

Integrable Structure of the Overlaps for Integrable Non-Hermitian Random Matrices and Zeros of Random Power Series with Finitely Dependent Gaussian Coefficients

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Doctoral dissertation

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Integrable Non-Hermitian Random Matrices and
Zeros of Random Power Series with
Finitely Dependent Gaussian Coefficients**

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ABSTRACT

This thesis consists of five parts, and the outline in this thesis is as follows.

First, we will study the determinantal structure of the multi-point correlation function of the overlaps for integrable non-Hermitian random matrices. The overlap is the quantity defined by left and right eigenvectors of a non-normal matrix, introduced by Chalker and Mehlig in 1998. It plays a crucial role in analyzing the instability of eigenvalue of matrix and a probabilistic analysis of eigenvalues in non-Hermitian matrix-valued Brownian motion. Bourgade and Dubach computed the probability distribution of the diagonal overlap and the scaling limits of 2-point correlations for the Ginibre unitary ensemble. Akemann, Tribe, Tsareas, and Zaboronski computed the multi-points correlation function of the overlaps for Ginibre unitary ensemble, and they showed the scaling limits based on planar orthogonal polynomials. From the later perspective, we will study the multi-point correlation function of the overlaps for the induced Ginibre/spherical ensembles. The former model is regarded as the one-parameter generalization of the Ginibre unitary ensemble, and the later model has the similar structure, but it is a non-Gaussian integrable non-Hermitian random matrix. Depending on parameters, it is known that the macroscopic behavior of the eigenvalues of the induced Ginibre/spherical unitary ensembles is drastically different. They are classified into three regimes, strongly non-unitary, weakly non-unitary, and singular origin regimes. The local statistics for eigenvalues of the induced Ginibre/spherical unitary ensemble in these regimes have been studied, but the overlap statistics in these three regimes have been not studied. We will show the universality for the overlaps in the strongly non-unitary regime via the multi-point correlation function of the overlaps for both models, and we will find new scaling limits for the conditional expectation of the overlaps in the weakly non-unitary and singular origin regimes. In subsection 2.2, we present our results about the determinantal structure of the multi-point correlation of the overlaps for the induced Ginibre/spherical ensembles based on [113] and [114]. In part 2 and part 3, we prove their results.

Second, we will study the Pfaffian structure of the multi-point correlation function of the diagonal overlap for the Ginibre symplectic ensemble. Ginibre symplectic ensemble belongs to a different class of non-Hermitian random matrices. Indeed, the eigenvalues of GinSE form pairs with their complex conjugate, constituting a logarithmic Coulomb gas model that symmetrically repels along the real axis. And also, the multi-points correlation function of the eigenvalues of Ginibre symplectic ensemble forms Pfaffian. Due to this structure, it is natural to expect that the multi-point correlation function of the overlaps for Ginibre symplectic ensemble also forms the Pfaffian structure (cf. [11]), but it remained unsolved. We will solve this problem for the diagonal overlap conditioned on the real line. Our result is still restricted to the specific situation, but we need to develop a new tools to resolve this problem, and we need many complicated computations. So, we emphasize that our result is extremely non-trivial, and we believe that our result is a cornerstone in studies for skew-orthogonal polynomials in random matrix theory. Indeed, we have established a new method for constructing skew-orthogonal polynomials beyond the framework in [10]. In particular, to analyze the limiting skew kernel, we study on a specific second order differential equation. As a consequence, we demonstrate bulk and edge scaling limits for the conditional expectation of the diagonal overlap for the Ginibre symplectic ensemble conditioned on the real line. For the bulk case, this result can be regarded as the generalization of the result in [59], and the edge case is the new result to our best knowledge. In subsection 3.3, we present our results and new techniques. Their results are based on the on-going joint work with Gernot Akemann and Sung-Soo Byun. In part 4, we prove the results in subsection 3.3.

Peres and Virág studied a random power series with independent, identically distributed (i.i.d.) Gaussian coefficients. Remarkably, they showed that its zeros point process forms the determinantal point process on the unit disk. Various extensions of the Gaussian random power series have been known since the breakthrough work by Peres and Virág, but almost extensions are assumed to be i.i.d. Gaussian coefficients. For i.i.d. Gaussian random power series case, there are few works, but they were focused on the specific models. We will study the asymptotic behavior of the expected number of zeros of random power series with finitely dependent Gaussian coefficients, and we made the general structure to determine the sub-leading order for the asymptotic behavior clear. For the specific model, we could determine the constant term and the sub-sub-leading order. In subsection 4.2, we present the results on the zeros point processes of random power series with finitely dependent Gaussian coefficients based on the published paper [115], which is the joint work with Tomoyuki Shirai. In part 5, we prove Theorem 4.2, Theorem 4.3, and Theorem 4.5 together with some preliminaries and some examples in subsection 4.2.

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Part 1. Introduction and summary of contributions

1. INTRODUCTION: DETERMINANTAL STRUCTURE OF THE MULTI-POINTS CORRELATION FUNCTION OF THE OVERLAPS FOR INTEGRABLE NON-HERMITIAN RANDOM MATRICES

1.1. Eigenvalues of random matrices. In 1945, Wigner initiated the spectral analysis of random matrix, [141–143]. He considered three Gaussian orthogonal/unitary/symplectic ensembles, which are shortly denoted by GOE/GUE/GSE. Here, we focus on GUE with size N , H_N , which is precisely defined by

$$H_N = \sqrt{\frac{N}{2}}(G_N + G_N^\dagger),$$

where $G_N = (G_{i,j})_{i,j=1}^N$ is a non-Hermitian random matrix whose elements are independent, identically distributed complex Gaussian random variables with mean 0 and variance $1/N$. \dagger means the complex conjugate transpose of matrix, i.e., $M^\dagger = \overline{M}^t$ for $M \in \mathbb{C}^{n \times m}$. Here, G_N is called the Ginibre unitary ensemble (GinUE), which plays a central role in this thesis as we will see later. Wigner showed that the empirical spectral distribution weakly converges to the Wigner's semi-circle law in large size limit of matrix dimension, i.e., for the eigenvalues $(x_1, \dots, x_N) \in \mathbb{R}^N$ of GUE with size N ,

$$(1.1) \quad \frac{1}{N} \sum_{j=1}^N \delta_{x_j/\sqrt{N}} \rightarrow \rho_{\text{sc}}(x) = \frac{1}{2\pi} \sqrt{4 - x^2}, \quad \text{as } N \rightarrow \infty \text{ in weak sense,}$$

with probability one. Also, the joint probability distribution function of the eigenvalues of GUE with size N is

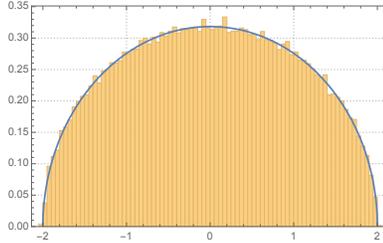


FIGURE 1. Histogram of eigenvalues of GUE with size 250 and graph of $\rho_{\text{sc}}(x)$.

give by

$$d\mathbb{P}_N^{(\text{GUE})}(x_1, \dots, x_N) = \frac{1}{Z_N^{(\text{GUE})}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{j=1}^N e^{-\frac{1}{2}x_j^2} dx_j, \quad (x_1, \dots, x_N) \in \mathbb{R}^N,$$

where $Z_N^{(\text{GUE})}$ is the partition function, and dx is the Lebesgue measure on the real line. To study the particle system of $(x_1, \dots, x_N) \in \mathbb{R}^N$, it is natural to study the k -th correlation function

$$(1.2) \quad \mathbf{R}_{N,k}^{(\text{GUE})}(x_1, \dots, x_k) = \frac{N!}{(N-k)!} \int_{\mathbb{R}^{N-k}} \mathbb{P}_N^{(\text{GUE})}(x_1, \dots, x_N) \prod_{j=k+1}^N dx_j.$$

This quantity essentially contains all the information about the particle system and is used to analyze statistical properties such as linear statistics and the extreme value distribution for the largest eigenvalue. In particular, (1.2) can be written as the determinantal form

$$(1.3) \quad \mathbf{R}_{N,k}^{(\text{GUE})}(x_1, \dots, x_k) = \det(K_N^{(\text{GUE})}(x_i, x_j))_{i,j=1}^k,$$

where $K_N^{(\text{GUE})}$ is called the correlation kernel, and (1.3) is called the determinantal point process (DPP), [70,108]. For GUE, the correlation kernel $K_N^{(\text{GUE})}$ can be written in terms of the Hermite polynomials:

$$(1.4) \quad K_N^{(\text{GUE})}(x, y) = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(y) = \sqrt{N} \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x - y},$$

where

$$\psi_k(x) = \frac{e^{-x^2/4} H_k(x)}{(\sqrt{2\pi k!})^{1/2}}, \quad H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad \text{for } k = 0, 1, 2, \dots$$

The last identity in (1.4) is called Christoffel-Darboux identity [124]. This identity is crux in one-dimensional integrable random matrix theory. Indeed, by the strong asymptotics of the Hermite polynomials in [124], we can show that for $p \in (-2, 2)$,

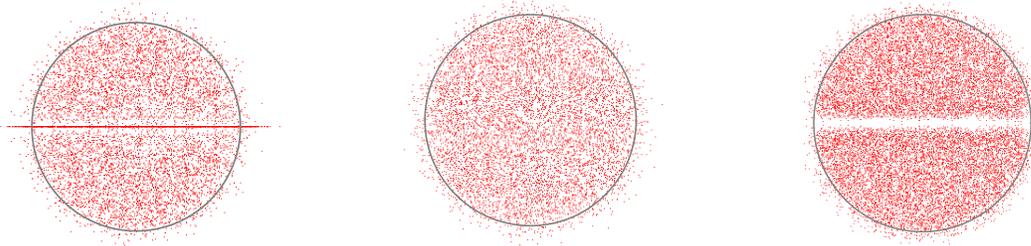
$$(1.5) \quad \frac{1}{\sqrt{N} \rho_{\text{sc}}(p)} K_N^{(\text{GUE})} \left(\sqrt{N}p + \frac{x}{\sqrt{N} \rho_{\text{sc}}(p)}, \sqrt{N}p + \frac{x}{\sqrt{N} \rho_{\text{sc}}(p)} \right) \rightarrow K_{\text{sin}}(x, y), \quad \text{as } N \rightarrow \infty,$$

uniformly for x, y in compact subsets of \mathbb{R} . The correlation kernel (1.5) plays the essential role to describe the level spacing distribution in the bulk, and Gaudin and Mehta [107] showed that for any compact subset A of \mathbb{R} ,

$$(1.6) \quad \lim_{N \rightarrow \infty} \mathbb{P}_N^{(\text{GUE})} \left(\bigcap_{j=1}^N \{ \sqrt{N}x_j \notin A \} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_A \cdots \int_A \det(K_{\text{sin}}(x_i, x_j))_{i,j=1}^n \prod_{j=1}^n dx_j,$$

where the right hand side is understood as the Fredholm determinant. GUE is one of the integrable Hermitian random matrices in the sense that the local statistics such as k -th correlation function can be explicitly written in terms of the orthogonal polynomials. Apart from the integrable Hermitian random matrices, global and local statistics of eigenvalues of random matrices have been still studied by many mathematicians and physicists. In face, for symmetric/Hermitian random matrices and for wider classes of the one-dimensional logarithmic Coulomb gases, the universality of the macroscopic and microscopic statistics of eigenvalues have been shown in [57, 65, 70, 129]. The most important phenomenon of eigenvalues of random matrices is *universality*. Roughly speaking, the universality means that statistics of eigenvalues of random matrices with size N such as the empirical distribution (1.1), point processes (1.4) and (1.5), and level spacing distribution (1.6) do not depend on random matrix models with some assumptions in a suitable large N -limit. Such universality can be also confirmed in very wide fields such as the Coulomb gases [101], random tiling [82], infinite interacting particle systems [85], number theory [91], and KPZ equation [56]. Beyond mathematics and physics, such universality of random matrices can be applied into other scientific fields in nowadays. In fact, results and techniques for spectral analysis in random matrix theory can be applied into machine learning [95], biology [3], economics [35], and quantum optics [23].

Eigenvalue statistics of non-Hermitian random matrices have also received much attention. Jean Ginibre initiated the spectral analysis of the eigenvalues of non-Hermitian random matrices, specifically the Ginibre orthogonal/unitary/symplectic ensembles, in 1965 [79]. In this thesis, we focus on the Ginibre unitary ensemble



(A) Plot of eigenvalues of GinOE (B) Plot of eigenvalues of GinUE (C) Plot of eigenvalues of GinSE

FIGURE 2. (A) is the plot of eigenvalues of Ginibre orthogonal ensemble (GinOE), which is the random matrix with i.i.d. real Gaussian random variables of mean 0 and variance 1 entries. (B) is the plot of eigenvalues of Ginibre unitary ensemble (GinUE), which is the random matrix with i.i.d. complex Gaussian random variables of mean 0 and variance 1 entries. (C) is the plot of the eigenvalues of GinSE.

and the Ginibre symplectic ensemble. The detailed introduction for the latter model and related topics are left to section 3, and so, we focus on the Ginibre unitary ensemble (GinUE). As we already mentioned, GinUE G_N

with size N is defined by the non-Hermitian random matrix whose elements are i.i.d. complex Gaussian random variables with mean 0 and variance $1/N$, and its joint probability function of the eigenvalues $(z_1, \dots, z_N) \in \mathbb{C}^N$ of GinUE is given by

$$d\mathbb{P}_N^{(\text{GinUE})}(z_1, \dots, z_N) = \frac{1}{Z_N^{(\text{GinUE})}} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \prod_{j=1}^N e^{-N|z_j|^2} dA(z_j),$$

where $Z_N^{(\text{GinUE})}$ is the partition function, and $dA(z)$ is the planar Lebesgue measure on \mathbb{C} . Different from (1.1), the empirical distribution of $(z_1, \dots, z_N) \in \mathbb{C}^N$ of GinUE weakly converges to the circular law in a large N -limit, i.e.,

$$(1.7) \quad \frac{1}{N} \sum_{j=1}^N \delta_{z_j} \rightarrow \mathbf{1}_{|z| < 1}, \quad \text{as } N \rightarrow \infty \text{ in weak sense,}$$

with probability one. Similar to (1.2), the k -th correlation function of the point process for GinUE forms DPP given by

$$(1.8) \quad \mathbf{R}_{N,k}^{(\text{GinUE})}(z_1, \dots, z_k) = \det(\mathbf{K}_N^{(\text{GinUE})}(z_i, z_j))_{i,j=1}^k,$$

where

$$(1.9) \quad \mathbf{K}_N^{(\text{GinUE})}(z, w) = e^{-\frac{N}{2}(|z|^2 + |w|^2)} N \sum_{k=0}^{N-1} \frac{(Nz\bar{w})^k}{k!} = e^{N(z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2)} Q(N, Nz\bar{w}).$$

Here, $Q(n, x)$ is the incomplete gamma function defined by $Q(n, x) = e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$ for $x \in \mathbb{C}$. In order to motivate our analysis in subsection 2.2 and 3.3, let us explain two ways of the asymptotic analysis for (1.9). The first is to use the uniform asymptotic expansion for the incomplete gamma function [117, 132, 133]. For $z = e^{i\theta}(p + \frac{\zeta}{\sqrt{N}})$ and $w = e^{i\theta}(p + \frac{\eta}{\sqrt{N}})$ with $p \in \text{clo}(\mathbb{D})$, ζ, η in compact subsets of \mathbb{C} , and $\theta \in [0, 2\pi)$, from [117, section 8.12], we have

$$(1.10) \quad \frac{1}{N} \mathbf{K}_N^{(\text{GinUE})}\left(e^{i\theta}\left(p + \frac{\zeta}{\sqrt{N}}\right), e^{i\theta}\left(p + \frac{\eta}{\sqrt{N}}\right)\right) \rightarrow \begin{cases} e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \zeta\bar{\eta}}, & \text{if } p \in \text{int}(\mathbb{D}), \\ e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \zeta\bar{\eta}} \frac{1}{2} \text{erfc}\left(\frac{\zeta + \bar{\eta}}{\sqrt{2}}\right) & \text{if } p \in \partial(\mathbb{D}), \end{cases}$$

as $N \rightarrow \infty$, uniformly for ζ, η in compact subsets of \mathbb{C} . The former limit is called the Ginibre bulk point process, and the later is called the Ginibre boundary point process. Another approach we will show is to make a differential equation. (1.9) does not have Christoffel-Darboux identity as in (1.4), and so, we need to directly deal with the sum itself. As we already showed, we can use the uniform asymptotic expansion for the incomplete gamma function, but in many situations, we do not have such exact expression by the special function. The spirit to use Christoffel-Darboux identity is that to directly analyze the infinite sum in a scaling limit, but to analyze some single functions is easier than the summation if we know the uniform asymptotic expansion for their single functions. Hence, let $\widehat{\mathbf{K}}_N^{(\text{GinUE})}(z, w) = e^{-Nz\bar{w}} \mathbf{K}_N^{(\text{GinUE})}(z, w)$, and then, we have

$$(1.11) \quad \partial_z \widehat{\mathbf{K}}_N^{(\text{GinUE})}(z, w) = -e^{-Nz\bar{w}} N^2 \bar{w} \frac{(Nz\bar{w})^{N-1}}{(N-1)!}.$$

Since we can compute the asymptotic behavior of the right-hand side above, we can easily recover (1.10) by solving the limiting differential equation with a suitable initial condition. The two approaches demonstrated here are important points in this thesis. The first approach is applied to the results in subsection 2.2 with some additional efforts, and the second approach is applied to the results in subsection 3.3. The origin of the differential equation approach is [97] for the elliptic Ginibre unitary ensemble, and this implementation was applied to other ensembles [6, 36–39, 42, 89]. Similar to integrable Hermitian random matrices and more general Hermitian random matrices, the macroscopic and local statistics for wider integrable non-Hermitian random matrices, more general non-Hermitian random matrices, and those related to two-dimensional logarithmic Coulomb gases have been studied. We refer to [7, 45, 47] for counting statistics, [46, 81, 96, 136, 137] for large gap probability, [49, 50, 119–121] for linear statistics and edge statistics, and the universality of the correlation

function [16–21, 48, 83, 84]. We emphasize that the spirit and many objects of studies for non-Hermitian random matrices are similar to those of studies for Hermitian random matrices, but the difficulties to overcome and technical aspects are drastically different from those of Hermitian random matrices, as we have demonstrated using the simplest models GUE and GinUSE in Hermitian and non-Hermitian random matrix theory, respectively. For further topics and details, we refer to recent instructive and comprehensive reviews for non-Hermitian random matrices [40] for topics related to Ginibre unitary ensembles and [41] for topics related to Ginibre orthogonal/symplectic ensembles.

1.2. Eigenvectors-overlaps for non-Hermitian random matrices. In the previous subsection, we quickly reviewed the studies on the eigenvalues of Hermitian and non-Hermitian random matrices. Eigenvectors for Hermitian random matrices have also been studied, but we emphasize that eigenvectors do not play an important role in Wigner’s origin theory. Indeed, GUE is the invariant ensemble under conjugation by unitary matrices, and hence, each eigenvector of GUE is distributed according to the Haar measure on the unitary group. Since it is well known that the entries of the unitary group with size N multiplied by \sqrt{N} converge to the Gaussian distribution in law as $N \rightarrow \infty$, which was shown by Borel in 1906 in [110, Theorem 2.4], the eigenvector statistics for GUE or GOE are not particularly interesting objects. However, we emphasize that eigenvector statistics for general Hermitian random matrices, called Wigner matrices or generalized Wigner matrices, are intensively investigated from the perspective of universality [30, 92, 130]. For an instructive and comprehensive survey of eigenvectors for symmetric/Hermitian random matrices, we refer to [122].

Compared to the eigenvector statistics for Hermitian random matrices, the situation for non-Hermitian random matrices dramatically changes. Let us consider a non-Hermitian random matrix G_N with size N assumed to have N simple eigenvalues. Then, G_N is diagonalizable as $G_N = SD_N S^{-1}$, where $D_N = \text{diag}(\lambda_1, \dots, \lambda_N)$ is the diagonal matrix with eigenvalues elements. If G_N is non-normal, S is not a unitary matrix. Hence, $S^\dagger \neq S^{-1}$. The columns of S are associated with the right eigenvectors, and the rows of S^{-1} are associated with the left eigenvectors, which satisfy the following and denoted by

$$(1.12) \quad G_N R_j = \lambda_j R_j, \quad L_j^\dagger G_N = \lambda_j L_j^\dagger,$$

and

$$S = (S_{j,k})_{j,k=1}^N = (R_1 \ R_2 \ \cdots \ R_N) \in \mathbb{C}^{N \times N},$$

and

$$S^{-1} = (L_1 \ L_2 \ \cdots \ L_N)^\dagger \in \mathbb{C}^{N \times N}.$$

For instance, the right eigenvector R_k is expressed as

$$R_k = \begin{pmatrix} R_{k,1} \\ R_{k,2} \\ \vdots \\ R_{k,N} \end{pmatrix} = \begin{pmatrix} S_{1,k} \\ S_{2,k} \\ \vdots \\ S_{N,k} \end{pmatrix}.$$

These left and right eigenvectors form bi-orthogonal basis, and we normalize these as

$$(1.13) \quad \langle L_i, \bar{R}_j \rangle = \delta_{i,j},$$

but they are not orthogonal

$$\langle L_i, L_j \rangle \neq \delta_{i,j}, \quad \langle R_i, R_j \rangle \neq \delta_{i,j}$$

where $\langle x, y \rangle = \sum_{k=1}^d x_k \bar{y}_k$ for $x = (x_1, x_2, \dots, x_d) \in \mathbb{C}^d$ and $y = (y_1, y_2, \dots, y_d)^\dagger \in \mathbb{C}^d$ denotes the Euclidean inner product on \mathbb{C}^d . By the bi-orthogonality condition (1.13), we have

$$(S^{-1}S)_{j,k} = \sum_{i=1}^N L_{j,i} R_{i,k} = \langle L_j, \bar{R}_k \rangle = \delta_{j,k}.$$

We now define

$$(1.14) \quad \begin{aligned} A &= (A_{j,k})_{j,k=1}^N = S^\dagger S, \quad \text{with } A_{k,j} = \langle R_j, R_k \rangle, \\ A^{-1} &= (A_{j,k}^{-1})_{j,k=1}^N = S^{-1} S^{-1\dagger}, \quad \text{with } A_{j,k}^{-1} = \langle L_j, L_k \rangle, \end{aligned}$$

With the notation (1.14), we define

$$(1.15) \quad \mathcal{O} = (\mathcal{O}_{j,k})_{j,k=1}^N, \quad \mathcal{O}_{j,k} = A_{j,k}^{-1} A_{k,j} = \langle L_j, L_k \rangle \langle R_j, R_k \rangle.$$

The expression (1.15) is called an overlap matrix, and its elements are referred to as overlaps. In particular, $\mathcal{O}_{j,j}$ for $j = 1, 2, \dots, N$ are called the diagonal overlaps, and $\mathcal{O}_{i,j}$ for $i \neq j = 1, 2, \dots, N$ are called the off-diagonal overlaps. Note that the overlap is invariant under the scale transformation $L_j \mapsto c_j^{-1} L_j$ and $R_j \mapsto c_j R_j$ due to the normalization (1.13). By the definition in (1.15), for normal matrices G_N , the corresponding overlap matrix becomes the identity matrix since the left eigenvectors coincide with the right eigenvectors, and hence, they form orthogonal systems. This implies that the overlap is used to measure the non-normality of matrices, which plays a central role in the instability of eigenvalues of non-normal matrices. In fact, as discussed in [135, sections 35 and 52], [29, section 1], and [77, 144, section 1] (the discussion below was done in [29]), when we consider the eigenvalues $(\lambda_j(t))_{j=1}^N \in \mathbb{C}^N$ of the perturbed matrix $X + tE$ given a non-normal matrix X whose eigenvalues $(\lambda_j)_{j=1}^N \in \mathbb{C}^N$ are all distinct, from the first-order perturbation theory, we have

$$\sqrt{\mathcal{O}_{j,j}} = \lim_{t \rightarrow 0} \sup_{\|E\|=1} \frac{|\lambda_j(t) - \lambda_j|}{t}, \quad \|E\| = \sup_{\|x\|_2=1} \|Ex\|_2.$$

$\sqrt{\mathcal{O}_{j,j}}$ is called the condition number, which measures the sensitivity of eigenvalues by the overlap and is used to estimate the pseudo-eigenvalue [135]. Also, the motivation to study the instability of eigenvalues dates back to May in 1972, focusing on the stability of complex ecological webs [106]. For additional physical motivations, refer to [73, 74, 78]. In [74, 78], two interesting non-Hermitian random matrices related to open or scattering quantum chaotic systems were proposed. They obtained the exact expressions of the overlaps for the two models. These are also related to non-Hermitian rank one perturbed random matrices.

The first systematic attempt to understand the non-orthogonality of eigenvectors in non-Hermitian random matrices was originally initiated by Chalker and Mehlig [53, 54]. In particular, they focused on the simplest non-normal random operator, GinUE. They studied the one-point correlation function and the two-point correlation function

$$D_{1,1}^{(N,1)}(z) = \mathbb{E} \left[\sum_{k=1}^N \mathcal{O}_{k,k} \delta(z - \lambda_k) \right], \quad D_{1,2}^{(N,2)}(z_1, z_2) = \mathbb{E} \left[\sum_{j \neq k}^N \mathcal{O}_{j,k} \delta(z_1 - \lambda_j) \delta(z_2 - \lambda_k) \right]$$

In [53, 54], they estimated that for any $|z_1|, |z_2| < 1$ of GinUE,

$$D_{1,1}^{(N,1)}(z_1) \sim N(1 - |z_1|^2), \quad D_{1,2}^{(N,2)}(z_1, z_2) \sim -\frac{1 - z_1 \bar{z}_2}{|z_1 - z_2|^4},$$

as $N \rightarrow \infty$, and they rigorously showed them for $z_1 = 0$. The former quantity was essentially shown for $|z_1| < 1$ in [54]. Later, rigorous calculations for the diagonal overlap were done in [139]. Here, we note that $D_{1,1}^{(N,1)}(z)$ is not an exact conditional expectation. Since the seminal works [53, 54], the overlaps have been studied in [139] and from the connection with free probability based on a diagrammatic approach [116], but mathematically systematic and unified studies for the overlap had not been done by 2018. In such circumstances, three breakthrough papers [13, 29, 75] appeared. In these three papers, the overlaps for GinUE were studied using three different approaches. To motivate our work in this thesis, let us briefly mention their results and techniques.

1.2.1. *Probabilistic results in [29].* In [29, Theorem 1.1], Bourgade and Dubach showed that the diagonal overlap of the Ginibre unitary ensemble conditioned at $z_1 = a$ converges to the inverse gamma distribution in law, that is, conditionally on $z_1 = a \in \text{int}(\mathbb{D})$,

$$\frac{\mathcal{O}_{1,1}}{N(1 - |a|^2)} \xrightarrow{d} \gamma_2^{-1}, \quad \text{as } N \rightarrow \infty,$$

where γ_α is the gamma distribution with shape α . Their proof is based on a strong probabilistic method. Indeed, their method relies on the rigidity estimate for the eigenvalues of the Ginibre unitary ensemble. Moreover, they showed the asymptotic behaviors of the correlation for the overlaps. For instance, in [29, Theorem 1.3, 1.4],

they showed that for any $\delta \in (0, 1/2)$, $\epsilon > 0$, and $C > 0$, uniformly in z_1, z_2 such that $|z_1| < 1 - N^{-1/2+\delta}$, and $\sqrt{N}|z_1 - z_2| \in [N^{-C}, N^{\delta-\epsilon}]$,

$$(1.16) \quad \mathbb{E}[\mathcal{O}_{1,2}|\lambda_1 = z_1, \lambda_2 = z_2] = -N \frac{1 - z_1 \bar{z}_2}{N^2 |z_1 - z_2|^4} \frac{1 - (1 + N|z_1 - z_2|^2)e^{-N|z_1 - z_2|^2}}{1 - e^{-N|z_1 - z_2|^2}} (1 + O(N^{-2\delta+\epsilon})).$$

Also, for any $\delta \in (0, 1/2)$, $\kappa \in (0, \delta)$, $\epsilon > 0$, uniformly in z_1, z_2 such that $|z_1| < 1 - N^{-1/2+\delta}$, and $\sqrt{N}|z_1 - z_2| \in [N^{-\delta/2+\epsilon}, N^\kappa]$, they showed that

$$\begin{aligned} \mathbb{E}[|\mathcal{O}_{1,2}|^2|\lambda_1 = z_1, \lambda_2 = z_2] &= \frac{N^2(1 - |z_1|^2)(1 - |z_2|^2)}{N^2|z_1 - z_2|^4} (1 + O(N^{2(\sigma-\delta)+\epsilon})), \\ \mathbb{E}[\mathcal{O}_{1,1}\mathcal{O}_{1,2}|\lambda_1 = z_1, \lambda_2 = z_2] &= \frac{N^2(1 - |z_1|^2)(1 - |z_2|^2)}{N^2|z_1 - z_2|^4} \frac{1 + N^2|z_1 - z_2|^4 - e^{-N|z_1 - z_2|^2}}{1 - e^{-N|z_1 - z_2|^2}} (1 + O(N^{2(\sigma-\delta)+\epsilon})). \end{aligned}$$

The results and techniques in [29] are very robust and probabilistic, and the complicated correlations for the overlaps can be computed with a rate of convergence at both micro and mesoscopic scales. They also derived a system of stochastic differential equations for the eigenvalue processes of the matrix-valued non-Hermitian complex Ornstein-Uhlenbeck process, as we will see in the next sub-subsection.

1.2.2. *Integrable results in [13].* In [13], Akemann, Tribe, Tsareas, and Zaboronski considered the determinantal structure of the multi-points correlation function of the overlaps for the GinUE. Their motivation to study such object is to understand a stochastic dynamics for eigenvalues processes of the matrix valued non-Hermitian Brownian motion for a certain time scale. In physical paper [80] or mathematical paper [29], a family of the stochastic process of eigenvalues $(z_1(t), \dots, z_N(t))$ of matrix-valued non-Hermitian Ornstein-Uhlenbeck process $G(t) = (G_{i,j}(t))_{i,j=1}^N$ with entries

$$dG_{i,j}(t) = dB_{i,j}(t) - \frac{1}{2}G_{i,j}(t)dt,$$

where $(B_{i,j})_{i,j=1}^N$ are independent standard complex Brownian motions, satisfies the system of stochastic differential equations

$$(1.17) \quad dz_k(t) = dM_k(t) - \frac{1}{2}z_k(t)dt, \quad \text{for } k = 1, \dots, N.$$

Here, the quadratic variation for (1.17) is given by

$$(1.18) \quad d\langle z_i(t), \overline{z_j(t)} \rangle_t = \mathcal{O}_{i,j}(t)dt.$$

This means that in order to understand the family of the stochastic processes $(z_k(t))_{k=1}^N$, we need to make the structure of the overlaps more clear. Also, (1.18) implies that the dynamical eigenvalues are complex martingales evolving at the scale $O(N^{-1})$. This suggests us to formally study that

$$(1.19) \quad \lim_{N \rightarrow \infty} \mathbb{E}[\mathcal{O}_{i,j}|z_1, \dots, z_N]dt, \quad \text{for } 1 \leq i, j, \leq N.$$

This setting is time-evolution setting. It seems to be difficult to directly analyze and make an estimate for (1.19) without any results in a static case. Thus, they studied the multi-points correlation functions of the overlaps $\mathcal{O}_{1,1}, \mathcal{O}_{1,2}$ for the Ginibre unitary ensemble conditioned at multiple points, namely, for $q = 1, 2$,

$$(1.20) \quad \begin{aligned} D_{1,q}^{(N,k)}(z_1, \dots, z_k) &= \frac{1}{Z_N} \frac{N!}{(N-k)!} \\ &\times \int_{\mathbb{C}^{N-k}} \mathbb{E}[\mathcal{O}_{i,j}|z_1, \dots, z_N] |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N e^{-|z_j|^2} \prod_{j=k+1}^N dA(z_j). \end{aligned}$$

We need the explicit expression for $\mathbb{E}[\mathcal{O}_{i,j}|\lambda_1 = z_1, \dots, \lambda_N = z_N]$, but it is known that

$$(1.21) \quad \mathbb{E}[\mathcal{O}_{1,1}|\lambda_1 = z_1, \dots, \lambda_N = z_N] = \prod_{i=2}^N \left(1 + \frac{1}{|z_1 - z_i|^2} \right),$$

and

$$(1.22) \quad \mathbb{E}[\mathcal{O}_{1,2} | \lambda_1 = z_1, \dots, \lambda_N = z_N] = -\frac{1}{|z_1 - z_2|^2} \prod_{i=3}^N \left(1 + \frac{1}{(z_1 - z_i)(z_2 - z_i)}\right).$$

For the proof, see [29, 53, 54]. Therefore, (1.20) can be rewritten as

$$(1.23) \quad D_{1,1}^{(N,k)}(z_1, \dots, z_k) = \frac{1}{Z_N} \frac{N!}{(N-k)!} \times \int_{\mathbb{C}^{N-k}} \prod_{i=2}^N \left(1 + \frac{1}{|z_1 - z_i|^2}\right) |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N e^{-|z_j|^2} \prod_{j=k+1}^N dA(z_j),$$

and

$$(1.24) \quad D_{1,2}^{(N,k)}(z_1, \dots, z_k) = -\frac{1}{Z_N} \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \frac{1}{|z_1 - z_2|^2} \prod_{i=3}^N \left(1 + \frac{1}{(z_1 - z_i)(z_2 - z_i)}\right) \times |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N e^{-|z_j|^2} \prod_{j=k+1}^N dA(z_j).$$

They showed the scaling limits for (1.23) and (1.24) at the bulk and edge points of the spectral droplet for the GinUE. Here, we will not discuss their main results since we aim to recover their results for induced Ginibre/spherical ensembles, which are the main focus of this thesis.

1.2.3. *The diagonal overlap for the Ginibre orthogonal/unitary ensembles [75, 144] and the elliptic Ginibre orthogonal ensemble [77, 131].* In [75], Fyodorov studied the full conditional probability law of the diagonal overlap for GinUE and the diagonal overlap for the GinOE conditioned at a real point. His method is based on the incomplete Schur decomposition and the super-symmetric method. In particular, his result holds for any finite N , allowing us to take bulk and edge scaling limits. Also, his result is consistent with [29]. In fact, he showed the inverse cubic tail decay of the full conditional probability density of the diagonal overlap. The method he developed can also be applied to the diagonal overlap for the elliptic Ginibre orthogonal ensemble (eGinOE).

In [77], based on the same method in [75], they calculated $\mathcal{P}_N(z, t) = \mathbb{E}[\sum_k \delta(\mathcal{O}_{k,k} - 1 - t) \delta(z - \lambda_k)]$ for $z \in \mathbb{R}$ for eGinOE, and hence, they derived the bulk scaling limits in the strong non-Hermiticity and the weakly non-Hermiticity regimes. Remarkably, they showed the inverse quadratic tail decay of the full conditional probability density of the diagonal overlap. They also computed the edge scaling limit in the strong non-Hermiticity regime. Later, Tarnowski [131] showed the edge scaling limit for the diagonal overlap conditioned at a real point in the weakly non-Hermiticity regime.

Furthermore, in [144], Würfel, Crumpton, and Fyodorov calculated the one-point correlation function of the diagonal overlap for GinOE conditioned at a complex point, which is valid for any finite N based on the incomplete Schur decomposition and using Grassmann integration. As a consequence, they confirmed the universality for the diagonal overlap in the bulk and edge scaling limits conditioned at a complex point, which are indeed same as the ones of GinUE. This result that the diagonal overlap conditioned at a complex point in the upper half plane in a large N -limit is coincident with the one of the GinUE is consistent with the phenomenon that the eigenvalues statistics for GinOE in the only upper half plane are coincident with the ones of GinUE, [28] and [41, section 3].

1.2.4. *Other works and our motivation.* Lastly, let us mention other works of the overlaps for non-Hermitian random matrices. Recently, [50, 64] studied the diagonal overlap for generic non-Hermitian random matrices, and then they showed the lower and upper bounds. Their results and techniques are strongly probabilistic, but they would still need continuous works to reveal the overlaps for generic non-Hermitian random matrices.

The major motivation for studying the overlap in non-Hermitian random matrices within probabilistic communities is to apply results about the overlaps to the stochastic analysis of eigenvalues and eigenvector-overlap processes for matrix-valued non-Hermitian Brownian motions. In the seminal work of the Krakow School [80], they derived the system of stochastic evolution equations for eigenvalues and eigenvectors. Later,

Bourgade and Dubach [29, Appendix A] derived the stochastic differential equations for the eigenvalue processes in the matrix-valued non-Hermitian Ornstein-Uhlenbeck process, as discussed in sub-subsection 1.2.2, and they demonstrated the non-collision property for the eigenvalue processes. Similarly, they also discussed the case of the dynamical GinOE. Yabuoku [145] studied the eigenvalue processes of the elliptic Ginibre ensemble with complex Brownian motion entries. Remarkably, he showed that a Dyson drift term appears in the stochastic differential equation for the eigenvalue processes. Esaki, Katori, and Yabuoku [66] rigorously derived the system of stochastic differential equations for the coupled system of the eigenvalue process and the eigenvector-overlap process. Also, they studied the time-dependent point process of eigenvalues and its variation weighted by the diagonal elements, and they discussed the connection between the Brown measure, the time-dependent point process, and initial conditions problems related to non-normal matrices.

The overlaps and non-Hermitian random matrices studied in this thesis are rather focused on the specific models, induced Ginibre/spherical ensembles and the Ginibre symplectic ensemble. And, we will confirm the universality of the conditional expectation of the diagonal and off-diagonal overlaps for the former two models. It is worth stressing that our results are consistent with [116]. Furthermore, the induced Ginibre/spherical unitary ensemble have the weakly non-unitary regime, which is similar regime with the weak non-Hermiticity for eGinOE, and the singular origin regime. For detailed discussions, refer to sub-subsection 2.1.2. In these regimes, we will find new scaling limits for both diagonal and off-diagonal overlaps. Our analysis is strongly inspired by [13], and hence, our results are based on the exact calculation of the planar orthogonal polynomials and valid for any finite N . Also, as we will discuss in detail in part 4, we will analyze the Pfaffian structure of the diagonal overlap for GinSE and derived bulk and edge scaling limits for the diagonal overlap for GinSE conditioned on the real line.

2. DETERMINANTAL STRUCTURE OF THE OVERLAPS FOR INDUCED GINIBRE/SPHERICAL ENSEMBLE

2.1. Ginibre/spherical unitary ensembles and their overlaps. In order to state the results in this thesis, we introduce induced Ginibre/spherical unitary ensembles.

2.1.1. Induced Ginibre/spherical unitary ensembles. The below tells us a way to construct induced Ginibre/spherical unitary ensembles.

Proposition 2.1 ([68], [40]). *For $n, N \in \mathbb{N}$ with $n \geq N$, let $g : \mathbb{C}^{N \times N} \rightarrow \mathbb{R}_{\geq 0}$ and \mathbf{G} be a bi-unitary invariant rectangular $n \times N$ random matrix with joint matrix element distribution proportional to $g(\mathbf{G}^\dagger \mathbf{G})$.*

Then, the joint probability distribution of the matrix $\mathbf{A} = (\mathbf{G}^\dagger \mathbf{G})^{\frac{1}{2}} \mathbf{U}$ with a Haar unitary matrix \mathbf{U} with size N is proportional to

$$(2.1) \quad (\det \mathbf{A}^\dagger \mathbf{A})^{n-N} g(\mathbf{A}^\dagger \mathbf{A}).$$

In this Proposition 2.1, we set $g(X) = e^{-\text{Tr}(X)}$ for $X \in \mathbb{C}^{N \times N}$, and we fix \mathbf{G} as $n \times N$ rectangular complex Ginibre ensemble, whose elements are i.i.d. standard complex Gaussian random variables mean 0 and variance 1. Then, we can construct the random matrix \mathbf{A}_N with joint probability distribution function (2.1). We call such random matrix \mathbf{A}_N the *induced Ginibre unitary ensemble*. In short, we denote the induced Ginibre unitary ensemble by IGINUE. By change of variables from the matrix elements of \mathbf{A}_N to the eigenvalues $\mathbf{z}_{(N)} = (z_1, \dots, z_N) \in \mathbb{C}^N$ of \mathbf{A}_N , we can find that the joint probability distribution function of the eigenvalues of IGINUE \mathbf{A}_N is given by

$$(2.2) \quad d\mathbb{P}_N^{(\text{IGinUE})}(\mathbf{z}_{(N)}) = \frac{1}{Z_N} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \prod_{k=1}^N |z_k|^{2\alpha} e^{-|z_k|^2} dA(z_k), \quad \alpha = n - N,$$

where the partition function $Z_N^{(\text{IGinUE})}$ is given by

$$(2.3) \quad Z_N^{(\text{IGinUE})} = N! \prod_{k=0}^{N-1} \Gamma(k + \alpha + 1).$$

Here, $dA(z)$ is the planar Lebesgue measure on the complex plane \mathbb{C} , and α is a non-negative integer valued parameter when we consider the random matrix model \mathbf{A}_N . However, we can still consider the general $\alpha > -1$.

It is well-known that the k -th correlation function of (2.2) forms the determinantal point process. When we consider the rescaling $z \mapsto \sqrt{Na_N}z$, the corresponding to correlation kernel is given by

$$\mathbf{R}_{N,k}^{(\text{IGinUE})}(\mathbf{z}^{(k)}) = \det(\mathbf{K}_N^{(\text{IGUE})}(z_i, z_j))_{i,j=1}^k,$$

where

$$\mathbf{K}_N^{(\text{IGUE})}(z, w) = Na_N \sum_{j=0}^{N-1} \frac{(Na_N z \bar{w})^{j+\alpha}}{\Gamma(j+\alpha+1)} e^{-\frac{Na_N}{2}(|z|^2+|w|^2)}.$$

From the basic fact of the logarithmic potential theory [44, 123], the spectral droplet associated with (2.2) (after rescaling $z \mapsto \sqrt{Na_N}z$) tends to

$$(2.4) \quad S_g = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\} \quad \text{with } r_1 = \sqrt{\frac{b_N}{Na_N}} \text{ and } r_2 = \sqrt{\frac{b_N + N}{Na_N}}.$$

On the other hand, Proposition 2.1 can be applied to the induced spherical unitary ensemble. In short, we denote the induced spherical unitary ensemble by ISUE. Indeed, the *induced spherical unitary ensemble* can be realized as a random matrix $\mathbf{G}_N = \mathbf{U}_N(\mathbf{Y}_N^\dagger \mathbf{Y}_N)^{\frac{1}{2}}$, where \mathbf{U}_N is a circular unitary matrix uniformly distributed to the Haar measure on the unitary group, and \mathbf{Y}_N is an $n \times N$ rectangular complex Ginibre ensemble. Hence, for $n \geq N$ and $L \geq 0$, the joint probability distribution function of the induced spherical unitary ensemble \mathbf{G}_N is given by

$$(2.5) \quad d\mathbb{P}_N^{(\text{ISUE})}(\mathbf{G}_N) = C_N \frac{\det(\mathbf{G}_N \mathbf{G}_N^\dagger)^L}{\det(\mathbf{1}_N + \mathbf{G}_N \mathbf{G}_N^\dagger)^{n+N+L}} d[\mathbf{G}_N], \quad C_N := \frac{1}{\pi^{N^2}} \prod_{k=1}^N \frac{\Gamma(k)\Gamma(n+N+k)}{\Gamma(L+k)\Gamma(n-N+k)},$$

where parameters n and L satisfy $n \geq L$ and $L \geq 0$, and they may depend on N . By change of variables from the matrix space to the eigenvalues, the joint probability distribution function of the eigenvalues $\mathbf{z}_{(N)} = (z_1, \dots, z_N) \in \mathbb{C}^N$ associated with (2.5) is given by

$$(2.6) \quad d\mathbb{P}_N^{(\text{ISUE})}(\mathbf{z}_{(N)}) = \frac{1}{Z_N^{(\text{ISUE})}} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \prod_{j=1}^N e^{-NQ(z_j)} dA(z_j),$$

where $Z_N^{(\text{ISUE})}$ is the partition function, and

$$(2.7) \quad Q(z) = \frac{n+L+1}{N} \log(1+|z|^2) - \frac{2L}{N} \log|z|.$$

Again, from the basic of the logarithmic potential theory [83, 123], the empirical measure of the eigenvalues for ISUE tends to be distributed on the spectral droplet

$$(2.8) \quad S_s = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}, \quad r_1 = \sqrt{\frac{L}{n}}, \quad r_2 = \sqrt{\frac{N+L}{n-N}},$$

with the density

$$\Delta Q(z) \mathbf{1}_{S_s} \sim \frac{n+L}{N} \frac{1}{(1+|z|^2)^2}.$$

As in IGINUE, it is well-known that the point process of eigenvalues distributed to (2.6) forms the determinantal point process [40, 69]. The k -th correlation function of the point process for (2.6) is given by

$$(2.9) \quad \mathbf{K}_N^{(\text{ISUE})}(z, w) = e^{-\frac{N}{2}(Q(z)+Q(w))} \sum_{k=0}^{N-1} \frac{\Gamma(n+L+1)}{\Gamma(n-k)\Gamma(k+L+1)} (\bar{z}w)^k.$$

Here, we only discussed the construction for the unitary case, but their construction can be also applied to the orthogonal/symplectic ensembles. For the detailed construction of induced Ginibre/spherical orthogonal or symplectic ensemble ensembles, we refer to [41, 68, 69, 104, 105].

2.1.2. *Spectral droplets for the induced Ginibre/spherical unitary ensembles.* We have discussed the joint probability distribution functions of the eigenvalues for the I GinUE/ISUE and their spectral droplets in the previous sub-subsection. Here, (2.4) and (2.8) tell us that macroscopic shapes of the spectral droplets for I GinUE and ISUE may change depending on the parameters a_N, b_N for I GinUE na n, L for ISUE. Indeed, following [21, 44], we introduce the three regimes: **strongly non-unitary**, **weakly non-unitary regimes**, and **the singular origin regime**. These spectral regimes play essential roles in this thesis.

(1) **Strongly non-unitary regime:** For I GinUE, we set $a_N = 1$ and $b_N = Nb$ for $b > 0$. Then, the inner and outer radii of (2.4) satisfy

$$r_1 = \sqrt{b}, \quad r_2 = \sqrt{1+b}.$$

For ISUE, we set the parameters L, n . Then, for a fixed $a, b \geq 0$, and

$$L = aN, \quad n = (b+1)N,$$

then the inner and outer radii of (2.8) satisfy

$$r_1 = \sqrt{\frac{a}{b+1}} + O(N^{-1}), \quad r_2 = \sqrt{\frac{a+1}{b}} + O(N^{-1}), \quad \text{as } N \rightarrow \infty.$$

Roughly speaking, the strongly non-unitary regime means that the width of spectral droplet is $O(1)$ regime. For instance, when we look at the scaling limits for (2.9), we have

$$(2.10) \quad \lim_{N \rightarrow \infty} \frac{1}{N\delta_N(p)} \mathbf{K}_N^{(\text{ISUE})} \left(e^{i\theta} \left(p + \mathfrak{s} \frac{\zeta}{\sqrt{N\delta_N(p)}} \right), e^{i\theta} \left(p + \mathfrak{s} \frac{\eta}{\sqrt{N\delta_N(p)}} \right) \right) \\ = \begin{cases} G(\bar{\zeta}, \eta), & \text{if } p \in \text{int}(S_s), \theta = 0, \text{ and } \mathfrak{s} = 1, \\ G(\bar{\zeta}, \eta)F(\zeta + \bar{\eta}), & \text{if } p \in \partial S_s, \theta \in [0, 2\pi), \text{ and } \mathfrak{s} = 1 \text{ (outer edge) or } -1 \text{ (inner edge)}, \end{cases}$$

uniformly for ζ, η in compact subsets of \mathbb{C} , where

$$G(\bar{\zeta}, \eta) = e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \bar{\zeta}\eta},$$

and

$$(2.11) \quad F(x) = \frac{1}{2} \operatorname{erfc} \left(\frac{x}{\sqrt{2}} \right), \quad x \in \mathbb{C}.$$

For the detailed proof, refer to [69], and these limiting kernels are coincident with (1.10). These bulk and edge scaling limits similarly hold for I GinUE. Recently, for more wide classes of normal random matrix models with an external potential with some nice assumptions, it is known that their scaling limits hold. However, for random matrix models or two-dimensional Coulomb gases with a N -dependent external potential such as I GinUE and ISUE, we emphasize that the universality result by Hedenmalm and Wennman [84] can be not applied.

(2) **Weakly non-unitary regime:** For I GinUE, we set

$$a_N = \frac{N}{\rho^2}, \quad b_N = N \left(\frac{N}{\rho^2} - \frac{1}{2} \right)$$

with a fixed $\rho > 0$. Then, the inner and outer radii of (2.4) satisfy

$$r_1 = 1 - \frac{\rho^2}{4N} + o(N^{-1}), \quad r_2 = 1 + \frac{\rho^2}{4N} + o(N^{-1}), \quad \text{as } N \rightarrow \infty.$$

For ISUE, we set the parameters as

$$L = \frac{N^2}{\rho^2} - N, \quad n = \frac{N^2}{\rho^2}.$$

Then, the radii of the spectral droplet (2.8) satisfy

$$r_1 = 1 - \frac{\rho^2}{2N} + O(N^{-2}) \quad r_2 = 1 + \frac{\rho^2}{2N} + O(N^{-2}), \quad \text{as } N \rightarrow \infty.$$

In the weakly non-unitary regime, the spectral droplet S accumulates the unit circle with width $O(N^{-1})$, and roughly speaking, the macroscopic eigenvalue distribution in the weakly non-unitary regime seems to be the circular unitary ensemble. Due to this macroscopic spectral property, the local statistics of IGINUE/ISUE in the weakly non-unitary regime can be formally expected to interpolate the local statistics between the Ginibre unitary ensemble and the circular unitary ensemble. Indeed, this observation is true, by the result in [44], we have

$$(2.12) \quad \lim_{N \rightarrow \infty} \frac{1}{N\delta_N(1)} \mathbf{K}_N^{(\text{ISUE})} \left(1 + \frac{\zeta}{\sqrt{N\delta_N(1)}}, 1 + \frac{\eta}{\sqrt{N\delta_N(1)}} \right) = e^{-\frac{1}{2}(|z|^2 + |w|^2) + \bar{\zeta}\eta} L_\rho(\bar{\zeta} + \eta),$$

uniformly for ζ, η in compact subsets of \mathbb{C} . Here, $L_\rho(z)$ is defined by

$$(2.13) \quad L_\rho(z) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\rho}{\sqrt{2}}}^{\frac{\rho}{\sqrt{2}}} e^{-\frac{1}{2}(z-\xi)^2} d\xi,$$

for $z \in \mathbb{C}$ and $\rho > 0$. Since the setting in [44] dealt two-dimensional Coulomb gases with a radially symmetric external potential, the above scaling limit in the weakly non-unitary regime also holds for IGINUE.

(3) At the singular origin: For IGINUE, we set

$$a_N = 1, \quad b_N = b > 0.$$

Then, the inner and outer radii of (2.4) satisfy

$$r_1 = \frac{b}{\sqrt{N}}, \quad r_2 = 1 + O\left(\frac{1}{N}\right), \quad \text{as } N \rightarrow \infty.$$

For ISUE, we set the parameters as

$$L > 0, \quad n = (b+1)N.$$

Then, the inner and outer radii of (2.8) satisfy

$$r_1 = O(N^{-1}), \quad r_2 = \frac{1}{\sqrt{b}} + O(N^{-1}), \quad \text{as } N \rightarrow \infty.$$

The limiting reproducing kernel of the point process at the singular origin is characterized by the two-parametric Mittag-Leffler function defined by

$$(2.14) \quad \lim_{N \rightarrow \infty} \frac{1}{N\delta_N(0)} \mathbf{K}_N^{(\text{ISUE})} \left(\frac{\zeta}{\sqrt{N\delta_N(0)}}, \frac{\eta}{\sqrt{N\delta_N(0)}} \right) = E_{1,1+L}(\bar{\zeta}\eta) |\zeta\eta|^L e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)},$$

where

$$(2.15) \quad E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}.$$

For further discussion in this regime, we refer to [21, 39, 40].

We have discussed the macroscopic pictures and the local statistics for the eigenvalues of induced spherical unitary ensemble so far. In next sub-subsection, we shall explain the overlaps for IGINUE and ISUE.

2.1.3. Overlaps of IGINUE and ISUE. In this sub-subsection, we summarize the results of the overlaps for IGINUE and ISUE. In order to compute the similar quantity with (1.23) and (1.24), we need the conditional expectation of the overlaps conditioned at the full points.

Proposition 2.2. *Conditionally on $\{\lambda_{(N)} = \mathbf{z}_{(N)}\}$, the diagonal overlap $\mathcal{O}_{1,1}$ of the induced Ginibre unitary ensemble is distributed as*

$$(2.16) \quad \mathcal{O}_{1,1} \stackrel{d}{=} \prod_{j=2}^N \left(1 + \frac{|X_j|^2}{|z_1 - z_j|^2} \right),$$

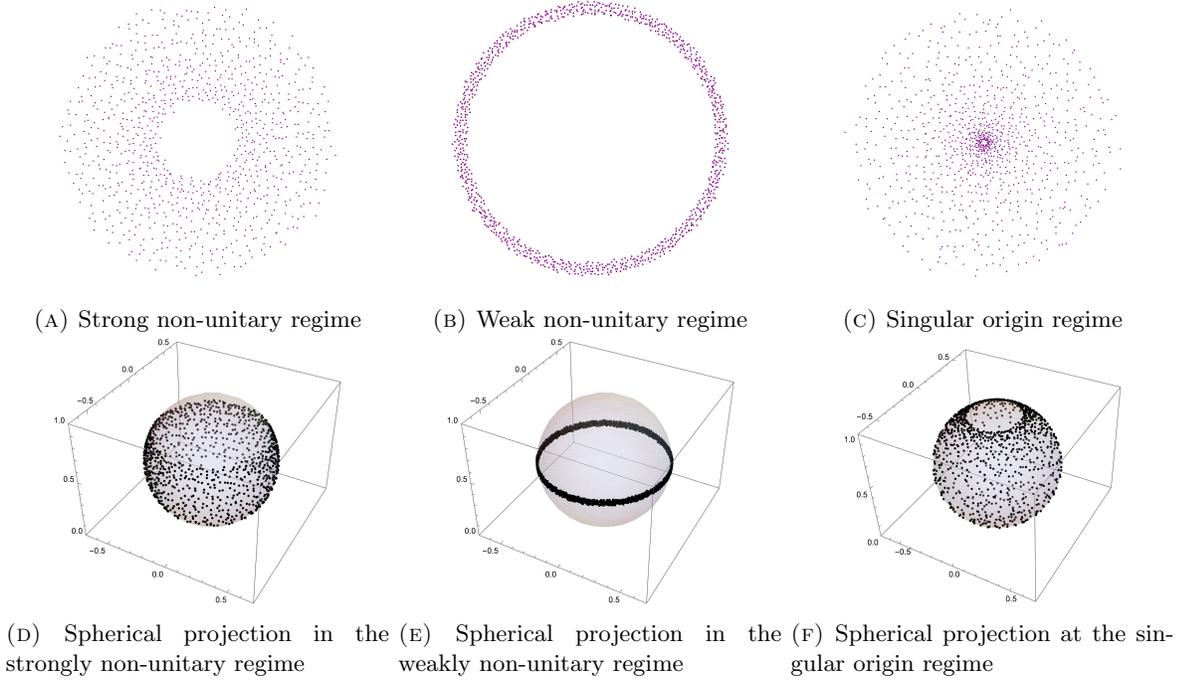


FIGURE 3. (A), (B), and (C) are the plots of the eigenvalues of induced spherical unitary ensemble in each regime on the complex plane. (D), (E), and (F) are the plots of the spherical projection for their eigenvalues.

where X_j 's are independent complex Gaussian random variables with variance 1. In particular, conditionally on $\{\lambda_{(N)} = \mathbf{z}_{(N)}\}$,

$$(2.17) \quad \mathbb{E} [\mathcal{O}_{1,1} | \lambda_{(N)} = \mathbf{z}_{(N)}] = \prod_{j=2}^N \left(1 + \frac{1}{|z_1 - z_j|^2} \right).$$

Similarly, conditionally on $\{\lambda_{(N)} = \mathbf{z}_{(N)}\}$,

$$(2.18) \quad \mathcal{O}_{1,2} \stackrel{d}{=} -\frac{1}{|z_1 - z_2|^2} \prod_{k=3}^N \left(1 + \frac{|X_k|^2}{(z_1 - z_k)(z_2 - z_k)} \right),$$

and we also have

$$(2.19) \quad \mathbb{E} [\mathcal{O}_{1,2} | \lambda_{(N)} = \mathbf{z}_{(N)}] = -\frac{1}{|z_1 - z_2|^2} \prod_{j=3}^N \left(1 + \frac{1}{(z_1 - z_j)(z_2 - z_j)} \right).$$

Note that these quantities are same as (1.21) and (1.22) although the potential is different from the case of Ginibre unitary ensemble. Here, we focus on (2.17). For the other quantities, (2.17), (2.18), and (2.19), we can verify them similar to the below discussions.

Sketch of proof of (2.17). The sketch of proof follows the strategy in [29, 53, 54]. First, we recall that from Proposition 2.1 and (2.1), the joint probability distribution function of the matrix elements for I GinUE is proportional to

$$d\mathbb{P}^{(\text{IGinUE})}(\mathbf{A}_N) \propto \det(\mathbf{A}_N \mathbf{A}_N^\dagger)^\alpha \exp(-\text{Tr} \mathbf{A}_N \mathbf{A}_N^\dagger) d[\mathbf{A}_N],$$

where $d[\mathbf{A}_N]$ is the planar Lebesgue measure on the $N \times N$ matrix space $M_N(\mathbb{C})$ with the field \mathbb{C} . Then, from the Schur decomposition, we have

$$\mathbf{A}_N = U(\Lambda_N + T_N)U^\dagger,$$

where $\Lambda_N = \text{diag}(\lambda_1, \dots, \lambda_N)$ is the diagonal matrix with eigenvalues elements of \mathbf{A}_N and T_N is the upper triangular elements. Here, $U \in \mathbf{U}(N)$ is $N \times N$ a unitary matrix. Note that $\det(\mathbf{A}_N \mathbf{A}_N^\dagger)^\alpha = \det(\Lambda_N \Lambda_N^\dagger)^\alpha = \prod_{j=1}^N |\lambda_j|^{2\alpha}$, and notice also $\text{Tr} \mathbf{A}_N \mathbf{A}_N^\dagger = \text{Tr}(\Lambda_N + T_N)(\Lambda_N + T_N)^\dagger = \sum_{j=1}^N |z_j|^2 - \text{Tr} T_N T_N^\dagger$. Hence, by change of variables, we have

$$d\mathbb{P}^{\text{(IGinUE)}}(\mathbf{A}_N) \propto \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 \prod_{j=1}^N |\lambda_j|^{2\alpha} e^{-|\lambda_j|^2} dA(\lambda_j) \cdot e^{-\text{Tr} T_N T_N^\dagger} d[T_N] d[S_N],$$

where $d[S_N]$ is the induced Haar measure of the coset for the $\mathbf{U}(N) \setminus \mathbf{U}(1)^N$. Then, by (1.12), (1.13), and from the upper triangular structure T_N , for the eigenvalue λ_1 of \mathbf{A}_N , $L_1 = (1, b_2, \dots, b_N)^\dagger$, and $R_1 = (1, 0, \dots, 0)^\dagger$, we can recursively determine $b_1 = 1$ and $b_k = \frac{1}{\lambda_1 - \lambda_k} \sum_{j=1}^{k-1} b_j T_{k,j}$ for $k = 1$. Form this recursion, if we write the truncation of the left eigenvector $L_1^{(k)} = (1, b_2, \dots, b_k)^\dagger$ with $k < N$, then we have

$$\|L_1^{(k+1)}\|^2 = \|L_1^{(k)}\|^2 \left(1 + \frac{1}{|\lambda_1 - \lambda_{k+1}|^2} \left| \sum_{j=1}^{k+1} \frac{b_j T_{k+1,j}}{\sqrt{\sum_{p=1}^k |b_p|^2}} \right|^2 \right)$$

When we integrate the elements of the upper triangular matrix T_N , it is nothing from the complex Gaussian integral. Therefore, similar to [29], we obtain (2.17). \square

The key point that we have demonstrated here is that the conditional expectation of the overlaps for IGinUE is same as the ones of Ginibre unitary ensemble since the part of Gaussian integration does not depend on the point insertion at the origin.

Next, we discuss the case of ISUE. Similar to the the proof of the case for $L = 0$ in [58], we can show the followings:

Proposition 2.3. *Conditionally on $\mathbf{z}_{(N)} = \boldsymbol{\lambda}_{(N)}$, the on-diagonal overlap $\mathcal{O}_{1,1}$ of the induced spherical unitary ensemble is distributed as*

$$(2.20) \quad \mathcal{O}_{1,1} \stackrel{d}{=} \prod_{k=2}^N \left(1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} \right),$$

where $X_m^{(k)}$ is i.i.d. distributed to a real random variable with density

$$\frac{m+1}{(1+x)^{m+2}} \mathbf{1}_{\mathbb{R}^+}.$$

In particular, the quenched expectation is given by

$$(2.21) \quad \mathbb{E}[\mathcal{O}_{1,1} | \mathbf{z}_{(N)} = \boldsymbol{\lambda}_{(N)}] = \prod_{k=2}^N \left(1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_k|^2)}{(n+L)|\lambda_1 - \lambda_k|^2} \right).$$

Proposition 2.4. *The quenched expectation of the off-diagonal overlap $\mathcal{O}_{1,2}$ for the induced spherical unitary ensemble is given by*

$$(2.22) \quad \mathbb{E}[\mathcal{O}_{1,2} | \mathbf{z}_{(N)} = \boldsymbol{\lambda}_{(N)}] = -\frac{1}{(n+L)|\lambda_1 - \lambda_2|^2} \prod_{k=3}^N \left(1 + \frac{(1 + \lambda_1 \bar{\lambda}_2)(1 + |\lambda_k|^2)}{(n+L)(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right).$$

We omit the proof since the proof is same as the spherical unitary ensemble ($L = 0$) case as we have already demonstrated for IGinUE.

2.2. Our contributions [113, 114]. In this thesis, we will study the determinantal structures of the multi-points correlation function of the overlaps for IGinUE and ISUE. The main motivations to study them are to confirm the universality of the scaling limits for the determinantal structure in the strongly non-unitary regime shown in [13] and to find new scaling limits in the weakly non-unitary and the singular origin regime. This is also interesting from the perspective of the point processes. Indeed, the scaling limit for the correlation kernel associated with the weight function for each IGinUE and ISUE deformed by the diagonal overlap is Let us mention that such attempt has been already seen in the prior works. Fyodorov and Tarnowski [77]

studied the diagonal overlap for the elliptic Ginibre orthogonal ensemble, which is the real version of (16.1), in the strongly and weakly non-unitary regimes. They derived a full-probability distribution function for the diagonal overlap conditioned at a real point via the super-symmetric method. However, the analysis based on the super-symmetric method is limited to the diagonal overlap. Hence, the off-diagonal overlaps have been not covered. In [144], they also studied a complex conditional point case, but their result is still limited to the diagonal overlap. Our main contribution in this thesis is to study the multi-points correlation function of the overlaps for both IGINUE and ISUE and to cover the off-diagonal case in the weakly non-unitary regime

2.2.1. *Our objects: the multi-points correlation function of the overlaps for IGINUE/ISUE.* By Proposition 2.2 and Proposition 2.3, for $\mathbf{z}^{(k)} = (z_1, z_2, \dots, z_k) \in \mathbb{C}^k$, we define

$$(2.23) \quad D_{1,1,(g)}^{(N,k)}(\mathbf{z}^{(k)}) = \frac{N!}{(N-k)!} \frac{1}{Z_N^{(g)}} \int_{\mathbb{C}^{N-k}} \prod_{j=2}^N \left(1 + \frac{1}{|z_1 - z_j|^2}\right) |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N |z_j|^{2\alpha} e^{-|z_j|^2} \prod_{j=k+1}^N dA(z_j),$$

$$(2.24) \quad D_{1,2,(g)}^{(N,k)}(\mathbf{z}^{(k)}) = -\frac{N!}{(N-k)!} \frac{1}{Z_N^{(g)}} \int_{\mathbb{C}^{N-k}} \frac{1}{|z_1 - z_2|^2} \prod_{j=3}^N \left(1 + \frac{1}{(z_1 - z_j)(\bar{z}_2 - z_j)}\right) \\ \times |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N |z_j|^{2\alpha} e^{-|z_j|^2} \prod_{j=k+1}^N dA(z_j),$$

$$(2.25) \quad D_{1,1,(s)}^{(N,k)}(\mathbf{z}^{(k)}) = \frac{N!}{(N-k)!} \frac{1}{Z_N^{(s)}} \int_{\mathbb{C}^{N-k}} \prod_{j=2}^N \left(1 + \frac{(1 + |z_1|^2)(1 + |z_j|^2)}{(n+L)|z_1 - z_j|^2}\right) |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N e^{-NQ(z_j)} \prod_{j=k+1}^N dA(z_j),$$

and

$$(2.26) \quad D_{1,2,(s)}^{(N,k)}(\mathbf{z}^{(k)}) = -\frac{N!}{(N-k)!} \frac{1}{Z_N^{(s)}} \int_{\mathbb{C}^{N-k}} \frac{1}{(n+L)|z_1 - z_2|^2} \\ \times \prod_{j=3}^N \left(1 + \frac{(1 + z_1 \bar{z}_2)(1 + |z_j|^2)}{(n+L)(z_1 - z_j)(\bar{z}_2 - z_j)}\right) |\Delta_N(\mathbf{z}_{(N)})|^2 \prod_{j=1}^N e^{-NQ(z_j)} \prod_{j=k+1}^N dA(z_j).$$

Here, $\Delta_N(\mathbf{z}_{(N)})$ is the Vandermonde determinant for $\mathbf{z}_{(N)} \in \mathbb{C}^N$. (2.23) and (2.24) correspond to the multi-points correlation function of the overlaps for IGINUE, and (2.25) and (2.26) correspond to the multi-points correlation function of the overlaps for ISUE. In order to show the determinantal structure, i.e., in order to construct the correlation kernel for (2.23), (2.24), (2.25) and (2.26), we introduce two new weight functions: for $z, w, u, v \in \mathbb{C}$, we define

$$(2.27) \quad \omega_{(g)}(z, \bar{z}|u, v) = (1 + (z - u)(\bar{z} - v))(z\bar{z})^\alpha e^{-z\bar{z}},$$

and

$$(2.28) \quad \omega_{(s)}(z, \bar{z}|u, v) = \left((z - u)(\bar{z} - v) + \frac{(1 + uv)(1 + z\bar{z})}{n + L}\right) e^{-NQ(z)},$$

with respect to the planar Lebesgue measure $dA(z)$.

Here, the relationship between (2.23) and (2.24) and the relationship between (2.25) and (2.26) can be associated by the following decoupling lemma:

Lemma 2.5. *Let $T : \mathbb{C}^{2k} \rightarrow \mathbb{C}^{2k}$ be a map given by*

$$(2.29) \quad Tf(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \dots) = f(z_1, \bar{z}_2, z_2, \bar{z}_1, z_3, \dots)$$

for any function f on \mathbb{C}^{2k} , which is regarded as the real surface. Then, we have

$$(2.30) \quad D_{1,2,(g)}^{(N,k)}(\mathbf{z}^{(k)}) = -\frac{e^{-|z_1 - z_2|^2}}{1 - |z_1 - z_2|^2} \widehat{T} D_{1,1,(g)}^{(N,k)}(\mathbf{z}^{(k)}),$$

and

$$(2.31) \quad D_{1,2,(s)}^{(N,k)}(\mathbf{z}^{(k)}) = -\frac{|1+z_1\bar{z}_2|^{2(n+L+1)}}{(1+|z_1|^2)^{n+L+1}(1+|z_2|^2)^{n+L+1}} \frac{TD_{1,1,(s)}^{(N,k)}(\mathbf{z}^{(k)})}{\frac{|1+z_1\bar{z}_2|^2}{n+L} - |z_1 - z_2|^2}.$$

The proof of Lemma 2.5 is almost same as the Ginibre unitary ensemble case [13]. Hence, we omit the proof. This lemma allows us to focus on the diagonal overlap case. Indeed, all proofs in this thesis will be done for the diagonal overlap case, and the results for the off-diagonal overlap case will be obtained as a simple application of Lemma 2.5. In order to present a finite N -kernel associated with (2.27) and (2.28), we now introduce some functions.

(1) (**IGinUE case**) We denote the truncated generalized exponential polynomial defined on \mathbb{C} by

$$e_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{x^k}{\Gamma(k+\alpha+1)}, \quad \text{for } \alpha > -1.$$

We also denote

$$\mathbf{e}_n^{(\alpha)}(z|x) = e_n^{(\alpha)}(z) + \frac{1}{\Gamma(\alpha)} \frac{1}{x-\alpha}, \quad \text{for } z, x \in \mathbb{C}.$$

We write

$$(2.32) \quad \omega_\alpha(z, \bar{z}|\lambda, \bar{\lambda}) = \varpi(z, \bar{z}|\lambda, \bar{\lambda}) \widehat{\omega}_\alpha(z, \bar{z}),$$

where

$$(2.33) \quad \widehat{\omega}_\alpha(z, w) = (zw)^\alpha e^{-\frac{1}{2}(|z|^2+|w|^2)}, \quad \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) = (1+|\zeta-\chi|^2).$$

Finally, we define

$$(2.34) \quad f_n^{(\alpha)}(x) = \frac{x-\alpha}{x} \left\{ (n+\alpha+1) \mathbf{e}_n^{(\alpha)}(x|x) - x \mathbf{e}_{n-1}^{(\alpha)}(x|x) \right\}.$$

(2) (**ISUE case**) We write

$$(2.35) \quad g_m^{(n,L)}(x) = \frac{x-L/n}{x} \sum_{k=0}^m (m+1-k) \binom{L+n}{L+k} x^k + \frac{L(m+1)}{nx} \binom{L+n}{L},$$

and

$$(2.36) \quad \widehat{g}_N^{(n,L)}(x) = (L+N+1) \widehat{q}_{N+1}^{(n,L)}(x|x) - x(n-N-1) \widehat{q}_N^{(n,L)}(x|x),$$

where

$$(2.37) \quad \widehat{q}_N^{(n,L)}(y|x) = q_N^{(n,L)}(y) + \frac{1}{nx-L} \frac{\Gamma(L+n+1)}{\Gamma(n+1)\Gamma(L)},$$

and

$$(2.38) \quad q_N^{(n,L)}(x) = \sum_{k=0}^N \frac{\Gamma(L+n+1)}{\Gamma(k+L+1)\Gamma(n+1-k)} x^k.$$

Then, the relationship between (2.35) and (2.36) is given by

$$g_N^{(n,L)}(x) = \frac{(x-L/n)}{x(1+x)} \widehat{g}_N^{(n,L)}(x).$$

Let us also denote

$$(2.39) \quad \widehat{\omega}^{(n,L)}(z, w) = \frac{(zw)^L}{((1+|z|^2)(1+|w|^2))^{\frac{n+L+1}{2}}},$$

and

$$(2.40) \quad \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = |z-\lambda|^2 + \frac{(1+\lambda\bar{\lambda})(1+z\bar{z})}{n+L}.$$

Now, we construct a finite N -kernel associated with (2.27) and (2.28).

Proposition 2.6. (1) **(IGinUE case)** We write

$$(2.41) \quad G_{N-1}^{(\alpha)}(x|y, z) = \sum_{n,m=0}^{N-1} f_n^{(\alpha)}(x) f_m^{(\alpha)}(x) y^n z^m \sum_{k=\max\{n,m\}}^{N-1} \frac{1}{\Gamma(k+\alpha+2)} \frac{x^k}{f_k^{(\alpha)}(x) f_{k+1}^{(\alpha)}(x)},$$

where $f_p^{(\alpha)}(x)$ is defined in (2.34). Then, for any $2 \leq k \leq N$, we have

$$(2.42) \quad D_{1,1,(\mathbf{g})}^{(N,k)}(\mathbf{z}(k)) = f_{N-1}^{(\alpha)}(z_1 \bar{z}_1) \widehat{\omega}_\alpha(z_1, \bar{z}_1) \det_{2 \leq i, j \leq k} \left(K_{1,1,(\mathbf{g})}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_1) \right),$$

where the correlation kernel $K_{1,1,(\mathbf{g})}^{(N)}$ is given by

$$(2.43) \quad K_{1,1,(\mathbf{g})}^{(N)}(z, \bar{z}, w, \bar{w} | \lambda, \bar{\lambda}) = \left(\frac{|zw|}{\bar{z}w} \right)^\alpha \mathcal{K}_N^{(\alpha)}(\bar{z}, w | \lambda, \bar{\lambda}) \omega_\alpha(\bar{z}, w | \lambda, \bar{\lambda}),$$

and

$$(2.44) \quad \mathcal{K}_N^{(\alpha)}(\bar{z}, w | \lambda, \bar{\lambda}) = G_{N-1}^{(\alpha)} \left(\lambda \bar{\lambda} \left| \frac{\bar{z}}{\lambda}, \frac{w}{\lambda} \right. \right).$$

Here, $\omega(\bar{z}, z | \lambda, \bar{\lambda})$ is defined by (2.32). Furthermore, for $2 \leq k \leq N$, we have

$$(2.45) \quad D_{1,2,(\mathbf{g})}^{(N,k)}(\mathbf{z}(k)) = -\widehat{\omega}_\alpha(z_1, \bar{z}_1) \widehat{\omega}_\alpha(z_2, \bar{z}_2) \mathcal{K}_{N-1}^{(\alpha)}(\bar{z}_1, z_2 | z_1, \bar{z}_2) f_{N-1}^{(\alpha)}(z_1 \bar{z}_2) \\ \times \det_{3 \leq i, j \leq N} \left(K_{1,2,(\mathbf{g})}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_2) \right),$$

where

$$K_{1,2,(\mathbf{g})}^{(N)}(z, \bar{z}, w, \bar{w} | u, \bar{v}) = \frac{\omega_\alpha(z, \bar{z} | u, \bar{v})}{\mathcal{K}_N^{(\alpha)}(\bar{u}, v | u, \bar{v})} \det \begin{pmatrix} \mathcal{K}_N^{(\alpha)}(\bar{u}, v | u, \bar{v}) & \mathcal{K}_N^{(\alpha)}(\bar{u}, w | u, \bar{v}) \\ \mathcal{K}_N^{(\alpha)}(\bar{z}, v | u, \bar{v}) & \mathcal{K}_N^{(\alpha)}(\bar{z}, w | u, \bar{v}) \end{pmatrix}.$$

(2) **(ISUE case)** We write

$$(2.46) \quad G_N^{(n,L)}(x|y, z) = \sum_{s,t=0}^{N-1} g_s^{(n,L)}(x) y^s g_t^{(n,L)}(x) z^t \\ \times \sum_{k=\max\{s,t\}}^{N-1} \frac{\Gamma(n+L+1)}{\Gamma(k+L+2)\Gamma(n-k-1)} \frac{x^k}{g_{k+1}^{(n,L)}(x) g_k^{(n,L)}(x)}.$$

Then, for any $2 \leq k \leq N$, we have

$$(2.47) \quad D_{1,1,(\mathbf{s})}^{(N,k)}(\mathbf{z}(k)) = \frac{n(z_1 \bar{z}_1 - \frac{L}{n})}{z_1 \bar{z}_1 (1 + z_1 \bar{z}_1)} \widehat{g}_{N-1}^{(n,L)}(z_1 \bar{z}_1) \widehat{\omega}^{(n,L)}(\bar{z}_1, z_1) \det_{2 \leq i, j \leq N} \left(K_{1,1,(\mathbf{s})}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_1) \right),$$

where $K_{1,1,(\mathbf{s})}^{(N)}$ is the correlation kernel given by

$$(2.48) \quad K_{1,1,(\mathbf{s})}^{(N)}(z, \bar{z}, w, \bar{w} | \lambda, \bar{\lambda}) = \mathcal{K}_N^{(n,L)}(\bar{z}, w | \lambda, \bar{\lambda}) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) \widehat{\omega}^{(n,L)}(\bar{z}, z).$$

Here, $\mathcal{K}_N^{(n,L)}$ is the polynomial kernel given by

$$(2.49) \quad \mathcal{K}_N^{(n,L)}(\bar{z}, w | \lambda, \bar{\lambda}) = G_N^{(n,L)} \left(\lambda \bar{\lambda} \left| \frac{\bar{z}}{\lambda}, \frac{w}{\lambda} \right. \right).$$

Furthermore, for $2 \leq k \leq N$, we have

$$(2.50) \quad D_{1,2,(\mathbf{s})}^{(N,k)}(\mathbf{z}(k)) = -\frac{n(z_1 \bar{z}_2 - \frac{L}{n})}{z_1 \bar{z}_2 (1 + z_1 \bar{z}_2)} \widehat{g}_{N-1}^{(n,L)}(z_1 \bar{z}_2) \widehat{\omega}^{(n,L)}(\bar{z}_2, z_1) \\ \times \mathcal{K}_{N-1}^{(n,L)}(\bar{z}_1, z_2 | z_1, \bar{z}_2) \widehat{\omega}^{(n,L)}(\bar{z}_1, z_2) \det_{3 \leq i, j \leq N} \left(K_{1,2,(\mathbf{s})}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_2) \right),$$

where

$$K_{1,2,(\mathbf{s})}^{(N)}(z, \bar{z}, w, \bar{w} | u, \bar{v}) = \frac{\varpi^{(n,L)}(z, \bar{z} | u, \bar{v}) \widehat{\omega}^{(n,L)}(z, w)}{\mathcal{K}_N^{(n,L)}(\bar{u}, v | u, \bar{v})} \det \begin{pmatrix} \mathcal{K}_N^{(n,L)}(\bar{u}, v | u, \bar{v}) & \mathcal{K}_N^{(n,L)}(\bar{u}, w | u, \bar{v}) \\ \mathcal{K}_N^{(n,L)}(\bar{z}, v | u, \bar{v}) & \mathcal{K}_N^{(n,L)}(\bar{z}, w | u, \bar{v}) \end{pmatrix}.$$

In particular, since we get $K_{1,2,(g)}^{(N)}$ from $K_{1,1,(g)}^{(N)}$ and $K_{1,2,(s)}^{(N)}$ from $K_{1,1,(s)}^{(N)}$ via Lemma 2.5, we mainly focus on the diagonal overlap case as we already announced. To directly analyze the finite N -kernels (2.43) and (2.48) in Proposition 2.6 is difficult. Indeed, (2.41) and (2.46) constitute many summations and complicated terms in the denominator. Therefore, in order to compute the scaling limits of (2.43) and (2.48), we need to decompose (2.43) and (2.48) into nice functions. The below result is the building block to analyze a large N -limit.

Theorem 2.7. (1) (**IGinUE case**) For $n \in \mathbb{N}$, we define

$$(2.51) \quad \mathfrak{W}_n^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) = \left\{ \mathbf{e}_n^{(\alpha)}(\lambda\bar{z}|\lambda\bar{\lambda})\mathbf{e}_n^{(\alpha)}(w\bar{\lambda}|\lambda\bar{\lambda}) \right. \\ \left. - (1 - (\bar{z} - \bar{\lambda})(w - \lambda))\mathbf{e}_n^{(\alpha)}(w\bar{z}|\lambda\bar{\lambda})\mathbf{e}_n^{(\alpha)}(\lambda\bar{\lambda}|\lambda\bar{\lambda}) \right\} \widehat{\omega}_\alpha(\bar{z}, \lambda)\widehat{\omega}_\alpha(\bar{\lambda}, w),$$

$$(2.52) \quad \mathfrak{H}_n^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) = \frac{\lambda\bar{\lambda} - \alpha(n + \alpha + 1)\mathfrak{W}_{n+1}^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) - \lambda\bar{\lambda}\mathfrak{W}_n^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda)}{\lambda\bar{\lambda} f_n^{(\alpha)}(\lambda\bar{\lambda})\widehat{\omega}_\alpha(\bar{z}, w)\widehat{\omega}_\alpha(\bar{\lambda}, \lambda)(\bar{z} - \bar{\lambda})^2(w - \lambda)^2},$$

and

$$(2.53) \quad \mathfrak{F}_n^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) = -\frac{\lambda\bar{\lambda} - \alpha}{\lambda\bar{\lambda}} \frac{1}{(\bar{z} - \bar{\lambda})(w - \lambda)} \frac{(\bar{z}w)^{n+1}}{\Gamma(n + \alpha + 1)} \frac{\mathbf{e}_{n+1}^{(\alpha)}(\lambda\bar{\lambda}|\lambda\bar{\lambda})}{f_n^{(\alpha)}(\lambda\bar{\lambda})} - \frac{1}{\Gamma(\alpha)} \frac{1}{\lambda\bar{\lambda} - \alpha} \frac{1}{(\bar{z} - \bar{\lambda})(w - \lambda)}.$$

Then, we can express $\mathcal{K}_N^{(\alpha)}$ in terms of (2.52) and (2.53):

$$(2.54) \quad \mathcal{K}_N^{(\alpha)}(\bar{z}, w|\lambda, \bar{\lambda}) = \mathfrak{H}_N^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) + \mathfrak{F}_N^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda).$$

(2) (**ISUE case**) For $N \in \mathbb{N}$, we define

$$(2.55) \quad \mathfrak{H}_N^{(n,L)}(\bar{z}, w, \lambda) = \frac{(1 + \lambda\bar{\lambda})}{(\bar{z} - \bar{\lambda})^2(w - \lambda)^2\widehat{q}_N^{(n,L)}(\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)} \\ \times \left((N + L + 1)Q_{N+1}^{(n,L)}(\bar{z}, w, \lambda) - (n - N - 1)\lambda\bar{\lambda}Q_N^{(n,L)}(\bar{z}, w, \lambda) \right),$$

where

$$(2.56) \quad Q_N^{(n,L)}(\bar{z}, w, \lambda) = \widehat{q}_N^{(n,L)}(\bar{\lambda}w|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, w)\widehat{q}_N^{(n,L)}(\bar{z}\lambda|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{z}, \lambda) \\ - \left(1 - \frac{(\bar{z} - \bar{\lambda})(w - \lambda)}{1 + \lambda\bar{\lambda}} \right) \widehat{q}_N^{(n,L)}(\lambda\bar{\lambda}|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)\widehat{q}_N^{(n,L)}(\bar{z}w|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{z}, w).$$

We also define

$$(2.57) \quad \mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) = \mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) + \mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) + \mathfrak{J}\mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda),$$

where

$$(2.58) \quad \mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) \\ = \frac{(L + N)\widehat{q}_N^{(n,L)}(\bar{z}w|\bar{\lambda}\lambda)\widehat{\omega}^{(n,L)}(\bar{z}, w) + (n - N - 1 - \bar{z}w)\widehat{q}_{N-1}^{(n,L)}(\bar{z}w|\bar{\lambda}\lambda)\widehat{\omega}^{(n,L)}(\bar{z}, w)}{(1 + \bar{z}w)(\bar{z} - \bar{\lambda})(w - \lambda)} \\ - \frac{n(1 + \lambda\bar{\lambda})}{n\lambda\bar{\lambda} - L} \frac{\Gamma(L + n + 1)}{\Gamma(n + 1)\Gamma(L)} \frac{\widehat{\omega}^{(n,L)}(\bar{z}, w)}{(1 + \bar{z}w)(\bar{z} - \bar{\lambda})(w - \lambda)},$$

$$(2.59) \quad \mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) = -\frac{(n - N - 1)\Gamma(L + n + 1)}{\Gamma(L + N + 1)\Gamma(n - N)} \frac{(\bar{z}w)^N\widehat{\omega}^{(n,L)}(\bar{z}, w)}{(\bar{z} - \bar{\lambda})(w - \lambda)\widehat{q}_N^{(n,L)}(\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)} \\ \times \left(\widehat{q}_{N-1}^{(n,L)}(\lambda\bar{\lambda}|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda) - \frac{\widehat{q}_N^{(n,L)}(\lambda\bar{\lambda}|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)}{n - N} \right),$$

and

$$(2.60) \quad \mathfrak{J}\mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w, \lambda) = \frac{(n-N-1)\Gamma(L+n+1)}{\Gamma(L+N+1)\Gamma(n-N)} \frac{(\bar{z}w)^N \widehat{\omega}^{(n,L)}(\bar{z}, w)}{(\bar{z}-\bar{\lambda})(w-\lambda) \widehat{g}_N^{(n,L)}(\lambda\bar{\lambda}) \widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)} \\ \times \left(\frac{L+N+1}{n-N} \widehat{q}_N^{(n,L)}(\lambda\bar{\lambda}|\lambda\bar{\lambda}) \widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda) - \frac{\bar{z}w \widehat{q}_{N+1}^{(n,L)}(\lambda\bar{\lambda}|\lambda\bar{\lambda}) \widehat{\omega}^{(n,L)}(\bar{\lambda}, \lambda)}{n-N-1} \right).$$

Then, we can express

$$(2.61) \quad K_{1,1,(s)}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) = C_K(z, w) (\mathfrak{H}_N^{(n,L)}(\bar{z}, w, \lambda) + \mathfrak{F}_N^{(n,L)}(\bar{z}, w, \lambda)) \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}),$$

where $C_K(z, w)$ is the conjugation factor given by

$$C_K(z, w) = \frac{\sqrt{\widehat{\omega}^{(n,L)}(\bar{z}, z) \widehat{\omega}^{(n,L)}(\bar{w}, w)}}{\widehat{\omega}^{(n,L)}(\bar{z}, w)} = \left(\frac{|zw|}{\bar{z}w} \right)^L.$$

Remark 2.8. If we take $\alpha = 0$, then this result recovers the result for the Ginibre unitary ensemble shown in [13]. Hence, our results should be regarded as the generalization of the case of the Ginibre unitary ensemble.

Remark 2.9. As we will see later, the remainder terms (2.59) and (2.60) do not play essential roles in a large N -limit. Indeed, the main contributions come from (2.55) and (2.58).

With help of Theorem 2.7, we can compute the scaling limits of the kernel in the strongly non-unitary, the weakly non-unitary, and the singular origin regimes. To this end, let us introduce some functions and rescaled points for IGINUE and ISUE. We write

$$K_{1,1}^{(*)}(\zeta, \bar{\zeta}, \eta, \bar{\eta}|\chi, \bar{\chi}) = \mathcal{K}_{1,1}^{(*)}(\bar{\zeta}, \eta|\chi, \bar{\chi}) \omega^{(*)}(\bar{\zeta}, \zeta|\chi, \bar{\chi}), \text{ for } \zeta, \eta, \chi \in \mathbb{C},$$

where the symbols $\mathbf{b}, \mathbf{e}, \mathbf{w}, \mathbf{s}$ are assigned to $*$ for the bulk and edge cases in strongly non-unitary regime, in the weakly non-unitary regime, and at the singular origin regime, respectively. In the same rule, we also write

$$K_{1,2}^{(*)}(\zeta, \bar{\zeta}, \eta, \bar{\eta}|u, \bar{v}) = \frac{\omega^{(*)}(\bar{\zeta}, \zeta|u, \bar{v})}{\mathcal{K}_{1,1}^{(*)}(\bar{u}, v|u, \bar{v})} \det \begin{pmatrix} \mathcal{K}_{1,1}^{(*)}(\bar{u}, v|u, \bar{v}) & \mathcal{K}_{1,1}^{(*)}(\bar{u}, \eta|u, \bar{v}) \\ \mathcal{K}_{1,1}^{(*)}(\bar{\zeta}, v|u, \bar{v}) & \mathcal{K}_{1,1}^{(*)}(\bar{\zeta}, \eta|u, \bar{v}) \end{pmatrix}, \text{ for } \zeta, \eta, u, v \in \mathbb{C}.$$

For IGINUE case, for ζ_j in compact subsets of \mathbb{C} for $j = 1, 2, \dots, k$ and for $\theta \in [0, 2\pi)$, let

$$(2.62) \quad z_j = \begin{cases} \sqrt{N}p + \zeta_j & \text{if } p \in \text{int}(S_g) \text{ and the strongly non-unitary regime,} \\ e^{i\theta}(\sqrt{N(1+b)} + \zeta_j) & \text{if the outer edge and the strongly non-unitary regime,} \\ e^{i\theta}(\sqrt{Nb} - \zeta_j) & \text{if the inner edge and the strongly non-unitary regime,} \\ e^{i\theta}(\sqrt{Na_N} + \zeta_j) & \text{if the weakly non-unitary regime,} \\ \zeta_j & \text{if the singular origin regime.} \end{cases}$$

For ISUE case, for $p \in \text{clo}(S_s)$, we set

$$(2.63) \quad \delta_N(p) := \frac{n+L+1}{N} \frac{1}{(1+|p|^2)^2}.$$

For ζ_j in compact subsets of \mathbb{C} for $j = 1, 2, \dots, k$ and for $\theta \in [0, 2\pi)$, let

$$(2.64) \quad z_j = \begin{cases} p + \frac{\zeta_j}{\sqrt{N\delta(p)}} & \text{if } p \in \text{int}(S_s) \text{ and the strongly non-unitary regime,} \\ e^{i\theta} \left(p + \frac{\zeta_j}{\sqrt{N\delta(p)}} \right) & \text{if } p \text{ is on the outer boundary of } \partial S_s \text{ and the strongly non-unitary regime,} \\ e^{i\theta} \left(p - \frac{\zeta_j}{\sqrt{N\delta(p)}} \right) & \text{if } p \text{ is on the inner boundary of } \partial S_s \text{ and the strongly non-unitary regime,} \\ e^{i\theta} \left(1 + \frac{\zeta_j}{\sqrt{N\delta(1)}} \right) & \text{if the weakly non-unitary regime,} \\ \frac{\zeta_j}{\sqrt{N\delta(0)}} & \text{if the singular origin regime.} \end{cases}$$

For I GinUE with rescaled points (2.62), we define

$$\mathfrak{D}_{1,m,(g)}^{(N,k)}(\zeta_{(k)}) = D_{1,m,(g)}^{(N,k)}(\mathbf{z}_{(k)}), \quad \text{for } m = 1, 2.$$

For ISUE with rescaled points (2.64), we define

$$\mathfrak{D}_{1,m,(s)}^{(N,k)}(\zeta_{(k)}) = \frac{1}{(N\delta_N(p))^k} D_{1,m,(s)}^{(N,k)}(\mathbf{z}_{(k)}), \quad \text{for } m = 1, 2.$$

Theorem 2.10. (I) (**Strongly non-unitary regime: bulk case**) *We write*

$$\mathcal{K}_{1,1}^{(\mathbf{b})}(\bar{\zeta}, \eta | \chi, \bar{\chi}) = \frac{d}{dx} \left(\frac{e^x - 1}{x} \right) \Big|_{x=(\bar{\zeta}-\bar{\chi})(\eta-\chi)}, \quad \omega^{(\mathbf{b})}(\zeta, \bar{\zeta} | \chi, \bar{\chi}) = \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) e^{-(\bar{\zeta}-\bar{\chi})(\zeta-\chi)}.$$

Let

$$\mathfrak{D}_{1,1}^{(k,\mathbf{b})}(\zeta_{(k)}) = \det_{i,j=2}^k (K_{1,1}^{(\mathbf{b})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_1)), \quad \mathfrak{D}_{1,2}^{(k,\mathbf{b})}(\zeta_{(k)}) = \mathcal{K}_{1,1}^{(\mathbf{b})}(\bar{\zeta}_1, \zeta_2 | \zeta_1, \bar{\zeta}_2) \det_{i,j=3}^k (K_{1,2}^{(\mathbf{b})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_2)).$$

Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,1,(g)}^{(N,k)}(\zeta_{(k)}) = \frac{(|p|^2 - b)(1 + b - |p|^2)}{|p|^2} \mathfrak{D}_{1,1}^{(k,\mathbf{b})}(\zeta_{(k)}),$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,1,(s)}^{(N,k)}(\zeta_{(k)}) = \frac{b(b+1)}{a+b+1} \frac{(|p|^2 - \frac{a}{b+1}) \left(\frac{a+1}{b} - |p|^2 \right)}{|p|^2} \mathfrak{D}_{1,1}^{(k,\mathbf{b})}(\zeta_{(k)}),$$

uniformly for ζ_k in compact subsets of \mathbb{C} . Moreover, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,2,(g)}^{(N,k)}(\zeta_{(k)}) = -\frac{(|p|^2 - b)(1 + b - |p|^2)}{|p|^2} \mathcal{K}_{1,1}^{(\mathbf{b})}(\bar{\zeta}_1, \zeta_2 | \zeta_1, \bar{\zeta}_2) \mathfrak{D}_{1,2}^{(k,\mathbf{b})}(\zeta_{(k)}),$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,2,(s)}^{(N,k)}(\zeta_{(k)}) = -\frac{b(b+1)}{a+b+1} \frac{(|p|^2 - \frac{a}{b+1}) \left(\frac{a+1}{b} - |p|^2 \right)}{|p|^2} \mathcal{K}_{1,1}^{(\mathbf{b})}(\bar{\zeta}_1, \zeta_2 | \zeta_1, \bar{\zeta}_2) \mathfrak{D}_{1,2}^{(k,\mathbf{b})}(\zeta_{(k)}),$$

uniformly for ζ_k in compact subsets of \mathbb{C} .

(II) (**Strongly non-unitary regime: edge case**) *For $x \in \mathbb{C}$, we define*

$$(2.65) \quad \mathcal{F}(x) = e^{-\frac{1}{2}x^2} - \sqrt{2\pi}x\mathcal{F}(x),$$

and we write

$$\mathbf{c}_b = \begin{cases} \sqrt{\frac{1}{2\pi(1+b)}} & \text{if the outer edge case,} \\ \sqrt{\frac{1}{2\pi b}} & \text{if the inner edge case,} \end{cases} \quad \mathbf{c}_s = \begin{cases} \sqrt{\frac{a+b+1}{2\pi(a+1)b}} & \text{if the outer edge case,} \\ \sqrt{\frac{a+b+1}{2\pi a(b+1)}} & \text{if the inner edge case.} \end{cases}$$

We also write

$$(2.66) \quad \begin{aligned} & H(a, b, c, d, f) \\ & = -\frac{\sqrt{2\pi} \frac{d}{dx} \left[e^{\frac{(a+x)^2}{2}} \left(e^{-f} F(b+x)F(c+x) - F(d+x)F(a+x) + fF(d)F(a+x) \right) \right] \Big|_{x=0}}{e^{\frac{1}{2}a^2} \mathcal{F}(a)}, \end{aligned}$$

and

$$\mathcal{K}_{1,1}^{(\mathbf{e})}(\bar{\zeta}, \eta | \chi, \bar{\chi}) = \frac{e^{\bar{\zeta}\eta} H(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \eta, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2}, \quad \omega^{(\mathbf{e})}(\bar{\zeta}, \zeta | \chi, \bar{\chi}) = \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) e^{-\bar{\zeta}\zeta}.$$

Let

$$\mathfrak{D}_{1,1}^{(k,\mathbf{e})}(\zeta_{(k)}) = \mathcal{F}(\zeta_1 + \bar{\zeta}_1) \det_{i,j=2}^k (K_{1,1}^{(\mathbf{e})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_1)),$$

and

$$\mathfrak{D}_{1,2}^{(k,\mathbf{e})}(\zeta_{(k)}) = e^{-|\zeta_1 - \zeta_2|^2} \mathcal{F}(\zeta_1 + \bar{\zeta}_2) \frac{H(\bar{\zeta}_2 + \zeta_1, \bar{\zeta}_1 + \zeta_1, \bar{\zeta}_2 + \zeta_2, \bar{\zeta}_2 + \zeta_1, -(\bar{\zeta}_1 - \bar{\zeta}_2)(\zeta_1 - \eta_2))}{(\bar{\zeta}_1 - \bar{\zeta}_2)^2 (\zeta_1 - \zeta_2)^2}$$

$$\times \det_{i,j=3}^k (K_{1,2}^{(\mathbf{e})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_2)).$$

Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathfrak{D}_{1,1,(g)}^{(N,k)}(\zeta_{(k)}) = \mathbf{c}_6 \mathfrak{D}_{1,1}^{(k,\mathbf{e})}(\zeta_{(k)}), \quad \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathfrak{D}_{1,1,(s)}^{(N,k)}(\zeta_{(k)}) = \mathbf{c}_5 \mathfrak{D}_{1,1}^{(k,\mathbf{e})}(\zeta_{(k)}),$$

uniformly for ζ_k in compact subsets of \mathbb{C} . Moreover, we have

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathfrak{D}_{1,2,(g)}^{(N,k)}(\zeta_{(k)}) = -\mathbf{c}_6 \mathfrak{D}_{1,1}^{(k,\mathbf{e})}(\zeta_{(k)}), \quad \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathfrak{D}_{1,2,(s)}^{(N,k)}(\zeta_{(k)}) = -\mathbf{c}_5 \mathfrak{D}_{1,2}^{(k,\mathbf{e})}(\zeta_{(k)}),$$

uniformly for ζ_k in compact subsets of \mathbb{C} .

(III) **(Weakly non-unitary regime)** For $z \in \mathbb{C}$ and $\rho > 0$, we define

$$(2.67) \quad \mathcal{L}_\rho(z) = \frac{1}{\sqrt{2\pi}} \left\{ \left(z + \frac{\rho}{\sqrt{2}} \right) e^{-\frac{1}{2}(z - \frac{\rho}{\sqrt{2}})^2} - \left(z - \frac{\rho}{\sqrt{2}} \right) \left(\sqrt{2\pi} \left(z + \frac{\rho}{\sqrt{2}} \right) L_\rho(z) + e^{-\frac{1}{2}(z + \frac{\rho}{\sqrt{2}})^2} \right) \right\}.$$

We write

$$\mathcal{A}_\rho(a, b, c, d, f) = \left(\frac{\rho}{\sqrt{2}} + a \right) \left(\frac{\rho}{\sqrt{2}} - a \right) \left(e^f (f-1) L_\rho(a) L_\rho(d) + L_\rho(b) L_\rho(c) \right),$$

$$\mathcal{B}_\rho(a, b, c) = L_\rho(b) \left\{ \left(a + \frac{\rho}{\sqrt{2}} \right) \frac{e^{-\frac{1}{2}(c - \frac{\rho}{\sqrt{2}})^2}}{\sqrt{2\pi}} - \left(a - \frac{\rho}{\sqrt{2}} \right) \frac{e^{-\frac{1}{2}(c + \frac{\rho}{\sqrt{2}})^2}}{\sqrt{2\pi}} \right\},$$

and

$$\mathcal{C}_\rho(a, b) = \frac{e^{-\frac{1}{2}(a - \frac{\rho}{\sqrt{2}})^2 - \frac{1}{2}(b + \frac{\rho}{\sqrt{2}})^2}}{\sqrt{2\pi}^2} + \frac{e^{-\frac{1}{2}(a + \frac{\rho}{\sqrt{2}})^2 - \frac{1}{2}(b - \frac{\rho}{\sqrt{2}})^2}}{\sqrt{2\pi}^2}.$$

We also write

$$(2.68) \quad \begin{aligned} \mathcal{H}_\rho(a, b, c, d, f) = & \mathcal{A}_\rho(a, b, c, d, f) + \mathcal{B}_\rho(a, b, c) + \mathcal{B}_\rho(a, c, b) + f e^f \mathcal{B}_\rho(a, d, a) - e^f \mathcal{B}_\rho(a, d, a) \\ & - e^f \mathcal{B}_\rho(a, a, d) + \mathcal{C}_\rho(b, c) - e^f \mathcal{C}_\rho(a, d). \end{aligned}$$

We write

$$\mathcal{K}_{1,1}^{(\mathbf{w})}(\bar{\zeta}, \eta | \chi, \bar{\chi}) = \frac{\mathcal{H}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \eta, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{L}_\rho(\chi + \bar{\chi})},$$

and

$$\omega^{(\mathbf{w})}(\bar{\zeta}, \zeta | \chi, \bar{\chi}) = \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) e^{-(\bar{\zeta} - \bar{\chi})(\zeta - \chi)}.$$

Let

$$\mathfrak{D}_{1,1}^{(k,\mathbf{w})}(\zeta_{(k)}) = \mathcal{L}_\rho(\zeta_1 + \bar{\zeta}_1) \det_{i,j=2}^k (K_{1,1}^{(\mathbf{w})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_1)),$$

and

$$\mathfrak{D}_{1,2}^{(k,\mathbf{w})}(\zeta_{(k)}) = \mathcal{L}_\rho(\zeta_1 + \bar{\zeta}_2) \mathcal{K}_{1,1}^{(\mathbf{w})}(\bar{\zeta}_1, \zeta_2 | \zeta_1, \bar{\zeta}_2) \det_{i,j=3}^k (K_{1,2}^{(\mathbf{w})}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_2)).$$

Then, we have

$$\lim_{N \rightarrow \infty} \mathfrak{D}_{1,1,(g)}^{(N,k)}(\zeta_{(k)}) = \mathfrak{D}_{1,1}^{(k,\mathbf{w})}(\zeta_{(k)}) \Big|_{\rho \mapsto \rho/\sqrt{2}}, \quad \lim_{N \rightarrow \infty} \mathfrak{D}_{1,1,(s)}^{(N,k)}(\zeta_{(k)}) = \mathfrak{D}_{1,1}^{(k,\mathbf{w})}(\zeta_{(k)}),$$

and

$$\lim_{N \rightarrow \infty} \mathfrak{D}_{1,2,(g)}^{(N,k)}(\zeta_{(k)}) = -\mathfrak{D}_{1,2}^{(k,\mathbf{w})}(\zeta_{(k)}) \Big|_{\rho \mapsto \rho/\sqrt{2}}, \quad \lim_{N \rightarrow \infty} \mathfrak{D}_{1,2,(s)}^{(N,k)}(\zeta_{(k)}) = -\mathfrak{D}_{1,2}^{(k,\mathbf{w})}(\zeta_{(k)}),$$

uniformly for ζ_k in compact subsets of \mathbb{C} .

(IV) **(Singular origin regime)** We also write

$$(2.69) \quad \mathcal{E}_{1,c}(z|x) = (x-c)E_{1,c+1}(z) + \frac{1}{\Gamma(c)},$$

and we define

$$(2.70) \quad \begin{aligned} & \mathcal{S}_L(\bar{\zeta}\chi, \bar{\chi}\eta, \bar{\zeta}\eta, \bar{\chi}\chi, (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \\ &= (\chi\bar{\chi} - L) \left(E_{1,L+1}(\bar{\zeta}\chi) E_{1,L+1}(\bar{\chi}\eta) - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) E_{1,L+1}(\bar{\zeta}\eta) E_{1,L+1}(\bar{\chi}\chi) \right) \\ &+ \frac{1}{\Gamma(L)} \left(E_{1,L+1}(\bar{\zeta}\chi) + E_{1,L+1}(\bar{\chi}\eta) - E_{1,L+1}(\bar{\zeta}\eta) - E_{1,L+1}(\bar{\chi}\chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi) E_{1,L+1}(\bar{\chi}\chi) \right). \end{aligned}$$

We write

$$\mathcal{K}_{1,1}^{(s)}(\bar{\zeta}, \eta | \chi, \bar{\chi}) = \frac{\mathcal{S}_L(\bar{\zeta}\chi, \bar{\chi}\eta, \bar{\zeta}\eta, \bar{\chi}\chi, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{E}_{1,L}(\bar{\chi}\chi | \bar{\chi}\chi)},$$

and

$$\omega^{(s)}(\bar{\zeta}, \zeta | \chi, \bar{\chi}) = \varpi(\zeta, \bar{\zeta}, \eta, \bar{\eta} | \chi, \bar{\chi}) (\bar{\zeta}\eta)^L e^{-|\zeta|^2}.$$

Let

$$\mathfrak{D}_{1,1}^{(k,s)}(\zeta_k) = |\zeta_1|^{2L-2} \mathcal{E}_{1,L}(|\zeta_1|^2) e^{-|\zeta_1|^2} \det_{i,j=2}^k (K_{1,1}^{(s)}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_1)),$$

and

$$\mathfrak{D}_{1,2}^{(k,s)}(\zeta_k) = \frac{\mathcal{E}_{1,L}(\zeta_1 \bar{\zeta}_2) |\zeta_1|^{2L} |\zeta_2|^{2L}}{\zeta_1 \bar{\zeta}_2} e^{-|\zeta_1|^2 - |\zeta_2|^2} \mathcal{K}_{1,1}^{(s)}(\bar{\zeta}_1, \zeta_2 | \zeta_1, \bar{\zeta}_2) \det_{i,j=3}^k (K_{1,2}^{(s)}(\zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j | \zeta_1, \bar{\zeta}_2)).$$

Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,1,(g)}^{(N,k)}(\zeta_k) = \mathfrak{D}_{1,1}^{(k,s)}(\zeta_k) \Big|_{L \rightarrow b}, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,1,(s)}^{(N,k)}(\zeta_k) = \mathfrak{D}_{1,1}^{(k,s)}(\zeta_k),$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,2,(g)}^{(N,k)}(\zeta_k) = -\mathfrak{D}_{1,2}^{(k,s)}(\zeta_k) \Big|_{L \rightarrow b}, \quad \lim_{N \rightarrow \infty} \frac{1}{N} \mathfrak{D}_{1,2,(s)}^{(N,k)}(\zeta_k) = -\mathfrak{D}_{1,2}^{(k,s)}(\zeta_k),$$

uniformly for ζ_k in compact subsets of \mathbb{C} .

Remark 2.11. By (2.67), note that

$$\mathcal{L}_\rho(x) \sim \frac{2}{\rho^2}, \quad \text{as } \rho \rightarrow \infty.$$

From the definition of (2.68), we have

$$\frac{2}{\rho^2} \mathfrak{D}_{1,1}^{(k,w)}(\zeta_k) \rightarrow \mathfrak{D}_{1,1}^{(k,b)}(\zeta_k), \quad \text{as } \rho \rightarrow \infty.$$

This implies that the weak non-unitary regime recovers the strongly non-unitary regime.

From $D_{1,1}^{(N,1)}(z_1)$ and $D_{1,2}^{(N,2)}(z_1, z_2)$, we can derive the scaling limits of the conditional expectation of the diagonal and off-diagonal overlaps. This fact was not explicitly pointed out and not computed in [13]. We introduce some functions. We fix $z_1, z_2 \in \mathbb{C}$. For the diagonal case of IGINUE, we write

$$(2.71) \quad Z_{N,(g)}^{(z_1)} = \widehat{\omega}_\alpha(z_1, \bar{z}_1) \mathcal{B}_N^{(\alpha)}(z_1, z_1).$$

For the off-diagonal case of IGINUE, we also write

$$(2.72) \quad Z_{N,(g)}^{(z_1, z_2)} = \widehat{\omega}_\alpha(z_1, \bar{z}_1) \widehat{\omega}_\alpha(z_2, \bar{z}_2) \det \left[\begin{pmatrix} \mathcal{B}_N^{(\alpha)}(z_1, z_1) & \mathcal{B}_N^{(\alpha)}(z_1, z_2) \\ \mathcal{B}_N^{(\alpha)}(z_2, z_1) & \mathcal{B}_N^{(\alpha)}(z_2, z_2) \end{pmatrix} \right],$$

where

$$\mathcal{B}_N^{(\alpha)}(z, w) = \sum_{k=0}^{N-1} \frac{(z\bar{w})^k}{\Gamma(k + \alpha + 1)}.$$

For the diagonal case of ISUE, we write

$$(2.73) \quad Z_{N,(s)}^{(z_1)} = \widehat{\omega}^{(n,L)}(\bar{z}_1, z_1) \mathcal{B}_N^{(n,L)}(z_1, z_1).$$

For the off-diagonal case of ISUE, we also write

$$(2.74) \quad Z_{N,(s)}^{(z_1, z_2)} = \widehat{\omega}^{(n,L)}(\bar{z}_1, z_1) \widehat{\omega}^{(n,L)}(\bar{z}_2, z_2) \det \left[\begin{pmatrix} \mathcal{B}_N^{(n,L)}(z_1, z_1) & \mathcal{B}_N^{(n,L)}(z_1, z_2) \\ \mathcal{B}_N^{(n,L)}(z_2, z_1) & \mathcal{B}_N^{(n,L)}(z_2, z_2) \end{pmatrix} \right],$$

where

$$\mathcal{B}_N^{(n,L)}(z, w) = \sum_{k=0}^{N-1} \frac{\Gamma(n+L+1)}{\Gamma(n-k)\Gamma(k+L+1)} (z\bar{w})^k.$$

First, let us consider the conditional expectation of the diagonal overlap for IGINUE conditioned at $z_1 \in \mathbb{C}$. Then, for $f \in L^2(\widehat{\omega}_\alpha(\lambda, \bar{\lambda}))$, we have

$$\mathbb{E}_N \left[\prod_{j=2}^N f(\lambda_j) \mid \lambda_1 = z_1 \right] = \frac{\mathbb{E}_{N-1} \left[\prod_{j=2}^N f(\lambda_j) \prod_{j=2}^N |\lambda_j - z_1|^2 \cdot \widehat{\omega}_\alpha(z_1, \bar{z}_1) \right]}{\mathbb{E}_{N-1} \left[\prod_{j=2}^N |\lambda_j - z_1|^2 \cdot \widehat{\omega}_\alpha(z_1, \bar{z}_1) \right]},$$

where \mathbb{E}_N denote the expectation with respect to IGINUE with size N . The denominator $\mathbb{E}_{N-1} \left[\prod_{j=2}^N |\lambda_j - z_1|^2 \right]$ is be regarded as IGINUE with a point insertion, which can be computed by [14, Theorem 1]:

$$\mathbb{E}_{N-1} \left[\prod_{j=2}^N |\lambda_j - z_1|^2 \cdot \widehat{\omega}_\alpha(z_1, \bar{z}_1) \right] = Z_{N,(g)}^{(z_1)}.$$

Note that the numerator is nothing but $D_{1,1}^{(N,1)}(z_1)$ when we consider f as (2.17). On the other hand, when we consider the off-diagonal case, for $g \in L^2(\widehat{\omega}_\alpha(\lambda, \bar{\lambda}))$, we have

$$\mathbb{E}_N \left[\prod_{j=3}^N f(\lambda_j) \mid \lambda_1 = z_1 \right] = \frac{\mathbb{E}_{N-1} \left[\prod_{j=3}^N g(\lambda_j) \prod_{j=3}^N |\lambda_j - z_1|^2 \cdot |z_1 - z_2|^2 \widehat{\omega}_\alpha(z_1, \bar{z}_1) \widehat{\omega}_\alpha(z_2, \bar{z}_2) \right]}{\mathbb{E}_{N-1} \left[\prod_{j=3}^N |\lambda_j - z_1|^2 \prod_{j=3}^N |\lambda_j - z_2|^2 \cdot |z_1 - z_2|^2 \widehat{\omega}_\alpha(z_1, \bar{z}_1) \widehat{\omega}_\alpha(z_2, \bar{z}_2) \right]}.$$

By [14, Theorem 1] again, we have

$$\mathbb{E}_{N-1} \left[\prod_{j=3}^N |\lambda_j - z_1|^2 \prod_{j=3}^N |\lambda_j - z_2|^2 \cdot |z_1 - z_2|^2 \widehat{\omega}_\alpha(z_1, \bar{z}_1) \widehat{\omega}_\alpha(z_2, \bar{z}_2) \right] = Z_{N,(g)}^{(z_1, z_2)}.$$

Similarly, note that the numerator is nothing but $D_{1,2}^{(N,2)}(z_1, z_1)$ when we consider g as (2.19). The similar discussion holds for ISUE. We omit the index N in the expectation from now on. We summarize each conditional expectation of the diagonal and off-diagonal overlaps for IGINUE and ISUE as follows: for IGINUE and $z_1, z_2 \in \mathbb{C}$,

$$\mathbb{E}[\mathcal{O}_{1,1} \mid \lambda_1 = z_1] = \frac{D_{1,1}^{(N,1)}(z_1)}{Z_{N,(g)}^{(z_1)}}, \quad \mathbb{E}[\mathcal{O}_{1,2} \mid \lambda_1 = z_1, \lambda_2 = z_2] = \frac{D_{1,2}^{(N,2)}(z_1, z_2)}{Z_{N,(g)}^{(z_1, z_2)}},$$

and for ISUE and $z_1, z_2 \in \mathbb{C}$,

$$\mathbb{E}[\mathcal{O}_{1,1} \mid \lambda_1 = z_1] = \frac{D_{1,1}^{(N,1)}(z_1)}{Z_{N,(s)}^{(z_1)}}, \quad \mathbb{E}[\mathcal{O}_{1,2} \mid \lambda_1 = z_1, \lambda_2 = z_2] = \frac{D_{1,2}^{(N,2)}(z_1, z_2)}{Z_{N,(s)}^{(z_1, z_2)}}.$$

Similar to (2.10), (2.12), and (2.14), we have the scaling limits of the correlation kernel for IGINUE. The only difference is that for the weakly non-unitary regime of IGINUE, we replace ρ with $\rho \mapsto \rho/\sqrt{2}$, and for the singular origin regime, we replace L with $L \mapsto b$. To state the result, we introduce some functions. For the bulk case and for $z \in \mathbb{C}$, we define

$$\Psi_{1,2}^{(b)}(z) = -\frac{1}{|z|^4} \frac{1 - (1 + |z|^2)e^{-|z|^2}}{1 - e^{-|z|^2}}.$$

For the edge case and for $z, a, b, c, d, f \in \mathbb{C}$, we define

$$\Psi_{1,1}^{(\mathbf{e})}(z) = \frac{\mathcal{F}(z)}{F(z)}, \quad \Psi_{1,2}^{(\mathbf{e})}(a, b, c, d, f) = -\frac{H(a, b, c, d, -|f|^2)e^{-|f|^2}\mathcal{F}(a)}{|f|^4(F(b)F(c) - e^{-|f|^2}F(a)F(d))}.$$

For the weakly non-unitary regime and for $z, a, b, c, d, f \in \mathbb{C}$, we define

$$\Psi_{1,1}^{(\rho)}(z) = \frac{\mathcal{L}_\rho(z)}{L_\rho(z)}, \quad \Psi_{1,2}^{(\rho)}(a, b, c, d, f) = -\frac{\mathcal{H}_\rho(a, b, c, d, -|f|^2)}{|f|^4(L_\rho(b)L_\rho(c) - e^{-|f|^2}L_\rho(a)L_\rho(d))}.$$

For the singular origin regime and for $z, a, b, c, d, f \in \mathbb{C}$ and $L \geq 0$, we define

$$\Psi_{1,1}^{(L)}(z) = \frac{\mathcal{E}_L(|z|^2)}{|z|^2 E_{1,1+L}(|z|^2)}, \quad \Psi_{1,2}^{(L)}(a, b, c, d, f) = -\frac{\mathcal{S}_L(a, b, c, d, -|f|^2)}{a|f|^2(E_{1,1+L}(b)E_{1,1+L}(c) - E_{1,1+L}(a)E_{1,1+L}(d))}.$$

Combining (2.10), (2.12), and (2.14) with (2.71), (2.72), (2.73), and (2.74), we obtain the following.

Corollary 2.12. (I) **(Strongly non-unitary regime: bulk case)**

(IGinUE case) : *Conditionally on $\{z_1 = \sqrt{N}p + \zeta_1\}$ for $p \in \text{int}(S_g)$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,1} \middle| z_1 = \sqrt{N}p + \zeta_1 \right] = \frac{(|p|^2 - b)(1 + b - |p|^2)}{|p|^2},$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = \sqrt{N}p + \zeta_1, z_2 = \sqrt{N}p + \zeta_2\}$ for $p \in \text{int}(S_g)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,2} \middle| z_1 = \sqrt{N}p + \zeta_1, z_2 = \sqrt{N}p + \zeta_2 \right] = -\frac{(|p|^2 - b)(1 + b - |p|^2)}{|p|^2} \Psi_{1,2}^{(\mathbf{b})}(\zeta_1 - \zeta_2),$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(ISUE case) : *Conditionally on $\{z_1 = p + \zeta_1 / \sqrt{N\delta(p)}\}$ for $\text{int}(S_s)$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,1} \middle| z_1 = p + \frac{\zeta_1}{\sqrt{N\delta(p)}} \right] = \frac{b(b+1)}{a+b+1} \frac{\left(|p|^2 - \frac{a}{b+1}\right) \left(\frac{a+1}{b} - |p|^2\right)}{|p|^2},$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = p + \zeta_1 / \sqrt{N\delta(p)}, z_2 = p + \zeta_2 / \sqrt{N\delta(p)}\}$ for $\text{int}(S_s)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,2} \middle| z_1 = p + \frac{\zeta_1}{\sqrt{N\delta(p)}}, z_2 = p + \frac{\zeta_2}{\sqrt{N\delta(p)}} \right] = -\frac{b(b+1)}{a+b+1} \frac{\left(|p|^2 - \frac{a}{b+1}\right) \left(\frac{a+1}{b} - |p|^2\right)}{|p|^2} \Psi_{1,2}^{(\mathbf{b})}(\zeta_1 - \zeta_2),$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(II) **(Strongly non-unitary regime: edge case)**

(IGinUE case) : *Conditionally on $\{z_1 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_1)\}$ or $\{z_1 = e^{i\theta}(\sqrt{Nb} - \zeta_1)\}$ for $\theta \in [0, 2\pi)$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,1} \middle| z_1 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_1) \right] = \mathbf{c}_b \Psi_{1,1}^{(\mathbf{e})}(\zeta_1 + \bar{\zeta}_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} , and

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,1} \middle| z_1 = e^{i\theta}(\sqrt{Nb} - \zeta_1) \right] = \mathbf{c}_b \Psi_{1,1}^{(\mathbf{e})}(\zeta_1 + \bar{\zeta}_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_1), z_2 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_2)\}$ or $\{z_1 = e^{i\theta}(\sqrt{Nb} - \zeta_1), z_2 = e^{i\theta}(\sqrt{Nb} - \zeta_2)\}$ for $\theta \in [0, 2\pi)$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,2} \middle| z_1 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_1), z_2 = e^{i\theta}(\sqrt{N(1+b)} + \zeta_2) \right] \\ &= \mathbf{c}_b \Psi_{1,2}^{(\mathbf{e})}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, \zeta_1 - \zeta_2), \end{aligned}$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} , and

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,2} \middle| z_1 = e^{i\theta}(\sqrt{Nb} - \zeta_1), z_2 = e^{i\theta}(\sqrt{Nb} - \zeta_2) \right]$$

$$= \mathbf{c}_b \Psi_{1,2}^{(e)}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, \zeta_1 - \zeta_2),$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(ISUE case) : Conditionally on $\{z_1 = e^{i\theta}(p + \mathfrak{s}\zeta_1/\sqrt{N\delta(p)})\}$ for $p \in \partial S_s$ with $\theta \in [0, 2\pi)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = e^{i\theta} \left(p + \mathfrak{s} \frac{\zeta_1}{\sqrt{N\delta(p)}} \right) \right] = \mathbf{c}_s \Psi_{1,1}^{(e)}(\zeta_1 + \bar{\zeta}_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = p + \mathfrak{s}\zeta_1/\sqrt{N\delta(p)}, z_2 = p + \mathfrak{s}\zeta_2/\sqrt{N\delta(p)}\}$ for $p \in \partial S_s$, we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = p + \mathfrak{s} \frac{\zeta_1}{\sqrt{N\delta(p)}}, z_2 = p + \mathfrak{s} \frac{\zeta_2}{\sqrt{N\delta(p)}} \right] \\ &= \mathbf{c}_s \Psi_{1,2}^{(e)}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, \zeta_1 - \zeta_2), \end{aligned}$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(III) (Weakly non-unitary regime)

(IGinUE case) : Conditionally on $\{z_1 = e^{i\theta}(\sqrt{Na_N} + \zeta_1)\}$ for $\theta \in [0, 2\pi)$, we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = e^{i\theta}(\sqrt{Na_N} + \zeta_1) \right] = \Psi_{1,1}^{(\rho/\sqrt{2})}(\zeta_1 + \bar{\zeta}_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = e^{i\theta}(\sqrt{Na_N} + \zeta_1), z_2 = e^{i\theta}(\sqrt{Na_N} + \zeta_2)\}$ for $\theta \in [0, 2\pi)$,

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathcal{O}_{1,2} \Big| z_1 = e^{i\theta}(\sqrt{Na_N} + \zeta_1), z_2 = e^{i\theta}(\sqrt{Na_N} + \zeta_2) \right] \\ &= \Psi_{1,2}^{(\rho/\sqrt{2})}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, \zeta_1 - \zeta_2), \end{aligned}$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(ISUE case) : Conditionally on $\{z_1 = e^{i\theta}(1 + \zeta_1/\sqrt{N\delta(1)})\}$ for $p = 1$ with $\theta \in [0, 2\pi)$, we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = e^{i\theta} \left(1 + \frac{\zeta_1}{\sqrt{N\delta(1)}} \right) \right] = \Psi_{1,1}^{(\rho)}(\zeta_1 + \bar{\zeta}_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = 1 + \zeta_1/\sqrt{N\delta(1)}, z_2 = 1 + \zeta_2/\sqrt{N\delta(1)}\}$, we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathcal{O}_{1,2} \Big| z_1 = 1 + \frac{\zeta_1}{\sqrt{N\delta(1)}}, z_2 = 1 + \frac{\zeta_2}{\sqrt{N\delta(1)}} \right] = \Psi_{1,2}^{(\rho)}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, \zeta_1 - \zeta_2)$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(IV) (Singular origin regime)

(IGinUE case) : Conditionally on $\{z_1 = \zeta_1\}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = \zeta_1 \right] = \Psi_{1,1}^{(b)}(\zeta_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = \zeta_1, z_2 = \zeta_2\}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,2} \Big| z_1 = \zeta_1, z_2 = \zeta_2 \right] = \Psi_{1,2}^{(b)}(\zeta_1 \bar{\zeta}_2, \zeta_1 \bar{\zeta}_1, \zeta_2 \bar{\zeta}_2, \zeta_2 \bar{\zeta}_1, \zeta_1 - \zeta_2),$$

uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} .

(ISUE case) : Conditionally on $\{z_1 = \zeta_1/\sqrt{N\delta(p)}\}$ for $p \in \partial S$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,1} \Big| z_1 = \frac{\zeta_1}{\sqrt{N\delta(0)}} \right] = \Psi_{1,1}^{(L)}(\zeta_1),$$

uniformly for ζ_1 in a compact subset of \mathbb{C} . Moreover, conditionally on $\{z_1 = p + \zeta_1/\sqrt{N\delta(0)}, z_2 = \zeta_2/\sqrt{N\delta(0)}\}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \left[\mathcal{O}_{1,2} \Big| z_1 = \zeta_1, z_2 = \zeta_2 \right] = \Psi_{1,2}^{(L)}(\zeta_1 \bar{\zeta}_2, \zeta_1 \bar{\zeta}_1, \zeta_2 \bar{\zeta}_2, \zeta_2 \bar{\zeta}_1, \zeta_1 - \zeta_2),$$

uniformly for ζ_1, ζ_2 in a compact subset of \mathbb{C} .

Remark 2.13. In Corollary 2.12, the results for the bulk case in the strongly non-unitary regime are universal. Indeed, their results are same as (1.16) with the rescaled points $z_1 = p + \zeta_1/\sqrt{N}$ and $z_2 = p + \zeta_2/\sqrt{N}$ for $z_1, z_2 \in \text{int}(\mathbb{D})$ up to the geometric factors. However, we would like to emphasize that the fact (1.16) is macroscopic and gives the more precise estimate with the convergence rate than our results. To improve our results is left to the future work.

3. GINIBRE SYMPLECTIC ENSEMBLE, PFAFFIAN POINT PROCESS, AND OVERLAPS

3.1. Ginibre symplectic ensemble and Pfaffian point process. In this thesis, we will also study the Pfaffian structure of the multi-points correlation function of the diagonal overlap for Ginibre symplectic ensemble as an analogue of the multi-points correlation function of the overlaps for Ginibre unitary and induced Ginibre/spherical ensembles studied in this thesis. Nevertheless we study the same object, due to the Pfaffian structure of the joint probability distribution function for the eigenvalues of Ginibre symplectic ensemble, we need to develop a new method to study the Pfaffian structure of the multi-points correlation function of the diagonal overlap. To explain our results, we firstly explain the details for the Ginibre symplectic ensemble. Ginibre symplectic ensemble is $2N \times 2N$ non-Hermitian matrix with symplectic structure defined by

$$(3.1) \quad \mathbf{G}_{2N} := \begin{pmatrix} \mathbf{A}_N & \mathbf{B}_N \\ -\mathbf{B}_N & \mathbf{A}_N \end{pmatrix},$$

where $\mathbf{A}_N, \mathbf{B}_N$ are $N \times N$ independent Ginibre unitary ensembles. From now on, in short, we denote the Ginibre symplectic ensemble by GinSE. Due to the Kramer degeneracy, (3.1) has N -complex conjugate pairs of simple complex eigenvalues

$$(3.2) \quad \mathbf{Z}_N = \text{diag} \left[\begin{pmatrix} z_1 & \\ & \bar{z}_1 \end{pmatrix}, \dots, \begin{pmatrix} z_N & \\ & \bar{z}_N \end{pmatrix} \right].$$

This implies that there is no eigenvalue on the real line for (3.1), one eigenvalue of (3.1) is repulsive with complex conjugate pair of oneself with respect to the real line. Indeed, we figure out this picture from the joint probability distribution function of the eigenvalues for (3.1). Coming back to (3.1) and (3.2), (3.1) can be diagonalized in the sense of Schur decomposition, that is to say, there exists a unitary symplectic matrix $\mathbf{U} \in \mathbf{Usp}(2N) \setminus \mathbf{U}(1)^N$ such that

$$(3.3) \quad \mathbf{G}_{2N} = \mathbf{U} \widehat{\mathbf{G}}_{2N} \mathbf{U}^\dagger, \quad \widehat{\mathbf{G}}_{2N} = \mathbf{Z}_N + \mathbf{T}_N,$$

where \mathbf{Z}_N is given by (3.2), and \mathbf{T}_N is an upper-triangular matrix with elements

$$\begin{pmatrix} T_{i,j+1} & T_{i,\bar{j}+1} \\ T_{\bar{i},j+1} & T_{\bar{i},\bar{j}+1} \end{pmatrix}.$$

Here, $T_{i,j+1}$ are independent complex Gaussian random variables. The Jacobian of the above transform $\mathbf{G}_{2N} \mapsto \widehat{\mathbf{G}}_{2N}$ is well-known, and hence, we find that the joint probability distribution function of the elements of \mathbf{G}_{2N} becomes

$$(3.4) \quad d\mathbb{P}_N(\mathbf{G}_{2N}) = C_N \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{j=1}^N |z_j - \bar{z}_j|^2 e^{-2|z_j|^2} e^{-\text{Tr} \mathbf{T}_N \mathbf{T}_N^\dagger} dA(z_j) d\mathbf{T}_N \mathbf{U}^\dagger d\mathbf{U},$$

where C_N is a normalization constant. Furthermore, by integrating \mathbf{U} and \mathbf{T}_N , the joint probability distribution function of the eigenvalues for (3.1) is given by

$$(3.5) \quad d\mathbb{P}_N^{(\text{GinSE})}(\mathbf{z}_{(N)}) = \frac{1}{Z_N^{(\text{GinSE})}} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{j=1}^N |z_j - \bar{z}_j|^2 e^{-2|z_j|^2} dA(z_j),$$

for $\mathbf{z}_{(N)} = (z_1, \dots, z_N) \in \mathbb{C}^N$, and $Z_N^{(\text{GinSE})}$ is the partition function is given by

$$(3.6) \quad Z_N^{(\text{GinSE})} = N! \prod_{k=0}^{N-1} \frac{(2k+1)!}{2^{2k+1}}.$$

We write the k -th correlation function of (3.5) by

$$\mathbf{R}_{N,k}^{(\text{GinSE})}(z_1, \dots, z_k) = \frac{N!}{(N-k)!} \int_{\mathbb{C}} \mathbb{P}_N^{(\text{GinSE})}(z_1, \dots, z_N) \prod_{j=k+1}^N dA(z_j).$$

Then, it can be written as

$$(3.7) \quad \mathbf{R}_{N,k}^{(\text{GinSE})}(z_1, \dots, z_k) = \text{Pf}_{i,j=1}^k \left[\begin{pmatrix} \chi_N^{(\mathbf{g})}(z_i, z_j) & \chi_N^{(\mathbf{g})}(z_i, \bar{z}_j) \\ \chi_N^{(\mathbf{g})}(\bar{z}_i, z_j) & \chi_N^{(\mathbf{g})}(\bar{z}_i, \bar{z}_j) \end{pmatrix} e^{-|z_i|^2 - |z_j|^2} \right] \prod_{j=1}^k (\bar{z}_j - z_j),$$

where $\chi_N^{(\mathbf{g})}(z, w)$ is called a skew pre-kernel defined given by

$$(3.8) \quad \chi_N^{(\mathbf{g})}(z, w) = \mathbf{G}_N^{(\mathbf{g})}(z, w) - \mathbf{G}_N^{(\mathbf{g})}(w, z), \quad \mathbf{G}_N^{(\mathbf{g})}(z, w) = \sum_{k=0}^{N-1} \frac{q_{2k+1}^{(\mathbf{g})}(z) q_{2k}^{(\mathbf{g})}(w)}{r_k^{(\mathbf{g})}}.$$

where $\{q_k^{(\mathbf{g})}\}_k$ is called the skew-orthogonal polynomial (for the precise definition, see Definition 3.1). In short, it is called SOPs. Here, we emphasize that $\{q_k^{(\mathbf{g})}\}_k$ and $\chi_N^{(\mathbf{g})}(z, w)$ are the skew-orthogonal polynomials and the skew pre-kernel for Ginibre symplectic ensemble (3.5) by the subscript (\mathbf{g}) , respectively. It is known that the skew-orthogonal polynomials and skew pre-kernel for GinSE are given by

$$(3.9) \quad q_{2k+1}^{(\mathbf{g})}(z) = z^{2k+1}, \quad q_{2k}^{(\mathbf{g})}(z) = \sum_{\ell=0}^k \frac{k!}{\ell!} z^{2\ell}, \quad r_k^{(\mathbf{g})} = \frac{(2k+1)!}{2^{2k+1}}.$$

Then, by (3.8), we have

$$(3.10) \quad \begin{aligned} \chi_N^{(\mathbf{g})}(z, w) &= \sum_{k=0}^{N-1} \frac{q_{2k+1}^{(\mathbf{g})}(z) q_{2k}^{(\mathbf{g})}(w) - q_{2k}^{(\mathbf{g})}(z) q_{2k+1}^{(\mathbf{g})}(w)}{r_k^{(\mathbf{g})}} \\ &= \sqrt{2} \left[\sum_{k=0}^{N-1} \sum_{\ell=0}^k \left(\frac{(\sqrt{2}z)^{2k+1}}{(2k+1)!!} \frac{(\sqrt{2}w)^{2\ell}}{(2\ell)!!} - \frac{(\sqrt{2}w)^{2k+1}}{(2k+1)!!} \frac{(\sqrt{2}z)^{2\ell}}{(2\ell)!!} \right) \right]. \end{aligned}$$

These were originally derive in [89, 108]. Apart from GinSE, when we consider more general planar Pfaffian Coulomb gases with a nice potentials Q

$$(3.11) \quad d\mathbb{P}_N(\mathbf{z}_{(N)}) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{j=1}^N |z_j - \bar{z}_j|^2 e^{-2NQ(z_j)} dA(z_j),$$

how do we construct the corresponding to the skew-orthogonal polynomials and the skew pre-kernels associated with the weight function $e^{-2NQ(z)}$? [10] answered this question under a certain assumption. Their result is extremely essential in this thesis, and to more precisely explain skew-orthogonal polynomials, let us recall [10, Theorem 3.1]. First, in general, let us denote D by a symmetric planar domain with respect to the real line. Let us also denote a positive Borel measure on \mathbb{C} , which has an infinite number of points in its support D and satisfies a moment condition $\int_D |z|^m d\mu(z) < \infty$ for any $m \in \mathbb{N}$. From on on, we always assume that a Borel measure μ satisfies this assumption from now on. For any $f, g \in \mathbb{C}[z]$, we define the skew-inner product

$$(3.12) \quad \langle f, g \rangle_s := \int_{\mathbb{C}} (f(z)g(z) - f(z)g(\bar{z}))(z - \bar{z}) d\mu(z).$$

Definition 3.1 ([10]). *A family of polynomials $\{q_k\}_k$ with degree $\deg q_k = k$ is called **skew-orthogonal polynomials** associated with the Borel measure μ , if they satisfy for all non-negative integers $k, \ell \in \mathbb{Z}$,*

$$(3.13) \quad \begin{aligned} \langle q_{2k}, q_{2\ell} \rangle_s &= \langle q_{2k+1}, q_{2\ell+1} \rangle_s = 0, \\ \langle q_{2k}, q_{2\ell+1} \rangle_s &= -\langle q_{2\ell+1}, q_{2k} \rangle_s = r_k \delta_{k,\ell}, \end{aligned}$$

with r_k being their skew-norming constants.

Given a Borel measure μ , by Gram-Schmidt process, there exists a unique family of planar polynomials

$$p_n(z) = \gamma_n z^n + O(z^{n-1}),$$

satisfying the orthogonal condition with respect to the inner product

$$(3.14) \quad \langle p_n, p_m \rangle = \int_{\mathbb{C}} p_n(z) \overline{p_m(z)} d\mu(z) = h_n \delta_{n,m}.$$

Now, we recall [10, Theorem 3.1]:

Theorem 3.2 (Akemann, Ebke, and Parra in [10]). *We write a sequence of real numbers by $(\mu_{k,j})_{k,j}$ with $\mu_{k,k} = 1$ and $\mu_{k,j} = \lambda_{k-1} \mu_{k-1,j}$, where λ_{k-1} is independent of $j < k$. Let (p_n) be a family of planar monic orthogonal polynomials with respect to (3.14). We assume that the family of $(q_n)_n$ constructed via*

$$(3.15) \quad q_{2k}(z) = \sum_{j=0}^k \mu_{k,j} p_{2j}(z), \quad q_{2k+1}(z) = p_{2k+1}(z),$$

satisfies the skew-orthogonal conditions (3.13). Then, we have that (p_n) satisfy a classical three term recurrence

$$(3.16) \quad zp_k(z) = p_{k+1}(z) + b_k p_k(z) + c_k p_{k-1}(z), \quad b_k, c_k \in \mathbb{R}.$$

Conversely, the family of the planar monic orthogonal polynomials $(p_n)_n$ satisfies a three term-recurrence (3.16), then the sequence of planar monic polynomials (3.15) forms a family of skew-orthogonal polynomials, and the coefficients $\mu_{k,j}$ in (3.15) can be computed as

$$(3.17) \quad r_k = 2(h_{2k+1} - c_{2k+1}h_{2k}) > 0, \quad \mu_{k,j} = \prod_{\ell=j}^{k-1} \lambda_{\ell} = \frac{h_{2\ell+2} - c_{2\ell+2}h_{2\ell+1}}{h_{2\ell+1} - c_{2\ell+1}h_{2\ell}} \quad (j < k).$$

Clearly, the monic orthogonal polynomials $p_k(z) = z^k$ satisfies a trivial three-term recurrence. Therefore, we can construct the skew-orthogonal polynomials (3.9) associated with the Gaussian potential $e^{-2|z|^2}$ using Theorem 3.2. Once we know skew-orthogonal polynomials and skew pre-kernel associated with (3.11), next step is to analyze a scaling limit for the skew pre-kernel as a large N -limit. Here, we focus on (3.10). First, note that to directly analyze (3.10) seems to be difficult since (3.10) contains the double sum. [89, 108] derived the origin scaling limit for (3.10), and they showed that the limiting correlation function still forms the Pfaffian point proces. See also [10]. However, the arguments in [10, 89, 108] can be not applied into more general zooming points in bulk and edge regimes. [6] proposed a differential equation method (ODE method) to analyze a skew pre-kernel from (3.8) or a skew-pre-kernel associated with (3.11). Their method is robust, and ODE method can be applied to other planar symplectic ensembles such as the Mittag Leffler ensemble [6], the elliptic Ginibre symplectic ensemble [37, 38], and induced Ginibre/spherical symplectic ensembles [36, 39]. For the detailed review, we refer to [41]. Coming back to (3.10), by [40, Eq (5.29)], we have

$$(3.18) \quad \begin{aligned} \partial_z \mathfrak{K}_k^{(\mathbf{g})}(z, w) &= 2z \mathfrak{K}_k^{(\mathbf{g})}(z, w) + 2 \sum_{\ell=0}^{2k-1} \frac{(2zw)^\ell}{\ell!} - 2 \frac{(2z^2)^k}{(2k-1)!!} \sum_{p=0}^{k-1} \frac{w^{2p}}{p!} \\ &= 2z \mathfrak{K}_k^{(\mathbf{g})}(z, w) + 2e_{2k-1}(2zw) - 2 \frac{(2z^2)^k}{(2k-1)!!} e_{k-1}(w^2), \end{aligned}$$

where e_k is the truncated exponential polynomial defined by

$$(3.19) \quad e_n(x) = \sum_{k=0}^n \frac{x^k}{k!}.$$

We define

$$(3.20) \quad \widehat{\mathfrak{K}}_k^{(\mathbf{g})}(z, a) = e^{-2za} \mathfrak{K}_k^{(\mathbf{g})}(z, a),$$

and then by (3.18)

$$(3.21) \quad \partial_z \widehat{\mathfrak{K}}_k^{(\mathbf{g})}(z, a) = 2(z-a) \widehat{\mathfrak{K}}_k^{(\mathbf{g})}(z, a) + 2e_{2k-1}(2za) e^{-2za} - 2 \frac{(2z^2)^k}{(2k-1)!!} e_{k-1}(a^2) e^{-2za}.$$

Let us mention two key points here. (3.18) means that the polynomial kernel (3.10) satisfies the first order differential equation (3.18). This can be regarded as an analogue of the Christoffel-Darboux type identity for the reproducing kernels for the elliptic Ginibre unitary ensemble [97] or the Lemniscate ensemble [42]. In the inhomogeneous terms, the first term corresponds to the polynomial reproducing kernel for the Ginibre unitary ensemble, and so, we know the asymptotic behavior of that term well. On the other hand, there is no interpretation for the second inhomogeneous term, but we can still analyze the that term via the uniform asymptotic expansion for the incomplete gamma function. However, when we work on scaling limits for (3.18), there are some subtle problems. Indeed, the polynomial kernel in general diverges in a large N -limit, we need to rescale the polynomial kernel with some factor. Moreover, if we take rescaled points $z = \sqrt{N}p + \zeta$ and $w = \sqrt{N}p + \eta$ for $p \in [-1, 1]$, it is inconvenient to compute (3.18). In order to overcome such subtle problems, as in (3.20), it is better to work on the differential equation for the rescaled kernel (3.21). As a summary, key points to analyze a pre-skew-kernel for planar symplectic ensembles in general are to derive a differential equation, which may not be a first order differential equation, and we need to find an appropriate factor function to derive a scaling limit. This viewpoint is also important for our problem later. Finally, we only mention the scaling limits for (3.10) or (3.20), and we conclude this section. [6] and later [37] showed the following (we only mention the Ginibre symplectic ensemble, and the below statement is cited from [37, Theorem 1.2]):

Theorem 3.3 ([6, 37]). *For ζ_j in compact subsets of \mathbb{C} for $j = 1, \dots, k$ and $p \in [-1, 1]$, let*

$$z_j = \sqrt{N}p + \mathfrak{s} \zeta_j \quad \text{for} \quad \begin{cases} \mathfrak{s} = 0, & \text{if } p \in (-1, 1), \\ \mathfrak{s} = 1, & \text{if } p = 1, \\ \mathfrak{s} = -1, & \text{if } p = -1. \end{cases}$$

We write

$$(3.22) \quad R_{N,k}(\zeta_1, \dots, \zeta_k) = \mathbf{R}_{N,k}(z_1, \dots, z_k).$$

Then, there exists a pre-kernel $\kappa_N^{(*)}$ such that

$$R_{N,k}(\zeta_1, \dots, \zeta_k) = \text{Pf}_{i,j=1}^k \left[\begin{pmatrix} \kappa_N(\zeta_i, \zeta_j) & \kappa_N(\zeta_i, \bar{\zeta}_j) \\ \kappa_N(\bar{\zeta}_i, \zeta_j) & \kappa_N(\bar{\zeta}_i, \bar{\zeta}_j) \end{pmatrix} e^{-|\zeta_i|^2 - |\zeta_j|^2} \right] \prod_{j=1}^k (\bar{\zeta}_j - \zeta_j),$$

and it satisfies the following asymptotics as $N \rightarrow \infty$.

$$\kappa_N^{(*)}(\zeta, \eta) = \begin{cases} \kappa_{\mathbb{R}}^{(b)}(\zeta, \eta) + O(e^{-\epsilon N}), & \text{if } p \in (-1, 1), \\ \kappa_{\mathbb{R}}^{(e)}(\zeta, \eta) + \frac{1}{\sqrt{2N}} \kappa_{\mathbb{R}}^{(e,1/2)}(\zeta, \eta) + O(N^{-1+\epsilon}) & \text{if } p = |1|, \end{cases}$$

where

$$\begin{aligned} \kappa_{\mathbb{R}}^{(b)}(\zeta, \eta) &= \frac{e^{\zeta^2 + \eta^2}}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-2(\zeta-u)^2} \text{erfc}(\sqrt{2}(\eta-u)) - e^{-2(\eta-u)^2} \text{erfc}(\sqrt{2}(\zeta-u)) du, \\ \kappa_{\mathbb{R}}^{(e)}(\zeta, \eta) &= \frac{e^{\zeta^2 + \eta^2}}{\sqrt{2}} \int_{-\infty}^0 e^{-2(\zeta-u)^2} \text{erfc}(\sqrt{2}(\eta-u)) - e^{-2(\eta-u)^2} \text{erfc}(\sqrt{2}(\zeta-u)) du, \end{aligned}$$

and

$$\kappa_{\mathbb{R}}^{(e,1/2)}(\zeta, \eta) = \frac{e^{\zeta^2 + \eta^2}}{12\sqrt{2}} \left[\left((2\zeta^2 + 1)e^{-2\zeta^2} \text{erfc}(\sqrt{2}\eta) + 2\sqrt{\frac{2}{\pi}} \eta e^{-2(\zeta^2 + \eta^2)} \right) - (\zeta \leftrightarrow \eta) \right].$$

Here, the convergence is uniform for ζ, η in compact subsets of \mathbb{C} , and \leftrightarrow means that we exchange the variables ζ and η .

3.2. Overlaps for Ginibre symplectic ensemble. In this thesis, as an analogue of the determinantal structure of the overlaps for Ginibre unitary and induced Ginibre/spherical unitary ensembles, we study the Pfaffian structure of the diagonal overlap for GinSE. To proceed with that, we need to introduce the overlaps for GinSE. The difference between non-Hermitian matrices and non-Hermitian symplectic matrices is that we need to consider the overlaps for complex conjugate pairs of eigenvalues λ_j for $j = 1, \dots, N$. Let us introduce the conjugate linear operator $\mathcal{I} : \mathbb{C}^{2d} \rightarrow \mathbb{C}^{2d}$ defined by

$$\mathcal{I} : u_{2k+1} \mapsto -\bar{u}_{2k+2}, \quad u_{2k+2} \mapsto \bar{u}_{2k+1} \quad \text{for } \mathbf{u} = (u_1, \dots, u_{2d})^t \in \mathbb{C}^{2d}.$$

This map was introduced by [102], and see also [11]. More precisely, this map means

$$\mathcal{I} \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2d-1} \\ u_{2d} \end{bmatrix} \right) = \begin{bmatrix} -\overline{u_2} \\ \overline{u_1} \\ \vdots \\ -\overline{u_{2d}} \\ \overline{u_{2d-1}} \end{bmatrix}.$$

Then, we have $\langle \mathbf{u} | \mathcal{I} \mathbf{u} \rangle = 0$ with respect to the Euclidean inner product on \mathbb{C}^d . We consider the right and left eigenvectors induced by the eigenvalue λ_j denoted by $R_j = (R_{j,k})_{k=1,\dots,2N}$ and $L_j^\dagger = (L_{j,k})_{k=1,\dots,2N}$, respectively, more precisely, they are defined by

$$(3.23) \quad GR_j = \lambda_j R_j, \quad L_j^\dagger G = \lambda_j L_j^\dagger.$$

The left and right eigenvectors corresponding to $\overline{\lambda_j}$ are defined by

$$G\mathcal{I}R_j = \overline{\lambda_j} \mathcal{I}R_j, \quad (\mathcal{I}L_j)^\dagger G = \overline{\lambda_j} (\mathcal{I}L_j)^\dagger.$$

Then, we have the bi-orthogonality relationships

$$\langle L_i | \overline{R_j} \rangle = \delta_{i,j}, \quad \langle L_i | \mathcal{I}R_j \rangle = 0, \quad \langle \mathcal{I}L_i | \overline{R_j} \rangle = 0, \quad \langle \mathcal{I}L_i | \mathcal{I}R_j \rangle = \delta_{i,j}.$$

We define $S = (R_1 \quad \mathcal{I}R_1 \quad \cdots \quad R_N \quad \mathcal{I}R_N)$. More precisely,

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & \cdots & S_{1,2N-1} & S_{1,2N} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & \cdots & S_{2,2N-1} & S_{2,2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{2N-1,1} & S_{2N-1,2} & S_{2N-1,3} & S_{2N-1,4} & \cdots & S_{2N-1,2N-1} & S_{2N-1,2N} \\ S_{2N,1} & S_{2N,2} & S_{2N,3} & S_{2N,4} & \cdots & S_{2N,2N-1} & S_{2N,2N} \end{pmatrix} \\ = \begin{pmatrix} R_{1,1} & -\overline{R_{1,2}} & R_{2,1} & -\overline{R_{2,2}} & \cdots & R_{N,1} & -\overline{R_{N,2}} \\ R_{1,2} & \overline{R_{1,1}} & R_{2,2} & \overline{R_{2,1}} & \cdots & R_{N,2} & \overline{R_{N,1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{1,2N-1} & -\overline{R_{1,2N}} & R_{2,2N-1} & -\overline{R_{2,2N}} & \cdots & R_{N,2N-1} & -\overline{R_{N,2N}} \\ R_{1,2N} & \overline{R_{1,2N-1}} & R_{2,2N} & \overline{R_{2,2N-1}} & \cdots & R_{N,2N} & \overline{R_{N,2N-1}} \end{pmatrix}.$$

Its inverse matrix is given by

$$S^{-1} = (L_1 \quad \mathcal{I}L_1 \quad \cdots \quad L_N \quad \mathcal{I}L_N)^\dagger \\ = \begin{pmatrix} \frac{L_{1,1}}{-L_{1,2}} & \frac{L_{1,2}}{L_{1,1}} & \frac{L_{1,3}}{-L_{1,4}} & \frac{L_{1,4}}{L_{1,3}} & \cdots & \frac{L_{1,2N-1}}{-L_{1,2N}} & \frac{L_{1,2N}}{L_{1,2N-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{L_{N,1}}{-L_{N,2}} & \frac{L_{N,2}}{L_{N,1}} & \frac{L_{N,3}}{-L_{N,4}} & \frac{L_{N,4}}{L_{N,3}} & \cdots & \frac{L_{N,2N-1}}{-L_{N,2N}} & \frac{L_{N,2N}}{L_{N,2N-1}} \end{pmatrix}.$$

Clearly, we have

$$(S^{-1}S)_{i,j} = \delta_{i,j} \quad \text{for } 1 \leq i, j \leq 2N.$$

Also, we have

$$G = SAS^{-1}.$$

We define

$$\begin{aligned}
A := (A_{i,j})_{i,j=1}^{2N} &= S^\dagger S = \begin{pmatrix} \overline{R_1}^t \\ (\mathcal{I}R_1)^t \\ \cdots \\ \overline{R_N}^t \\ (\mathcal{I}R_N)^t \end{pmatrix} \begin{pmatrix} R_1 & (\mathcal{I}R_1) & \cdots & R_N & (\mathcal{I}R_N) \end{pmatrix} \\
&= \begin{pmatrix} \overline{R_1}^t R_1 & \overline{R_1}^t (\mathcal{I}R_1) & \cdots & \overline{R_1}^t R_N & \overline{R_1}^t (\mathcal{I}R_N) \\ (\mathcal{I}R_1)^t R_1 & (\mathcal{I}R_1)^t (\mathcal{I}R_1) & \cdots & (\mathcal{I}R_1)^t R_N & (\mathcal{I}R_1)^t (\mathcal{I}R_N) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{R_N}^t R_1 & \overline{R_N}^t (\mathcal{I}R_1) & \cdots & \overline{R_N}^t R_N & \overline{R_N}^t (\mathcal{I}R_N) \\ (\mathcal{I}R_N)^t R_1 & (\mathcal{I}R_N)^t (\mathcal{I}R_1) & \cdots & (\mathcal{I}R_N)^t R_N & (\mathcal{I}R_N)^t (\mathcal{I}R_N) \end{pmatrix} \\
&= \begin{pmatrix} \langle R_1|R_1 \rangle & \langle R_1|\mathcal{I}R_1 \rangle & \cdots & \langle R_1|R_N \rangle & \langle R_1|\mathcal{I}R_N \rangle \\ \langle \mathcal{I}R_1|R_1 \rangle & \langle \mathcal{I}R_1|\mathcal{I}R_1 \rangle & \cdots & \langle \mathcal{I}R_1|R_N \rangle & \langle \mathcal{I}R_1|\mathcal{I}R_N \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle R_N|R_1 \rangle & \langle R_N|\mathcal{I}R_1 \rangle & \cdots & \langle R_N|R_N \rangle & \langle R_N|\mathcal{I}R_N \rangle \\ \langle \mathcal{I}R_N|R_1 \rangle & \langle \mathcal{I}R_N|\mathcal{I}R_1 \rangle & \cdots & \langle \mathcal{I}R_N|R_N \rangle & \langle \mathcal{I}R_N|\mathcal{I}R_N \rangle \end{pmatrix}^t,
\end{aligned}$$

and we regard $A = (A_{i,j})_{i,j=1}^{2N}$ as

$$A = \begin{pmatrix} A_{i,j} & A_{i,\bar{j}} \\ A_{\bar{i},j} & A_{\bar{i},\bar{j}} \end{pmatrix}_{i,j=1}^N.$$

Similarly, we define

$$\begin{aligned}
A^{-1} &= (A_{i,j}^{-1})_{i,j=1}^{2N} = S^{-1}(S^{-1})^\dagger \\
&= (L_1 \quad (\mathcal{I}L_1) \quad \cdots \quad L_N \quad (\mathcal{I}L_N))^t \begin{pmatrix} \overline{L_1} \\ (\mathcal{I}L_1) \\ \vdots \\ \overline{L_N} \\ (\mathcal{I}L_N) \end{pmatrix} \\
&= \begin{pmatrix} \langle L_1|L_1 \rangle & \langle L_1|\mathcal{I}L_1 \rangle & \cdots & \langle L_1|L_N \rangle & \langle L_1|\mathcal{I}L_N \rangle \\ \langle \mathcal{I}L_1|L_1 \rangle & \langle \mathcal{I}L_1|\mathcal{I}L_1 \rangle & \cdots & \langle \mathcal{I}L_1|L_N \rangle & \langle \mathcal{I}L_1|\mathcal{I}L_N \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle L_N|L_1 \rangle & \langle L_N|\mathcal{I}L_1 \rangle & \cdots & \langle L_N|L_N \rangle & \langle L_N|\mathcal{I}L_N \rangle \\ \langle \mathcal{I}L_N|L_1 \rangle & \langle \mathcal{I}L_N|\mathcal{I}L_1 \rangle & \cdots & \langle \mathcal{I}L_N|L_N \rangle & \langle \mathcal{I}L_N|\mathcal{I}L_N \rangle \end{pmatrix}.
\end{aligned}$$

and we regard $A^{-1} = (A_{i,j}^{-1})_{i,j=1}^{2N}$ as

$$A^{-1} = \begin{pmatrix} A_{i,j}^{-1} & A_{i,\bar{j}}^{-1} \\ A_{\bar{i},j}^{-1} & A_{\bar{i},\bar{j}}^{-1} \end{pmatrix}_{i,j=1}^N.$$

Then, we define the overlap matrix $\mathcal{O} = (\mathcal{O}_{i,j})_{i,j=1}^{2N}$ defined by

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_{i,j} & \mathcal{O}_{i,\bar{j}} \\ \mathcal{O}_{\bar{i},j} & \mathcal{O}_{\bar{i},\bar{j}} \end{pmatrix}_{i,j=1}^N$$

with elements given by

$$\begin{aligned}
\mathcal{O}_{i,j} &= A_{i,j}^{-1} A_{j,i} = \langle L_i|L_j \rangle \langle R_i|R_j \rangle, & \mathcal{O}_{i,\bar{j}} &= A_{i,\bar{j}}^{-1} A_{\bar{j},i} = \langle L_i|\mathcal{I}L_j \rangle \langle R_i|\mathcal{I}R_j \rangle, \\
\mathcal{O}_{\bar{i},j} &= A_{\bar{i},j}^{-1} A_{j,\bar{i}} = \langle \mathcal{I}L_i|L_j \rangle \langle \mathcal{I}R_i|R_j \rangle, & \mathcal{O}_{\bar{i},\bar{j}} &= A_{\bar{i},\bar{j}}^{-1} A_{\bar{j},\bar{i}} = \langle \mathcal{I}L_i|\mathcal{I}L_j \rangle \langle \mathcal{I}R_i|\mathcal{I}R_j \rangle.
\end{aligned}$$

Hence, it suffices to essentially consider $\mathcal{O}_{i,j}$ and $\mathcal{O}_{i,\bar{j}}$. We expect that this general structure would be useful to derive a system of stochastic differential equations for the eigenvalues processes and the overlap processes of Ginibre symplectic ensemble. In [11], Akemann, Förster, and Kieburg showed the following:

Theorem 3.4 ([11]). *For the diagonal overlap of GinSE, conditionally on $\lambda_{(N)} = \mathbf{z}_{(N)} \in \mathbb{C}^N$, we have*

$$(3.24) \quad \mathbb{E}[\mathcal{O}_{1,1} | \lambda_{(N)} = \mathbf{z}_{(N)}] = \prod_{j=2}^N \left(1 + \frac{1}{2|z_1 - z_2|^2} + \frac{1}{2|z_1 - \bar{z}_2|^2} \right).$$

Moreover, for the off-diagonal overlap of GinSE, conditionally on $\lambda_{(N)} = \mathbf{z}_{(N)} \in \mathbb{C}^N$, we have

$$\begin{aligned} \mathbb{E}[\mathcal{O}_{1,2} | \lambda_{(N)} = \mathbf{z}_{(N)}] &= -\frac{1}{2|z_1 - z_2|^2} \prod_{j=3}^N \left(1 + \frac{1}{2(z_1 - z_2)(\bar{z}_2 - \bar{z}_k)} + \frac{1}{2(z_1 - \bar{z}_k)(\bar{z}_2 - z_k)} \right), \\ \mathbb{E}[\mathcal{O}_{1,\bar{2}} | \lambda_{(N)} = \mathbf{z}_{(N)}] &= -\frac{1}{2|z_1 - \bar{z}_2|^2} \prod_{j=3}^N \left(1 + \frac{1}{2(z_1 - z_2)(z_2 - \bar{z}_k)} + \frac{1}{2(z_1 - \bar{z}_k)(z_2 - z_k)} \right). \end{aligned}$$

The spirit of the proof of Theorem 3.4 is same as Proposition 2.2. In fact, we can show Theorem 3.4 via a recurrence equation obtained from the structure of the above overlap for the non-Hermitian symplectic matrix and the relationship between eigenvalues-eigenvectors of GinSE and the upper triangular matrix of GinSE in the Schur decomposition. For the details, we refer to [11] and [59]. As in the determinantal case, we will study the multi-points correlation function of the diagonal overlap for Ginibre symplectic ensemble conditioned on the real line as the generalization of [11] and [59].

3.3. Our contribution [8]: joint work with Gernot Akemann and Sung-Soo Byun. In this subsection, we present our results for the multi-points correlation function of the diagonal overlap for Ginibre symplectic ensemble conditioned on the real line. The reason why we consider only a real conditional point and the diagonal overlap is due to a technical reason. It would be a future project for an extension to more general case for any complex points and the off-diagonal overlaps. Now, we consider the multi-points correlation function of the diagonal overlap for GinSE defined by

$$(3.25) \quad D_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) = \mathbb{E} \left[\sum_{i_1 \neq \dots \neq i_k = 1}^N \mathcal{O}_{1,1} \delta(z_{i_1} - \lambda_{i_1}) \cdots \delta(z_{i_k} - \lambda_{i_k}) \right], \quad \text{for } \mathbf{z}_{(k)} \in \mathbb{C}^k.$$

We can explicitly express (3.25) as

$$(3.26) \quad \begin{aligned} D_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) &= \frac{|z_1 - \bar{z}_1|^2 e^{-2|z_1|^2}}{Z_N^{(\text{GinSE})}} \frac{N!}{(N-k)!} \\ &\times \int_{\mathbb{C}^{N-k}} \prod_{2 \leq i < j \leq N} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{j=2}^N |z_j - \bar{z}_j|^2 \omega_{(\text{over})}(z_j | z_1) \prod_{\ell=k+1}^N dA(z_\ell), \end{aligned}$$

where

$$(3.27) \quad \omega_{(\text{over})}(z|a) = |z - \bar{a}|^2 (1 + |z - a|^2) e^{-2|z|^2}, \quad a \in \mathbb{C}.$$

Let us also write

$$(3.28) \quad \omega_{(\text{pre})}(z|a) = (1 + |z - a|^2) e^{-2|z|^2}, \quad a \in \mathbb{C}.$$

We call (3.27) the **overlap weight function** and (3.28) the **pre-overlap weight function**. We write

$$(3.29) \quad \begin{aligned} \widehat{D}_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) &= \frac{D_{1,1}^{(N,k)}(\mathbf{z}_{(k)})}{|z_1 - \bar{z}_1|^2} \\ &= \frac{e^{-2|z_1|^2}}{Z_N^{(\text{GinSE})}} \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \prod_{2 \leq i < j \leq N} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{j=2}^N |z_j - \bar{z}_j|^2 \omega_{(\text{over})}(z_j | z_1) \prod_{\ell=k+1}^N dA(z_\ell). \end{aligned}$$

Then, (3.26) is expressed as the Pfaffian form, which is characterized by a skew-symmetric pre-kernel denoted by $\mathfrak{K}_{N-1}^{(\text{over})}(z, w|z_1)$. We define

$$(3.30) \quad \begin{aligned} & \mathbf{R}_{N,k}^{(\text{over})}(z_2, \dots, z_k|z_1) \\ &= \lim_{\text{Im } z_1 \rightarrow 0} \widehat{D}_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) \\ &= \frac{e^{-2z_1^2} N! Z_{N-1}^{(\text{over})}(z_1)}{Z_N^{(\text{GinSE})}} \text{Pf}_{i,j=2,\dots,k} \left[\begin{pmatrix} \mathfrak{K}_{N-1}^{(\text{over})}(z_i, z_j|z_1) & \mathfrak{K}_{N-1}^{(\text{over})}(z_i, \bar{z}_j|z_1) \\ \mathfrak{K}_{N-1}^{(\text{over})}(\bar{z}_i, z_j|z_1) & \mathfrak{K}_{N-1}^{(\text{over})}(\bar{z}_i, \bar{z}_j|z_1) \end{pmatrix} \right] \prod_{j=2}^k (\bar{z}_j - z_j) \omega_{(\text{over})}(z_j|z_1). \end{aligned}$$

In particular, we are interested in the rescaled multi-points correlation function of the diagonal overlap along the real line

$$(3.31) \quad R_{N,k}^{(\text{over},*)}(\zeta_2, \dots, \zeta_k|\zeta_1) = \mathbf{R}_{N,k}^{(\text{over})}(\sqrt{N}p + \zeta_2, \dots, \sqrt{N}p + \zeta_k | \sqrt{N}p + \zeta_1).$$

for $p \in [-1, 1]$ (we only consider $p = 1$), $\zeta_1 \in \mathbb{R}$ and ζ_2, \dots, ζ_k in compact subsets of \mathbb{C} , and $*$ = b if $p \in (-1, 1)$ and $*$ = e if $p = \pm 1$. Here, $Z_{N-1}^{(\text{over})}(z_1)$ denotes the partition function with respect to (3.26), which depends on the variable z_1 .

3.3.1. Recurrence formula and finite N -kernel associated with (3.27) and (3.28). In order to construct $\mathfrak{K}_{N-1}^{(\text{over})}(z, w|z_1)$, we decompose (3.27) into a **pre-overlap weight function part** and a **point insertion part**. The pre-overlap weight function corresponds to (3.28), and the point insertion part corresponds to $|z - a|^2$. The below result is not a main result in this paper. However, it implies that we can not apply [10, Theorem 2.1] to our setting.

Proposition 3.5 (Planar orthogonal polynomials associated with (3.28)). *We write a family of planar orthogonal polynomials associated with (3.28) by $\{P_k(\cdot|a)\}_k$. Then, we have*

$$(3.32) \quad P_k(z|a) = \sum_{j=0}^k a^{k-j} \frac{F_j(2a\bar{a})}{F_k(2a\bar{a})} z^j,$$

where

$$F_p(x) = (p+2)(p+1)e_p(x) - 2(p+1)xe_{p-1}(x) + x^2e_{p-2}(x), \quad p = 0, 1, 2, \dots,$$

and

$$(3.33) \quad e_p(x) = \sum_{j=0}^p \frac{x^j}{j!}.$$

Remark 3.6. *We can easily check $\{P_k(\cdot|a)\}_k$ satisfy the non-standard three-term recurrence in the sense of [42, Remark 1.3]:*

$$(3.34) \quad zP_k(z|a) = P_{k+1}(z|a) + b_kP_k(z|a) + zc_kP_{k-1}(z|a),$$

where

$$b_k = -a \frac{F_k(2a\bar{a})}{F_{k+1}(2a\bar{a})}, \quad c_k = a \frac{F_{k-1}(2a\bar{a})}{F_k(2a\bar{a})}.$$

By Proposition 3.5 and Remark 3.6, we can not directly use [10, Theorem 5.1] to construct the family of skew-orthogonal polynomials associated with (3.28). The only fact which we have is that once we could construct that, we can apply the family of skew-orthogonal polynomials associated with (3.27) via [10, Theorem 5.1]. Hence, our problem which we should overcome is to construct the skew-orthogonal polynomials associated with (3.28) and hence to construct a skew symmetric pre-kernel associated with (3.28), which is denoted by $\mathfrak{K}_N^{(\text{pre})}(z, w|a)$ for $z, w \in \mathbb{C}$, $a \in \mathbb{R}$.

Now, we shall develop a new construction of skew-orthogonal polynomials (SOPs). Let μ be a positive Borel measure on \mathbb{C} with an infinite number of points in its support on a domain D . We define the inner product and the skew-inner product with respect μ on D by

$$(3.35) \quad \langle f, g \rangle := \int_D f(z) \overline{g(z)} d\mu(z), \quad \langle f, g \rangle_s := \int_D (f(z) \overline{g(z)} - g(z) \overline{f(z)})(z - \bar{z}) d\mu(z).$$

Let

$$(3.36) \quad m_{j,k} = \langle z^j, z^k \rangle \quad \widehat{m}_{j,k} = \operatorname{Re}(m_{j,k}),$$

and let $\mathbf{m} = (m_{j,k})$ be a moment matrix satisfying $\mathbf{m}^* = \mathbf{m}$, where $*$ denotes the complex conjugate transpose, and $m_{k,k+1}, m_{k,k} \neq 0 \in \mathbb{C}$ for all k , and $m_{j,k} = 0$ otherwise. Hence, \mathbf{m} is a tridiagonal moment matrix. We denote

$$(3.37) \quad g_{j,k} = \langle z^j, z^k \rangle_s = 2(\widehat{m}_{j+1,k} - \widehat{m}_{j,k+1}), \quad G_k = (g_{i,j})_{i,j=0}^{2k-1}.$$

We define

$$(3.38) \quad \Delta_{-1} = 0, \quad \Delta_k = \operatorname{Pf}(G_{k+1}), \quad \mathcal{Z}_k = \frac{1}{2} \frac{\Delta_k}{\Delta_{k-1}}.$$

The below is our new construction for SOPs.

Theorem 3.7. *Under the setting of (3.36) satisfying a tridiagonal Hermitian moment matrix, (3.37), and (3.38), we define SOPs associated with a Borel measure μ as*

$$(3.39) \quad q_{2k}(z) = \sum_{j=0}^{2k} \alpha_{2k,j} z^j, \quad q_{2k+1}(z) = \sum_{j=0}^{2k+1} \beta_{2k+1,j} z^j,$$

with $\alpha_{2k,2k} = 1$ and $\beta_{2k+1,2k+1} = 1, \beta_{2k+1,2k} = 0$. Then, the coefficients $\{\alpha_{2k,j}\}_{j=0}^{2k}$ and $\{\beta_{2k+1,j}\}_{j=0}^{2k+1}$ are determined by the following recurrence relationships: for $j = 1, 2, \dots, k$,

$$(3.40) \quad \begin{aligned} \mathcal{Z}_j \alpha_{2k,2j-1} &= -\widehat{m}_{2j-1,2j} \alpha_{2k,2j}, & \mathcal{Z}_j \alpha_{2k,2j-2} &= \widehat{m}_{2j,2j+1} \alpha_{2k,2j+1} + \widehat{m}_{2j,2j} \alpha_{2k,2j}, \\ \mathcal{Z}_k \alpha_{2k,2k-1} &= -\widehat{m}_{2k-1,2k}, & \mathcal{Z}_k \alpha_{2k,2k-2} &= \widehat{m}_{2k,2k}, \end{aligned}$$

and for $j = 1, 2, \dots, k$,

$$(3.41) \quad \begin{aligned} \mathcal{Z}_j \beta_{2k+1,2j-1} &= -\widehat{m}_{2j-1,2j} \beta_{2k+1,2j}, & \mathcal{Z}_j \beta_{2k+1,2j-2} &= \widehat{m}_{2j,2j+1} \beta_{2k+1,2j+1} + \widehat{m}_{2j,2j} \beta_{2k+1,2j}, \\ \mathcal{Z}_{k-1} \beta_{2k+1,2k-3} &= -\widehat{m}_{2k-3,2k-2} \beta_{2k+1,2k-2}, & \mathcal{Z}_{k-1} \beta_{2k+1,2k-4} &= -\widehat{m}_{2k-2,2k-2} \beta_{2k+1,2k-2}, \\ \beta_{2k+1,2k-1} &= 0, & \mathcal{Z}_k \beta_{2k+1,2k-2} &= \widehat{m}_{2k,2k+1}. \end{aligned}$$

Remark 3.8. *The initial condition $\beta_{2k+1,2k} = 0$ in Theorem 3.7 uniquely determines odd SOPs, see [10, Lemma 2.2].*

We denote the SOPs associated with (3.28) by

$$(3.42) \quad q_{2k}^{(\text{pre})}(z|a) = \sum_{j=0}^{2k} \alpha_{2k,j}^{(\text{pre})} z^j, \quad q_{2k+1}^{(\text{pre})}(z|a) = \sum_{j=0}^{2k+1} \beta_{2k+1,j}^{(\text{pre})} z^j.$$

As a consequence of Theorem 3.7, we have the following:

Theorem 3.9. *The coefficients $\{\alpha_{2k,j}^{(\text{pre})}\}_{j=0}^{2k}$ and $\{\beta_{2k+1,j}^{(\text{pre})}\}_{j=0}^{2k+1}$ are given by*

$$(3.43) \quad \alpha_{2k,2j}^{(\text{pre})} = \frac{k!}{2^k} \frac{2^j}{j!} \sum_{\ell=j}^k \left((\ell+1-j) \frac{(2k+3)!!}{(2\ell+3)!!} - (\ell-j) \frac{(2k+1)!!}{(2\ell+1)!!} \right) (2a^2)^{\ell-j} \frac{f_j(a^2)}{f_k(a^2)},$$

$$(3.44) \quad \alpha_{2k,2j+1}^{(\text{pre})} = 2a \frac{k!}{2^k} \frac{2^j}{j!} \sum_{\ell=j}^{k-1} \left((\ell+1-j) \frac{(2k+3)!!}{(2\ell+5)!!} - (\ell-j) \frac{(2k+1)!!}{(2\ell+5)!!} \right) (2a^2)^{\ell-j} \frac{f_j(a^2)}{f_k(a^2)},$$

$$(3.45) \quad \beta_{2k+1,2j}^{(\text{pre})} = a \frac{k!}{2^k} \frac{2^j}{j!} \left((2a^2)^{k-j} + 2 \sum_{\ell=j}^{k-1} (\ell+1-j) \frac{(2k+1)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} \right) \frac{f_j(a^2)}{f_k(a^2)},$$

$$(3.46) \quad \beta_{2k+1,2j+1}^{(\text{pre})} = \frac{k!}{2^k} \frac{2^j}{j!} \left((2a^2)^{k-j} + 2 \sum_{\ell=j}^{k-1} (\ell-j) \frac{(2k+1)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} \right) \frac{f_j(a^2)}{f_k(a^2)},$$

where

$$(3.47) \quad f_k(x) = \sum_{j=0}^k (k+1-j) \frac{k!}{j!} x^j = (k+1)e_k(x) - x e_{k-1}(x),$$

and the skew norming constant associated with (3.28) is given by

$$(3.48) \quad r_k^{(\text{pre})} = r_k^{(\text{pre})}(a) = \frac{(2k+1)!}{2^{2k+1}} \frac{f_{k+1}(a^2)}{f_k(a^2)}.$$

Then, we can express the skew symmetric pre-kernel associated with (3.28) in terms of (3.42) as

$$(3.49) \quad \mathfrak{K}_N^{(\text{pre})}(z, w|a) = \sum_{k=0}^{N-1} \frac{q_{2k+1}^{(\text{pre})}(z|a)q_{2k}^{(\text{pre})}(w|a) - q_{2k+1}^{(\text{pre})}(w|a)q_{2k}^{(\text{pre})}(z|a)}{r_k^{(\text{pre})}}.$$

As we already announced, by [10, Theorem 5.1], we obtain $\mathfrak{K}_N^{(\text{over})}(z, w|a)$.

Corollary 3.10. *We write the SOPs associated with (3.27) by $\{q_k^{(\text{over})}(z|a)\}_k$. Then, we have*

$$\begin{aligned} q_{2k}^{(\text{over})}(z|a) &= \frac{r_k^{(\text{pre})} \mathfrak{K}_{k+1}^{(\text{pre})}(a, z|a)}{(a-z)q_{2k}^{(\text{pre})}(a|a)}, \\ q_{2k+1}^{(\text{over})}(z|a) &= \frac{q_{2k+2}^{(\text{pre})}(a|a)q_{2k}^{(\text{pre})}(z|a) - q_{2k}^{(\text{pre})}(a|a)q_{2k+2}^{(\text{pre})}(z|a)}{(a-z)q_{2k}^{(\text{pre})}(a|a)}, \\ r_k^{(\text{over})} &= r_k^{(\text{pre})} \frac{q_{2k+2}^{(\text{pre})}(a|a)}{q_{2k}^{(\text{pre})}(a|a)}. \end{aligned}$$

Furthermore, we have

$$(3.50) \quad \mathfrak{K}_N^{(\text{over})}(z, w|a) = \frac{\mathfrak{K}_N^{(\text{pre})}(z, w|a)q_{2N}^{(\text{pre})}(a|a) - \mathfrak{K}_N^{(\text{pre})}(z, a|a)q_{2N}^{(\text{pre})}(w|a) + \mathfrak{K}_N^{(\text{pre})}(w, a|a)q_{2N}^{(\text{pre})}(z|a)}{(a-z)(a-w)q_{2N}^{(\text{pre})}(a|a)}.$$

Remark 3.11. *In Corollary 3.10,*

$$q_{2k}^{(\text{over})}(z|a) = \frac{r_k^{(\text{pre})} \mathfrak{K}_{k+1}^{(\text{pre})}(a, z|a)}{(a-z)q_{2k}^{(\text{pre})}(a|a)}$$

can be regarded as the limit

$$\frac{r_k^{(\text{pre})} \mathfrak{K}_{k+1}^{(\text{pre})}(a, z|a)}{(a-z)q_{2k}^{(\text{pre})}(a|a)} = \lim_{u \rightarrow a} \frac{r_k^{(\text{pre})} \mathfrak{K}_{k+1}^{(\text{pre})}(u, z|a)}{(a-z)q_{2k}^{(\text{pre})}(u|a)},$$

which can be computed by the l'Hôpital's rule. Similarly, we should understand the other terms in the sense of the l'Hôpital's rule.

3.3.2. Integrable structure of the finite N -kernel: Differential equation. By Corollary 3.10, it suffices to consider scaling limits of (3.42) and (3.49). However, it is not easy to compute the scaling limits of (3.42) and (3.49). Indeed, (3.49) constitutes many summations in both numerators and denominators. Hence, to directly analyze the scaling limit for (3.49) is difficult. In order to overcome this difficulty, we derive a differential equation for (3.49) as we demonstrated in the case of Ginibre symplectic ensemble (3.20) and (3.18). The following Lemma is useful to write the differential equation, and it tells us that the SOPs and the skew pre-kernel associated with (3.49) can be written in terms of (3.20).

Lemma 3.12. *Let us write*

$$(3.51) \quad \mathcal{L}_k(z, a) = \left(2(z-a)^2 - 1\right) \mathfrak{K}_k^{(\mathbf{g})}(z, a) + 2(z-a)e_{2k-1}(2za) - 2(z-a) \frac{2^k z^{2k}}{(2k-1)!!} e_{k-1}(a^2).$$

Then, we have

$$(3.52) \quad (z-a)^3 q_{2k+1}^{(\text{pre})}(z|a) = (z-a)^2 z^{2k+2} + a \frac{(2k+1)!!}{2^{k+2} f_k(a^2)} \mathcal{L}_{k+1}(z, a),$$

and

$$(3.53) \quad (z-a)^3 q_{2k}^{(\text{pre})}(z|a) = a(z-a)^2 \frac{e_{k+1}(a^2)}{f_k(a^2)} z^{2k+2} + \frac{(2k+3)!!}{2^{k+3} f_k(a^2)} \mathcal{L}_{k+2}(z, a) - a^2 \frac{(2k+1)!!}{2^{k+2} f_k(a^2)} \mathcal{L}_{k+1}(z, a).$$

We define

$$\widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) = \widehat{G}_N^{(\text{pre})}(z, w|a) - \widehat{G}_N^{(\text{pre})}(w, z|a),$$

where

$$\begin{aligned} \widehat{G}_N^{(\text{pre})}(z, w|a) &= (z-a)^3 (w-a)^3 e^{-2za-2wa} \sum_{k=0}^{N-1} \frac{q_{2k+1}^{(\text{pre})}(z|a) q_{2k}^{(\text{pre})}(w|a)}{r_k^{(\text{pre})}} \\ &= \frac{1}{2^3} \sum_{k=0}^{N-1} \frac{\widehat{q}_{2k+1}^{(\text{pre})} \widehat{q}_{2k}^{(\text{pre})}(w|a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)}, \end{aligned}$$

Let us also write

$$(3.54) \quad \widehat{q}_{2k+1}^{(\text{pre})}(z|a) = e^{-2za} (z-a)^3 q_{2k+1}^{(\text{pre})}(z|a),$$

and

$$(3.55) \quad \widehat{q}_{2k}^{(\text{pre})}(w|a) = e^{-2wa} (w-a)^3 q_{2k}^{(\text{pre})}(w|a).$$

Theorem 3.13. *Let us denote*

$$(3.56) \quad \widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, w|a) = e^{2a^2-2za-2wa} (z-a)^3 (w-a)^3 \mathfrak{X}_N^{(\text{pre})}(z, w|a).$$

We define the second order differential operator

$$(3.57) \quad \mathfrak{D}_z^{(a)} = (z-a)\partial_z^2 - (2(z-a)^2 + 2)\partial_z - 2(z-a), \quad \text{for } z \in \mathbb{C} \text{ and } a \in \mathbb{R}.$$

Then, we have

$$(3.58) \quad \begin{aligned} &\mathfrak{D}_z^{(a)} \widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, w|a) \\ &= 4(z-a)^3 (w-a)^3 e^{2(z-a)(w-a)} \left(Q(2N+1, 2zw) - \frac{1}{2a} \partial_z Q(2N+1, 2zw) \right) \\ &\quad - 2(z-a)^3 \frac{(2z)^{2N}}{a(2N)!} e^{-2a(z-a)} \left(2\widehat{q}_{2N+1}^{(\text{pre})}(w|a) + \frac{f_{N-1}(a^2)}{f_N(a^2)} a(2N+1-2za) \widehat{q}_{2N-2}^{(\text{pre})}(w|a) \right), \end{aligned}$$

where Q is the incomplete gamma function. Furthermore, let us also denote

$$\widetilde{\mathfrak{X}}_N^{(\text{over})}(z, w|a) = (z-a)^4 (w-a)^4 e^{2a^2-2zw-2wa} \mathfrak{X}_N^{(\text{over})}(z, w|a).$$

Then, by Corollary 3.10, we have

$$(3.59) \quad \widetilde{\mathfrak{X}}_N^{(\text{over})}(z, w|a) = \widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, w|a) - \widehat{q}_{2N}^{(\text{pre})}(w, a) \lim_{u \rightarrow a} \frac{\widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, u|a)}{\widehat{q}_{2N}^{(\text{pre})}(u|a)} + \widehat{q}_{2N}^{(\text{pre})}(z, a) \lim_{u \rightarrow a} \frac{\widetilde{\mathfrak{X}}_N^{(\text{pre})}(w, u|a)}{\widehat{q}_{2N}^{(\text{pre})}(u|a)}.$$

3.3.3. Bulk scaling limits. We fix $p \in (-1, 1)$, and let

$$(3.60) \quad z = \sqrt{N}p + \zeta, \quad w = \sqrt{N}p + \eta, \quad a = \sqrt{N}p + \chi, \quad \zeta, \eta \in \mathbb{C}, \chi \in \mathbb{R}.$$

We denote

$$(3.61) \quad \widetilde{\kappa}_b^{(\text{over})}(\zeta, \eta|\chi) = \lim_{N \rightarrow \infty} \widetilde{\mathfrak{X}}_N^{(\text{over})}(z, w|a).$$

Then, the below is our main result.

Theorem 3.14. *For $p \in (-1, 1)$, we write*

$$R_k^{(\text{over}, b)}(\zeta_2, \dots, \zeta_k | \zeta_1) = \lim_{N \rightarrow \infty} \frac{1}{N} R_{N, k}^{(\text{over}, b)}(\zeta_2, \dots, \zeta_k | \zeta_1).$$

Then, we have

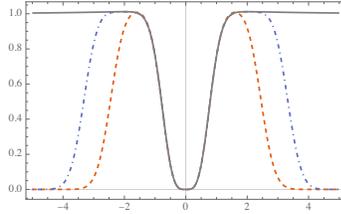
$$(3.62) \quad R_k^{(\text{over}, b)}(\zeta_2, \dots, \zeta_k | \zeta_1) = \frac{4}{3} (1-p^2) \text{Pf} \left[\begin{pmatrix} \kappa_b^{(\text{over})}(\zeta_i, \zeta_j | \zeta_1) & \kappa_b^{(\text{over})}(\zeta_i, \bar{\zeta}_j | \zeta_1) \\ \kappa_b^{(\text{over})}(\bar{\zeta}_i, \zeta_j | \zeta_1) & \kappa_b^{(\text{over})}(\bar{\zeta}_i, \bar{\zeta}_j | \zeta_1) \end{pmatrix} \right]_{i, j=2}^k \prod_{j=2}^k (\bar{\zeta}_j - \zeta_j) \omega_b(\zeta_j | \zeta_1)$$

uniformly for ζ_1 in a compact subset of \mathbb{R} and for ζ_2, \dots, ζ_k in compact subsets of \mathbb{C} , where

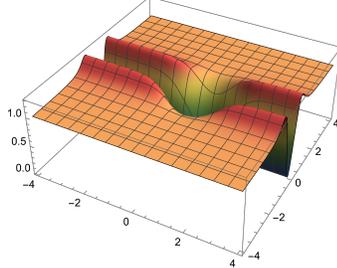
$$(3.63) \quad \omega_b(z|a) = |z - a|^2(1 + |z - a|^2)e^{-2|z - a|^2}, \quad a \in \mathbb{R},$$

and for $\zeta, \eta \in \mathbb{C}$ and $\chi \in \mathbb{R}$,

$$\begin{aligned} \kappa_b^{(\text{over})}(\zeta, \eta|\chi) &= \frac{1}{2} \frac{(\zeta - \eta)(1 + (\zeta - \chi)(\eta - \chi) - e^{2(\zeta - \chi)(\eta - \chi)})}{((\zeta - \chi)(\eta - \chi))^4} \\ &+ \frac{\sqrt{\pi}}{4} \frac{(2(\zeta - \chi)^2 - 1)(2(\eta - \chi)^2 - 1)e^{(\zeta - \chi) + (\eta - \chi)^2} \operatorname{erf}(\zeta - \eta)}{((\zeta - \chi)(\eta - \chi))^4} \\ &+ \frac{\sqrt{\pi}}{4} \frac{(2(\eta - \chi)^2 - 1)((\zeta - \chi)^2 - 1)e^{(\eta - \chi)^2} \operatorname{erf}(\eta - \chi)}{((\zeta - \chi)(\eta - \chi))^4} \\ &- \frac{\sqrt{\pi}}{4} \frac{(2(\zeta - \chi)^2 - 1)((\eta - \chi)^2 - 1)e^{(\zeta - \chi)^2} \operatorname{erf}(\zeta - \chi)}{((\zeta - \chi)(\eta - \chi))^4}. \end{aligned}$$



(A) The gray curve is the curve of $R_1^{(\text{over},b)}(x + iy|0.5)$, the red dot curve is the curve of $R_{N,1}^{(\text{over})}(0.5 + iy|0.5)$ with $N = 5$, and the blue dot curve is the curve of $R_{N,1}^{(\text{over})}(0.5 + iy|0.5)$ with $N = 10$.



(B) The graph of the one point density $R_1^{(\text{over},b)}(x + iy|0.5)$.

FIGURE 4. Comparison between the numerical plots and the analytic plot and the plot of the one-point density $R_1^{(\text{over},b)}(x + iy|0.5)$.

As a consequence of Theorem 3.14, we can derive the conditional expectation of the diagonal overlap conditioned at a real point. By the definition of the conditional expectation, we need the normalization factor. From [4, Theorem 1], the normalization factor for the conditional expectation at $a \in \mathbb{C}$ is given by

$$Z_N^{(\text{GinSE})}(a) = |a - \bar{a}|^2 e^{-2|a|^2} \frac{\mathfrak{K}_N^{(\mathbf{g})}(a, \bar{a})}{a - \bar{a}}$$

Then, for $a = \sqrt{N}p + \xi$ for $p \in (-1, 1)$ and $\xi \in \mathbb{C}$, by [37, Theorem 1.2] and the l'Hôpital's rule with help of the uniform asymptotic behavior, we have

$$Z_N^{(\text{GinSE})}(a) \sim 2|\xi - \bar{\xi}|^2, \quad \text{as } N \rightarrow \infty \text{ and } \operatorname{Im}(\xi) \rightarrow 0,$$

uniformly for ξ in a compact subset of \mathbb{C} . The factor $|\xi - \bar{\xi}|^2$ here corresponds to the factor divided in advance $|z_1 - \bar{z}_1|^2$. Therefore, we have the following:

Theorem 3.15. *Conditionally on $\{z_1 = \sqrt{N}p + \chi\}$ for $p \in (-1, 1)$, we have*

$$(3.64) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\mathcal{O}_{1,1} | z_1 = \sqrt{N}p + \chi] = \mathbb{E}[(\gamma_4/2)^{-1}](1 - p^2),$$

uniformly for χ in a compact subset of \mathbb{R} , and γ_β is the Gamma distribution with a parameter β .

Remark 3.16. *This result is consistent with the conditional expectation of the diagonal overlap in [59, Theorem 3.5]. The constant term except of the boundary affect $(1 - p^2)$ for $p \in (-1, 1)$ is universal, and our result can be regarded as the generalization of [59, Theorem 3.5].*

We can show the edge scaling limit for the conditional expectation of the diagonal overlap. Due to the symmetry, we only consider the edge point $p = 1$.

Theorem 3.17. *Conditionally on $\{z_1 = \sqrt{N} + \chi\}$,*

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E}[\mathcal{O}_{1,1} | z = \sqrt{N} + \chi] = \frac{e^{-4\chi^2}}{3\sqrt{\pi}} \frac{1 + \sqrt{2\pi}e^{2\chi^2} \chi \operatorname{erfc}(\sqrt{2}\chi) - 4\sqrt{\pi}e^{4\chi^2} \chi \operatorname{erfc}(2\chi)}{\operatorname{erfc}(2\chi) - \frac{1}{\sqrt{2}}e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\chi)},$$

uniformly for χ in a compact subset of \mathbb{R} .

4. CONTRIBUTIONS OF ZEROS OF GAUSSIAN ANALYTIC FUNCTIONS

4.1. Gaussian analytic function by Peres and Virág. In a breakthrough paper [118], Peres and Virág considered a random power series with coefficients of independent, identically distributed (i.i.d.) to standard complex Gaussian random variables, and they showed that zeros point processes of that forms a determinantal point process on the unit disk. More precisely, we define a random power series

$$(4.1) \quad f_{\text{PV}}(z) = \sum_{k=0}^{\infty} \zeta_k z^k,$$

where $\{\zeta_k\}_{k=0}^{\infty}$ are i.i.d. standard complex Gaussian random variables denoted by $\zeta \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}_{\mathbb{C}}(0, 1)$. We write the zeros point process of (4.1) by $\mathcal{Z}_{f_{\text{PV}}}$. Then, they showed the following:

Theorem 4.1 ([118]). *The zeros of point process $\mathcal{Z}_{f_{\text{PV}}}$ is the determinantal point process with the correlation function*

$$(4.2) \quad K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D}.$$

Namely, the k -th correlation function of (4.1) denoted by $\rho_k^{(\text{PV})}$ forms

$$(4.3) \quad \rho_k^{(\text{PV})}(z_1, z_2, \dots, z_k) = \det(K_{\mathbb{D}}(z_i, z_j))_{i,j=1}^k$$

This remarkable determinantal point process allows us to simplify to compute some quantities such as the expected number of zeros and the number variance of zeros inside the unit disk \mathbb{D} . Indeed, for a domain $D \subset \mathbb{C}$, let f be a random variable on a probability space which takes values in the space of analytic functions on D , and we write the number of zeros for f on a domain D by

$$(4.4) \quad N_f(D) = \#\{z \in D : f(z) = 0\}.$$

If we consider a disk with radius r centered at the origin, we simply write

$$(4.5) \quad N_f(r) := N_f(\mathbb{D}_r) \# \{z \in \mathbb{D}_r : f(z) = 0\}, \quad \mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}.$$

Then, in [118], it was shown that

$$(4.6) \quad \mathbb{E} \binom{N_{f_{\text{PV}}}(r)}{k} = \frac{r^{k(k+1)}}{(1 - r^2)(1 - r^4) \cdots (1 - r^{2k})}.$$

In particular, we have

$$(4.7) \quad \mathbb{E} N_{f_{\text{PV}}}(r) = \frac{r^2}{1 - r^2}, \quad \text{Var } N_{f_{\text{PV}}}(r) = \frac{r^2}{1 - r^4}.$$

Also, since the one-point density of $\mathcal{Z}_{f_{\text{PV}}}$ by Theorem 4.1 is given by

$$(4.8) \quad \rho_1^{(\text{PV})}(z) = \frac{1}{\pi(1 - |z|^2)^2},$$

the zeros point process $\mathcal{Z}_{f_{\text{PV}}}$ of (4.1) can be regarded as the uniform distribution on the Poincaré hyperbolic disk. Although we do not mention the detailed properties of the determinantal point processes, for interested readers, we refer to [86] and [125]. Initiated by Peres and Virág' breakthrough work, studies for zeros of Gaussian

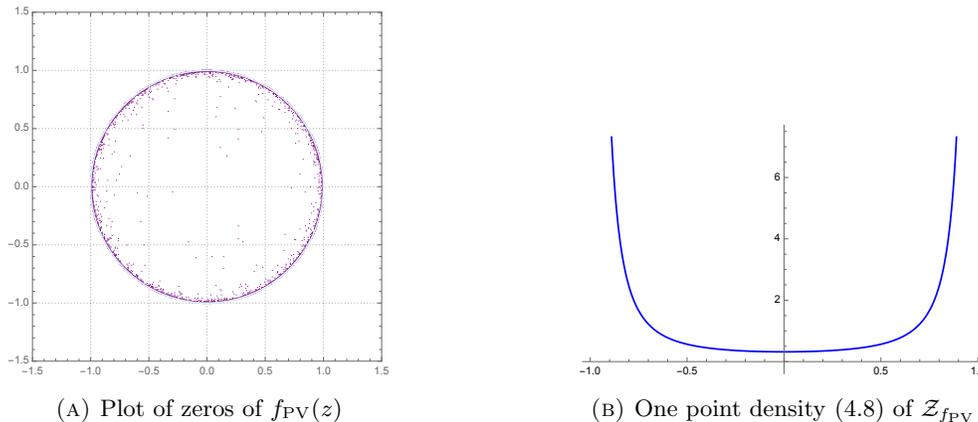


FIGURE 5. The figure (A) is the plot of zeros of f_{PV} approximated by random polynomial with degree 500 and sample times 20. The figure (B) is the graph of one point density (4.8) of $\mathcal{Z}_{f_{\text{PV}}}$ along $[-1, 1]$. Since the one-point density (4.8) is radial symmetric, the graph is symmetric.

analytic function have been drastically developed by many researchers. Here, Gaussian analytic function f , shortly denoted by GAF, is a complex Gaussian distribution on a probability space which takes values in the space of analytic functions on a domain D , and it is uniquely characterized by a covariance function

$$(4.9) \quad K_f(z, w) = \mathbb{E}[f(z)\overline{f(w)}].$$

For the detailed properties of GAF, we refer to [86]. Before we explain our contributions in this thesis, let us mention the recent developments for studies of zeros of Gaussian analytic functions, in particular, GAF defined on the unit disk.

In [86, Chapter 5], the one-parameter generalization of (4.1) was introduced:

$$(4.10) \quad f_L(z) = \sum_{k=0}^{\infty} \zeta_k \sqrt{\frac{\Gamma(L+k)}{\Gamma(L)\Gamma(k+1)}} z^k.$$

where $\{\zeta_k\}_{k=0}^{\infty}$ are i.i.d. standard complex Gaussian random variables. Then, the radius of convergence of (4.10) is also one, and (4.10) is also GAF defined on the unit disk \mathbb{D} . In particular, when $L = 1$, (4.10) agree with (4.1). In general, it was shown that the zeros point processes \mathcal{Z}_{f_L} except $L = 1$ can not form the determinantal point processes, see [86, Chapter 5, Subsection 5.1]. Although the zeros point processes of f_L are not determinantal point processes, we can still analyze the various properties of zeros point processes of (4.10) by using probabilistic and complex analysis arguments. Indeed,

Finally, we only mentioned the studies of zeros point processes of the hyperbolic type GAFs, but as in [86], the zeros point processes of entire GAFs and spherical GAFs have been also studied. Those topics are not dealt with in this review, but we refer to [86, Chapter 2] for interested readers.

4.2. Our contribution [115]. In the previous subsection 4.1, we reviewed the study by Peres and Virág and the recent developments. In any cases, the coefficients are assumed to be i.i.d. complex Gaussian random variables. As a natural question, what happens if we replace the i.i.d. complex Gaussian coefficients with dependent complex Gaussian random variables, in particular, a stationary complex Gaussian process coefficients? The one of our aim in this thesis is to answer to this question, and in fact, from the perspective of the asymptotic

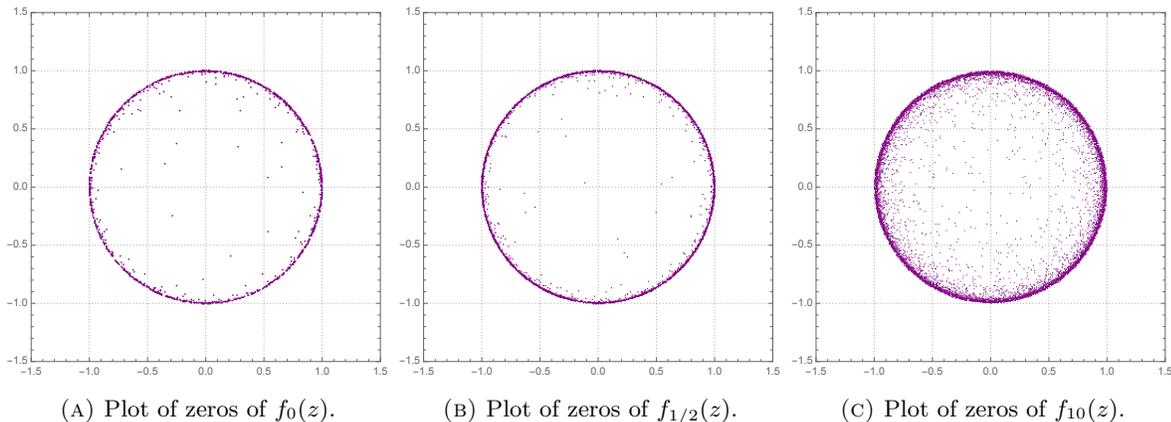


FIGURE 6. The figures (A), (B), and (C) are plots of zeros of $f_L(z)$ for $L = 0, 1/2, 10$. In all cases, f_L is approximated by a random polynomial with degree 500, and sample times are 20.

behaviors for the expected number of zeros, we will make the difference between zeros point processes of random power series with stationary complex Gaussian processes coefficients and (4.1) clear.

To describe our results, we now introduce our settings. Let $\Xi = \{\xi_k\}_{k \in \mathbb{Z}}$ be a stationary complex Gaussian process with mean 0, unit variance and covariance function given by

$$(4.11) \quad \mathbb{E}[\xi_j \overline{\xi_k}] = \gamma(k - j), \quad n, m \in \mathbb{Z},$$

where $\gamma(0) = 1$ (unit variance) and $\overline{\gamma(k)} = \gamma(-k)$. We study the following random power series

$$(4.12) \quad f_{\Xi}(z) = \sum_{k=0}^{\infty} \xi_k z^k.$$

To lighten the notation, we often omit the subscript Ξ in f_{Ξ} . Then, the covariance function of (4.12) is given by

$$(4.13) \quad K_f(z, w) = \mathbb{E}[f(z) \overline{f(w)}] = S(z, w) G_f(z, w),$$

where

$$(4.14) \quad S(z, w) = \frac{1}{1 - z \overline{w}}$$

is the Szegő kernel and $G_f(z, w)$ defined by

$$(4.15) \quad G_f(z, w) = 1 + G(z) + \overline{G(w)}, \quad G(z) = \sum_{k=1}^{\infty} \overline{\gamma(k)} z^k.$$

From the positive definiteness, since $|\gamma(k)| \leq \gamma(0) = 1$, the radius of convergence of $G(z)$ is more than or equal to 1. Here, the covariance function $\gamma(k)$ can be represented as

$$\gamma(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} d\Delta(\theta),$$

where $\Delta(\theta)$ is called the spectral function of Ξ , and $i = \sqrt{-1}$ is the pure imaginary number. When $\Delta(\theta)$ is absolutely continuous with respect to the Lebesgue measure, the density $\Delta'(\theta) = d\Delta(\theta)/d\theta$ is called the spectral density of Ξ (cf. [63]). Here, note that $G_f(e^{i\theta}, e^{i\theta})$ is the spectral density of the Gaussian process Ξ if $G(z)$ is analytic in a neighborhood of \mathbb{D} . When $\{\xi_k\}_{k \in \mathbb{Z}}$ are i.i.d., $\gamma(k) = \delta_{k,0}$, where $\delta_{k,m}$ is Kronecker's delta function, and $K_f(z, w) = S(z, w)$. Hence, (4.1) is contained in a class of (4.12). In this thesis, we focus on (4.12) with finitely dependent Gaussian coefficients, which will be explained later, and we compare the expected number of zeros of that GAF with zeros of (4.1).

We will firstly deal with the case of 2-dependent stationary complex Gaussian processes with covariance function

$$(4.16) \quad \gamma_{a,b}(k) = \begin{cases} 1 & \text{if } k = 0, \\ a & \text{if } |k| = 1, \\ b & \text{if } |k| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can verify that $\{\gamma_{a,b}(k)\}_{k \in \mathbb{Z}}$ is positive definite if and only if (a, b) is in the region $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ with

$$(4.17) \quad \begin{aligned} \mathcal{P}_1 &= \left\{ (a, b) \in \mathbb{R}^2 : \frac{a^2}{8} + \left(b - \frac{1}{4}\right)^2 \leq \frac{1}{16} \right\}, \\ \mathcal{P}_2 &= \left\{ (a, b) \in \mathbb{R}^2 : \frac{a^2}{8} + \left(b - \frac{1}{4}\right)^2 \geq \frac{1}{16}, |a| - \frac{1}{2} \leq |b| \leq \frac{1}{6} \right\}. \end{aligned}$$

See Figure 7 for the region of positive definiteness. We write the GAF $f_{a,b}$ associated with the covariance

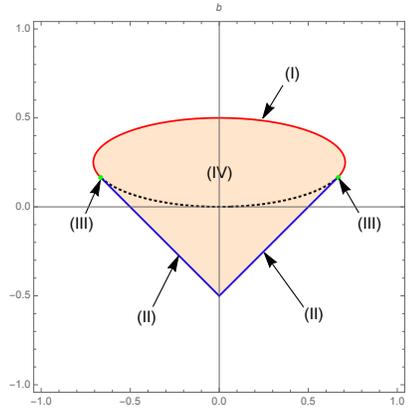


FIGURE 7. The region \mathcal{P} of positive definiteness of $\gamma(k)$ defined in (4.17). The red and black dashed ellipse are the boundaries of \mathcal{P}_1 , and the green points are $(a, b) = (\pm 2/3, 1/6)$. The blue line segments are $b = |a| - 1/2$ for $-1/2 \leq b \leq 1/6$. The similar figure can be seen in [31]. In fact, the region \mathcal{P} is equivalent to the invertibility conditions for moving boundary average processes (MA(2) processes).

function (4.16). Then, since the variance of ξ_k with the covariance function (4.16) is normalized to be 1, the radius of convergence of $f_{a,b}(z)$ is 1 a.s. for any $(a, b) \in \mathcal{P}$. Then, our first result is as follows:

Theorem 4.2. *Let $f_{a,b}$ be the GAF defined in (4.12) with the covariance function (4.16). Then, the asymptotic behaviors of the expected number of zeros of $f_{a,b}$ are as follows:*

(I) *If (a, b) satisfies $a^2/8 + (b - 1/4)^2 = 1/16$ and $1/6 < b \leq 1/2$, then*

$$(4.18) \quad \mathbb{E}N_{f_{a,b}}(r) = \frac{r^2}{1-r^2} - \sqrt{\frac{2b}{6b-1}} \frac{1}{(1-r^2)^{1/2}} + O(1), \quad \text{as } r \rightarrow 1.$$

(II) *If (a, b) satisfies $b = |a| - 1/2$ and $-1/2 \leq b < 1/6$, then*

$$(4.19) \quad \mathbb{E}N_{f_{a,b}}(r) = \frac{r^2}{1-r^2} - \frac{1}{2} \sqrt{\frac{1-2b}{1-6b}} \frac{1}{(1-r^2)^{1/2}} + O(1), \quad \text{as } r \rightarrow 1.$$

(III) *If $(a, b) = (\pm 2/3, 1/6)$, then*

$$(4.20) \quad \mathbb{E}N_{f_{a,b}}(r) = \frac{r^2}{1-r^2} - \frac{1}{2^{5/4}} \frac{1}{(1-r^2)^{3/4}} + O\left(\frac{1}{(1-r^2)^{1/4}}\right), \quad \text{as } r \rightarrow 1.$$

(IV) If (a, b) is in the interior of \mathcal{P} , then there exists a non-negative constant $C(a, b)$ such that

$$(4.21) \quad \mathbb{E}N_{f_{a,b}}(r) = \frac{r^2}{1-r^2} - C(a, b) + O(1-r^2), \quad \text{as } r \rightarrow 1.$$

The constant $C(a, b)$ is positive except for $(a, b) = (0, 0)$. The numbers (I)–(IV) correspond to those in Figure 7.

Since

$$\mathbb{E}N_{f_{0,0}}(r) = \mathbb{E}N_{f_{\text{PV}}}(r) = \frac{r^2}{1-r^2},$$

from Theorem 4.2, we can readily find that the expected number of zeros is less than that of (4.1) at least in the limit as $r \rightarrow 1$ for all cases. We can show the following stronger result.

Theorem 4.3. *Let f be a GAF defined in (4.12). Let $D \subset \mathbb{D}$ be a domain with smooth boundaries and $N_f(D)$ be the number of zeros of f inside D . Then, $\mathbb{E}N_f(D)$ is always less than or equal to $\mathbb{E}N_{f_{\text{PV}}}(D)$. Moreover, the equality holds for some domain D if and only if $f(z)$ is equal to f_{PV} in law.*

Remark 4.4. *We emphasize that our setting is different from [112]. In [112], they considered the Gaussian random power series with stationary complex Gaussian processes whose covariance function is the inverse covariance function of the original Gaussian processes coefficients. Translating their setting into our setting, their setting means that (4.15) has no zeros inside the unit disk. From such special setting, there exists a conformal map such that the zeros distribution of the random power series in [112] is equal to ones of Peres and Virág GAF (4.1) in law via the conformal map.*

As we already saw in the above, the asymptotic behavior at $(a, b) = (\pm 2/3, 1/6)$ corresponding to Theorem 4.2 (III) is special since $G_f(z, z)$ is the most degenerated in the sense that

$$G_f(z, z) = 1 \pm \frac{2}{3}(z + z^{-1}) + \frac{1}{6}(z^2 + z^{-2}) = \frac{1}{6}z^{-2}(z \pm 1)^4, \quad \text{for } z \in \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

The above $G_f(z, z)$ has the degenerated zero at $z = \mp 1$. Now, we shall generalize the above phenomena into the n -dependent stationary complex Gaussian process Ξ_n with the covariance function $\{\gamma_n(k)\}_{k \in \mathbb{Z}}$, which is the most degenerated in the sense above, that is to say,

$$(4.22) \quad \gamma_n(k) = \begin{cases} \binom{2n}{n}^{-1} \binom{2n}{n+k} & \text{if } |k| = 0, 1, 2, \dots, n, \\ 0 & \text{otherwise,} \end{cases}$$

which is normalized as $\gamma(0) = 1$. Then, we can easily find that

$$(4.23) \quad G_f(z, z) = \sum_{k=-n}^n \gamma_n(k) z^k = \binom{2n}{n}^{-1} z^{-n} (z+1)^{2n} \quad \text{for } z \in \partial\mathbb{D}.$$

Note that $z = -1$ is the zero of order $2n$. As in Figure 7, for this n -dependent Gaussian process $\Xi_n = \{\xi_k^{(n)}\}_{k \in \mathbb{Z}}$, we have the following moving-average representation:

$$\xi_k^{(n)} = \binom{2n}{n}^{-1/2} \sum_{j=0}^n \binom{n}{j} \zeta_{n-j}, \quad \text{for } k = 0, 1, 2, \dots,$$

where $\{\zeta_j\}_{j \in \mathbb{Z}}$ are i.i.d. standard complex Gaussian random variables. Then, we have the following asymptotics, which include Theorem 4.2 (III) as the spacial case $n = 2$.

Theorem 4.5. *Let $\gamma_n(k)$ be defined as (4.22) and $\Xi_n = \{\xi_k^{(n)}\}_{k \in \mathbb{Z}}$ be the stationary, centered, complex Gaussian process with the covariance function $\{\gamma_n(k)\}_{k \in \mathbb{Z}}$. Then, the expected number of zeros of the power series $f_n(z) = \sum_{k=0}^{\infty} \xi_k^{(n)} z^k$ within \mathbb{D}_r is given by*

$$(4.24) \quad \mathbb{E}N_{f_n}(r) = \frac{r^2}{1-r^2} - D_n(1-r^2)^{-\frac{2n-1}{2n}} + O((1-r^2)^{-\frac{2n-3}{2n}}), \quad \text{as } r \rightarrow 1,$$

where

$$(4.25) \quad D_n = \frac{1}{2n \sin \frac{\pi}{2n}} \binom{2(n-1)}{n-1}^{\frac{1}{2n}}.$$

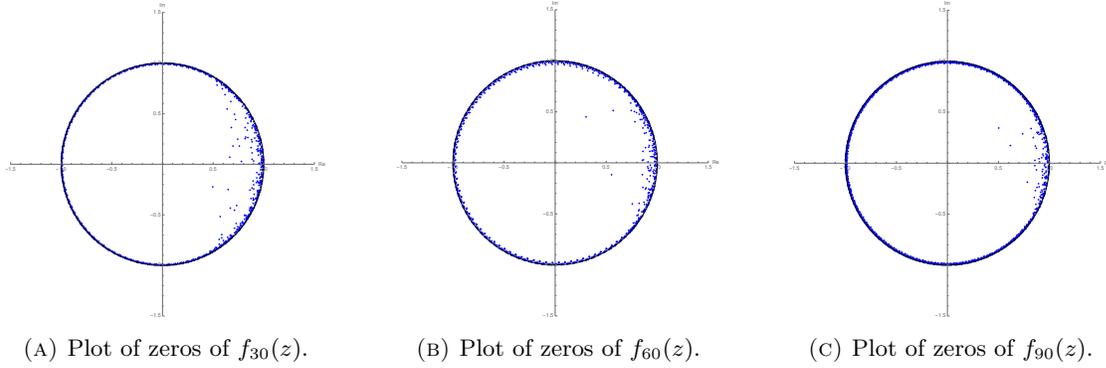


FIGURE 8. The figures (A), (B), and (C) are plots of zeros of $f_n(z)$ defined in Theorem 4.5.

Remark 4.6. *As we will see in the proof of Theorem 4.5, the sub-sub-leading order $(1 - r^2)^{-\frac{2n-2}{2n}}$ in (4.24) vanishes by a cancellation. Hence, the correct sub-sub-leading order is $O((1 - r^2)^{-\frac{2n-3}{2n}})$.*

As we will see in the proof of Theorem 4.2 and Theorem 4.5, the sub-leading order in the asymptotic expansion comes from the behavior of the zeros of $G_f(z, z)$. If $G_f(z, z)$ has a zero with multiplicity $2k$ on $\partial\mathbb{D}$, then the sub-leading order $(1 - r^2)^{-\frac{2k-1}{2k}}$ appears in the asymptotic expansion of $\mathbb{E}N_f(r)$ as $r \rightarrow 1$. Hence, the zeros of the spectral density with the most multiplicity determines the main sub-leading order term. Therefore, in general, we obtain the following result for finitely dependent stationary complex Gaussian process coefficients.

Corollary 4.7. *Let $\Xi = \{\xi_k\}_{k \in \mathbb{Z}}$ be the stationary, centered, finitely dependent complex Gaussian process. When the spectral density of Ξ has zeros θ_j with multiplicity $2k_j$ for $j = 1, 2, \dots, p$, we set $\alpha = (2k - 1)/(2k)$ with $k = \max_{1 \leq j \leq p} k_j$ and $\alpha = 0$ otherwise. Then, there exists a positive constant C_Ξ such that the expected number of zeros of the GAF f with coefficients Ξ within \mathbb{D}_r is given by*

$$(4.26) \quad \mathbb{E}N_f(r) = \frac{r^2}{1 - r^2} - C_\Xi(1 - r^2)^{-\alpha} + o((1 - r^2)^{-\alpha}), \quad \text{as } r \rightarrow 1.$$

For instance, the Gaussian process with the spectral density $G_f(z, z) = C_f \prod_{j=1}^p |z + a_j|^{2k_j}$ for $z, a_1, \dots, a_p \in \partial\mathbb{D}$ and $k_1, \dots, k_p \geq 1$ with the normalization constant C_f gives an example of the GAF described in Corollary 4.7.

Part 2. Proof of determinantal structure of the overlaps for induced Ginibre unitary ensemble

5. FINITE N -KERNEL: PROOF OF THE CASE OF IGINUE IN THEOREM 2.7

5.1. Proof of the case of IGINUE in Proposition 2.6.

5.1.1. *Moment method.* We regard complex variables z and \bar{z} as independent variables. Then, there exists a family of bi-orthogonal polynomials associated with the weight function (2.27) or (2.28) denoted by $\{P_k(\cdot|a, \bar{a}), Q_k(\cdot|a, \bar{a})\}_{k=0}^{\infty}$ on \mathbb{C} such that for $a \in \mathbb{C}$,

$$\langle P_j(\cdot|a, \bar{a}), Q_k(\cdot|a, \bar{a}) \rangle_{\omega_{g/s}} = \int_{\mathbb{C}} \overline{P_j(z|a, \bar{a})} Q_k(z|a, \bar{a}) \omega_{g/s}(z, \bar{z}|a, \bar{a}) dA(z) = h_j \delta_{j,k},$$

where h_j is the norming constant for (2.27) or (2.28). From the elementary linear algebra, we have

$$\prod_{2 \leq j < k \leq N} |z_j - z_k|^2 \prod_{j=2}^N \omega_{g/s}(z_j, \bar{z}_j | z_1, \bar{z}_1) = \prod_{j=0}^{N-2} h_j \times \det_{2 \leq j, k \leq N} \left(K_{1,1,(g/s)}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_1) \right),$$

where $K_{1,1,(g/s)}^{(N-1)}$ is an integral kernel defined by

$$K_{1,1,(g/s)}^{(N)}(z, \bar{z}, w, \bar{w} | a, \bar{a}) = \sum_{k=0}^{N-1} \frac{\overline{P_k(z|a, \bar{a})} Q_k(w|a, \bar{a})}{h_k} \omega_{g/s}(z, \bar{z} | a, \bar{a}).$$

Here, it would be convenient to define the reduced polynomial kernel $\mathcal{K}_{1,1,(g/s)}^{(N)}$ via

$$K_{1,1,(g/s)}^{(N)}(z, \bar{z}, w, \bar{w} | a, \bar{a}) = \mathcal{K}_{1,1,(g/s)}^{(N)}(\bar{z}, w | a, \bar{a}) \omega(z, \bar{z} | a, \bar{a}),$$

where

$$\mathcal{K}_{1,1,(g/s)}^{(N)}(\bar{z}, w | a, \bar{a}) = \sum_{k=0}^{N-1} \frac{\overline{P_k(z|a, \bar{a})} Q_k(w|a, \bar{a})}{h_k}.$$

$\mathcal{K}_{1,1,(g/s)}^{(N)}(\bar{z}, w | a, \bar{a})$ correspond to (2.44) or (2.49). Now, we shall prove the case of IGINUE in Proposition 2.6.

Proof of the case of IGINUE in Proposition 2.6. First, we define the moment matrix M with entries $M_{i,j} := \langle z^i, z^j \rangle_{\omega_{(g)}}$ with respect to the inner product (2.27). Then, we have

$$M_{i,j} = \Gamma(i + \alpha + 1) \{ \delta_{i,j} (\alpha + i + 2 + |\lambda|^2) - \delta_{i+1,j} \bar{\lambda} (i + \alpha + 1) - \delta_{i,j+1} \lambda \} = \Gamma(i + \alpha + 1) \mu_{i,j},$$

where

$$\mu_{i,j} = \delta_{i,j} (\alpha + i + 2 + |\lambda|^2) - \delta_{i+1,j} \bar{\lambda} (i + \alpha + 1) - \delta_{i,j+1} \lambda.$$

Similar to [13], we work on the LDU decomposition for the matrix $\mu = (\mu_{i,j})_{i,j}$. When we have the LDU decomposition as $\mu = LDU$, where

$$(5.1) \quad D_{p,q} = d_p \delta_{p,q}, \quad L_{p,q} = \delta_{p,q} + \ell_p \delta_{p,q+1}, \quad U_{p,q} = \delta_{p,q} + u_q \delta_{q,p+1}, \quad \text{for } p, q \geq 0,$$

then we see that

$$(5.2) \quad d_p = -d_{p-1} \ell_{p-1} u_p \mathbf{1}_{p \geq 1} + 2 + \alpha + \lambda \bar{\lambda} + p, \quad u_{p+1} = -\frac{\lambda(p + \alpha + 1)}{d_p}, \quad \ell_{p+1} = -\frac{\bar{\lambda}}{d_p}.$$

Hence, we have the following recurrence equation

$$d_p = 2 + \alpha + \lambda \bar{\lambda} + p - \frac{\lambda \bar{\lambda} (p + \alpha)}{d_{p-1}}, \quad p \geq 1, \quad d_0 = \alpha + 2 + \lambda \bar{\lambda}.$$

Let $x = |\lambda|^2$ and $d_p = \frac{r_{p+1}}{r_p}$ with $r_0 = 1$. Then, we can express

$$r_{p+1} + (p + \alpha) x r_{p-1} = (2 + \alpha + x + p) r_p, \quad r_1 = d_0 = \alpha + 2 + x.$$

From an induction argument, we find that the unique solution of the above recurrence equation is given by

$$r_p = \mathcal{D}_{p-1} = \Gamma(p + \alpha + 1) h_p^{(\alpha)}(x),$$

where

$$h_p^{(\alpha)}(x) = \frac{x-\alpha}{x} \sum_{k=0}^p \frac{p+1-k}{\Gamma(k+\alpha+1)} x^k + \frac{\alpha(p+1)}{\Gamma(\alpha+1)} \frac{1}{x} \quad \text{with } h_0^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha+1)}.$$

Before completing the proof, it is convenient to simplify $h_p^{(\alpha)}(x)$. Note that

$$\begin{aligned} h_p^{(\alpha)}(x) &= \frac{x-\alpha}{x} (p+1) \sum_{k=0}^p \frac{x^k}{\Gamma(k+\alpha+1)} - (x-\alpha) \sum_{k=0}^p \frac{kx^{k-1}}{\Gamma(k+\alpha+1)} + \frac{1}{x} \frac{\alpha(p+1)}{\Gamma(\alpha+1)} \\ &= \frac{x-\alpha}{x} \left\{ (p+1)e_p^{(\alpha)}(x) - xe_{p-1}^{(\alpha)}(x) + \alpha xe_{p-1}^{(\alpha+1)}(x) + \frac{\alpha}{x-\alpha} \frac{p+1}{\Gamma(\alpha+1)} \right\}. \end{aligned}$$

Here, recall that $e_{N-1}^{(\alpha)}(x)$ can be written in terms of the incomplete Gamma function:

$$e_{N-1}^{(\alpha)}(z) = \frac{e^z}{z^\alpha} (Q(N+\alpha, z) - Q(\alpha, z)).$$

Form this, we have that

$$h_N^{(\alpha)}(x) = \frac{x-\alpha}{x} \left((N+\alpha+1)e_N^{(\alpha)}(x) - xe_{N-1}^{(\alpha)}(x) + \frac{\alpha}{\Gamma(\alpha+1)} \frac{N+\alpha+1-x}{x-\alpha} \right) = f_N^{(\alpha)}(x),$$

where $f_N^{(\alpha)}(x)$ is defined as (2.34). Coming back to the LDU decomposition step and from the definition of d_p , we obtain

$$d_p = \frac{r_{p+1}}{r_p} = \frac{\Gamma(p+\alpha+2)f_{p+1}^{(\alpha)}(x)}{\Gamma(p+\alpha+1)f_p^{(\alpha)}(x)} = (p+\alpha+1) \frac{f_{p+1}^{(\alpha)}(x)}{f_p^{(\alpha)}(x)}.$$

By LDU decomposition, we have $\langle P_k(\cdot|\lambda, \bar{\lambda}), Q_\ell(\cdot|\lambda, \bar{\lambda}) \rangle_{\omega_{(\mathfrak{g})}} = D_{kk} \delta_{k,\ell}$, $P_k(z|\lambda, \bar{\lambda}) = \sum_{m=0}^k (\bar{L}^{-1})_{km} z^m$, and $Q_k(z|\lambda, \bar{\lambda}) = \sum_{m=0}^k z^m (U^{-1})_{mk}$ for $k \in \mathbb{Z}_{\geq 0}$. Then, we have $\mathcal{K}^{(N)}(z, \bar{z}|\lambda, \bar{\lambda}) = \sum_{i,j=0}^{N-1} z^i C_{i,j}^{(N-1)} \bar{z}^j$, where $C_{i,j}^{(N)} = \sum_{k=0}^N (U^{-1})_{ik} \frac{1}{D_{kk}} (L^{-1})_{kj}$ for $i, j \geq 0$. From (5.1) and (5.2), multiplying the diagonal matrix $\text{diag}(\Gamma(i+\alpha+1))$ by $\mu = LDU$ and updating notations, we have

$$L_{pm} = \delta_{pm} - \bar{\lambda} \frac{f_{p-1}^{(\alpha)}(x)}{f_p^{(\alpha)}(x)} \delta_{p,m+1}, \quad U_{mq} = \delta_{mq} - \lambda \frac{f_{m-1}^{(\alpha)}(x)}{f_m^{(\alpha)}(x)} \delta_{q,m+1}, \quad (m, q \geq 0),$$

and

$$D_m = \langle P_m, Q_m \rangle_{\omega_{(\mathfrak{g})}} = \Gamma(m+\alpha+2) \frac{f_{m+1}^{(\alpha)}(x)}{f_m^{(\alpha)}(x)}, \quad m \geq 0.$$

Note that

$$\prod_{q=0}^{N-2} \langle P_q, Q_q \rangle_{\omega_{(\mathfrak{g})}} = \prod_{m=0}^{N-2} \Gamma(m+\alpha+2) \frac{f_{m+1}^{(\alpha)}(x)}{f_m^{(\alpha)}(x)} = \prod_{m=1}^{N-1} \Gamma(m+\alpha+1) \cdot \frac{f_{N-1}^{(\alpha)}(x)}{f_0^{(\alpha)}(x)},$$

and hence we have

$$\frac{N!}{Z_N} \prod_{q=0}^{N-2} \langle P_q, Q_q \rangle_{\omega_{(\mathfrak{g})}} \cdot x^\alpha e^{-x} = \frac{N!}{N!} \prod_{j=1}^N \frac{1}{\Gamma(j+\alpha)} \prod_{m=1}^{N-1} \Gamma(m+\alpha+1) \cdot \frac{f_{N-1}^{(\alpha)}(x)}{f_0^{(\alpha)}(x)} \cdot x^\alpha e^{-x} = f_{N-1}^{(\alpha)}(x) x^\alpha e^{-x},$$

where we used the fact $f_0^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha+1)}$. We recall that the inverse matrix of the lower triangular matrix L is a lower triangular matrix and the inverse matrix of the upper triangular matrix U is an upper triangular matrix. As a consequence, we see that

$$(L^{-1})_{pq} = \begin{cases} 0 & q > p, \\ 1 & q = p, \\ (\bar{\lambda})^{p-q} \frac{f_q^{(\alpha)}(x)}{f_p^{(\alpha)}(x)} & q < p, \end{cases} \quad (U^{-1})_{pq} = \begin{cases} (\lambda)^{q-p} \frac{f_p^{(\alpha)}(x)}{f_q^{(\alpha)}(x)} & q > p, \\ 1 & q = p, \\ 0 & q < p. \end{cases}$$

Finally, we put

$$G_{N-1}^{(\alpha)}(x|y, z) = \sum_{n,m=0}^{N-1} f_n^{(\alpha)}(x) f_m^{(\alpha)}(x) y^n z^m \sum_{k=\max\{n,m\}}^{N-1} \frac{1}{\Gamma(k+\alpha+2)} \frac{x^k}{f_k^{(\alpha)}(x) f_{k+1}^{(\alpha)}(x)},$$

then this completes the proof of the case of IGINUE in Proposition 2.6. \square

Remark 5.1. If $\alpha = 0$, then

$$r_p = \Gamma(p+1) f_p^{(0)}(x) = p! \sum_{k=0}^p \frac{p+1-k}{k!} x^k = (p+1)! e_p(x) - p! x e_{p-1}(x),$$

where $e_p(x) = \sum_{k=0}^p \frac{x^k}{k!}$. This is consistent with the Ginibre unitary ensemble case [13].

Remark 5.2. From the proof of the case in Proposition 2.6, we find that the planar orthogonal polynomials $\{P_k(\cdot|\lambda, \bar{\lambda})\}_k$ with respect to the weight function (2.27) are given by

$$(5.3) \quad P_k(z|\lambda, \bar{\lambda}) = \sum_{j=0}^k \lambda^{k-j} \frac{f_j^{(\alpha)}(\lambda \bar{\lambda})}{f_k^{(\alpha)}(\lambda \bar{\lambda})} z^j.$$

Then, we can confirm that (5.3) satisfies the non-standard three-term recurrence in the sense of [42, Remark 1.3]. More precisely, we have

$$(5.4) \quad z P_k(z|\lambda, \bar{\lambda}) = P_{k+1}(z|\lambda, \bar{\lambda}) + b_k P_k(z|\lambda, \bar{\lambda}) + c_k z P_{k-1}(z|\lambda, \bar{\lambda}),$$

where

$$(5.5) \quad b_k = \bar{\lambda} \frac{f_k^{(\alpha)}(\lambda \bar{\lambda})}{f_{k+1}^{(\alpha)}(\lambda \bar{\lambda})} \quad c_k = -\bar{\lambda} \frac{f_{k-1}^{(\alpha)}(\lambda \bar{\lambda})}{f_k^{(\alpha)}(\lambda \bar{\lambda})}.$$

It would be interesting to study a differential equation of the finite N -kernel (2.44) based on (5.3), and we expect that the corresponding differential equation would satisfy a second order differential equation. Indeed, as the simplest case, we now consider $\lambda = 0$ case here. Let us denote the finite N -kernel associated with (2.27) with $u = v = 0$ by $\mathcal{K}_N(z, w|0)$. Then, it is easy to see that

$$\mathcal{K}_N^{(\alpha)}(z, w|0) = \sum_{k=0}^{N-1} \frac{(z\bar{w})^k}{(k+\alpha+2)\Gamma(k+\alpha+2)}.$$

We write

$$\widehat{\mathcal{K}}_N^{(\alpha)}(z, w|0) = (z\bar{w})^{\alpha+2} e^{-z\bar{w}} \mathcal{K}_N^{(\alpha)}(z, w|0).$$

By differentiating the above with respect to z , we can derive

$$\left[\partial_z^2 + \frac{z\bar{w}-1}{z} \partial_z - \frac{\bar{w}}{z} \right] \widehat{\mathcal{K}}_N^{(\alpha)}(z, w|0) = -\bar{w}^2 e^{-z\bar{w}} \left(\frac{(z\bar{w})^{N+\alpha}}{\Gamma(N+\alpha)} - \frac{(z\bar{w})^\alpha}{\Gamma(\alpha)} \right).$$

It would require more tasks to generalize the general case.

5.2. Simplification step. In this subsection, we complete the proof of IGINUE case of Theorem 2.7. To this end, firstly, note that (2.41) can be rewritten as

$$G_{N-1}^{(\alpha)}(x|y, z) = \sum_{n,m=0}^{N-1} f_n^{(\alpha)}(x) f_m^{(\alpha)}(x) y^n z^m \left(\Phi_{N-1}^{(\alpha)}(x) - \Phi_{\max\{n,m\}-1}^{(\alpha)}(x) \right),$$

where

$$(5.6) \quad \Phi_n^{(\alpha)}(x) := \sum_{k=0}^n \frac{1}{\Gamma(k+\alpha+2)} \frac{x^k}{f_k^{(\alpha)}(x) f_{k+1}^{(\alpha)}(x)}, \quad \Phi_{-1}^{(\alpha)}(x) = 0.$$

Lemma 5.3. (5.6) can be written as

$$(5.7) \quad \Phi_n^{(\alpha)}(x) = \Gamma(\alpha+1) \frac{x-(\alpha+1)}{(x-\alpha)x} + \frac{1}{x^2 f_{n+1}^{(\alpha)}(x)} \frac{x(n+\alpha+2-x)}{x-\alpha}.$$

Proof. The proof is done by induction. For fixed x , we have $\Phi_{n+1}^{(\alpha)}(x) = \Phi_n^{(\alpha)}(x) + \frac{1}{\Gamma(n+\alpha+3)} \frac{x^{n+1}}{f_{n+1}^{(\alpha)}(x)f_{n+2}^{(\alpha)}(x)}$ with $\Phi_0(x) = \frac{1}{x+\alpha+2}$. From $f_1^{(\alpha)}(x) = \frac{x+\alpha+2}{\Gamma(\alpha+2)}$, the initial condition is satisfied. Now, we assume that (5.7) holds for $n \in \mathbb{N}$. Then, we have

$$\Phi_{n+1}^{(\alpha)}(x) = \Gamma(\alpha+1) \frac{x - (\alpha+1)}{(x-\alpha)x} + \frac{1}{x^2 f_{n+1}^{(\alpha)}(x) f_{n+2}^{(\alpha)}(x)} \left\{ \frac{x(n+\alpha+2-x)f_{n+2}^{(\alpha)}(x)}{x-\alpha} + \frac{x^{n+3}}{\Gamma(n+\alpha+3)} \right\}.$$

Here, it is easy to see that

$$\frac{x(n+\alpha+2-x)f_{n+2}^{(\alpha)}(x)}{x-\alpha} + \frac{x^{n+3}}{\Gamma(n+\alpha+3)} = \frac{x(n+\alpha+3-x)}{x-\alpha} f_{n+1}^{(\alpha)}(x),$$

which completes the proof. \square

Remark 5.4. If $\alpha = 0$, then we clearly have $\Phi_n(x) := \Phi_n^{(0)}(x) = \frac{x-1}{x^2} + \frac{1}{x^2 f_{n+1}(x)}(n+2-x)$, which is consistent with [13].

Combining (5.7) with (2.41), we have

$$\begin{aligned} & G_{N-1}^{(\alpha)}(x|y, z) \\ &= \frac{1}{x^2 f_N^{(\alpha)}(x)} \frac{x(N+\alpha+1-x)}{x-\alpha} \mu_{N-1}^{(\alpha)}(x, y) \mu_{N-1}^{(\alpha)}(x, z) + \frac{1}{x^2} \frac{x(x-\alpha-1-\omega\partial_\omega)}{x-\alpha} \sum_{n=0}^{N-1} f_n^{(\alpha)}(x) \omega^n \Big|_{\omega=yz} \\ &+ \frac{1}{x^2} \frac{x(x-\alpha-1-z\partial_z)}{x-\alpha} \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} f_n^{(\alpha)}(x) y^n z^m + \frac{1}{x^2} \frac{x(x-\alpha-1-y\partial_y)}{x-\alpha} \sum_{m=0}^{N-1} \sum_{n=m+1}^{N-1} f_m^{(\alpha)}(x) z^m y^n, \end{aligned}$$

where we defined

$$(5.8) \quad \mu_n^{(\alpha)}(x, y) = \sum_{k=0}^n f_k^{(\alpha)}(x) y^k, \quad n \in \mathbb{N}.$$

5.2.1. *Complete the proof of the case of IGINUE in Theorem 2.7.* Now, we are ready to prove IGINUE case in Theorem 2.7.

Proof of the case of IGINUE case in Theorem 2.7. By elementary but involved computations, we have the following identities:

$$\sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} f_n^{(\alpha)}(x) y^n z^m = \frac{z}{1-z} \mu_{N-1}^{(\alpha)}(x, yz) - \frac{z^N}{1-z} \mu_{N-1}^{(\alpha)}(x, y),$$

where

$$(5.9) \quad \begin{aligned} & \mu_{N-1}^{(\alpha)}(x, t) \\ &= \frac{x-\alpha}{x} \left\{ \frac{\mathbf{e}_{N-1}^{(\alpha)}(x\omega|x)}{(1-t)^2} - \frac{1}{1-t} \frac{(xt)^N}{\Gamma(N+\alpha)} - \frac{t^N(N+\alpha+1-x-(N+\alpha-x)t)}{(1-t)^2} \mathbf{e}_{N-1}^{(\alpha)}(x|x) \right\}, \end{aligned}$$

$$(5.10) \quad \begin{aligned} & \partial_t \mu_{N-1}^{(\alpha)}(x, t) \\ &= \left(\frac{N}{t} + \frac{2}{1-t} \right) \mu_{N-1}^{(\alpha)}(x, t) - \frac{x-\alpha}{x} \frac{N+\alpha-x}{t(1-t)^2} e_{N-1}^{(\alpha)}(xt) + \frac{x-\alpha}{x} \frac{t^N(N+\alpha-x)e_{N-1}^{(\alpha)}(x)}{(1-t)^2} \\ & - \frac{x-\alpha}{x} \frac{1}{t(1-t)^2} \left\{ \frac{(xt)^N(1-t)}{\Gamma(N+\alpha)} + \frac{\alpha}{\Gamma(\alpha+1)} \frac{(N+\alpha-x)(1-t^{N+1})}{x-\alpha} \right\}, \end{aligned}$$

and hence, we have

$$\begin{aligned}
& x^2 G_{N-1}^{(\alpha)}(x|y, z) \\
&= \frac{x}{x-\alpha} \left\{ \frac{(N+\alpha+1-x)}{f_N^{(\alpha)}(x)} \mu_{N-1}^{(\alpha)}(x, y) \mu_{N-1}^{(\alpha)}(x, z) \right. \\
(5.11) \quad &+ (x-\alpha-1) \left(\frac{1-yz}{(1-y)(1-z)} \mu_{N-1}^{(\alpha)}(x, yz) - \frac{y^N}{1-y} \mu_{N-1}^{(\alpha)}(x, z) - \frac{z^N}{1-z} \mu_{N-1}^{(\alpha)}(x, y) \right) \\
&+ \frac{Nz^N - (N-1)z^{N+1}}{(1-z)^2} \mu_{N-1}^{(\alpha)}(x, y) + \frac{Ny^N - (N-1)y^{N+1}}{(1-y)^2} \mu_{N-1}^{(\alpha)}(x, z) \\
&\left. - \frac{1-yz}{(1-y)(1-z)} t \partial_t \mu_{N-1}^{(\alpha)}(x, t) \Big|_{t=yz} - \left(\frac{y}{(1-y)^2} + \frac{z}{(1-z)^2} \right) \mu_{N-1}^{(\alpha)}(x, yz) \right\}.
\end{aligned}$$

Notice also that

$$(5.12) \quad \left(\frac{Nz^N - (N-1)z^{N+1}}{(1-z)^2} - (x-\alpha-1) \frac{z^N}{1-z} \right) \mu_{N-1}^{(\alpha)}(x, y) = z^N \left(\frac{N+\alpha-x}{1-z} + \frac{1}{(1-z)^2} \right) \mu_{N-1}^{(\alpha)}(x, y).$$

Using (5.9), (5.10), (5.11), (5.12), and grouping by the denominators, we have

$$x(x-\alpha)G_{N-1}^{(\alpha)}(x|y, z) = \frac{T_A^{(\alpha)}(x, y, z)}{(1-y)^2(1-z)^2} + \frac{T_B^{(\alpha)}(x, y, z)}{(1-y)(1-z)} + \frac{T_C^{(\alpha)}(x, y, z)}{(1-y)^2(1-z)} + \frac{T'_C^{(\alpha)}(x, z, y)}{(1-y)(1-z)^2},$$

where

$$\begin{aligned}
\left(\frac{x}{x-\alpha} \right)^2 f_N^{(\alpha)}(x) T_A^{(\alpha)}(x, y, z) &= (N+\alpha+1) \left\{ \mathbf{e}_N^{(\alpha)}(xy|x) \mathbf{e}_N^{(\alpha)}(xz|x) - \mathbf{e}_{N-1}^{(\alpha)}(xyz|x) \mathbf{e}_{N-1}^{(\alpha)}(x|x) \right\} \\
&\quad - x \left\{ \mathbf{e}_{N-1}^{(\alpha)}(xy|x) \mathbf{e}_{N-1}^{(\alpha)}(xz|x) - \mathbf{e}_{N-1}^{(\alpha)}(xyz|x) \mathbf{e}_{N-1}^{(\alpha)}(x|x) \right\}, \\
\left(\frac{x}{x-\alpha} \right)^2 f_N^{(\alpha)}(x) T_B^{(\alpha)}(x, y, z) &= x \frac{(xy)^N (xz)^N}{\Gamma(N+\alpha+1)\Gamma(N+\alpha)} + x(N+\alpha-x) \mathbf{e}_{N-1}^{(\alpha)}(x|x) \frac{(xyz)^N}{\Gamma(N+\alpha+1)} \\
&\quad + x \mathbf{e}_{N-1}^{(\alpha)}(xyz) \left\{ (N+\alpha+1-x) \mathbf{e}_{N-1}^{(\alpha)}(x|x) + \frac{(N+\alpha+1)x^N}{\Gamma(N+\alpha+1)} \right\},
\end{aligned}$$

and

$$\left(\frac{x}{x-\alpha} \right)^2 f_N^{(\alpha)}(x) T_C^{(\alpha)}(x, y, z) = x \frac{(xyz)^N}{\Gamma(N+\alpha+1)} \mathbf{e}_{N-1}^{(\alpha)}(x|x) - x \frac{(xz)^N}{\Gamma(N+\alpha+1)} \mathbf{e}_{N-1}^{(\alpha)}(xy|x).$$

Then, we obtain

$$\begin{aligned}
& x^2 \frac{x}{x-\alpha} f_N^{(\alpha)}(x) G_{N-1}^{(\alpha)}(x|y, z) \\
&= \frac{(N+\alpha+1)W_N^{(\alpha)}(x, y, z) - xW_{N-1}^{(\alpha)}(x, y, z)}{(1-y)^2(1-z)^2} + x \frac{(xyz)^N \mathbf{e}_{N-1}^{(\alpha)}(x|x) - (xz)^N \mathbf{e}_{N-1}^{(\alpha)}(xy|x)}{\Gamma(N+\alpha+1)(1-y)^2(1-z)} \\
&+ x \frac{(xyz)^N \mathbf{e}_{N-1}^{(\alpha)}(x|x) - (xy)^N \mathbf{e}_{N-1}^{(\alpha)}(xz|x)}{\Gamma(N+\alpha+1)(1-y)(1-z)^2} - x \frac{(xyz)^N}{\Gamma(N+\alpha+1)} \frac{\mathbf{e}_N^{(\alpha)}(x|x) + x \mathbf{e}_{N-1}^{(\alpha)}(x|x)}{(1-y)(1-z)} \\
&- \frac{1}{\Gamma(\alpha)} \frac{x}{x-\alpha} \frac{(N+\alpha+1) \mathbf{e}_N^{(\alpha)}(x|x) - x \mathbf{e}_{N-1}^{(\alpha)}(x|x)}{(1-y)(1-z)},
\end{aligned}$$

where we set

$$W_N^{(\alpha)}(x, y, z) = \mathbf{e}_N^{(\alpha)}(xy|x) \mathbf{e}_N^{(\alpha)}(xz|x) - (1-x(1-y)(1-z)) \mathbf{e}_N^{(\alpha)}(xyz|x) \mathbf{e}_N^{(\alpha)}(x|x).$$

Using $e_{n-1}^{(\alpha)}(x) = e_n^{(\alpha)}(x) - \frac{x^n}{\Gamma(n+\alpha+1)}$, we see that

$$W_N^{(\alpha)}(x, y, z) = W_{N+1}^{(\alpha)}(x, y, z) - \frac{(xz)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(xy|x)}{\Gamma(N+\alpha+2)} - \frac{(xy)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(xz|x)}{\Gamma(N+\alpha+2)} + \frac{x(1-y)(1-z)(x^2yz)^{N+1}}{\Gamma(N+\alpha+2)^2}$$

$$+ (1-x)(1-y)(1-z) \left\{ \frac{x^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(xyz|x)}{\Gamma(N+\alpha+2)} + \frac{(xyz)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(x|x)}{\Gamma(N+\alpha+2)} \right\}.$$

Therefore, we can find that

$$\begin{aligned} & x^2 \frac{x}{x-\alpha} f_N^{(\alpha)}(x) G_{N-1}^{(\alpha)}(x|y, z) \\ &= \frac{(N+\alpha+1)W_{N+1}^{(\alpha)}(x, y, z) - xW_N^{(\alpha)}(x, y, z)}{(1-y)^2(1-z)^2} - \frac{x}{(1-y)(1-z)} \frac{(xyz)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(x|x)}{\Gamma(N+\alpha+1)} \\ & - \frac{1}{\Gamma(\alpha)} \frac{x}{x-\alpha} \frac{(N+\alpha+1) \mathbf{e}_N^{(\alpha)}(x|x) - x \mathbf{e}_{N-1}^{(\alpha)}(x|x)}{(1-y)(1-z)}. \end{aligned}$$

Here, the third term can be written as

$$- \frac{1}{\Gamma(\alpha)} \frac{x}{x-\alpha} \frac{(N+\alpha+1) \mathbf{e}_N^{(\alpha)}(x|x) - x \mathbf{e}_{N-1}^{(\alpha)}(x|x)}{(1-y)(1-z)} = - \frac{1}{\Gamma(\alpha)} \left(\frac{x}{x-\alpha} \right)^2 \frac{f_N^{(\alpha)}(x)}{(1-y)(1-z)}.$$

Therefore, we have

$$\begin{aligned} G_{N-1}^{(\alpha)}(x|y, z) &= \frac{(N+\alpha+1)W_{N+1}^{(\alpha)}(x, y, z) - xW_N^{(\alpha)}(x, y, z)}{x^2(1-y)^2(1-z)^2 g_N^{(\alpha)}(x)} \\ & - \frac{1}{x(1-y)(1-z) g_N^{(\alpha)}(x)} \frac{(xyz)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(x|x)}{\Gamma(N+\alpha+1)} - \frac{1}{\Gamma(\alpha)} \frac{1}{x(x-\alpha)} \frac{1}{(1-y)(1-z)}. \end{aligned}$$

Letting $x = \lambda \bar{\lambda}$, $y = \frac{\bar{z}}{\lambda}$, $z = \frac{w}{\lambda}$, then we have

$$\begin{aligned} G_{N-1}^{(\alpha)} \left(\lambda \bar{\lambda} \middle| \frac{\bar{z}}{\lambda}, \frac{w}{\lambda} \right) &= \frac{\lambda \bar{\lambda} - \alpha}{\lambda \bar{\lambda}} \frac{(N+\alpha+1) \mathfrak{W}_{N+1}^{(\alpha)}(\bar{z}, w|\lambda, \bar{\lambda}) - \lambda \bar{\lambda} \mathfrak{W}_N^{(\alpha)}(\bar{z}, w|\lambda, \bar{\lambda})}{(\bar{z} - \bar{\lambda})^2 (w - \lambda)^2 f_N(\lambda \bar{\lambda}) \varpi(\bar{z}, w) \varpi(\bar{\lambda}, \lambda)} \\ & - \frac{\lambda \bar{\lambda} - \alpha}{\lambda \bar{\lambda}} \frac{(\bar{z}w)^{N+1} \mathbf{e}_{N+1}^{(\alpha)}(\lambda \bar{\lambda}|\lambda \bar{\lambda})}{\Gamma(N+\alpha+1)(\bar{z} - \bar{\lambda})(w - \lambda) f_N^{(\alpha)}(\lambda \bar{\lambda})} - \frac{1}{\Gamma(\alpha)} \frac{1}{(\lambda \bar{\lambda} - \alpha)(\bar{z} - \bar{\lambda})(w - \lambda)} \\ & = \mathfrak{H}_N^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda) + \mathfrak{F}_N^{(\alpha)}(\bar{z}, w|\bar{\lambda}, \lambda). \end{aligned}$$

This completes the proof. \square

6. SCALING LIMITS: PROOF OF THE CASE OF IGINUE IN THEOREM 2.10

In below all proofs, a constant $\epsilon > 0$ is independent of N , but it may be different in each line. It does not affect the our desired result. Also, note that a co-cycle factor of kernel the $K(x, y)$, $K(x, y) \mapsto \phi(x)K(x, y)\phi^{-1}(y)$, does not change the value of the determinant. Hence, for the simplicity of the notation, we omit a co-cycle factor. Indeed, from the expression of (2.52), (2.53) and (2.54), we can find that a co-cycle factor does not affect the resulting value, and we do not mention that in each line.

6.1. Proof of the case of IGINUE in the strongly non-unitary regime in Theorem 2.10. We fix $a_N = 1$ and $\alpha = b_N = Nb$ in this subsection.

Proof of the case of IGINUE in Theorem 2.10 in the strongly non-unitary regime: bulk case. For $p \in \text{int}(S)$, let $z = \sqrt{N}p + \zeta$, $w = \sqrt{N}p + \eta$, $\lambda = \sqrt{N}p + \chi$. It is straightforward to see that

$$\mathbf{e}_N^{(b_N)}(\bar{z}w|\lambda \bar{\lambda}) \widehat{\omega}_{b_N}(\bar{z}, w) = e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \bar{\zeta}\eta} + O(e^{-\epsilon N}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Therefore, we obtain

$$f_N^{(b_N)}(\lambda \bar{\lambda}) \widehat{\omega}_{b_N}(\lambda, \bar{\lambda}) = N \frac{(|p|^2 - b)(1 + b - |p|^2)}{|p|^2} (1 + O(e^{-\epsilon N})),$$

uniformly for χ in a compact subset of \mathbb{C} . With help of these asymptotic expansions, we have

$$(6.1) \quad \begin{aligned} & \mathfrak{H}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda})\omega_{b_N}(\bar{z}, z|\lambda, \bar{\lambda}) \\ &= \frac{d}{du} \left(\frac{e^u - 1}{u} \right) \Big|_{u=(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\bar{\zeta}\eta} e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} (1 + O(N^{-1})) \\ &= c(\zeta, \eta, \chi) \mathcal{K}_{1,1}^{(b)}(\bar{\zeta}, \eta|\chi, \bar{\chi}) \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) e^{-(\bar{\zeta}-\bar{\chi})(\zeta-\chi)} (1 + O(N^{-1})), \end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} , where $c(\zeta, \eta, \chi)$ is the conjugation factor. Here, by Stirling formula, it is easy to see that

$$(6.2) \quad \mathfrak{F}_N^{(b_N)}(\bar{z}, z|\lambda, \bar{\lambda})\omega_{b_N}(\bar{z}, z|\lambda, \bar{\lambda}) = O(e^{-\epsilon N}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Combining (6.1) with (6.2), we have

$$(6.3) \quad K_{1,1,(g)}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) = c_N(\zeta, \eta, \chi) \mathcal{K}_{1,1}^{(b)}(\bar{\zeta}, \eta|\chi, \bar{\chi}) \omega^{(b)}(\bar{\zeta}, \zeta|\chi, \bar{\chi}) + O(e^{-\epsilon N}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Here, $c_N(\zeta, \eta, \chi)$ is a conjugation factor, which is explicitly written, but it is not important. For the proof of the off-diagonal case, since we can easily calculate the scaling limit similar to [13], we omit the details here. \square

Next, we prove the edge scaling limit.

Proof of the case of IGINUE in the strongly non-unitary regime in Theorem 2.10: edge case. We fix $a_N = 1$ and $\alpha = b_N = Nb$ again. We use the following uniform asymptotic behaviour of the incomplete Gamma function [117, equation (8.8.9)]:

$$(6.4) \quad Q(s+1, s+\sqrt{sz}) = \frac{1}{2} \operatorname{erfc} \left(\frac{z}{\sqrt{2}} \right) + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi s}} \frac{2+z^2}{3} + O\left(\frac{1}{s}\right) \quad \text{as } s \rightarrow \infty,$$

uniformly for z in a compact subset of \mathbb{C} . Here, we recall that

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi z}} (1 + O(z^{-2})) \quad \text{as } z \rightarrow \infty \text{ with } |\arg(z)| < \frac{3}{4}\pi,$$

and by the relation $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$,

$$\operatorname{erfc}(-z) = 2 - \frac{e^{-z^2}}{\sqrt{\pi z}} (1 + O(z^{-2})) \quad \text{as } z \rightarrow \infty \text{ with } |\arg(z)| < \frac{3}{4}\pi.$$

For the outer edge case, we set $z = e^{i\theta}(\sqrt{N(b+1)} + \zeta)$, $w = e^{i\theta}(\sqrt{N(b+1)} + \eta)$, $\lambda = e^{i\theta}(\sqrt{N(b+1)} + \chi)$. For the inner edge case, we set $z = e^{i\theta}(\sqrt{Nb} - \zeta)$, $w = e^{i\theta}(\sqrt{Nb} - \eta)$, $\lambda = e^{i\theta}(\sqrt{Nb} - \chi)$. First, we consider the outer edge case. Using (6.4), we have

$$(6.5) \quad \mathfrak{e}_{N+k-1}^{(b_N)}(z\bar{w}|\lambda\bar{\lambda})\widehat{\omega}_{b_N}(z, \bar{w}) = e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\zeta\bar{\eta}} \left(F(\zeta + \bar{\eta}) + \frac{e^{-\frac{(\zeta+\bar{\eta})^2}{2}}}{\sqrt{2\pi N(1+b)}} \left(\frac{(\zeta + \bar{\eta})^2 + 2}{3} + k \right) + O(N^{-1}) \right),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . For the inner edge case, similar to the above, we have

$$(6.6) \quad \mathfrak{e}_{N+k-1}^{(b_N)}(z\bar{w}|\lambda\bar{\lambda})\widehat{\omega}_{b_N}(z, \bar{w}) = e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\zeta\bar{\eta}} \left(F(\zeta + \bar{\eta}) - \frac{e^{-\frac{(\zeta+\bar{\eta})^2}{2}}}{\sqrt{2\pi}(\chi + \bar{\chi})} + O(N^{-1/2}) \right),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Here, $F(u)$ is the complementary error function defined in (2.11). Using (6.5) and (6.6), we can find that for the outer edge case,

$$(6.7) \quad f_N^{(b_N)}(\lambda\bar{\lambda})\widehat{\omega}_{b_N}(\lambda, \bar{\lambda}) = \sqrt{\frac{N}{2\pi(1+b)}} \mathcal{F}(\chi + \bar{\chi}) (1 + O(N^{-1/2})),$$

and for the inner edge case,

$$(6.8) \quad f_N^{(b_N)}(\lambda\bar{\lambda})\widehat{\omega}_{b_N}(\lambda, \bar{\lambda}) = \sqrt{\frac{N}{2\pi b}} \mathcal{F}(\chi + \bar{\chi}) (1 + O(N^{-1/2})),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Using (6.5), we have

$$\begin{aligned} \mathfrak{W}_{N+k}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) &= G(\bar{\zeta}, \eta) \left[e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \left\{ F(\bar{\zeta}+\chi)F(\bar{\chi}+\eta) + \frac{1}{\sqrt{N}} \left(C_N(\eta, \bar{\chi})F(\chi+\bar{\zeta}) + C_N(\chi, \bar{\zeta})F(\eta+\bar{\chi}) \right) \right. \right. \\ &\quad \left. \left. + \frac{k}{\sqrt{2\pi N(1+b)}} \left(e^{-\frac{1}{2}(\eta+\bar{\chi})^2} F(\chi+\bar{\zeta}) + e^{-\frac{1}{2}(\chi+\bar{\zeta})^2} F(\eta+\bar{\chi}) \right) + O(N^{-1}) \right\} \right]. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &(N+b_N+1)\mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \lambda\bar{\lambda}\mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \\ &= -\sqrt{\frac{N(1+b)}{2\pi}} G(\bar{\zeta}, \eta) \left[-e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \left(e^{-\frac{1}{2}(\eta+\bar{\chi})^2} F(\chi+\bar{\zeta}) + e^{-\frac{1}{2}(\chi+\bar{\zeta})^2} F(\eta+\bar{\chi}) \right) \right. \\ &\quad \left. + (1-(\bar{\zeta}-\bar{\chi})(\eta-\chi)) \left(e^{-\frac{1}{2}(\chi+\bar{\chi})^2} F(\eta+\bar{\zeta}) + e^{-\frac{1}{2}(\eta+\bar{\zeta})^2} F(\chi+\bar{\chi}) \right) \right. \\ &\quad \left. + \sqrt{2\pi}(\chi+\bar{\chi}) \left(e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} F(\chi+\bar{\zeta})F(\eta+\bar{\chi}) - (1-(\bar{\zeta}-\bar{\chi})(\eta-\chi))F(\chi+\bar{\chi})F(\eta+\bar{\zeta}) \right) + O(N^{-1/2}) \right] \\ &= \sqrt{\frac{N(1+b)}{2\pi}} G(\bar{\zeta}, \eta) \mathcal{F}(\chi+\bar{\chi})H(a, b, c, d, f) + \sqrt{\frac{N(1+b)}{2\pi}} G(\bar{\zeta}, \eta)(\bar{\zeta}-\bar{\chi})(\eta-\chi) e^{-\frac{1}{2}(\eta+\bar{\zeta})^2} F(\chi+\bar{\chi}) + O(1), \end{aligned}$$

with $a = \chi + \bar{\chi}, b = \chi + \bar{\zeta}, c = \eta + \bar{\chi}, d = \eta + \bar{\zeta}, f = (\bar{\zeta} - \bar{\chi})(\eta - \chi)$. Here, H is defined by (2.66). By (2.52), we obtain

$$\begin{aligned} \mathfrak{H}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda})\omega_{b_N}(z, \bar{z}|\lambda, \bar{\lambda}) &= \frac{G(\bar{\zeta}, \eta)H(a, b, c, d, f)}{(\bar{\zeta}-\bar{\chi})^2(\eta-\chi)^2} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z)}{\widehat{\omega}_{b_N}(\bar{z}, w)} \\ &\quad + \frac{G(\bar{\zeta}, \eta)}{(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2} F(\chi+\bar{\chi})}{\mathcal{F}(\chi+\bar{\chi})} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z)}{\widehat{\omega}_{b_N}(\bar{z}, w)} + O(N^{-1/2}). \end{aligned}$$

On the other hand, by (2.53) and Stirling formula, we have

$$\mathfrak{F}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda})\omega_{b_N}(\bar{z}, w|\lambda, \bar{\lambda}) = -\frac{G(\bar{\zeta}, \eta)}{(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2} F(\chi+\bar{\chi})}{\mathcal{F}(\chi+\bar{\chi})} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z)}{\widehat{\omega}_{b_N}(\bar{z}, w)} + O(N^{-1/2}).$$

Therefore, there exists a conjugation factor $c_N(\zeta, \eta, \chi)$, which is explicitly written (but it is not important), such that

$$K_{1,1,(\mathfrak{g})}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) = c_N(\zeta, \eta, \chi) \mathcal{K}_{1,1}^{(\mathfrak{e})}(\bar{\zeta}, \eta|\chi, \bar{\chi}) \omega^{(\mathfrak{e})}(\zeta, \bar{\zeta}|\chi, \bar{\chi}) + O(N^{-1/2}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . This completes the proof of the outer edge case. Next, we consider the inner edge case, but almost computations are same as the outer edge case. Indeed, using (6.6) and by similar computations, we have that with same convention,

$$\begin{aligned} &\mathfrak{H}_N^{(b_N)}(\bar{z}, w|\bar{\lambda}, \lambda)\omega_{b_N}(\bar{z}, z|\lambda, \bar{\lambda}) \\ &= G(\bar{\zeta}, \eta) \left\{ \frac{H(a, b, c, d, f)}{(\bar{\zeta}-\bar{\chi})^2(\eta-\chi)^2} + \frac{1}{\sqrt{2\pi}(\bar{\chi}+\chi)} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2}}{(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \right\} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z)}{\widehat{\omega}_{b_N}(\bar{z}, w)} (1 + O(N^{-1/2})), \end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . On the other hand,

$$\mathfrak{F}_N^{(b_N)}(\bar{z}, w|\bar{\lambda}, \lambda)\omega_{b_N}(\bar{z}, z|\lambda, \bar{\lambda}) = -\frac{G(\bar{\zeta}, \eta)}{\sqrt{2\pi}(\chi+\bar{\chi})} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2}}{(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z)}{\widehat{\omega}_{b_N}(\bar{z}, w)} (1 + O(N^{-1/2})),$$

Therefore, there exists a conjugation factor $c_N(\zeta, \eta, \chi)$, which is explicitly written (but it is not important), such that

$$K_{1,1,(\mathfrak{g})}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) = c_N(\zeta, \eta, \chi) \mathcal{K}_{1,1}^{(\mathfrak{e})}(\bar{\zeta}, \eta|\chi, \bar{\chi}) \omega^{(\mathfrak{e})}(\zeta, \bar{\zeta}|\chi, \bar{\chi}) + O(N^{-1/2}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . For the off-diagonal case, by applying Lemma 2.5 and just performing the similar computations, we get the desired result. \square

6.2. Proof of the case of IGINUE in the weakly non-unitary regime in Theorem 2.10.

Proof of the case of IGINUE in the weakly non-unitary regime in Theorem 2.10. We fix $a_N = \frac{N}{\rho^2}$ and $\alpha = b_N = N \left(\frac{N}{\rho^2} - \frac{1}{2} \right)$. As in the strong non-unitary regime, we use (6.4) many times. For $\theta \in [0, 2\pi)$ and χ, ζ, η in a compact subset of complex plane \mathbb{C} , let $z = e^{i\theta}(\sqrt{Na_N} + \zeta)$, $w = e^{i\theta}(\sqrt{Na_N} + \eta)$, $\lambda = e^{i\theta}(\sqrt{Na_N} + \chi)$. Let

$$(6.9) \quad \mathcal{J}_{\rho/\sqrt{2}}(\zeta, \bar{\eta}|\chi, \bar{\chi}) = L_{\rho/\sqrt{2}}(\zeta + \bar{\eta}) + \frac{e^{-\frac{1}{2}(\zeta + \bar{\eta} + \frac{\rho}{2})^2}}{\sqrt{2\pi}(\chi + \bar{\chi} + \frac{\rho}{2})}.$$

Here, L_ρ is defined by (2.13). By (6.4), we have

$$(6.10) \quad \begin{aligned} & \mathfrak{e}_{N+k-1}^{(b_N)}(z\bar{w}|\lambda\bar{\lambda})\widehat{\omega}_{b_N}(z, \bar{w}) \\ &= e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \zeta\bar{\eta}} \left\{ \mathcal{J}_{\rho/\sqrt{2}}(\zeta, \bar{\eta}|\chi, \bar{\chi}) + \frac{\rho}{N} \left(k \frac{e^{-\frac{1}{2}(\zeta + \bar{\eta} - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + C_g(\zeta, \bar{\eta}) \right) \right\} (1 + O(N^{-2})), \end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Here, we defined

$$C_g(\zeta, \bar{\eta}) = \frac{1}{\sqrt{2\pi}} \left\{ \frac{2 + (\zeta + \bar{\eta} - \frac{\rho}{2})^2}{3} e^{-\frac{1}{2}(\zeta + \bar{\eta} - \frac{\rho}{2})^2} - \frac{2 + (\zeta + \bar{\eta} + \frac{\rho}{2})^2}{3} e^{-\frac{1}{2}(\zeta + \bar{\eta} + \frac{\rho}{2})^2} \right\},$$

which does not depend on k . From (6.10), we see that

$$(6.11) \quad f_N^{(b_N)}(\lambda\bar{\lambda})\widehat{\omega}_{b_N}(\lambda, \bar{\lambda}) = \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(1 + O(N^{-1})),$$

uniformly for χ in a compact subset of \mathbb{C} . Here, \mathcal{L}_ρ is defined by (2.67). First, we observe that

$$(6.12) \quad \begin{aligned} & (N + b_N + 1)\mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \lambda\bar{\lambda}\mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \\ &= \frac{N^2}{\rho^2} (\mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda})) + \left(\frac{N}{2} \mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \frac{N(\chi + \bar{\chi})}{\rho} \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \right) \\ &+ \mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - |\chi|^2 \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}). \end{aligned}$$

In order to see the asymptotic behavior of (6.12), we observe that for $k = 0, 1$,

$$\mathfrak{W}_{N+k}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) = G(\bar{\zeta}, \eta) (e^{-\bar{\zeta}(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathfrak{w}_{N,k}^{(1)} - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \mathfrak{w}_{N,k}^{(2)}),$$

where

$$\begin{aligned} \mathfrak{w}_{N,k}^{(1)} &= \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) + \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \frac{\rho}{N} \left(\frac{k+1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\eta + \bar{\chi} - \frac{\rho}{2})^2} + C_g(\eta, \bar{\chi}) \right) \\ &+ \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) \frac{\rho}{N} \left(\frac{k+1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\chi + \bar{\zeta} - \frac{\rho}{2})^2} + C_g(\chi, \bar{\zeta}) \right) + O(N^{-2}), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{w}_{N,k}^{(2)} &= \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) + \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\rho}{N} \left(\frac{k+1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\chi + \bar{\chi} - \frac{\rho}{2})^2} + C_g(\chi, \bar{\chi}) \right) \\ &+ \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) \frac{\rho}{N} \left(\frac{k+1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\eta + \bar{\zeta} - \frac{\rho}{2})^2} + C_g(\eta, \bar{\zeta}) \right) + O(N^{-2}), \end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Since

$$\begin{aligned} \mathfrak{w}_{N,1}^{(1)} - \mathfrak{w}_{N,0}^{(1)} &= \frac{\rho}{N} \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta + \bar{\chi} - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + \frac{\rho}{N} \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\chi + \bar{\zeta} - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + O(N^{-2}), \\ \mathfrak{w}_{N,1}^{(2)} - \mathfrak{w}_{N,0}^{(2)} &= \frac{\rho}{N} \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\chi + \bar{\chi} - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + \frac{\rho}{N} \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta + \bar{\zeta} - \frac{\rho}{2})^2}}{\sqrt{2\pi}} + O(N^{-2}), \end{aligned}$$

we have

$$\frac{N^2}{\rho^2} (\mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}))$$

$$\begin{aligned}
&= \frac{N^2}{\rho^2} G(\bar{\zeta}, \eta) (e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} (\mathfrak{w}_{N,1}^{(1)} - \mathfrak{w}_{N,0}^{(1)}) - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) (\mathfrak{w}_{N,1}^{(2)} - \mathfrak{w}_{N,0}^{(2)})) \\
&= \frac{N}{\rho} \frac{G(\bar{\zeta}, \eta)}{\sqrt{2\pi}} \left\{ e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} (\mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) e^{-\frac{1}{2}(\eta+\bar{\chi}-\frac{\rho}{2})^2} + \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) e^{-\frac{1}{2}(\chi+\bar{\zeta}-\frac{\rho}{2})^2}) \right. \\
&\quad \left. - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) (\mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) e^{-\frac{1}{2}(\chi+\bar{\chi}-\frac{\rho}{2})^2} + \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) e^{-\frac{1}{2}(\eta+\bar{\zeta}-\frac{\rho}{2})^2}) \right\} (1 + O(N^{-1})),
\end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Next, we observe that

$$\begin{aligned}
&\frac{N}{2} \mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \frac{N}{\rho} (\chi + \bar{\chi}) \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \\
&= - (1 + O(N^{-1})) \frac{N}{\rho} (\chi + \bar{\chi} - \frac{\rho}{2}) G(\bar{\zeta}, \eta) \\
&\quad \times \left(e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) \right),
\end{aligned}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Then, we obtain

$$(N + b_N + 1) \mathfrak{W}_{N+1}^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) - \lambda \bar{\lambda} \mathfrak{W}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) = \frac{N}{\rho} G(\bar{\zeta}, \eta) \mathfrak{w}^{(3)} (1 + O(N^{-1})),$$

where we set

$$\begin{aligned}
\mathfrak{w}^{(3)} &= e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} (\mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta+\bar{\chi}-\frac{\rho}{2})^2}}{\sqrt{2\pi}} + \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\chi+\bar{\zeta}-\frac{\rho}{2})^2}}{\sqrt{2\pi}}) \\
&\quad - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) (\mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\chi+\bar{\chi}-\frac{\rho}{2})^2}}{\sqrt{2\pi}} + \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}-\frac{\rho}{2})^2}}{\sqrt{2\pi}}) \\
&\quad - (\chi + \bar{\chi} - \frac{\rho}{2}) \left(e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\chi}|\chi, \bar{\chi}) \right. \\
&\quad \left. - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \mathcal{J}_{\rho/\sqrt{2}}(\eta, \bar{\zeta}|\chi, \bar{\chi}) \mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi}) \right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
(6.13) \quad &\mathfrak{H}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \omega_{b_N}(\bar{z}, z|\lambda, \bar{\lambda}) \\
&= \left(\chi + \bar{\chi} + \frac{\rho}{2} \right) \frac{G(\bar{\zeta}, \eta) \mathfrak{w}^{(3)} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi})}{\mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi}) (\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2} \frac{\widehat{\omega}_{b_N}(\bar{z}, z|\lambda, \bar{\lambda})}{\widehat{\omega}_{b_N}(\bar{z}, w|\lambda, \bar{\lambda})} (1 + O(N^{-1})) \\
&= \frac{G(\bar{\zeta}, \eta) \mathfrak{w}^{(4)}}{\mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi}) (\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z|\lambda, \bar{\lambda})}{\widehat{\omega}_{b_N}(\bar{z}, w|\lambda, \bar{\lambda})} (1 + O(N^{-1})),
\end{aligned}$$

where

$$\begin{aligned}
(6.14) \quad &\mathfrak{w}^{(4)} \\
&= e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)} \mathcal{H}_{\rho/\sqrt{2}}(\chi + \bar{\chi}, \chi + \bar{\zeta}, \eta + \bar{\chi}, \eta + \bar{\zeta}, (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \\
&\quad + (\bar{\zeta} - \bar{\chi})(\eta - \chi) \left\{ (\chi + \bar{\chi} + \frac{\rho}{2}) L_{\rho/\sqrt{2}}(\chi + \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}-\frac{\rho}{2})^2}}{\sqrt{2\pi}} - (\chi + \bar{\chi} - \frac{\rho}{2}) L_{\rho/\sqrt{2}}(\chi + \bar{\chi}) \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}+\frac{\rho}{2})^2}}{\sqrt{2\pi}} \right. \\
&\quad \left. + \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}+\frac{\rho}{2})^2} - \frac{1}{2}(\chi+\bar{\chi}-\frac{\rho}{2})^2}{\sqrt{2\pi}^2} + \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}-\frac{\rho}{2})^2} - \frac{1}{2}(\chi+\bar{\chi}+\frac{\rho}{2})^2}{\sqrt{2\pi}^2} - \frac{\chi + \bar{\chi} - \frac{\rho}{2}}{\chi + \bar{\chi} + \frac{\rho}{2}} \frac{e^{-\frac{1}{2}(\eta+\bar{\zeta}+\frac{\rho}{2})^2} - \frac{1}{2}(\chi+\bar{\chi}+\frac{\rho}{2})^2}}{\sqrt{2\pi}^2} \right\}.
\end{aligned}$$

Using asymptotics (6.10) and (6.11), note that

$$\begin{aligned}
&\mathfrak{F}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \omega_{b_N}(\bar{z}, w|\lambda, \bar{\lambda}) \\
&= -G(\bar{\zeta}, \eta) \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z|\lambda, \bar{\lambda})}{\widehat{\omega}_{b_N}(\bar{z}, w|\lambda, \bar{\lambda})}
\end{aligned}$$

$$\times \left\{ \frac{(\chi + \bar{\chi} + \frac{\rho}{2})}{\sqrt{2\pi}} \frac{\mathcal{J}_{\rho/\sqrt{2}}(\chi, \bar{\chi}|\chi, \bar{\chi})}{\mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} e^{-\frac{1}{2}(\bar{\zeta} + \eta - \frac{\rho}{2})^2} + \frac{e^{-\frac{1}{2}(\eta + \bar{\zeta} + \frac{\rho}{2})^2}}{\sqrt{2\pi}(\chi + \bar{\chi} + \frac{\rho}{2})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \right\} (1 + O(N^{-1})),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . The above can be also written as follows.

$$(6.15) \quad \begin{aligned} & \mathfrak{F}_N^{(b_N)}(\bar{z}, w|\lambda, \bar{\lambda}) \omega_{b_N}(z, \bar{z}|\lambda, \bar{\lambda}) \\ &= -G(\bar{\zeta}, \eta) \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) \frac{\widehat{\omega}_{b_N}(\bar{z}, z|\lambda, \bar{\lambda})}{\widehat{\omega}_{b_N}(\bar{z}, w|\lambda, \bar{\lambda})} \left\{ \frac{e^{-\frac{1}{2}(\bar{\zeta} + \eta - \frac{\rho}{2})^2}}{\sqrt{2\pi} \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} (\chi + \bar{\chi} + \frac{\rho}{2}) L_{\rho/\sqrt{2}}(\chi + \bar{\chi}) \right. \\ &+ \frac{e^{-\frac{1}{2}(\bar{\zeta} + \eta - \frac{\rho}{2})^2} e^{-\frac{1}{2}(\chi + \bar{\chi} + \frac{\rho}{2})^2}}{\sqrt{2\pi}^2 \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} - \frac{e^{-\frac{1}{2}(\eta + \bar{\zeta} + \frac{\rho}{2})^2} e^{-\frac{1}{2}(\chi + \bar{\chi} + \frac{\rho}{2})^2}}{\sqrt{2\pi}^2 \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)(\chi + \bar{\chi} + \frac{\rho}{2})} \\ &\left. + \frac{e^{-\frac{1}{2}(\eta + \bar{\zeta} + \frac{\rho}{2})^2} e^{-\frac{1}{2}(\chi + \bar{\chi} - \frac{\rho}{2})^2}}{\sqrt{2\pi}^2 \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} - \frac{(\chi + \bar{\chi} - \frac{\rho}{2}) e^{-\frac{1}{2}(\eta + \bar{\zeta} + \frac{\rho}{2})^2} L_{\rho/\sqrt{2}}(\chi + \bar{\chi})}{\sqrt{2\pi} \mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \right\} (1 + O(N^{-1})). \end{aligned}$$

Combining (6.13) and (6.14) with (6.15), there exists a conjugation factor $c_N(\zeta, \eta, \chi)$ such that we have

$$K_{1,1,(g)}^{(N)}(\zeta, \bar{\zeta}, \eta, \bar{\eta}|\bar{\chi}, \chi) = c_N(\zeta, \eta, \chi) \frac{\mathcal{H}_{\rho/\sqrt{2}}(\chi + \bar{\chi}, \chi + \bar{\zeta}, \eta + \bar{\chi}, \eta + \bar{\zeta}, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{\mathcal{L}_{\rho/\sqrt{2}}(\chi + \bar{\chi})(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) e^{-(\bar{\zeta} - \bar{\chi})(\zeta - \chi)} + o(1),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . This completes the first part of the proof of the weakly non-unitary regime. Next, we consider the off-diagonal overlap case. For the computation of the kernel, we just apply Lemma 2.5 into our setting. Using (6.10) and (6.11) and from the above discussions, we see that for $z_1 = \sqrt{Na_N} + \zeta_1, z_2 = \sqrt{Na_N} + \zeta_2$,

$$\begin{aligned} & f_{N-1}^{(b_N)}(z_1 \bar{z}_2) \widehat{\omega}_{b_N}(z_1, \bar{z}_2) \widehat{\omega}_{b_N}(\bar{z}_1, z_2) \mathcal{K}_{1,1,(g)}^{(N-1)}(\bar{z}_1, z_2|z_1, \bar{z}_2) \\ &= \frac{\mathcal{H}_{\rho/\sqrt{2}}(\zeta_1 + \bar{\zeta}_2, \zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_2 + \bar{\zeta}_1, -(\bar{\zeta}_1 - \bar{\zeta}_2)(\zeta_1 - \zeta_2))}{(\bar{\zeta}_1 - \bar{\zeta}_2)^2(\zeta_1 - \zeta_2)^2} (1 + o(1)), \end{aligned}$$

as $N \rightarrow \infty$, uniformly for ζ_1, ζ_2 in compact subsets of \mathbb{C} . This completes the proof. \square

6.3. Proof of the case of I GinUE in the singular origin regime in Theorem 2.10. We fix $a_N = 1$ and $b_N = b > 0$ as the parameters. Let $z = \zeta, w = \eta, \lambda = \chi$. Then, observe that

$$(6.16) \quad \mathbf{e}_{N+k-1}^{(b)}(z\bar{w}|\lambda\bar{\lambda}) \widehat{\omega}_b(z, \bar{w}) = \left((\zeta\bar{\eta})^b E_{1,b+1}(\zeta\bar{\eta}) + \frac{1}{\Gamma(b)} \frac{(\zeta\bar{\eta})^b}{|\chi|^2 - b} \right) e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)}, \quad \text{as } N \rightarrow \infty,$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Based on this asymptotic expansion, we have

$$(6.17) \quad f_N^{(b)}(\lambda\bar{\lambda}) \widehat{\omega}_b(\lambda, \bar{\lambda}) = \frac{N}{\chi\bar{\chi}} \mathcal{E}_b(\chi\bar{\chi}|\chi\bar{\chi})(\chi\bar{\chi})^b e^{-|\chi|^2} (1 + O(N^{-1})), \quad \text{as } N \rightarrow \infty,$$

uniformly for χ in a compact subset of \mathbb{C} , and \mathcal{E}_b is defined by (2.69). With these asymptotics, we obtain

$$\begin{aligned} \mathfrak{W}_N^{(b)}(\bar{z}, w|\lambda, \bar{\lambda}) &= \left(\mathcal{E}_b(\bar{\zeta}\chi|\chi\bar{\chi}) \mathcal{E}_b(\bar{\chi}\eta|\chi\bar{\chi}) - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \mathcal{E}_b(\bar{\zeta}\eta|\chi\bar{\chi}) \mathcal{E}_b(\bar{\chi}\chi|\chi\bar{\chi}) \right) \\ &\quad \times \frac{(\bar{\zeta}\eta)^b (\chi\bar{\chi})^b e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) - |\chi|^2}}{(|\chi|^2 - b)^2} (1 + O(N^{-1})), \end{aligned}$$

uniformly for ζ, η, χ in a compact subset of \mathbb{C} . Therefore, we have

$$\mathfrak{H}_N^{(b)}(\bar{z}, w|\lambda, \bar{\lambda}) = \frac{\mathcal{E}_b(\bar{\zeta}\chi|\chi\bar{\chi}) \mathcal{E}_b(\bar{\chi}\eta|\chi\bar{\chi}) - (1 - (\bar{\zeta} - \bar{\chi})(\eta - \chi)) \mathcal{E}_b(\bar{\zeta}\eta|\chi\bar{\chi}) \mathcal{E}_b(\bar{\chi}\chi|\chi\bar{\chi})}{(\chi\bar{\chi} - b)(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2 \mathcal{E}_b(\chi\bar{\chi}|\chi\bar{\chi})} (1 + O(N^{-1})).$$

Similarly, we have

$$\mathfrak{F}_N^{(b)}(\bar{z}, w|\lambda, \bar{\lambda}) = -\frac{1}{\Gamma(b)} \frac{1}{(\chi\bar{\chi} - b)(\bar{\zeta} - \bar{\chi})(\eta - \chi)} (1 + O(N^{-1})).$$

Hence, we obtain

$$\mathcal{K}_N^{(b)}(\bar{z}, w|\lambda, \bar{\lambda}) = \frac{\mathcal{S}_b(\bar{\zeta}\chi, \bar{\chi}\eta, \bar{\zeta}\eta, \bar{\chi}\chi, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2\mathcal{E}_b(\chi\bar{\chi})}(1 + O(N^{-1})),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Here, \mathcal{S}_b is defined by (2.70). Then, the claim immediately follows. For the off-diagonal case, since Lemma 2.5, it suffices to consider

$$f_N^{(\alpha)}(z_1\bar{z}_2)\widehat{\omega}_b(z_1, \bar{z}_2)\widehat{\omega}_b(\bar{z}_1, z_2)\mathcal{K}_{N-1}^{(b)}(\bar{z}_1, z_2|z_1, \bar{z}_2).$$

However, it is straightforward to see that

$$f_N^{(b)}(z_1\bar{z}_2) = \frac{N}{\zeta_1\bar{\zeta}_2}\mathcal{E}_b(\zeta_1\bar{\zeta}_2)(1 + O(N^{-1})), \quad \widehat{\omega}_b(z_1, \bar{z}_2)\widehat{\omega}_b(\bar{z}_1, z_2) = |\zeta_1|^{2b}|\zeta_2|^{2b}e^{-(|\zeta_1|^2+|\zeta_2|^2)},$$

and

$$\begin{aligned} \mathcal{K}_{N-1}^{(b)}(\bar{z}_1, z_2|z_1, \bar{z}_2) &= \mathcal{K}_{1,1}^{(s)}(\bar{\zeta}_1, \zeta_2|\zeta_1, \bar{\zeta}_2)(1 + O(N^{-1})). \\ &= \frac{\mathcal{S}_b(\bar{\zeta}_1\zeta_1, \bar{\zeta}_2\zeta_2, \bar{\zeta}_1\zeta_2, \bar{\zeta}_2\zeta_1, (\bar{\zeta}_1 - \bar{\zeta}_2)(\zeta_2 - \zeta_1))}{(\bar{\zeta}_1 - \bar{\zeta}_2)^2(\zeta_2 - \zeta_1)^2\mathcal{E}_b(\zeta_1\bar{\zeta}_2)}(1 + o(1)). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &f_N^{(b)}(z_1\bar{z}_2)\widehat{\omega}_b(z_1, \bar{z}_2)\widehat{\omega}_b(\bar{z}_1, z_2)\mathcal{K}_{N-1}^{(b)}(\bar{z}_1, z_2|z_1, \bar{z}_2) \\ &= \frac{\mathcal{S}_b(\bar{\zeta}_1\zeta_1, \bar{\zeta}_2\zeta_2, \bar{\zeta}_1\zeta_2, \bar{\zeta}_2\zeta_1, (\bar{\zeta}_1 - \bar{\zeta}_2)(\zeta_2 - \zeta_1))}{\zeta_1\bar{\zeta}_2(\bar{\zeta}_1 - \bar{\zeta}_2)^2(\zeta_2 - \zeta_1)^2}|\zeta_1|^{2b}|\zeta_2|^{2b}e^{-(|\zeta_1|^2+|\zeta_2|^2)}(1 + O(N^{-1})). \end{aligned}$$

This completes the proof of the case of IGINUE in Theorem 2.10 in the singular origin regime.

Part 3. Proof of determinantal structure of the overlaps for induced spherical unitary ensemble

7. FINITE N -KERNEL ANALYSIS: PROOFS OF PROPOSITION 2.6 AND THEOREM 2.7

In this section, we prove Proposition 2.6 and Theorem 2.7. Our strategy to construct a family of the planar orthogonal polynomial associated with the weight function (2.28) follows the moment method as in [13]. By the similar discussion in Part 2, we find that the multi-points correlation function of the diagonal overlap is given by

$$D_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) = \frac{N!}{Z_N} \frac{|z_1|^{2L}}{(1 + |z_1|^2)^{n+L+1}} \prod_{j=0}^{N-2} h_j \times \det_{2 \leq i, j \leq k} \left(K_{1,1}^{(N-1)}(z_i, \bar{z}_i, z_j, \bar{z}_j | z_1, \bar{z}_1) \right)$$

As we already mentioned, from now on, we focus on the diagonal overlap case. The step to get from (2.47) to (2.50) is done similar to [13, p.13]. Now, we shall prove Proposition 2.6.

Proof of Proposition 2.6. We define a moment matrix

$$M_{i,j} = \langle z^i, z^j \rangle_\omega = \int_{\mathbb{C}} \bar{z}^i z^j \omega_{(s)}(z, \bar{z} | a, \bar{a}) dA(z),$$

where the weight function is defined by (2.28). Since (recall (2.7))

$$\int_{\mathbb{C}} |z|^{2k} e^{-NQ(z)} dA(z) = \frac{\Gamma(k+L+1)\Gamma(n-k)}{\Gamma(L+n+1)},$$

we have

$$(7.1) \quad M_{i,j} = \frac{\Gamma(n-i-1)\Gamma(i+L+1)}{\Gamma(n+L+1)} \mu_{i,j},$$

where

$$(7.2) \quad \mu_{i,j} = (i+L+2+(n-i)|a|^2\delta_{i,j} - (i+L+1)a\delta_{i+1,j} - (n-i-1)\bar{a}\delta_{i,j+1}).$$

By the LDU decomposition of $\mu = LDU$, where $D_{p,q} = d_p\delta_{p,q}$, $L_{p,q} = \delta_{p,q} + \ell_p\delta_{p,q+1}$, and $U_{p,q} = \delta_{p,q} + u_q\delta_{q,p+1}$ for $p, q \in \mathbb{N} \cup \{0\}$, we have

$$d_p = -d_{p-1}\ell_p u_p \mathbf{1}_{p \geq 1} + p + L + 2 + |a|^2(n-p), \quad u_{p+1} = -\frac{a(p+L+1)}{d_p}, \quad \ell_{p+1} = -\frac{\bar{a}(n-p-2)}{d_p}.$$

This implies that

$$d_p = -\frac{x(p+L)(n-p-1)}{d_{p-1}} + p + L + 2 + x(n-p), \quad p \geq 1,$$

with $d_0 = L + 2 + nx$ for $x = |a|^2$. We define a sequence $\{r_p\}_{p=0}^\infty$ by $d_p = \frac{r_{p+1}}{r_p}$. Then, we have

$$(7.3) \quad r_{p+1} = ((n-p)x + p + L + 2)r_p - x(p+L)(n-p-1)r_{p-1}, \quad r_1 = L + 2 + nx, \quad r_0 = 1.$$

By the induction argument, the unique solution of (7.3) is given by

$$r_p = \frac{n!(L+p)!}{(L+n)!} g_p^{(n,L)}(x),$$

where $g_p^{(n,L)}(x)$ is defined by (2.35). After multiplying μ by $\text{diag}(\Gamma(n-i-1)\Gamma(i+L+1)/\Gamma(n+L+1))_{i=0,1,2,\dots}$, and with the same notation, we find the LDE decomposition of $M = LDU$,

$$L_{p,k} = \delta_{p,k} - \bar{a} \frac{g_{k-1}^{(n,L)}(x)}{g_k^{(n,L)}(x)} \delta_{p,k+1}, \quad U_{k,p} = \delta_{k,p} - a \frac{g_{k-1}^{(n,L)}(x)}{g_k^{(n,L)}(x)} \delta_{p,k+1}, \quad D_{k,k} = \frac{\Gamma(k+L+2)\Gamma(n-k-1)}{\Gamma(n+L+1)} \frac{g_{k+1}^{(n,L)}(x)}{g_k^{(n,L)}(x)},$$

for $p, k \geq 0$. Here, similar to the same discussion in [13, p.14], we have

$$h_k = D_{k,k}, \quad P_k(z|a, \bar{a}) = \sum_{m=0}^k \bar{L}_{k,m}^{-1} z^m, \quad Q_k(z|a, \bar{a}) = \sum_{m=0}^k U_{m,k}^{-1} z^m.$$

Then, the reduced polynomial kernel (2.49) can be written in terms of L, D, U :

$$\mathcal{K}_{1,1}^{(N)}(z, \bar{z}|a, \bar{a}) = \sum_{i,j=0}^{N-1} \bar{z}^i C_{i,j}^{(N-1)} z^j, \quad C_{i,j}^{(N)} = \sum_{k=0}^N U_{i,k}^{-1} \frac{1}{D_{k,k}} L_{k,j}^{-1}.$$

Note that

$$\frac{N!}{Z_N} \prod_{j=0}^{N-2} h_j = \frac{g_{N-1}^{(n,L)}(x) \Gamma(L+n+1)}{g_0^{(n,L)}(x) \Gamma(L+1)\Gamma(n)} = n g_{N-1}^{(n,L)}(x),$$

where we used $g_0^{(n,L)}(x) = \frac{\Gamma(L+N+1)}{\Gamma(L+1)\Gamma(n+1)}$. Notice also that

$$L_{p,q}^{-1} = \begin{cases} 0 & q > p, \\ 1 & q = p, \\ \bar{a}^{p-q} \frac{g_q^{(n,L)}(x)}{g_p^{(n,L)}(x)} & q < p, \end{cases} \quad U_{p,q}^{-1} = \begin{cases} a^{q-p} \frac{g_p^{(n,L)}(x)}{g_q^{(n,L)}(x)} & q > p, \\ 1 & q = p, \\ 0 & q < p. \end{cases}$$

Finally, we define

$$G_N^{(n,L)}(x|y, z) = \sum_{j,k=0}^{N-1} g_j^{(n,L)}(x) y^j g_k^{(n,L)}(x) z^k \sum_{m=\max(j,s)}^{N-1} \frac{\Gamma(n+L+1)}{\Gamma(m+L+2)\Gamma(n-m-1)} \frac{x^m}{g_{m+1}^{(n,L)}(x) g_m^{(n,L)}(x)},$$

which completes the proof. \square

Remark 7.1. *The planar orthogonal polynomials associated with (2.28) for $a = 0$ are monomials with with a norming constant*

$$h_k(0) = (L+k+2) \frac{\Gamma(n-k-1)\Gamma(L+k+1)}{\Gamma(n+L+1)}.$$

The corresponding to the finite N -kernel is given by

$$(7.4) \quad \mathcal{K}_{1,1}^{(N)}(z, w|0) = \sum_{k=0}^{N-1} \frac{\Gamma(n+L+1)}{(L+k+2)\Gamma(n-k-1)\Gamma(L+k+1)} (z\bar{w})^k.$$

Remark 7.2. *From the proof of Proposition 2.6, we can readily find that the planar orthogonal polynomials associated with (2.28) are explicitly written as*

$$(7.5) \quad P_k(z|a, \bar{a}) = \sum_{j=0}^k a^{k-j} \frac{g_j^{(n,L)}(a\bar{a})}{g_k^{(n,L)}(a\bar{a})} z^j$$

Notice also that as in [42], we can confirm that $\{P_k(\cdot|a, \bar{a})\}_k$ satisfy the non-standard three term recurrence relationship:

$$z P_k(z|a, \bar{a}) = P_{k+1}(z|a, \bar{a}) + b_k P_k(z|a, \bar{a}) + z c_k P_{k-1}(z|a, \bar{a}),$$

where

$$b_k = -a \frac{g_k^{(n,L)}(a\bar{a})}{g_{k+1}^{(n,L)}(a\bar{a})}, \quad c_k = a \frac{g_{k-1}^{(n,L)}(a\bar{a})}{g_k^{(n,L)}(a\bar{a})}.$$

Then, the finite N -kernel can be also written in terms of $\{P_k(\cdot|a, \bar{a})\}_k$:

$$\begin{aligned} \mathcal{K}_{1,1}^{(N)}(z, w|a, \bar{a}) &= \sum_{k=0}^{N-1} \frac{P_k(z|a, \bar{a}) \overline{P_k(w|a, \bar{a})}}{h_k} \\ &= \sum_{k=0}^{N-1} \frac{\Gamma(n+L+1)}{\Gamma(k+L+2)\Gamma(n-k-1)} \frac{g_k^{(n,L)}(a\bar{a})}{g_{k+1}^{(n,L)}(a\bar{a})} P_k(z|a, \bar{a}) \overline{P_k(w|a, \bar{a})}. \end{aligned}$$

Although we do not pursue here, it would be interesting to find a Christoffel-Darboux type identity as in [42] and to analyze the corresponding differential equation. In this case, we expect that the corresponding differential equation is the second order differential equation different form [42]. Indeed, using (7.4), we write

$$\widehat{\mathcal{K}}_{1,1}^{(N)}(z, w|0) = \frac{(z\bar{w})^{L+2}\mathcal{K}_{1,1}^{(N)}(z, w|0)}{(1+z\bar{w})^{n+L-1}}.$$

Then, the above satisfies the following differential equation:

$$\begin{aligned} & \left\{ z(1+z\bar{w})\partial_z^2 + (z\bar{w}(n+L-1)-1)\partial_z - (n+L-1)\bar{w} \right\} \widehat{\mathcal{K}}_{1,1}^{(N)}(z, w|0) \\ &= \frac{(L+n)\bar{w}\Gamma(n+L+1)(z\bar{w})^{L+1}}{(1+z\bar{w})^{n+L}\Gamma(n)\Gamma(L)} - \frac{(L+n)\bar{w}\Gamma(n+L+1)(z\bar{w})^{N+L+1}}{(1+z\bar{w})^{n+L}\Gamma(n-N)\Gamma(N+L)}. \end{aligned}$$

This suggests that for the general case, as we already mentioned, we would have a second order differential equation depending on the parameter $a \in \mathbb{C}$.

7.1. Simplification step for the finite N -kernel. In this subsection, we will find the simplified representation of (2.46), that is, we will prove Theorem 2.7. First, we can rewrite (2.46) as

$$(7.6) \quad G_N^{(n,L)}(x|y, z) = \sum_{s,t=0}^{N-1} g_s^{(n,L)}(x) y^s g_t^{(n,L)}(x) z^t \left(\Phi_{N-1}^{(n,L)}(x) - \Phi_{\max\{s,t\}-1}^{(n,L)}(x) \right),$$

where

$$\Phi_\ell^{(n,L)}(x) := \sum_{j=0}^{\ell} \frac{\Gamma(n+L+1)}{\Gamma(j+L+2)\Gamma(n-j-1)} \frac{x^j}{g_{j+1}^{(n,L)}(x)g_j^{(n,L)}(x)}.$$

Let us denote

$$\Phi_{\ell,m}^{(n,L)}(x) := \sum_{j=m}^{\ell} \frac{\Gamma(n+L+1)}{\Gamma(j+L+2)\Gamma(n-j-1)} \frac{x^j}{g_{j+1}^{(n,L)}(x)g_j^{(n,L)}(x)}.$$

Lemma 7.3. *We have*

$$\Phi_q^{(n,L)}(x) = \frac{\Gamma(n+1)\Gamma(L+1)}{\Gamma(n+L+1)} \frac{(n-1)x - (L+1)}{x(x-L/n)} + \frac{-(n-q-2)x + L + q + 2}{x(x-L/n)g_{q+1}^{(n,L)}(x)}.$$

Proof. By the induction argument, we can show that

$$\begin{aligned} \Phi_{m+q,m}^{(n,L)}(x) &= \frac{\Gamma(n+L+1)}{\Gamma(L+q+m+2)\Gamma(n-m-1)} \frac{x^m}{g_{m+q+1}^{(n,L)}(x)g_m^{(n,L)}(x)} \\ &\quad \times \left\{ \sum_{k=0}^q \frac{(q+1-k)\Gamma(n-m)\Gamma(L+q+m+2)}{\Gamma(n-m-k)\Gamma(L+k+m+2)} x^k \right. \\ &\quad \left. - (L+m+1) \sum_{k=0}^q \frac{(q-k)\Gamma(n-m-1)\Gamma(L+q+m+2)}{\Gamma(n-m-k-1)\Gamma(L+k+m+3)} x^k \right\}. \end{aligned}$$

By taking $m=0$ and rearranging the summations, we have

$$\begin{aligned} \Phi_q^{(n,L)}(x) &= \frac{\Gamma(n+1)\Gamma(L+1)}{\Gamma(n+L+1)g_{q+1}^{(n,L)}(x)} \\ &\quad \times \left\{ \frac{(n-1)x - (L+1)}{x} \sum_{k=0}^q \frac{(q+1-k)\Gamma(n+L+1)}{\Gamma(n-k)\Gamma(L+k+2)} x^k + \frac{q+1}{x} \frac{\Gamma(n+L+1)}{\Gamma(n)\Gamma(L+1)} \right\} \\ &= \frac{\Gamma(n+1)\Gamma(L+1)}{\Gamma(n+L+1)g_{q+1}^{(n,L)}(x)} \\ &\quad \times \left\{ \frac{(n-1)x - (L+1)}{x} \frac{g_{q+1}^{(n,L)}(x)}{x - \frac{L}{n}} + \frac{\Gamma(n+L+1)}{\Gamma(n+1)\Gamma(L+1)} \frac{-(n-q-2)x + L + q + 2}{x(x - \frac{L}{n})} \right\}, \end{aligned}$$

which completes the proof. \square

We are ready to show Theorem 2.7

Proof of Theorem 2.7. We write

$$(7.7) \quad \alpha_m^{(n,L)}(x, \omega) := \sum_{s=0}^m g_s^{(n,L)}(x) \omega^s.$$

Then, using (2.35), we find that (7.7) can be expressed as

$$(7.8) \quad \alpha_m^{(n,L)}(x, \omega) = \frac{x - \frac{L}{n}}{x} \frac{1}{(1-\omega)^2} \widehat{q}_m^{(n,L)}(x\omega|x) - \frac{x - \frac{L}{n}}{x} \frac{(x\omega)^{m+1} \Gamma(L+n+1)}{(1-\omega)(1+x)\Gamma(L+m+1)\Gamma(n-m)} \\ - \frac{x - \frac{L}{n}}{x} \frac{\omega^{m+1} (L+m+1 - (n-m-1)x) \widehat{q}_m^{(n,L)}(x|x)}{(1+x)(1-\omega)} - \frac{x - \frac{L}{n}}{x} \frac{\omega^{m+1} \widehat{q}_m^{(n,L)}(x|x)}{(1-\omega)^2}.$$

By differentiating (7.8) with respect to ω , we have

$$(7.9) \quad \omega \partial_\omega \alpha_m^{(n,L)}(x, \omega) \\ = \left(m+1 + \frac{2\omega}{1-\omega} \right) \alpha_m^{(n,L)}(x, \omega) + \frac{x - \frac{L}{n}}{x} \frac{\omega^{m+2} (L+m+1 - (n-m-1)x) \widehat{q}_m^{(n,L)}(x|x)}{(1-\omega)^2(1+x)} \\ - \frac{x - \frac{L}{n}}{x} \frac{(m+1) \widehat{q}_m^{(n,L)}(x\omega|x)}{(1-\omega)^2} + \frac{x - \frac{L}{n}}{x} \frac{(nx\omega - L) \widehat{q}_m^{(n,L)}(x\omega|x\omega)}{(1+x\omega)(1-\omega)^2} \\ - \frac{x - \frac{L}{n}}{x} \frac{(x\omega)^{m+1} (1+x+x\omega)}{(1-\omega)(1+x)(1+x\omega)} \frac{\Gamma(L+n+1)}{\Gamma(L+m+1)\Gamma(n-m)} \\ = \left(m+1 + \frac{2\omega}{1-\omega} \right) \alpha_m^{(n,L)}(x, \omega) + R_m(x, \omega),$$

where

$$(7.10) \quad R_m(x, \omega) = \frac{x - \frac{L}{n}}{x} \frac{\omega^{m+2} (L+m+1 - (n-m-1)x) \widehat{q}_m^{(n,L)}(x|x)}{(1-\omega)^2(1+x)} \\ + \frac{x - \frac{L}{n}}{x} \frac{(nx\omega - L) \widehat{q}_m^{(n,L)}(x\omega|x\omega)}{(1+x\omega)(1-\omega)^2} - \frac{x - \frac{L}{n}}{x} \frac{m+1}{(1-\omega)^2} \widehat{q}_m^{(n,L)}(x\omega|x) \\ - \frac{x - \frac{L}{n}}{x} \frac{(x\omega)^{m+1} (1+x+x\omega)}{(1-\omega)(1+x)(1+x\omega)} \frac{\Gamma(L+n+1)}{\Gamma(L+m+1)\Gamma(n-m)}.$$

To compute the summation for the index $\max\{s, t\} - 1$, we need

$$(7.11) \quad \sum_{0 \leq s < t \leq N-1} g_s^{(n,L)}(x) y^s z^t = \frac{z}{1-z} \alpha_{N-1}^{(n,L)}(x, yz) - \frac{z^N}{1-z} \alpha_{N-1}^{(n,L)}(x, y),$$

$$(7.12) \quad \sum_{0 \leq s < t \leq N-1} g_s^{(n,L)}(x) t y^s z^t = \frac{z}{(1-z)^2} \alpha_{N-1}^{(n,L)}(x, yz) + \frac{z^2}{1-z} \partial_z \alpha_{N-1}^{(n,L)}(x, yz) \\ + \frac{(N-1)z^{N+1} - Nz^N}{(1-z)^2} \alpha_{N-1}^{(n,L)}(x, y).$$

These follow from the straightforward calculations. Together with (7.8), (7.9), (7.10), (7.11), and (7.12), we can express (7.6) as

$$G_N^{(n,L)}(x|y, z) = \frac{-(n-N-1)x + L + N + 1}{x(x - \frac{L}{n})} g_N^{(n,L)}(x) \alpha_{N-1}^{(n,L)}(x, y) \alpha_{N-1}^{(n,L)}(x, z) \\ - \frac{x+1}{x(x - \frac{L}{n})} \frac{1-yz}{(1-y)(1-z)} R_{N-1}(x, y, z)$$

$$\begin{aligned}
& - \frac{(1-yz)\left\{(1+x)(1-yz) + (N+L-(n-N)x)(1-y)(1-z)\right\}\alpha_{N-1}^{(n,L)}(x,yz)}{x\left(x-\frac{L}{n}\right)(1-y)^2(1-z)^2} \\
& + \frac{z^N((N+L-(n-N)x)(1-z) + 1+x)\alpha_{N-1}^{(n,L)}(x,y)}{(1-z)^2x\left(x-\frac{L}{n}\right)} \\
& + \frac{y^N((N+L-(n-N)x)(1-y) + 1+x)\alpha_{N-1}^{(n,L)}(x,z)}{(1-y)^2x\left(x-\frac{L}{n}\right)}.
\end{aligned}$$

After long and involved, but simple computations, we have

$$(7.13) \quad x^2 G_N^{(n,L)}(x|y,z) = \frac{T_A^{(n,L)}(x,y,z)}{(1-y)^2(1-z)^2} + \frac{T_B^{(n,L)}(x,y,z)}{(1-y)(1-z)} + \frac{T_C^{(n,L)}(x,y,z)}{(1-y)^2(1-z)} + \frac{T_C^{(n,L)}(x,z,y)}{(1-y)(1-z)^2},$$

where we set

$$\begin{aligned}
(7.14) \quad & \widehat{g}_N^{(n,L)}(x)T_A^{(n,L)}(x,y,z) \\
& = (1+x)(N+L+1+x)\left\{\widehat{q}_N^{(n,L)}(xy|x)\widehat{q}_N^{(n,L)}(xz|x) - \widehat{q}_N^{(n,L)}(xyz|x)\widehat{q}_N^{(n,L)}(x|x)\right\} \\
& - (1+x)(n-N)x\left\{\widehat{q}_{N-1}^{(n,L)}(xy|x)\widehat{q}_{N-1}^{(n,L)}(xz|x) - \widehat{q}_{N-1}^{(n,L)}(xyz|x)\widehat{q}_{N-1}^{(n,L)}(x|x)\right\},
\end{aligned}$$

$$\begin{aligned}
(7.15) \quad & \widehat{g}_N^{(n,L)}(x)T_B^{(n,L)}(x,y,z) \\
& = \frac{(n-N)\Gamma(L+n+1)^2}{\Gamma(L+N+1)\Gamma(L+N)\Gamma(n+1-N)^2}x(xy)^N(xz)^N \\
& + \frac{(n-N)(L+N-(n-N)x)\Gamma(L+n+1)}{\Gamma(L+N+1)\Gamma(n+1-N)}x(xyz)^N\widehat{q}_{N-1}^{(n,L)}(x|x) \\
& + \left\{\frac{(L+n)x\widehat{q}_{N-1}^{(n,L)}(xyz)}{1+xyz} - \frac{x\Gamma(L+n+1)}{(1+xyz)\Gamma(n+1)\Gamma(L)} + \frac{x(xyz)^N\Gamma(L+n+1)}{(1+xyz)\Gamma(L+N)\Gamma(n+1-N)}\right\} \\
& \times \left\{(N+L+1+x)\widehat{q}_N^{(n,L)}(x|x) - (n-N)x\widehat{q}_{N-1}^{(n,L)}(x|x)\right\},
\end{aligned}$$

and

$$\begin{aligned}
(7.16) \quad & \widehat{g}_N^{(n,L)}(x)T_C^{(n,L)}(x,y,z) \\
& = \frac{(1+x)x(xyz)^N\Gamma(L+n+1)}{\Gamma(L+N+1)\Gamma(n-N)}\widehat{q}_N^{(n,L)}(x|x) - \frac{(1+x)x(xz)^N\Gamma(L+n+1)}{\Gamma(L+N+1)\Gamma(n-N)}\widehat{q}_N^{(n,L)}(xy|x).
\end{aligned}$$

We write

$$(7.17) \quad W_N^{(n,L)}(x,y,z) = \widehat{q}_N^{(n,L)}(xy|x)\widehat{q}_N^{(n,L)}(xz|x) - \left(1 - \frac{x(1-y)(1-z)}{1+x}\right)\widehat{q}_N^{(n,L)}(x|x)\widehat{q}_N^{(n,L)}(xyz|x).$$

Then, we have

$$\begin{aligned}
W_N^{(n,L)}(x,y,z) & = W_{N+1}^{(n,L)}(x,y,z) - \frac{\Gamma(L+n+1)(xy)^{N+1}\widehat{q}_{N+1}^{(n,L)}(xz|x)}{\Gamma(L+N+2)\Gamma(n-N)} - \frac{\Gamma(L+n+1)(xz)^{N+1}\widehat{q}_{N+1}^{(n,L)}(xy|x)}{\Gamma(L+N+2)\Gamma(n-N)} \\
& + \left(1 - \frac{x(1-y)(1-z)}{1+x}\right)\frac{\Gamma(L+n+1)}{\Gamma(L+N+2)\Gamma(n-N)}(xyz)^{N+1}\widehat{q}_{N+1}^{(n,L)}(x|x) \\
& + \left(1 - \frac{x(1-y)(1-z)}{1+x}\right)\frac{\Gamma(L+n+1)}{\Gamma(L+N+2)\Gamma(n-N)}x^{N+1}\widehat{q}_{N+1}^{(n,L)}(xyz|x) \\
& + \frac{x(1-y)(1-z)\Gamma(L+n+1)^2(xy)^{N+1}(xz)^{N+1}}{1+x\Gamma(L+N+2)^2\Gamma(n-N)^2}.
\end{aligned}$$

Together with the above identity, (7.13) with (7.14), (7.15), (7.16), and (7.17),

$$x^2 \widehat{g}_N^{(n,L)}(x)G_N^{(n,L)}(x,y,z)$$

$$= \frac{(1+x)}{(1-y)^2(1-z)^2} \left((L+N+1)W_{N+1}^{(n,L)}(x,y,z) - x(n-N-1)W_N^{(n,L)}(x,y,z) \right) + \frac{R_N^{(n,L)}(x,y,z)}{(1-y)(1-z)},$$

where

$$\begin{aligned} & R_N^{(n,L)}(x,y,z) \\ &= \widehat{g}_N^{(n,L)}(x)r_N^{(n,L)}(x,y,z) \\ &= \frac{x^2(n-N-1)\Gamma(L+n+1)(xyz)^N \widehat{q}_{N-1}^{(n,L)}(x|x)}{\Gamma(L+N+1)\Gamma(n-N)} + \frac{x^2(n-N-1)\Gamma(L+n+1)(xyz)^N \widehat{q}_N^{(n,L)}(x|x)}{\Gamma(L+N+1)\Gamma(n-N+1)} \\ &= \frac{x\Gamma(L+n+1)(xyz)^{N+1} \widehat{q}_{N+1}^{(n,L)}(x|x)}{\Gamma(L+N+1)\Gamma(n-N)} + \frac{x(L+N+1)(n-N-1)\Gamma(L+n+1)(xyz)^N \widehat{q}_N^{(n,L)}(x|x)}{\Gamma(L+N+1)\Gamma(n-N+1)}, \end{aligned}$$

and

$$\begin{aligned} r_N^{(n,L)}(x,y,z) &= x(n-N-1)\widehat{q}_{N-1}^{(n,L)}(xyz|x) \\ &+ \frac{x}{(1+xyz)} \left\{ (L+N)\widehat{q}_N^{(n,L)}(xyz|xyz) - xyz(n-N)\widehat{q}_{N-1}^{(n,L)}(xyz|xyz) \right. \\ &\left. - \left(\frac{N(1+xyz)}{nxyz-L} + \frac{(n-N)(1+xyz)}{nx-L} \right) \frac{\Gamma(n+L+1)}{\Gamma(n+1)\Gamma(L)} \right\}. \end{aligned}$$

By taking $x \mapsto \lambda\bar{\lambda}$, $y \mapsto \bar{z}/\bar{\lambda}$, $z \mapsto w/\lambda$ and some computations, we complete the proof. \square

8. PROOF OF MAIN RESULTS

In this section, we conclude the proof of the main result in this note. As already seen in [13], once we know the scaling limit of the joint averaged diagonal overlap, we can readily find the joint averaged off-diagonal overlap via Lemma 2.5. Our proof is strongly inspired by [39]. We first collect the asymptotic behavior of (2.37).

Lemma 8.1. *Let*

$$\delta_N(p) = \frac{n+L+1}{N} \frac{1}{(1+|p|^2)^2}, \quad p \in \text{clo}(S_s).$$

(I) Strong non-unitarity regime: *Let $L = aN$ and $n = (b+1)N$ with fixed $a \geq 0$ and $b > 0$. For $k = 0, 1, 2$, we have*

$$(8.1) \quad \begin{aligned} & \widehat{q}_{N+k-1}^{(n,L)}(\bar{z}w|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{z},w) \\ &= \begin{cases} \frac{e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\bar{\zeta}\eta}}{1+|p|^2} (1 + O(e^{-\epsilon N})) & \text{(bulk),} \\ \frac{e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\bar{\zeta}\eta}}{1+p^2} \left(F(\bar{\zeta}+\eta) + \frac{C_{\text{out}}(\zeta,\eta)}{\sqrt{N}} + \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} k e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2} + O\left(\frac{1}{N}\right) \right) & \text{(outer),} \\ \frac{e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\bar{\zeta}\eta}}{1+p^2} \left(F(\bar{\zeta}+\eta) - \frac{1}{\chi+\bar{\chi}} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2}}{\sqrt{2\pi}} + \frac{C_{\text{out}}(\zeta,\eta)}{\sqrt{N}} + O\left(\frac{1}{N}\right) \right) & \text{(inner),} \end{cases} \end{aligned}$$

as $N \rightarrow \infty$, uniformly for ζ, η, χ in compact subsets of \mathbb{C} . Here, F is defined by (2.11).

(II) Weakly non-unitarity regime: *Let $L = \frac{N^2}{\rho^2} - N$ and $n = \frac{N^2}{\rho^2}$ with fixed $\rho > 0$. For $z =$*

$$e^{i\theta} \left(1 + \frac{\zeta}{\sqrt{N\delta_N(1)}} \right), \quad w = e^{i\theta} \left(1 + \frac{\eta}{\sqrt{N\delta_N(1)}} \right) \quad \text{with } \theta \in [0, 2\pi), \text{ we have that for } k = 0, 1, 2,$$

$$(8.2) \quad \begin{aligned} & \widehat{q}_{N-1+k}^{(n,L)}(\bar{z}w|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(\bar{z},w) \\ &= \frac{e^{-\frac{1}{2}(|\zeta|^2+|\eta|^2)+\bar{\zeta}\eta}}{2} \left(\mathcal{J}_\rho(\bar{\zeta}, \eta|\bar{\chi}, \chi) + k \frac{\rho}{N} \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta-\frac{\rho}{\sqrt{2}})^2}}{\sqrt{\pi}} + \frac{C_w(\zeta, \eta)}{N} + O(N^{-2}) \right), \end{aligned}$$

as $N \rightarrow \infty$, where some constant $C_w(\zeta, \eta)$ depends on ζ, η, ρ , and

$$(8.3) \quad \mathcal{J}_\rho(\bar{\zeta}, \eta|\bar{\chi}, \chi) = L_\rho(\bar{\zeta}+\eta) + \frac{e^{-\frac{1}{2}(\bar{\zeta}+\eta+\frac{\rho}{\sqrt{2}})^2}}{\sqrt{2\pi(\chi+\bar{\chi}+\frac{\rho}{\sqrt{2}})}}.$$

Here, the convergence is uniform for ζ, η, χ in compact subsets of \mathbb{C} , and $L_\rho(z)$ is defined by (2.13).

(III) At singular points regime: Let $L \geq 0$ be fixed and $n = (b+1)N$ with fixed $b > 0$. For $z = \frac{\zeta}{\sqrt{N\delta_N(0)}}$, $w = \frac{\eta}{\sqrt{N\delta_N(0)}}$ with ζ, η in a compact subset of \mathbb{C} , we have that for $k = 0, 1, 2$,

$$(8.4) \quad \widehat{q}_{N+k-1}^{(n,L)}(\bar{z}w|\lambda\bar{\lambda})\widehat{\omega}_N^{(n,L)}(\bar{z}, w) = \frac{1}{\chi\bar{\chi} - L}(\bar{\zeta}\eta)^L \mathcal{E}_{1,L}(\bar{\zeta}\eta|\bar{\chi}\chi) e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)} (1 + o(1)),$$

uniformly for ζ, η, χ in a compact subset of \mathbb{C} and we omitted the cocycle factor. Here, $E_{a,b}(z)$ is defined by (2.15), and $\mathcal{E}_{1,c}(z|x)$ is defined by (2.69).

Remark 8.2. In Lemma 8.1, we have omitted a co-cycle factor for the simplicity of the notation. Indeed, it does not affect the results in this note since we can readily find that a co-cycle factor is canceled out or put together in a leading term from the form of (2.55), (2.56), and (2.57).

Proof. Our proof is based on [39, proof of Lemma 3.5.]. Therefore, we firstly assume that $z, w \in \mathbb{R}$, and we also assume that the parameters L, n are integers. Then, following the same strategy by [39, proof of Lemma 3.5.], we prove Lemma 8.1 by the probabilistic argument. And after that, we extend the validity to the complex variables $z, w \in \mathbb{C}$ by Vitali's Lemma. For the detailed discussion of the extension to complex variables $z, w \in \mathbb{C}$, we refer to [39, proof of Lemma 3.5.]. Also, for the notational convenience, we still use the complex conjugate notation. Moreover, we in sequel omit the a co-cycle factor in each resulting asymptotic behavior to lighten the notation. Now, we shall prove the claim in (I). Through proof, we also assume that $p \in \mathbb{R}$ from the perspective of Vitali's argument. Let

$$\mathfrak{p} = \frac{z\bar{w}}{1 + z\bar{w}}, \quad \text{for } z, w \in \mathbb{R}.$$

Note that

$$\begin{aligned} \widehat{q}_{N-1}^{(n,L)}(z\bar{w})\widehat{\omega}_N^{(n,L)}(z, \bar{w}) &= \mathfrak{p}^{-L}(1 - \mathfrak{p})^{-n}\widehat{\omega}_N^{(n,L)}(z, \bar{w}) \sum_{k=0}^{N-1} \binom{n+L}{k+L} \mathfrak{p}^{k+L}(1 - \mathfrak{p})^{n-k} \\ &= \mathfrak{p}^{-L}(1 - \mathfrak{p})^{-n}\widehat{\omega}_N^{(n,L)}(z, \bar{w})\mathfrak{q}_{N-1}^{(n,L)}(\mathfrak{p}), \end{aligned}$$

where

$$\mathfrak{q}_{N-1}^{(n,L)}(\mathfrak{p}) = \sum_{k=0}^{N-1} \binom{n+L}{k+L} \mathfrak{p}^{k+L}(1 - \mathfrak{p})^{n-k}.$$

Let $X \stackrel{d}{\sim} B(n+L, \mathfrak{p})$ be the binomial distribution. Then, we can express

$$\begin{aligned} \mathfrak{q}_{N-1}^{(n,L)}(\mathfrak{p}) &= \mathbb{P}(L \leq X \leq N+L-1) \\ &= \mathbb{P}\left(\frac{L - (n+L)\mathfrak{p}}{\sqrt{(n+L)\mathfrak{p}(1-\mathfrak{p})}} \leq \frac{X - (n+L)\mathfrak{p}}{\sqrt{(n+L)\mathfrak{p}(1-\mathfrak{p})}} \leq \frac{N+L-1 - (n+L)\mathfrak{p}}{\sqrt{(n+L)\mathfrak{p}(1-\mathfrak{p})}}\right). \end{aligned}$$

(I) Strongly non-unitary regime: under the setting of (I), by Taylor expansion, we have

$$\begin{aligned} \frac{N+L-1 - (n+L)\mathfrak{p}}{\sqrt{(n+L)\mathfrak{p}(1-\mathfrak{p})}} &= \frac{a+1 - b\mathfrak{p}^2}{\sqrt{a+b+1}\mathfrak{p}}\sqrt{N} - \frac{(1+p^2)(a+1+b\mathfrak{p}^2)}{2(a+b+1)\mathfrak{p}^2}(\zeta + \bar{\eta}) + O\left(\frac{1}{\sqrt{N}}\right), \\ \frac{L - (n+L)\mathfrak{p}}{\sqrt{(n+L)\mathfrak{p}(1-\mathfrak{p})}} &= \frac{a - (b+1)\mathfrak{p}^2}{\sqrt{a+b+1}\mathfrak{p}}\sqrt{N} - \frac{(1+p^2)(a+(1+b)\mathfrak{p}^2)}{2(a+b+1)\mathfrak{p}^2}(\zeta + \bar{\eta}) + O\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

Then, for $p \in \text{int}S_{(\mathfrak{r})}$, by the Gaussian approximation of the binomial distribution, we have

$$\mathbb{P}(L \leq X \leq N+L-1) = \mathbb{P}(-\infty \leq \mathcal{N} \leq \infty) + O(e^{-cN}) = 1 + O(e^{-cN}),$$

where \mathcal{N} denotes the standard Gaussian distribution and c is a positive constant. On the other hand, for the outer edge case, we have

$$\mathbb{P}(L \leq X \leq N+L-1) = \mathbb{P}(-\infty \leq \mathcal{N} \leq -(\zeta + \bar{\eta})) + \frac{C_{\text{out}}(\zeta, \bar{\eta})}{\sqrt{N}} + O\left(\frac{1}{N}\right).$$

Similarly, for the inner edge case, we have

$$\mathbb{P}(L \leq X \leq N + L - 1) = \mathbb{P}(-\infty \leq \mathcal{N} \leq -(\zeta + \bar{\eta})) + \frac{C_{\text{in}}(\zeta, \bar{\eta})}{\sqrt{N}} + O\left(\frac{1}{N}\right),$$

where

$$\mathbb{P}(-\infty \leq \mathcal{N} \leq -(\zeta + \bar{\eta})) = \frac{1}{2} \operatorname{erfc}\left(\frac{\zeta + \bar{\eta}}{\sqrt{2}}\right) = F(\zeta + \bar{\eta}).$$

Here, $C_{\text{out}}(\zeta, \bar{\eta})$ and $C_{\text{in}}(\zeta, \bar{\eta})$ only depend on $\zeta, \bar{\eta}, a, b$. By Taylor expansion, we have

$$\begin{aligned} \widehat{\omega}^{(n,L)}(z, \bar{w}) &= \left(1 + O(N^{-1/2})\right) \frac{p^{2aN}}{(1+p^2)^{(a+b+1)N+1}} \exp\left(\pm \frac{a - (b+1)p^2}{2p\sqrt{a+b+1}}(\zeta + \bar{\zeta} + \eta + \bar{\eta})\sqrt{N}\right) \\ &\quad \times \exp\left(\frac{a - (b+1)p^2}{2(a+b+1)} \frac{1+p^2}{p^2} (|\zeta|^2 + |\eta|^2)\right) \\ &\quad \times \exp\left(-\frac{a(1+p^2) + (a - (b+1)p^2)p^2}{4(a+b+1)p^2} ((\zeta + \bar{\zeta})^2 + (\eta + \bar{\eta})^2)\right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{p}^{-L}(1-\mathbf{p})^{-n} &= \left(1 + O(N^{-1/2})\right) p^{-2aN} (1+p^2)^{(a+b+1)N} \exp\left(\pm \frac{(-a + (b+1)p^2)}{p\sqrt{a+b+1}}(\zeta + \bar{\eta})\sqrt{N}\right) \\ &\quad \times \exp\left(-\frac{(a - (b+1)p^2)(1+p^2)}{(a+b+1)p^2} \zeta \bar{\eta} (\zeta + \bar{\eta}) + \frac{(1+p^2)a + (a - (b+1)p^2)p^2}{2(a+b+1)p^2} (\zeta + \bar{\eta})^2\right). \end{aligned}$$

Here, we adopt the sign $+$ if the outer edge, and we adopt the sign $-$ if the inner edge. Together with these asymptotic behaviors, we have

$$(8.5) \quad \mathbf{p}^{-L}(1-\mathbf{p})^{-n} \widehat{\omega}^{(n,L)}(z, \bar{w}) = c(\zeta, \eta) \frac{1}{1+p^2} G(\zeta, \bar{\eta}) (1 + o(1)),$$

where $c(\zeta, \eta)$ is a cocycle factor, which does not affect the value of determinant. Although we can explicitly write it, we do not need its explicit form here. We need the asymptotic expansion of the remainder term in the summation. Observe that by the Stirling formula,

$$\begin{aligned} \binom{n+L}{N+L} &= \frac{\Gamma(n+L+1)}{\Gamma(N+L+1)\Gamma(n-N+1)} = \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} \left(\frac{(a+b+1)^{a+b+1}}{(a+1)^{a+1}b^b}\right)^N (1 + O(N^{-1})), \\ \binom{n+L}{N+1+L} &= \frac{\Gamma(n+L+1)}{\Gamma(N+L+2)\Gamma(n-N)} = \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} \left(\frac{(a+b+1)^{a+b+1}}{(a+1)^{a+1}b^b}\right)^N \frac{b}{a+1} (1 + O(N^{-1})). \end{aligned}$$

Also, observe that

$$\begin{aligned} \mathbf{p}^{N+L}(1-\mathbf{p})^{n-N} &= \begin{cases} \left(\frac{(a+1)^{a+1}b^b}{(a+b+1)^{a+b+1}}\right)^N e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} (1 + O(N^{-1/2})) & \text{(outer),} \\ \left(\frac{a^{a+1}(b+1)^b}{(a+b+1)^{a+b+1}}\right)^N (1 + O(N^{-1})) & \text{(inner),} \end{cases} \\ \mathbf{p}^{N+L+1}(1-\mathbf{p})^{n-N-1} &= \begin{cases} \left(\frac{(a+1)^{a+1}b^b}{(a+b+1)^{a+b+1}}\right)^N \frac{a+1}{b} e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} (1 + O(N^{-1/2})) & \text{(outer),} \\ \left(\frac{a^{a+1}(b+1)^b}{(a+b+1)^{a+b+1}}\right)^N \frac{a}{b+1} (1 + O(N^{-1})) & \text{(inner).} \end{cases} \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \binom{n+L}{N+L} \mathbf{p}^{N+L}(1-\mathbf{p})^{n-N} &= \begin{cases} \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} (1 + O(N^{-1/2})) & \text{(outer),} \\ O(e^{-N}) & \text{(inner),} \end{cases} \\ \binom{n+L}{N+1+L} \mathbf{p}^{N+L+1}(1-\mathbf{p})^{n-N-1} &= \begin{cases} \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} (1 + O(N^{-1/2})) & \text{(outer),} \\ O(e^{-N}) & \text{(inner),} \end{cases} \end{aligned}$$

for some constant $c > 1$. Furthermore, by Stirling formula, we have

$$(8.6) \quad \frac{\Gamma(n+L+1)}{\Gamma(n+1)\Gamma(L)} \widehat{\omega}^{(n,L)}(z, w) = \begin{cases} O(e^{-\epsilon N}) & \text{(bulk),} \\ O(e^{-\epsilon N}) & \text{(outer),} \\ -\frac{1}{1+p^2} \frac{e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} G(\zeta, \bar{\eta})}{\sqrt{2\pi}(\chi+\bar{\chi})} \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) & \text{(inner).} \end{cases}$$

Here, ϵ is a positive constant independent of N , and note that we omitted a co-cycle factor. As a consequence, by combining these asymptotic behaviours, we obtain that for $k = 0, 1, 2$,

$$\widehat{q}_{N+k-1}^{(n,L)}(z\bar{w}|\lambda\bar{\lambda})\widehat{\omega}^{(n,L)}(z, w) = \begin{cases} \frac{G(\zeta, \bar{\eta})}{1+p^2} (1 + O(e^{-\epsilon N})) & \text{(bulk),} \\ \frac{G(\zeta, \bar{\eta})}{1+p^2} \left(F(\zeta + \bar{\eta}) + \frac{C_{\text{out}}(\zeta, \eta)}{\sqrt{N}} + \sqrt{\frac{a+b+1}{2\pi(a+1)bN}} k e^{-\frac{1}{2}(\zeta+\bar{\eta})^2} + O\left(\frac{1}{N}\right) \right) & \text{(outer),} \\ \frac{G(\zeta, \bar{\eta})}{1+p^2} \left(F(\zeta + \bar{\eta}) - \frac{1}{\chi+\bar{\chi}} \frac{e^{-\frac{1}{2}(\zeta+\bar{\eta})^2}}{\sqrt{2\pi}} + \frac{C_{\text{out}}(\zeta, \eta)}{\sqrt{N}} + O\left(\frac{1}{N}\right) \right) & \text{(inner),} \end{cases}$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} .

(II) Weakly non-unitary regime: under the setting of (II), by Taylor expansion, we have

$$\begin{aligned} \frac{N+L-1-(n+L)\mathbf{p}}{\sqrt{(n+L)\mathbf{p}(1-\mathbf{p})}} &= -\left(\zeta + \bar{\eta} - \frac{\rho}{\sqrt{2}}\right) + O(N^{-1}), \\ \frac{L-(n+L)\mathbf{p}}{\sqrt{(n+L)\mathbf{p}(1-\mathbf{p})}} &= -\left(\zeta + \bar{\eta} + \frac{\rho}{\sqrt{2}}\right) + O(N^{-1}). \end{aligned}$$

Then, by Gaussian approximation, we have

$$\begin{aligned} \mathfrak{q}_{N-1}^{(n,L)}(\mathbf{p}) &= \mathbb{P}(L \leq X \leq N+L-1) \\ &= \mathbb{P}\left(-\left(\eta + \bar{\zeta} + \frac{\rho}{\sqrt{2}}\right) \leq \mathcal{N} \leq -\left(\eta + \bar{\zeta} - \frac{\rho}{\sqrt{2}}\right)\right) + O(N^{-1}), \quad \text{as } N \rightarrow \infty. \end{aligned}$$

By Taylor expansion, we have

$$(8.7) \quad \widehat{\omega}^{(n,L)}(\bar{z}, w) = c_N(\zeta, \eta) 2^{N-\frac{2N^2}{\rho^2}} \frac{1}{2} G(\eta, \bar{\zeta}) e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2 - \frac{\sqrt{2}}{2}\rho(\bar{\zeta}+\eta)} (1 + O(N^{-1})),$$

where $c_N(\zeta, \eta)$ is a co-cycle factor. Also, notice that

$$\begin{aligned} \mathbf{p}^{-L}(1-\mathbf{p})^{-n} &= 2^{\frac{2N^2}{\rho^2}-N} e^{\frac{\sqrt{2}}{2}\rho(\bar{\zeta}+\eta) + \frac{1}{2}(\bar{\zeta}+\eta)^2} (1 + O(N^{-1})), \\ \mathbf{p}^{N+L}(1-\mathbf{p})^{n-N} &= 2^{-\frac{2N^2}{\rho^2}+N} e^{\frac{\sqrt{2}}{2}\rho(\bar{\zeta}+\eta) - \frac{1}{2}(\bar{\zeta}+\eta)^2} (1 + O(N^{-1})), \end{aligned}$$

and

$$\mathbf{p}^{N+L+1}(1-\mathbf{p})^{n-N-1} = 2^{-\frac{2N^2}{\rho^2}+N} e^{\frac{\sqrt{2}}{2}\rho(\bar{\zeta}+\eta) - \frac{1}{2}(\bar{\zeta}+\eta)^2} (1 + O(N^{-1})),$$

which give that for $k = 1, 2$,

$$\widehat{\omega}^{(n,L)}(\bar{z}, w) \mathbf{p}^{-L}(1-\mathbf{p})^{-n} \mathbf{p}^{N+L+(k-1)}(1-\mathbf{p})^{n-N-(k-1)} = 2^{N-\frac{2N^2}{\rho^2}} \frac{1}{2} G(\eta, \bar{\zeta}) e^{-\frac{1}{2}(\bar{\zeta}+\eta)^2 + \frac{\sqrt{2}}{2}\rho(\bar{\zeta}+\eta)} (1 + O(N^{-1})).$$

Also, by Stirling formula, we have

$$\begin{aligned} \binom{n+L}{N+L} &= 2^{\frac{2N^2}{\rho^2}-N} e^{-\frac{\rho^2}{4}} \frac{\rho}{\sqrt{\pi N}} \left(1 + \frac{2\rho^2 - \rho^4}{8N} + O(N^{-2})\right), \\ \binom{n+L}{N+L+1} &= 2^{\frac{2N^2}{\rho^2}-N} e^{-\frac{\rho^2}{4}} \frac{\rho}{\sqrt{\pi N}} \left(1 - \frac{6\rho^2 + \rho^4}{8N} + O(N^{-2})\right). \end{aligned}$$

Therefore, we have that for $k = 1, 2$,

$$\begin{aligned} &\mathbf{p}^{-L}(1-\mathbf{p})^{-n} \widehat{\omega}^{(n,L)}(\bar{z}, w) \sum_{j=1}^k \binom{n+L}{N+L+j-1} \mathbf{p}^{N+L+j-1} (1-\mathbf{p})^{n-N-(j-1)} \\ &= k \frac{\rho}{2\sqrt{\pi N}} G(\eta, \bar{\zeta}) e^{-\frac{1}{2}(\bar{\zeta}+\eta - \frac{\rho}{\sqrt{2}})^2} (1 + O(N^{-1})). \end{aligned}$$

Here, note that

$$\frac{1}{nx-L} \frac{\Gamma(L+n+1)}{\Gamma(n+1)\Gamma(L)} \widehat{\omega}^{(n,L)}(\bar{z}, w) = \frac{G(\eta, \bar{\zeta})}{2\sqrt{2\pi}(\chi + \bar{\chi} + \frac{\rho}{\sqrt{2}})} e^{-\frac{1}{2}(\bar{\zeta} + \eta + \frac{\rho}{\sqrt{2}})^2} (1 + O(N^{-1})).$$

Together with these asymptotic expansions, we obtain

$$\widehat{q}_{N-1+k}^{(n,L)}(\bar{z}w|\lambda\bar{\lambda}) \widehat{\omega}^{(n,L)}(\bar{z}, w) = \frac{G(\eta, \bar{\zeta})}{2} \left(\mathcal{J}_\rho(\bar{\zeta}, \eta|\bar{\chi}, \chi) + k \frac{\rho}{N} \frac{e^{-\frac{1}{2}(\bar{\zeta} + \eta - \frac{\rho}{\sqrt{2}})^2}}{\sqrt{\pi}} + \frac{C_w(\zeta, \eta)}{N} + O(N^{-2}) \right),$$

as $N \rightarrow \infty$, where some constant $C_w(\zeta, \eta)$ depends on ζ, η, ρ . Here, the convergence is uniform for ζ, η, χ in compact subsets of \mathbb{C} .

(III) Singular origin regime: under the setting of (III), by Taylor expansion, we have

$$z\bar{w} = \frac{\zeta\bar{\eta}}{1+b} \frac{1}{N} + O\left(\frac{1}{N^2}\right).$$

By Poisson approximation, we have

$$\mathbb{P}(L \leq X \leq N+k+L-1) = P(L, \zeta\bar{\eta}) (1 + o(1)), \quad \text{as } N \rightarrow \infty,$$

where

$$P(c, z) := \frac{1}{\Gamma(c)} \int_0^z t^{c-1} e^{-t} dt = e^{-z} z^c E_{1,1+c}(z) \quad \text{for } c > 0.$$

On the other hand, since

$$\mathbf{p}^{-L} (1-\mathbf{p})^{-n} \widehat{\omega}_N^{(n,L)}(z, w) = c(\zeta, \eta) e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2) + \zeta\bar{\eta}},$$

where $c(\zeta, \eta)$ is a co-cycle factor, and hence, by simple computations, we have

$$\widehat{q}_{N+k-1}^{(n,L)}(z\bar{w}) \widehat{\omega}_N^{(n,L)}(z, w) = \frac{1}{\chi\bar{\chi} - L} (\zeta\bar{\eta})^L \mathcal{E}_{1,L}(\zeta\bar{\eta}|\bar{\chi}\chi) e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)} (1 + o(1)),$$

uniformly for ζ, η, χ in a compact subset of \mathbb{C} and we omitted the cocycle factor. \square

Remark 8.3. *Our claim can be also proved by using the asymptotics for the incomplete beta function in [132].*

With help of Lemma 8.1, we can complete the proof of Theorem 2.10. Now, we shall finish the proof for each case.

8.1. Proof of the case of ISUE in the strongly non-unitary regime in Theorem 2.10. In this subsection, we assume that $L = aN$ and $n = (b+1)N$ with $a, b \geq 0$.

Proof of the case of ISUE in the strongly non-unitary regime in Theorem 2.10: bulk case. Through the proof of the bulk case in the case of ISUE in the strongly non-unitary regime in Theorem 2.10, let

$$z = p + \frac{\zeta}{\sqrt{N\delta_N(p)}}, \quad w = p + \frac{\eta}{\sqrt{N\delta_N(p)}}, \quad \lambda = p + \frac{\chi}{\sqrt{N\delta_N(p)}}.$$

First, note that

$$N\delta_N(p) = N \frac{a+b+1}{(1+|p|^2)^2} (1 + O(N^{-1})).$$

Second, note that

$$\varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = \frac{(1+|p|^2)^2}{(a+b+1)N} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) (1 + O(N^{-1})),$$

and $C_K(z, w)$ is the co-cycle factor. Notice also that

$$\widehat{g}_N^{(n,L)}(\lambda\bar{\lambda}) \widehat{\omega}^{(n,L)}(\lambda, \bar{\lambda}) = N \frac{a+1-b|p|^2}{1+|p|^2} (1 + O(N^{-1})).$$

Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \frac{1}{N\delta_N(p)} \frac{n(z\bar{z} - \frac{L}{n})}{z\bar{z}(1+z\bar{z})} g_{N-1}^{(n,L)}(z\bar{z}) \widehat{\omega}^{(n,L)}(\bar{z}, z) = \frac{b(b+1)}{a+b+1} \frac{(|p|^2 - \frac{a}{b+1}) \left(\frac{a+1}{b} - |p|^2\right)}{|p|^2}.$$

Since

$$\begin{aligned} & (L + N + 1)Q_{N+1}^{(n,L)}(\bar{z}, w, \lambda) - \lambda\bar{\lambda}(n - N - 1)Q_{N+1}^{(n,L)}(\bar{z}, w, \lambda) \\ &= N \frac{a+1-b|p|^2}{(1+|p|^2)^2} G(\bar{\zeta}, \eta) e^{-\bar{\zeta}-\bar{\eta}(\eta-\chi)} (1 - e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)}) (1 + O(N^{-1})), \end{aligned}$$

we have

$$\mathfrak{H}_N^{(n,L)}(z, w, \lambda) = (N\delta_N(p))^2 \frac{1 - e^{-(\bar{\zeta}-\bar{\chi})(\eta-\chi)}}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2} G(\bar{\zeta}, \eta) e^{-\bar{\zeta}-\bar{\chi}(\eta-\chi)} (1 + O(N^{-1})).$$

On the other hand, we have

$$\mathfrak{F}_N^{(n,L)}(z, w, \lambda) = \frac{(N\delta_N(p))^2}{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} G(\bar{\zeta}, \eta) (1 + O(N^{-1})).$$

Hence, there exists a conjugation factor $c_N(\zeta, \eta, \chi)$ such that we have

$$\begin{aligned} K_{1,1,(s)}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) &= c_N(\zeta, \eta, \chi) N\delta_N(p) \frac{1 + (\bar{\zeta} - \bar{\chi})(\eta - \chi) e^{-\bar{\zeta}-\bar{\chi}(\eta-\chi)} - e^{-\bar{\zeta}-\bar{\chi}(\eta-\chi)}}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2} \\ &\quad \times \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) e^{-\bar{\zeta}-\bar{\chi}(\zeta-\chi)} (1 + O(N^{-1})), \end{aligned}$$

which gives

$$\lim_{N \rightarrow \infty} \frac{1}{N\delta_N(p)} K_{1,1,(s)}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}) = K_{1,1}^{(\mathbf{b})}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda}),$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . This completes the proof for the diagonal case. By decoupling Lemma 2.5, it suffices to consider the front factor. From the proof of the diagonal case and by (2.50), we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \frac{-1}{(N\delta_N(p))^2} \frac{n(z\bar{w} - \frac{L}{n})}{z\bar{w}(1+z\bar{w})} \hat{g}_{N-1}^{(n,L)}(z\bar{w}) \hat{\omega}^{(n,L)}(\bar{w}, z) \mathcal{K}_{N-1}^{(n,L)}(\bar{z}, w|z, \bar{w}) \hat{\omega}^{(n,L)}(\bar{z}, w) \\ &= -\frac{b(b+1)}{a+b+1} \frac{\left(|p|^2 - \frac{a}{b+1}\right) \left(\frac{a+1}{b} - |p|^2\right)}{|p|^2} \mathcal{K}_{1,1}^{(\mathbf{b})}(\bar{\zeta}, \eta|\zeta, \bar{\eta}). \end{aligned}$$

This completes the proof. \square

Proof of the case of ISUE in the strongly non-unitary regime in Theorem 2.10: edge case. Through the proof of the edge case in the case of ISUE in the strongly non-unitary regime in Theorem 2.10, we often omitted a conjugation factor. for $\theta \in [0, 2\pi)$, let

$$z = e^{i\theta} \left(p + \mathfrak{s} \frac{\zeta}{\sqrt{N\delta_N(p)}} \right), \quad w = e^{i\theta} \left(p + \mathfrak{s} \frac{\eta}{\sqrt{N\delta_N(p)}} \right), \quad \lambda = e^{i\theta} \left(p + \mathfrak{s} \frac{\chi}{\sqrt{N\delta_N(p)}} \right).$$

Here, when we consider the outer edge case, then $p = \sqrt{(a+1)/b}$, and when we consider the inner edge case, then $p = \sqrt{a/(b+1)}$. Also, when we consider the outer edge case, then $\mathfrak{s} = 1$, and when we consider the inner edge case, then $\mathfrak{s} = -1$. First, note that

$$\varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = \frac{(1+|p|^2)^2}{(a+b+1)N} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi}) (1 + O(N^{-1})),$$

and

$$\hat{g}_N^{(n,L)}(\lambda\bar{\lambda}) \hat{\omega}^{(n,L)}(\bar{\lambda}, \lambda) = \begin{cases} \frac{\sqrt{N}}{\sqrt{2\pi}} \sqrt{\frac{(a+1)b}{a+b+1}} \mathcal{F}(\chi + \bar{\chi}) (1 + O(N^{-1/2})), & \text{if } \mathfrak{s} = 1, \\ -\frac{N}{\sqrt{2\pi}(\chi + \bar{\chi})} \mathcal{F}(\chi + \bar{\chi}) (1 + O(N^{-1/2})), & \text{if } \mathfrak{s} = -1, \end{cases}$$

where $\mathcal{F}(x)$ is given by (2.65). Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \frac{1}{N\delta_N(p)} \frac{n(z\bar{z} - \frac{L}{n})}{z\bar{z}(1+z\bar{z})} g_{N-1}^{(n,L)}(z\bar{z}) \hat{\omega}^{(n,L)}(\bar{z}, z) = \begin{cases} \sqrt{\frac{a+b+1}{2\pi(a+1)b}} \mathcal{F}(\zeta + \bar{\zeta}) (1 + O(N^{-1/2})) & \text{if } \mathfrak{s} = 1, \\ \sqrt{\frac{a+b+1}{2\pi a(b+1)}} \mathcal{F}(\zeta + \bar{\zeta}) (1 + O(N^{-1/2})) & \text{if } \mathfrak{s} = -1. \end{cases}$$

(1) outer edge case: By Lemma 8.1, we have

$$\begin{aligned} & (N+L+1)Q_{N+1}^{(n,L)}(z, w, \lambda) - \lambda\bar{\lambda}(n-N-1)Q_N^{(n,L)}(z, w, \lambda) \\ &= \frac{\sqrt{N}\sqrt{a+1}b^{3/2}}{\sqrt{2\pi}(a+b+1)^{3/2}} H_1(\zeta, \eta, \chi) G(\bar{\zeta}, \eta) (1 + O(N^{-1/2})), \end{aligned}$$

where we set

$$\begin{aligned} H_1(\zeta, \eta, \chi) &= \sqrt{2\pi}(\chi + \bar{\chi})F(\bar{\zeta} + \eta)F(\bar{\chi} + \chi) - F(\bar{\zeta} + \eta)e^{-\frac{1}{2}(\chi + \bar{\chi})^2} - F(\bar{\chi} + \chi)e^{-\frac{1}{2}(\bar{\zeta} + \eta)^2} \\ &+ e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \left(F(\bar{\zeta} + \chi)e^{-\frac{1}{2}(\bar{\chi} + \eta)^2} + F(\bar{\chi} + \eta)e^{-\frac{1}{2}(\bar{\zeta} + \chi)^2} - \sqrt{2\pi}(\chi + \bar{\chi})F(\bar{\zeta} + \chi)F(\bar{\chi} + \eta) \right). \end{aligned}$$

Then, we have

$$\mathfrak{H}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = N\delta_N(p) \frac{H_1(\zeta, \eta, \chi) G(\bar{\zeta}, \eta)}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{F}(\chi + \bar{\chi})} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) (1 + O(N^{-1/2})).$$

Similarly, we have

$$\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = N\delta_N(p) \frac{F(\bar{\zeta} + \eta) G(\bar{\zeta}, \eta)}{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) (1 + O(N^{-1/2})).$$

For (2.59) and (2.60), it is straightforward to see that

$$\mathfrak{J}\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = o(N\delta_N(p)),$$

and

$$\mathfrak{J}\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = o(N\delta_N(p)),$$

as $N \rightarrow \infty$. Here, we recall that $\partial_x F(a+x) = e^{-\frac{1}{2}(a+x)^2} / \sqrt{2\pi}$, and

$$\begin{aligned} & e^{-\frac{1}{2}a^2} \frac{d}{dx} \left[e^{\frac{(a+x)^2}{2}} \left(e^{-f} F(b+x)F(c+x) - F(d+x)F(a+x) + fF(d)F(a+x) \right) \right] \Big|_{x=0} \\ &= \frac{1}{\sqrt{2\pi}} \left[\sqrt{2\pi}a \left(e^{-f} F(b)F(c) - F(d)F(a) + fF(d)F(a) \right) \right. \\ & \left. - \left(e^{-f} e^{-\frac{1}{2}b^2} F(c) + e^{-f} e^{-\frac{1}{2}c^2} F(b) - e^{-\frac{1}{2}d^2} F(a) - e^{-\frac{1}{2}a^2} F(d) + f e^{-\frac{1}{2}a^2} F(d) \right) \right]. \end{aligned}$$

Here, we note that

$$\begin{aligned} & H_1(\zeta, \eta + \chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi)F(\bar{\zeta} + \eta)\mathcal{F}(\bar{\chi} + \chi) \\ &= \sqrt{2\pi}(\chi + \bar{\chi})F(\bar{\zeta} + \eta)F(\bar{\chi} + \chi) - e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \sqrt{2\pi}(\chi + \bar{\chi})F(\bar{\zeta} + \chi)F(\bar{\chi} + \eta) \\ & - \sqrt{2\pi}(\bar{\zeta} - \bar{\chi})(\eta - \chi)(\chi + \bar{\chi})F(\bar{\zeta} + \eta)F(\bar{\chi} + \chi) - F(\bar{\zeta} + \eta)e^{-\frac{1}{2}(\chi + \bar{\chi})^2} - F(\bar{\chi} + \chi)e^{-\frac{1}{2}(\bar{\zeta} + \eta)^2} \\ & + e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi)e^{-\frac{1}{2}(\bar{\chi} + \eta)^2} + e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\chi} + \eta)e^{-\frac{1}{2}(\bar{\zeta} + \chi)^2} + (\bar{\zeta} - \bar{\chi})(\eta - \chi)e^{-\frac{1}{2}(\chi + \bar{\chi})^2} F(\bar{\zeta} + \eta) \\ & = -\sqrt{2\pi}(\chi + \bar{\chi}) \left(e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi)F(\bar{\chi} + \eta) - F(\bar{\zeta} + \eta)F(\bar{\chi} + \chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi)F(\bar{\zeta} + \eta)F(\bar{\chi} + \chi) \right) \\ & + \left(e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi)e^{-\frac{1}{2}(\bar{\chi} + \eta)^2} + e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\chi} + \eta)e^{-\frac{1}{2}(\bar{\zeta} + \chi)^2} \right. \\ & \left. - F(\bar{\zeta} + \eta)e^{-\frac{1}{2}(\chi + \bar{\chi})^2} - F(\bar{\chi} + \chi)e^{-\frac{1}{2}(\bar{\zeta} + \eta)^2} + (\bar{\zeta} - \bar{\chi})(\eta - \chi)e^{-\frac{1}{2}(\chi + \bar{\chi})^2} F(\bar{\zeta} + \eta) \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & H_1(\zeta, \eta + \chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi)F(\bar{\zeta} + \eta)\mathcal{F}(\bar{\chi} + \chi) \\ &= -\sqrt{2\pi}e^{-\frac{1}{2}a^2} \frac{d}{dx} \left[e^{\frac{(a+x)^2}{2}} \left(e^{-f} F(b+x)F(c+x) - F(d+x)F(a+x) + fF(d)F(a+x) \right) \right] \Big|_{x=0}. \end{aligned}$$

where $a = \bar{\chi} + \chi$, $b = \bar{\zeta} + \chi$, $c = \bar{\chi} + \eta$, $d = \bar{\zeta} + \eta$, $f = (\bar{\zeta} - \bar{\chi})(\eta - \chi)$. This completes the proof for the outer edge case.

(2) inner edge case: Next, we consider the outer edge case. By Lemma 8.1, we have

$$(N+L+1)Q_{N+1}^{(n,L)}(z, w, \lambda) - \lambda\bar{\lambda}(n-N-1)Q_N^{(n,L)}(z, w, \lambda)$$

$$= N \frac{b+1}{a+b+1} \frac{G(\bar{\zeta}, \eta)}{\sqrt{2\pi}(\bar{\chi} + \chi)} H_2(\zeta, \eta, \chi) (1 + O(N^{-1/2})),$$

where we set

$$\begin{aligned} H_2(\zeta, \eta, \chi) &= \sqrt{2\pi}(\bar{\chi} + \chi) (e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi) F(\bar{\chi} + \eta) - F(\bar{\chi} + \chi) F(\bar{\zeta} + \eta)) \\ &\quad + e^{-\frac{1}{2}(\bar{\chi} + \chi)^2} F(\bar{\zeta} + \eta) + e^{-\frac{1}{2}(\bar{\zeta} + \eta)^2} F(\bar{\chi} + \chi) \\ &\quad - e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi) e^{-\frac{1}{2}(\bar{\chi} + \eta)^2} - e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\chi} + \eta) e^{-\frac{1}{2}(\bar{\zeta} + \chi)^2}. \end{aligned}$$

Then, we have

$$\mathfrak{H}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = -N\delta_N(p) \frac{H_2(\zeta, \eta, \chi) G(\bar{\zeta}, \eta)}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{F}(\bar{\chi} + \chi)} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) (1 + O(N^{-1/2})).$$

Similarly, we have

$$\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = N\delta_N(p) \frac{F(\bar{\zeta} + \eta) G(\bar{\zeta}, \eta)}{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) (1 + O(N^{-1/2})).$$

As in the outer edge case, for (2.59) and (2.60), it is straightforward to see that as $N \rightarrow \infty$,

$$\mathfrak{J}\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = o(N\delta_N(p)),$$

and

$$\mathfrak{J}\mathfrak{J}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) = o(N\delta_N(p)).$$

Here, we note that

$$\begin{aligned} & -H_2(\zeta, \eta, \chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi) F(\bar{\zeta} + \eta) \mathcal{F}(\bar{\chi} + \chi) \\ &= -\sqrt{2\pi}(\bar{\chi} + \chi) \left(e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi) F(\bar{\chi} + \eta) - F(\bar{\chi} + \chi) F(\bar{\zeta} + \eta) + (\bar{\zeta} - \bar{\chi})(\eta - \chi) F(\bar{\chi} + \chi) F(\bar{\zeta} + \eta) \right) \\ &+ e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\zeta} + \chi) e^{-\frac{1}{2}(\bar{\chi} + \eta)^2} + e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} F(\bar{\chi} + \eta) e^{-\frac{1}{2}(\bar{\zeta} + \chi)^2} \\ &- F(\bar{\zeta} + \eta) e^{-\frac{1}{2}(\bar{\chi} + \chi)^2} - F(\bar{\chi} + \chi) e^{-\frac{1}{2}(\bar{\zeta} + \eta)^2} + (\bar{\zeta} - \bar{\chi})(\eta - \chi) F(\bar{\zeta} + \eta) e^{-\frac{1}{2}(\chi + \bar{\chi})^2} \\ &= -\sqrt{2\pi} e^{-\frac{1}{2}a^2} \frac{d}{dx} \left[e^{\frac{(a+x)^2}{2}} \left(e^{-f} F(b+x) F(c+x) - F(d+x) F(a+x) + f F(d) F(a+x) \right) \right] \Big|_{x=0}. \end{aligned}$$

where $a = \bar{\chi} + \chi$, $b = \bar{\zeta} + \chi$, $c = \bar{\chi} + \eta$, $d = \bar{\zeta} + \eta$, $f = (\bar{\zeta} - \bar{\chi})(\eta - \chi)$. This completes the proof of the diagonal case. It suffices to consider the front factor due the decoupling Lemma 2.5. From the proof of the diagonal case and by (2.50),

$$\begin{aligned} & -\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \frac{1}{(N\delta_N(p))^2} \frac{n(z\bar{w} - \frac{L}{n})}{z\bar{w}(1+z\bar{w})} \widehat{\mathfrak{g}}_{N-1}^{(n,L)}(z\bar{w}) \widehat{\omega}^{(n,L)}(\bar{w}, z) \mathcal{K}_{N-1}^{(n,L)}(\bar{z}, w | z, \bar{w}) \widehat{\omega}^{(n,L)}(\bar{z}, w). \\ &= -\mathfrak{c}_s e^{-|\zeta - \eta|^2} \mathcal{F}(\zeta + \bar{\eta}) \frac{H(\bar{\eta} + \zeta, \bar{\zeta} + \zeta, \bar{\eta} + \eta, \bar{\eta} + \zeta, -(\bar{\zeta} - \bar{\eta})(\zeta - \eta))}{(\bar{\zeta} - \bar{\eta})^2 (\zeta - \eta)^2}. \end{aligned}$$

This completes the proof. \square

8.2. Proof of the case of ISUE in the weakly non-unitary regime in Theorem 2.10. In this section, we assume that $L = N^2/\rho^2 - N$ and $n = N^2/\rho^2$.

Proof of the case of ISUE in the weakly non-unitary regime in Theorem 2.10. Through the proof of the case of ISUE in the weakly non-unitary regime in Theorem 2.10, for $\theta \in [0, 2\pi)$, we set

$$z = e^{i\theta} \left(1 + \frac{\zeta}{\sqrt{N\delta_N(1)}} \right), \quad w = e^{i\theta} \left(1 + \frac{\eta}{\sqrt{N\delta_N(1)}} \right), \quad \lambda = e^{i\theta} \left(1 + \frac{\chi}{\sqrt{N\delta_N(1)}} \right).$$

Note that

$$N\delta_N(1) = \frac{N^2}{2\rho^2} \left(1 - \frac{\rho^2}{2N} + \frac{\rho^2}{2N^2} + O(N^{-3}) \right).$$

Notice also that

$$\varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = \frac{1}{N\delta_N(1)} \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi})(1 + O(N^{-1})),$$

and

$$\hat{g}_N^{(n,L)}(\lambda\bar{\lambda})\hat{\omega}^{(n,L)}(\bar{\lambda}, \lambda) = \frac{N}{\sqrt{2\rho}} \frac{1}{\chi + \bar{\chi} + \frac{\rho}{\sqrt{2}}} \mathcal{L}_\rho(\chi + \bar{\chi})(1 + O(N^{-1})).$$

Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N\delta_N(1)} \frac{n(\lambda\bar{\lambda} - L/n)}{\lambda\bar{\lambda}(1 + \lambda\bar{\lambda})} \hat{g}_N^{(n,L)}(\lambda\bar{\lambda})\hat{\omega}^{(n,L)}(\lambda, \bar{\lambda}) = \mathcal{L}_\rho(\chi + \bar{\chi}).$$

Since

$$\begin{aligned} & Q_{N+1}^{(n,L)}(z, w, \lambda) - Q_N(z, w, \lambda) \\ &= \frac{\rho}{N} \frac{G(\bar{\zeta}, \eta)}{2^2 \sqrt{\pi}} e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \left\{ \mathcal{J}_\rho(\bar{\zeta}, \chi|\bar{\chi}, \chi) e^{-\frac{1}{2}(\bar{\chi} + \eta - \frac{\rho}{\sqrt{2}})^2} + \mathcal{J}_\rho(\bar{\chi}, \eta|\bar{\chi}, \chi) e^{-\frac{1}{2}(\bar{\zeta} + \chi - \frac{\rho}{\sqrt{2}})^2} \right. \\ & \quad \left. - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{J}_\rho(\bar{\zeta}, \eta|\bar{\eta}, \chi) e^{-\frac{1}{2}(\bar{\chi} + \chi - \frac{\rho}{\sqrt{2}})^2} - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{J}_\rho(\bar{\chi}, \chi|\bar{\chi}, \chi) e^{-\frac{1}{2}(\bar{\zeta} + \eta - \frac{\rho}{\sqrt{2}})^2} \right\} (1 + O(N^{-1})), \end{aligned}$$

we have

$$(N + L + 1)Q_{N+1}^{(n,L)}(z, w, \lambda) - \lambda\bar{\lambda}(n - N - 1)Q_N^{(n,L)}(z, w, \lambda) = \frac{\sqrt{2}N}{\rho} \frac{G(\bar{\zeta}, \eta)}{2^2} \frac{e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)}}{\chi + \bar{\chi} + \frac{\rho}{\sqrt{2}}} H_3(\zeta, \eta, \chi)(1 + O(N^{-1})),$$

where

$$\begin{aligned} H_3(\zeta, \eta, \chi) &= \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta) + \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \chi) \\ & \quad - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \eta, \bar{\chi} + \chi) - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\chi} + \chi, \bar{\zeta} + \eta) \\ & \quad + \left(\bar{\chi} + \chi + \frac{\rho}{\sqrt{2}} \right) \left(\bar{\chi} + \chi - \frac{\rho}{\sqrt{2}} \right) \left(e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} L_\rho(\bar{\zeta} + \eta) L_\rho(\bar{\chi} + \chi) - L_\rho(\bar{\zeta} + \chi) L_\rho(\bar{\chi} + \eta) \right) \\ & \quad + \mathcal{C}_\rho(\bar{\zeta} + \chi, \bar{\chi} + \eta) - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{C}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \eta). \end{aligned}$$

Then, we have

$$\mathfrak{H}_N^{(n,L)}(z, w, \lambda) \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = N\delta_N(1) \frac{H_3(\zeta, \eta, \chi) G(\bar{\zeta}, \eta) \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi})}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{L}_\rho(\bar{\chi} + \chi)} e^{-(\bar{\zeta} - \bar{\chi})(\eta - \chi)} (1 + O(N^{-1})).$$

On the other hand, similarly, we have

$$\mathfrak{J}_N^{(n,L)}(\bar{z}, w|\bar{\lambda}, \lambda) \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = N\delta_N(1) \frac{L_\rho(\bar{\zeta} + \eta) G(\bar{\zeta}, \eta) \varpi(\zeta, \bar{\zeta}|\chi, \bar{\chi})}{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} (1 + O(N^{-1})).$$

For (2.59) and (2.60), it is straightforward to see that

$$\mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w|\bar{\lambda}, \lambda) \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = o(N\delta_N(1)),$$

and

$$\mathfrak{J}\mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w|\bar{\lambda}, \lambda) \varpi^{(n,L)}(z, \bar{z}|\lambda, \bar{\lambda}) = o(N\delta_N(1)).$$

Here, we note that

$$\begin{aligned} & H_3(\zeta, \eta, \chi) + (\bar{\zeta} - \bar{\chi})(\eta - \chi) L_\rho(\bar{\zeta} + \eta) e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{L}_\rho(\bar{\chi} + \chi) \\ &= \mathcal{A}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \eta, (\bar{\zeta} - \bar{\chi})(\eta - \chi)) + \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta) + \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \chi) \\ & \quad + (\bar{\zeta} - \bar{\chi})(\eta - \chi) e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \eta, \bar{\chi} + \chi) - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \eta, \bar{\chi} + \chi) \\ & \quad - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{B}_\rho(\bar{\chi} + \chi, \bar{\chi} + \chi, \bar{\zeta} + \eta) + \mathcal{C}_\rho(\bar{\zeta} + \chi, \bar{\chi} + \eta) - e^{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \mathcal{C}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \eta). \end{aligned}$$

Therefore, there exists a co-cycle factor $c_N(\zeta, \eta, \chi)$ such that we have

$$\lim_{N \rightarrow \infty} \frac{1}{N\delta_N(1)} K_{1,1,(s)}^{(N)}(z, \bar{z}, w, \bar{w}|\lambda, \bar{\lambda})$$

$$= c_N(\zeta, \eta, \chi) \frac{\mathcal{H}_\rho(\bar{\chi} + \chi, \bar{\zeta} + \chi, \bar{\chi} + \eta, \bar{\zeta} + \eta, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2 \mathcal{L}_\rho(\chi + \bar{\chi})} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) e^{-(\bar{\zeta} - \bar{\chi})(\zeta - \chi)},$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . This completes the proof of the diagonal case. Similarly, it suffices to consider the front factor for (2.50). From the proof of the diagonal case and by (2.50), we have

$$\begin{aligned} & - \lim_{N \rightarrow \infty} \frac{1}{(N\delta_N(1))^2} \frac{n(z\bar{w} - \frac{L}{n})}{z\bar{w}(1+z\bar{w})} \hat{g}_{N-1}^{(n,L)}(z\bar{w}) \hat{\omega}^{(n,L)}(\bar{w}, z) \mathcal{K}_{N-1}^{(n,L)}(\bar{z}, w | z, \bar{w}) \hat{\omega}^{(n,L)}(\bar{z}, w) \\ & = - \frac{\mathcal{H}_\rho(\bar{\eta} + \zeta, \bar{\zeta} + \zeta, \bar{\eta} + \eta, \bar{\eta} + \zeta, -(\bar{\zeta} - \bar{\eta})(\zeta - \eta))}{(\bar{\zeta} - \bar{\eta})^2(\zeta - \eta)^2}. \end{aligned}$$

This completes the proof. \square

8.3. Proof of the case of ISUE in the singular origin regime in Theorem 2.10.

Proof of the case of ISUE in the singular origin regime in Theorem 2.10. For ζ, η, χ of compact subsets of \mathbb{C} , we write

$$z = \frac{\zeta}{\sqrt{N\delta_N(0)}}, \quad w = \frac{\eta}{\sqrt{N\delta_N(0)}}, \quad \lambda = \frac{\chi}{\sqrt{N\delta_N(0)}}.$$

First, note that

$$z\bar{w} = \frac{\zeta\bar{\eta}}{N(b+1)} (1 + O(N^{-1})).$$

Then, by Lemma 8.1, we have

$$(8.8) \quad \hat{g}_N^{(n,L)}(\lambda\bar{\lambda}) \hat{\omega}^{(n,L)}(\lambda, \bar{\lambda}) = N \frac{(\chi\bar{\chi})^L \mathcal{E}_{1,L}(\chi\bar{\chi}) e^{-|\chi|^2}}{\chi\bar{\chi} - L} (1 + O(N^{-1})),$$

which gives

$$\frac{n(\lambda\bar{\lambda} - L/n)}{\lambda\bar{\lambda}(1 + \lambda\bar{\lambda})} \hat{g}_N^{(n,L)}(\lambda\bar{\lambda}) \hat{\omega}^{(n,L)}(\lambda, \bar{\lambda}) = N^2(b+1) |\chi|^{2L-2} \mathcal{E}_{1,L}(\chi\bar{\chi}) e^{-|\chi|^2} (1 + O(N^{-1})).$$

Similar to the other case, we compute the asymptotic behavior for each term. For $\mathfrak{H}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda)$, we have

$$(8.9) \quad \begin{aligned} \mathfrak{H}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) &= (N(b+1))^2 (\bar{\zeta}\eta)^L e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)} \\ &\times \frac{\mathcal{E}_{1,L}(\bar{\zeta}\chi|\bar{\chi}\chi) \mathcal{E}_{1,L}(\bar{\chi}\eta|\bar{\chi}\chi) - \mathcal{E}_{1,L}(\bar{\zeta}\eta|\bar{\chi}\chi) \mathcal{E}_{1,L}(\bar{\chi}\chi|\bar{\chi}\chi)}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2(\bar{\chi}\chi - L) \mathcal{E}_{1,L}(\chi\bar{\chi}|\bar{\chi}\chi)} (1 + O(N^{-1})). \end{aligned}$$

For $\mathfrak{F}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda)$, since

$$\mathfrak{J}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) = (N(b+1))^2 \frac{(\bar{\zeta}\eta)^L e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)}}{(\bar{\zeta} - \bar{\chi})(\eta - \chi)} \left(\frac{\mathcal{E}_{1,L}(\bar{\zeta}\eta|\bar{\chi}\chi)}{\bar{\chi}\chi - L} - \frac{1}{\Gamma(L)} \frac{1}{\chi\bar{\chi} - L} \right) (1 + O(N^{-1})),$$

For (2.59) and (2.60), similarly, it is easy to see that

$$\begin{aligned} \mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) &= o(1), \\ \mathfrak{J}\mathfrak{J}\mathfrak{J}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) \varpi^{(n,L)}(z, \bar{z} | \lambda, \bar{\lambda}) &= o(1), \end{aligned}$$

and

$$\varpi^{(n,L)}(\bar{z}, z | \lambda, \bar{\lambda}) = \frac{\varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi})}{(b+1)N} (1 + O(N^{-1})).$$

Since

$$\begin{aligned} & \mathfrak{H}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) + \mathfrak{F}_N^{(n,L)}(\bar{z}, w | \bar{\lambda}, \lambda) \\ &= (N(b+1))^2 \frac{\mathcal{S}_L(\bar{\zeta}\chi, \bar{\chi}\eta, \bar{\zeta}\eta, \bar{\chi}\chi, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2(\eta - \chi)^2 \mathcal{E}_{1,L}(\bar{\chi}\chi|\bar{\chi}\chi)} (\bar{\zeta}\eta)^L e^{-\frac{1}{2}(|\zeta|^2 + |\eta|^2)} (1 + O(N^{-1})), \end{aligned}$$

there exists a conjugation factor $c(\zeta, \eta)$ such that

$$\lim_{N \rightarrow \infty} \frac{1}{N \delta_N(0)} K_{1,1,(s)}^{(N)}(z, \bar{z}, w, \bar{w} | \lambda, \bar{\lambda}) = c(\zeta, \eta) \frac{\mathcal{S}_L(\bar{\zeta}\chi, \bar{\chi}\eta, \bar{\zeta}\eta, \bar{\chi}\chi, (\bar{\zeta} - \bar{\chi})(\eta - \chi))}{(\bar{\zeta} - \bar{\chi})^2 (\eta - \chi)^2 \mathcal{E}_{1,L}(\bar{\chi}\chi | \bar{\chi}\chi)} \varpi(\zeta, \bar{\zeta} | \chi, \bar{\chi}) |\zeta|^{2L} e^{-|\zeta|^2},$$

uniformly for ζ, η, χ in compact subsets of \mathbb{C} . This completes the proof for the diagonal case. Similarly, it suffices to consider the front factor due to the decoupling Lemma 2.5. From the proof of the diagonal case and by (2.50), we have

$$\begin{aligned} & - \lim_{N \rightarrow \infty} \frac{1}{N} \frac{1}{(N \delta_N(0))^2} \frac{n(z\bar{w} - \frac{L}{n})}{z\bar{w}(1 + z\bar{w})} \hat{g}_{N-1}^{(n,L)}(z\bar{w}) \hat{\omega}^{(n,L)}(\bar{w}, z) \mathcal{K}_{N-1}^{(n,L)}(\bar{z}, w | z, \bar{w}) \hat{\omega}^{(n,L)}(\bar{z}, w) \\ & = \frac{\mathcal{E}_{1,L}(\zeta\bar{\eta}) |\zeta|^{2L} |\eta|^{2L}}{\zeta_1 \bar{\zeta}_2} e^{-|\zeta|^2 - |\eta|^2} \mathcal{K}_{1,1}^{(s)}(\bar{\zeta}, \eta | \zeta, \bar{\eta}). \end{aligned}$$

This completes the proof. □

Part 4. Proofs Pfaffian structure of the on-diagonal overlap of the Ginibre symplectic ensembles

9. CONSTRUCTION OF SKEW-ORTHOGONAL POLYNOMIALS

9.1. **Proof of Theorem 3.7.** In this subsection, we prove Theorem 3.7.

Proof of Theorem 3.7. The proof is done by the induction argument. Through the proof, for the simplicity of the notation, we omit the notation $\hat{\cdot}$ for $\widehat{m}_{j,k}$. In the inner product in this proof, (3.35) is used. We assume that the coefficients of q_{2k} satisfy (3.40) and the ones of q_{2k+1} satisfy (3.41). Then, the coefficients of q_{2k}, q_{2k+1} also satisfy (3.40) and (3.41), respectively. We write

$$(9.1) \quad q_{2k+2}(z) = \sum_{i=0}^{2k+2} \alpha_{2k+2,i} z^i.$$

Then, for $1 \leq j \leq k-1$, we have

$$\langle q_{2j-2}, q_{2k+2} \rangle_s = \langle z^{2j-2} + \cdots, \alpha_{2k+2,2k-1} z^{2k-1} + \cdots \rangle_s, \quad \langle q_{2j-1}, q_{2k+2} \rangle_s = \langle z^{2j-1} + \cdots, \alpha_{2k+2,2k-1} z^{2k-1} + \cdots \rangle_s.$$

By the assumption and replacing $2k$ with $2k+2$, it follows that for $1 \leq j \leq k-1$,

$$\mathcal{Z}_j \alpha_{2k+2,2j-1} = -m_{2j-1,2j} \alpha_{2k+2,2j}, \quad \mathcal{Z}_j \alpha_{2k+2,2j-2} = m_{2j,2j+1} \alpha_{2k+2,2j+1} + m_{2j,2j} \alpha_{2k+2,2j}.$$

Therefore, it suffices to consider q_{2j-2}, q_{2j-1} for $j = k, k+1$. First, note that

$$\begin{aligned} \langle q_{2k-2}, q_{2k+2} \rangle_s &= 2\alpha_{2k+2,2k} m_{2k-1,2k} + 2\alpha_{2k+2,2k-1} m_{2k-1,2k-1} \\ &\quad - 2\alpha_{2k+2,2k-3} m_{2k-2,2k-2} - 2\alpha_{2k+2,2k-4} m_{2k-2,2k-3} + \left\langle \sum_{j=0}^{2k-3} \alpha_{2k-2,j} z^j, \sum_{j=0}^{2k-1} \alpha_{2k+2,j} z^j \right\rangle_s. \end{aligned}$$

To see (3.40), it suffices to show that

$$\begin{aligned} &2\alpha_{2k+2,2k-1} m_{2k-1,2k-1} - 2\alpha_{2k+2,2k-3} m_{2k-2,2k-2} \\ &- 2\alpha_{2k+2,2k-4} m_{2k-2,2k-3} + \left\langle \sum_{j=0}^{2k-3} \alpha_{2k-2,j} z^j, \sum_{j=0}^{2k-1} \alpha_{2k+2,j} z^j \right\rangle_s = 2\mathcal{Z}_k \alpha_{2k+2,2k-1}. \end{aligned}$$

However, since

$$\begin{aligned} &2\alpha_{2k+2,2k-1} m_{2k-1,2k-1} - 2\alpha_{2k+2,2k-3} m_{2k-2,2k-2} - 2\alpha_{2k+2,2k-4} m_{2k-2,2k-3} \\ &= 2\alpha_{2k+2,2k-1} \left(m_{2k-1,2k-1} - \frac{m_{2k-3,2k-2} m_{2k-2,2k-1}}{\mathcal{Z}_{k-1}} \right) = 2\mathcal{Z}_k \alpha_{2k+2,2k-1}, \end{aligned}$$

our task is to show that

$$\left\langle \sum_{j=0}^{2k-3} \alpha_{2k-2,j} z^j, \sum_{j=0}^{2k-1} \alpha_{2k+2,j} z^j \right\rangle_s = 0.$$

Since the left hand side in the above is expanded as

$$\begin{aligned} \left\langle \sum_{j=0}^{2k-3} \alpha_{2k-2,j} z^j, \sum_{j=0}^{2k-1} \alpha_{2k+2,j} z^j \right\rangle_s &= 2 \sum_{j=3}^{2k-2} \alpha_{2k-2,2k-j} u_{k,j} \\ &\quad + 2\alpha_{2k-2,1} (\alpha_{2k+2,3} m_{2,3} + \alpha_{2k+2,2} m_{2,2} - \alpha_{2k+2,0} m_{1,1}) \\ &\quad + 2\alpha_{2k-2,0} (\alpha_{2k+2,2} m_{1,2} + \alpha_{2k+2,1} m_{1,1}), \end{aligned}$$

where

$$(9.2) \quad \begin{aligned} u_{k,j} &= \alpha_{2k+2,2k+2-j} m_{2k+1-j,2k+2-j} + \alpha_{2k+2,2k+1-j} m_{2k+1-j,2k+1-j} \\ &\quad - \alpha_{2k+2,2k-j-1} m_{2k-j,2k-j} - \alpha_{2k+2,2k-j-2} m_{2k-j,2k-j-1}, \end{aligned}$$

for $j = 3, 4, \dots, 2k-2$, it is enough to show that $u_{k,j} = 0$ for $3 \leq j \leq 2k-2$. By the assumption of the induction,

$$u_{k,j} = \alpha_{2k+2,2k-2j+2} m_{2k-2j+1,2k-2j+2} + \alpha_{2k+2,2k-2j+1} m_{2k-2j,2k-2j+1} + m_{2k-2j,2k-2j} \frac{m_{2k-2j-1,2k-2j} \alpha_{2k+2,2k-2j}}{\mathcal{Z}_{k-j}}$$

$$- m_{2k-2j-1, 2k-2j} \frac{m_{2k-2j-1, 2k-2j} \alpha_{2k+2, 2k-2j+1} + m_{2k-2j, 2k-2j} \alpha_{2k+2, 2k-2j}}{\mathcal{Z}_{k-j}}$$

$$= m_{2k-2j+1, 2k-2j+2} \alpha_{2k+2, 2k+2-2j} + \mathcal{Z}_{k-j+1} \alpha_{2k+2, 2k-2j+1} = 0.$$

In the same spirit,

$$u_{k, 2j+1} = \alpha_{2k+2, 2(k-j)+1} m_{2(k-j), 2(k-j)+1} + \alpha_{2k+2, 2(k-j)} m_{2(k-j), 2(k-j)}$$

$$- m_{2(k-j)-1, 2(k-j)-1} \frac{m_{2(k-j), 2(k-j)+1} \alpha_{2k+2, 2(k-j)+1} + m_{2(k-j), 2(k-j)} \alpha_{2k+2, 2(k-j)}}{\mathcal{Z}_{k-j}}$$

$$+ \frac{m_{2(k-j)-3, 2(k-j)-2} m_{2(k-j)-2, 2(k-j)-1}}{\mathcal{Z}_{k-j-1}} \frac{m_{2(k-j), 2(k-j)+1} \alpha_{2k+2, 2(k-j)+1} + m_{2(k-j), 2(k-j)} \alpha_{2k+2, 2(k-j)}}{\mathcal{Z}_{k-j}}$$

$$= \alpha_{2k+2, 2(k-j)+1} m_{2(k-j), 2(k-j)+1} + \alpha_{2k+2, 2(k-j)} m_{2(k-j), 2(k-j)}$$

$$- \frac{m_{2(k-j), 2(k-j)+1} \alpha_{2k+2, 2(k-j)+1} + m_{2(k-j), 2(k-j)} \alpha_{2k+2, 2(k-j)}}{\mathcal{Z}_{k-j}}$$

$$\times \left(m_{2(k-j)-1, 2(k-j)-1} - \frac{m_{2(k-j)-3, 2(k-j)-2} m_{2(k-j)-2, 2(k-j)-1}}{\mathcal{Z}_{k-j-1}} \right) = 0.$$

Hence, we have shown that $u_{k, j} = 0$ for all $3 \leq j \leq 2k-2$. Next, we impose that $\langle q_{2k-1}, q_{2k+2} \rangle_s = 0$. Then, observe that

$$\langle q_{2k-1}, q_{2k+2} \rangle_s = 2\alpha_{2k+2, 2k+1} m_{2k, 2k+1} + 2\alpha_{2k+2, 2k} m_{2k, 2k} - 2\mathcal{Z}_k \alpha_{2k+2, 2k-2} + \left\langle \sum_{j=0}^{2k-4} \beta_{2k-1, j} z^j, \sum_{j=0}^{2k-2} \beta_{2k+2, j} z^j \right\rangle_s.$$

Therefore, it suffices to show that

$$\left\langle \sum_{j=0}^{2k-4} \beta_{2k-1, j} z^j, \sum_{j=0}^{2k-2} \beta_{2k+2, j} z^j \right\rangle_s = 0.$$

Similar to the even coefficients case, it is reduced to show $u_{k, j} = 0$ for $4 \leq j \leq 2k-2$, but this argument has been already shown. Hence, we have verified the induction argument for q_{2j-2}, q_{2j-1} with $j = k$. The remainder task is to verify the induction argument for q_{2j-2}, q_{2j-1} with $j = k+1$. We shall impose that $\langle q_{2k}, q_{2k+2} \rangle_s = 0, \langle q_{2k+1}, q_{2k+2} \rangle_s = 0$. For (3.40), we can recycle computations since the highest degree term of q_{2k+2} is z^{2k+2} . Hence, there is no additional task to show the induction argument for (3.40). For (3.41), similarly, we can recycle computations so far. This completes the proof. \square

9.2. Proof of Theorem 3.9. With help of Theorem 3.7, we prove Theorem 3.9.

Proof of Theorem 3.9. We define

$$(9.3) \quad m_{i, j}^{(\text{pre})} = \int_{\mathbb{C}} z^i \bar{z}^j \omega_{(\text{pre})}(z|a) dA(z) \quad \text{for } a \in \mathbb{R}.$$

Then, we have

$$(9.4) \quad m_{i, j}^{(\text{pre})} = \begin{cases} 0 & \text{if } i < j-1, \\ -a \frac{\Gamma(j+1)}{2^{j+1}} & \text{if } i = j-1, \\ \frac{\Gamma(j+1)}{2^{j+2}} (2a^2 + j + 3) & \text{if } i = j, \\ -a \frac{\Gamma(j+2)}{2^{j+2}} & \text{if } i = j+1, \\ 0 & \text{if } i > j+1. \end{cases}$$

Let us also write

$$(9.5) \quad g_{i, j}^{(\text{pre})} = \int_{\mathbb{C}} (z^i \bar{z}^j - \bar{z}^j z^i) (z - \bar{z}) \omega_{(\text{pre})}(z|a) dA(z), \quad G_k^{(\text{pre})} = (g_{i, j}^{(\text{pre})})_{i, j=0}^{2k-1}.$$

We set

$$(9.6) \quad \Delta_{-1}^{(\text{pre})} = 1, \quad \Delta_k^{(\text{pre})} = \text{Pf}(G_{k+1}^{(\text{pre})}), \quad \mathcal{Z}_{k+1}^{(\text{pre})} = \frac{1}{2} \frac{\Delta_k^{(\text{pre})}}{\Delta_{k-1}^{(\text{pre})}}.$$

Then, by (9.4) and (9.5), (9.6) satisfies

$$\mathcal{Z}_k^{(\text{pre})} = m_{2k,2k+1}^{(\text{pre})} - \frac{m_{2k-1,2k}^{(\text{pre})} m_{2k,2k+1}^{(\text{pre})}}{\mathcal{Z}_{k-1}^{(\text{pre})}},$$

which is equivalent to

$$(9.7) \quad \Delta_k^{(\text{pre})} = 2\Delta_{k-1}^{(\text{pre})} m_{2k+1,2k+1}^{(\text{pre})} - 4m_{2k-1,2k}^{(\text{pre})} m_{2k,2k+1}^{(\text{pre})} \Delta_{k-2}^{(\text{pre})}.$$

Then, the unique solution of (9.7) is given by

$$\Delta_k^{(\text{pre})} = \frac{2^{k+1} \prod_{i=1}^{k+1} \Gamma(2i)}{2^{(k+1)(k+2)}} f_{k+1}(a^2).$$

Therefore, we obtain

$$\mathcal{Z}_k^{(\text{pre})} = \frac{(2k-1)!}{2^{2k}} \frac{f_k(a^2)}{f_{k-1}(a^2)}.$$

By Theorem 3.7, we have

$$\begin{cases} \mathcal{Z}_n \alpha_{2n,2n-1}^{(\text{pre})} = -m_{2n-1,2n}^{(\text{pre})}, \\ \mathcal{Z}_n \alpha_{2n,2n-2}^{(\text{pre})} = m_{2n,2n}^{(\text{pre})}, \\ \mathcal{Z}_j \alpha_{2n,2j-1}^{(\text{pre})} = -m_{2j-1,2j}^{(\text{pre})} \alpha_{2n,2j}^{(\text{pre})}, \\ \mathcal{Z}_j \alpha_{2n,2j-2}^{(\text{pre})} = m_{2j,2j+1}^{(\text{pre})} \alpha_{2n,2j+1}^{(\text{pre})} + m_{2j,2j}^{(\text{pre})} \alpha_{2n,2j}^{(\text{pre})}, \end{cases} \begin{cases} \mathcal{Z}_n \beta_{2n+1,2n}^{(\text{pre})} = 0, \\ \mathcal{Z}_n \beta_{2n+1,2n-1}^{(\text{pre})} = m_{2n,2n+1}^{(\text{pre})}, \\ \mathcal{Z}_j \beta_{2n+1,2n-2}^{(\text{pre})} = m_{2n,2n+1}^{(\text{pre})}, \\ \mathcal{Z}_j \beta_{2n+1,2n-3}^{(\text{pre})} = -m_{2n-3,2n-2}^{(\text{pre})} \beta_{2n+1,2n-2}^{(\text{pre})}, \\ \mathcal{Z}_j \beta_{2n+1,2n-4}^{(\text{pre})} = -m_{2n-2,2n-2}^{(\text{pre})} \beta_{2n+1,2n-2}^{(\text{pre})}, \\ \mathcal{Z}_j \beta_{2n+1,2j-1}^{(\text{pre})} = -m_{2j-1,2j}^{(\text{pre})} \beta_{2n+1,2j}^{(\text{pre})}, \\ \mathcal{Z}_j \beta_{2n+1,2j-2}^{(\text{pre})} = m_{2j,2j+1}^{(\text{pre})} \beta_{2n+1,2j+1}^{(\text{pre})} + m_{2j,2j}^{(\text{pre})} \beta_{2n+1,2j}^{(\text{pre})}, \end{cases}$$

with $\alpha_{2n,2n} = \beta_{2n+1,2n+1} = 1$. Then, we shall look at some coefficients:

$$\alpha_{2n,2n-1}^{(\text{pre})} = an \frac{f_{n-1}(a^2)}{f_n(a^2)}, \quad \alpha_{2n,2n-2}^{(\text{pre})} = n \frac{2a^2 + 2n + 3}{2} \frac{f_{n-1}(a^2)}{f_n(a^2)}, \quad \alpha_{2n,2n-3}^{(\text{pre})} = an(n-1) \frac{2a^2 + 2n + 3}{2} \frac{f_{n-2}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-4}^{(\text{pre})} = \frac{n(n-1)}{2^2} (4a^4 + 2a^2(2n+5) + (2n+3)(2n+1)) \frac{f_{n-2}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-5}^{(\text{pre})} = \frac{an(n-1)(n-2)}{2^2} (4a^4 + 2a^2(2n+5) + (2n+3)(2n+1)) \frac{f_{n-3}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-6}^{(\text{pre})} = \frac{n(n-1)(n-2)}{2^3} (8a^6 + 4a^4(2n+7) + 2a^2(2n+7)(2n+1) + (2n+3)(2n+1)(2n-1)) \frac{f_{n-3}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-7}^{(\text{pre})} = \frac{an(n-1)(n-2)(n-3)}{2^3} (8a^6 + 4a^4(2n+7) + 2a^2(2n+7)(2n+1) + (2n+3)(2n+1)(2n-1)) \frac{f_{n-4}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-8}^{(\text{pre})} = \frac{n(n-1)(n-2)(n-3)}{2^4} \left(16a^8 + 8a^6(2n+9) + 4a^4(2n+11)(2n+1) \right. \\ \left. + 2a^2(2n+9)(2n+1)(2n-1) + (2n+3)(2n+1)(2n-1)(2n-3) \right) \frac{f_{n-4}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-9}^{(\text{pre})} = \frac{an(n-1)(n-2)(n-3)(n-4)}{2^4} \left(16a^8 + 8a^6(2n+9) + 4a^4(2n+11)(2n+1) \right. \\ \left. + 2a^2(2n+9)(2n+1)(2n-1) + (2n+3)(2n+1)(2n-1)(2n-3) \right) \frac{f_{n-5}(a^2)}{f_n(a^2)},$$

$$\alpha_{2n,2n-10}^{(\text{pre})} = \frac{n(n-1)(n-2)(n-3)(n-4)}{2^5} \left(32a^{10} + 16a^8(2n+11) + 8a^6(2n+15)(2n+1) \right. \\ \left. + 4a^4(2n+15)(2n+1)(2n-1) + 2a^2(2n+11)(2n+1)(2n-1)(2n-3) \right)$$

$$+ (2n+3)(2n+1)(2n-1)(2n-3)(2n-5) \frac{f_{n-5}(a^2)}{f_n(a^2)}, \dots,$$

and

$$\begin{aligned} \beta_{2n+1,2n+1}^{(\text{pre})} &= 1, \quad \beta_{2n+1,2n}^{(\text{pre})} = a, \quad \beta_{2n+1,2n-1}^{(\text{pre})} = a^2 n \frac{f_{n-1}(a^2)}{f_n(a^2)}, \quad \beta_{2n+1,2n-2}^{(\text{pre})} = an(2^{-1}(2a^2)^1 + 1) \frac{f_{n-1}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-3}^{(\text{pre})} &= a^2 n(n-1)(2^{-1}(2a^2)^1 + 1) \frac{d_{n-2}(a^2)}{f_n(a^2)}, \quad \beta_{2n+1,2n-4}^{(\text{pre})} = \frac{an(n-1)}{2} (2^{-1}(2a^2)^2 + 2(2a^2) + (2n+1)) \frac{f_{n-2}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-5}^{(\text{pre})} &= \frac{a^2 n(n-1)(n-2)}{2} (2^{-1}(2a^2)^2 + 2(2a^2) + (2n+1)) \frac{f_{n-3}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-6}^{(\text{pre})} &= \frac{an(n-1)(n-2)}{2^2} (2^{-1}(2a^2)^3 + 3(2a^2)^2 + 2(2n+1)2a^2 + (2n+1)(2n-1)) \frac{f_{n-3}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-7}^{(\text{pre})} &= \frac{a^2 n(n-1)(n-2)(n-3)}{2^2} (2^{-1}(2a^2)^3 + 3(2a^2)^2 + 2(2n+1)2a^2 + (2n+1)(2n-1)) \frac{f_{n-4}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-8}^{(\text{pre})} &= \frac{an(n-1)(n-2)(n-3)}{2^3} (2^{-1}(2a^2)^4 + 4(2a^2)^3 + 3(2n+1)(2a^2)^2 \\ &\quad + 2(2n+1)(2n-1)2a^2 + (2n+1)(2n-1)(2n-3)) \frac{f_{n-4}(a^2)}{f_n(a^2)}, \\ \beta_{2n+1,2n-9}^{(\text{pre})} &= \frac{a^2 n(n-1)(n-2)(n-3)(n-4)}{2^3} (2^{-1}(2a^2)^4 + 4(2a^2)^3 + 3(2n+1)(2a^2)^2 \\ &\quad + 2(2n+1)(2n-1)2a^2 + (2n+1)(2n-1)(2n-3)) \frac{f_{n-5}(a^2)}{f_n(a^2)}, \dots \end{aligned}$$

Form these coefficients lists, we claim that

$$(9.8) \quad \alpha_{2n,2j}^{(\text{pre})} = \sum_{\ell=j}^n \left((\ell+1-j) \frac{(2n+3)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} - (\ell-j) \frac{(2n+1)!!}{(2\ell+1)!!} (2a^2)^{\ell-j} \right) \frac{2^j n!}{2^n j!} \frac{f_j(a^2)}{f_n(a^2)},$$

$$(9.9) \quad \alpha_{2n,2j+1}^{(\text{pre})} = a \sum_{\ell=j}^{n-1} \left((\ell+1-j) \frac{(2n+3)!!}{(2\ell+5)!!} (2a^2)^{\ell-j} - (\ell-j) \frac{(2n+1)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} \right) \frac{2^{j+1} n!}{2^n j!} \frac{f_j(a^2)}{f_n(a^2)},$$

$$(9.10) \quad \beta_{2n+1,2j+1}^{(\text{pre})} = \left((2a^2)^{n-j} + 2 \sum_{\ell=j}^{n-1} (\ell-j) \frac{(2n+1)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} \right) \frac{n! 2^j}{j! 2^n} \frac{f_j(a^2)}{f_n(a^2)},$$

$$(9.11) \quad \beta_{2n+1,2j}^{(\text{pre})} = a \left((2a^2)^{n-j} + 2 \sum_{\ell=j}^{n-1} (\ell+1-j) \frac{(2n+1)!!}{(2\ell+3)!!} (2a^2)^{\ell-j} \right) \frac{n! 2^j}{j! 2^n} \frac{f_j(a^2)}{f_n(a^2)}.$$

Here, we implicitly have used the fact that under the transformation $q_{2k+1}(z) \mapsto q_{2k+1}(z) + d_k q_{2k}(z)$ for any constants d_k , the value for the skew-inner product does not change. We prove (9.8) and (9.9) by the induction argument. By exchanging the indices, we express (9.8) and (9.9) as

$$(9.12) \quad \alpha_{2n,2(n-j)}^{(\text{pre})} = \sum_{\ell=0}^j \left(\frac{(\ell+1)(2n+3)!!}{(2\ell+2n-2j+3)!!} (2a^2)^\ell - \frac{\ell(2n+1)!!}{(2\ell+2n-2j+1)!!} (2a^2)^\ell \right) \frac{2^{n-j} n!}{2^n (n-j)!} \frac{f_{2(n-j)}(a^2)}{f_n(a^2)},$$

$$(9.13) \quad \alpha_{2n,2(n-j)+1}^{(\text{pre})} = a \sum_{\ell=0}^{j-1} \left(\frac{(\ell+1)(2n+3)!!}{(2\ell+2n-2j+5)!!} (2a^2)^\ell - \frac{\ell(2n+1)!!}{(2\ell+2n-2j+3)!!} (2a^2)^\ell \right) \frac{2^{n-j+1} n!}{2^n (n-j)!} \frac{f_{2(n-j)}(a^2)}{f_n(a^2)}.$$

We assume that (9.12) and (9.13) hold, then we shall compute $\alpha_{2n,2(n-j-1)}$ and $\alpha_{2n,2(n-j-1)+1}$. Note that

$$\mathcal{Z}_{(n-j-1)+1}^{(\text{pre})} \alpha_{2n,2(n-j-1)+1}^{(\text{pre})} = -m_{2(n-j-1)+1,2(n-j-1)+2}^{(\text{pre})} \alpha_{2n,2(n-j-1)+2}^{(\text{pre})}$$

$$\begin{aligned}
&= a \left(\sum_{\ell=0}^j \frac{(2n+3)!!}{(2\ell+2n-2j+3)!!} (\ell+1)(2a^2)^\ell - \sum_{\ell=0}^j \frac{(2n+1)!!}{(2\ell+2n-2j+1)!!} \ell(2a^2)^\ell \right) \\
&\quad \times \frac{(2n-2j)!}{2^{2n-2j+1}} \frac{2^{n-j}}{2^n} \frac{n!}{(n-j)!} \frac{f_{n-j}(a^2)}{f_n(a^2)}.
\end{aligned}$$

Since $\mathcal{Z}_{(n-j-1)+1}^{(\text{pre})} = \frac{(2n-2j-1)!}{2^{2n-2j}} \frac{f_{n-j}(a^2)}{f_{n-j-1}(a^2)}$, we have

$$\begin{aligned}
\alpha_{2n,2(n-j-1)+1}^{(\text{pre})} &= a \left(\sum_{\ell=0}^j \frac{(2n+3)!!}{(2\ell+2n-2(j+1)+5)!!} (\ell+1)(2a^2)^\ell - \sum_{\ell=0}^j \frac{(2n+1)!!}{(2\ell+2n-2(j+1)+3)!!} \ell(2a^2)^\ell \right) \\
&\quad \times \frac{2^{n-(j+1)+1}}{2^n} \frac{n!}{(n-j-1)!} \frac{f_{n-j-1}(a^2)}{f_n(a^2)}.
\end{aligned}$$

Next, we shall compute $\alpha_{2n,2(n-j-1)}^{(\text{pre})}$.

$$\begin{aligned}
\mathcal{Z}_{(n-j-1)+1}^{(\text{pre})} \alpha_{2n,2(n-j-1)}^{(\text{pre})} &= m_{2(n-j-1)+2,2(n-j-1)+3}^{(\text{pre})} \alpha_{2n,2(n-j-1)+3}^{(\text{pre})} + m_{2(n-j-1)+2,2(n-j-1)+2}^{(\text{pre})} \alpha_{2n,2(n-j-1)+2}^{(\text{pre})} \\
&= -a \frac{(2n-2j+1)!}{2^{2n-2j+2}} \alpha_{2n,2n-2j+1}^{(\text{pre})} + \frac{(2n-2j)!(2a^2+2n-2j+3)}{2^{2n-2j+2}} \alpha_{2n,2n-2j}^{(\text{pre})},
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha_{2n,2(n-j-1)}^{(\text{pre})} &= \frac{1}{2^{j+1}} \frac{n!}{(n-j-1)!} \frac{f_{n-j-1}(a^2)}{f_n(a^2)} \\
&\quad \times \left\{ (2a^2+2n-2j+3) \left(\sum_{\ell=0}^j \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=0}^j \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \right) \right. \\
&\quad \left. - 2a^2(2n-2j+1) \left(\sum_{\ell=0}^{j-1} \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+5)!!} - \sum_{\ell=0}^{j-1} \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+3)!!} \right) \right\}.
\end{aligned}$$

We write

$$\begin{aligned}
\mathfrak{A}_1 &= (2a^2+2n-2j+3) \left(\sum_{\ell=0}^j \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=0}^j \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \right), \\
\mathfrak{A}_2 &= -2a^2(2n-2j+1) \left(\sum_{\ell=0}^{j-1} \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+5)!!} - \sum_{\ell=0}^{j-1} \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+3)!!} \right).
\end{aligned}$$

We will show

$$\mathfrak{A}_1 + \mathfrak{A}_2 = \sum_{\ell=0}^{j+1} \frac{(2n+3)!!}{(2\ell+2n-2j+1)!!} (\ell+1)(2a^2)^\ell - \sum_{\ell=0}^{j+1} \frac{(2n+1)!!}{(2\ell+2n-2j-1)!!} \ell(2a^2)^\ell.$$

First, note that

$$\begin{aligned}
\mathfrak{A}_1 &= \sum_{\ell=0}^{j+1} \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+1)!!} - \sum_{\ell=0}^{j+1} \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j-1)!!} \\
&\quad - \sum_{\ell=0}^{j+1} \frac{(2n+3)!!(2a^2)^\ell}{(2\ell+2n-2j+1)!!} + \sum_{\ell=1}^{j+1} \frac{(2n+1)!!(2a^2)^\ell}{(2\ell+2n-2j-1)!!} + (2n-2j+3) \frac{(2n+3)!!}{(2n-2j+3)!!} \\
&\quad + (2n-2j+3) \left(\sum_{\ell=1}^j \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=1}^j \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \right).
\end{aligned}$$

Notice also that

$$\mathfrak{A}_2 = -(2n - 2j + 1) \left(\sum_{\ell=1}^j \frac{(2n+3)!!\ell(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=1}^j \frac{(2n+1)!!(\ell-1)(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \right).$$

Hence, we have

$$\begin{aligned} & \mathfrak{A}_1 + \mathfrak{A}_2 \\ &= \sum_{\ell=0}^{j+1} \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+1)!!} - \sum_{\ell=0}^{j+1} \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j-1)!!} - \sum_{\ell=1}^j \frac{(2n+3)!!(2a^2)^\ell}{(2\ell+2n-2j+1)!!} + \sum_{\ell=1}^j \frac{(2n+1)!!(2a^2)^\ell}{(2\ell+2n-2j-1)!!} \\ &+ \sum_{\ell=1}^j \frac{(2n+3)!!(2\ell+2n-2j+3-(2n-2j+1))(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=1}^j \frac{(2n+1)!!(2\ell+2n-2j+1-(2n-2j+1))(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \\ &+ (2n-2j+1) \left(\sum_{\ell=1}^j \frac{(2n+3)!!(2a^2)^\ell}{(2\ell+2n-2j+3)!!} - \sum_{\ell=1}^j \frac{(2n+1)!!(2a^2)^\ell}{(2\ell+2n-2j+1)!!} \right) \\ &= \sum_{\ell=0}^{j+1} \frac{(2n+3)!!(\ell+1)(2a^2)^\ell}{(2\ell+2n-2j+1)!!} - \sum_{\ell=0}^{j+1} \frac{(2n+1)!!\ell(2a^2)^\ell}{(2\ell+2n-2j-1)!!}, \end{aligned}$$

which completes the proof of the induction argument for (9.12) and (9.13). Similarly, we can prove (9.10) and (9.11). The proof is left to interested readers. \square

10. FINITE N -ANALYSIS

With help of Theorem 3.9, we define (3.49). Our next task is to analyze an asymptotic behavior of (3.49) as $N \rightarrow \infty$. To this end, we need to find a simplified expression of (3.49). Before we directly moving ahead of analysis for (3.49), it is instructive to analyze (3.49) for $a = 0$.

10.1. The conditional origin case: $a = 0$.

10.1.1. *Differential equation for the conditional origin case.* Let us denote

$$(10.1) \quad \omega^{(\text{pre})}(z) = \omega^{(\text{pre})}(z|0) = (1 + |z|^2)e^{-2|z|^2}.$$

Lemma 10.1. *Let us denote the skew pre-kernel associated with the weight function (10.1) by $\mathfrak{K}_N^{(\text{pre})}(z, w)$. Then, the pre-kernel $\mathfrak{K}_N^{(\text{pre})}(z, w)$ is given by*

$$(10.2) \quad \mathfrak{K}_N^{(\text{pre})}(z, w) = G_N^{(\text{pre})}(z, w) - G_N^{(\text{pre})}(w, z),$$

where

$$(10.3) \quad G_N^{(\text{pre})}(z, w) = \sqrt{2}^3 \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{(2k+3)(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+1} (\sqrt{2}w)^{2\ell}.$$

Proof. First, note that

$$h_k = \int_{\mathbb{C}} |z|^{2k} \omega^{(\text{pre})}(z) dA(z) = \frac{(k+3)\Gamma(k+1)}{2^{k+2}},$$

where $dA(z)$ is the area measure on \mathbb{C} . By [10, Theorem 3.1], we have

$$\prod_{j=0}^{k-\ell-1} \frac{h_{2\ell+2j+2}}{h_{2\ell+2j+1}} = \frac{(\ell+1)\Gamma(k+\frac{5}{2})}{(k+1)\Gamma(\ell+\frac{5}{2})},$$

and

$$q_{2k+1}^{(\text{pre})}(z) = z^{2k+1}, \quad q_{2k}^{(\text{pre})}(z) = \sum_{\ell=0}^k \frac{(\ell+1)\Gamma(k+\frac{5}{2})}{(k+1)\Gamma(\ell+\frac{5}{2})} z^{2\ell}.$$

Here, the skew-norm is given by

$$r_k^{(\text{pre})} = 2h_{2k+1} = \frac{(2k+4)\Gamma(2k+2)}{2^{2k+2}}.$$

Then, we set

$$G_N^{(\text{pre})}(z, w) = \sum_{k=0}^{N-1} \frac{q_{2k+1}^{(\text{pre})}(z)q_{2k}^{(\text{pre})}(w)}{r_k^{(\text{pre})}} = \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{2^{2k+1}(\ell+1)\Gamma(k+\frac{5}{2})}{(2k+4)\Gamma(2k+2)(k+1)\Gamma(\ell+\frac{5}{2})} z^{2k+1} w^{2\ell},$$

which completes the proof of the lemma. \square

We now define

$$(10.4) \quad \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) = (zw)^3 \mathfrak{X}_N^{(\text{pre})}(z, w),$$

and

$$\widehat{G}_N^{(\text{pre})}(z, w) = (zw)^3 G_N^{(\text{pre})}(z, w) = \frac{1}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{(2k+3)(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+4} (\sqrt{2}w)^{2\ell+3}.$$

We now derive the differential equation of (10.4):

Proposition 10.2. *We have*

$$(10.5) \quad \begin{aligned} & \left[z\partial_z^2 - (2z^2 + 2)\partial_z - 2z \right] \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) \\ &= \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{1}{k!} (2zw)^{k+3} - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{(2N+1)(2N+3)}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}. \end{aligned}$$

Proof.

$$\partial_z \widehat{G}_N^{(\text{pre})}(z, w) = \frac{\sqrt{2}}{\sqrt{2}^3} \frac{1}{0!!1!!} (2zw)^3 + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-2} \sum_{\ell=0}^{k+1} \frac{(2k+5)(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+5} (\sqrt{2}w)^{2\ell+3}$$

By rearranging the last term, we see that

$$\begin{aligned} & \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-2} \sum_{\ell=0}^{k+1} \frac{(2k+5)(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+5} (\sqrt{2}w)^{2\ell+3} \\ &= \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{(2k+5)(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+5} (\sqrt{2}w)^{2\ell+3} + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-2} \frac{(2zw)^{2k+5}}{(2k+2)!!(2k+3)!!} \\ & \quad - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{2N+3}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}, \end{aligned}$$

which leads to

$$\begin{aligned} \partial_z \widehat{G}_N^{(\text{pre})}(z, w) &= 2z \widehat{G}_N^{(\text{pre})}(z, w) + 4z^2 \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{(2\ell+2)}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+3} (\sqrt{2}w)^{2\ell+3} \\ & \quad + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \frac{(2zw)^{2k+3}}{(2k+1)!} - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{2N+3}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}. \end{aligned}$$

Similarly, we observe that

$$\begin{aligned} \partial_z \widehat{G}_N^{(\text{pre})}(w, z) &= 2z \widehat{G}_N^{(\text{pre})}(w, z) + 4z^2 \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{2k+3}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}w)^{2k+4} (\sqrt{2}z)^{2\ell+2} \\ & \quad + 2 \frac{\sqrt{2}}{\sqrt{2}^3} (\sqrt{2}z)^2 \sum_{k=0}^{N-1} \frac{2k+3}{(2k+4)!!} (\sqrt{2}w)^{2k+4} - \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \frac{(2zw)^{2k+4}}{(2k+2)!}. \end{aligned}$$

Here, since

$$\int \frac{1}{z^2} \widehat{G}_N^{(\text{pre})}(z, w) dz = \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{2\ell+2}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}z)^{2k+3} (\sqrt{2}w)^{2\ell+3}$$

and

$$\int \frac{1}{z^2} \widehat{G}_N^{(\text{pre})}(w, z) dz = \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \sum_{\ell=0}^k \frac{2k+3}{(2k+4)!!(2\ell+3)!!} (\sqrt{2}w)^{2k+4} (\sqrt{2}z)^{2\ell+2},$$

we have

$$\begin{aligned} \partial_z \widehat{G}_N^{(\text{pre})}(z, w) &= 2z \widehat{G}_N^{(\text{pre})}(z, w) + 4z^2 \int_0^z \frac{\widehat{G}_N^{(\text{pre})}(t, w)}{t^2} dt \\ &\quad + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \frac{(2zw)^{2k+3}}{(2k+1)!} - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{2N+3}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}, \end{aligned}$$

and

$$\begin{aligned} \partial_z \widehat{G}_N^{(\text{pre})}(w, z) &= 2z \widehat{G}_N^{(\text{pre})}(w, z) + 4z^2 \int_0^z \frac{\widehat{G}_N^{(\text{pre})}(w, t)}{t^2} dt \\ &\quad + 2 \frac{\sqrt{2}}{\sqrt{2}^3} (\sqrt{2}z)^2 \sum_{k=0}^{N-1} \frac{2k+3}{(2k+4)!!} (\sqrt{2}w)^{2k+4} - \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \frac{(2zw)^{2k+4}}{(2k+2)!}. \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} \partial_z \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) &= 2z \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) + 4z^2 \int_0^z \frac{\widehat{\mathfrak{X}}_N^{(\text{pre})}(t, w)}{t^2} dt + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{(2zw)^{k+3}}{(k+1)!} \\ &\quad - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{2N+3}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3} \\ &\quad - 2 \frac{\sqrt{2}}{\sqrt{2}^3} (\sqrt{2}z)^2 \sum_{k=0}^{N-1} \frac{2k+3}{(2k+4)!!} (\sqrt{2}w)^{2k+4}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{1}{z^2} \partial_z \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) &= 2 \frac{1}{z} \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) + 4 \int_0^z \frac{\widehat{\mathfrak{X}}_N^{(\text{pre})}(t, w)}{t^2} dt + \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{2^{k+3} z^{k+1} w^{k+3}}{(k+1)!} \\ &\quad - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{2N+3}{(2N+2)!!} \sqrt{2}^{2N+3} z^{2N+1} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3} \\ &\quad - \frac{4\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{N-1} \frac{2k+3}{(2k+4)!!} (\sqrt{2}w)^{2k+4}. \end{aligned}$$

By differentiating the above with respect to z ,

$$\begin{aligned} & z \left[z \partial_z^2 - (2z^2 + 2) \partial_z - 2z \right] \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) \\ &= \frac{\sqrt{2}z}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{1}{k!} (2zw)^{k+3} - \frac{\sqrt{2}z}{\sqrt{2}^3} \frac{(2N+1)(2N+3)}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}, \end{aligned}$$

which leads to

$$\begin{aligned} & \left[z \partial_z^2 - (2z^2 + 2) \partial_z - 2z \right] \widehat{\mathfrak{X}}_N^{(\text{pre})}(z, w) \\ &= \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{1}{k!} (2zw)^{k+3} - \frac{\sqrt{2}}{\sqrt{2}^3} \frac{(2N+1)(2N+3)}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3}. \end{aligned}$$

This completes the proof. \square

10.1.2. *Bulk limiting kernel for the conditional origin case.* Finally, we show the bulkscaling limit of (10.4) conditioned at the origin based on (10.5).

Proposition 10.3. *The pre-kernel $\mathfrak{K}_N^{(\text{pre})}$ converges uniformly on compact subsets of \mathbb{C} to $\mathfrak{K}_\infty^{(\text{pre})}$ as $N \rightarrow \infty$, which is given by*

$$(10.6) \quad \begin{aligned} \mathfrak{K}_\infty^{(\text{pre})}(z, w) &= \frac{\sqrt{\pi}}{4(zw)^3} (2z^2 - 1)(2w^2 - 1) e^{z^2+w^2} \operatorname{erf}(z-w) \\ &+ \frac{\sqrt{\pi}}{4(zw)^3} (2z^2 - 1)(2w^2 - 1) e^{z^2+w^2} (\operatorname{erf}(w) - \operatorname{erf}(z)) \\ &- \frac{1}{2(zw)^3} z(2w^2 - 1)e^{w^2} + \frac{1}{2(zw)^3} w(2z^2 - 1)e^{z^2} - \frac{1}{2(zw)^3} (z-w)e^{2zw}. \end{aligned}$$

Proof. By (10.5), since

$$\lim_{N \rightarrow \infty} \frac{\sqrt{2}}{\sqrt{2}^3} \sum_{k=0}^{2N-1} \frac{1}{k!} (2zw)^{k+3} = 4(zw)^3 e^{2zw}$$

and

$$\lim_{N \rightarrow \infty} -\frac{\sqrt{2}}{\sqrt{2}^3} \frac{(2N+1)(2N+3)}{(2N+2)!!} (\sqrt{2}z)^{2N+3} \sum_{\ell=0}^{N-1} \frac{2\ell+2}{(2\ell+3)!!} (\sqrt{2}w)^{2\ell+3} = 0,$$

one can show that $\widehat{\mathfrak{K}}_\infty^{(\text{pre})}(z, w) = \lim_{N \rightarrow \infty} \widehat{\mathfrak{K}}_N^{(\text{pre})}(z, w)$ satisfies

$$(10.7) \quad \left[z \partial_z^2 - (2z^2 + 2) \partial_z - 2z \right] \widehat{\mathfrak{K}}_\infty^{(\text{pre})}(z, w) = 4(zw)^3 e^{2zw}.$$

By solving this ODE with the initial conditions,

$$(10.8) \quad \widehat{\mathfrak{K}}_\infty^{(\text{pre})}(w, w) = 0, \quad \partial_z \widehat{\mathfrak{K}}_\infty^{(\text{pre})}(z, w)|_{z=0} = 0,$$

we obtain

$$(10.9) \quad \begin{aligned} \widehat{\mathfrak{K}}_\infty^{(\text{pre})}(z, w) &= \frac{\sqrt{\pi}}{4} (2z^2 - 1)(2w^2 - 1) e^{z^2+w^2} \operatorname{erf}(z-w) + \frac{\sqrt{\pi}}{4} (2z^2 - 1)(2w^2 - 1) e^{z^2+w^2} (\operatorname{erf}(w) - \operatorname{erf}(z)) \\ &- \frac{1}{2} z(2w^2 - 1)e^{w^2} + \frac{1}{2} w(2z^2 - 1)e^{z^2} - \frac{1}{2} (z-w)e^{2zw}. \end{aligned}$$

By dividing the above by $(zw)^3$, this completes the proof. \square

To study the origin case is important since if we assume that the bulk limiting kernel has translational invariance under the shift $z \mapsto z - a$ for $a \in \mathbb{R}$ with respect to the real line, we could expect that a candidate differential operator to simplify the general finite N -kernel (3.49) would be (3.57) with some exponential factors. In fact, this heuristic observation will be correct as we will see later.

10.2. Proof of Theorem 3.13. In this subsection, we prove Theorem 3.13. To this end, we firstly show Lemma 3.12.

Proof of Lemma 3.12. First, we observe that

$$\begin{aligned} & f_k(a^2) q_{2k}^{(\text{pre})}(z|a) \\ &= \sum_{\ell=0}^k \sum_{p=0}^{\ell} \sum_{j=p}^{\ell} \left\{ (\ell+1-j) \frac{(2k+3)!!}{(2\ell+3)!!} - (\ell-j) \frac{(2k+1)!!}{(2\ell+1)!!} \right\} 2^{\ell-k} a^{2\ell+2p} \frac{(j+1-p)}{p!} \left(\frac{z}{a}\right)^{2j} \\ &+ \sum_{\ell=0}^k \sum_{p=0}^{\ell} \sum_{j=p}^{\ell-1} \left\{ (\ell-j) \frac{(2k+3)!!}{(2\ell+3)!!} - (\ell-1-j) \frac{(2k+1)!!}{(2\ell+1)!!} \right\} 2^{\ell-k} a^{2\ell+2p} \frac{(j+1-p)}{p!} \left(\frac{z}{a}\right)^{2j+1} \\ &= \sum_{\ell=0}^k \sum_{p=0}^{\ell} \frac{(2k+3)!!}{(2\ell+3)!!} \frac{2^{\ell-k} a^{2\ell+2p}}{p!} \left\{ \sum_{j=p}^{\ell} (\ell+1-j)(j+1-p) \left(\frac{z}{a}\right)^{2j} + \sum_{j=p}^{\ell-1} (\ell-j)(j+1-p) \left(\frac{z}{a}\right)^{2j+1} \right\} \end{aligned}$$

$$- \sum_{\ell=0}^k \sum_{p=0}^{\ell} \frac{(2k+1)!!}{(2\ell+1)!!} \frac{2^{\ell-k} a^{2\ell+2p}}{p!} \left\{ \sum_{j=p}^{\ell} (\ell-j)(j+1-p) \left(\frac{z}{a}\right)^{2j} + \sum_{j=p}^{\ell-1} (\ell-1-j)(j+1-p) \left(\frac{z}{a}\right)^{2j+1} \right\}.$$

Observe that for any $x \in \mathbb{C}$,

$$\sum_{j=p}^{\ell} (\ell+1-j)(j+1-p)x^{2j} + \sum_{j=p}^{\ell-1} (\ell-j)(j+1-p)x^{2j+1} = \frac{(\ell+2-p)(x^{2p+1} - x^{2\ell+3}) - (\ell+1-p)(x^{2p} - x^{2\ell+4})}{(x+1)(x-1)^3}$$

and

$$\sum_{j=p}^{\ell} (\ell-j)(j+1-p)x^{2j} + \sum_{j=p}^{\ell-1} (\ell-1-j)(j+1-p)x^{2j+1} = \frac{(\ell+1-p)(x^{2p+1} - x^{2\ell+1}) - (\ell-p)(x^{2p} - x^{2\ell+2})}{(x+1)(x-1)^3}.$$

Hence, we have

$$(10.10) \quad 2^k(z+a)(z-a)^3 f_k(a^2) q_{2k}^{(\text{pre})}(z|a) = (2k+3)!! \mathcal{M}_1 + (2k+1)!! \mathcal{M}_2,$$

where

$$\begin{aligned} \mathcal{M}_1 &= \sum_{\ell=0}^k \frac{2^{\ell} a^{2\ell+3}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+2-p)z^{2p+1}}{p!} - \sum_{\ell=0}^k \frac{2^{\ell} z^{2\ell+3}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+2-p)a^{2p+1}}{p!} \\ &\quad - \sum_{\ell=0}^k \frac{2^{\ell} a^{2\ell+4}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p)z^{2p}}{p!} + \sum_{\ell=0}^k \frac{2^{\ell} z^{2\ell+4}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p)a^{2p}}{p!}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_2 &= - \sum_{\ell=0}^k \frac{2^{\ell} a^{2\ell+3}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p)z^{2p+1}}{p!} + \sum_{\ell=0}^k \frac{2^{\ell} z^{2\ell+1}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p)a^{2p+3}}{p!} \\ &\quad + \sum_{\ell=0}^k \frac{2^{\ell} a^{2\ell+4}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{(\ell-p)z^{2p}}{p!} - \sum_{\ell=0}^k \frac{2^{\ell} z^{2\ell+2}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{(\ell-p)a^{2p+2}}{p!}. \end{aligned}$$

By elementary but involved computations,

$$\begin{aligned} \mathcal{M}_1 &= \frac{(z-a)^2(z+a)}{4} \chi_{k+1}^{(\mathbf{g})}(z, a) - (z-a)z^2 \frac{2^k a^{2k+3}}{(2k+3)!!} \sum_{p=0}^k \frac{z^{2p}}{p!} - (z-a)a^2 \frac{2^k z^{2k+3}}{(2k+3)!!} \sum_{p=0}^k \frac{a^{2p}}{p!} \\ &\quad - \frac{(z+a)}{8} \chi_{k+1}^{(\mathbf{g})}(z, a) + \frac{(z+a)}{2} \left\{ \frac{2^k a^{2k+3}}{(2k+3)!!} \sum_{p=0}^k \frac{z^{2p}}{p!} - \frac{2^k z^{2k+3}}{(2k+3)!!} \sum_{p=0}^k \frac{a^{2p}}{p!} \right\} \\ &\quad - \frac{(z-a)(z+a)}{4} \left\{ \frac{2^{2k+2}(za)^{2k+2}}{(2k+3)!} - \sum_{\ell=0}^{2k+2} \frac{(2za)^{\ell}}{\ell!} \right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathcal{M}_2 &= - \frac{a^2(z-a)^2(z+a)}{2} \chi_{k+1}^{(\mathbf{g})}(z, a) + \frac{a^2(z+a)}{4} \chi_{k+1}^{(\mathbf{g})}(z, a) - \frac{a^2(z-a)(z+a)}{2} \sum_{\ell=0}^{2k+1} \frac{(2za)^{\ell}}{\ell!} \\ &\quad + (z-a) \frac{2^k a^{2k+5}}{(2k+1)!!} \sum_{p=0}^k \frac{z^{2p}}{p!} + (z-a) \frac{2^k z^{2k+3} a^2}{(2k+1)!!} \sum_{p=0}^k \frac{a^{2p}}{p!}. \end{aligned}$$

By the explicit expressions for \mathcal{M}_1 and \mathcal{M}_2 , we have

$$\begin{aligned} 2^k f_k(a^2)(z-a)^3 q_{2k}^{(\text{pre})}(z|a) &= (2(z-a)^2 - 1) \frac{(2k+3)!!}{8} \chi_{k+1}^{(\mathbf{g})}(z, a) - a^2(2(z-a)^2 - 1) \frac{(2k+1)!!}{4} \chi_{k+1}^{(\mathbf{g})}(z, a) \\ &\quad - \frac{(z-a)}{4(2k+2)!!} \left\{ (2za)^{2k+3} - (2k+3)! \sum_{\ell=0}^{2k+3} \frac{(2za)^{\ell}}{\ell!} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{a^2(z-a)}{2(2k+2)!!} \left\{ (2za)^{2k+2} - (2k+2)! \sum_{\ell=0}^{2k+2} \frac{(2za)^\ell}{\ell!} \right\} \\
& + 2^k \frac{a^{2k+3}}{2} \sum_{p=0}^{k+1} \frac{z^{2p}}{p!} - 2^k \frac{z^{2k+3}}{2} \sum_{p=0}^{k+1} \frac{a^{2p}}{p!} - (z-a)^2 2^k a^{2k+3} \sum_{p=0}^k \frac{z^{2p}}{p!}.
\end{aligned}$$

By some elementary computations, we arrive at (3.53). For the odd coefficients, we define

$$\begin{aligned}
\mathfrak{N}_1 & := \frac{2^k}{2k+3} \sum_{j=0}^k (2j+1) a^{2k+1} f_j(a^2) \left(\frac{z}{a}\right)^{2j}, & \mathfrak{N}_3 & := \sum_{j=0}^k \sum_{\ell=j}^k \sum_{p=0}^j \frac{2^{\ell+1} a^{2\ell+2p+1}}{(2\ell+3)!! p!} (\ell+1-j)(j+1-p) \left(\frac{z}{a}\right)^{2j}, \\
\mathfrak{N}_2 & := \frac{2^k}{2k+3} \sum_{j=0}^k (2j+3) a^{2k+1} f_j(a^2) \left(\frac{z}{a}\right)^{2j+1}, & \mathfrak{N}_4 & := \sum_{j=0}^k \sum_{\ell=j}^k \sum_{p=0}^j \frac{2^{\ell+1} a^{2\ell+2p+1}}{(2\ell+3)!!} (\ell-j)(j+1-p) \left(\frac{z}{a}\right)^{2j+1}.
\end{aligned}$$

Then we have

$$\begin{aligned}
2^k f_k(a^2) q_{2k+1}^{(\text{pre})}(z|a) & = \frac{2^k}{2k+3} \sum_{j=0}^k (2j+1) a^{2k+1} f_j(a^2) \left(\frac{z}{a}\right)^{2j} + \frac{2^k}{2k+3} \sum_{j=0}^k (2j+3) a^{2k+1} f_j(a^2) \left(\frac{z}{a}\right)^{2j+1} \\
& + (2k+1)!! \sum_{j=0}^k \sum_{\ell=j}^k \sum_{p=0}^j \frac{2^{\ell+1} a^{2\ell+2p+1}}{(2\ell+3)!! p!} (\ell+1-j)(j+1-p) \left(\frac{z}{a}\right)^{2j} \\
& + (2k+1)!! \sum_{j=0}^k \sum_{\ell=j}^k \sum_{p=0}^j \frac{2^{\ell+1} a^{2\ell+2p+1}}{(2\ell+3)!!} (\ell-j)(j+1-p) \left(\frac{z}{a}\right)^{2j+1} \\
& = \mathfrak{N}_1 + \mathfrak{N}_2 + (2k+1)!! \mathfrak{N}_3 + (2k+1)!! \mathfrak{N}_4.
\end{aligned}$$

Since

$$\sum_{j=p}^{\ell} (\ell+1-j)(j+1-p) x^{2j} + \sum_{j=p}^{\ell} (\ell-j)(j+1-p) x^{2j+1} = \frac{(\ell+2-p)(x^{2p+1} - x^{2\ell+3}) - (\ell+1-p)(x^{2p} - x^{2\ell+4})}{(x+1)(x-1)^3},$$

we have

$$\mathfrak{N}_3 + \mathfrak{N}_4 = \frac{1}{(z+a)(z-a)^3} \sum_{\ell=0}^k \sum_{p=0}^{\ell} \frac{2^{\ell+1} a^{2\ell+2p+5}}{(2\ell+3)!! p!} \left\{ (\ell+2-p) \left(\left(\frac{z}{a}\right)^{2p+1} - \left(\frac{z}{a}\right)^{2\ell+3} \right) - (\ell+1-p) \left(\left(\frac{z}{a}\right)^{2p} - \left(\frac{z}{a}\right)^{2\ell+4} \right) \right\}.$$

We write

$$\mathfrak{N}_a = \sum_{\ell=0}^k \frac{2^{\ell+1} a^{2\ell+4}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+2-p) z^{2p+1}}{p!} - \sum_{\ell=0}^k \frac{2^{\ell+1} z^{2\ell+3}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+2-p) a^{2p+2}}{p!}$$

and

$$\mathfrak{N}_b = - \sum_{\ell=0}^k \frac{2^{\ell+1} a^{2\ell+5}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p) z^{2p}}{p!} + \sum_{\ell=0}^k \frac{2^{\ell+1} z^{2\ell+4}}{(2\ell+3)!!} \sum_{p=0}^{\ell} \frac{(\ell+1-p) a^{2p+1}}{p!}$$

Then, we see that

$$(z+a)(z-a)^3 (\mathfrak{N}_3 + \mathfrak{N}_4) = \mathfrak{N}_a + \mathfrak{N}_b.$$

By long and involved, but simple computations, we have

$$\begin{aligned}
\mathfrak{N}_a + \mathfrak{N}_b & = a(z-a)^2 (z+a) \left\{ \sum_{\ell=0}^{k+1} \frac{2^\ell z^{2\ell+1}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{a^{2p}}{p!} - \sum_{\ell=0}^{k+1} \frac{2^\ell a^{2\ell+1}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{z^{2p}}{p!} \right\} \\
& + \frac{a(z-a)(z+a)}{2} \left\{ \sum_{\ell=0}^{k+1} \frac{(2za)^{2\ell+1}}{(2\ell+1)!} + \sum_{\ell=0}^{k+1} \frac{(2za)^{2\ell}}{(2\ell+1)!} + \sum_{\ell=0}^k \frac{(2za)^{2\ell+2}}{(2\ell+2)!} - \sum_{\ell=0}^k \frac{(2za)^{2\ell+2}}{(2\ell+3)!} \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{a(z+a)}{2} \left\{ \sum_{\ell=0}^{k+1} \frac{2^\ell z^{2\ell+1}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{a^{2p}}{p!} - \sum_{\ell=0}^{k+1} \frac{2^\ell a^{2\ell+1}}{(2\ell+1)!!} \sum_{p=0}^{\ell} \frac{z^{2p}}{p!} \right\} \\
& - \frac{2^{k+1} a^{2k+6} (z-a)}{(2k+3)!!} \sum_{p=0}^{k+1} \frac{z^{2p}}{p!} - \frac{2^{k+1} z^{2k+5} a (z-a)}{(2k+3)!!} \sum_{p=0}^{k+1} \frac{a^{2p}}{p!} \\
& = \frac{a(z-a)^2(z+a)}{2} \mathfrak{N}_{k+2}^{(\mathbf{g})}(z, a) + \frac{a(z-a)(z+a)}{2} \sum_{\ell=0}^{2k+3} \frac{(2za)^\ell}{\ell!} - \frac{a(z+a)}{4} \mathfrak{N}_{k+2}^{(\mathbf{g})}(z, a) \\
& - \frac{2^{k+1} a^{2k+6} (z-a)}{(2k+3)!!} \sum_{p=0}^{k+1} \frac{z^{2p}}{p!} - \frac{2^{k+1} z^{2k+5} a (z-a)}{(2k+3)!!} \sum_{p=0}^{k+1} \frac{a^{2p}}{p!}.
\end{aligned}$$

Coming back to $\mathfrak{N}_1 + \mathfrak{N}_2$, we have

$$\begin{aligned}
& (z+a)(z-a)^3(\mathfrak{N}_1 + \mathfrak{N}_2) \\
& = \frac{2^k}{2k+3} \left\{ - (2k+5)z^{2k+4} \left((k+1)ae_k(a^2) - a^3e_{k-1}(a^2) \right) + (2k+3)z^{2k+5} \left((k+1)e_k(a^2) - a^2e_{k-1}(a^2) \right) \right. \\
& \quad + (2k+3)z^{2k+2} \left((k+2)a^3e_k(a^2) - a^5e_{k-1}(a^2) \right) - (2k+1)z^{2k+3} \left((k+2)a^2e_k(a^2) - a^4e_{k-1}(a^2) \right) \\
& \quad \left. + 2(z-a)a^{2k+4}z^2e_{k-1}(z^2) - (z+a)a^{2k+4}e_k(z^2) \right\} =: \frac{2^k}{2k+3} \mathfrak{N}_c.
\end{aligned}$$

After long computations, we see that

$$\begin{aligned}
\mathfrak{N}_c & = (2k+3)z^{2k+2}(z+a)(z-a)^2f_k(a^2) - (2k+3)z^{2k+2}a(z-a)(z+a)e_k(a^2) \\
& \quad - \frac{2a(z-a)(z+a)(za)^{2k+2}}{k!} - \frac{2a(z-a)(z+a)(za)^{2k+3}}{(k+1)!} \\
& \quad + a(z+a) \left(z^{2k+3}e_k(a^2) - a^{2k+3}e_k(z^2) \right) + 2z^{2k+3}a^3(z-a)e_{k+1}(a^2) + 2z^2a^{2k+4}(z-a)e_{k+1}(z^2).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\frac{2^k}{2k+3} \mathfrak{N}_c & = 2^k z^{2k+2} (z+a)(z-a)^2 f_k(a^2) - 2^k z^{2k+2} a (z-a)(z+a) e_k(a^2) \\
& \quad - \frac{2^{k+1} a (z-a)(z+a)(za)^{2k+2}}{(2k+3)k!} - \frac{2^{k+1} a (z-a)(z+a)(za)^{2k+3}}{(2k+3)(k+1)!} \\
& \quad + \frac{2^k a (z+a)}{2k+3} \left(z^{2k+3} e_k(a^2) - a^{2k+3} e_k(z^2) \right) \\
& \quad + \frac{2^{k+1} z^{2k+3} a^3 (z-a) e_{k+1}(a^2)}{2k+3} + \frac{2^{k+1} z^2 a^{2k+4} (z-a) e_{k+1}(z^2)}{2k+3}.
\end{aligned}$$

Finally, after long and simple computations again, we have

$$\begin{aligned}
& (z+a)(z-a)^3 2^k f_k(a^2) q_{2k+1}^{(\text{pre})}(z|a) \\
& = (2k+1)!! (\mathfrak{N}_a + \mathfrak{N}_b) + \frac{2^k}{2k+3} \mathfrak{N}_c \\
& = (2k+1)!! \frac{a(z-a)^2(z+a)}{2} \mathfrak{N}_{k+2}^{(\mathbf{g})}(z, a) - (2k+1)!! \frac{a(z+a)}{4} \mathfrak{N}_{k+2}^{(\mathbf{g})}(z, a) \\
& \quad + (2k+1)!! \frac{a(z-a)(z+a)}{2} e_{2k+1}(2za) + 2^k z^{2k+2} (z+a)(z-a)^2 f_k(a^2) - 2^k z^{2k+2} a (z-a)(z+a) e_k(a^2) \\
& \quad + \frac{2^{2k+2} a (z-a)(z+a)(za)^{2k+2}}{2(2k+3)(2k+2)!!} + \frac{2^k a (z+a)}{2k+3} \left(z^{2k+3} e_k(a^2) - a^{2k+3} e_k(z^2) \right) \\
& \quad + \frac{2^{k+1} a (z-a)^2 (z+a)}{2k+3} \left(a^{2k+3} e_{k+1}(z^2) - z^{2k+3} e_{k+1}(a^2) \right).
\end{aligned}$$

Here, note that

$$\begin{aligned} & \frac{2^{2k+2}a(z-a)(z+a)(za)^{2k+2}}{2(2k+3)(2k+2)!!} + \frac{2^k a(z+a)}{2k+3} \left(z^{2k+3} e_k(a^2) - a^{2k+3} e_k(z^2) \right) \\ &= \frac{2^k a(z+a)}{2k+3} \left(z^{2k+3} e_{k+1}(a^2) - a^{2k+3} e_{k+1}(z^2) \right). \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} & (z+a)(z-a)^3 2^k f_k(a^2) q_{2k+1}^{(\text{pre})}(z|a) \\ &= (2k+1)!! \frac{a(z-a)^2(z+a)}{2} \mathfrak{K}_{k+2}^{(\mathbf{g})}(z, a) - (2k+1)!! \frac{a(z+a)}{4} \mathfrak{K}_{k+2}^{(\mathbf{g})}(z, a) \\ & \quad + (2k+1)!! \frac{a(z-a)(z+a)}{2} e_{2k+1}(2za) + 2^k z^{2k+2} (z+a)(z-a)^2 f_k(a^2) - 2^k z^{2k+2} a(z-a)(z+a) e_k(a^2) \\ & \quad + \frac{2^k a(z+a)}{2k+3} \left(z^{2k+3} e_{k+1}(a^2) - a^{2k+3} e_{k+1}(z^2) \right) + \frac{2^{k+1} a(z-a)^2(z+a)}{2k+3} \left(a^{2k+3} e_{k+1}(z^2) - z^{2k+3} e_{k+1}(a^2) \right). \end{aligned}$$

By simple but long computations, we finally arrive at (3.52). \square

Lemma 10.4. *We have*

$$\begin{aligned} & \mathfrak{D}_z^{(a)} \widehat{G}_N(z, w|a) \\ (10.11) \quad &= \frac{(z-a)^3 (w-a)^3 e^{-2wa}}{2^3} \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} q_{2k}^{(\text{pre})}(w|a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} \\ & \quad \times \left(2^2 f_k(a^2) \frac{(2k+2)!}{(2a)^{2k+3}} e_{2k+2}(2az) e^{-2za} + (2k+2) f_{k+1}(a^2) z^{2k+1} e^{-2za} - 2 f_k(a^2) z^{2k+3} e^{-2za} \right), \end{aligned}$$

and

$$\begin{aligned} & \mathfrak{D}_z^{(a)} \widehat{G}_N(w, z|a) \\ (10.12) \quad &= \frac{(z-a)^3 (w-a)^3 e^{-2wa}}{2^3} \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} q_{2k+1}^{(\text{pre})}(w|a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} \\ & \quad \times \left(2^3 a e_{k+1}(a^2) \frac{(2k+2)!}{(2a)^{2k+3}} e^{-2za} e_{2k+2}(2az) - 2^2 f_{k+1}(a^2) z^{2k+2} e^{-2za} \right). \end{aligned}$$

Lemma 10.5. *We have*

$$\begin{aligned} \mathfrak{D}_z^{(a)} \widehat{G}_N(z, w|a) &= \frac{(z-a)^3 e^{-2za-2wa}}{2^3} \left\{ 2^5 w (w-a)^2 \sum_{k=0}^{N-1} \frac{(2zw)^{2k}}{(2k)!} - 2^5 a (w-a)^2 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+1}}{(2k+1)!} \right. \\ & \quad - 2^5 a (w-a)^2 - a (w-a)^2 \frac{2^{2N+4} (2N+1-2za)}{(2N)! f_N(a^2)} e_N(a^2) (zw)^{2N} \\ & \quad - \frac{2^{N+2} (2N+1-2za) ((2N+1) \mathcal{L}_{N+1}(w, a) - 2a^2 \mathcal{L}_N(w, a))}{(2N)!! f_N(a^2)} z^{2N} \\ & \quad - \frac{2^{N+3} (2N+1) \mathcal{L}_{N+1}(w, a)}{(2N)!! f_N(a^2)} z^{2N} - a (w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+6} e_{k+1}(a^2)}{(2k+2)! f_{k+1}(a^2)} (zw)^{2k+2} \\ & \quad \left. + a^2 \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_{k+1}(a^2)} z^{2k+2} - a \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k+1} \right\}. \end{aligned}$$

Lemma 10.6. *We have*

$$\begin{aligned} \mathfrak{D}_z^{(a)} \widehat{G}_N(w, z|a) &= \frac{(z-a)^3 e^{-2za-2wa}}{2^3} \left\{ a(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k}}{(2k)!} - w(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+1}}{(2k+1)!} \right. \\ &\quad + a(w-a)^2 2^5 \frac{(2zw)^{2N}}{(2N)!} - 2^5 a(w-a)^2 - a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+6} e_{k+1}(a^2)}{(2k+2)! f_{k+1}(a^2)} (zw)^{2k+2} \\ &\quad \left. + a^2 \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)! f_{k+1}(a^2)} z^{2k+2} - a \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k)! f_k(a^2)} z^{2k+1} \right\}. \end{aligned}$$

We postpone the proof of Lemmata 10.4, 10.5, and 10.6. Now, we will complete the proof of Theorem 3.13.

Proof of Theorem 3.13. By Lemma 10.5 and 10.6 and using the identities (3.52) and (3.53), we have

$$\begin{aligned} &\mathfrak{D}_z^{(a)} \widehat{G}_N(z, w|a) - \mathfrak{D}_z^{(a)} \widehat{G}_N(w, z|a) \\ &= (z-a)^3 e^{-2za-2wa} \\ &\times \left\{ 2^2 (w-a)^3 \sum_{k=0}^{2N-1} \frac{(2zw)^k}{k!} - \frac{2^{N-1} (2N+1-2za) ((2N+1)! \mathcal{L}_{N+1}(w, a) - 2(2N-1)! a^2 \mathcal{L}_N(w, a))}{(2N)! f_N(a^2)} z^{2N} \right. \\ &\quad \left. - a(w-a)^2 \frac{2^{2N+1} (2N+1-2za)}{(2N)! f_N(a^2)} e_N(a^2) (zw)^{2N} - \frac{2^N (2N+1)! \mathcal{L}_{N+1}(w, a)}{(2N)! f_N(a^2)} z^{2N} - a(w-a)^2 2^2 \frac{(2zw)^{2N}}{(2N)!} \right\} \\ &= 4(z-a)^3 (w-a)^3 e^{-2za-2wa} \\ &\times \left\{ \sum_{k=0}^{2N-1} \frac{(2zw)^k}{k!} - \frac{(2N+1-2za) f_{N-1}(a^2) (2z)^{2N}}{2 f_N(a^2) (2N)!} q_{2N-2}^{(\text{pre})}(w|a) - \frac{(2z)^{2N}}{a (2N)!} \left(q_{2N+1}^{(\text{pre})}(w|a) - w^{2N} (w+a) \right) \right\} \\ &= 4(z-a)^3 (w-a)^3 e^{2zw-2za-2wa} \left\{ Q(2N+1, 2zw) + \mathcal{I}_N(z, w, a) - \mathcal{II}_N(z, w, a) \right\}, \end{aligned}$$

where

$$\mathcal{I}_N(z, w, a) = \frac{w (2zw)^{2N}}{a (2N)!} e^{-2zw} = -\frac{1}{2a} \partial_z Q(2N+1, 2zw),$$

and

$$\mathcal{II}_N(z, w, a) = \frac{(2z)^{2N}}{(2N)!} e^{-2zw} \left(\frac{q_{2N+1}^{(\text{pre})}(w|a)}{a} + \frac{f_{N-1}(a^2)}{2 f_N(a^2)} (2N+1-2za) q_{2N-2}^{(\text{pre})}(w|a) \right).$$

Let us denote $\widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, w|a) = e^{2a^2} \widehat{\mathfrak{X}}_N(z, w|a)$. Then, we have

$$\mathfrak{D}_z^{(a)} \widetilde{\mathfrak{X}}_N^{(\text{pre})}(z, w|a) = 4(z-a)^3 (w-a)^3 e^{2(z-a)(w-a)} \left\{ Q(2N+1, 2zw) + \mathcal{I}_N(z, w, a) - \mathcal{II}_N(z, w, a) \right\}.$$

The right hand side can be written as

$$\begin{aligned} &4(z-a)^3 (w-a)^3 e^{2(z-a)(w-a)} \left\{ Q(2N+1, 2zw) + \mathcal{I}_N(z, w, a) - \mathcal{II}_N(z, w, a) \right\} \\ &= 4(z-a)^3 (w-a)^3 e^{2(z-a)(w-a)} \left(Q(2N+1, 2zw) - \frac{1}{2a} \partial_z Q(2N+1, 2zw) \right) \\ &\quad - 2(z-a)^3 \frac{(2z)^{2N}}{a (2N)!} e^{-2a(z-a)} \left(2 \widehat{q}_{2N+1}^{(\text{pre})}(w|a) + \frac{f_{N-1}(a^2)}{f_N(a^2)} a (2N+1-2za) \widehat{q}_{2N-2}^{(\text{pre})}(w|a) \right). \end{aligned}$$

This completes the proof. \square

Our remainder task is to show Lemma 10.4, Lemma 10.5, and Lemma 10.6.

Proof of Lemma 10.4. Let us denote

$$(10.13) \quad \widehat{\mathcal{L}}_k(z, a) = e^{-2za} \mathcal{L}_k(z, a)$$

By (3.20), we have

$$(10.14) \quad \widehat{\mathcal{L}}_k(z, a) = -\widehat{\kappa}_k^{(\mathbf{g})}(z, a) + (z - a)\partial_z \widehat{\kappa}_k^{(\mathbf{g})}(z, a),$$

which also gives

$$(10.15) \quad \frac{e^{-2za} \mathcal{L}_{k+1}(z, a)}{(z - a)^2} = \partial_z \frac{\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z, a)}{z - a}.$$

By (10.15) and simple computations, we have

$$(10.16) \quad \begin{aligned} \partial_z \widehat{\mathcal{L}}_{k+1}(z, a) &= 4(z - a)\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z, a) + 2(z - a)(e^{-2za} \mathcal{L}_{k+1}(z, a)) \\ &+ \frac{2^{k+3}}{(2k + 1)!!} (z - a)(ae_k(a^2)(z - a) - f_k(a^2))z^{2k+1}e^{-2za}. \end{aligned}$$

Then, by (3.52), (3.54), and (10.16), we have

$$\begin{aligned} \frac{\partial_z \widehat{q}_{2k+1}^{(\text{pre})}(z|a)}{(z - a)^2} &= 2 \frac{\widehat{q}_{2k+1}^{(\text{pre})}(z|a)}{z - a} + 4(2k + 1)!! a \frac{\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z, a)}{z - a} \\ &+ 2^{k+3} a^2 e_k(a^2) z^{2k+1} e^{-2za} + 2^{k+3} (k + 2) f_k(a^2) z^{2k+1} e^{-2za} - 2^{k+3} f_k(a^2) z^{2k+3} e^{-2za}. \end{aligned}$$

Here, we note that

$$\int_a^z \left(\frac{\widehat{q}_{2k+1}^{(\text{pre})}(t|a)}{(t - a)^2} - 2^{k+2} f_k(a^2) t^{2k+2} e^{-2ta} \right) dt = a(2k + 1)!! \frac{\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z, a)}{z - a},$$

and then, we have

$$(10.17) \quad \begin{aligned} \frac{\partial_z \widehat{q}_{2k+1}^{(\text{pre})}(z|a)}{(z - a)^2} &= 2 \frac{\widehat{q}_{2k+1}^{(\text{pre})}(z|a)}{z - a} + 4 \int_a^z \left(\frac{\widehat{q}_{2k+1}^{(\text{pre})}(t|a)}{(t - a)^2} - 2^{k+2} f_k(a^2) t^{2k+2} e^{-2ta} \right) dt \\ &+ \left(2^{k+3} a^2 e_k(a^2) z^{2k+1} + 2^{k+3} (k + 2) f_k(a^2) z^{2k+1} - 2^{k+3} f_k(a^2) z^{2k+3} \right) e^{-2za}. \end{aligned}$$

Since

$$\int_a^z t^{2k+2} e^{-2ta} dt = \frac{(2k + 2)!}{(2a)^{2k+3}} (e^{-2a^2} e_{2k+2}(2a^2) - e^{-2az} e_{2k+2}(2az)),$$

we have

$$-4 \cdot 2^{k+2} f_k(a^2) z^{2k+2} e^{-2za} = 2^{k+4} f_k(a^2) \frac{(2k + 2)!}{(2a)^{2k+3}} \partial_z (e^{-2za} e_{2k+2}(2az)).$$

Combining this identity with (10.17), we have

$$\begin{aligned} \mathfrak{D}_z^{(a)} \widehat{q}_{2k+1}^{(\text{pre})}(z|a) &= (z - a)^3 \partial_z \left(2^{k+4} f_k(a^2) \frac{(2k + 2)!}{(2a)^{2k+3}} e_{2k+2}(2az) + 2^{k+3} a^2 e_k(a^2) z^{2k+1} \right. \\ &\quad \left. + 2^{k+3} (k + 2) f_k(a^2) z^{2k+1} - 2^{k+3} f_k(a^2) z^{2k+3} \right) e^{-2z}. \end{aligned}$$

Using the following identity

$$a^2 e_k(a^2) + (k + 2) f_k(a^2) = (k + 1) f_{k+1}(a^2),$$

we obtain

$$\begin{aligned} \mathfrak{D}_z^{(a)} \widehat{G}_N(z, w|a) &= \frac{1}{2^3} \sum_{k=0}^{N-1} \frac{\mathfrak{D}_z^{(a)} \widehat{q}_{2k+1}^{(\text{pre})}(z|a) \widehat{q}_{2k}^{(\text{pre})}(w|a)}{(2k + 2)! f_{k+1}(a^2) f_k(a^2)} \\ &= \frac{(z - a)^3}{2^3} \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} \widehat{q}_{2k}^{(\text{pre})}(w|a)}{(2k + 2)! f_{k+1}(a^2) f_k(a^2)} \\ &\quad \times \left(4 f_k(a^2) \frac{(2k + 2)!}{(2a)^{2k+3}} e_{2k+2}(2az) + (2k + 2) f_{k+1}(a^2) z^{2k+1} - 2 f_k(a^2) z^{2k+3} \right) e^{-2za}, \end{aligned}$$

which shows the first part of the claim in Lemma 10.4. Similarly, we show the second part of the claim in Lemma 10.4. By similar computations, we firstly note that

$$\begin{aligned} \frac{\partial_z \widehat{q}_{2k}^{(\text{pre})}(z|a)}{(z-a)^2} &= 2 \frac{\widehat{q}_{2k}^{(\text{pre})}(z|a)}{z-a} + 4(2k+3)!! \frac{\widehat{\kappa}_{k+2}^{(\mathbf{g})}(z,a)}{z-a} - 4 \cdot 2a^2(2k+1)!! \frac{\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z,a)}{z-a} \\ &\quad + (2k+3)!! \cdot \frac{2^{k+4}}{(2k+3)!!} (ae_{k+1}(a^2)(z-a) - f_{k+1}(a^2)) \frac{z^{2k+3}e^{-2za}}{z-a} \\ &\quad - 2a^2(2k+1)!! \cdot \frac{2^{k+3}}{(2k+1)!!} (ae_k(a^2)(z-a) - f_k(a^2)) \frac{z^{2k+1}e^{-2za}}{z-a} \\ &\quad + 2^{k+4}ae_{k+1}(a^2) \frac{z^{2k+2}e^{-2za}}{z-a} + 2^{k+4}ae_{k+1}(a^2)(k+1)z^{2k+1}e^{-2za} - 2^{k+4}ae_{k+1}(a^2)z^{2k+3}e^{-2za}. \end{aligned}$$

Using the similar identity,

$$4(2k+3)!! \partial_z \frac{\widehat{\kappa}_{k+2}^{(\mathbf{g})}(z,a)}{z-a} - 4 \cdot 2a^2(2k+1)!! \partial_z \frac{\widehat{\kappa}_{k+1}^{(\mathbf{g})}(z,a)}{z-a} = 4 \int_a^z \left(\frac{\widehat{q}_{2k}^{(\text{pre})}(t|a)}{(t-a)^2} - 2^{k+3}ae_{k+1}(a^2)t^{2k+2}e^{-2ta} \right) dt,$$

we have

$$\frac{\partial_z \widehat{q}_{2k}^{(\text{pre})}(z|a)}{(z-a)^2} = 2 \frac{\widehat{q}_{2k}^{(\text{pre})}(z|a)}{z-a} + 4 \int_a^z \left(\frac{\widehat{q}_{2k}^{(\text{pre})}(t|a)}{(t-a)^2} - 2^{k+3}ae_{k+1}(a^2)t^{2k+2}e^{-2ta} \right) dt - 2^{k+4}f_{k+1}(a^2)z^{2k+2}e^{-2za}.$$

Then, we obtain

$$\mathfrak{D}_z^{(a)} \widehat{q}_{2k}^{(\text{pre})}(z|a) = (z-a)^3 \partial_z \left(2^{k+5}a(f_{k+1}(a^2) - f_k(a^2)) \frac{(2k+2)!}{(2a)^{2k+3}} e^{-2za} e_{2k+2}(2az) - 2^{k+4}f_{k+1}(a^2)z^{2k+2}e^{-2za} \right).$$

As a consequence, we get

$$\begin{aligned} \mathfrak{D}_z^{(a)} \widehat{G}_N(w, z|a) &= \frac{(z-a)^3}{2^3} \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} \widehat{q}_{2k+1}^{(\text{pre})}(w|a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} \\ &\quad \times \left(2^3 a (f_{k+1}(a^2) - f_k(a^2)) \frac{(2k+2)!}{(2a)^{2k+3}} e^{-2za} e_{2k+2}(2az) - 2^2 f_{k+1}(a^2) z^{2k+2} e^{-2za} \right). \end{aligned}$$

This completes the proof. \square

We are ready to show Lemma 10.5 and Lemma 10.6.

Proof of Lemma 10.5. We shall expand (10.11). We define

$$\begin{aligned} \mathcal{M}_1 &= \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} a f_k(a^2) e_{k+1}(a^2) (w-a)^2 w^{2k+2}}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^{k+5} \frac{(2k+2)!}{(2a)^{2k+3}} e_{2k+2}(2az) e^{-2az} \\ &= -a(w-a)^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{2k+6} e_{k+1}(a^2)}{(2k+2)! f_{k+1}(a^2)} (zw)^{2k+2}, \\ \mathcal{M}_2 &= \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} a e_{k+1}(a^2) (w-a)^2 w^{2k+2}}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^{k+3} \left((2k+2) f_{k+1}(a^2) z^{2k+1} e^{-2za} - 2 f_k(a^2) z^{2k+3} e^{-2za} \right), \\ \mathcal{M}_3 &= \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} \left((2k+3)!! \mathcal{L}_{k+2}(w, a) - 2a^2(2k+1)!! \mathcal{L}_{k+1}(w, a) \right)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^2 f_k(a^2) \frac{(2k+2)!}{(2a)^{2k+3}} e_{2k+2}(2az) e^{-2az}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_4 &= \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} \left((2k+3)!! \mathcal{L}_{k+2}(w, a) - 2a^2(2k+1)!! \mathcal{L}_{k+1}(w, a) \right)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} \\ &\quad \times \left((2k+2) f_{k+1}(a^2) z^{2k+1} e^{-2za} - 2 f_k(a^2) z^{2k+3} e^{-2za} \right). \end{aligned}$$

Then, we have

$$\mathfrak{D}_z^{(a)} \widehat{G}_N(z, w|a) = \frac{(z-a)^3 e^{-2wa}}{2^3} (\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4).$$

We shall calculate \mathcal{M}_j for $j = 2, 3, 4$. By simple computations, we have

$$\begin{aligned} e^{2za} \mathcal{M}_2 &= a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+2}}{(2k)! f_k(a^2)} e_{k+1}(a^2) - a^2(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+6} z^{2k+1} w^{2k+2}}{(2k+1)! f_k(a^2)} e_{k+1}(a^2) \\ &\quad - a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+4} (2k+1) z^{2k} w^{2k}}{(2k)! f_k(a^2)} e_k(a^2) + a^2(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k+1} w^{2k}}{(2k)! f_k(a^2)} e_k(a^2) \\ &\quad + a \frac{(w-a)^2 2^4 e_0(a^2)}{f_0(a^2)} - a^2 \frac{(w-a)^2 2^5 e_0(a^2) z}{f_0(a^2)} - a \frac{(w-a)^2 2^{2N+4} (2N+1) e_N(a^2) z^{2N} w^{2N}}{(2N)! f_N(a^2)} \\ &\quad + a^2 \frac{(w-a)^2 2^{2N+5} e_N(a^2) z^{2N+1} w^{2N}}{(2N)! f_N(a^2)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} e^{2za} \mathcal{M}_3 &= - \sum_{k=0}^{N-1} \frac{2^{k+3} (2k+1) \mathcal{L}_{k+1}(w, a)}{(2k)! f_k(a^2)} z^{2k} - \frac{2^{N+3} (2N+1) \mathcal{L}_{N+1}(w, a)}{(2N)! f_N(a^2)} z^{2N} + \frac{2^3 \mathcal{L}_1(w, a)}{f_0(a^2)} \\ &\quad + 2a^2 \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k+2)! f_{k+1}(a^2)} z^{2k+2}. \end{aligned}$$

The most hard part is to compute \mathcal{M}_4 . First note that \mathcal{M}_4 can be written as

$$\mathcal{M}_4 = e^{-2za} (\mathcal{M}_4^{(1)} + \mathcal{M}_4^{(2)}),$$

where

$$\begin{aligned} \mathcal{M}_4^{(1)} &= \sum_{k=0}^{N-1} \frac{2^{k+2} (2k+1) \left((2k+3) \mathcal{L}_{k+2}(w, a) - (2k+1) \mathcal{L}_{k+1}(w, a) \right)}{(2k)! f_k(a^2)} z^{2k} \\ &\quad - a^2 \sum_{k=0}^{N-1} \frac{2^{k+3} (2k+1) \left(\mathcal{L}_{k+1}(w, a) - \mathcal{L}_k(w, a) \right)}{(2k)! f_k(a^2)} z^{2k} + \frac{2^2 \mathcal{L}_1(w, a) - 2^3 a^2 \mathcal{L}_0(w, a)}{f_0(a^2)} \\ &\quad - \frac{2^{N+2} (2N+1) \left((2N+1)! \mathcal{L}_{N+1}(w, a) - 2a^2 (2N-1)! \mathcal{L}_N(w, a) \right)}{(2N)! f_N(a^2)} z^{2N}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_4^{(2)} &= -a \sum_{k=0}^{N-1} \frac{2^{k+3} \left((2k+3) \mathcal{L}_{k+2}(w, a) - (2k+1) \mathcal{L}_{k+1}(w, a) \right)}{(2k)! f_k(a^2)} z^{2k+1} \\ &\quad + a^3 \sum_{k=0}^{N-1} \frac{2^{k+4} (\mathcal{L}_{k+1}(w, a) - \mathcal{L}_k(w, a))}{(2k)! f_k(a^2)} z^{2k+1} + a \frac{2^{N+3} (2N+1) \mathcal{L}_{N+1}(w, a)}{(2N)! f_N(a^2)} z^{2N+1} \\ &\quad - 2a^3 \frac{2^{N+3} \mathcal{L}_N(w, a)}{(2N)! f_N(a^2)} z^{2N+1} - a \frac{2^3 \mathcal{L}_1(w, a)}{f_0(a^2)} z + 2a^3 \frac{2^3 \mathcal{L}_0(w, a)}{f_0(a^2)} z. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{L}_{k+1}(w, a) - \mathcal{L}_k(w, a) &= - \left(2(w-a)^2 - 1 \right) \frac{2^{k+1} a^{2k+1} e_k(w^2)}{(2k+1)!} - a(w-a) \frac{2^{k+2} e_{k-1}(a^2)}{(2k+1)!} w^{2k+1} \\ &\quad - \frac{2^{k+1} e_k(a^2)}{(2k+1)!} w^{2k+1} + (w-a) \frac{2^{k+1} e_k(a^2)}{(2k-1)!} w^{2k}, \end{aligned}$$

and

$$\begin{aligned} & (2k+3)\mathcal{L}_{k+2}(w, a) - (2k+1)\mathcal{L}_{k+1}(w, a) \\ &= 2\mathcal{L}_{k+1}(w, a) - (2(w-a)^2 - 1) \frac{2^{k+2} a^{2k+3} e_{k+1}(w^2)}{(2k+1)!!} - a(w-a) \frac{2^{k+3} e_k(a^2) w^{2k+3}}{(2k+1)!!} \\ & \quad - \frac{2^{k+2} e_{k+1}(a^2)}{(2k+1)!!} w^{2k+3} + (w-a)(2k+3) \frac{2^{k+2} e_{k+1}(a^2)}{(2k+1)!!} w^{2k+2}, \end{aligned}$$

we have

$$\begin{aligned} & \sum_{k=0}^{N-1} \frac{2^{k+2}(2k+1) \left((2k+3)\mathcal{L}_{k+2}(w, a) - (2k+1)\mathcal{L}_{k+1}(w, a) \right)}{(2k)!! f_k(a^2)} z^{2k} \\ & - a^2 \sum_{k=0}^{N-1} \frac{2^{k+3}(2k+1) \left(\mathcal{L}_{k+1}(w, a) - \mathcal{L}_k(w, a) \right)}{(2k)!! f_k(a^2)} z^{2k} \\ & - a \sum_{k=0}^{N-1} \frac{2^{k+3} \left((2k+3)\mathcal{L}_{k+2}(w, a) - (2k+1)\mathcal{L}_{k+1}(w, a) \right)}{(2k)!! f_k(a^2)} z^{2k+1} + a^3 \sum_{k=0}^{N-1} \frac{2^{k+4} \left(\mathcal{L}_{k+1}(w, a) - \mathcal{L}_k(w, a) \right)}{(2k)!! f_k(a^2)} z^{2k+1} \\ &= \sum_{k=0}^{N-1} \frac{2^{k+3}(2k+1)\mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k} - a \sum_{k=0}^{N-1} \frac{2^{k+4}\mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k+1} + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_1 &= -(2(w-a)^2 - 1) \sum_{k=0}^{N-1} \frac{2^{2k+4} a^{2k+3} z^{2k} w^{2k+2}}{(2k)!! f_k(a^2) (k+1)!}, \\ \mathcal{P}_2 &= -a(w-a) \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k}}{(2k)!! f_k(a^2)} (w^2 e_k(a^2) - a^2 e_{k-1}(a^2)) w^{2k+1}, \\ \mathcal{P}_3 &= - \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k}}{(2k)!! f_k(a^2)} (w^2 e_{k+1}(a^2) - a^2 e_k(a^2)) w^{2k+1} \\ \mathcal{P}_4 &= (w-a) \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k}}{(2k)!! f_k(a^2)} \left((2k+3) e_{k+1}(a^2) w^2 - (2k+1) e_k(a^2) a^2 \right) w^{2k}, \\ \mathcal{Q}_1 &= a(2(w-a)^2 - 1) \sum_{k=0}^{N-1} \frac{2^{2k+5} a^{2k+3} z^{2k+1} w^{2k+2}}{(2k+1)!! f_k(a^2) (k+1)!} \\ \mathcal{Q}_2 &= a^2(w-a) \sum_{k=0}^{N-1} \frac{2^{2k+6} z^{2k+1}}{(2k+1)!! f_k(a^2)} (e_k(a^2) w^2 - e_{k-1}(a^2) a^2) w^{2k+1}, \\ \mathcal{Q}_3 &= a \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k+1}}{(2k+1)!! f_k(a^2)} (e_{k+1}(a^2) w^2 - e_k(a^2) a^2) w^{2k+1}, \\ \mathcal{Q}_4 &= -a(w-a) \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k+1}}{(2k+1)!! f_k(a^2)} \left((2k+3) e_{k+1}(a^2) w^2 - (2k+1) e_k(a^2) a^2 \right) w^{2k}. \end{aligned}$$

By simple computations, we have

$$\begin{aligned} \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 &= -a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+2}}{(2k)!! f_k(a^2)} e_{k+1}(a^2) - (w-a) \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} w^{2k+2}}{(2k)!! f_k(a^2)} e_{k+1}(a^2) \\ & \quad - a(w-a) \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} w^{2k+1}}{(2k)!! f_k(a^2)} e_k(a^2) - a(w-a) \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+1} a^{2k+2}}{(2k)!! k! f_k(a^2)} \end{aligned}$$

$$-a^2(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+1} e_k(a^2)}{(2k)! f_k(a^2)} e_k(a^2),$$

and

$$\begin{aligned} \mathcal{P}_4 &= a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+4} (2k+1) z^{2k} w^{2k}}{(2k)! f_k(a^2)} e_k(a^2) + (w-a) \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} w^{2k+2} e_{k+1}(a^2)}{(2k)! f_k(a^2)} e_{k+1}(a^2) \\ &+ (w-a) \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+2} a^{2k+2}}{(2k)! k! f_k(a^2)} + (w-a) \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} w^{2k+2}}{(2k)! f_k(a^2)} e_k(a^2) \\ &+ (w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+4} (2k+1) z^{2k} w^{2k+1}}{(2k)! f_k(a^2)} e_k(a^2). \end{aligned}$$

Since

$$\sum_{k=0}^{N-1} \frac{2^{2k+4} (2k+1) z^{2k} w^{2k+1}}{(2k)! f_k(a^2)} e_k(a^2) = \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+1}}{(2k)! f_k(a^2)} (k+1) e_k(a^2) - \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} w^{2k+1}}{(2k)! f_k(a^2)} e_k(a^2),$$

and using $f_k(a^2) = (k+1)e_k(a^2) - a^2 e_{k-1}(a^2)$, we have

$$\begin{aligned} \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 &= -a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k} w^{2k+2}}{(2k)! f_k(a^2)} e_{k+1}(a^2) + a(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+4} z^{2k} (2k+1) w^{2k}}{(2k)! f_k(a^2)} e_k(a^2) \\ &+ w(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k}}{(2k)!}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 &= a^2(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+6} z^{2k+1} w^{2k+2}}{(2k+1)! f_k(a^2)} e_{k+1}(a^2) - a^2(w-a)^2 \sum_{k=0}^{N-1} \frac{2^{2k+5} z^{2k+1} w^{2k}}{(2k)! f_k(a^2)} e_k(a^2) \\ &- a(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+1}}{(2k+1)!}. \end{aligned}$$

By simple, but long computations, we have

$$\begin{aligned} &e^{2za}(\mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4) \\ &= 2^5 w(w-a)^2 \sum_{k=0}^{N-1} \frac{(2zw)^{2k}}{(2k)!} - 2^5 a(w-a)^2 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+1}}{(2k+1)!} \\ &- \frac{2^{N+2} (2N+1)!! \left((2N+1) \mathcal{L}_{N+1}(w, a) - 2a^2 \mathcal{L}_N(w, a) \right)}{(2N)! f_N(a^2)} z^{2N} - \frac{2^{N+3} (2N+1) \mathcal{L}_{N+1}(w, a)}{(2N)!! f_N(a^2)} z^{2N} \\ &+ a \frac{2^{N+3} \left((2N+1) \mathcal{L}_{N+1}(w, a) - 2a^2 \mathcal{L}_N(w, a) \right)}{(2N)!! f_N(a^2)} z^{2N+1} - 2^5 e^{-2za} a(w-a)^2 \\ &+ a^2 \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_{k+1}(a^2)} z^{2k+2} - a \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k+1} \\ &- a(w-a)^2 \frac{2^{2N+4} (2N+1) e_N(a^2) (zw)^{2N}}{(2N)! f_N(a^2)} + a^2(w-a)^2 \frac{2^{2N+5} e_N(a^2) z^{2N+1} w^{2N}}{(2N)! f_N(a^2)}. \end{aligned}$$

This complete the proof. \square

Similarly, we prove Lemma 10.6.

Proof of Lemma 10.6. The proof is done similar to Lemma 10.5.

$$\mathcal{M}_5 = \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} a f_k(a^2) e_{k+1}(a^2) (w-a)^2 w^{2k+2}}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^{k+5} \frac{(2k+2)!}{(2a)^{2k+3}} e_{2k+2}(2az) e^{-2za},$$

$$\mathcal{M}_6 = \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} a^2 e_{k+1}(a^2) (2k+1)!! \mathcal{L}_{k+1}(w, a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^3 \frac{(2k+2)!}{(2a)^{2k+3}} e_{2k+2}(2za) e^{-2za},$$

$$\mathcal{M}_7 = \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} (w-a)^2 w^{2k+2} f_{k+1}(a^2) f_k(a^2)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^{k+4} z^{2k+2} e^{-2za},$$

and

$$\mathcal{M}_8 = \partial_z \sum_{k=0}^{N-1} \frac{2^{k+2} a f_{k+1}(a^2) (2k+1)!! \mathcal{L}_{k+1}(w, a)}{(2k+2)! f_{k+1}(a^2) f_k(a^2)} 2^2 z^{2k+2} e^{-2za}.$$

Then, we can express

$$\mathfrak{D}_z^{(a)} \widehat{G}_N(w, z|a) = \frac{(z-a)^3 e^{-2wa}}{2^3} (\mathcal{M}_5 + \mathcal{M}_6 - \mathcal{M}_7 - \mathcal{M}_8).$$

First, note that $\mathcal{M}_5 = \mathcal{M}_1$. Hence, it suffices to expand \mathcal{M}_j for $j = 6, 7, 8$, but it is straightforward to see that

$$\mathcal{M}_6 = 2a^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_{k+1}(a^2)} z^{2k+2} - a^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_k(a^2)} z^{2k+2},$$

$$\mathcal{M}_7 = w(w-a)^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{2k+6} (zw)^{2k+1}}{(2k+1)!} - a(w-a)^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{2k+7} (zw)^{2k+2}}{(2k+2)!},$$

$$\mathcal{M}_8 = a e^{-2za} \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k+1} - a^2 e^{-2za} \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_k(a^2)} z^{2k+2}.$$

Combining these identities, we obtain

$$\begin{aligned} e^{2za} (\mathcal{M}_6 - \mathcal{M}_7 - \mathcal{M}_8) &= a^2 \sum_{k=0}^{N-1} \frac{2^{k+5} \mathcal{L}_{k+1}(w, a)}{(2k+2)!! f_{k+1}(a^2)} z^{2k+2} - a \sum_{k=0}^{N-1} \frac{2^{k+4} \mathcal{L}_{k+1}(w, a)}{(2k)!! f_k(a^2)} z^{2k+1} \\ &\quad - w(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+1}}{(2k+1)!} + a(w-a)^2 2^5 \sum_{k=0}^{N-1} \frac{(2zw)^{2k+2}}{(2k+2)!}. \end{aligned}$$

This completes the proof. \square

11. BULK SCALING LIMITS

In this section, we complete the proof of Theorem 3.14.

11.1. Bulk scaling limits of the skew pre-kernel associated with (3.28). First, let us recall (3.58) and (3.59):

$$\widetilde{\mathfrak{K}}_N^{(\text{over})}(z, w|a) = (z-a)^4 (w-a)^4 e^{2a^2 - 2zw - 2wa} \mathfrak{K}_N^{(\text{over})}(z, w|a),$$

and

$$\widetilde{\mathfrak{K}}_N^{(\text{over})}(z, w|a) = \widetilde{\mathfrak{K}}_N^{(\text{pre})}(z, w|a) - \widehat{q}_{2N}^{(\text{pre})}(w, a) \lim_{u \rightarrow a} \frac{\widetilde{\mathfrak{K}}_N^{(\text{pre})}(z, u|a)}{\widehat{q}_{2N}^{(\text{pre})}(u|a)} + \widehat{q}_{2N}^{(\text{pre})}(z, a) \lim_{u \rightarrow a} \frac{\widetilde{\mathfrak{K}}_N^{(\text{pre})}(w, u|a)}{\widehat{q}_{2N}^{(\text{pre})}(u|a)}.$$

From this expression, our main task is to compute the uniform asymptotic behavior of (3.56) using (3.58).

Proposition 11.1. For $p \in (-1, 1)$, $\zeta, \eta \in \mathbb{C}$, and $\chi \in \mathbb{R}$, we set

$$z = \sqrt{N}p + \zeta, \quad \eta = \sqrt{N}p + \eta, \quad \chi = \sqrt{N}p + a.$$

Then, we have

$$\lim_{N \rightarrow \infty} \tilde{\mathfrak{z}}_N^{(\text{pre})}(z, w|a) = \tilde{\kappa}_b^{(\text{pre})}(\zeta, \eta|\chi),$$

uniformly for ζ, η in compact subsets of \mathbb{C} and for χ in a compact subset of \mathbb{R} , where

$$\begin{aligned} \tilde{\kappa}_b^{(\text{pre})}(\zeta, \eta|\chi) &= \frac{\sqrt{\pi}}{4} (2(\zeta - \chi)^2 - 1)(2(\eta - \chi)^2 - 1) e^{(\zeta - \chi)^2 + (\eta - \chi)^2} \operatorname{erf}(\zeta - \eta) \\ &\quad + \frac{\sqrt{\pi}}{4} (2(\zeta - \chi)^2 - 1)(2(\eta - \chi)^2 - 1) e^{(\zeta - \chi)^2 + (\eta - \chi)^2} (\operatorname{erf}(\eta - \chi) - \operatorname{erf}(\zeta - \chi)) \\ (11.1) \quad &\quad - \frac{1}{2} (\zeta - \chi)(2(\eta - \chi)^2 - 1) e^{(\eta - \chi)^2} + \frac{1}{2} (\eta - \chi)(2(\zeta - \chi)^2 - 1) e^{(\zeta - \chi)^2} \\ &\quad - \frac{1}{2} (\zeta - \eta) e^{2(\zeta - \chi)(\eta - \chi)}. \end{aligned}$$

Proof of Proposition 11.1. Let us recall (3.52), (3.53), (3.54), (3.55). By (10.13) and (10.15), we have

$$\widehat{\mathcal{L}}_N(w, a) = (w - a)^2 \partial_w \frac{\widehat{\mathfrak{z}}_N^{(\mathbf{g})}(w, a)}{w - a}.$$

Then, we have

$$\begin{aligned} \widehat{q}_{2N}^{(\text{pre})}(w|a) &= a(w - a)^2 \frac{e_{N+1}(a^2)}{f_N(a^2)} e^{-2wa} w^{2N+2} \\ (11.2) \quad &\quad + \frac{(2N+1)!!(w - a)^2}{2^{N+3} f_N(a^2)} \partial_w \frac{(2N+3) \widehat{\mathfrak{z}}_{N+2}^{(\mathbf{g})}(w, a) - 2a^2 \widehat{\mathfrak{z}}_{N+1}^{(\mathbf{g})}(w, a)}{w - a}, \end{aligned}$$

and

$$(11.3) \quad \widehat{q}_{2N+1}^{(\text{pre})}(w|a) = (w - a)^2 e^{-2wa} w^{2N+2} + a \frac{(2N+1)!!(w - a)^2}{2^{N+2} f_N(a^2)} \partial_w \frac{\widehat{\mathfrak{z}}_{N+1}^{(\mathbf{g})}(w, a)}{w - a}.$$

By [6, Eq. (2.6)] and [37, Theorem 1.2], we have

$$\widehat{\mathcal{L}}_N(z, a) = \left((2(\zeta - \chi)^2 - 1) \sqrt{\pi} e^{(\zeta - \chi)^2} \operatorname{erf}(\zeta - \chi) + 2(\zeta - \chi) \right) (1 + O(e^{-\epsilon N}))$$

On the other hand, by the Stirling formula, we have

$$\frac{(2N+3)!!}{2^{N+3} f_N(a^2)} \widehat{\mathcal{L}}_{N+2}(z, a) - a^2 \frac{(2N+1)!!}{2^{N+2} f_N(a^2)} \widehat{\mathcal{L}}_{N+1}(z, a) = e^{-a^2} \widehat{\mathcal{L}}_{N+1}(z, a) \frac{N^{N+1}}{\sqrt{2}} e^{-N} (1 + O(N^{-1/2})),$$

and

$$(11.4) \quad \widehat{q}_{2N}^{(\text{pre})}(z|a) = \frac{e^{-a^2} \widehat{\mathcal{L}}_{N+1}(z, a)}{\sqrt{2}} N^{N+1} e^{-N} (1 + O(e^{-\epsilon N})).$$

Hence, by the translational invariance of $\mathfrak{D}_z^{(a)}$, we have

$$(11.5) \quad \mathfrak{D}_\zeta^{(\chi)} \tilde{\kappa}_b^{(\text{pre})}(\zeta, \eta|\chi) = 4(\zeta - \chi)^3 (\eta - \chi)^3 e^{2(\zeta - \eta)(\eta - \chi)},$$

where the convergence is uniform for ζ, η in compact subsets of \mathbb{C} and χ in a compact subsets of \mathbb{R} . By solving the above second order differential equation, we have

$$\begin{aligned} \tilde{\kappa}_b^{(\text{pre})}(\zeta, \eta|\chi) &= \frac{\sqrt{\pi}}{4} (2(\zeta - \chi)^2 - 1)(2(\eta - \chi)^2 - 1) e^{(\zeta - \chi)^2 + (\eta - \chi)^2} \operatorname{erf}(\zeta - \eta) \\ &\quad + \frac{\sqrt{\pi}}{4} (2(\zeta - \chi)^2 - 1)(2(\eta - \chi)^2 - 1) e^{(\zeta - \chi)^2 + (\eta - \chi)^2} (\operatorname{erf}(\eta - \chi) - \operatorname{erf}(\zeta - \chi)) \\ &\quad - \frac{1}{2} (\zeta - \chi)(2(\eta - \chi)^2 - 1) e^{(\eta - \chi)^2} + \frac{1}{2} (\eta - \chi)(2(\zeta - \chi)^2 - 1) e^{(\zeta - \chi)^2} - \frac{1}{2} (\zeta - \eta) e^{2(\zeta - \chi)(\eta - \chi)}. \end{aligned}$$

This completes the proof. \square

11.2. **Proof of Theorem 3.14.** Now, we are ready to finish the proof of Theorem 3.14.

Proof of Theorem 3.14. By the uniform convergence of (11.4) and Proposition 11.1, we can exchange the large N -limit and the limit $u \rightarrow a$, i.e., $\xi \rightarrow \chi$. Note that

$$\begin{aligned} & \lim_{\xi \rightarrow \chi} \frac{\tilde{\kappa}_b^{(\text{pre})}(\zeta, \xi|\chi) \left((2(\eta - \chi)^2 - 1) \sqrt{\pi} e^{\eta^2 + \chi^2 - 2\eta\chi} \text{erf}(\eta - \chi) + 2(\eta - \chi) \right)}{(2(\xi - \chi)^2 - 1) \sqrt{\pi} e^{\xi^2 + \chi^2 - 2\xi\chi} \text{erf}(\xi - \chi) + 2(\xi - \chi)} \\ &= \frac{1}{4} \left\{ 1 - (\zeta - \chi)^2 + e^{(\zeta - \chi)^2} (2(\zeta - \chi)^2 - 1) \right\} \left\{ (2(\eta - \chi)^2 - 1) \sqrt{\pi} e^{(\eta - \chi)^2} \text{erf}(\eta - \chi) + 2(\eta - \chi) \right\}. \end{aligned}$$

By combining the above with (3.50) and by long computations, we obtain

$$\begin{aligned} \tilde{\kappa}_b^{(\text{over})}(z, w|a) &:= \frac{1}{2} \left((z - w)(1 + (z - a)(w - a) - e^{2(z-a)(w-a)}) \right) \\ &+ \frac{\sqrt{\pi}}{4} \left((2(z - a)^2 - 1)(2(w - a)^2 - 1) e^{(z-a)^2 + (w-a)^2} \text{erf}(z - w) \right) \\ &+ \frac{\sqrt{\pi}}{4} \left((z - a)^2 - 1)(2(w - a)^2 - 1) e^{(w-a)^2} \text{erf}(w - a) \right) \\ &- \frac{\sqrt{\pi}}{4} \left((w - a)^2 - 1)(2(z - a)^2 - 1) e^{(z-a)^2} \text{erf}(z - a) \right). \end{aligned}$$

Therefore, using these uniform asymptotic behavior, we get

$$\begin{aligned} & \lim_{\text{Im}(z_1) \rightarrow 0} \widehat{D}_{1,1}^{(N,k)}(\mathbf{z}_{(k)}) \\ &= \frac{e^{-2z_1^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} \text{Pf}_{i,j=2,\dots,k} \left[\begin{pmatrix} \mathfrak{X}_{N-1}^{(\text{over})}(z_i, z_j|z_1) & \mathfrak{X}_{N-1}^{(\text{over})}(z_i, \bar{z}_j|z_1) \\ \mathfrak{X}_{N-1}^{(\text{over})}(\bar{z}_i, z_j|z_1) & \mathfrak{X}_{N-1}^{(\text{over})}(\bar{z}_i, \bar{z}_j|z_1) \end{pmatrix} \right] \prod_{j=2}^k (\bar{z}_j - z_j) |z_j - z_1|^2 (1 + |z_j - z_1|^2) e^{-2|z_j|^2} \\ &= \frac{e^{-2z_1^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} \text{Pf}_{i,j=2,\dots,k} \left[\begin{pmatrix} \frac{\phi_{z_1}(z_i) \mathfrak{X}_{N-1}^{(\text{over})}(z_i, z_j|z_1) \phi_{z_1}(z_j)}{((z_i - z_1)(z_j - z_1))^4} & \frac{\phi_{z_1}(z_i) \mathfrak{X}_{N-1}^{(\text{over})}(z_i, \bar{z}_j|z_1) \phi_{z_1}^{-1}(z_j)}{((z_i - z_1)(\bar{z}_j - z_1))^4} \\ \frac{\phi_{z_1}^{-1}(z_i) \mathfrak{X}_{N-1}^{(\text{over})}(\bar{z}_i, z_j|z_1) \phi_{z_1}(z_j)}{((\bar{z}_i - z_1)(z_j - z_1))^4} & \frac{\phi_{z_1}^{-1}(z_i) \mathfrak{X}_{N-1}^{(\text{over})}(\bar{z}_i, \bar{z}_j|z_1) \phi_{z_1}^{-1}(z_j)}{((\bar{z}_i - z_1)(\bar{z}_j - z_1))^4} \end{pmatrix} \right] \prod_{j=2}^k (\bar{z}_j - z_j) \omega_b(z_j|z_1), \end{aligned}$$

where $\omega_b(z|a)$ is defined by (3.63), and $\phi_a(z) = e^{2ia \text{Im}(z)}$ is a co-cycle factor, which does not affect the value of the Pfaffian. The remainder task is to compute the quantity $\frac{Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}}$. Note that

$$\begin{aligned} Z_{N-1}^{(\text{over})} &= (N-1)! \prod_{k=0}^{N-2} r_k^{(\text{over})} = (N-1)! \lim_{u \rightarrow a} \prod_{k=0}^{N-2} r_k^{(\text{pre})} \frac{q_{2k+2}^{(\text{pre})}(u|a)}{q_{2k}^{(\text{pre})}(u|a)} \\ &= \frac{f_{N-1}(a^2)}{f_0(a^2)} \lim_{u \rightarrow a} \frac{q_{2N-2}^{(\text{pre})}(u|a)}{q_0^{(\text{pre})}(u|a)} (N-1)! \prod_{k=0}^{N-2} \frac{(2k+1)!}{2^{2k+1}}. \end{aligned}$$

Notice also that

$$Z_N^{(\text{GinSE})} = N! \prod_{k=0}^{N-1} \frac{(2k+1)!}{2^{2k+1}} = (N-1)! \frac{(2N)!}{2^{2N}} \prod_{k=0}^{N-2} \frac{(2k+1)!}{2^{2k+1}}.$$

Then, we have

$$\frac{e^{-2a^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} = \frac{N! 2^{2N}}{(2N)!} e^{-a^2} f_{N-1}(a^2) e^{-a^2} \lim_{u \rightarrow a} q_{2N-2}^{(\text{pre})}(u|a)$$

Using the uniform asymptotic expansion of (11.4), we obtain

$$\frac{e^{-2a^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} \sim \frac{N! 2^{2N}}{(2N)!} e^{-a^2} f_{N-1}(a^2) e^{-a^2} \lim_{u \rightarrow a} q_{2N-2}^{(\text{pre})}(u|a) \sim \frac{4}{3} (1 - p^2) N, \quad \text{as } N \rightarrow \infty.$$

Therefore, we obtain

$$R_k^{(\text{over,b})}(\zeta_2, \dots, \zeta_k | \zeta_1) = \lim_{N \rightarrow \infty} \left(\lim_{\text{Im}(z_1) \rightarrow 0} \frac{1}{N} R_{N,k}^{(\text{over,b})}(\zeta_2, \dots, \zeta_k | \zeta_1) \right)$$

$$= \frac{4}{3}(1-p^2) \operatorname{Pf}_{i,j=2,\dots,k} \left[\left(\begin{array}{cc} \kappa_b^{(\text{over})}(\zeta_i, \zeta_j | \zeta_1) & \kappa_b^{(\text{over})}(\zeta_i, \bar{\zeta}_j | \zeta_1) \\ \kappa_b^{(\text{over})}(\bar{\zeta}_i, \zeta_j | \zeta_1) & \kappa_b^{(\text{over})}(\bar{\zeta}_i, \bar{\zeta}_j | \zeta_1) \end{array} \right) \right] \prod_{j=2}^k (\bar{\zeta}_j - \zeta_j) \omega_b(\zeta_j | \zeta_1),$$

uniformly for ζ_j in compact subsets of \mathbb{C} for $j = 2, 3, \dots, k$ and ζ_1 in a compact subset of \mathbb{R} . This completes the proof. \square

We finally prove Theorem 3.15.

Proof of Theorem 3.15. In general, we have

$$\mathbb{E}[\mathcal{O}_{1,1} | z_1 = a] = \frac{1}{Z_N^{(\text{GinSE})}(a)} |a - \bar{a}|^2 \frac{e^{-2|a|^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} = \frac{a - \bar{a}}{\mathfrak{K}_N^{(\mathbf{g})}(a, \bar{a})} \frac{N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}}.$$

By the uniformity of the asymptotic expansion, when we consider $a = \sqrt{N}p + \chi$ for $p \in (-1, 1)$ and χ in a compact subset of \mathbb{C} , from the proof of Theorem 3.14, we obtain

$$\lim_{N \rightarrow \infty} \lim_{\operatorname{Im}(\chi) \rightarrow 0} \frac{1}{N} \mathbb{E}[\mathcal{O}_{1,1} | z_1 = \sqrt{N}p + \chi] = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

This completes the proof. \square

12. EDGE SCALING LIMIT

In this section, we prove the scaling limit for the conditional expectation at the edge regime. Through this section, let us fix $z = \sqrt{N} + \zeta, w = \sqrt{N} + \eta$ for ζ, η in compact subsets of \mathbb{C} .

Proof of Theorem 3.17. From the proof of Theorem 3.15, we have

$$\mathbb{E}[\mathcal{O}_{1,1} | z_1 = a] = \frac{a - \bar{a}}{\widehat{\mathfrak{K}}_N^{(\mathbf{g})}(a, \bar{a})} \frac{e^{-2|a|^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}}.$$

From [6, Theorem 2.1] and [37, Theorem 1.2], recall that

$$(12.1) \quad \widehat{\mathfrak{K}}_N^{(\mathbf{g})}(z, w) = e^{(\zeta - \chi)^2} f_{(\text{e})}^{(\mathbf{g})}(\zeta, \eta) + O(N^{-1/2}),$$

where

$$f_{(\text{e})}^{(\mathbf{g})}(\zeta, \eta) = \frac{1}{\sqrt{2}} \int_{-\infty}^0 e^{-2(\zeta - u)^2} \operatorname{erfc}(\sqrt{2}(\eta - u)) - e^{-2(\eta - u)^2} \operatorname{erfc}(\sqrt{2}(\zeta - u)) du.$$

Note that

$$\partial_\zeta f_{(\text{edge})}^{(\mathbf{g})}(\zeta, \chi) = e^{-(\zeta - \chi)^2} \operatorname{erfc}(\zeta + \chi) - \frac{e^{-2\zeta^2}}{\sqrt{2}} \operatorname{erfc}(\sqrt{2}\chi).$$

Therefore, we have that $a = \sqrt{N} + \chi$ for $\chi \in \mathbb{C}$,

$$\lim_{N \rightarrow \infty} \lim_{\operatorname{Im}(\chi) \rightarrow 0} \frac{\widehat{\mathfrak{K}}_N^{(\mathbf{g})}(a, \bar{a})}{a - \bar{a}} = \operatorname{erfc}(2\chi) - \frac{1}{\sqrt{2}} e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\chi), \quad \text{for } \chi \in \mathbb{R}.$$

To compute the remainder uniform asymptotic behavior, we need the uniform asymptotic expansion (3.47):

$$(12.2) \quad f_N((\sqrt{N} + \chi)^2) = e^{(\sqrt{N} + \chi)^2} \sqrt{\frac{N}{2\pi}} F_1(\chi) \left(1 - \sqrt{\frac{2\pi}{N}} \frac{F_2(\chi)}{F_1(\chi)} + O(N^{-1}) \right),$$

where

$$F_1(\chi) = e^{-2\chi^2} - \sqrt{2\pi}\chi \operatorname{erfc}(\sqrt{2}\chi).$$

From (11.2), we recall

$$\widehat{q}_{2N-2}^{(\text{pre})}(z|a) = a(z-a)^2 \frac{e^{-2za} e_N(a^2) z^{2N}}{f_{N-1}(a^2)} + \frac{(2N-1)!!}{2^{N+2} f_{N-1}(a^2)} \mathcal{D}_z^{(a)} \left[(2N+1) \widehat{\mathfrak{K}}_{N+1}^{(\mathbf{g})}(z, a) - 2a^2 \widehat{\mathfrak{K}}_N^{(\mathbf{g})}(z, a) \right],$$

where $\mathcal{D}_z^{(a)} := (z-a)\partial_z - 1$. We divide two parts

$$(2N+1) \widehat{\mathfrak{K}}_{N+1}^{(\mathbf{g})}(z, a) - 2a^2 \widehat{\mathfrak{K}}_N^{(\mathbf{g})}(z, a) = (2N+1-2a^2) \widehat{\mathfrak{K}}_N^{(\mathbf{g})}(z, a) + (2N+1) \widehat{R}_N^{(\mathbf{g})}(z, a),$$

where

$$\widehat{R}_N^{(\mathbf{g})}(z, a) = e^{-2za} \frac{q_{2N+1}^{(\mathbf{g})}(z)q_{2N}^{(\mathbf{g})}(a) - q_{2N+1}^{(\mathbf{g})}(a)q_{2N}^{(\mathbf{g})}(z)}{r_N^{(\mathbf{g})}} = e^{-2za} \frac{2^{2N+1}N!}{(2N+1)!} \left(z^{2N+1}e_N(a^2) - a^{2N+1}e_N(z^2) \right).$$

Under $z = \sqrt{N} + \zeta$ and $a = \sqrt{N} + \chi$ for χ in a compact subset of \mathbb{R} , first note that

$$(12.3) \quad a(z-a)^2 \frac{e^{-2za}e_N(a^2)z^{2N}}{f_{N-1}(a^2)} = N^N e^{-2N-2\sqrt{N}\chi-\chi^2} e^{(\zeta-\chi)^2} \sqrt{\frac{\pi}{2}} (\zeta-\chi)^2 e^{-2\zeta^2} \frac{\operatorname{erfc}(\sqrt{2}\chi)}{F_1(\chi)} (1+o(1)),$$

uniformly for ζ, χ in compact subsets of \mathbb{C} and \mathbb{R} , respectively. Combining (12.2) with Stirling formula, we have

$$(12.4) \quad \frac{(2N-1)!!}{2^{N+2}f_{N-1}(a^2)} = \frac{1}{2} N^N e^{-2N-2\sqrt{N}\chi-\chi^2} \sqrt{\frac{\pi}{N}} \frac{1}{F_1(\chi)} (1+o(1)).$$

Using (12.1) and Stirling formula, we have

$$(12.5) \quad (2N+1-2a^2)\widehat{\mathfrak{R}}_N^{(\mathbf{g})}(z, a) = -4\sqrt{N}\chi e^{(\zeta-\chi)^2} f_{(\mathbf{e})}^{(\mathbf{g})}(\zeta, \chi)(1+o(1)).$$

and

$$(12.6) \quad (2N+1)\widehat{R}_N^{(\mathbf{g})}(z, a) = \frac{\sqrt{N}}{\sqrt{2}} e^{(\zeta-\chi)^2} \left(e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) - e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) \right) (1+o(1)),$$

uniformly for ζ, χ in compact subsets of \mathbb{C} and \mathbb{R} , respectively. Let us denote

$$\mathfrak{g}_1(\zeta, \chi) = e^{(\zeta-\chi)^2} \sqrt{\frac{\pi}{2}} (\zeta-\chi)^2 e^{-2\zeta^2} \frac{\operatorname{erfc}(\sqrt{2}\chi)}{F_1(\chi)}$$

and

$$\mathfrak{g}_2(\zeta, \chi) = \frac{\sqrt{\pi}}{F_1(\chi)} \mathcal{D}_\zeta^{(\chi)} \left[-2\chi e^{(\zeta-\chi)^2} f_{(\text{edge})}^{(\mathbf{g})}(\zeta, \chi) + \frac{e^{(\zeta-\chi)^2}}{2\sqrt{2}} \left(e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) - e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) \right) \right].$$

Then, by (12.3), (12.4), (12.5), and (12.6), we obtain

$$(12.7) \quad \widehat{q}_{2N-2}^{(\text{pre})}(z|a) = N^N e^{-2N-2\sqrt{N}\chi-\chi^2} (\mathfrak{g}_1(\zeta, \chi) + \mathfrak{g}_2(\zeta, \chi))(1+o(1)).$$

uniformly for ζ, χ in compact subsets of \mathbb{C} and \mathbb{R} , respectively.

$$\lim_{N \rightarrow \infty} \frac{e^{-2a^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} = \lim_{N \rightarrow \infty} \lim_{z \rightarrow a} \frac{N! 2^{2N}}{(2N)!} e^{-a^2} f_{N-1}(a^2) e^{-a^2} e^{2za} \frac{\widehat{q}_{2N-2}^{(\text{pre})}(z|a)}{(z-a)^3}.$$

By the uniformity of the asymptotic expansion, we can exchange the limits $N \rightarrow \infty$ and $z \rightarrow a$. Therefore, we have

$$\begin{aligned} \frac{e^{-2a^2} N! Z_{N-1}^{(\text{over})}}{Z_N^{(\text{GinSE})}} &= \lim_{z \rightarrow a} \frac{N! 2^{2N}}{(2N)!} e^{-a^2} f_{N-1}(a^2) e^{-a^2} e^{2za} \frac{1}{(z-a)^3} \widehat{q}_{2N-2}^{(\text{pre})}(z|a) \\ &\sim \frac{1}{\sqrt{2}} \sqrt{\frac{N}{2\pi}} F_1(\chi) \lim_{\zeta \rightarrow \chi} \frac{\mathfrak{g}_1(\zeta, \chi) + \mathfrak{g}_2(\zeta, \chi)}{(\zeta-\chi)^3}. \end{aligned}$$

Now, we compute $\mathfrak{g}_1(\zeta, \chi) + \mathfrak{g}_2(\zeta, \chi)$. Since

$$\begin{aligned} &\partial_\zeta \left[-2\chi e^{(\zeta-\chi)^2} f_{(\text{edge})}^{(\mathbf{g})}(\zeta, \chi) + \frac{e^{(\zeta-\chi)^2}}{2\sqrt{2}} \left(e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) - e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) \right) \right] \\ &= -2\chi e^{(\zeta-\chi)^2} \left(2\chi f_{(\text{edge})}^{(\mathbf{g})}(\zeta, \chi) + e^{-(\zeta-\chi)^2} \operatorname{erfc}(\zeta+\chi) - \frac{e^{-2\zeta^2}}{\sqrt{2}} \operatorname{erfc}(\sqrt{2}\chi) \right) \\ &\quad - \frac{e^{(\zeta-\chi)^2}}{\sqrt{2}} \left((\zeta+\chi) e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) + (\zeta-\chi) e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) - \sqrt{\frac{2}{\pi}} e^{-2\zeta^2-2\chi^2} \right), \end{aligned}$$

we have

$$\mathfrak{g}_2(\zeta, \chi) = -\frac{\sqrt{\pi}}{F_1(\chi)} \left[2\chi e^{(\zeta-\chi)^2} \left\{ (2(\zeta-\chi)^2 - 1) f_{(\text{edge})}^{(\mathbf{g})}(\zeta, \chi) \right. \right.$$

$$\begin{aligned}
& + (\zeta - \chi)e^{-(\zeta-\chi)^2} \operatorname{erfc}(\zeta + \chi) - \frac{(\zeta - \chi)e^{-2\zeta^2}}{\sqrt{2}} \operatorname{erfc}(\sqrt{2}\chi) \Big\} \\
& + \frac{e^{(\zeta-\chi)^2}}{2\sqrt{2}} \left\{ (2(\zeta - \chi)^2 - 1)e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) \right. \\
& \left. + (2(\zeta + \chi)(\zeta - \chi) + 1)e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) - 2\sqrt{\frac{2}{\pi}}(\zeta - \chi)e^{-2\zeta^2 - 2\chi^2} \right\},
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \mathfrak{g}_1(\zeta, \chi) + \mathfrak{g}_2(\zeta, \chi) \\
& = \frac{\sqrt{\pi}}{F_1(\chi)} \left[e^{(\zeta-\chi)^2} (\zeta - \chi)^2 \frac{e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi)}{\sqrt{2}} \right. \\
& - 2\chi e^{(\zeta-\chi)^2} \left((2(\zeta - \chi)^2 - 1)f_{(e)}^{(\mathfrak{g})}(\zeta, \chi) + (\zeta - \chi)e^{-(\zeta-\chi)^2} \operatorname{erfc}(\zeta + \chi) - \frac{(\zeta - \chi)e^{-2\zeta^2}}{\sqrt{2}} \operatorname{erfc}(\sqrt{2}\chi) \right) \\
& \left. - \frac{e^{(\zeta-\chi)^2}}{2\sqrt{2}} \left((2(\zeta - \chi)^2 - 1)e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\zeta) + (2(\zeta + \chi)(\zeta - \chi) + 1)e^{-2\zeta^2} \operatorname{erfc}(\sqrt{2}\chi) - 2\sqrt{\frac{2}{\pi}}(\zeta - \chi)e^{-2\zeta^2 - 2\chi^2} \right) \right].
\end{aligned}$$

By Taylor expansion, we have that as $\zeta \rightarrow \chi$,

$$\mathfrak{g}_1(\zeta, \chi) + \mathfrak{g}_2(\zeta, \chi) = \frac{\sqrt{\pi}}{F_1(\chi)} \frac{2e^{-4\chi^2}}{3\sqrt{\pi}} \left(1 + \sqrt{2\pi}e^{2\chi^2} \chi \operatorname{erfc}(\sqrt{2}\chi) - 4\sqrt{\pi}e^{4\chi^2} \chi \operatorname{erfc}(2\chi) \right) (\zeta - \chi)^3 + O((\zeta - \chi)^4).$$

As a consequence, we obtain

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \mathbb{E}[\mathcal{O}_{1,1} | z = \sqrt{N} + \chi] = \frac{e^{-4\chi^2}}{3\sqrt{\pi}} \frac{1 + \sqrt{2\pi}e^{2\chi^2} \chi \operatorname{erfc}(\sqrt{2}\chi) - 4\sqrt{\pi}e^{4\chi^2} \chi \operatorname{erfc}(2\chi)}{\operatorname{erfc}(2\chi) - \frac{1}{\sqrt{2}}e^{-2\chi^2} \operatorname{erfc}(\sqrt{2}\chi)},$$

uniformly for χ in a compact subset of \mathbb{R} . This complete the proof. \square

Part 5. Proof of Expected number of zeros of random power series with finitely dependent Gaussian coefficients

13. THE FORMULA OF THE EXPECTED NUMBER OF ZEROS AND EXAMPLES

13.1. The formula for the expected number of zeros. Our key implement to show Theorem 4.2 and Theorem 4.5 is to rewrite the Edelman-Kostlan formula for the expected number of zeros of GAF into the contour integral.

Proposition 13.1. *Let $D \subset \mathbb{C}$ be a domain with smooth boundaries, f a GAF defined in a neighborhood of D , and $N_f(D)$ be the number of zeros of f inside D . Then,*

$$(13.1) \quad \mathbb{E}N_f(D) = \frac{1}{4\pi} \int_D \Delta \log K_f(z, z) dm(z) = \frac{1}{2\pi i} \oint_{\partial D} \partial_z \log K_f(z, z) dz,$$

assuming that no singularity lies on ∂D for the second equality, where $dm(z)$ is the Lebesgue measure on the complex plane \mathbb{C} and $i = \sqrt{-1}$ is the imaginary unit.

For the proof of the first equality, see [86]. For the second equality, the Stokes theorem is used as in [32, 93]. In our setting, we have much simpler expressions for $\mathbb{E}N_f(r)$.

Corollary 13.2. *Let f be a GAF defined in (4.12) with (4.13) and (4.15). Let $D \subset \mathbb{D}$ be a domain with smooth boundaries and $N_f(D)$ be the number of zeros inside D . Then,*

$$(13.2) \quad \mathbb{E}N_f(D) = \frac{1}{2\pi i} \oint_{\partial D} \frac{\bar{z}}{1 - |z|^2} dz + \mathcal{J}(D),$$

where $\mathcal{J}(D)$ has two expressions as follows:

$$(13.3) \quad \mathcal{J}(D) = \frac{1}{2\pi i} \oint_{\partial D} \frac{G'(z)}{G_2(z, z)} dz$$

and

$$(13.4) \quad \mathcal{J}(D) = -\frac{1}{\pi} \int_D \left(\frac{|G'(z)|}{G_2(z, z)} \right)^2 dm(z).$$

In particular, when $D = \mathbb{D}(r)$, (13.2) becomes

$$(13.5) \quad \mathbb{E}N_f(r) = \frac{r^2}{1 - r^2} + \mathcal{J}(r),$$

where we simply write $\mathcal{J}(r)$ for $\mathcal{J}(\mathbb{D}(r))$.

Proof. The first expression (13.3) directly follows from (4.13), (4.15) and the second equality in Proposition 13.1. For the second expression (13.4), since $\partial_z \overline{\partial_z G(z)} = \partial_{\bar{z}}(\overline{G(z)})$, it is easy to see from the first equality in Proposition 13.1 that

$$\mathcal{J}(D) = \frac{1}{\pi} \int_D \partial_z \partial_{\bar{z}} \log G_2(z, z) dm(z) = -\frac{1}{\pi} \int_D \frac{|\partial_z G(z)|^2}{(1 + G(z) + \overline{G(z)})^2} dm(z).$$

This completes the proof. \square

Here, let us mention that the expression (13.4) essentially appeared in [111]. However, the corresponding expression is not explicit. They derived a similar expression from one-point correlation and used to evaluate the expected number of zeros in the case of fractional Gaussian noise.

Remark 13.3. *Note that $G(z)$ is a polynomial in this thesis. By the change of variables $z \mapsto rz$ in (13.3) with $D = \mathbb{D}(r)$, we have*

$$(13.6) \quad \mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial \mathbb{D}} \frac{G'(rz)}{\Theta(r, z)} dz,$$

where $\Theta(r, z)$ is the rational function of z obtained from $G_2(rz, rz)$ by putting $\bar{z} = z^{-1}$ on $\partial\mathbb{D}$. In particular, when $\gamma(k)$ is real for every $k \in \mathbb{Z}$, we have

$$\Theta(r, z) = \sum_{k \in \mathbb{Z}} \gamma(k) r^{|k|} z^k.$$

Note that $\Theta(1, e^{i\theta})$ is the spectral density at least for finitely dependent Gaussian processes. Then, one can apply the residue theorem, and from this point of view, the behavior of zeros of $\Theta(r, z)$ as $r \rightarrow 1$ is essential for the order of $\mathcal{J}(r)$.

Theorem 4.3 is a direct consequence of the second expression (13.4) of $\mathcal{J}(D)$. Now, we show Theorem 4.3.

Proof of Theorem 4.3. The error term $\mathcal{J}(D)$ is clearly non-positive from (13.4). Moreover, the right-hand side of (13.4) is zero if and only if $G'(z) = 0$ m -a.e. D . It follows from the uniqueness theorem that $G'(z)$ is identically zero on \mathbb{D} , and thus so is $G(z)$ since $G(0) = 0$. Therefore, $f(z)$ is equal to $f_{\text{PV}}(z)$ in law. This completes the proof. \square

We have seen the Edelman-Kostlan formula for the expected number of zeros. To see the usefulness of our reformulation Corollary 13.2 and philosophy to show Theorem 4.2 and Theorem 4.5, we shall look at some examples.

13.2. Examples.

Example (Ornstein-Uhlenbeck process). Let $\gamma(k) = \rho^{|k|}$ ($0 < \rho < 1$). The corresponding stationary Gaussian process is the (discrete time) Ornstein-Uhlenbeck process. In this case, we see that $G(z) = \rho z(1 - \rho z)^{-1}$ and

$$G_2(z, w) = \frac{1 - \rho^2 z \bar{w}}{(1 - \rho z)(1 - \rho \bar{w})}.$$

By using $\bar{z} = z^{-1}$ for $z \in \partial\mathbb{D}$, we see that

$$\Theta(r, z) = \frac{z(1 - \rho^2 r^2)}{(1 - \rho r z)(z - \rho r)}.$$

We apply (13.3) to this case. The only zero $z = 0$ of $\Theta(r, z)$, which does not move in r , contributes to the residue as the only pole. Hence, we have

$$\mathbb{E}N_f(r) = \frac{r^2}{1 - r^2} - \frac{\rho^2 r^2}{1 - \rho^2 r^2} = \frac{r^2}{1 - r^2} - \frac{\rho^2}{1 - \rho^2} + O(1 - r^2), \quad r \rightarrow 1.$$

In this case, $G(z)$ is analytic in $\mathbb{D}(1/\rho)$ and $\Theta(1, z)$, or equivalently $G_2(z, z)$, does not vanish on $\partial\mathbb{D}$.

Remark 13.4. *As was seen in this example, the second term $\mathcal{J}(r)$ is $O(1)$ as $r \rightarrow 1$ whenever $G(z)$ is analytic in a neighborhood of $\bar{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ and $\Theta(r, z)$ does not vanish on $\partial\mathbb{D}$.*

Example. For $0 < \rho < 1$, let ζ and $\{\eta_k\}_{k \in \mathbb{Z}}$ be i.i.d. complex standard normal random variables and define the Gaussian process $\Xi = \{\xi_k\}_{k \in \mathbb{Z}}$ by

$$\xi_k = \sqrt{\rho} \zeta + \sqrt{1 - \rho} \eta_k \quad \text{for } k \in \mathbb{Z}.$$

Then, the corresponding GAF is equal in law to

$$(13.7) \quad \sqrt{\rho} \frac{\zeta}{1 - z} + \sqrt{1 - \rho} f_{\text{PV}}(z)$$

and its covariance function is given by

$$\gamma(k) = \begin{cases} 1 & k = 0, \\ \rho & \text{otherwise.} \end{cases}$$

In this case, $G(z) = \rho z(1 - z)^{-1}$ and

$$G_2(z, z) = \frac{1 - (1 - \rho)(z + \bar{z}) + (1 - 2\rho)|z|^2}{(1 - z)(1 - \bar{z})},$$

and hence

$$\Theta(r, z) = -\frac{(1-\rho)rz^2 - (1+(1-2\rho)r^2)z + (1-\rho)r}{(1-rz)(z-r)}$$

The zeros of $\Theta(r, z)$ are ν and ν^{-1} , where $\nu = \frac{\delta - \sqrt{\delta^2 - 4}}{2}$ and $\delta = \frac{1+(1-2\rho)r^2}{(1-\rho)r}$. Note that ν (resp., ν^{-1}) is inside (resp., outside) \mathbb{D} . By using (13.3) and the residue theorem, we have

$$\mathbb{E}N_f(r) = \frac{r^2}{1-r^2} - \frac{\rho}{1-\rho} \frac{\nu-r}{(\nu-\nu^{-1})(1-\nu r)}.$$

As $r \rightarrow 1$, we have

$$\mathbb{E}N_f(r) = \frac{r^2}{1-r^2} - \frac{1}{2} \sqrt{\frac{\rho}{1-\rho}} \frac{1}{\sqrt{1-r^2}} + O(1).$$

Remark 13.5. (i) The convergence radius of $G(z)$ is 1 and its singularity is located only at $z = 1$. The zeros of $\Theta(r, z)$ are ν and ν^{-1} given above, where ν (resp., ν^{-1}) is inside (resp., outside) $\mathbb{D}(r)$. Both ν and ν^{-1} converge to 1 as $r \rightarrow 1$ and the second term of $O((1-r^2)^{-1/2})$ comes from $(\nu-\nu^{-1})^{-1}$ as the residue at $z = \nu$. (ii) From (13.7) we intuitively observe that near $z = 1$, the first term $\zeta/(1-z)$ pushes up the absolute values of $\sqrt{1-\rho}f_{\text{PV}}(z)$ and decreases the number of zeros.

The examples in this subsection help us to understand the fine asymptotic behavior of (13.3) as $r \rightarrow 1$. Indeed, to find the fine asymptotic expansion for (13.3) as $r \rightarrow 1$, we firstly find zeros of $\Theta(r, z)$. After that, we exactly or asymptotically express (13.3) by the residue theorem.

14. 2-DEPENDENT CASE: PROOF OF THEOREM 4.2

We now show Theorem 4.2.

14.1. The proof of case (I). First we show case (I).

Proof of Case (I) in Theorem 4.2. First we note that $G(z) = az + bz^2$, and then we have $\Theta(r, z) = 1 + ar(z + z^{-1}) + br^2(z^2 + z^{-2})$. From (13.3), we have

$$(14.1) \quad \mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial\mathbb{D}} \frac{a + 2brz}{1 + ar(z + z^{-1}) + br^2(z^2 + z^{-2})} dz,$$

We suppose $(a, b) \in \partial\mathcal{P}_1 \cap \partial\mathcal{P}$, i.e., $a = \pm 2\sqrt{b(1-2b)}$ with $1/6 \leq b \leq 1/2$. By the symmetry, it is enough to consider the case $a > 0$. Since the denominator is reciprocal, if γ is one of its roots, then the roots are given as $\gamma, \gamma^{-1}, \bar{\gamma}, \bar{\gamma}^{-1}$. Here we suppose $\gamma \in \mathbb{D}$ and in the upper-half plane. Thus, $\gamma, \bar{\gamma}$ (resp., $\gamma^{-1}, \bar{\gamma}^{-1}$) are inside (resp., outside) \mathbb{D} . By taking the residues at γ and $\bar{\gamma}$, we see that

$$\begin{aligned} \mathcal{J}(r) &= \frac{1}{2\pi i br} \oint_{\partial\mathbb{D}} \frac{z^2(a + 2brz)}{(z-\gamma)(z-\bar{\gamma})(z-\gamma^{-1})(z-\bar{\gamma}^{-1})} dz \\ &= \frac{2}{br} \operatorname{Re} \left(\frac{\gamma^2(a + 2br\gamma)}{(\gamma-\bar{\gamma})(\gamma-\gamma^{-1})(\gamma-\bar{\gamma}^{-1})} \right). \end{aligned}$$

Let $X = z + z^{-1}$ and rewrite the denominator as $br^2X^2 + arX + 1 - 2br^2$, whose roots are distinct and given by $X_{\pm} = (-a \pm i2\sqrt{2b}\sqrt{1-r^2})/(2br)$. It is easy to see that

$$\gamma = \frac{X_- + \sqrt{X_-^2 - 4}}{2}, \quad \bar{\gamma} = \frac{X_+ + \sqrt{X_+^2 - 4}}{2}, \quad \gamma^{-1} = \frac{X_- - \sqrt{X_-^2 - 4}}{2}, \quad \bar{\gamma}^{-1} = \frac{X_+ - \sqrt{X_+^2 - 4}}{2}.$$

Here we take the branch of \sqrt{z} such that $\sqrt{1} = 1$ and analytic in $\mathbb{C} \setminus (-\infty, 0]$. Note that

$$(14.2) \quad \gamma - \gamma^{-1} = \sqrt{X_-^2 - 4} = \frac{1}{br} \left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \right)$$

with $\alpha = b - 2b^2(r^2 + 2)$ and $\beta = 2b\sqrt{2(1-r^2)b(1-2b)}$. It is easy to see that

$$(\gamma - \bar{\gamma})(\gamma - \bar{\gamma}^{-1}) = \gamma(X_- - X_+) = -\gamma \frac{2\sqrt{2(1-r^2)}}{r}i$$

and hence

$$(14.3) \quad \mathcal{J}(r) = -\frac{1}{b\sqrt{2(1-r^2)}} \operatorname{Im} \left(\frac{\gamma(a + 2br\gamma)}{\gamma - \gamma^{-1}} \right).$$

We note that $\gamma = (X_- + \gamma - \gamma^{-1})/2$. Substituting it to the numerator and expanding it by $Y := \gamma - \gamma^{-1}$, we have

$$(14.4) \quad \begin{aligned} \frac{\gamma(a + 2br\gamma)}{\gamma - \gamma^{-1}} &= \frac{1}{2Y} \left(X_-(a + brX_-) + (a + 2brX_-)Y + brY^2 \right) \\ &= \frac{2br^2 - 1}{2r} Y^{-1} - ib\sqrt{2(1-r^2)} + \frac{brY}{2}. \end{aligned}$$

Here we used the fact that X_- is a solution of the equation $br^2X^2 + arX + 1 - 2br^2 = 0$. Since $\alpha = -b(6b - 1) + O(1 - r^2)$ and $\beta = 2b\sqrt{2b(1-2b)}\sqrt{1-r^2}$, we see that

$$(14.5) \quad \operatorname{Im} Y = \sqrt{\frac{6b-1}{b}} + O(1-r^2), \quad \operatorname{Im} Y^{-1} = -\sqrt{\frac{b}{6b-1}} + O(1-r^2), \quad r \rightarrow 1.$$

Hence it follows from (14.3), (14.4) and (14.5) that

$$\mathcal{J}(r) = -\sqrt{\frac{2b}{6b-1}} \frac{1}{\sqrt{1-r^2}} + O(1), \quad r \rightarrow 1.$$

This completes the proof of Case (I). □

14.2. Proof of case (II). Next we prove Case (II).

Proof of Case (II) in Theorem 4.2. By the symmetry, it is enough to consider the case $b = a - 1/2$ ($-1/2 \leq b \leq 1/6$). We divide the proof of Case (II) into two cases, i.e., (i) $0 < b \leq 1/6$ and (ii) $-1/2 \leq b \leq 0$. In this subsection, we always consider the situation for r sufficiently close to 1 depending on b .

First we prove the case (i). The roots of $br^2X^2 + arX + 1 - 2br^2 = 0$ are real and given by $X_{\pm} = (-a \pm \lambda)/2br \in \mathbb{R}$ with $\lambda = \sqrt{a^2 - 4b^2 + 8b^2r^2}$. Note that $X_{\pm}^2 - 4 \geq 0$, and $X_+ \rightarrow -2$ and $X_- \rightarrow (2b-1)/(2b)$ as $r \rightarrow 1$. As in Case (I), by (14.1), since the denominator is reciprocal, if two real roots γ and κ lie inside \mathbb{D} such that $\gamma < \kappa < 0$, then all the roots are given as $\gamma, \gamma^{-1}, \kappa, \kappa^{-1}$. Here γ, κ (resp. γ^{-1}, κ^{-1}) are in $\mathbb{D} \cap \mathbb{R}$ (resp. in $\mathbb{D}^c \cap \mathbb{R}$), which are given by

$$(14.6) \quad \gamma = \frac{X_+ + \sqrt{X_+^2 - 4}}{2}, \quad \gamma^{-1} = \frac{X_+ - \sqrt{X_+^2 - 4}}{2}, \quad \kappa = \frac{X_- + \sqrt{X_-^2 - 4}}{2}, \quad \kappa^{-1} = \frac{X_- - \sqrt{X_-^2 - 4}}{2}.$$

By (14.1) and the residue theorem, we see that

$$(14.7) \quad \begin{aligned} \mathcal{J}(r) &= \frac{1}{2\pi i br} \oint_{\partial \mathbb{D}} \frac{z^2(a + 2brz)}{(z - \gamma)(z - \gamma^{-1})(z - \kappa)(z - \kappa^{-1})} dz \\ &= \frac{1}{br} \left\{ \frac{\gamma^2(a + 2br\gamma)}{(\gamma - \gamma^{-1})(\gamma - \kappa)(\gamma - \kappa^{-1})} + \frac{\kappa^2(a + 2br\kappa)}{(\kappa - \gamma)(\kappa - \gamma^{-1})(\kappa - \kappa^{-1})} \right\} \\ &= \frac{1}{\lambda} \left\{ \frac{\gamma(a + 2br\gamma)}{\gamma - \gamma^{-1}} - \frac{\kappa(a + 2br\kappa)}{\kappa - \kappa^{-1}} \right\}. \end{aligned}$$

Here we used

$$(\gamma - \kappa)(\gamma - \kappa^{-1}) = \gamma(X_+ - X_-) = \frac{\gamma\lambda}{br}, \quad (\kappa - \gamma)(\kappa - \gamma^{-1}) = \kappa(X_- - X_+) = -\frac{\kappa\lambda}{br}.$$

Since $(\kappa - \kappa^{-1})^{-1} = O(1)$, it suffices to focus on the first term of (14.7). We again use the expansion in (14.4) and have

$$Y = \gamma - \gamma^{-1} = 2\sqrt{\frac{1-2b}{1-6b}}\sqrt{1-r^2} + O(1-r^2), \quad r \rightarrow 1.$$

Therefore,

$$\mathcal{J}(r) = -\frac{1}{2}\sqrt{\frac{1-2b}{1-6b}}\frac{1}{\sqrt{1-r^2}} + O(1), \quad r \rightarrow 1.$$

Next we prove the case (ii) of (II). Computation is almost the same as in the case (i) of (II), but we only need to change the roles of $\gamma, \gamma^{-1}, \kappa, \kappa^{-1}$. Indeed, γ and κ^{-1} (resp. γ^{-1}, κ) in (14.6) are in $\mathbb{D} \cap \mathbb{R}$ (resp. in $\mathbb{D}^c \cap \mathbb{R}$). By (14.1), (14.6) and

$$(\kappa^{-1} - \gamma)(\kappa^{-1} - \gamma^{-1}) = \kappa^{-1}(X_- - X_+) = -\frac{\kappa^{-1}\lambda}{2br},$$

we see that

$$\begin{aligned} \mathcal{J}(r) &= \frac{1}{br} \left\{ \frac{\gamma^2(a + 2br\gamma)}{(\gamma - \gamma^{-1})(\gamma - \kappa)(\gamma - \kappa^{-1})} + \frac{\kappa^{-2}(a + 2br\kappa^{-1})}{(\kappa^{-1} - \gamma)(\kappa^{-1} - \gamma^{-1})(\kappa^{-1} - \kappa)} \right\} \\ &= \frac{2}{\lambda} \left\{ \frac{\gamma(a + 2br\gamma)}{\gamma - \gamma^{-1}} - \frac{\kappa^{-1}(a + 2br\kappa^{-1})}{\kappa^{-1} - \kappa} \right\} \\ &= -\frac{1}{2}\sqrt{\frac{1-2b}{1-6b}}\frac{1}{\sqrt{1-r^2}} + O(1), \quad r \rightarrow 1. \end{aligned}$$

This completes the proof of case (II). \square

Remark 14.1. *By the continuity, we have the same asymptotic in Case (II), but the behavior of roots $\gamma, \gamma^{-1}, \kappa, \kappa^{-1}$ in (II) is completely different from Case (I). Indeed, $\gamma, \gamma^{-1} \rightarrow -1$ and $\kappa, \kappa^{-1} \rightarrow (2b-1)/4b \pm \sqrt{(1-6b)(1+2b)}/2|b|$ as $r \rightarrow 1$ in Case (II). That is, there is only one pair of roots toward the boundary $\partial\mathbb{D}$ as $r \rightarrow 1$ except $b = -1/2$. This implies that the asymptotic order is affected by the degeneracy of roots of $\Theta(1, z)$ located on the boundary $\partial\mathbb{D}$.*

14.3. Proof of case (III). We give a proof of Case (III).

Proof of Case (III) in Theorem 4.2. Suppose $(a, b) = (2/3, 1/6)$. Since $\alpha = \frac{1}{18}(1-r^2)$ and $\beta = \frac{1}{9}\sqrt{2(1-r^2)}$, by (14.2), we have

$$Y = \gamma - \gamma^{-1} = \frac{1}{r} \left(\sqrt{(1-r^2) + \sqrt{(1-r^2)(9-r^2)}} + i\sqrt{-(1-r^2) + \sqrt{(1-r^2)(9-r^2)}} \right).$$

It easily follows from this expression that $\text{Im} Y = O((1-r^2)^{1/4})$ and

$$\text{Im} Y^{-1} = -2^{-7/4}(1-r^2)^{-1/4} + O((1-r^2)^{1/4}), \quad r \rightarrow 1.$$

Hence, from (14.3) and (14.4), we can conclude that

$$\mathcal{J}(r) = -2^{-5/4}(1-r^2)^{-3/4} + O((1-r^2)^{-1/4}), \quad r \rightarrow 1.$$

This completes the proof of Case (III). \square

14.4. Proof of case (IV). Finally, we give a sketch of the proof of Case (IV). Since all zeros of $\Theta(r, z)$ stay away from $\partial\mathbb{D}$ as $r \rightarrow 1$ when (a, b) is in the interior of \mathcal{P} , any singularity contributing to the asymptotic behavior do not appear on the boundary $\partial\mathbb{D}$, and hence it suffices to consider as r equals to 1. Here we only consider the interior of \mathcal{P}_1 and $a > 0$. We use the same notations in the proof of Case (I). In this case, $X_{\pm} = (-a \pm i\lambda(a, b))/(2b)$ with $\lambda(a, b) = \sqrt{4b - 8b^2 - a^2}$ and we see that $(\gamma - \bar{\gamma})(\gamma - \bar{\gamma}^{-1}) = -\gamma b^{-1}\lambda(a, b)i$. Hence,

$$C(a, b) = -\mathcal{J}(1) = \frac{2}{\lambda(a, b)} \text{Im} \left(\frac{\gamma(a + 2b\gamma)}{\gamma - \gamma^{-1}} \right).$$

A little more computation shows that

$$C(a, b) = \frac{\mu(a, b) - (2b - 1)}{2\lambda(a, b)\mu(a, b)} \sqrt{4b^2 + 2b - a^2 + 2b\mu(a, b)} - 1,$$

where $\mu(a, b) = \sqrt{(1 + 2b)^2 - 4a^2}$ and that $C(a, b) > 0$ unless $(a, b) = (0, 0)$. We omit the other cases since we obtain the results just by repeating the similar computation.

15. FINITE-DEPENDENT CASE: PROOF OF THEOREM 4.5

In this section, we show Theorem 4.5. As in the proof of Theorem 4.2, our strategy is to compute the asymptotics for the residue theorem. However, different from the 2-dependent case, to explicitly write down the zeros of the spectral density for the n -dependent case seems to be difficult. From (13.3), we have

$$\mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial\mathbb{D}} \frac{G'(z)}{\Theta(r, z)} dz = \frac{r}{2\pi i} \oint_{\partial\mathbb{D}} \frac{p_n(r, z)}{q_n(r, z)} dz$$

where $p_n(r, z) = z^n \binom{2n}{n} G'(w)|_{w=rz}$ and

$$q_n(r, z) := z^n \binom{2n}{n} \Theta(r, z) = z^n \sum_{k=-n}^n \binom{2n}{n+k} r^{|k|} z^k.$$

We note from (4.23) that

$$q_n(1, z) = (z + 1)^{2n}.$$

To see the asymptotic behavior of $\mathbb{E}N_f(r)$ as $r \rightarrow 1$, we need that of $z(r)$ for $q_n(r, z(r)) = 0$.

15.1. Behavior of the root $z(r)$ as $r \rightarrow 1$. We first note that $q_n(1, -1) = 0$ and $\partial_z q_n(r, z)|_{(r, z)=(1, -1)} = 0$. Hence, we cannot apply the implicit function theorem in the variable z to $q_n(r, z)$. Alternatively, we follow a strategy of using Puiseux series expansion and Newton polygon method (cf. [138]).

First we note that

$$\begin{aligned} \partial_r q_n(r, z)|_{(r, z)=(1, -1)} &= 2 \sum_{k=1}^n k (-1)^{n+k} \binom{2n}{n+k} \\ &= (-1)^{n+1} \frac{n+1}{2n-1} \binom{2n}{n+1} \neq 0. \end{aligned}$$

By shifting $(r, z) \rightarrow (1 - r, z + 1)$ in $q_n(r, z)$, we consider

$$(15.1) \quad Q_n(x, y) := \sum_{l=0}^{2n} \binom{2n}{l} (1-x)^{|l-n|} (y-1)^l.$$

Note that $Q_n(0, y) = y^{2n}$. Following [138], we denote by $\mathbb{C}\{x, y\}$ (resp., $\mathbb{C}\{x\}$) the ring of convergent power series defined by two variables x, y (resp., one variable x). If $f \in \mathbb{C}\{x, y\}$ satisfies $f(0, y) = y^m A(y)$ with $A(0) \neq 0$, then we say f is *regular in y of order m* [138, p.20]. In our setting, $Q_n(x, y)$ is regular in y of order $2n$. We can use the following theorem from [138, p.20, Theorem 2.2.6] to guarantee the existence of $2n$ distinct solutions to the equation $Q_n(x, y) = 0$ around $(x, y) = (0, 0)$.

Theorem 15.1 ([138]). (i) Any equation $f(x, y) = 0$ where $f \in \mathbb{C}\{x, y\}$ with $f(0, 0) = 0$, $f(0, y) \neq 0$ admits at least one solution of the form $y = g(x^{1/m_1}) \in \mathbb{C}\{x\}$.

(ii) If f is regular in y of order m , and we write $f = UF$ with U a unit and F a monic polynomial of degree m in y , there are m such solutions $g_j(x^{1/m_j})$, all distinct unless the discriminant of F vanishes identically, and $F(y) \equiv \prod_{j=1}^m (y - g_j(x^{1/m_j}))$.

For our purpose, we need more explicit form of g_j 's so that we directly perform the Newton polygon method below. The solution $y(x)$ to $Q_n(x, y) = 0$ around the neighborhood of the origin $(0, 0)$ is described by this theorem since $Q_n(x, y)$ is a bivariate polynomial. Now we will compute the asymptotic expansion of $y = y(x)$ in $Q_n(x, y(x)) = 0$ at the origin $(0, 0)$ following the Newton polygon method [138, p.15, Theorem 2.1.1]. Here we give a brief description of the algorithm following [138].

- (1) First, when we are provided a equation $f(x, y) = 0$, then we plot a point (r, s) of exponents for each term $c_{r,s}x^r y^s$ of $f(x, y)$ on \mathbb{R}^2 plane if $c_{r,s} \neq 0$. Then, we have the convex hull containing all point plotted.
- (2) The boundary of its convex hull is drawn down by a straight line. Their segments do not line on the (r, s) -coordinates. We call its boundary *Newton polygon*.
- (3) We write one of the reciprocal numbers of the negative of a slope among these segments by m_1 . Then we consider $f(x, x^{m_1}(a_1 + y_1))$ and solve a_1 by focusing on the terms of the lowest degrees in x due to $f(x, y) = 0$.
- (4) We set $f^{(1)}(x, y_1) = x^{-l}f(x, x^{m_1}(a_1 + y_1))$, where l is the intersection of s -axes. Repeat the above process, and then we can obtain the solution $y = a_1 x^{m_1} + a_2 x^{m_1+m_2} + \dots$ of $f(x, y) = 0$ for $f \in \mathbb{C}\{x, y\}$.

For $Q_n(x, y)$, its Newton polygon joins $(1, 0)$ and $(0, 2n)$ as shown in Figure 9 for $n = 4$. Thus, it is guaranteed

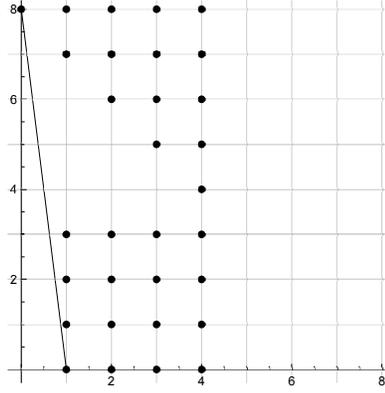


FIGURE 9. Newton polygon of $Q_n(x, y)$ for $n = 4$. A point (r, s) is marked when the coefficient $x^r y^s$ of $Q_n(x, y)$ is nonzero.

that $Q_n(x, y) = 0$ has the solution of the form

$$y = x^{1/(2n)}(a_1 + y_1),$$

where $y_1 = x^{m_2}(a_2 + y_2)$ with $m_2 \in \mathbb{Q}$ being positive. Setting $t = x^{1/(2n)}$ (equivalently $x = t^{2n}$) in (15.1) for simplicity, we have

$$Q_n(t^{2n}, t(a_1 + y_1)) = \sum_{l=0}^{2n} \binom{2n}{l} (1 - t^{2n})^{|l-n|} (t(a_1 + y_1) - 1)^l = 0$$

and the left-hand side can be expanded as follows:

$$\begin{aligned} & Q_n(t^{2n}, t(a_1 + y_1)) \\ &= \left(\sum_{l=0}^{2n} \binom{2n}{l} |l-n| (-1)^{l+1} + a_1^{2n} + 2na_1^{2n-1}y_1 + \binom{2n}{2} a_1^{2n-2}y_1^2 \right) t^{2n} \\ & \quad + \sum_{l=0}^{2n} \binom{2n}{l} |l-n| (-1)^l (a_1 + y_1) t^{2n+1} + \sum_{l=0}^{2n} \binom{2n}{l} |l-n| \binom{l}{2} (-1)^{l-1} (a_1 + y_1)^2 t^{2n+2} + O(t^{2n+3}) \end{aligned}$$

Since $y_1 = O(x^{m_2}) = O(t^{2nm_2})$ for positive $m \in \mathbb{Q}$, the leading term is of order t^{2n} and its coefficient is given by

$$a_1^{2n} + \sum_{l=0}^{2n} \binom{2n}{l} |l-n| (-1)^{l+1} = a_1^{2n} + 2(-1)^n \binom{2(n-1)}{n-1}.$$

Thus, a_1 is characterized by the solution of the equation

$$(15.2) \quad a_1^{2n} + 2(-1)^n \binom{2(n-1)}{n-1} = 0.$$

For this a_1 , the term of the lowest order t^{2n} in $Q_n(t^{2n}, t(a_1 + y_1))$ vanishes and we have

$$(15.3) \quad \begin{aligned} Q_n^{(1)}(t, y_1) &:= t^{-2n} Q_n(t^{2n}, t(a_1 + y_1)) \\ &= 2na_1^{2n-1}y_1 + n(2n-1)a_1^{2n-2}y_1^2 \\ &\quad + c(a_1 + y_1)t + \sum_{l=0}^{2n} \binom{2n}{l} |l-n| \binom{l}{2} (-1)^{l-1} (a_1 + y_1)^2 t^2 + O(t^3), \end{aligned}$$

where

$$c = \sum_{l=0}^{2n} \binom{2n}{l} |l-n| l (-1)^l = (-1)^{n+1} 2n \binom{2(n-1)}{n-1} \neq 0,$$

which implies $y_1 = O(t)$. Now we repeat the same procedure for $Q_n^{(1)}(t, y_1)$. We substitute $y_1 = t(a_2 + y_2)$ in $Q_n^{(1)}(t, y_1)$ and compare the term of order t to obtain

$$ca_1 + 2na_1^{2n-1}a_2 = 0,$$

and hence

$$(15.4) \quad a_2 = -\frac{ca_1^{-2(n-1)}}{2n} = -\frac{1}{2}a_1^2.$$

Putting $y_1 = t(a_2 + y_2)$ in (15.3) and using (15.2) and (15.4) yields

$$t^{-1}Q_n^{(1)}(t, t(a_2 + y_2)) = 2na_1^{2n-1}y_2 + (c' + cy_2 + \dots)t + O(t^2),$$

where

$$c' = n(2n-1)a_1^{2n-2}a_2^2 + ca_2 + \sum_{l=0}^{2n} \binom{2n}{l} |l-n| \binom{l}{2} (-1)^{l-1} a_1^2 \neq 0,$$

which implies $y_2 = O(t)$. In summary, by taking (15.2), (15.4) and $y = t\{a_1 + t(a_2 + O(t))\}$ into account, the solutions to the equation $Q_n(x, y) = 0$ around $x = 0$ are of the form

$$(15.5) \quad y_j^{(n)}(x) = b_j^{(n)} x^{1/(2n)} - \frac{1}{2}(b_j^{(n)})^2 x^{1/n} + O(x^{3/(2n)}), \quad \text{as } x \rightarrow 0,$$

for $j = 0, 1, \dots, 2n-1$, where $\{b_j^{(n)}\}_{j=0}^{2n-1}$ are the solutions of (15.2).

Proposition 15.2. *Let $q_n(r, z) = z^n \sum_{k=-n}^n \binom{2n}{n+k} r^{|k|} z^k$. Then, the solutions $z = z_j^{(n)}(r)$ to the equation $q_n(r, z) = 0$ are of the form*

$$(15.6) \quad z_j^{(n)}(r) = -1 + b_j^{(n)}(1-r)^{\frac{1}{2n}} - \frac{1}{2}(b_j^{(n)})^2(1-r)^{\frac{1}{n}} + O((1-r)^{\frac{3}{2n}}), \quad r \rightarrow 1,$$

where

$$(15.7) \quad b_j^{(n)} = \left\{ 2 \binom{2(n-1)}{n-1} \right\}^{1/(2n)} \exp\left(\frac{2j-n+1}{2n}\pi i\right) \quad (j = 0, 1, \dots, 2n-1).$$

Proof. Since $z_j^{(n)}(r) = -1 + y_j^{(n)}(1-r)$, putting $x = 1-r$ and $y = z+1$ in (15.5) yields

$$z_j^{(n)}(r) + 1 = b_j^{(n)}(1-r)^{\frac{1}{2n}} - \frac{1}{2}(b_j^{(n)})^2(1-r)^{\frac{1}{n}} + O((1-r)^{\frac{3}{2n}}),$$

as $r \rightarrow 1$. We obtain the assertion. \square

15.2. **Proof of Theorem 4.5.** We first observe the following asymptotics.

Lemma 15.3. For $k = 0, 1, \dots, 2n - 1$, as $r \rightarrow 1$,

$$\prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (z_k^{(n)}(r) - z_j^{(n)}(r)) = (2n)(-1)^{n-1} (e_k^{(n)})^{-1} \left\{ \binom{2(n-1)}{n-1} \right\}^{\frac{2n-1}{2n}} (1-r^2)^{\frac{2n-1}{2n}} \\ \times \left\{ 1 - C_n e_k^{(n)} (1-r^2)^{\frac{1}{2n}} + O\left((1-r^2)^{\frac{1}{n}}\right) \right\},$$

where C_n is a constant depending only on n and

$$(15.8) \quad e_k^{(n)} = \exp\left(\frac{2k-n+1}{2n}\pi i\right) \quad (k = 0, 1, \dots, 2n-1).$$

Proof. From Proposition 15.2, we have

$$\prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (z_k^{(n)}(r) - z_j^{(n)}(r)) = \prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (b_k^{(n)} - b_j^{(n)}) \cdot (1-r)^{\frac{2n-1}{2n}} \\ - \frac{1}{2} \sum_{\substack{l=0 \\ l \neq k}}^{2n-1} \prod_{\substack{j=0 \\ j \neq k, l}}^{2n-1} (b_k^{(n)} - b_j^{(n)}) \cdot \left\{ (b_k^{(n)})^2 - (b_l^{(n)})^2 \right\} \cdot (1-r)^{\frac{2n}{2n}} \\ + O\left((1-r)^{\frac{2n+1}{2n}}\right).$$

Since $\prod_{j=0}^{2n-1} (z - e^{\frac{j-k}{n}\pi i}) = z^{2n} - 1$, by differentiating both sides and putting $z = 1$, we obtain $\prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (1 - e^{\frac{j-k}{n}\pi i}) = 2n$

for every $k = 0, 1, \dots, 2n - 1$. Hence, we have

$$\prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (e_k^{(n)} - e_j^{(n)}) = (e_k^{(n)})^{2n-1} \prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (1 - e^{\frac{j-k}{n}\pi i}) = 2n(-1)^{n-1} (e_k^{(n)})^{-1}.$$

and thus, by (15.7),

$$\prod_{\substack{j=0 \\ j \neq k}}^{2n-1} (b_k^{(n)} - b_j^{(n)}) = \left\{ 2 \binom{2(n-1)}{n-1} \right\}^{\frac{2n-1}{2n}} 2n(-1)^{n-1} (e_k^{(n)})^{-1}.$$

Similarly,

$$\sum_{\substack{l=0 \\ l \neq k}}^{2n-1} \prod_{\substack{j=0 \\ j \neq k, l}}^{2n-1} (b_k^{(n)} - b_j^{(n)}) \cdot \left\{ (b_k^{(n)})^2 - (b_l^{(n)})^2 \right\} = 2 \binom{2(n-1)}{n-1} 2n(-1)^{n-1} (e_k^{(n)})^{-1} \sum_{\substack{l=0 \\ l \neq k}}^{2n-1} (e_k^{(n)} + e_l^{(n)}) \\ = (-1)^{n-1} 8n(n-1) \binom{2(n-1)}{n-1}.$$

Since $1-r = \frac{1-r^2}{2} + O((1-r^2)^2)$, we obtain the assertion. \square

Now we give a proof of Theorem 4.5. First we remark that the constant $b_j^{(n)}$ in (15.7) lies in the right-half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ for $j = 0, 1, \dots, n-1$ and the left-half plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ for $j = n, n+1, \dots, 2n-1$. Thus, if r is sufficiently close to 1, $z_j^{(n)}(r)$ for $j = 0, 1, \dots, n-1$ lie inside \mathbb{D} and $z_j^{(n)}(r)$ for $j = n+1, n+2, \dots, 2n-1$ lie outside \mathbb{D} . Therefore, we have

$$\mathcal{J}(r) = \frac{r}{2\pi i} \oint_{\partial \mathbb{D}} \frac{p_n(r, z)}{q_n(r, z)} dz \\ = r \sum_{k=0}^{n-1} \operatorname{res} \left(\frac{p_n(r, z)}{\prod_{j=0}^{2n-1} (z - z_j^{(n)}(r))}; z = z_k^{(n)}(r) \right)$$

$$(15.9) \quad = r \sum_{k=0}^{n-1} \frac{p_n(r, z_k^{(n)}(r))}{\prod_{j=0, j \neq k}^{2n-1} (z_k^{(n)}(r) - z_j^{(n)}(r))}.$$

Since $p_n(1, -1) = (-1)^n \binom{2(n-1)}{n-1}$, from Lemma 15.3 and

$$p_n(r, z_k^{(n)}(r)) = p_n(1, -1) \left\{ 1 + C'_n e_k^{(n)} (1-r^2)^{1/(2n)} + O((1-r^2)^{1/n}) \right\},$$

we have

$$\begin{aligned} \frac{p_n(r, z_k^{(n)}(r))}{\prod_{j=0, j \neq k}^{2n-1} (z_k^{(n)}(r) - z_j^{(n)}(r))} &= \frac{-1}{2n} \binom{2(n-1)}{n-1}^{\frac{1}{2n}} e_k^{(n)} (1-r^2)^{-\frac{2n-1}{2n}} \\ &\quad \times \left\{ 1 + (C_n + C'_n) e_k^{(n)} (1-r^2)^{\frac{1}{2n}} + O((1-r^2)^{\frac{2}{2n}}) \right\}, \end{aligned}$$

where C'_n is a constant depending only on n . It is easy to see that

$$(15.10) \quad \sum_{k=0}^{n-1} e_k^{(n)} = \left(\sin \frac{\pi}{2n} \right)^{-1}, \quad \sum_{k=0}^{n-1} (e_k^{(n)})^2 = 0.$$

Therefore, from (15.9), we obtain

$$\mathcal{J}(r) = \frac{-1}{2n \sin(\frac{\pi}{2n})} \binom{2(n-1)}{n-1}^{\frac{1}{2n}} (1-r^2)^{-\frac{2n-1}{2n}} \left(1 + O((1-r^2)^{\frac{2}{2n}}) \right).$$

This completes the proof.

Remark 15.4. *A naive computation gives only the error term $O((1-r)^{-(n-1)/n})$. Here we saw the cancellation as the second equality in (15.10) to obtain $O((1-r)^{-(2n-3)/(2n)})$, which matches the direct computation in Case (III) for $n = 2$.*

Remark 15.5. *This method can be applied to all cases of finitely dependent Gaussian processes. Indeed, the zero of $\Theta(1, e^{i\theta})$ of order $2k$ contributes to $\mathcal{J}(r)$ as constant multiple of $(1-r^2)^{-\frac{2k-1}{2k}}$.*

Concluding remarks

16. ON THE DETERMINANTAL STRUCTURE OF THE MULTI-POINTS CORRELATION FUNCTION OF THE OVERLAPS FOR IGINUSE AND ISUE

In both cases, we have shown the universality for the multi-points correlation function of the overlaps for IGINUE and ISUE in the strongly non-unitary regime. Indeed, we showed the same results with the Ginibre unitary ensemble shown by Akemann et al. [13] up to the constant. This difference up to the constant is due to the geometric boundary affect. We expect that macroscopic structure of the diagonal overlap for non-Hermitian random matrices at a bulk point of spectral droplet is a common property for widely non-Hermitian random matrices. For instance, we expect that there appears a boundary affect due to the ellipse curve for the elliptic Ginibre ensemble or the non-Hermitian Wishart ensemble [5, 9, 12, 16, 40]. On the other hand, we found the new scaling limits for the overlaps in the weakly and singular origin regimes, which have not been studied so far in this context. Potentially, we expect that the result in the weakly non-unitary regime would be applied to study the difference between non-normal matrices and circular unitary ensemble from the perspective of the eigenvectors and the overlaps in detail. Furthermore, coming back to our original motivation, their quantities may be applied to study the scaling limits for overlap processes and hydrodynamic limits for the stochastic differential equations for the induced Ginibre/spherical ensembles. Finally, we conclude this section by making lists for future directions of the overlaps for integrable non-Hermitian random matrices.

- (1) The most important model in random matrices with non-radially symmetric potential is elliptic Ginibre unitary ensemble defined by

$$(16.1) \quad G_N^{(\tau)} = \sqrt{\frac{1+\tau}{2}} H_N^{(1)} + \sqrt{\frac{1-\tau}{2}} H_N^{(2)}, \quad \text{for } \tau \in [-1, 1],$$

where $H_N^{(1)}, H_N^{(2)}$ are identically, independent Gaussian unitary ensembles and anti-Hermitian Gaussian unitary ensemble, respectively. The joint probability distribution function for the eigenvalues of elliptic Ginibre unitary ensemble is given by

$$d\mathbb{P}_N^{(\tau)}(\zeta_{(N)}) = \frac{1}{Z_N^{(\tau)}} \prod_{1 \leq i < j \leq N} |\zeta_i - \zeta_j|^2 \prod_{j=1}^N e^{-\frac{N}{1-\tau^2} (|\zeta_j|^2 - \tau \operatorname{Re} \zeta_j^2)} dA(\zeta_j),$$

where $Z_N^{(\tau)}$ is the partition function. Similar to the discussion in [11, 29, 58], it is straightforward to see that conditionally on $\{\lambda_{(N)} = \mathbf{z}_{(N)} \in \mathbb{C}^N\}$, we have

$$\mathbb{E}_N[\mathcal{O}_{1,1} | \lambda_{(N)} = \mathbf{z}_{(N)}] = \prod_{j=2}^N \left(1 + \frac{1-\tau^2}{N} \frac{1}{|z_j - z_1|^2} \right).$$

The conditional expectation of the off-diagonal overlap can be similarly computed. Then, in order to study the similar determinantal structure of the multi-points correlation function of the overlaps for elliptic Ginibre unitary ensemble as shown in this thesis, we need to construct a family of planar orthogonal polynomials associated with the following weight function:

$$\omega_\tau(z, \bar{z} | a, \bar{a}) = \left(\frac{1-\tau^2}{N} + |z-a|^2 \right) e^{-\frac{N}{1-\tau^2} (|z|^2 - \tau \operatorname{Re} z^2)}, \quad \text{for } z, a \in \mathbb{C}.$$

However, the corresponding to the moment matrix associated with the above weight function as in (7.1) is not a tridiagonal moment matrix, and hence, it would be difficult to work on LDE decomposition. Possibly, we need to choose another nice basis such as Hermite polynomial in advance.

- (2) We only studied the induced Ginibre/spherical unitary ensemble. It is natural to study the Pfaffian structure of the overlaps for induced Ginibre/spherical orthogonal/symplectic ensemble, cf. [11] and subsection 3.3.
- (3) The overlap plays the essential role to analyze a dynamical model for non-Hermitian random matrices with Brownian motion entries. Indeed, in [29, 66, 145], the dynamical extension of Ginibre unitary ensemble, that is to say, the eigenvalues and the eigenvector-overlap processes for the non-Hermitian random matrix with complex Brownian motion entries were investigated. It would be interesting to

derive the eigenvalues and overlap processes for the spherical unitary ensemble $S(t) = G_1^{-1}(t)G_2(t)$ with two independent non-Hermitian random matrices with complex Brownian motions $G_1(t), G_2(t)$ or the dynamical extension for the I GinUE/ISUE.

17. ON THE PFAFFIAN STRUCTURE OF THE MULTI-POINTS CORRELATION FUNCTION OF THE DIAGONAL OVERLAP FOR GINSE

In this thesis, we have showed the new method for constructing skew-orthogonal polynomials, Theorem 3.7 and analyze the second order differential equation for the skew pre-kernel (3.58). Also, in order to prove Theorem 3.13, 3.14, 3.15, and 3.17, we worked on enormous exact calculations. However, our result is restricted to a real conditional point. Due to this reason, our analysis is only applied to the diagonal overlap, and we could not analyze the off-diagonal overlap case. The below list is the future direction for studies of the overlaps for the Ginibre symplectic ensemble and related models.

- (1) We have showed the bulk scaling limit Theorem 3.14 for the multi-points correlation function for the diagonal overlap. We also have the edge scaling limit, but we did not show the result because we could not find an explicit and a closed representation as in Theorem 3.14. To find a nice and a closed expression for the edge limiting pre-kernel is left to the continued work.
- (2) In Remark 3.6, we mentioned the non-standard three term recurrence (3.34). Also, note that in Theorem 3.7, we have chosen the basis as the monomial z^k . It is natural to consider the basis in Theorem 3.7 as $P_k(z) - c_k P_{k-1}(z)$. Indeed, since

$$\langle z(P_k - c_k P_{k-1}), P_m - c_m P_{m-1} \rangle_s = \langle P_{k+1}, P_m \rangle - \bar{c}_m \langle P_{k+1}, P_{m-1} \rangle + b_k \langle P_k, P_m \rangle - b_k \bar{c}_m \langle P_k, P_{m-1} \rangle,$$

the resulting skew-moment matrix becomes the tridiagonal skew-symmetric matrix, where $\langle \cdot, \cdot \rangle_s$ is defined by (3.12) and $\{P_k\}$ satisfies the non-standard three-term recurrence in the sense of (3.34). Hence, it may be possible to construct a more general family of skew-orthogonal polynomials $\{q_k\}_k$, where $Q_k(z) = P_k(z) - c_k P_{k-1}(z)$. However, a subtle problem is to compute \mathcal{Z}_k in (3.38) or to find an explicit formula \mathcal{Z}_k . If we overcome this problem in a unified form, it may be possible to construct a family of skew-orthogonal polynomials associated with the weight function $|z - a|^{2c} e^{-|z|^2}$ for $z, a \in \mathbb{C}$. For the recent developments for the planar orthogonal polynomials associated with the weight function $|z - a|^{2c} e^{-|z|^2}$, we refer to [42, 98–100, 140].

- (3) In the future work, we plan to generalize the assumption that a conditional point is real into the case of complex conditional point case. The main difficulty here is that we can not use [10, Theorem 5.1]. Therefore, we need to directly study the weight function (3.27) for any $a \in \mathbb{C}$, which essentially corresponds to two point insertions for the two-dimensional Coulomb gases since the weight function in this setting becomes $|z - \bar{a}|^2 (1 + |z - a|^2) e^{-2|z|^2}$ for $z, a \in \mathbb{C}$.
- (4) The one of the motivation to analyze the overlap in a probabilistic community is to apply to the stochastic calculus for the eigenvalues and eigenvector-overlap processes for GinSE. Hence, in the future work, as a separate work of the multi-point correlation function of the overlaps, we plan to derive a system of stochastic differential equations for the eigenvalues and eigenvector-overlap processes for the dynamical Ginibre symplectic ensemble whose elements are complex Brownian motions, and we will analyze the their stochastic differential equations.

18. ON THE ZEROS OF RANDOM POWER SERIES WITH FINITELY DEPENDENT GAUSSIAN COEFFICIENTS

We have studied the asymptotic behavior of the expected number of zeros of the random power series with finitely dependent stationary complex Gaussian process coefficients. We would like to emphasize that our work is strongly inspired by [111], where they only studied the zeros of random power series $f_H(z)$ with fractional Gaussian processes coefficients with Hurst index H . More precisely, they considered

$$(18.1) \quad f_H(z) = \sum_{k=0}^{\infty} \xi_k^{(H)} z^k,$$

where $\{\xi_k^{(H)}\}_{k \in \mathbb{Z}}$ are stationary complex Gaussian processes with mean 0 and unit variance, and its covariance function is given by

$$(18.2) \quad \mathbb{E}[\xi_k^{(H)} \overline{\xi_{n+k}^{(H)}}] = \frac{1}{2}(|n+1|^{2H} + |n-1|^{2H} - 2|n|^{2H}), \quad 0 < H < 1.$$

Then, they showed that there exist non-explicit constants $C_{H,1}, C_{H,2} \geq 0$ depending on H such that

$$(18.3) \quad \frac{r^2}{1-r^2} - C_{H,1} \left(\frac{1}{2\sqrt{1-r^2}} - \frac{1}{2} \right) \leq \mathbb{E}N_{f_H}(r) \leq \frac{r^2}{1-r^2} - C_{H,2} \left(\frac{1}{2\sqrt{1-r^2}} - \frac{1}{2} \right).$$

Their result is not fine asymptotic behavior, but it would be interesting to find explicit constants $C_{H,1}, C_{H,2}$ following our strategy.

Also, let us mention [112]. In that paper, Mukeru and Mulaudzi studied a random power series with stationary Gaussian process coefficients with inverse covariance function of the original stationary complex Gaussian process. Then, the zeros point process for that model forms the determinantal point process. In such model, a covariance function in that model does not have zeros, and hence, we emphasize that their result is different from our setting. Finally, let us mention one future direction. We have investigated the asymptotic behavior of the expected number as we have already seen. It is natural to study the asymptotic behavior of the number variance in our setting. The one of the difficulties is that we need to appropriately compute residue due to the double contour integrals. This situation is different from the expected number of zeros case, and the problem is subtle.

REFERENCES

- [1] M. Adler, P. J. Forrester, T. Nagao and P. van Moerbeke, *Classical skew orthogonal polynomials and random matrices*, J. Stat. Phys. **99** (2000), 141–170.
- [2] G. Akemann, *The complex Laguerre symplectic ensemble of non-Hermitian matrices*, Nuclear Phys. B **730** (2005), 253–299.
- [3] G. Akemann, M. Baake, N. Chakarov, O. Krger, A. Mielke, M. Ottensmann and R. Werdehausen, *Territorial behaviour of buzzards versus random matrix spacing distributions*, J. Theor. Biol. **509** (2021), 110475.
- [4] G. Akemann and F. Basile, *Massive partition functions and complex eigenvalue correlations in matrix models with symplectic symmetry*, Nuclear Phys. B **766** (2007), 150–177.
- [5] G. Akemann and M. Bender, *Interpolation between Airy and Poisson statistics for unitary chiral non-Hermitian random matrix ensembles*, J. Math. Phys. **51** (2010), 103524.
- [6] G. Akemann, S.-S. Byun and N.-G. Kang, *Scaling limits of planar symplectic ensembles*, SIGMA Symmetry Integrability Geom. Methods Appl. **18** (2022), Paper No. 007, 40pp.
- [7] G. Akemann, S.-S. Byun, M. Ebke and G. Schehr, *Universality in the number variance and counting statistics of the real and symplectic Ginibre ensemble*, J. Phys. A **56** (2023), 495202.
- [8] G. Akemann, S.-S. Byun and K. Noda, *Pfaffian structure of the diagonal overlap for Ginibre symplectic ensemble conditioned on the real line*, in preparation.
- [9] G. Akemann, M. Cikovic and M. Venker, *Universality at weak and strong non-Hermiticity beyond the elliptic Ginibre ensemble*, Comm. Math. Phys. **362** (2018), 1111–1141.
- [10] G. Akemann, M. Ebke and I. Parra, *Skew-orthogonal polynomials in the complex plane and their Bergman-like kernels*, Comm. Math. Phys. **389** (2022), 621–659.
- [11] G. Akemann, Y. Förster and M. Kieburg, *Universal eigenvector correlations in quaternionic Ginibre ensembles*, J. Phys. A., **53**, (2020), 145201.
- [12] G. Akemann and M. J. Phillips, *The interpolating Airy kernels for the $\beta = 1$ and $\beta = 4$ elliptic Ginibre ensembles*, J. Stat. Phys. **155** (2014), 421–465.
- [13] G. Akemann, R. Tribe, A. Tsareas, and O. Zeitouni, *On the determinantal structure of conditional overlaps for the complex Ginibre ensemble*, Random Matrices: Theory and Applications 9 (2020), no. 04, 2050015. MR4133071
- [14] G. Akemann and G. Vernizzi, *Characteristic polynomials of complex random matrix models*, Nuclear Physics B 660.3 (2003), pp. 532–556.
- [15] G. W. Anderson, A. Guionnet and O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press (2005).
- [16] Y. Ameur and S.-S. Byun, *Almost-Hermitian random matrices and bandlimited point processes*, Anal. Math. Phys. **13** (2023), 52.
- [17] Y. Ameur, H. Hedenmalm, and N. Makarov, *Fluctuations of eigenvalues of random normal matrices*, Duke Math. J. **159** (2011), 31–81.
- [18] Y. Ameur, N.-G. Kang, and N. Makarov, *Rescaling Ward identities in the random normal matrix model*, Constr. Approx. **50** (2019), 63–127.
- [19] Y. Ameur, N.-G. Kang, N. Makarov, and A. Wennman, *Scaling limits of random normal matrix processes at singular boundary points*, J. Funct. Anal. **278** (2020), 108340.
- [20] Y. Ameur, N.-G. Kang, and S.-M. Seo, *On boundary confinements for the Coulomb gas*, Anal. Math. Phys. **10**(68) (2020).

- [21] Y. Ameur, N.-G. Kang, and S.-M. Seo, *The random normal matrix model: insertion of a point charge*, Potential Anal. (online), 2021.
- [22] R. Bardenet and A. Hardy, *Time-frequency transforms of white noises and Gaussian analytic functions*, Appl. Comput. Harmon. Anal. **50** (2021), 73–104.
- [23] R. Bardenet, A. Feller, J. Bouttier, P. Degiovanni, A. Hardy, A. Rancon, B. Roussel, G. Schehr, and C. I. Westbrook, *From point processes to quantum optics and back*, arXiv:2210.05522, 2022.
- [24] S. Belinchi, M.A. Nowak, R. Speicher and W. Tarnowski, *Squared eigenvalue condition numbers and eigenvector correlations from the single ring theorem*, J. Phys. A **50** (2017), 105204.
- [25] F. Benaych-Georges and O. Zeitouni, *Eigenvectors of non normal random matrices*, Electron. Commun. Probab. (2018) **23**, 1–12.
- [26] O. Bohigas, M.-J. Giannoni, and C. Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52** (1984), no. 1, 1–4.
- [27] A. Borodin, *Biorthogonal ensembles*, Nucl. Phys. B 536 (1998), no. **3**, 704–732.
- [28] A. Borodin and C.D. Sinclair, *The Ginibre ensemble of real random matrices and its scaling limit*, Commun. Math. Phys. **291** (2009), 177.
- [29] P. Bourgade and G. Dubach, *The distribution of overlaps between eigenvectors of Ginibre matrices*, Probab. Theory Relat. Fields **177** (2020), 397–464.
- [30] P. Bourgade and H.-T. Yau, *The eigenvector moment flow and local quantum unique ergodicity*. Communications in Mathematical Physics, **350**(1):231–278, 2017.
- [31] G.E. Box, G.C. Reinsel, G.M. Jenkins, G.M. Ljung, *Time series analysis: Forecasting and control*, John Wiley & Sons, 2015.
- [32] J. Buckley, *Fluctuations in the zero set of the hyperbolic Gaussian analytic function*, Int. Math. Res. Not. IMRN **6** (2015), 1666–1687.
- [33] J. Buckley, A. Nishry, R. Peled, and M. Sodin, *Hole probability for zeros of Gaussian Taylor series with finite radii of convergence*, Probability Theory and Related Fields, **171** (2018), 377–430.
- [34] Z. Burda, B.J. Spisak and P. Vivo, *Eigenvector statistics of the product of Ginibre matrices*, Phys. Rev. E, **95** (2017), 022134.
- [35] A. Bykhovskaya and V. Gorin, *Cointegration in large VARs*, Annals of Statistics, **50**(3), (2022) 1593–1617.
- [36] S.-S. Byun and C. Charlier, *On the almost-circular symplectic induced Ginibre ensemble*, Stud. Appl. Math. **150** (2023), 184–217.
- [37] S.-S. Byun and M. Ebke, *Universal scaling limits of the symplectic elliptic Ginibre ensembles*, Random Matrices Theory Appl. **12** (2023), 2250047.
- [38] S.-S. Byun, M. Ebke and S.-M. Seo, *Wronskian structures of planar symplectic ensembles*, Nonlinearity **36** (2023), 809–844.
- [39] S.-S. Byun and P.J. Forrester, *Spherical induced ensembles with symplectic symmetry*, SIGMA Symmetry Integrability Geom. Methods Appl. **19** (2023), 033, 28pp.
- [40] S.-S. Byun and P. J. Forrester, *Progress on the study of the Ginibre ensembles I: GinUE*, arXiv:2211.16223.
- [41] S.-S. Byun and P. J. Forrester, *Progress on the study of the Ginibre ensembles II: GinOE and GinSE*, arXiv:2301.05022.
- [42] S.-S. Byun, S.-Y. Lee and M. Yang, *Lemniscate ensembles with spectral singularity*, preprint arXiv:2107.07221, 2022.
- [43] S.-S. Byun, N.-G. Kang, and S.-M. Seo, *Partition Functions of Determinantal and Pfaffian Coulomb Gases with Radially Symmetric Potentials*, Commun. Math. Phys. (2023). <https://doi.org/10.1007/s00220-023-04673-1>
- [44] S.-S. Byun and S.-M. Seo, *Random normal matrices in the almost-circular regime*, Bernoulli **29** (2) 1615–1637, May 2023. <https://doi.org/10.3150/22-BEJ1514>
- [45] C. Charlier, *Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles*. Adv. Math., 408:36, 2022.
- [46] C. Charlier, *Large gap asymptotics on annuli in the random normal matrix model*, Math. Ann. (Online), 2023.
- [47] C. Charlier and J. Lenells. *Exponential moments for disk counting statistics of random normal matrices in the critical regime*, Nonlinearity, **36**(3):1593, 2023.
- [48] G. Cipolloni, L. Erdős, and D. Schröder, *Edge universality for non-Hermitian random matrices*, Probab. Theory Rel. Fields **179** (2021), 1–28.
- [49] G. Cipolloni, L. Erdős, and D. Schröder, *Fluctuation around the circular law for random matrices with real entries*, Electron. J. Probab., 26:1, 2021.
- [50] G. Cipolloni, L. Erdős, and D. Schröder, *Central limit theorem for linear eigenvalue statistics of non-Hermitian random matrices*, Comm. Pure Appl. Math., **76**(5):946, 2023.
- [51] C. Cipolloni and D. Schröder, *On the condition number of the shifted real Ginibre ensemble*, SIAM J. Matr. Anal. Applic. (2022) **43**, 1469–1487.
- [52] C. Cipolloni, L. Erdős, J. Henheik, and D. Schröder.: *Optimal Lower Bound on Eigenvector Overlaps for non-Hermitian Random Matrices*, preprint arXiv:2301.03549, 2023.
- [53] J. T. Chalker and B. Mehlis, *Eigenvector statistics in non-Hermitian random matrix ensembles*, Phys. Rev. Lett. **81** (1998), 3367–3370.
- [54] J. T. Chalker and B. Mehlis, *Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles*, J. Math. Phys. **41** (2000), 3233–3256.
- [55] N. Crawford and R. Rosenthal, *Eigenvector correlations in the complex Ginibre ensemble*, Ann. Appl. Probab. (2022) **32**, 2706–2754.
- [56] I. Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl, **1**(1):1130001, 76, 2012.

- [57] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of Courant Lecture Notes in Mathematics. New York University Courant Institute of Mathematical Sciences, New York, 1999.
- [58] G. Dubach, *On eigenvector statistics in the spherical and truncated unitary ensembles*, Elec. J. Probab. **26** (2021), 1–29.
- [59] G. Dubach, *Symmetries of the quaternionic Ginibre ensemble*, Random Matrices Theory Appl., **10** (2021), 2150013.
- [60] G. Dubach, *Explicit formulas concerning eigenvectors of weakly non-unitary matrices*, Electron. Commun. Probab. (2023) 28 1–11.
- [61] G. Dubach and L. Erdős, *Dynamics of a rank-one perturbation of a Hermitian matrix*, Electron. Commun. Probab. **28**: 1–13 (2023).
- [62] G. Dubach and J. Reker, *Dynamics of a rank-one multiplicative perturbation of a unitary matrix*, arXiv:2212.14638
- [63] H. Dym and H.P. McKean, *Gaussian Process, Function Theory and the Inverse Spectral Problem*, Probability and Mathematical Statistics, vol. 31. Academic, New York, 1976.
- [64] L. Erdős and H.C. Ji, *Wegner estimate and upper bound on the eigenvalue condition number of non-Hermitian random matrices*, preprint, arXiv:2301.04981. (2023)
- [65] L. Erdős and H.-T. Yau, *A dynamical approach to random matrix theory*, volume 28 of Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2017.
- [66] S. Esaki, M. Katori, and S. Yabuoku, *Eigenvalues, eigenvector-overlaps, and regularized Fuglede-Kadison determinant of the non-Hermitian matrix-valued Brownian motion*, preprint, arXiv:2306.00300, 2023.
- [67] J.A. Fischmann, *Eigenvalue distributions on a single ring*, Ph.D. thesis, Queen Mary, University of London, 2012.
- [68] J. Fischmann, W. Bruzda, B. A. Khoruzhenko, H.-J. Sommers, and K. Zyczkowski, *Induced Ginibre ensemble of random matrices and quantum operations*, J. Phys. A, **45**(7):075203, 31, 2012.
- [69] J. Fischmann and P. J. Forrester, *One-component plasma on a spherical annulus and a random matrix ensemble*, J. Stat. Mech.: Theor. Exp., 2011(10):P10003, 2011.
- [70] P.J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ, 2010.
- [71] P. J. Forrester. *Analogies between random matrix ensembles and the one-component plasma in two-dimensions*. Nucl. Phys. B, **904**:253–281, 2016.
- [72] Y. V. Fyodorov, B. A. Khoruzhenko and H.-J. Sommers, *Universality in the random matrix spectra in the regime of weak non-Hermiticity*, Ann. Inst. H. Poincaré Phys. Théor., **68** (1998), 449–489.
- [73] Y.V. Fyodorov and B. Mehlrig, *Statistics of resonances and nonorthogonal eigenfunctions in a model for single-channel chaotic scattering*, Phys. Rev. E. (2002) **66**, 045202.
- [74] Y.V. Fyodorov and D.V. Savin, *Statistics of resonance width shifts as a signature of eigenfunction non-orthogonality*, Phys. Rev. Lett. (2012) **108**, 184101.
- [75] Y. V. Fyodorov, *On statistics of bi-orthogonal eigenvectors in real and complex Ginibre ensembles: combining partial Schur decomposition with supersymmetry*, Commun. Math. Phys. **363** (2018), 579–603.
- [76] Y. V. Fyodorov, B. A. Khoruzhenko, and M. Poplavskiy, *Extreme eigenvalues and the emerging outlier in rank-one non-Hermitian deformations of the Gaussian unitary ensemble*, Entropy **25** (2023), no. 1, 74. MR4539184
- [77] Y.V. Fyodorov and W. Tarnowski, *Condition numbers for real eigenvalues in the real elliptic Gaussian ensemble*, Ann. Henri Poincaré, **22**, (2021), 309–330.
- [78] Y.V. Fyodorov and H.-J. Sommers, *Random matrices close to hermitian or unitary: overview of methods and results*, J. Phys. A **36** (2003), 3303–3347
- [79] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices*, J. Math. Phys. **6**, 440–449 (1965)
- [80] J. Grela and P. Warchol, *Full Dysonian dynamics of the complex Ginibre ensemble*, J. Phys. A **51** (2018), 42.
- [81] S. Ghosh and A. Nishry, *Point process, hole events, and large deviations: random complex zeros and Coulomb gases*, Constr. Approx. **48** (2018), no. 1, 101–136.
- [82] V. Gorin, *Lectures on random lozenge tilings*, Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2021.
- [83] H. Hedenmalm, N. Makarov.: *Coulomb gas ensembles and Laplacian growth*, Proc. London Math Soc. **106** (2013), 859–907.
- [84] H. Hedenmalm and A. Wennman, *Planar orthogonal polynomials and boundary universality in the random normal matrix model*, Acta Math., **227** (2):309–406, 2021.
- [85] O. Hirofumi, *Interacting Brownian Motions in Infinite Dimensions with Logarithmic Interaction Potentials*, Ann. Probab., **41**(1):1–49, 2013.
- [86] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Zeros of Gaussian analytic functions and determinantal point process*, University Lecture Series, vol. 51, American Mathematical Society, Providence, RI, 2009.
- [87] R.A. Janik, W. Nörenberg, M.A. Nowak, G. Papp, and I. Zahed.: *Correlations of eigenvectors for non Hermitian random matrix models*, Phys. Rev. E. (1999) **60**, 2699–2705
- [88] S. Janson, *Gaussian Hilbert space*, Cambridge Tracts in Mathematics, vol. 129, Cambridge University Press, Cambridge, 1997.
- [89] E. Kanzieper, *Eigenvalue correlations in non-Hermitian symplectic random matrices*, J. Phys. A **35** (2002), 6631–6644.
- [90] M. Katori and T. Shirai, *Zeros of the i.i.d. Gaussian Laurent series on an annulus: weighted Szegő kernels and permanental-determinantal point process*, arXiv:2008.04177[math.PR]
- [91] J. P. Keating and N. C. Snaitch, *Random matrix theory and $\zeta(1/2 + it)$* , Comm. Math. Phys. **214** (2000), no. **1**, 57–89.
- [92] A. Knowles and J. Yin, *Eigenvector distribution of Wigner matrices*, Probab. Theory Related Fields, **155**(3-4):543–582, 2013.
- [93] A. Kiro and A. Nishry, *Fluctuations for Zeros of Gaussian analytic Taylor series*, arXiv:1911.13051[math.PR]

- [94] M. Krishnapur, *From random matrices to random analytic functions*, Ann. Probab. **37** (2009), no. 1, 314–346.
- [95] A. Kulesza, B. Taskar, et al, *Determinantal point processes for machine learning*, Foundations and Trends in Machine Learning, 5(2–3): **123**, 2012.
- [96] B. Lacroix-A-Chez-Toine, S. N. Majumdar, and G. Schehr, *Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance*, Phys. Rev. A, **99**(2): 021602, 2019.
- [97] S.-Y. Lee and R. Riser, *Fine asymptotic behavior for eigenvalues of random normal matrices: Ellipse case*, J. Math. Phys. **57** (2016), 023302.
- [98] S.-Y. Lee and M. Yang, *Discontinuity in the asymptotic behavior of planar orthogonal polynomials under a perturbation of the Gaussian weight*, Comm. Math. Phys., **355** (1):303–338, 2017.
- [99] S.-Y. Lee and M. Yang, *Planar orthogonal polynomials as Type II multiple orthogonal polynomials*, J. Phys. A, 52(27):275202, 14, 2019.
- [100] S.-Y. Lee and M. Yang, *Strong asymptotics of planar orthogonal polynomials: Gaussian weight perturbed by finite number of point charges*, Comm. Pure Appl. Math. **76** (2023), no.10, 2888–2956.
- [101] T. Leblé and S. Serfaty, *Large deviation principle for empirical fields of log and Riesz gases*, Invent. Math., 210(3):645–757, 2017.
- [102] T. A. Loring, *Factorization of matrices of quaternions*, Expo. Math. **30** (2012) 250.
- [103] S. Matsumoto and T. Shirai, *Correlation functions for zeros of a Gaussian power series and Pfaffians*, Electron. J. Probab. **18** (2013), 1–18.
- [104] A. Mays, *A real quaternion spherical ensemble of random matrices*, J. Stat. Phys., 153(1):48–69, 2013.
- [105] A. Mays and A. Ponsaing, *An induced real quaternion spherical ensemble of random matrices*, Random Matrices Theory Appl., 6(1):1750001, 29, 2017.
- [106] R.M. May, *Will a large complex system be stable?* Nature **238** (1972), 413–424.
- [107] M. L. Mehta and M. Gaudin, *On the density of eigenvalues of a random matrix*, Nuclear Physics, **18**: 420–427, 1960.
- [108] M. L. Mehta, *Random matrices*, 2nd ed., Academic Press, New York, 1991.
- [109] B. Mehlhig and M. Santer, *Universal eigenvector statistics in a quantum scattering ensemble*, Phys. Rev. E. (2001) **63**, 020105(R).
- [110] E. S. Meckes, *The random matrix theory of the classical compact groups*, volume 218 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2019.
- [111] S. Mukeru, M. P. Mulaudzi, J. Nzabanita, and M. M. Mpanda, *Zeros of Gaussian power series with dependent random variables*, Illinois J. Math. **64** (2020), no. 4, 569–582.
- [112] S. Mukeru and M. P. Mulaudzi, *Zeros of Gaussian power series, Hardy spaces and determinantal point processes*, arXiv:2103.111947[math.PR]
- [113] K. Noda, *Determinantal structure of the conditional expectation of the overlaps for the induced Ginibre unitary ensemble*, arXiv:2310.15362.
- [114] K. Noda, *Determinantal structure of the overlaps for induced spherical unitary ensemble*, arXiv:2312.12690.
- [115] K. Noda and T. Shirai, *Expected Number of Zeros of Random Power Series with Finitely Dependent Gaussian Coefficients*, J Theor Probab **36**, 1534–1554 (2023).
- [116] M. A. Nowak and W. Tarnowski, *Probing non-orthogonality of eigenvectors in non-Hermitian matrix models: diagrammatic approach*, JHEP 2018 **152** (2018).
- [117] F. W. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Editors). NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge, 2010.
- [118] Y. Peres and B. Virág, *Zeros of the i.i.d. gaussian power series: a conformally invariant determinantal process*, Acta Mathematica **194** (2005), no. 1, 1–35.
- [119] B. Rider, *A limit theorem at the edge of a non-Hermitian random matrix ensemble*, J. Phys. A **36** (2003), 3401–3410.
- [120] B. Rider and B. Virág, *Complex determinantal processes and H^1 noise*, Electron. J. Probab. **12** (2007), 1238–1257.
- [121] B. Rider and B. Virág, *The noise in the circular law and the Gaussian free field*, Int. Math. Res. Not. 2007 2007.
- [122] S. O’Rourke, V. Vu, and K. Wang.: *Eigenvectors of random matrices: a survey*. J. Combin. Theory. Ser. A **144** (2016), 361–442.
- [123] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Springer-Verlag, Berlin, 1997.
- [124] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, 1959, New York
- [125] T. Shirai and Y. Takahashi, *Random point fields associated with certain Fredholm determinants. I. fermion, Poisson and boson point processes*, J.Funct. Anal., **205** (2003) 414–463.
- [126] T. Shirai, *Limit theorems for random analytic functions and their zeros*, Functions in number theory and their probabilistic aspects, RIMS Kôkyûroku Bessatsu, **B34** (2012), 335–359.
- [127] M. Sodin, *Zeros of Gaussian analytic functions*, Math. Res. Lett., **7** (2000), no. 4, 371–381.
- [128] M. Sodin and B. Tsirelson, *Random complex zeros, I. Asymptotic normality*, Israel J. Math. **144** (2004), no. 1, 125–149.
- [129] T. Tao and V. Vu, *Random matrices: Universality of local eigenvalue statistics*, Acta Mathematica, 206(1):127–204, 2011.
- [130] T. Tao and V. Vu, *Random matrices: universal properties of eigenvectors*, Random Matrices: Theory and Applications, 01(01):1150001, 2012.
- [131] W. Tarnowski, *Condition numbers for real eigenvalues of real elliptic ensemble: weak non-normality at the edge*, arXiv:2401.03249.
- [132] N. M. Temme. *The asymptotic expansion of the incomplete gamma functions*, SIAM J. Math. Anal., 10(4):757–766,1979

- [133] N. M. Temme. *Special functions, An introduction to the classical functions of mathematical physics*, A WileyInterscience Publication. John Wiley & Sons, Inc., New York, 1996.
- [134] C. A. Tracy and H. Widom, *Level-Spacing Distributions and the Airy Kernel*, *Comm. Math. Phys.* **159**, 151–174 (1994)
- [135] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra, The Behavior of Nonnormal Matrices and Operators*, Princeton University Press (2005).
- [136] S. Torquato, *Hyperuniform states of matter*, *Phys. Rep.*, 745:1–95, 2018.
- [137] S. Torquato and F. H. Stillinger, *Local density fluctuations, hyperuniformity, and order metrics*, *Phys. Rev. E*, **68**(4):041113, 2003.
- [138] C.T.C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, vol. 63, Cambridge University Press, Cambridge, 2004.
- [139] M. Walters and S. Starr, *A note on mixed matrix moments for the complex Ginibre ensemble*, *J. Math. Phys.* **56** (2015), 013301.
- [140] C. Webb and M.D. Wong.: On the moments of the characteristic polynomial of a Ginibre random matrix. *Proc. Lond. Math. Soc.* **118** (2019), 1017–1056.
- [141] E. P. Wigner, *Characteristic Vectors of Bordered Matrices With Infinite Dimensions*, *Ann. of Math. (2)* **62**, 3, 548–564 (1955).
- [142] E. P. Wigner, *Statistical Properties of Real Symmetric Matrices with Many Dimensions*, *Can. Math. Congr. Proc.*, Univ. of Toronto Press, Toronto, Canada, 174–184 (1957).
- [143] E. P. Wigner, *On the Distribution of the Roots of Certain Symmetric Matrices*, *Ann. of Math. (2)* **67**, 2, 325–327 (1958).
- [144] T.R. Würfel, M.J. Crumpton, and Y.V. Fyodorov, *Mean left-right eigenvector self-overlap in the real Ginibre ensemble*, preprint, *arXiv:2310.04307*, 2023.
- [145] S. Yabuoku, *Eigenvalue processes of Elliptic Ginibre Ensemble and their Overlaps*, *Int. J. Math. Ind.* **12**, 1, 2050003 (2020)