\mathbb{S}^1_m$. Moreover we have the following proposition.

Lemma 1.4.6. *For $m \geqslant 2$ and a continuous map $f : [k,l]_{\mathbb{Z}} \to \mathbb{Z}/2m$, there exists a unique lift $\widetilde{f}$ of f which makes the following diagram commutative:*

$$\begin{tikzcd}
\{k\} \arrow[r, "\varepsilon_a"] \arrow[d, hook] & \mathbb{Z} \arrow[d, two heads, "q"] \\
{[k,l]_{\mathbb{Z}}} \arrow[r, "f"] \arrow[ur, dashed, "\exists! \widetilde{f}"] & \mathbb{Z}/2m
\end{tikzcd}$$

Proof. We inductively define $\widetilde{f}$ to satisfy $q \circ \widetilde{f} = f$ as follows:

(1) $\widetilde{f}(k) = a$ and

(2) for $z \in \mathbb{Z}$, $\widetilde{f}(z+1) = \begin{cases} \widetilde{f}(z)+1, & f(z+1) = f(z) + \overline{1}, \\ \widetilde{f}(z), & f(z+1) = f(z), \\ \widetilde{f}(z)-1, & f(z+1) = f(z) - \overline{1}. \end{cases}$

We have $q \circ \widetilde{f}(k) = q(a) = q \circ \varepsilon_a(k) = f(k)$, and if $q \circ \widetilde{f}(z) = f(z)$, then $q \circ \widetilde{f}(z+1) = f(z+1)$ because of the definition of $\widetilde{f}$. By induction, $\widetilde{f}$ is a lift of f.

Next, we will prove the continuity of $\widetilde{f}$. We assume that $\widetilde{f}$ is order-preserving on a subset $[k,a]$ of $[k,l]_{\mathbb{Z}}$. In the case when $a \leq a+1$, a is even. Since f is order-preserving, we have $f(a) \leq f(a+1)$. So $f(a+1) = f(a)$ or $f(a+1) = f(a) \pm \overline{1}$ and $f(a)$ is even. If $f(a+1) = f(a)$, then $\widetilde{f}(a+1) = \widetilde{f}(a)$. If $f(a+1) = f(a) \pm \overline{1}$, $\widetilde{f}(a+1) = \widetilde{f}(a) \pm \overline{1}$ and $\widetilde{f}(a)$ is even. Hence, $\widetilde{f}(a) \leq \widetilde{f}(a+1)$. In the case when $a \geq a+1$, we similarly get $\widetilde{f}(a) \geq \widetilde{f}(a+1)$. By induction, $\widetilde{f}$ is order-preserving, i.e. $\widetilde{f}$ is continuous.

Finally, we will prove the uniqueness of $\widetilde{f}$. Let $\widetilde{g}$ be another lift in the commutative diagram. We have $\widetilde{f}(k) = \widetilde{g}(k)$. Suppose that $\widetilde{f}|_{[k,a]_{\mathbb{Z}}} = \widetilde{g}|_{[k,a]_{\mathbb{Z}}}$. We have $\widetilde{f}(a) = \widetilde{g}(a)$. $|\widetilde{f}(a) - \widetilde{f}(a+1)| \leqslant 1$ and $|\widetilde{g}(a) - \widetilde{g}(a+1)| \leqslant 1$, so $|\widetilde{f}(a+1) - \widetilde{g}(a+1)| \leqslant 2$. Then $\widetilde{f}(a+1) = \widetilde{g}(a+1)$ because $q \circ \widetilde{f} = f = q \circ \widetilde{g}$ and $m \geqslant 2$. By induction, $\widetilde{f} = \widetilde{g}$. $\square$

LEMMA 1.4.7. *Let two maps* $f_0, f_1 : [k,l]_{\mathbb{Z}} \to \mathbb{Z}/2m$ *satisfy* $f_0 \sim f_1$. *If there exists the following commutative diagrams for* $i = 1, 2$*:*

$$\begin{tikzcd} \{k\} \arrow[r, "\varepsilon_a"] \arrow[d, hook] & \mathbb{Z} \arrow[d, two heads, "q"] \\ {[k,l]_{\mathbb{Z}}} \arrow[ur, "\widetilde{f_i}"] \arrow[r, "f_i"] & \mathbb{Z}/2m \end{tikzcd}$$

then, $\widetilde{f_0}(l) = \widetilde{f_1}(l)$.

PROOF. By the geometric realization, we have the following commutative diagrams for $i = 0, 1$:

$$\begin{tikzcd} \{k\} \arrow[r, "\varepsilon_a"] \arrow[d, hook] & {|\mathcal{K}(\mathbb{Z})| = \mathbb{R}} \arrow[d, two heads, "|\mathcal{K}(q)|"] \\ {I = |\mathcal{K}([k,l]_{\mathbb{Z}})|} \arrow[ur, "|\mathcal{K}(\widetilde{f_i})|"] \arrow[r, "|\mathcal{K}(f_i)|"] & {|\mathcal{K}(\mathbb{Z}/2m)| = S^1.} \end{tikzcd}$$

Since $|\mathcal{K}(f_0)|$ and $|\mathcal{K}(f_1)|$ are homotopic relative to endpoints, we obtain $\widetilde{f_0}(l) = |\mathcal{K}(\widetilde{f_0})|(l) = |\mathcal{K}(\widetilde{f_1})|(l) = \widetilde{f_1}(l)$. $\square$

From two above lemma, we obtain the following corollary.

COROLLARY 1.4.8. *Let* $m, n \geqslant 2$, *and* $p : [0, 2m]_{\mathbb{Z}} \to \mathbb{Z}/2m$ *be the quotient map* $z \mapsto \overline{z}$. *If the following diagram is commutative:*

$$\begin{tikzcd} & & \mathbb{Z} \arrow[d, two heads, "q"] \\ {[0,2m]_{\mathbb{Z}}} \arrow[urr, "F"] \arrow[r, two heads, "p"] & \mathbb{Z}/2m \arrow[r, "f"] & \mathbb{Z}/2n, \end{tikzcd}$$

then $\deg f = \{F(2m) - F(0)\}/2n$.

PROPOSITION 1.4.9. *If* $f : \mathbb{Z}/2m \to \mathbb{Z}/2n$ *satisfies* $|\deg f| < m/n$, *then it is homotopic to a continuous function* $g : \mathbb{Z}/2m \to \mathbb{Z}/2n$ *satisfying* $g(\overline{0}) = f(\overline{0}) + \overline{1}$.

PROOF. Without loss of generality, we may assume that $f(\overline{0}) = \overline{k}$ and $\deg f \geqslant 0$.

First, we consider the case when $\mathrm{height}(0) = \mathrm{height}(k)$, or equivalently, the case when k is even. There is a lift $\widetilde{f} : [0, 2m]_{\mathbb{Z}} \to \mathbb{Z}$ of $f \circ p$ satisfying $\widetilde{f}(0) = k$ by Lemma 1.4.6. Then we have $\widetilde{f}(0) \leqslant \widetilde{f}(2m)$ by Corollary 1.4.8, and $\widetilde{f} \sim h_{\widetilde{f}}$ by Proposition 1.4.5. Let $h : \mathbb{Z}/2m \to \mathbb{Z}/2n$

be a map defined by $h(\overline{z}) = q \circ h_{\widetilde{f}}(z)$. Now we have a homotopy H_t between $q \circ \widetilde{f}$ and $q \circ h_{\widetilde{f}}$ with $H_t(0) = H_t(2m) = \overline{k}$ for any $t \in I$. By taking the quotient of H_t, we obtain a homotopy between f and h.

From the hypothesis, we have $|\deg h| = |\deg f| < m/n$ and $h(\overline{2m-2}) = h(\overline{2m-1}) = h(\overline{0}) = \overline{k}$. Let $f_1, f_2 : \mathbb{Z}/2m \to \mathbb{Z}/2n$ be as follows:

$$f_1(\overline{z}) = \begin{cases} h(\overline{z}), & \overline{z} \neq \overline{2m-1}, \\ \overline{k} + \overline{1}, & \overline{z} = \overline{2m-1}. \end{cases}$$

$$f_2(\overline{z}) = \begin{cases} f_1(\overline{z}), & \overline{z} \neq \overline{0}, \\ \overline{k} + \overline{1}, & \overline{z} = \overline{0}. \end{cases}$$

In this case, we have $f_2 \simeq h$, since $h \geq f_1 \leq f_2$. Therefore, we obtain $f \simeq h \simeq f_2$ and hence $f_2(\overline{0}) = \overline{k} + \overline{1} = f(\overline{0}) + \overline{1}$.

Second, we consider the case when $\mathrm{height}(0) \neq \mathrm{height}(k)$, or equivalently, the case when k is odd. Then we have $\mathrm{height}(2m-1) = \mathrm{height}(k)$, and $f(\overline{2m-1}) = f(\overline{0}) = \overline{k}$ since k is odd.

In this case, there is a lift $\widetilde{f} : [2m-1, 4m-1]_{\mathbb{Z}} \to \mathbb{Z}$ of $f \circ q| : [2m-1, 4m-1]_{\mathbb{Z}} \to \mathbb{Z}/2n$ satisfying $\widetilde{f}(2m-1) = k$, which satisfies $\widetilde{f} \sim h_{\widetilde{f}}$. Then, for a map $h : \mathbb{Z}/2m \to \mathbb{Z}/2n$ defined by $h(\overline{z}) = q \circ h_{\widetilde{f}}(z)$, we obtain $f \simeq h$ since $\widetilde{f} \sim h_{\widetilde{f}}$.

Finally, we have $h(\overline{0}) = \overline{k} + \overline{1} = f(\overline{0}) + \overline{1}$, which completes the proof of this proposition. $\square$

Proposition 1.4.10. *Let $m, n \geqslant 2$. If $f_0, f_1 : \mathbb{Z}/2m \to \mathbb{Z}/2n$ are homotopic and $|\deg f_0| = m/n$, then $f_0 = f_1$.*

Proof. In the case when $\deg f_0 = -m/n$, by reversing the orientation $\mathbb{Z}/2m$, we may assume that $\deg f_0 = m/n$. Thus we consentrate in the case when $\deg f_0 = m/n$.

Firstly, we consider the special case when $f_0, f_1 : \mathbb{Z}/2m \to \mathbb{Z}/2n$ are homotopic relative to the base point $\overline{0} \in \mathbb{Z}/2m$. Let $a = f_0(\overline{0}) = f_1(\overline{0})$. By Proposition 1.4.6, there exists F_i which makes the following diagram commutative: for $i = 0, 1$:

$$\begin{tikzcd}
\{0\} \arrow[rr, hook, "\varepsilon_{f_i(\overline{0})}"] \arrow[d, hook] & & \mathbb{Z} \arrow[d, two heads, "q"] \\
{[0, 2m]} \arrow[r, two heads, "p"'] \arrow[urr, "F_i"] & \mathbb{Z}/2m \arrow[r, "f_i"'] & \mathbb{Z}/2n.
\end{tikzcd}$$

For any $z \in [0, 2m-1]$ and $i = 0, 1$, we have $F_i(z) \leq F_i(z+1)$ or $F_i(z) \geq F_i(z+1)$, and hence we obtain $|F_i(z) - F_i(z+1)| \leqslant 1$. However, we have $F_i(2m) - F_i(0) = 2m$ by Corollary 1.4.8, and hence we obtain $F_i(z) = F_i(0) + z = f_i(\overline{0}) + z = a + z$. Thus, we have $F_0 = F_1$. Since f_0 and f_1 are induced from F_0 and F_1, respectively, we have $f_0 = f_1$.

Next, we consider the general case. Suppose that $f_0, f_1 : \mathbb{Z}/2m \to \mathbb{Z}/2n$ are homotopic. By [**Bar11**, Lemma 2.1.1], there exists a sequence $g_0, g_1, \ldots, g_k$ such that $f_0 = g_0 \leq g_1 \geq g_2 \leq \cdots \geq (\leq) g_k = f_1$ and for $0 \leqslant i < k$, g_i and g_{i+1} coincide except a point which we denote by $x_i \in \mathbb{Z}/2m$. We select a point $y_i \in \mathbb{Z}/2m \setminus \{x_i\}$ for each $0 \leqslant i < k$. Then, we have $\deg g_i = m/n$ and g_i and g_{i+1} are homotopic relative to y_i. Thus, $f_0 = g_0 = g_1 = \cdots = g_k = f_1$. $\square$

THEOREM 1.4.11. *Let f, $g : \mathbb{Z}/2m \to \mathbb{Z}/2n$ be two distinct continuous functions. Then $f \simeq g$ if and only if $\deg f = \deg g$ and $|\deg f| < m/n$.*

PROOF. Firstly, we show ‘only if’ part. Since $f \simeq g$, we have $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$. Thus, $\deg f = \deg |\mathcal{K}(f)| = \deg |\mathcal{K}(g)| = \deg g$. We apply f to Corollary 1.4.8. Since $|F(2m) - F(0)| \leqslant 2m$, $\deg f = |F(2m) - F(0)|/2n \leqslant m/n$.

Secondly, we show ‘if’ part. So we assume that $\deg f = \deg g$ and $|\deg f| < m/n$. By reversing the orientation of Khalimsky circle, if necessary, we may assume that $0 \leqslant \deg f = \deg g < m/n$. Let $\overline{k} = f(\overline{0})$. Without loss of generality, we may assume that $f(\overline{0}) = g(\overline{0})$ and $\mathrm{height}(\overline{0}) = \mathrm{height}(\overline{k})$ by Proposition 1.4.9. By Lemma 1.4.6, there exist a lift $\widetilde{f}$ of $f \circ p$ and a lift $\widetilde{g}$ of $g \circ p$ which satisfy $\widetilde{f}(0) = \widetilde{g}(0) = k$. By Corollary1.4.8, $\widetilde{f}(2m) = \widetilde{g}(2m) = k + 2n \cdot \deg f$. Thus, $\widetilde{f} \sim h_{\widetilde{f}} = h_{\widetilde{g}} \sim \widetilde{g}$ by Proposition 1.4.5, so $f \simeq g$. $\square$

We get the pointed version of Theorem 1.4.11 as the following. We can easily confirm it in the proof of Theorem 1.4.11.

Corollary 1.4.12. *Let* f, $g : (\mathbb{Z}/2m, 0) \to (\mathbb{Z}/2n, 0)$ *be two distinct continuous functions. Then* f *and* g *are homotopic relative to base points if and only if* $\deg f = \deg g$ *and* $|\deg f| < m/n$.

CHAPTER 2

Counterexample to triangle inequality for contiguity distance

Homotopic distance was first introduced by Macías-Virgós and Mosquera-Lois in [**MVML18**], as a more general notion for two numerical important invariants: one is Lusternik-Schnirelmann category $\mathrm{cat}(X)$, and the other is topological complexity $\mathrm{TC}(X)$.

Definition. [**MVML18**] *The homotopic distance* $\mathrm{D}(f, g)$ *between two continuous map* $f, g : X \to Y$ *is the least integer* $n \geqslant 0$ *such that there exists an open cover* $\{U_0, U_1, \ldots, U_n\}$ *of* X *with the property that* $f|_{U_i}$ *and* $g|_{U_i}$ *are homotopic, for all* $i = 0, \ldots, n$.

In accordance with the name of homotopic distance, if the domain is normal, then triangle inequality holds, that is $\mathrm{D}(f, g) + \mathrm{D}(g, h) \leqslant \mathrm{D}(f, h)$ [**MVML18**]. Macías-Virgós and Mosquera-Lois showed that the above inequality implies Farber's result: $\mathrm{TC}(X) \leqslant 2\,\mathrm{cat}(X)$ for any normal space X [**Far03**].

Contiguity distance was also introduced in [**MVML18**] as the simplicial version of homotopic distance, and Ayse Borat, Mehmetcik Pamuk and Tane Vergili studied about it in more detail [**BPV23**]. This is defined by using the following concept of contiguity. For two simplicial maps $\phi, \psi : K \to L$, they are said be contiguous and denoted by $\phi \sim_c \psi$ if for any simplex $\sigma \in K$, $\phi(\sigma) \cup \psi(\sigma)$ is a simplex of L. Let $\sim$ denote the equivalence relation generated by $\sim_c$, i.e, $\phi \sim \psi$ if there exists a sequence of simplicial maps $\phi_0, \phi_1, \ldots, \phi_n$ such that $\phi = \phi_0 \sim_c \phi_1 \sim_c \cdots \sim_c \phi_n = \psi$.

Definition. [**MVML18**] *The contiguity distance* $\mathrm{SD}(\phi, \psi)$ *between two simplicial maps* $\phi, \psi : K \to L$ *is the least integer* $n \geqslant 0$ *such that there is a cover* $\{K_0, K_1, \ldots, K_n\}$ *of* K *consisting of subcomplexes with the property that* $\phi|_{K_i} \sim \psi|_{K_i} : K_i \to L$*, for all* $i = 0, \ldots, n$.

Actually, contiguity distance is also a more general notion for two combinatorial invariants: one is simplicial Lusternik-Schnirelmann category $\mathrm{scat}(K)$ [**FTMVV15**], and the other is discrete topological complexity $d\,\mathrm{TC}(K)$ [**FTMVMV18**]. As we have seen with the homotopic distance, if triangular inequalities hold for contiguity distance, we obtain $d\,\mathrm{TC}(K) \leqslant 2\,\mathrm{scat}(K)$. However, it was not known whether the triangular inequality holds in general.

In this chapter, we give a counterexample to triangle inequality for contiguity distance through the following theorem.

THEOREM 2.0.1. *Let $\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$ be the union of two copies of the boundary of a 2-simplex glueing two 1-simplexes Δ_1, i.e, the simplicial complex represented by the following figure:*

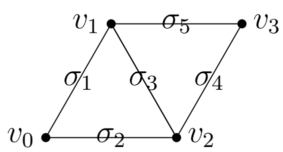


FIGURE 5. $\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$

Then, $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 1$ *and* $d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 3$.

Remarkably, since $|\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2| \simeq S^1 \vee S^1$, we have $\mathrm{cat}(|\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2|) = 1$. However, $\mathrm{TC}(|\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2|) = 2 < 3 = d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2)$. This means that $d\,\mathrm{TC}(K) \leqslant 2\,\mathrm{scat}(K)$ does not hold in general.

This chapter is organized as follows: In Section 2.1, we explain a counterexample to the triangle inequality for for contiguity distance from $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 1$ and $d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 3$. In Sections 2.2 and 2.3, we prove Theorem 2.0.1, but some combinatorial checks are done with computer.

2.1. Counterexample to the triangle inequality

In this section, we explane a counterexample to the triangle inequality for for contiguity distance. First, we recall the definition of simplicial Lusternik-Schnirelmann category $\mathrm{scat}(K)$.

Let K be a simplicial complex. A subcomplex $L \subset K$ is said to be categorical if there exists a vertex $v \in K$ such that the inclusion map $\iota : L \hookrightarrow K$ and the constant map c_v lie in the same contiguity class, i.e, $\iota \sim c_v$.

DEFINITION 2.1.1. [**FTMVV15**] *Let K be a simplicial complex. The simplicial LS-category* $\operatorname{scat}(K)$ *of a simplicial complex K is the least integer $n \geqslant 0$ such that there is a cover $\{K_0, K_1, \ldots, K_n\}$ of K consisting of categorical subcomplexes of K.*

Next, we recall the definition of discrete topological complexity $d\,\mathrm{TC}(K)$. For simplicial complexes K_1, K_2, the cartesian product $K_1 \prod K_2$ of K_1 and K_2 is defined as follows:

(1) (vertex): $\operatorname{vert}(K_1 \prod K_2) = \operatorname{vert}(K_1) \times \operatorname{vert}(K_2)$,

(2) (face) : $\sigma \in K_1 \prod K_2 \iff \mathrm{Pr}_1(\sigma) \in K_1$ and $\mathrm{Pr}_2(\sigma) \in K_2$,

where Pr_i is the i-th projection $\mathrm{Pr}_i : \operatorname{vert}(K_1) \times \operatorname{vert}(K_2) \to \operatorname{vert}(K_i)$ for $i = 1, 2$.

A subcomplex $L \subset K \prod K$ is said to be section-categorical if $\mathrm{Pr}_1|_L \sim \mathrm{Pr}_2|_L$.

DEFINITION 2.1.2. [**FTMVMV18**] *Let K be a simplicial complex. The discrete topological complexity $d\,\mathrm{TC}(K)$ of a simplicial complex K is the least integer $n \geqslant 0$ such that there is a cover $\{K_0, K_1, \ldots, K_n\}$ of $K \prod K$ consisting of section-categorical subcomplexes of $K \prod K$.*

REMARK 2.1.3. *In the original definition of* scat *and* $d\,\mathrm{TC}$*, they used 'subcomplexes', but we can replace 'subcomplexes' by 'full subcomplexes'.*

We give the following counterexample to the triangle inequality for contiguity distance.

PROPOSITION 2.1.4. *Let $K = \partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$ and $\mathrm{Pr}_i : K \prod K \to K$ be the i-th projection for $i = 1, 2$. Then,* $\mathrm{SD}(\mathrm{Pr}_1, c_v) + \mathrm{SD}(c_v, \mathrm{Pr}_2) < \mathrm{SD}(\mathrm{Pr}_1, \mathrm{Pr}_2)$.

PROOF. We have $\mathrm{SD}(\mathrm{Pr}_i, c_v) \leqslant \operatorname{scat}(K)$ for $i = 1, 2$. Thus $\mathrm{SD}(\mathrm{Pr}_1, c_v) + \mathrm{SD}(c_v, \mathrm{Pr}_2) \leqslant 2\operatorname{scat}(K) = 2$. However, $d\,\mathrm{TC}(K) = \mathrm{SD}(\mathrm{Pr}_1, \mathrm{Pr}_2) = 3$. Therefore, $\mathrm{SD}(\mathrm{Pr}_1, c_v) + \mathrm{SD}(c_v, \mathrm{Pr}_2) < \mathrm{SD}(\mathrm{Pr}_1, \mathrm{Pr}_2)$. $\square$

2.2. Simplicial homotopic and contiguity class

In this section, we prove the following lemma for the proof of $d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 3$.

LEMMA 2.2.1. *Let K be a simplicial complex, and L be a subcomplex of K with the property that the contiguity class of id_L is equal to $\{\mathrm{id}_L\}$. If any facet in L is also a facet in K, then the contiguity class of the inclusion $\iota : L \hookrightarrow K$ is equal to $\{\iota\}$.*

Before the proof of the above lemma, we introduce another concept as the discrete version of homotopy. For simplicial complexes K with a total order on the vertex set $\mathrm{vert}(K)$, let $\Delta(K{\times}I)$ be the simplicial complex defined as follows:

(1) (vertex): $\mathrm{vert}(\Delta(K{\times}I)) = \mathrm{vert}(K){\times}\{0,1\}$,

(2) (face) : $\sigma \in \Delta(K{\times}I) \iff \mathrm{Pr}_1(\sigma) \in K$ and σ is a totally ordered subset,

where Pr_1 is the 1-th projection, and the total order on $\mathrm{vert}(\Delta(K{\times}I))$ is the lexicographical order. For two simplicial maps $\phi, \psi : K \to L$, they are denoted by $\phi \sim_s \psi$ if there exist some total order on $\mathrm{vert}(K)$ and a simplicial map $\Phi : \Delta(K{\times}I) \to L$ such that $\Phi|_{(K\times 0)} = \phi$ and $\Phi|_{(K\times I)} = \psi$. Two simplicial maps $\phi, \psi : K \to L$ are said to be simplicial homotopic and denoted by $\phi \simeq \psi$ if there exists a sequence of simplicial maps $\phi_0, \phi_1, \ldots, \phi_n$ such that $\phi = \phi_0 \sim_s \phi_1 \sim_s \cdots \phi_n = \psi$. Actually, this definition is equal to contiguity.

PROPOSITION 2.2.2. *For simplicial maps $\phi, \psi : K \to L$, $\phi \sim \psi \iff \phi \simeq \psi$.*

PROOF. We assume that ϕ and ψ are contiguous. By choosing a total order on the vertex set of K, we get a simplicial complex $\Delta(K{\times}I)$ triangulated along the total order. We can define the simplicial map $\Phi : \Delta(K{\times}I) \to L$ by the vertex correspondence $\Phi((v,0)) = \phi(v)$ and $\Phi((v,1)) = \psi(v)$. Therefore, we have $\phi \sim_s \psi$ and $\phi \simeq \psi$.

Conversely, we assume that $\phi \sim_s \psi$. Then, we have $\Delta(K{\times}I)$ triangulated by along some total order $(v_0 < v_1 < \cdots < v_n)$ on the vertex set of K and a simplicial map $\Phi : \Delta(K{\times}I) \to L$

satisfying $\Phi|_{K\times\{0\}} = \phi$ and $\Phi|_{K\times\{1\}} = \psi$. For $0 \leqslant i \leqslant n{+}1$, we define $\phi_i : K \to L$ by

$$\phi_i(v_j) = \begin{cases} \Phi((v_j, 0)), & j \leqslant n-i \\ \Phi((v_j, 1)), & j > n-i. \end{cases}$$

Since $\phi_i \sim_c \phi_{i+1}$ for any $0 \leqslant i \leqslant n$, we have $\phi \sim \psi$. □

By the proof of the above proposition, we get the following corollary.

Corollary 2.2.3. *Suppose that $\phi \sim \psi : K \to L$. Then, there exists a sequence of simplicial maps $\phi = \phi_0, \phi_1, \ldots, \phi_n = \psi$ and $v_i \in \mathrm{vert}(K)$ for all $0 \leqslant i \leqslant n$ such that:*

(1) *ϕ_i and ϕ_{i+1} coincide on $\mathrm{vert}(K) \setminus \{v_i\}$, and*

(2) *for any facet σ containing v_i, $\phi_i(\sigma) \cup \phi_{i+1}(v_i) \in L$.*

Finally, we prove the lemma 2.2.1.

Proof of Lemma 2.2.1. Let $\phi : L \to K$ be a simplicial map satisfying that $\phi \sim \iota : L \to K$, where ι is the inclusion. By Corollary 2.2.3, there exists a sequence of simplicial maps $\iota = \phi_0, \phi_1, \ldots, \phi_n = \phi$ and $v_i \in \mathrm{vert}(L)$ for all $0 \leqslant i \leqslant n$ such that:

(1) ϕ_i and ϕ_{i+1} coincide on $\mathrm{vert}(L) \setminus \{v_i\}$, and

(2) for any facet $v_i \in \sigma \in L$, $\phi_i(\sigma) \cup \phi_{i+1}(v_i) \in K$.

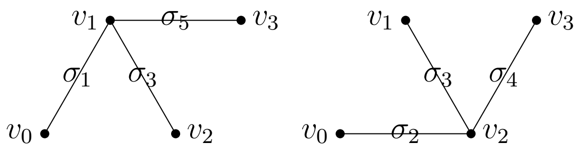
Firstly, we show $\iota = \phi_1$ if any facet in L is also a facet in K. We assume $\iota \neq \phi_1$. Then, $\phi_1(v_0) \neq v_0$. Since the contiguity class of id_L is $\{\mathrm{id}_L\}$, $\phi_1(v_0)$ is not a vertex on L. We taking a facet σ of L containing v, we have $\sigma \neq \sigma \cup \phi_1(v) \in K$. Hence, σ is not a facet of K, and this is a contradiction. Therefore $\iota = \phi_1$, and we get $\iota = \phi_1 = \phi_2 = \cdots = \phi_n = \phi$ by induction. □

2.3. The proof of $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 1$ and $d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 3$

First, we determine simplicial LS-category.

Theorem 2.3.1. $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) = 1$.

Proof. Since $|\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2| \simeq S^1 \vee S^1$ is not contractible, we have $0 < \mathrm{cat}(S^1 \vee S^1) \leqslant \mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2)$. It is sufficient to show that $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \leqslant 1$. Let K_0, K_1 be subcomplexes generated by respectively $\{\sigma_1, \sigma_3, \sigma_5\}$ and $\{\sigma_2, \sigma_3, \sigma_4\}$, i.e, subcomplexes represented by the following figure:



K_0 K_1

We can easily see $\mathrm{id}|_{K_0}$ and c_{v_1} are contiguous, and $\mathrm{id}|_{K_1}$ and c_{v_2} are contiguous. Thus we have $\mathrm{scat}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \leqslant 1$. $\square$

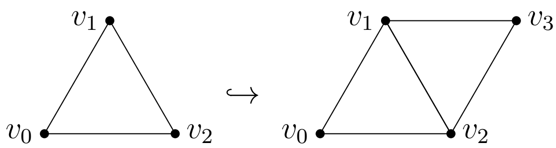
Next, we consider discrete topological complexity.

Proposition 2.3.2. *$d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \leqslant 3$.*

Proof. Let $K = \partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$. By [**FTMVMV18**, Theorem 4.5], we have $d\,\mathrm{TC}(K) \leqslant \mathrm{scat}(K \prod K)$. By [**FTMVV15**, Theorem 5.5], we have $\mathrm{scat}(K \prod K) \leqslant (\mathrm{scat}(K) + 1)^2 - 1$. Therefore, by Theorem 2.3.1, we have $d\,\mathrm{TC}(K) \leqslant (\mathrm{scat}(K) + 1)^2 - 1 = 3$. $\square$

The rest need only show $d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \geqslant 3$. Let us show that the cover consisting of 3 subcomplexes can't work.

Proposition 2.3.3. *The contiguity class of following inclusion ι is equal to $\{\iota\}$.*



Proof. By Lemma 2.2.1, the proof is done. $\square$

To simplify combinatorial descriptions of the cover $\{K_0, K_1, \dots, K_n\}$ of $(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \prod (\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2)$, let us represent the cover as a 5×5-matrix M. We define the matrix M by $M_{(i,j)}{=}c$ if the full subcomplex K_c contains $\sigma_i \prod \sigma_j$, where the label of simplexes are defined in Figure 5. Unlike the usual definition of matrix similarity, we say a matrix M is similar to another matrix M', if we obtain M by rearranging rows and rearranging columns from M' in this section.

Note that the subcomplex represented by any of the following colored submatrixes in Figure 6, is isomorphic to $\partial\Delta_2 \prod \partial\Delta_2$.

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

FIGURE 6.

PROPOSITION 2.3.4. *Suppose that a* 5×5*-matrix* M *represent a cover* $\{K_0, K_1, K_2\}$ *of* $(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \prod (\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2)$ *consisting of three section-categorical subcomplexes. For any small matrix* A *of* M *corresponding to a color in Figure 6 is similar to one of the following:*

$$\underset{(A1)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}} \underset{(A2)}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}} \underset{(A3)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underset{(A4)}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}}$$

$$\underset{(B1)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underset{(B2)}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}} \underset{(B3)}{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}} \underset{(B4)}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}} \underset{(B5)}{\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \underset{(B6)}{\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

PROOF. Let A be a 3×3-submatrix of M corresponding to a color in Figure 6. Since A is a 3×3-submatrix, there exists $n \in \{0, 1, 2\}$ such that n is in 3 or more squares in A. Then, the following property (1) hold.

(1) For $\forall i_0, i_1 \in \{1, 2, 3\}, \exists k \in \{1, 2, 3\}$ such that both $A_{i_0,k}$ and $A_{i_1,k}$ are not equal to n.

We show the above property. Suppose that there exists $i_0, i_1 \in \{1, 2, 3\}$ such that $A_{i_0,k} = n$ or $A_{i_1,k} = n$ for all $k \in \{1, 2, 3\}$. We take a vertex $v \in \sigma_{i_0} \cap \sigma_{i_1}$. Let $K = \partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$. Then, K_n includes $v \prod L$, where L is one of the $\partial\Delta_2$. Thus, $\mathrm{Pr}_1 |_{v \prod L} = c_v$, however, $\mathrm{Pr}_2 |_{v \prod L}$ is an inclusion ι. By Proposition 2.3.3, the contiguity class of ι is equal to $\{\iota\}$. This contradicts $\mathrm{Pr}_1|_{K_n} \sim \mathrm{Pr}_2|_{K_n}$. Similarly, the following property hold.

(2) For $\forall i_0, i_1 \in \{1,2,3\}, \exists k \in \{1,2,3\}$ such that both A_{k,i_0} and A_{k,i_1} are not equal to n.

By property (1) and property (2), A is similar to one of the following:

$$\underset{(i)}{\begin{bmatrix} n & n & * \\ n & * & * \\ * & * & * \end{bmatrix}} \underset{(ii)}{\begin{bmatrix} n & * & * \\ * & n & * \\ * & * & n \end{bmatrix}}$$

, where each square of $*$ contains some element.

First, we consider the case when 0, 1, and 2 are all in three squares. Then, by (i) and (ii), A is similar to one of the following:

$$\underset{(A1)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}} \underset{(A2)}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}} \underset{(A3)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underset{(A4)}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}}$$

Next, we consider the case when n is in four or more squares. By property (1) and property (2), A is similar to the below, and n is not in five squares.

$$\begin{bmatrix} n & n & * \\ n & n & * \\ * & * & * \end{bmatrix}$$

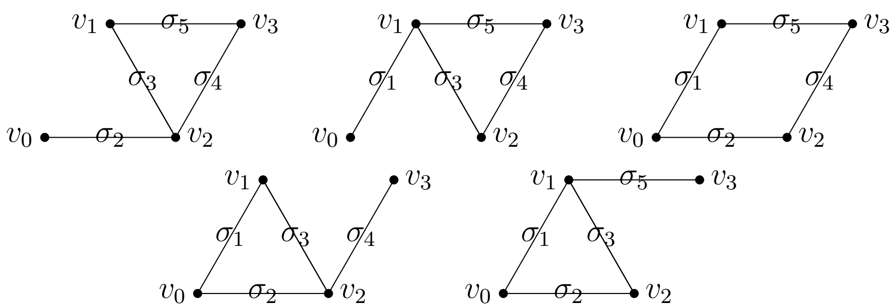
Moreover, by property (1) and property (2), A is similar to one of the following:

$$\underset{(B1)}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underset{(B2)}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}} \underset{(B3)}{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}} \underset{(B4)}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}} \underset{(B5)}{\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \underset{(B6)}{\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

□

PROPOSITION 2.3.5. *Let L be a subcomplex of $\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$. If L contains four different facets of $\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$, $|L|$ is not contractible.*

PROOF. We give all cases of L when L consists of four different facets as follows:



We can easily see that each subcomplex is not contractible. $\square$

Proposition 2.3.6. *Let $K = \partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2$. L is a subcomplex of $K \prod K$. If there exists v such that $L \cap \{v \prod K\}$ or $L \cap \{K \prod v\}$ has more than three facets, then $\mathrm{Pr}_1|_L$ and $\mathrm{Pr}_2|_L$ are not in the same contiguity class.*

Proof. We suppose that $L \cap \{v \prod K\}$ has more than three facets. Then, $\mathrm{Pr}_1|_{L\cap\{v\prod K\}} = c_v$. This contradict to Proposition 2.3.5. $\square$

By using Proposition 2.3.4 and Proposition 2.3.6, we can get the following proposition with computer.

Proposition 2.3.7. *$d\,\mathrm{TC}(\partial\Delta_2 \cup_{\Delta_1} \partial\Delta_2) \geqslant 3$.*

2.4. Code in Python

The following is the source code written in Python to check the Proposition 2.3.7.

```
import numpy as np
import itertools as it

#candidates of small block
A1 = np.array([[0,0,2],
               [0,1,1],
               [2,1,2]])
A2 = np.array([[0,0,1],
               [0,1,2],
               [1,2,2]])
A3 = np.array([[0,0,2],
               [0,2,1],
               [2,1,1]])
```

```
A4 = np.array([[1,1,0],
               [1,0,2],
               [0,2,2]])
B1 = np.array([[0,0,2],
                [0,0,1],
                [2,1,1]])
B2 = np.array([[0,0,1],
                [0,0,2],
                [1,2,2]])
B3 = np.array([[1,1,2],
                [1,1,0],
                [2,0,0]])
B4 = np.array([[1,1,0],
                [1,1,2],
                [0,2,2]])
B5 = np.array([[2,2,1],
                [2,2,0],
                [1,0,0]])
B6 = np.array([[2,2,0],
                [2,2,1],
                [0,1,1]])
def perm_matrix(A):
    Perm = [list(a) for a in it.permutations([0,1,2])]
    count = 0
    C = np.stack([A[q,:][:,p] for p in Perm for q in Perm], axis=2)
    return C
List1 = np.concatenate(
    [perm_matrix(A1),perm_matrix(A2),
    perm_matrix(A3),perm_matrix(A4),
    perm_matrix(B1),perm_matrix(B2),
    perm_matrix(B3),perm_matrix(B4),
    perm_matrix(B5),perm_matrix(B6)], axis=2)
num1 = List1.shape[2]

#candidates of large block
List2 = np.empty((3,5,0))
for i in range(num1):
    for j in range(i,num1):
        L1 = List1[:,:,i]
        L2 = List1[:,:,j]
        if np.all(L1[:,0]==L2[:,0]):
            L1_f = np.fliplr(L1)
            L = np.concatenate([L1_f,L2], axis=1)
            L = np.delete(L, 2, axis=1)
            L = L.reshape(3,5,1)
            List2 = np.concatenate([List2,L], axis=2)
num2 = List2.shape[2]
```

```
List3 = np.empty((5,5,0))
for i in range(num2):
    for j in range(i,num2):
        L1 = List2[:,:,i]
        L2 = List2[:,:,j]
        if np.all(L1[0,:]==L2[0,:]):
            L1_f = np.flipud(L1)
            L = np.concatenate([L1_f,L2], axis=0)
            L = np.delete(L, 2, axis=0)
            L = L.reshape(5,5,1)
            List3 = np.concatenate([List3,L], axis=2)
num3 = List3.shape[2]

#Are there 4 or more of the same number in the line?
def four(a):
    for i in range(3):
        if 4 <= np.count_nonzero(a==i):
            return True
    return False
def Four(M):
    raw, column = M.shape
    for i in range(raw):
        if four(M[i,:]) or four(M[:,i]):
            return True
    return False

#reducing candidates of large block
lis =[]
for i in range(num3):
    if Four(List3[:,:,i])==False:
        lis.append(i)
List4 = List3[:,:,lis]
num4 = List4.shape[2]

#Are there 4 or more of the same number in two lines?
def four_2(a,b):
    for i in range(3):
        if 4 <= np.count_nonzero((a==i)+(b==i)):
            return True
    return False

#If you find three section-categorical subcomplexes, print it
def Check(M):
    l = [[0,1],[0,2],[0,4],[1,2],[1,3],[2,3],[2,4],[3,4]]
    for i,j in l:
        if four_2(M[i,:],M[j,:]) or four_2(M[:,i],M[:,j]):
            return True
```

```
    print(M)
for i in range(num4):
    Check(List4[:,:,i])
print('finish')
```

CHAPTER 3

Fundamental monoid for finite spaces

As has been seen in Chapter 1, for any finite space X, there exists a simplicial complex $\mathcal{K}(X)$ which is weakly homotopy equivalent to X [**McC66**]. However, unlike the CW complex, a weak homotopy equivalence between finite spaces is generally not a homotopy equivalence. We are interested in this difference.

In this chapter, we introduce the fundamental monoid $\mathrm{M}(X, x_0)$ of a based finite topological space (X, x_0) as a homotopy invariant. The following theorem is our main theorem.

THEOREM. *For finite space X with a base point x_0, the group completion of the fundamental monoid $\mathrm{M}(X, x_0)$ is isomorphic to $\pi_1(X, x_0)\times\mathbb{Z}$.*

Note that we use 'group completion' for a non-commutative monoid, but this is generally used for a commutative monoid.

This chapter is organized as follows: In Section 3.1, we define the fundamental monoid. In Section 3.2, we consider two ways to obtain a group from the fundamental monoid. In Section 3.3, we introduce a combinatorial method for examining fundamental group under special conditions. In Section 3.4, we give an example of a simply-connected finite space with non-trivial fundamental monoid, where we say a fundamental monoid is trivial if it is isomorphic to the monoid of non-negative integers, the fundamental monoid of a point. Finally, in Sections 3.5, we explain the group completion of non-commutative monoid under some conditions.

To simplify combinatorial descriptions for finite spaces, we regard a finite T_0-space as a finite posets, unless otherwise stated (cf. Chapter 1). Throughout this chapter, we use two different notation of order. We denote the order in a poset by $\leq$, and denote the common order in the integer $\mathbb{Z}$ by $\leqslant$.

3.1. Fundamental monoid

In this section, we introduce fundamental monoid for finite spaces. We recall the definition of Khalimsky circles. Unlike in Chapter 1, $\mathbb{S}^1_0$ and $\mathbb{S}^1_1$ are defined for convenience.

DEFINITION 3.1.1. *The Khalimsky circle $\mathbb{S}^1_m$ is defined as the following: for $m \geqslant 1$,*

$$\mathbb{S}^1_m = \{0, 1, \dots, 2m{-}1\}$$

whose order is determined by $0 \leq 1 \geq \cdots \leq 2m{-}1 \geq 0$ and $\mathbb{S}^1_0 = \emptyset$.

REMARK 3.1.2. *In this chapter, we consider Khalimsky circle $\mathbb{S}^1_m$ with the base point $0 \in \mathbb{S}^1_m$. However, we can define the base point of $\mathbb{S}^1_m$ as a maximal point $1 \in \mathbb{S}^1_m$ instead of a minimal point $0 \in \mathbb{S}^1_m$. This difference is same as taking the opposite of posets.*

Let X be a finite T_0-space with a base point x_0. For a continuous map $f : \mathbb{S}^1_m \to X$, we denote $f(2m) = f(0)$. For given two continuous maps $f : (\mathbb{S}^1_m, 0) \to (X, x_0)$ and $g : (\mathbb{S}^1_n, 0) \to (X, x_0)$, we define the product $f \cdot g : (\mathbb{S}^1_{m+n}, 0) \to (X, x_0)$ as follows:

$$f \cdot g(t) = \begin{cases} f(t), & 0 \leqslant t \leqslant 2m{-}1, \\ g(t{-}2m), & 2m \leqslant t \leqslant 2m{+}2n{-}1. \end{cases}$$

PROPOSITION 3.1.3. *$f \cdot g$ is continuous.*

PROOF. It's enough to prove that $f \cdot g$ is order-preserving. We take two points s, t in $\mathbb{S}^1_{m+n}$ such that $s \leq t$. By the definition of the order on Khalimsky circle, s and t satisfy either of (i) : $0 \leqslant s, t \leqslant 2m{-}1$, (ii) : $2m \leqslant s, t \leqslant 2m{+}2n{-}1$, (iii) : $s = 2m$ and $t = 2m{-}1$, or (iv) : $s = 0$ and $t = 2m{+}2n{-}1$. In the case of (i), $f \cdot g(s) = f(s) \leq f(t) = f \cdot g(t)$. In the case of (ii), $f \cdot g(s) = g(s) \leq g(t) = f \cdot g(t)$. In the case of (iii), $f \cdot g(2m) = g(0) = x_0 = f(0) \leq f(2m{-}1) = f \cdot g(2m{-}1)$. In the case of (iv), $f \cdot g(0) = f(0) = x_0 = g(0) \leq g(2n{-}1) = f \cdot g(2m{+}2n{-}1)$. $\square$

PROPOSITION 3.1.4. *If $f \leq f'$ and $g \leq g'$, then $f \cdot g \leq f' \cdot g'$.*

PROOF. We have $f \cdot g(t) = f(t) \leq f'(t) = f' \cdot g'(t)$ for $0 \leqslant t \leqslant 2m{-}1$ and $f \cdot g(t) = g(t) \leq g'(t) = f' \cdot g'(t)$ for $2m \leqslant t \leqslant 2m{+}2n{-}1$. $\square$

We denote by $f \sim g$ if they are homotopic relative to the base point.

PROPOSITION 3.1.5. *If $f \sim f'$ and $g \sim g'$, then $f \cdot g \sim f' \cdot g'$.*

PROOF. There exists two sequence $f = f_0, f_1, \ldots, f_m = f'$ and $g = g_0, g_1, \ldots, g_n = g'$ satisfying $f_0 \leq f_1 \geq f_2 \leq \cdots \geq (\leq) f_m$ and $g_0 \leq g_1 \geq g_2 \leq \cdots \geq (\leq) g_n$. Without loss of generality, we may assume $m \leqslant n$. By defining $f_k := f_m$ for $k \geqslant m$, we have an extended sequence $f_0 \leq f_1 \geq f_2 \leq \cdots \geq (\leq) f_n$. By Proposition 3.1.4, we obtain $f \cdot g = f_0 \cdot g_0 \leq f_1 \cdot g_1 \geq f_2 \cdot g_2 \leq \cdots \geq (\leq) f_n \cdot g_n = f' \cdot g'$. Thus $f \cdot g \sim f' \cdot g'$. □

By the above proposition, the multiplication of homotopy classes $[f] \cdot [g] = [f \cdot g]$ is well-defined.

DEFINITION 3.1.6. *Let $\mathrm{M}(X, x_0)$ be a set $\bigcup_{0 \leqslant n} [\mathbb{S}^1_n, X]_*$ with the multiplication $\cdot$.*

PROPOSITION 3.1.7. *$\mathrm{M}(X, x_0)$ is a monoid.*

PROOF. For $f : (\mathbb{S}^1_m, 0) \to (X, x_0)$, $g : (\mathbb{S}^1_n, 0) \to (X, x_0)$ and $h : (\mathbb{S}^1_l, 0) \to (X, x_0)$, we have $(f \cdot g) \cdot h = f \cdot (g \cdot h) : (\mathbb{S}^1_{m+n+l}, 0) \to (X, x_0)$. Thus, $\mathrm{M}(X, x_0)$ satisfies associativity. The identity unit is c_0 by the definition. □

REMARK 3.1.8. *If we consider $(\mathbb{Z}_{\geqslant 0}, +)$ as a monoidal category, then we obtain the invariant of X as the functor from $(\mathbb{Z}_{\geqslant 0}, +)$ to the category of monoids.*

The constant map at the base point is denoted by $c_n : \mathbb{S}^1_n \to X$ for $n \geqslant 0$.

PROPOSITION 3.1.9. *$[c_m] \cdot [c_n] = [c_{m+n}]$ for $m, n \geqslant 0$.*

PROOF. $[c_m] \cdot [c_n] = [c_m \cdot c_n] = [c_{m+n}]$. □

PROPOSITION 3.1.10. *If X is contractible to x_0, then $\mathrm{M}(X, x_0) \cong \mathbb{Z}_{\geqslant 0}$.*

PROOF. If X is contractible to x_0, then $[\mathbb{S}^1_n, X]_* \cong \{[c_n]\}$ for all n. By Proposition 3.1.9, $[c_n] = [c_1]^n$ for all n. Thus, $[c_1]$ is the generator of $\mathrm{M}(X, x_0)$, and $\mathrm{M}(X, x_0) \cong \mathbb{Z}_{\geqslant 0}$. □

The fundamental monoid $\mathrm{M}(X, x_0)$ is said to be trivial if $\mathrm{M}(X, x_0) \cong \mathbb{Z}_{\geqslant 0}$.

Let us construct a map from the fundamental monoid to the fundamental group through geometric realization. We define the map $q : \mathrm{M}(X, x_0) \to \pi_1(|\mathcal{K}(X)|, x_0)$ by $q([f]) = [|\mathcal{K}(f)|]$.

PROPOSITION 3.1.11. *q is a monoid homomorphism.*

PROOF. For any $[f], [g] \in \mathrm{M}(X, x_0)$, we have $|\mathcal{K}(f) \cdot \mathcal{K}(g)| \sim |\mathcal{K}(f)| \cdot |\mathcal{K}(g)|$. Hence $q([f] \cdot [g]) = q([f]) \cdot q([g])$. $\square$

At the end of this section, we give an example of not trivial fundamental monoid. By Corollary 1.4.12, we immediately get the following results.

EXAMPLE 3.1.12. *For $n \geqslant 2$, $\mathrm{M}(\mathbb{S}^1_n, 0) \cong \{(d, l) \in \mathbb{Z} \times \mathbb{Z}_{\geqslant 0} \,;\, |d| \leqslant l/n\}$, where the product of the right-hand side is the addition of the vectors.*

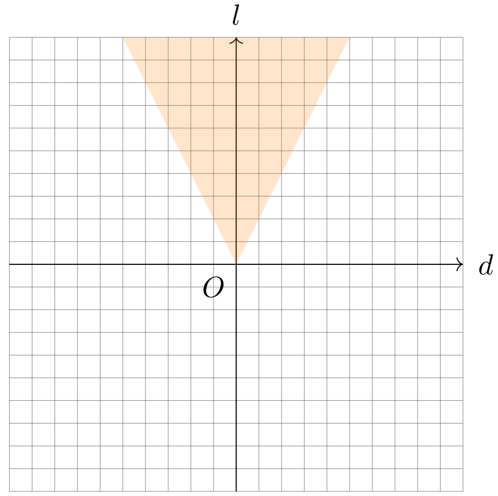


FIGURE 7. $\mathrm{M}(\mathbb{S}^1_2, 0)$

Thus, we know that the group completion of $\mathrm{M}(\mathbb{S}^1_n, 0)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

3.2. Groups obtained from fundamental monoid

In this section, we consider two ways to obtain a group from the fundamental monoid. First, we define the special element called *radical* of the monoid. We discuss some properties of radical elements in Section 3.5.

Definition 3.2.1. *An element z of M is said to be radical if z satisfies the following conditions:*

(1) $\forall a \in M \ az = za$.

(2) $\forall a \in M \ \exists a' \in M \ \exists n \in \mathbb{Z}_{\geqslant 0}$ *s.t.* $aa' = a'a = z^n$.

Proposition 3.2.2. $[c_1] \in \mathrm{M}(X, x_0)$ *is radical.*

Proof. Let f be a continuous map $(\mathbb{S}^1_n, 0) \to (X, x_0)$. We have $f \cdot c_1 = f \circ (\mathrm{id}_{\mathbb{S}^1_n} \cdot c_1)$ and $c_1 \cdot f = f \circ (c_1 \cdot \mathrm{id}_{\mathbb{S}^1_n})$. Since $\deg(\mathrm{id}_{\mathbb{S}^1_n} \cdot c_1) = \deg(c_1 \cdot \mathrm{id}_{\mathbb{S}^1_n})$, we have $\mathrm{id}_{\mathbb{S}^1_n} \cdot c_1 \sim c_1 \cdot \mathrm{id}_{\mathbb{S}^1_n}$ by Corollary 1.4.12. Hence $f \cdot c_1 = f \circ (\mathrm{id}_{\mathbb{S}^1_n} \cdot c_1) \sim f \circ (c_1 \cdot \mathrm{id}_{\mathbb{S}^1_n}) = c_1 \cdot f$, and $[c_1] \cdot [f] = [f] \cdot [c_1]$.

Let us take the reverse $\overline{f}$ of f. Since $\deg(\overline{f} \cdot f) = \deg(|\mathcal{K}(\overline{f}) \cdot \mathcal{K}(f)|) = 0$, we get $\overline{f} \cdot f \sim c_{2n}$ by Corollary 1.4.12. □

Definition 3.2.3. *For* $[f], [g] \in \mathrm{M}(X, x_0)$*, we define the relation* $\approx_c$ *as follows:*

$$[f] \approx_c [g] \iff \exists m, n > 0 \ s.t. \ [f] \cdot [c_m] = [g] \cdot [c_n] \in \mathrm{M}(X, x_0).$$

We define the quotient set $\widetilde{G}(X, x_0) := \mathrm{M}(X, x_0)/\approx_c$.

In Section 3.5, we shall show $\widetilde{G}(X, x_0)$ is a group. What is this group $\widetilde{G}(X, x_0)$? To find out, we prove a lemma.

Let τ be a continuous map $\tau : \mathrm{Sd}(X) \to X$ defined by $\tau(x_0 < x_1 < \cdots < x_n) = x_n$.

Proposition 3.2.4. [**Bar11**, Proposition 2.1, Proposition 2,2, and Theorem A.1.7] *Let* $f, g : X \to Y$ *be continuous maps between* T_0*-finite spaces. The following are equivalent.*

(1) There exists $k \geqslant 0$ *such that* $f \circ \tau^k, g \circ \tau^k : \mathrm{Sd}^k(X) \to Y$ *are homotopic.*

(2) $|\mathcal{K}(f)|, |\mathcal{K}(g)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ *are homotopic.*

Proof. Since $|\mathcal{K}(\tau)|$ is homotopic to the identity map, we can show (1) $\implies$ (2). By [**Bar11**, Proposition 2.1, Proposition 2,2, and Theorem A.1.7], we can show (2) $\implies$ (1). □

Proposition 3.2.5. *For any* $k \geqslant 0$, $\tau^k \sim \mathrm{id}_{\mathbb{S}^1_n} \cdot c_{(2^k-1)n} : \mathbb{S}^1_{2^k n} \to \mathbb{S}^1_n$.

Proof. Since $|\mathcal{K}(\tau^k)| \sim |\mathcal{K}(\mathrm{id}_{\mathbb{S}^1_{2^k n}})|$ and $|\mathcal{K}(\mathrm{id}_{\mathbb{S}^1_n} \cdot c_{(2^k-1)n})| = |\mathcal{K}(\mathrm{id}_{\mathbb{S}^1_n})| \cdot |\mathcal{K}(c_{(2^k-1)n})| \sim |\mathcal{K}(\mathrm{id}_{\mathbb{S}^1_{2^k n}})|$, we have $\deg \tau^k = \deg(1_{\mathbb{S}^1_n} \cdot c_{(2^k-1)n}) = 1$. By cororally 1.4.12, $\tau^k \sim 1_{\mathbb{S}^1_n} \cdot c_{(2^k-1)n}$. □

By the above two propositions, we get the following lemma.

Lemma 3.2.6. *For $[f], [g] \in \mathrm{M}(X, x_0)$, the following are equivalent.*

(1) $\exists m, n > 0$ s.t. $[f] \cdot [c_m] = [g] \cdot [c_n] \in \mathrm{M}(X, x_0)$,

(2) $[|\mathcal{K}(f)|] = [|\mathcal{K}(g)|] \in \pi_1(|\mathcal{K}(X)|, x_0)$.

Thus we obtain the following corollary.

Corollary 3.2.7. $\widetilde{G}(X, x_0) \cong \pi_1(|\mathcal{K}(X)|, x_0) \cong \pi_1(X, x_0)$.

From now on, we consider another group from the monoid $\mathrm{M}(X, x_0)$.

Definition 3.2.8. *For $[f], [g] \in \mathrm{M}(X, x_0)$, we define the relation $\sim_c$ as follows:*

$$([f], [c_m]) \sim_c ([g], [c_n]) \iff [f] \cdot [c_n] = [g] \cdot [c_m].$$

We define $G(X, x_0) := \mathrm{M}(X, x_0)/\sim_c$.

From now on, we use the word 'universal group' instead of 'group completion'.

Definition 3.2.9. [**BT05**] *Let M be a monoid, G a group and $\phi : M \to G$ a monoid morphism. We say (G, ϕ) is a universal group of M if for any group G' and any monoid homomorphism $f : M \to G'$, there exist a unique group homomorphism $f' : G \to G'$ which makes the following diagram commutative:*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "f"'] & G \arrow[d, dashed, "f'"] \\ & G' \end{tikzcd}$$

We define $\mathrm{len} : \mathrm{M}(X, x_0) \to \mathbb{Z}_{\geqslant 0}$ by $\mathrm{len}([f : (\mathbb{S}^1_n, 0) \to (X, x_0)]) = n$.

By Theorem 3.5.4 and Theorem 3.5.7 in Section 3.5, we get the following theorem.

Theorem 3.2.10. *$G(X,x_0) \cong \pi_1(X,x_0)\times\mathbb{Z}$. Moreover, $(G(X,x_0),\phi)$ is the universal group of $\mathrm{M}(X,x_0)$, where ϕ is the composition of $\Phi : \mathrm{M}(X,x_0) \to \pi_1(X,x_0)\times\mathbb{Z}; \Phi([f]) = (q([f]), \mathrm{len}([f]))$ and the group isomorphism from $\pi_1(X,x_0)\times\mathbb{Z}$ to $G(X,x_0)$.*

Example 3.2.11. *For $n \geqslant 2$, we have $\pi_1(\mathbb{S}^1_n) \cong \pi_1(S^1) \cong \mathbb{Z}$, thus $G(\mathbb{S}^1_n, 0) \cong \mathbb{Z}\times\mathbb{Z}$. On the other hand, since $\mathrm{M}(\mathbb{S}^1_n, 0) \cong \{(d,l) \in \mathbb{Z}\times\mathbb{Z}_{\geqslant 0}\ ;\ |d| \leqslant l/d\}$, the universal group of $\mathrm{M}(\mathbb{S}^1_n, 0)$ is also isomorphic to $\mathbb{Z}\times\mathbb{Z}$.*

3.3. Combinatorial method

The fundamental monoid $\mathrm{M}(X,x_0)$ is the disjoint union of the set of homotopy class of n-loops $[\mathbb{S}^1_n, X]_*$ as a set. So, in this section, we introduce a combinatorial method for examining the set of the homotopy class $[\mathbb{S}^1_n, X]_*$ for a finite space X with the base point x_0 which is minimal. More precisely, we reduce $[\mathbb{S}^1_n, X]_*$ to the homotopy class of extremally n-loops, and we define the combinatorial relation between extremally n-loops.

From now on, let X be a finite space with the base point x_0 which is minimal, unless otherwise stated. For simplification, we denote a continuous map $f : (\mathbb{S}^1_n, 0) \to (X, x_0)$ by a sequence of points $(f(0), f(1), \ldots, f(2n-1))$, and it is said to be an n-loop on (X,x_0). When $x_0 = y_0$, the product of m-loop $\underline{x} = (x_0, x_1, \ldots, x_{2m-1})$ and n-loop $\underline{y} = (y_0, y_1, \ldots, y_{2n-1})$ is equal to the $(m+n)$-loop $\underline{x}\cdot\underline{y} = (x_0, x_1, \ldots, x_{2m-1}, y_0, y_1, \ldots, y_{2n-1})$. We say that an n-loop $\underline{x} = (x_0, x_1, \ldots, x_{2n-1})$ is an extremally n-loop if x_{2i} is a minimal element and x_{2i+1} is a maximal element for any $0 \leqslant i \leqslant n-1$.

Proposition 3.3.1. *For an n-loop $\underline{x}$, there exists an extremally n-loop $\underline{x'}$ satisfying $\underline{x} \sim \underline{x'}$.*

Proof. Let $d : X \to \min(X)$ be a map defined by choosing a minimal element $d(x) \in X$ satisfying $d(x) \leq x$ for every $x \in X$. Similarly, let $u : X \to \max(X)$ be a map defined by choosing a maximal element $u(x) \in X$ satisfying $x \leq u(x)$ for every $x \in X$. As a remark, d and u cannot be continuous. For an n-loop $\underline{x}$, we define n-loops $\underline{y}$ and $\underline{x'}$ by $y_{2k} = x_{2k}$, $x'_{2k} = d(x_{2k})$

and $y_{2k+1} = x'_{2k+1} = u(x_{2k+1})$ for $0 \leqslant k \leqslant n{-}1$. We show that $\underline{y}$ and $\underline{x'}$ are n-loops. Since $\underline{x}$ is a n-loop, we have $x_{2k} \leq x_{2k+1}$ for any $0 \leqslant k \leqslant n{-}1$. We get $y_{2k} = x_{2k} \leq x_{2k+1} \leq u(x_{2k+1}) = y_{2k+1}$ and $x'_{2k} = d(x_{2k}) \leq x_{2k} \leq x_{2k+1} \leq u(x_{2k+1}) = x'_{2k+1}$ for any $0 \leqslant k \leqslant n{-}1$, hence $\underline{y}$ and $\underline{x'}$ are n-loops. Further, we have $x_i \leq y_i \geq x'_i$ for $0 \leq i \leq 2n$, thus $\underline{x} \sim \underline{y} \sim \underline{x'}$. □

Let $\mathrm{comp}(a)$ be the set of all comparable elements to a, that is, $\mathrm{comp}(a) = \{x \mid x \leq a \text{ or } x \geq a\}$. We then introduce two equivalence relations: one is on $X{\times}X{\times}X$, and the other is on the set of extremally n-loops on X using the first one.

Definition 3.3.2. *Let $\sim_r$ and $\sim_R$ be the equivalence relations respectively generated by the following relations ${}_r\!\rightsquigarrow$ and ${}_R\!\rightsquigarrow$.*

(1) For $x_0, x_1, x'_1, x_2 \in X$, the relation ${}_r\!\rightsquigarrow$ is defined by

$$(x_0, x_1, x_2) \,{}_r\!\rightsquigarrow (x_0, x'_1, x_2) \iff \exists a \in X,\ s.t.\ x_0, x_1, x'_1, x_2 \in \mathrm{comp}(a).$$

(2) For two extremally n-loops $\underline{x}$ and $\underline{y}$, the relation ${}_R\!\rightsquigarrow$ is defined by

$$\underline{x} \,{}_R\!\rightsquigarrow \underline{y} \iff \exists i \neq 0 \ s.t.\ x_j = y_j \text{ for any } j \neq i,\ \text{and } (x_{i-1}, x_i, x_{i+1}) \,{}_r\!\rightsquigarrow (y_{i-1}, y_i, y_{i+1}).$$

Theorem 3.3.3. *The set of homotopy class $[\mathbb{S}^1_n, X]_*$ of n-loops is equal to $\mathrm{EL}_n(X)/{\sim_R}$ as a set, where $\mathrm{EL}_n(X)$ is the set of extremally n-loops on X.*

For the proof of the above Theorem, we prepare two following proposition.

Proposition 3.3.4. *Let $\underline{x}$ and $\underline{y}$ be n-loops on X. Suppose that there exists $i \in \mathbb{S}^1_n$ such that:*

(1) $x_j = y_j$ for any $j \neq i$, and

(2) $x_i \leq y_i$ or $x_i \geq y_i$.

Then, there exists extremally n-loops $\underline{x'}$ and $\underline{y'}$ which have the relation ${}_R\!\rightsquigarrow$ and satisfies $\underline{x} \sim \underline{x'}$ and $\underline{y} \sim \underline{y'}$.

Proof. Without loss of generality, we may assume that $x_i \leq y_i$. By using maps u, d defined in the proof of Proposition 3.3.1, we obtain extremally n-loops $\underline{x'}$ defined by $x'_{2k} = d(x_{2k})$ and

$x'_{2k+1} = u(x_{2k+1})$ for $0 \leqslant k \leqslant n$. We similarly define y' by $y'_{2k} = d(y_{2k})$ and $y'_{2k+1} = u(y_{2k+1})$ for $0 \leqslant k \leqslant n$ In the case when i is odd, we can see that $d(x_{i-1}), u(x_i), u(y_i), d(x_{i+1}) \in \mathrm{comp}(x_i)$ in the left side of Figure 8. In the case when i is even, we can see that $u(x_{i-1}), d(x_i), d(y_i), u(x_{i+1}) \in \mathrm{comp}(x_i)$ in the right side of Figure 8.

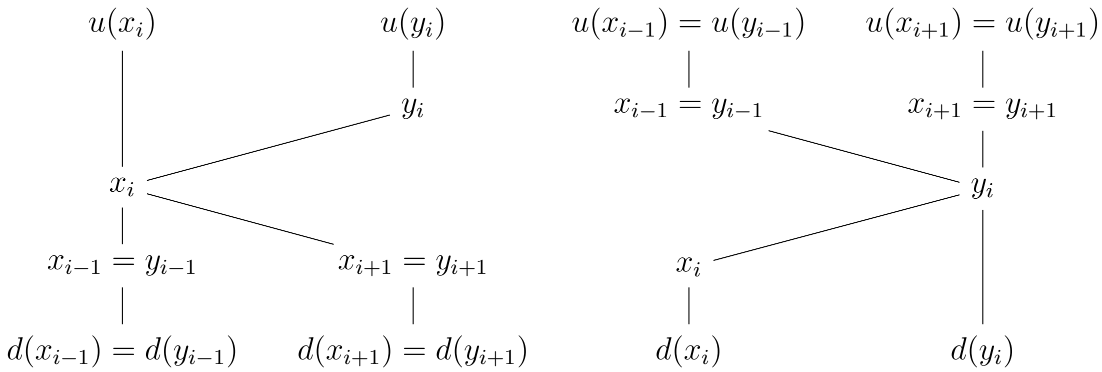


FIGURE 8.

Hence, we get $(x'_{i-1}, x'_i, x'_{i+1}) \;{}_r\!\rightsquigarrow (y'_{i-1}, y'_i, y'_{i+1})$ and $x' \;{}_R\!\rightsquigarrow y'$.

$\square$

PROPOSITION 3.3.5. *For two extremally n-loops $\underline{x}$ and $\underline{y}$, $\underline{x} \sim \underline{y}$ if and only if $\underline{x} \sim_R \underline{y}$.*

PROOF. Firstly, we show 'if' part. It is sufficient to show that $\underline{x} \;{}_R\!\rightsquigarrow \underline{y}$ implies $\underline{x} \sim \underline{y}$. We assume that $\underline{x} \;{}_R\!\rightsquigarrow \underline{y}$. By the definition of the reraltion ${}_R\!\rightsquigarrow$, there exists $i \neq 0$ and $a \in X$ such that $x_j = y_j$ for any $j \neq i$, and $x_{i-1}, x_i, y_i, x_{i+1} \in \mathrm{comp}(a)$. We define the n-loop $\underline{z}$ by $z_j = x_j = y_j$ for $j \neq i$ and $z_i = a$. Since $z_j = a$ and x_j are comparable and $z_i = x_i$ for $i \neq j$, we have $\underline{x} \leq \underline{z}$ or $\underline{z} \geq \underline{x}$. Hence, $\underline{x} \sim \underline{z}$. Similarly, we have $\underline{y} \sim \underline{z}$. Thus, $\underline{x} \sim \underline{y}$.

Secondly, we show 'only if' part. Let $\underline{x}$ and $\underline{y}$ be extremally n-loops satisfying $\underline{x} \sim \underline{y}$. By [], there exists a sequence of n-loops $\underline{x} = \underline{x^0}, \underline{x^1}, \ldots, \underline{x^k} = \underline{y}$ such that for every $0 \leqslant j \leqslant k$ there is a point $s_j \in \mathbb{S}^1_n$ with the following properties:

(1) $\underline{x^j}$ and $\underline{x^{j+1}}$ are coincide in $\mathbb{S}^1_n \setminus \{s_j\}$, and

(2) $x^j_{s_j} \leq x^{j+1}_{s_j}$ or $x^j_{s_j} \geq x^{j+1}_{s_j}$.

By Proposition 3.3.4, there exists a sequence of extremally n-loops $\underline{x} = \underline{x'^0}, \underline{x'^1}, \ldots, \underline{x'^k} = \underline{y}$ such that $\underline{x^i} \sim \underline{x'^i}$ for any i and $\underline{x'^0} \; {}_R\!\rightsquigarrow \underline{x'^1} \; {}_R\!\rightsquigarrow \cdots \; {}_R\!\rightsquigarrow \underline{x'^k}$. Thus, $\underline{x} \sim_R \underline{y}$. $\square$

PROOF OF THEOREM 3.3.3. The proof is done by Proposition 3.3.4 and Proposition 3.3.5. $\square$

3.4. Example

In this section, we give an example of a simply-connected finite space with non-trivial fundamental monoid.

Let $\mathbb{X}$ be the finite space expressed in the following Hasse diagram.

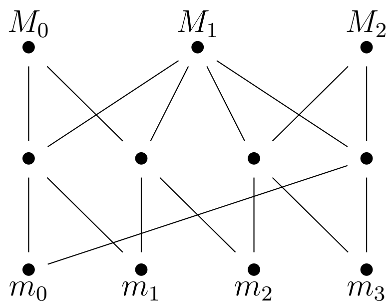


FIGURE 9. $\mathbb{X}$

Let $\mathbb{N}$ be the abelian monoid of non-negative integers with the usual multiplication as its monoid structure.

THEOREM 3.4.1. *The fundamental monoid* $\mathrm{M}(\mathbb{X}, m_0)$ *is isomorphic to the following submonoid of* $\mathbb{N}^3$*, where* $\mathbb{N}^3$ *is the abelian monoid with Hadamard product, i.e, the product is the entry-wise multiplication:*

$$\{(1, 0, 2^{2m}), (0, 1, 2^{2m}), (0, 0, 2^m), (1, 1, 1) \in \mathbb{N}^3 \mid 1 \leqslant m\}$$

The proof of the above theorem will be given at the end of this section. Remarkably, the universal group of $\mathrm{M}(\mathbb{X}, m_0)$ is isomorphic to $\mathbb{Z}$, hence $\pi_1(\mathbb{X}, m_0) \cong 1$. The following proposition is clear from the definition of the relation ${}_r\!\rightsquigarrow$.

PROPOSITION 3.4.2. *Let X be a finite T_0-space. For $m, m', M, M' \in X$,*

(1) $m \leq M \geq m' \implies (M, m, M) \,{}_r\!\!\rightsquigarrow (M, m', M)$,

(2) $M \geq m \leq M' \implies (m, M, m) \,{}_r\!\!\rightsquigarrow (m, M', m)$.

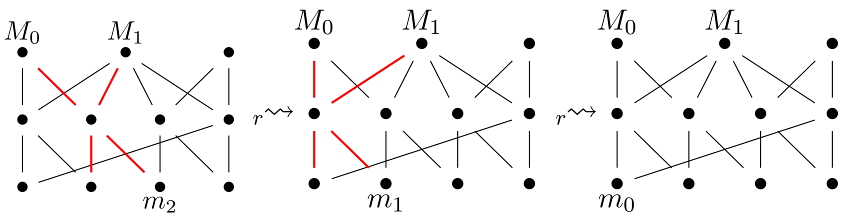
From here, we see some propositions for determining the fundamental monoid of $\mathbb{X}$.

PROPOSITION 3.4.3. *For a minimal element m and a maximal element M in $\mathbb{X}$,*

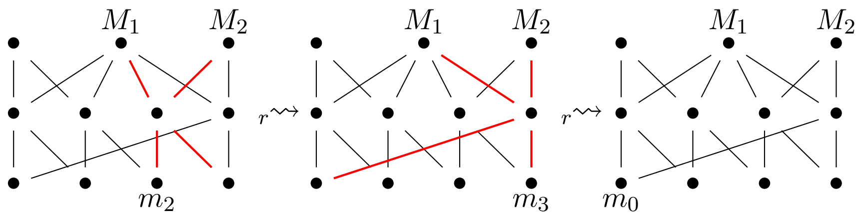
$$m \leq M \implies (M_1, m, M) \sim_r (M_1, m_0, M) \text{ and } (M, m, M_1) \sim_r (M, m_0, M_1).$$

PROOF. In the case when $M = M_1$, we have $(M_1, m, M_1) \sim_r (M_1, m_0, M_1)$ by Proposition 3.4.2.

In the case when $M = M_0$, we have $m = m_0, m_1$ or m_2. By the following figures, we get $(M_1, m_2, M_0) \,{}_r\!\!\rightsquigarrow (M_1, m_1, M_0) \,{}_r\!\!\rightsquigarrow (M_1, m_0, M_0)$.



In the case when $M = M_2$, we have $m = m_0, m_2$ or m_3. By the following figures, we get $(M_1, m_2, M_2) \,{}_r\!\!\rightsquigarrow (M_1, m_3, M_2) \,{}_r\!\!\rightsquigarrow (M_1, m_0, M_2)$.



□

PROPOSITION 3.4.4. *Every extremally n-loop through $M_1 \in \mathbb{X}$ is null-homotopic.*

Proof. Let $\underline{x}$ be n-loop through M_1. By Proposition 3.4.3, we can assume that there exist a subsequence (m_0, M_1, m_0) of $\underline{x}$.

By induction,

$$\begin{aligned}(\dots, m_0, M_1, m_0, M, m, \dots) &\sim_R (\dots, m_0, M, m_0, M, m, \dots)\\ &\sim_R (\dots, m_0, M, m, M, m, \dots)\\ &\sim_R (\dots, m_0, M, m, M_1, m, \dots)\\ &\sim_R (\dots, m_0, M, m_0, M_1, m_0, \dots)\\ &\sim_R (\dots, m_0, M_1, m_0, M_1, m_0, \dots)\end{aligned}$$

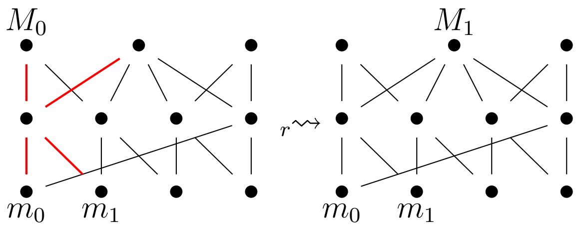
$\square$

Proposition 3.4.5. *Every extremally n-loop through m_1 or m_3 is null-homotopic.*

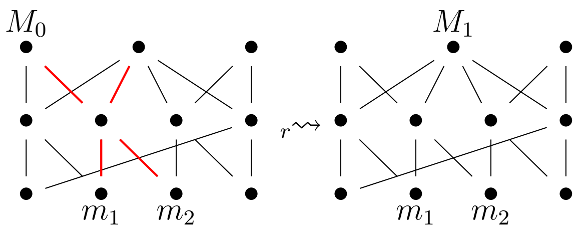
Proof. Let $\underline{x}$ be an extremally n-loop through m_1 or m_3. In other words, there exists i such that $x_i = m_1$ or $x_i = m_3$. If $x_i = x_{i+2}$, then $(x_i, x_{i+1}, x_i) \;{}_{r}\!\rightsquigarrow\; (x_i, M_1, x_i)$ by Proposition 3.4.2 and $\underline{x}$ is null-homotopic by Proposition 3.4.4.

Thus, we can assume $x_i \neq x_{i+2}$ and $x_{i+1} \neq M_1$. The pattern we have to consider are $(x_i, x_{i+1}, x_{i+2}) = (m_1, M_0, m_0), (m_1, M_0, m_2), (m_3, M_2, m_0)$ or (m_3, M_2, m_2). We have the following relation.

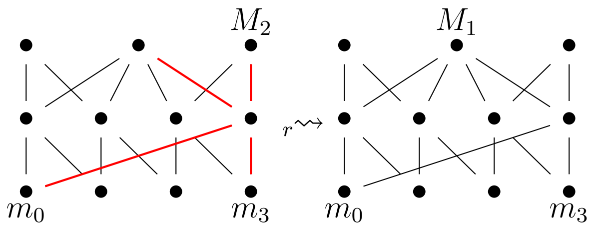
$(m_1, M_0, m_0) \;{}_{r}\!\rightsquigarrow\; (m_1, M_1, m_0)$



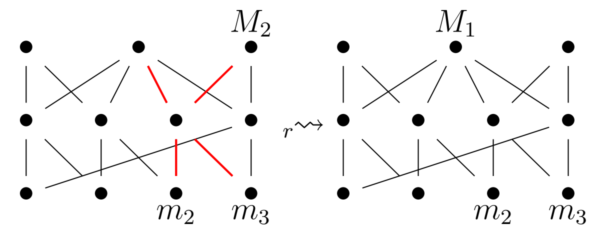
$(m_1, M_0, m_2) \;{}_{r}\!\rightsquigarrow\; (m_1, M_1, m_2)$



$(m_3, M_2, m_0) \;{}_r\!\rightsquigarrow (m_3, M_1, m_0)$



$(m_3, M_2, m_2) \;{}_r\!\rightsquigarrow (m_3, M_1, m_2)$



By Proposition 3.4.4, $\underline{x}$ is null-homotopic.

$\square$

Let $a = (m_0, M_0, m_2, M_2)$ and $\overline{a} = (m_0, M_2, m_2, M_0)$.

Proposition 3.4.6. *$[a^n] = \{a^n\}$ and $[\overline{a}^n] = \{\overline{a}^n\}$ for $n \geqslant 1$.*

Proof. In Figure 9, we can easily see that the object which has the relation ${}_r\!\rightsquigarrow$ to (m_0, M_0, m_2) is only itself. The same holds true for (M_0, m_2, M_2), (m_2, M_2, m_0), and (M_2, m_0, M_0). From the above together with [**Bar11**, Lemma 2.1.1], the homotopy class of a^n is $\{a^n\}$. Similarly, we have $[\overline{a}^n] = \{\overline{a}^n\}$. $\square$

Proposition 3.4.7. *Let $\underline{x}$ be an extremally n-loop on $\mathbb{X}$ which is not null-homotopic. Then, n is positive even, and $\underline{x} = a^{\frac{n}{2}}$ or $\underline{x} = \overline{a}^{\frac{n}{2}}$.*

Proof. Neither a^n nor $\overline{a}^n$ is null-homotopic for $n \geqslant 1$ by Proposition 3.4.6. Let $\underline{x}$ be an extremally n-loop which is not null-homotopic. We may assume that $\underline{x}$ only pass through points in $\{m_0, M_0, m_2, M_2\}$, by Proposition 3.4.4 and Proposition 3.4.5. Suppose that $\underline{x}$ has a repeated part (x, y, x). Then, there exists an extremally n-loop $\underline{x'}$ through M_1, m_1 or m_3 such that $\underline{x} \sim_R \underline{x'}$ by Proposition 3.4.2. However, by 3.4.4 and Proposition 3.4.5, $\underline{x'}$ and $\underline{x}$ are null-homotopic, and hence it is a contradiction. Thus, since the assumption that $\underline{x}$ has a repeated part is false, we obtain $\underline{x} = a^{\frac{n}{2}}$ or $\underline{x} = \overline{a}^{\frac{n}{2}}$. $\square$

Proof of Theorem 3.4.1. By Theorem 3.3.3 and Proposition 3.4.7, we have

$$\mathrm{M}(\mathbb{X}, m_0) = \{[c_0], [c_m], [a^m], [\overline{a}^m] \mid m \geqslant 1\}$$

as the set. Let c be an extremally 1-loop (m_0, M_0). Since c is homotopic to c_1, we have $[c_1]\cdot[a] = [c\cdot a]$ and $[c_1]\cdot[\overline{a}] = [c\cdot\overline{a}]$. From the above together with Proposition 3.4.7, $[c_1]\cdot[a] = [c_3] = [a]\cdot[c_1]$ and $[c_1] \cdot [\overline{a}] = [c_3] = [\overline{a}] \cdot [c_1]$. By Proposition 3.4.7, we also have $[a] \cdot [\overline{a}] = [c_4] = [\overline{a}] \cdot [a]$. We define a monoid homomorphism $\phi : \mathrm{M}(\mathbb{X}, m_0) \to \mathbb{Z}^3$ by $\phi(a) = (1, 0, 2^2)$, $\phi(\overline{a}) = (0, 1, 2^2)$, $\phi(c_1) = (0, 0, 2)$ and $\phi(c_0) = (1, 1, 2^0)$. We can easily see that ϕ is well-defined and injective. Therefore, $\mathrm{M}(\mathbb{X}, m_0) \cong \phi(\mathrm{M}(\mathbb{X}, m_0)) = \{(1, 0, 2^{2m}), (0, 1, 2^{2m}), (0, 0, 2^m), (1, 1, 1) \in \mathbb{Z}^3 \mid m \in \mathbb{Z}_{\geqslant 0}\}$. $\square$

3.5. Universal group with radical elements

The Grothendieck group is well known as the group completion of a commutative monoid. On the other hand, for non-commutative group, "Ore localization" is known as the localization of a subset called "Ore set". However, in this section, we will consider little simpler case than Ore localization. Note that the definition of 'radical' elements is in Definition 3.2.1.

Let M be a monoid. We define the relation $\sim_z$ on $M \times \mathbb{Z}_{\geqslant 0}$ is generated by

$$(a, m) \sim_z (b, n) \overset{\text{def}}{\iff} az^n = bz^m,$$

and we define $G_z(M)$ as the quotient set $M \times \mathbb{Z}_{\geqslant 0}/\sim_z$ with the product defined by $[(a, m)][(b, n)] = [(ab, m+n)]$.

Proposition 3.5.1. *If z commutes with every element of M, then $G_z(M)$ is a monoid.*

Proof. Supose that $(a_0, m_0) \sim_z (b_0, n_0)$ and $(a_1, m_1) \sim_z (b_1, n_1)$. Then, $(a_0a_1, m_0{+}m_1) \sim_z (b_0b_1, n_0{+}n_1)$ because $a_0a_1z^{n_0+n_1} = a_0z^{n_0}a_1z^{n_1} = b_0z^{m_0}b_1z^{m_1} = b_0b_1z^{m_0+m_1}$. Thus, we have $G_z(M)$ is a monoid. $\square$

We define a map $q : M \to G_z(M)$ by $q(a) = [(a, 0)]$. If z commutes with every element of M, q is a monoid homomorphism.

Remark 3.5.2. *The monoid generated by a radical element is an Ore set.*

Proposition 3.5.3. *If z is radical, then $G_z(M)$ is a group.*

Proof. For any $[(a, m)] \in G_z(M)$, there exists an element a' such that $aa' = a'a = z^n$ for some $n \geqslant 0$. Then, we obtain that $[(a'z^m, n)] \in G_z(M)$ is the inverse of $[(a, m)]$. In fact, we have $[(a, m)][(a'z^m, n)] = [(aa'z^m, m{+}n)] = [(z^{m+n}, m{+}n)] = [(z^0, 0)]$ and $[(a'z^m, n)][(a, m)] = [(a'z^ma, m{+}n)] = [(z^ma'a, m + n)] = [(z^{m+n}, m{+}n)] = [(z^0, 0)]$. $\square$

Theorem 3.5.4. *If z is radical, then $q : M \to G_z(M)$ is the universal group of M.*

Proof. Let G be a group, and $f : M \to G$ be a monoid map. We define $F : G_z(M) \to G$ by $F([(a, m)]) = f(a)f(z)^{-m}$. Firstly, we prove F is well-defined. Let $a, b \in M$ satisfy $az^n = bz^m$ for some $m, n \geqslant 0$. Then, we have $F([(a, m)]) = f(a)f(z)^{-m} = f(az^n)f(z)^{-n}f(z)^{-m} = f(bz^m)f(z)^{-m}f(z)^{-n} = f(b)f(z)^{-n} = F([(b, n)])$. So, F is well-defined. Next, we have $F \circ q(a) = F([(a, 0)]) = f(a)$.

Finally, we prove F is a group homomorphism. For any $a \in M$, $f(a)$ and $f(z)^{-1}$ are commutative in G, since $f(a)f(z)^{-1} = f(z)^{-1}f(z)f(a)f(z)^{-1} = f(z)^{-1}f(za)f(z)^{-1} = f(z)^{-1}f(az)f(z)^{-1}$ $= f(z)^{-1}f(a)f(z)f(z)^{-1} = f(z)^{-1}f(a)$. Thus, $F([(a, m)])F([(b, n)]) = f(a)f(z)^{-m}f(b)f(z)^{-n} = f(ab)f(z)^{-(m+n)} = F([(ab, m{+}n)])$. So, F is a group homomorphism. $\square$

To define a stabilized version of the universal group of M, we introduce the relation $\approx_z$ on M:

$$a \approx_z b \overset{\text{def}}{\iff} \exists m, n \in \mathbb{Z}_{\geqslant 0}\; az^m = bz^n.$$

We define $\widetilde{G}_z(M)$ as the quotient set $M/\approx_z$ with the product defined by $[a][b] = [ab]$.

PROPOSITION 3.5.5. *If z commutes with every element of M, then $\widetilde{G}_z(M)$ is a monoid.*

PROOF. Suppose that $a_0 \approx_z b_0$ and $a_1 \approx_z b_1$. That is, there exist m_0, n_0, m_1, n_1 such that $a_i z^{m_i} = b_i z^{n_i}$ for $i = 0, 1$. Hence, $a_0 z^{m_0} a_1 z^{m_1} = b_0 z^{n_0} b_1 z^{n_1}$. Since z commutes with every element of M, we have $a_0 a_1 z^{m_0+m_1} = b_0 b_1 z^{n_0+n_1}$, and hence, $a_0 a_1 \approx_z b_0 b_1$. Thus, $\widetilde{G}_z(M)$ is also a monoid. $\square$

PROPOSITION 3.5.6. *If z is radical, then $\widetilde{G}_z(M)$ is a group.*

PROOF. For any $[a] \in \widetilde{G}_z(M)$, there exist $a' \in M$ such that $aa' = z^n = a'a$ for some $n \geqslant 0$ because z is radical. Hence, $aa' \approx_z e \approx_z a'a$ where e is the unit. Thus, $[a]^{-1} = [a']$. Since the inverse element exists for $[a]$, $\widetilde{G}_z(M)$ is a group. $\square$

A monoid homomorphism $l : M \to Z_{\geqslant 0}$ is called a *length map* on M if $l(z) = 1$.

THEOREM 3.5.7. *If z is radical and M has a length map, then $G_z(M) \cong \widetilde{G}_z(M) \times \mathbb{Z}$.*

PROOF. We define $f : \widetilde{G}_z(M) \times \mathbb{Z} \to G_z(M)$ by $f([a], s) = [(az^s, l(a))]$ if $s \geqslant 0$ and $f([a], s) = f([a]^{-1}, -s)^{-1}$ if $s < 0$. Firstly, we show f is well-defined. Let $a, b \in M$. If $a \approx_z b$, then there exist $m, n \in \mathbb{Z}_{\geqslant 0}$ such that $az^m = bz^n$. Let $k := \max(0, m - l(b)) = \max(0, n - l(a))$. Since $az^{l(b)+k} = bz^{l(a)+k}$, we have $f([a], s) = [(az^s, l(a))] = [(az^{s+k}, l(a)+k)] = [(bz^{s+k}, l(b)+k)] = [(bz^s, l(b))] = f([b], s)$ for $s \geqslant 0$. Therefore, for $s < 0$, $f([a], s) = f([a]^{-1}, -s)^{-1}$ $= f([b]^{-1}, -s)^{-1} = f([b], s)$. Thus, f is well-defined.

Secondly, we show f is a group homomorphism. Let $([a], s), ([b], t) \in \widetilde{G}_z(M) \times \mathbb{Z}$. If $s, t \geqslant 0$, $f([ab], s+t) = [(abz^{s+t}, l(ab))] = [(az^s bz^t, l(a)+l(b))] = [(az^s, l(a))][(bz^t, l(b))] = f([a], s) f([b], t)$. In the case when $s, t < 0$, $f([ab], s+t) = f([ab]^{-1}, -s-t)^{-1} = \{f([b]^{-1}, -t) f([a]^{-1}, -s)\}^{-1} =$

$f([a]^{-1}, -s)^{-1} f([b]^{-1}, -t)^{-1} = f([a], s) f([b], t)$. If $s \geqslant 0 > t$, we take a element b' such that $bb' = z^n$. Then, $f([ab], s{+}t) = [(abz^{s+t}, l(a){+}l(b))] = [(abz^{s+t+\{l(b')-t\}}, l(a){+}l(b){+}\{l(b'){-}t\})] = [(abz^{s+l(b')}, l(a){+}n{-}t)] = [(az^s, l(a))][(bz^{l(b')}, n{-}t)] = [(az^s, l(a))] f([b'], -t)^{-1} = f([a], s) f([b], t)$. Similarly, if $t \geqslant 0 > s$, we have $f([ab], s{+}t) = f([a], s) f([b], t)$. For the unit, we have $f([e], 0) = [(e, 0)]$. Thus, f is a group homomorphism.

Thirdly, we show f is surjective. We take an element $[(a, k)] \in G_z(M)$. If $l(a){-}k \geqslant 0$, we have $f([az^k], l(a){-}k) = [(az^{l(a)}, l(a){+}k)] = [(a, k)]$. If $l(a){-}k < 0$, we take a' such that $aa' = a'a = z^n$. We have $f([a], l(a){-}k) {=} f([a'], k{-}l(a))^{-1} = [(a'z^{k-l(a)}, n{-}l(a)]^{-1} = [(a'z^k, n)]^{-1} = [(a, k)]$. Thus, f is surjective.

Finally, we show f is injective. Suppose that $f([a], s) = [(e, 0)]$. If $s \geqslant 0$, we have $f([a], s) = [(az^s, l(a))] = [(e, 0)]$. Hence, $az^{s+n} = z^{l(a)+n}$ for some n. For length reason, we have $s = 0$. Since $a \approx_z z$, we obtain $[a] = [z] = e$. If $s < 0$, $f([a], s) = f([a]^{-1}, -s)^{-1} = [(e, 0)]$. Therefore, $f([a]^{-1}, -s) = [(e, 0)]$, and hence $[a]^{-1} = e = e^{-1} = [a]$. Thus, f is injective.

$\square$

Bibliography

[Ale41] Paul Alexandroff. General combinatorial topology. *Trans. Amer. Math. Soc.*, 49:41–105, 1941.

[Bar11] Jonathan A. Barmak. *Algebraic topology of finite topological spaces and applications*, volume 2032 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[BPV23] AYŞE BORAT, MEHMETCİK PAMUK, and TANE VERGİLİ. Contiguity distance between simplicial maps. *Turkish Journal of Mathematics*, 47(2):664–677, 2023.

[BT05] László Babai and Evelin Toumpakari. A structure theory of the sandpile monoid for digraphs (extended abstract). 2005.

[Far03] Michael Farber. Topological complexity of motion planning. *Discrete Comput. Geom.*, 29(2):211–221, 2003.

[FTMVMV18] D. Fernández-Ternero, E. Macías-Virgós, E. Minuz, and J. A. Vilches. Discrete topological complexity. *Proc. Amer. Math. Soc.*, 146(10):4535–4548, 2018.

[FTMVV15] D. Fernández-Ternero, E. Macías-Virgós, and J. A. Vilches. Lusternik-Schnirelmann category of simplicial complexes and finite spaces. *Topology Appl.*, 194:37–50, 2015.

[Gon18] Jesús González. Simplicial complexity: piecewise linear motion planning in robotics. *New York J. Math.*, 24:279–292, 2018.

[Jam95] I. M. James. Lusternik-Schnirelmann category. In *Handbook of algebraic topology*, pages 1293–1310. North-Holland, Amsterdam, 1995.

[Kan19] Shelley Burrows Kandola. *The Topological Complexity of Spaces of Digital Images*. ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–University of Minnesota.

[Kİ18] İSMET KARACA and MELİH İS. Digital topological complexity numbers. *Turkish Journal of Mathematics*, 42(6):3173–3181, 2018.

[Kis00] Christer Oscar Kiselman. Digital jordan curve theorems. In *Discrete Geometry for Computer Imagery*, 2000.

[KKR90] Efim Khalimsky, Ralph Kopperman, and Paul R. Meyer. Computer graphics and connected topologies on finite ordered sets. *Topology and its Applications*, 36(1):1–17, 1990.

[Kov89] V.A Kovalevsky. Finite topology as applied to image analysis. *Computer Vision, Graphics, and Image Processing*, 46(2):141–161, 1989.

[McC66] Michael C. McCord. Singular homology groups and homotopy groups of finite topological spaces. *Duke Math. J.*, 33:465–474, 1966.

[MVML18] E. Macías-Virgós and D. Mosquera-Lois. Homotopic distance between maps. 10 2018.

[SSB15] Punam K. Saha, Robin Strand, and Gunilla Borgefors. Digital topology and geometry in medical imaging: A survey. *IEEE Transactions on Medical Imaging*, 34(9):1940–1964, 2015.

[Sto66] R. E. Stong. Finite topological spaces. *Trans. Amer. Math. Soc.*, 123:325–340, 1966.

[Tan18] Kohei Tanaka. A combinatorial description of topological complexity for finite spaces. *Algebr. Geom. Topol.*, 18(2):779–796, 2018.

[Udu92] Jayaram K. Udupa. Applications of digital topology in medical three-dimensional imaging. *Topology and its Applications*, 46(3):181–197, 1992.

[US96] Jayaram K. Udupa and Supun Samarasekera. Fuzzy connectedness and object definition: Theory, algorithms, and applications in image segmentation. *Graphical Models and Image Processing*, 58(3):246–261, 1996.