Strategyproof Matching Mechanism under General Constraints

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Strategyproof Matching Mechanism under General Constraints

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Abstract

The many-to-one matching problem is a fundamental issue involving the allocation of two sets of agents based on their preferences. Each agent belongs to one set (which we call students) and wishes to be assigned to at most one agent in the other set (which we call colleges), while multiple students can be assigned to one college. This problem encompasses practical applications in various contexts, such as students' enrollments in school choice programs or residents' placements in residency matching programs. As the theory of two-sided matching has been applied to increasingly diverse types of environments, researchers and practitioners have faced various forms of distributional constraints. A mechanism capable of handling a more general class of constraints allows for a more flexible assignment of students to colleges, thereby improving students' welfare with their allocation. However, a trade-off exists between students' welfare (efficiency) and fairness (ensuring no student has justified envy). This trade-off becomes sharper as the class of constraints becomes more general. The primary contribution of this thesis is to delineate the boundaries of whether a strategyproof mechanism exists that satisfies both fairness and specific efficiency properties for each general class of constraints: hereditary, hereditary M$^\natural$-convex set, and union of symmetric M-convex set. Building upon these theoretical results, we develop novel strategyproof mechanisms that operate for these general classes of constraints. Furthermore, we implement a comprehensive evaluation of their performance through computer simulations.

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Contents

Chapter 1

Introduction

1.1 Motivating Example

Envision yourself as a policymaker in a district tasked with deciding the implementation of a school choice program, an extensively debated topic in various states across the United States or prefectures in Japan. This program, providing students and families with educational alternatives beyond public schools, is scrutinized through the lens of mechanism design in scholarly work [1, 19, 47, 50]. A faulty assignment mechanism may result in dissatisfaction among them.

Delve into this issue with a simple example. Assume three students are deliberating whether to engage in a school choice program. In their residential area, there are two districts. Students in the western district typically attend the western school W, while those in the eastern district enroll in the east school E. School W surpasses school E in popularity among students and families in this area. Consequently, student Alice in district W is apprehensive about being relocated from school W to school E with the introduction of the school choice program. Another student, Becky, residing in district E, prefers school E but is concerned it might face closure if too many students migrate to school W. Furthermore, Carol in district E, residing near the border between schools, aspires to attend school W. How can a consensus be reached for the introduction of the school choice program?

A solution must adhere to specific criteria. Firstly, each student should be assigned to a school at least as good as their local or default school, which they would attend without the school choice program. This principle is known as individual rationality, ensuring Alice retains her place at school W. Secondly, the assignment must satisfy necessary distributional constraints, ensuring school E has a sufficient student population to operate, alleviating Becky's concerns about closure. This aspect is termed feasibility. Carol would be content if no better assignment for students exists within feasible and individually rational assignments. Even if she is not assigned to school W, a valid reason, such as violating feasibility, would justify it. A student might be jealous of another student attending a preferred school yet ranking lower based on a predetermined priority of a school. The priority often reflects

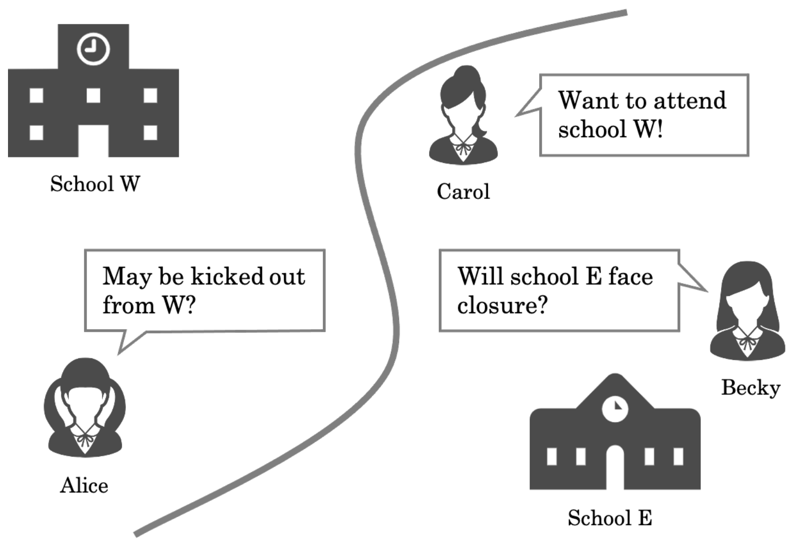


Figure 1.1: Introduction of a school choice program

social circumstances, such as the demand for the same attendance between siblings or distance to schools. This is defined as fairness in the matching theory. A student may attempt to manipulate a mechanism to endure a better school, for example, by falsifying information or colluding with affiliated families. To prevent manipulation, a mechanism must be strategyproof, meaning no student should have an incentive to manipulate it by misreporting their declared preference. Unfortunately, these criteria can be incompatible with individual rationality or feasibility constraints. Therefore, a mechanism should be devised to enhance students' welfare and uphold a certain level of fairness within possible strategyproof mechanisms.

1.2 Background

The theory of two-sided matching has undergone substantial development and has found diverse applications in many real-life domains (see Roth and Sotomayor [68] for a comprehensive survey in this literature). It has also garnered considerable attention from researchers in artificial intelligence [8, 6, 36, 40, 45, 80, 81].

As the theory has been applied to increasingly diverse types of environments, researchers and practitioners have encountered various forms of distributional constraints (see Aziz et al. [9] for a comprehensive survey on various distributional constraints). The standard model of two-sided matching considers only the maximum quota of each college [68], which we refer to as maximum quotas constraints.[1] Besides the maximum quotas constraints, regional maximum quotas restrict the total number of students assigned to a set of schools [41]. Minimum quotas guarantee that each

[1] Although this thesis is described in the context of a student-college matching problem, the obtained result is applicable to general matching problems.

school accepts a certain number of students [10, 22, 29, 73]. Diversity constraints ensure a balanced mix of different student types within a college [19, 32, 47, 50, 76]. More general classes of constraints are hereditary constraints [7, 30, 42] and hereditary $M^\natural$-convex set constraints [48]. An $M^\natural$-convex set is a discrete counterpart of a convex set in a continuous domain. Hereditary constraints require that if a matching between students and colleges is feasible, then any matching that places weakly fewer students at each college is also feasible.

The concept of stability was originally formulated by Gale and Shapley [27] for two-sided, one-to-one matching problems. In two-sided matching problems, it is characterized as a combination of fairness and nonwastefulness. Fairness ensures that when a student, denoted as s, is not accepted by a college, denoted as c, despite considering it to be better than her assigned college, then s is ranked lower than any student accepted by c, based on c's preference. Nonwastefulness is an efficiency notion that prohibits situations where a student can unilaterally move to a college she prefers without violating the underlying distributional constraints.

In standard matching markets with maximum quotas constraints, the Deferred Acceptance (DA) mechanism finds a stable matching [27]. However, it is well-known that in the presence of distributional constraints, a stable matching may not exist. This presents a trade-off between fairness and efficiency for mechanism designers.

In this thesis, we focus our attention on strategyproof mechanisms, which guarantee that students have no incentive to misreport their preference over colleges. From a theoretical standpoint, if we are interested in a property achieved in dominant strategies, strategyproof mechanisms can be exclusively considered without any loss of generality, as supported by the well-known revelation principle [28]. This principle states that if a certain property is satisfied in a dominant strategy equilibrium using a mechanism, it can also be achieved through a strategyproof mechanism. Strategyproof mechanisms are not only theoretically significant but also practically beneficial, as students do not need to speculate about the actions of others to achieve desirable outcomes; they only need to report their preferences truthfully.

1.3 Summary

The primary objective of this thesis is to clarify the tight boundaries on whether a strategyproof and fair mechanism can satisfy certain efficiency properties for each class of constraints (Table 1.1). Whether a strategyproof and fair mechanism can simultaneously satisfy weakly nonwastefulness has been an open question [42]. In this thesis, we show that under hereditary constraints, no strategyproof mechanism can simultaneously satisfy fairness and a very weak efficiency requirement called *no vacant college property*, a weaker notion than weak nonwastefulness. This impossibility result illustrates a critical dilemma: as we broaden/generalize the classes of constraints in pursuit of enhancing students' welfare, strict fairness requirements obstruct the guarantee of even a very weak requirement of students' welfare under general hereditary

Table 1.1: Existence of fair and strategyproof mechanism (✓ means such a mechanism exists, ✗ means such a mechanism does not exist, and ✖ means even without strategyproofness, a matching that satisfies fairness and the efficiency property may not exist. A red mark represents a new result obtained in this thesis)

	Maximum quotas	Hereditary & M$^\natural$-convex	Hereditary
Pareto efficiency	✖ [65]	✖	✖
nonwastefulness	✓ [66]	✖ [42]	✖
cut-off nonwastefulness	✓	✗ [Th. 1]	✗
weak nonwastefulness	✓	✓ [46]	✗ [Th. 2]
no vacant college	✓	✓	✗ [Th. 3]
no empty matching	✓	✓	✓ [Th. 4]

constraints. To exemplify, the matching where no student is assigned to any college is considered fair; however, a mere requirement of fairness proves inadequate. Faced with this dilemma, our subsequent objective is to establish a weaker fairness requirement. In this thesis, we propose a novel concept called *envy-freeness up to k peers* (EF-k). This concept is inspired by a criterion called envy-freeness up to k items, which is commonly used in the fair division of indivisible items [11]. EF-k guarantees that each student has justified envy towards at most k students. By varying k, EF-k can represent different levels of fairness. To the best of our knowledge, this thesis is the first to address the relaxed notion of fairness in two-sided, many-to-one matching.

Second, we introduce a new class of constraints denoted as *a union of symmetric M-convex sets*. M-convex set is a specialized notion of an M$^\natural$-convex set. Assuming all students are assigned to colleges, M$^\natural$-convex set becomes equivalent to an M-convex set. We posit this assumption to ensure the existence of a feasible matching under the union of symmetric M-convex set constraints. When distributional constraints form an M-convex set, a strategyproof and fair mechanism exists called Generalized Deferred Acceptance [48]. However, we cannot apply it to a union of symmetric M-convex sets since M-convexity is not closed under union and therefore a union of symmetric M-convex sets is *not* M-convex. Beyond M-convex, identifying constraints, for which we can develop a strategyproof and fair mechanism, is a challenging question. Under the union of symmetric M-convex sets, we develop a strategyproof and fair mechanism called Quota Reduction Deferred Acceptance, which repeatedly applies DA by sequentially reducing artificially introduced maximum quotas. To the best of our knowledge, we are the first to identify a general class of constraints beyond M-convex sets, wherein a non-trivial, strategyproof, and fair mechanism exists.

Third, we examine the attainability of efficiency and fairness properties under M-convex set constraints while ensuring *individual rationality*. This basic requirement ensures that a student can secure a seat in a college at least as good as her initial endowment. For efficiency, we analyze Pareto efficiency, a stronger welfare notion than nonwastefulness, which eliminates situations where students' welfare can be improved without detracting from others' welfare while satisfying distributional constraints. Regarding fairness, we consider a slightly relaxed requirement called fairness among Non-Initial Endowment students (NIE-fairness) and examine how it is achieved without sacrificing excessive efficiency. The question of whether a Pareto efficient mechanism can satisfy a fairness property under distributional constraints has generally remained open [41]. We introduce two strategyproof mechanisms that respect individual rationality: Top trading cycles under M-convex constraints which achieves Pareto efficiency, and DA based on ranks, which achieves NIE-fair. We show that these mechanisms strike a good balance between fairness and efficiency.

This thesis is organized as follows. Chapter 2 introduces the general two-sided matching model and defines the desirable properties for matching mechanisms. Chapter 3 introduces a very general class of constraints called *hereditary* constraints and studies the trade-off between fairness and efficiency. In Chapter 4, the family of M-convex sets is introduced, and we go beyond this family by defying the union of symmetric M-convex set constraints. Chapter 5 introduces an individually rational requirement and considers a strategyproof mechanism under M-convex set constraints. Finally, this thesis concludes in Section 6.

1.4 Related Work

This thesis is located at the intersection of discrete mathematics and economics. The insight from the former, and discrete convex analysis in particular, has been used in a broad range of applications on discrete optimization, including scheduling, facility location, and the structural analysis of engineering systems [49, 56, 70]. Recently, discrete convex analysis has been recognized as a powerful tool for analyzing economic or game theoretic applications, including exchange economies with indivisible objects [15, 55, 59, 74], systems analysis [55], inventory management [39, 82], and auctions [61] (see Murota [57] for an extensive survey on recent developments). Fujishige and Tamura [24, 25] and Murota and Yokoi [60] applied discrete convex analysis to study two-sided matching problems. Both works dealt with many-to-many matching problems, in which a doctor/employee can work at multiple hospitals/firms. Fujishige and Tamura [24, 25] addressed side payments as well.[2] Kojima et al. [48] applied the concept to two-sided matching problems with distributional constraints and showed that if the preferences of schools can be represented as an $M^\natural$-concave function, a general DA-based mechanism achieves a desirable outcome.

Several papers have investigated strategyproofness in matching models with quotas constraints, although they consider a setting distinct from ours [13, 14, 35, 63]. These papers addressed a student-course setting where only students express pref-

[2] See also an earlier contribution [20] that applied matroid theory to matching. This analysis is a special case of a more recent contribution by Fujishige and Tamura [25].

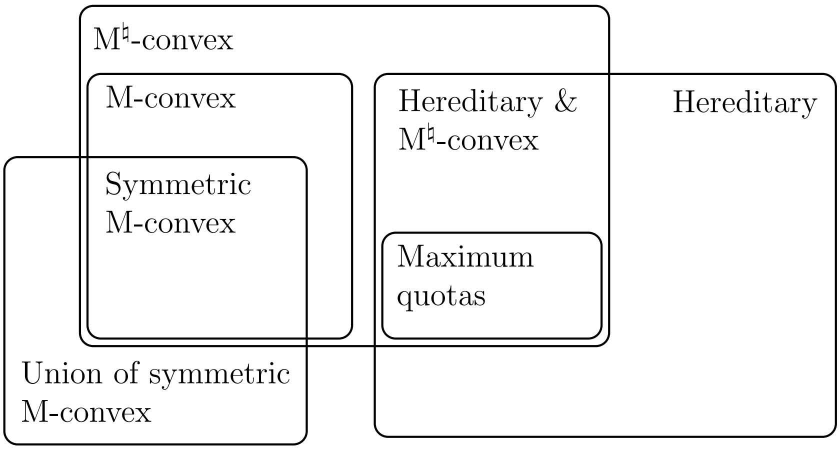


Figure 1.2: Inclusion of a general class of constraints

erences over courses, and both students and courses have quotas or capacities. In contrast, our setting involves preferences from both students and schools, with only schools having constraints. A study closer to our setting was conducted by Kamada and Kojima [43], who investigated student-school matchings and characterize the constraints referred to as *general upper bounds*, ensuring the existence of a student-optimal and fair matching. However, they consider constraints imposed on individual schools whereas our constraints are symmetric over the set of schools. Moreover, they allow constraints to depend on the identity of students, whereas our constraints are solely based on the number of students.

Two streams of works exist on matching with distributional constraints. One stream scrutinizes constraints that arise from real-life applications, such as regional maximum quotas [41], individual/regional minimum quotas [22, 30], affirmative actions [19, 50], and initial endowment settings [33, 75]. The other stream mathematically studies an abstract and general class of constraints: a substitute choice function [34], hereditary M$^\natural$-convex set constraints [48], hereditary constraints [7, 30, 42]. This thesis belongs to the second stream and Figure 1.2 illustrates the inclusion relationship among the concepts explored in this thesis. Note that M$^\natural$-convexity is a generalized notion of M-convexity, and heredity and M$^\natural$-convexity are independent properties. Since the set of unions of symmetric M-convex sets is symmetric, its intersection with the set of M-convex sets corresponds to the set of symmetric M-convex sets.

Since the presence of distributional constraints forces us to a trade-off between fairness and efficiency, several papers attempt to expose the boundaries of desirable mechanisms. Aziz et al. [7] introduced a weaker efficiency property called cut-off nonwastefulness, and proposed a fair and cut-off nonwasteful mechanism under hereditary constraints. Kimura et al. [46] showed that a general DA-based mechanism running under hereditary M$^\natural$-convex set constraints, is fair, weakly nonwasteful, and strategyproof. Kamada and Kojima [41] argued that Pareto efficiency is achievable under regional maximum quotas. Hamada et al. [33] developed two mechanisms: a Pareto efficiency and an NIE-fair mechanism when minimum (and maximum) quotas are imposed on each college. In the model, students are categorized into different types, Fragiadakis and Troyan [21] developed a strategyproof mechanism that fulfills type-specific minimum and maximum quotas and eliminates the justified envy among the same types.

The trade-off between fairness and efficiency has been also considered in fair allocation of divisible/indivisible items, where only agents have preferences on items (i.e., an item has no preference). Envy-freeness in this setting is defined as no agent strictly prefers the assignment of another agent. When items are indivisible, envy-freeness becomes too demanding: when the number of items equals the number of agents, it requires that each agent obtains her top item, which is impossible in general. Therefore, relaxations of envy-freeness were introduced such as envy-free up to at most one item (EF-1) [11], and envy-free up to at most any item (EF-x) [12].

Shapley and Scarf [71] first introduced the Top Trading Cycles (TTC). They investigated a housing market problem, where objects are initially owned by agents who have strict preferences over them, and there are no copies of an object in the market. TTC was further generalized to the Hierarchical Exchange mechanism [62] and the Trading Cycles mechanism [64]. TTC has been applied to a school choice problem [2] and also been attracting attention from AI researchers [26, 44, 72].

Abdulkadiroğlu and Sönmez [4] and Guillen and Kesten [31] investigated a housing market problem with existing tenants, where some agents may not initially own a house, and some houses are not initially owned by agents. The differences between their setting and ours are that we consider multiple copies of an object (college seat) and impose distributional constraints on them. Hamada et al. [33] considered individual minimum and maximum quotas where multiple copies of an object exist and agents have their initial endowments, although some objects are not initially owned by any agent. Our work is a strict extension of theirs.

Chapter 2

General Constrained Matching Problem

In this chapter, we present a formal definition of the constrained matching problem and introduce desirable properties of matching mechanisms.

2.1 Model

A matching market under distributional constraints is given by $I = (S, C, X, \omega, \succ_S, \succ_C, f)$. The meaning of each element is as follows.

- $S = \{s_1, \dots, s_n\}$ is a finite set of students. Let N denote $\{1, 2, \dots, n\}$.
- $C = \{c_1, \dots, c_m\}$ is a finite set of colleges. Let M denote $\{1, 2, \dots, m\}$.
- $X \subseteq S \times C$ is a finite set of contracts. Contract $x = (s, c) \in X$ represents the matching between student s and college c.

  For any $Y \subseteq X$, let $Y_s := \{(s, c) \in Y \mid c \in C\}$ and $Y_c := \{(s, c) \in Y \mid s \in S\}$ denote the sets of contracts in Y that involve s and c, respectively.
- $\omega : S \to C$ is an initial endowment function; $\omega(s) = c$ denotes the initial endowments college of s.
- $\succ_S = (\succ_{s_1}, \dots, \succ_{s_n})$ is a profile of the students' preferences. For each student s, $\succ_s$ represents the preference of s over $X_s \cup \{(s, \emptyset)\}$, where $(s, \emptyset)$ represents an outcome such that s is unmatched. We assume $\succ_s$ is strict for each s. Contract (s, c) is *acceptable* to s if $(s, c) \succ_s (s, \emptyset)$ holds. We sometimes use notations like $c \succ_s c'$ instead of $(s, c) \succ_s (s, c')$.
- $\succ_C = (\succ_{c_1}, \dots, \succ_{c_m})$ is a profile of the colleges' preferences. For each college c, $\succ_c$ represents the preference of c over $X_c \cup \{(\emptyset, c)\}$, where $(\emptyset, c)$ represents an outcome such that c is unmatched. We assume $\succ_c$ is strict for each c. Contract

(s, c) is *acceptable* to c if $(s, c) \succ_c (\emptyset, c)$ holds. We sometimes write $s \succ_c s'$ instead of $(s, c) \succ_c (s', c)$.

- $f : \mathbf{Z}_+^m \to \{-\infty, 0\}$ is a function that represents distributional constraints, where m is the number of colleges and $\mathbf{Z}_+^m$ is the set of vectors of m non-negative integers. For f, we call a family of vectors $F = \{\nu \in \mathbf{Z}_+^m \mid f(\nu) = 0\}$ *induced vectors* of f.

We assume each contract x in X_c is acceptable to c. This is without loss of generality because if some contract is unacceptable to a college, we can assume it is not included in X.

A set of contracts $Y \subseteq X$ is a *matching*, if for each $s \in S$, either (i) $Y_s = \{x\}$ and x is acceptable to s, or (ii) $Y_s = \emptyset$ holds.

For two m-element vectors $\nu, \nu' \in \mathbf{Z}_+^m$, $\nu \leq \nu'$ if for all $i \in M$, $\nu_i \leq \nu_i'$ holds. $\nu < \nu'$ if $\nu \leq \nu'$ and for some $i \in M$, $\nu_i < \nu_i'$ holds. Also, let $|\nu|$ denote the L_1 norm of ν, i.e., $|\nu| = \sum_{i \in M} \nu_i$.

Definition 1 (Feasibility with Distributional Constraints) Let ν be a vector of m non-negative integers. Vector ν is *feasible* in f if $f(\nu) = 0$. For $Y \subseteq X$, we define $\nu(Y)$ as $(|Y_{c_1}|, |Y_{c_2}|, \ldots, |Y_{c_m}|)$. Matching Y is *feasible* (in f) if $\nu(Y)$ is feasible in f.

We assume F is bounded, i.e., $|F|$ is finite. This is without loss of generality because we can assume each college c_i can accept at most $|X_{c_i}|$ students, i.e., $f(\nu) = -\infty$ holds when $\exists i \in M, \nu_i > |X_{c_i}|$.

With a slight abuse of notation, for two sets of contracts Y and Y', we denote $Y_s \succ_s Y_s'$ if either (i) $Y_s = \{x\}$, $Y_s' = \{x'\}$, and $x \succ_s x'$ for some $x, x' \in X_s$, or (ii) $Y_s = \{x\}$ for some $x \in X_s$ that is acceptable to s and $Y_s' = \emptyset$. Furthermore, we denote $Y_s \succeq_s Y_s'$ if either $Y_s \succ_s Y_s'$ or $Y_s = Y_s'$. We also use notations like $x \succ_s Y_s$ or $c \succ_s Y_s$, where x is a contract, Y is a matching, and c is a college. We indicate a college c by Y_s such that $Y_s = \{(s, c)\}$.

Mechanism φ is a function that takes a profile of student preferences $\succ_S$ as input and returns a set of contracts. Given that $\varphi(\succ_S) = Y$, let $\varphi_s(\succ_S)$ denote Y_s. Let $\succ_{S \setminus \{s\}}$ denote the profile of the preferences where all students' preferences are $\succ_S$ except s, whose preference is $\succ_s$.

2.2 Desirable Properties

We introduce several desirable properties of a matching and a mechanism. A mechanism satisfies property "A" if the mechanism produces a matching that satisfies property "A" in every possible matching market.

First, we define fairness which is defined through the notion of *justified envy* and it has been studied in constrained matching models [5, 79].

Definition 2 (Fairness) In matching Y, student s *has justified envy* toward another student s' if $(s, c) \in X$ is acceptable to s, $c \succ_s Y_s$, $(s', c) \in Y$, and $s \succ_c s'$ hold. Matching Y is *fair* if no student has justified envy.

Fairness implies that if student s is not assigned to college c (although she hopes to be assigned), then c prefers all students assigned to it over s.

Next, we define a series of properties on students' welfare (efficiency).

Definition 3 (Pareto Efficiency) Matching Y is Pareto dominated by another matching Y' if $\forall s \in S$, $Y'_s \succeq_s Y_s$, and $\exists s \in S$, $Y'_s \succ_s Y_s$ hold. Feasible matching Y is Pareto efficient if no other feasible matching Pareto dominates it.

In short, feasible matching Y is Pareto efficient if no other feasible matching Y' exists such that all students weakly prefer Y' over Y, and at least one student strictly prefers Y' over Y.

Definition 4 (Nonwastefulness) In matching Y, student s *claims an empty seat* in college c if (s, c) is acceptable to s, $c \succ_s Y_s$, and $(Y \setminus Y_s) \cup \{(s, c)\}$ is feasible. Feasible matching Y is *nonwasteful* if no student claims an empty seat.

Intuitively, nonwastefulness means that we cannot improve the matching of one student without affecting other students.

Aziz et al. [7] introduced a weaker efficiency concept called *cut-off nonwastefulness.*

Definition 5 (Cut-off Nonwastefulness) Feasible matching Y is *cut-off nonwasteful* if student s claims an empty seat in college c, then another student s' exists such that $c \succ_{s'} Y_{s'}$, $s' \succ_c s$, and $(Y \setminus Y_{s'}) \cup \{(s', c)\}$ is infeasible.

Intuitively, we consider the claim of student s to move her to college c from her current match is not considered legitimate if doing so, another student s' would have justified envy toward s.

Kamada and Kojima [42] proposed another weaker version of the nonwastefulness concept, which we refer to as *weak nonwastefulness.*

Definition 6 (Weak Nonwastefulness) In matching Y, student s *strongly claims an empty seat* in c if (s, c) is acceptable to s, $c \succ_s Y_s$, and $Y \cup \{(s, c)\}$ is feasible. Feasible matching Y is *weakly nonwasteful* if no student strongly claims an empty seat.

Student s can strongly claim an empty seat in c only when $Y \cup \{(s, c)\}$, i.e., the matching obtained by adding her to college c (without removing her from her current college), is feasible.

We define two more weaker efficiency properties.

Definition 7 (No Vacant College) Feasible matching Y satisfies *no vacant college* property if student s claims an empty seat in college c, then $Y_s \neq \emptyset$ or $Y_c \neq \emptyset$ holds.

Intuitively, no vacant college property means that the claim of student s to move her to college c from her current match is considered legitimate only when s is not matched to any college and no student is assigned to c.

Definition 8 (No Empty Matching) In matching Y, student s very strongly claims an empty seat in college c, if $Y = \emptyset$, $(s, c) \in X$, $c \succ_s \emptyset$, and $\{(s, c)\}$ is feasible. Feasible matching Y satisfies no empty matching property if no student very strongly claims an empty seat in any college.

Note that this series of efficiency properties becomes monotonically weaker in this order as long as distributional constraints are hereditary. More specifically, Pareto efficiency implies nonwastefulness, but not vice versa, nonwastefulness implies cut-off nonwastefulness, but not vice versa, and so on. Pareto efficiency means that we cannot improve the matching of a set of students without hurting other students, while nonwastefulness means that we cannot improve the matching of *one student* without affecting other students. Thus, Pareto efficiency implies nonwastefulness. If Y is nonwasteful, no student can claim an empty seat. If Y is cut-off nonwasteful, a student can claim an empty seat in some cases. Thus, cut-off nonwastefulness is weaker than nonwastefulness. Next, we show that cut-off nonwastefulness implies weak nonwastefulness by showing its contraposition. We assume student s strongly claims an empty seat in college c in Y. Then, we show that Y cannot be cut-off nonwasteful. The fact that s strongly claims an empty seat in c implies that s also claims an empty seat in c since if $Y \cup \{(s, c)\}$ is feasible, $(Y \setminus Y_s) \cup \{(s, c)\}$ is also feasible. Assume another student s' exists, where $c \succ_{s'} Y_{s'}$ and $s' \succ_c s$ hold. Then, since $Y \cup \{(s, c)\}$ is feasible, $(Y \setminus Y_{s'}) \cup \{(s', c)\}$ is also feasible. Thus, Y cannot be cut-off nonwasteful. Next, we show that weak nonwastefulness implies no vacant college property by showing its contraposition. No vacant college property means that the claim of student s to move her from the current matching to c is considered legitimate only when s is not matched to any college and no student is assigned to c. We assume Y does not satisfy no vacant college property, i.e., there exists student s who claims an empty seat in c when $Y_s = \emptyset$ and $Y_c = \emptyset$. Then, we show s strongly claims an empty seat in c in Y. Since $Y_s = \emptyset$, the fact that $(Y \setminus Y_s) \cup \{(s, c)\}$ is feasible implies that $Y \cup \{(s, c)\}$ is feasible. Thus, s also strongly claims an empty seat in c. Finally, we show that no vacant college property implies no empty matching property by showing its contraposition. We assume student s very strongly claims an empty seat in c in matching Y. Then, we show that Y does not satisfy no vacant college property. The fact that student s very strongly claims an empty seat in c implies $c \succ_s \emptyset$, $Y_s = \emptyset$, and $Y_c = \emptyset$ hold. Thus, Y does not satisfy no vacant college property.

Next, we introduce strategyproofness. It is a significant property in the field of mechanism design and has been the subject of extensive discourse in numerous existing studies [17, 21, 65, 80]. Strategyproof mechanisms are grounded in the concept from Wilson's theory [78], asserting that, when policymakers engage in institutional design, they should remain unrelated to the beliefs of individual students. Within strategyproof mechanisms, it becomes a weakly dominant strategy for students to declare their true preferences, irrespective of the beliefs they hold. If we are interested in a property achieved in dominant strategies, strategyproof mechanisms can be exclusively considered without any loss of generality, as supported by the well-known revelation principle [28]. This principle states that if a certain property is satisfied in a dominant strategy equilibrium using a mechanism, it can also be achieved through a strategyproof mechanism.

From a practical perspective, particularly for policymakers like school district management, it is anticipated that adopting strategyproof mechanisms will suggest that each student has no room for strategic deliberation. Consequently, even students not well-versed in the intricacies of the mechanism can be assigned to their most favored school among those potentially allocated, simply by offering the strategy of disclosing their sincere preferences. They also prevent from manipulating students who have less knowledge of the mechanism's properties, such as manipulating information.

For these reasons, in the design of school choice systems in many cities, including New York City, Boston, and New Orleans, strategyproof mechanisms are implemented.

Definition 9 (Strategyproofness) Mechanism φ is strategyproof if for all preference profile $\succ_S$, no student $s \in S$ and $\succ'_s$ (a possible misreport of s's preference) exist such that $\varphi_s((\succ'_s, \succ_{S\setminus\{s\}})) \succ_s \varphi_s((\succ_s, \succ_{S\setminus\{s\}}))$.

A mechanism satisfies strategyproofness if no profile exists where a student can benefit from misreporting information individually. Our focus is solely on strategic manipulations by students. It is well-known that even in the most basic model of one-to-one matching [27], satisfying strategyproofness (as well as basic fairness and efficiency requirements) for both sides is impossible [65]. One rationale for excluding the college side is that a college's preference must be presented objectively and cannot be manipulated arbitrarily.

A stronger requirement than strategyproofness is group strategyproofness, ensuring that no group of students can benefit from misreporting information collectively.

Definition 10 (Weak/Strong Group Strategyproofness) Mechanism φ is *weakly* group strategyproof if for all preference profile $\succ_S$, no group of students $S' \subseteq S$ and $\succ'_{S'}$ (a possible collusive misreport by S') exist such that for all $s \in S'$, $\varphi_s((\succ'_{S'}, \succ_{S\setminus S'})) \succ_s \varphi_s((\succ_{S'}, \succ_{S\setminus S'}))$. Mechanism φ is *strongly* group strategyproof if for all preference profile $\succ_S$, no group of students $S' \subseteq S$ and $\succ'_{S'}$ (a possible collusive misreport by S') exist such that for all $s \in S'$, $\varphi_s((\succ'_{S'}, \succ_{S\setminus S'})) \succeq_s \varphi_s((\succ_{S'}, \succ_{S\setminus S'}))$ and for some $s \in S'$, $\varphi_s((\succ'_{S'}, \succ_{S\setminus S'})) \succ_s \varphi_s((\succ_{S'}, \succ_{S\setminus S'}))$.

Chapter 3

Hereditary Constraints

In this chapter, we introduce a very general class of constraints called *hereditary* constraints and study the trade-off between fairness and efficiency. We assume $\omega(s) = \emptyset$ for every $s \in S$.

3.1 Model

We first define hereditary constraints.

Definition 11 (Heredity) A family of m-element vectors $F \subseteq \mathbf{Z}_+^m$ is *hereditary* if $e_0 \in F$ and for all $\nu, \nu' \in \mathbf{Z}_+^m$, if $\nu > \nu'$ and $\nu \in F$, then $\nu' \in F$ holds. Function f, representing distributional constraints, is *hereditary* if its induced vectors are hereditary.

Intuitively, heredity means that if Y is feasible in f, then any subset $Y' \subset Y$ is also feasible in f. Let e_i denote an m-element unit vector, where its i-th element is 1 and all other elements are 0. Let e_0 denote an m-element zero vector $(0, \ldots, 0)$.

3.2 Standard setting: Maximum Quotas Constraints

3.2.1 Model

We introduce a most basic model of hereditary constraints where only distributional constraints are colleges' maximum quotas.

Definition 12 (Maximum Quotas Constraints) A family of vectors $F \subseteq \mathbf{Z}_+^m$ is given as colleges' maximum quotas, when for each college $c_i \in C$, its maximum quota q_{c_i} is given, and $\nu \in F$ iff $\forall i \in M$, $\nu_i \leq q_{c_i}$ holds. Function f, representing distributional constraints, is given as colleges' maximum quotas if its induced vectors are given as colleges' maximum quotas.

Mechanism 1 Deferred Acceptance (DA)

Input: $S, C, X, \succ_S, \succ_C, f$

Output: matching Y

1: Each student s applies to her favorite college, according to $\succ_s$, among the colleges that have not rejected her so far.

2: Each college c provisionally accepts the top q_c students from the applying students based on $\succ_c$ and rejects the rest of them without any distinctions between newly applying and already provisionally accepted students.

3: Barring any rejected students, return the current matching. Otherwise, go to 1.

Maximum quotas constraints assume that each college adheres to a predetermined and externally given maximum quota. However, in real-world situations, especially in the context of school choice, if advance knowledge of a college's popularity is available, policymakers can strategically plan budgets and allocate additional human resources to popular colleges. Essentially, practical distributional constraints frequently have flexibility when considering an allocation, and this flexibility can naturally be reflected by hereditary constraints.

3.2.2 Existing Mechanisms

Gale and Shapley [27] developed the Deferred Acceptance mechanism (DA), which is presented in Mechanism 1. In DA, each student first applies to her favorite college and each college provisionally accepts students up to its capacity limit based on its preference and rejects the rest of them. A rejected student applies to her second choice. Each college provisionally accepts students who have applied without distinguishing between newly applied and already provisionally accepted students, and so forth. DA is not only fair and nonwasteful (called stable) but also strategyproof with maximum quotas constraints [17, 65]. Thanks to strategic robustness, DA has been pragmatically implemented by the school districts in New York City and Boston.

3.2.3 Impossibility Theorems

For maximum quotas constraints, fairness and Pareto efficiency are incompatible, i.e., even without strategyproofness, a matching that satisfies Pareto efficiency and fairness may not exist [65]. This implies an unattainable simultaneous demand for high degrees of fairness and efficiency. One potential avenue to address this incompatibility is through the adoption of more lenient requirements. DA shows a fair and nonwasteful matching can be found in a strategyproof manner. Nevertheless, the imposition of additional distributional constraints, beyond colleges' maximum quotas, typically results in the incompatibility of fairness and nonwastefulness. We explore this incompatibility under a general class of constraints in later sections.

3.3 Hereditary M$^\natural$-convex Set Constraints

3.3.1 Model

Next we introduce another general class of constraints. Kojima et al. [48] showed that when f is hereditary, and its induced vectors satisfy one additional condition called *M$^\natural$-convexity*, a general mechanism called Generalized Deferred Acceptance mechanism (GDA) exists, which satisfies several desirable properties.[1] We formally define an M$^\natural$-convex set.

Definition 13 (M$^\natural$-convex Set) A family of vectors $F \subseteq \mathbf{Z}_+^m$ forms an *M$^\natural$-convex set*, if for all $\nu, \nu' \in F$, for all i such that $\nu_i > \nu'_i$, there exists $j \in \{0\} \cup \{k \in M \mid \nu_k < \nu'_k\}$ such that $\nu - e_i + e_j \in F$ and $\nu' + e_i - e_j \in F$ hold. Function f, representing distributional constraints, satisfies M$^\natural$-convexity if its induced vectors form an M$^\natural$-convex set.

An M$^\natural$-convex set can be considered as the discrete analogue of a convex set in a continuous domain. Fundamentally, an M$^\natural$-convex set avoids any *hollow* spaces. This concept has been thoroughly investigated in discrete convex analysis, a branch of discrete mathematics. Recent advances in this area have yielded diverse applications in economics (see the survey paper by Murota [57]). Note that heredity and M$^\natural$-convexity are independent properties.

3.3.2 Pragmatic Examples

Next we show that several distributional constraints belong to the class of M$^\natural$-convex set. Note that the maximum quotas constraints (Definition 12) are one of the fundamental examples in it. We present other types of practical constraints that are a family of M$^\natural$-convex set.

Individual Minimum/Maximum Quotas Constraints [22, 33] In addition to the maximum quotas constraints, we assume for each college $c \in C$, a minimum quota p_c exists. The distributional constraints of this market can be expressed as F where

$$F = \{\nu \in \mathbb{Z}^m_{\geq 0} \mid p_c \leq \nu_c \leq q_c \ \forall c \in C\},$$

and it can be verified that F is M$^\natural$-convex.

[1] Kojima et al. [48] showed that to apply their framework, it is necessary that the family of feasible matchings forms a matroid. When distributional constraints are defined on $\nu(Y)$ rather than on contracts Y, the fact that the family of feasible contracts forms a matroid corresponds to the fact that (i) the family of feasible vectors forms an M$^\natural$-convex set, and (ii) it is hereditary [58].

Regional Minimum/Maximum Quotas constraints [29, 41] In addition to the individual minimum/maximum quotas constraints, capacity constraints are imposed on *regions*. Set of regions $R \subseteq 2^C \setminus \{\emptyset\}$ partitions set of colleges C into regions, and for each $r \in R$, there is a regional minimum quota p_r and a maximum quota q_r. The distributional constraints of this market can be expressed as F where

$$F = \{\nu \in \mathbb{Z}^m_{\geq 0} \mid p_c \leq \nu_c \leq q_c \text{ and } p_r \leq \sum_{c \in r} \nu_c \leq q_r \ \forall c \in C, \forall r \in R\},$$

and it can be verified that F is M$^\natural$-convex.

Regional minimum/maximum quotas can be extended to the case where regions have multiple layers (e.g., cities, counties, and states) [29, 41]. Assuming these regions have a laminar structure[2], distributional constraints can be represented as an M$^\natural$-convex set [48].

Type-specific Quotas Constraints [3, 21] We assume each student belongs to exactly one type. A student's type may represent her ethnicity, her gender, or her socioeconomic status. In addition to individual maximum quotas constraints, there are additional type-specific quotas. A set of types $T = \{t_1, \ldots, t_k\}$ exists. Each student belongs to exactly one type, and for each $c \in C$ and $t \in T$, type-specific minimum quota $p_{c,t}$ and maximum quota $q_{c,t}$ are defined. We assume distribution vector ν is represented as an $m \times |T|$ matrix, where $\nu_{c,t}$ denotes the number of type t students allocated to college c. The distributional constraints of this market can be expressed as F where

$$F = \{\nu \in \mathbb{Z}^{m \times |T|}_{\geq 0} \mid \sum_{t \in T} \nu_{c,t} \leq q_c \text{ and } p_{c,t} \leq \nu_{c,t} \leq q_{c,t} \ \forall c \in C,\ \forall t \in T\},$$

and it can be verified that F is M$^\natural$-convex. Note that F is a 2-dimensional matrix, which is an extension of the standard definition. More specifically, each element $\nu_{c,t}$ represents the number of type t students assigned to college c. We can assume each college c is divided into $|T|$ sub-colleges, and distributional constraints are imposed on these $m \times |T|$ sub-colleges. This model becomes equivalent to the standard model where distributional constraints are defined on a one-dimensional vector.

These series of minimum quotas constraints are M$^\natural$-convex set but not hereditary. There is a technique to handle them within heredity [48]. We define *semi-feasiblity*, i.e., a vector is semi-feasible if it is smaller than or equal to any feasible vector. Semi-feasible vectors become hereditary (and M$^\natural$-convex set). Assume all colleges are acceptable to all students and vice versa[3]. We can use any mechanism that can handle hereditary constraints by replacing feasibility with semi-feasibility. If

[2] R has a laminar structure if for any $r, r' \in R$ (where $r \neq r'$), one of the following conditions holds: $r \subset r'$, $r' \subset r$ or $r \cap r' = \emptyset$.

[3] We cannot guarantee the feasible matching when minimum quotas are imposed without this assumption.

the mechanism always obtains a matching where each student is assigned to some college, the obtained matching is guaranteed to be feasible. Hence, these constraints are considered as a family of hereditary M$^\natural$-convex set constraints.

Distance Constraints [48] When allocating n students among m colleges, suppose that an ideal distribution vector exists from the viewpoint of the mechanism designer who considers any distribution vector feasible if it is *close enough* to the ideal vector. More specifically, distance constraints are defined by an ideal vector ν^* and a distance k describing the maximum tolerable deviation from the ideal distribution. The set of feasible vectors for such distributional constraints is expressed as F, such that

$$F = \{\nu \in \mathbb{Z}^m_{\geq 0} \mid \delta(\nu, \nu^*) \leq k\},$$

where for any pair of m-dimensional vectors $\nu, \nu' \in \mathbb{Z}^m_{\geq 0}$, the distance function δ can be either (i) the Manhattan distance (or the L^1 distance), which is defined as $\delta(\nu, \nu') = \sum_{c \in C} |\nu_c - \nu'_c|$, or (ii) the Chebyshev distance (or the L^∞ distance), which is defined as $\delta(\nu, \nu') = \max_{c \in C} |\nu_c - \nu'_c|$. It can also be verified that F is M$^\natural$-convex.

3.3.3 Existing Mechanisms

In this section, we introduce existing mechanisms, which are strategyproof for hereditary M$^\natural$-convex set constraints. We introduce Generalized Deferred Acceptance mechanism (GDA), which works under hereditary M$^\natural$-convex set constraints [34]. As its name shows, it is a generalized version of DA [27]. To define GDA, we first introduce *choice functions* of students and colleges.

Definition 14 (Students' Choice Function) For each student s, her *choice function* Ch_s specifies her most preferred contract within each $Y \subseteq X$, i.e., $Ch_s(Y) = \{x\}$, where x is the most preferred acceptable contract in Y_s if one exists, and $Ch_s(Y) = \emptyset$ if no such contract exists. The choice function of all students is defined as $Ch_S(Y) := \bigcup_{s \in S} Ch_s(Y)$.

Definition 15 (Colleges' Choice Function) We assume each contract $(s, c) \in X$ is associated with its unique strictly positive weight $w((s, c))$. We assume these weights respect each college's preference $\succ_c$, i.e., if $(s, c) \succ_c (s', c)$, then $w((s, c)) > w((s', c))$ holds. For $Y \subseteq X$, let $w(Y)$ denote $\sum_{x \in Y} w(x)$. The choice function of all colleges is defined as $Ch_C(Y) := \arg\max_{Y' \subseteq Y} f(\nu(Y')) + w(Y')$.

As long as f induces a hereditary M$^\natural$-convex set, a unique subset Y' exists that maximizes the above formula. Furthermore, such a subset can be efficiently computed in the following greedy way. Let Y' denote the set of chosen contracts, which is initially $\emptyset$. Then, sort Y in the decreasing order of their weights. Then, choose contract x from Y one by one and add it to Y', as long as $Y' \cup \{x\}$ is feasible.

Using Ch_S and Ch_C, GDA is defined as Mechanism 2. Note that we describe the

Mechanism 2 Generalized Deferred Acceptance (GDA)

Input: X, Ch_S, Ch_C

Output: matching Y

1: $Re \leftarrow \emptyset$.

2: Each student s offers her most preferred contract (s, c) which has not been rejected before (i.e., $(s, c) \notin Re$). If no remaining contract is acceptable for s, s does not make any offer. Let Y be the set of contracts offered (i.e., $Y = Ch_S(X \setminus Re)$).

3: Colleges tentatively accept $Z = Ch_C(Y)$ and reject other contracts in Y (i.e., $Y \setminus Z$).

4: If all the contracts in Y are tentatively accepted at 3, then let Y be the final matching and terminate the mechanism. Otherwise, $Re \leftarrow Re \cup (Y \setminus Z)$, and go to 2.

mechanism using terms like "student s offers" to make the description more intuitive. In reality, GDA is a direct-revelation mechanism, where the mechanism first collects the preference of each student, and the mechanism chooses a contract on behalf of each student.

Kojima et al. [48] showed that when f induces a hereditary M$^\natural$-convex set, GDA is strategyproof, the obtained matching Y satisfies a property called Hatfield-Milgrom stability (HM-stability), and Y is the student-optimal matching within all HM-stable matchings (i.e., all students weakly prefer Y over any other HM-stable matching).

Definition 16 (HM-stability) Matching Y is HM-stable if $Y = Ch_S(Y) = Ch_C(Y)$, and no contract $x \in X \setminus Y$ exists, such that $x \in Ch_S(Y \cup \{x\})$ and $x \in Ch_C(Y \cup \{x\})$ hold.

Intuitively, HM-stability means that no contract exists in $X \setminus Y$ that is mutually preferred by students and colleges. Note that HM-stability implies fairness. If student s has justified envy in matching Y, there exists $(s, c) \in X \setminus Y$, $(s', c) \in Y$, s.t. $(s, c) \succ_s Y_s$ and $w((s, c)) > w((s', c))$ holds. Then, $(s, c) \in Ch_S(Y \cup \{(s, c)\})$ and $(s, c) \in Ch_C(Y \cup \{(s, c)\})$ hold, i.e., Y is not HM-stable.

For standard maximum quotas constraints, the only distributional constraints are $(q_c)_{c \in C}$, i.e., $f(Y) = 0$ iff for each $c \in C$, $|Y_c| \leq q_c$ holds. Then, $Ch_C(Y)$ is defined as $\bigcup_{c \in C} Ch_c(Y_c)$, where Ch_c is the choice function of each college c, which chooses top q_c contracts from Y_c based on $\succ_c$. When Ch_C is defined this way, GDA becomes equivalent to DA.

3.3.4 Impossibility Theorems

For hereditary M$^\natural$-convex set constraints, fairness, weak nonwastefulness, and strategyproofness are compatible, i.e., GDA satisfies these properties [46]. We pose the following question: Is there a strategyproof, fair, and cut-off nonwasteful mechanism

Table 3.1: Possible matchings for preference profiles (Theorem 1)

preference profile	s_1	s_2	possible matchings
$\succ_S^1$	c_1c_2	c_2	$[c_1, \emptyset]$
$\succ_S^2$	c_1	c_2c_1	$[\emptyset, c_2]$
$\succ_S^3$	c_1	c_2	$[c_1, \emptyset]$, $[\emptyset, c_2]$

under hereditary M$^\natural$-convex set constraints? Unfortunately, the answer is negative, as demonstrated in Theorem 1.

Theorem 1 No mechanism can simultaneously satisfy fairness, strategyproofness, and cut-off nonwastefulness under hereditary M$^\natural$-convex set constraints.

Proof Consider a matching market with two students $S = \{s_1, s_2\}$ and two colleges $C = \{c_1, c_2\}$. The colleges' preference profile $\succ_C$ are as follows:

$$c_1 : s_2 \succ_{c_1} s_1$$
$$c_2 : s_1 \succ_{c_2} s_2$$

To make the description concise, we denote a preference of students by a sequence of acceptable colleges. For example, we denote $c_1 \succ_s c_2 \succ_s \emptyset$ as c_1c_2, and $c_1 \succ_s \emptyset \succ_s c_2$ as c_1. Furthermore, we denote a matching as a pair of colleges assigned to s_1 and s_2. For example, we denote matching $\{(s_1, c_2), (s_2, c_1)\}$ as $[c_2, c_1]$.

Suppose $f(\nu) = 0$ if and only if $\nu \leq \nu'$ for some $\nu' \in \{(1, 0), (0, 1)\}$. This setting reflects the situation where the regional quotas constraints $|Y_{c_1}| + |Y_{c_2}| \leq 1$ are imposed, which form M$^\natural$-convex set constraints.

Assume, for the sake of contradiction, that there exists a fair, strategyproof, and cut-off nonwasteful mechanism. We examine three students' preference profiles: $\succ_S^1, \succ_S^2$, and $\succ_S^3$. These preference profiles and possible matchings that satisfy fairness and cut-off nonwastefulness are summarized in Table 3.1. First, for $\succ_S^1 = (c_1c_2, c_2)$, due to fairness, we cannot allocate s_2 to c_2. Also, due to cut-off nonwastefulness, we cannot allocate s_1 to c_2. Then, the mechanism must choose $[c_1, \emptyset]$.

Next, for $\succ_S^2 = (c_1, c_2c_1)$, due to fairness, we cannot allocate s_1 to c_1. Also, due to cut-off nonwastefulness, we cannot allocate s_2 to c_1. Then, the mechanism must choose $[\emptyset, c_2]$.

Finally, for $\succ_S^3 = (c_1, c_2)$, due to cut-off nonwastefulness and distributional constraints, exactly one student must be assigned to her acceptable college. Thus, there exist two possible matchings: (a) $[c_1, \emptyset]$ or (b) $[\emptyset, c_2]$. If (a) is chosen, then s_2 has an incentive to manipulate (to modify the profile to $\succ_S^2$) so that she is assigned to c_2. If (b) is chosen, then s_1 has an incentive to manipulate (to modify the profile to $\succ_S^1$) so that she is assigned to c_1. This fact violates our assumption that the mechanism is strategyproof. □

Mechanism 3 Serial Dictatorship (SD)

Input: $S, C, X, \succ_S, \succ_C, f$, master-list $L = (s_1, s_2, \ldots, s_n)$
Output: matching Y
1: $Y \leftarrow \emptyset$. $k \leftarrow 1$.
2: If $k > n$, return Y. Otherwise, choose $(s_k, c) \in X$, where c is favorite and acceptable college for s_k such that $Y \cup \{(s_k, c)\}$ is feasible. $Y \leftarrow Y \cup \{(s_k, c)\}$ (if no such c exists, s_k is not assigned to any college).
3: $k \leftarrow k + 1$. Go to 2.

3.4 Existing Mechanisms

Three general strategyproof mechanisms are introduced for hereditary constraints [30]. First, Serial Dictatorship mechanism (SD) uses a serial order among the students. Although the order can be arbitrary, it must be determined independently of student preferences to guarantee strategyproofness. W.l.o.g., we assume this order is $s_1, s_2, \ldots$.

Conceptually, SD can be described as follows. Let $\boldsymbol{Y}$ denote all the possible feasible matchings. The first student, s_1, chooses subset $\boldsymbol{Y}_1 \subseteq \boldsymbol{Y}$, such that she equally prefers any matching in $\boldsymbol{Y}_1$ and strictly prefers any matching in $\boldsymbol{Y}_1$ over any matching in $\boldsymbol{Y} \setminus \boldsymbol{Y}_1$. In other words, she chooses her favorite matchings in $\boldsymbol{Y}$. Since she is concerned with the college to which she is assigned and has no interest in the assignments of other students, her favorite matching is not unique, and her choice is a subset of $\boldsymbol{Y}$. SD is obviously strategyproof since each student can choose her favorite matchings from the exogenously determined possibilities. SD is also Pareto efficient for the following reason. Clearly, we cannot improve the assignment of s_1. Nor can we improve s_2 without hurting s_1, and so forth. Thus, it is impossible to improve the assignment of one student without hurting other students. Since SD is Pareto efficient, it is also nonwasteful.

The formal definition of SD is given in Mechanism 3. Unfortunately, SD is excessively *unfair*; many students could have justified envy in SD since it completely ignores college preferences.

The next mechanism is Artificial Caps Deferred Acceptance (ACDA), which is based on DA. ACDA deals with distributional constraints by transforming them into artificial maximum quotas, thereby ensuring that DA returns a matching that satisfies these constraints. ACDA is employed in Japanese medical resident matching programs [41] and serves as a benchmark mechanism in various works focused on distributional constraints [22, 29, 30, 80]. The detailed procedure of ACDA is presented in Mechanism 4. Although ACDA obtains a fair matching in polynomial-time, it can be excessively inefficient; many students may claim an empty seat since maximum quotas are chosen independently of their preferences.

The third mechanism is Adaptive Deferred Acceptance (ADA). Like SD, ADA uses a serial order among students. The order can be arbitrary, but it must be determined

Mechanism 4 Artificial Caps Deferred Acceptance (ACDA)

Input: $S, C, X, \succ_S, \succ_C, f$

Output: matching Y

1: Choose q_C independently of $\succ_S$ such that q_C guarantees the matching's feasibility.

2: Run DA with q_C.

Mechanism 5 Adaptive Deferred Acceptance (ADA)

Input: $S, C, X, \succ_S, \succ_C, f$, master-list $L = (s_1, s_2, \ldots, s_n)$, maximum quotas q_C

Output: matching Y

Initialization: $q^1_c \leftarrow q_c$ for each $c \in C$, $Y \leftarrow \emptyset$. Go to **Stage** 1.

Stage k: Proceed to **Round** 1.

Round t: Select t students from the top of L. Let Y' denote the matching obtained by DA for the selected students under $(q^k_c)_{c \in C}$.

(i) If all students in L are already selected, then $Y \leftarrow Y \cup Y'$, output Y and terminate the mechanism.

(ii) If some college $c_i \in C$ is forbidden, i.e., $|Y'_{c_i}| < q^k_{c_i}$ and $f(\nu(Y \cup Y') + e_i) = -\infty$ hold, then $Y \leftarrow Y \cup Y'$. Remove t students from the top of L. For each college c that is forbidden, set q^{k+1}_c to 0. For each $c \in C$, which is not forbidden, set q^{k+1}_c to $q^k_c - |Y'_c|$. Go to **Stage** $k + 1$.

(iii) Otherwise, go to **Round** $t + 1$.

independently of student preferences to guarantee strategyproofness. W.l.o.g., we assume this order is $s_1, s_2, \ldots$. As well as ACDA, ADA requires maximum quota q_c for each college c. If no maximum quota is given, i.e., if we assume $q_c = \infty$ for each $c \in M$, ADA obtains identical matching as SD. During the execution of ADA, college c is forbidden under (partial) matching Y if the following conditions hold: (i) $Y \cup \{(s, c)\}$ is infeasible, and (ii) $|Y_c| < q_c$. In other words, college c is forbidden if c cannot accept another student due to distributional constraints, even though it does not reach its maximum quota. Formally, ADA is defined in Mechanism 5. We can assume ADA combines SD and DA, in which student groups are sequentially allocated as SD, but within each group, students compete with each other by DA.

We next present the Sample and Deferred Acceptance (SDA) mechanism. SDA was developed by Liu et al. [51] for a special case for hereditary constraints where the maximum quota of each college is determined by allocating indivisible resources to each college. This setting is referred as Student-Project-Resource matching-allocation problem (SPR). In SPR, a set of indivisible resources $R = \{r_1, \ldots, r_{|R|}\}$ exists. Each resource r has its capacity $q_r \in \mathbb{N}_{>0}$. For each resource r, its college compatibility list T_r is defined; resource r can be allocated to exactly one college in $T_r \subseteq C$. Mapping μ denotes one possible allocation of resources to colleges, i.e., $\mu : R \to C$ maps each resource r to a college $\mu(r) \in T_r$. For given allocation μ, the maximum quota of college c is given as $q_\mu(c) = \sum_{r:\mu(r)=c} q_r$, i.e., the maximum quota of each college

Mechanism 6 Sample and Deferred Acceptance (SDA)

Input: $S, C, X, \succ_S, \succ_C, f$

Output: matching Y

1: Select $S' \subseteq S$ independently of $\succ_S$, which we call sampled students, and $S' \neq \emptyset$. We call $S \setminus S'$ the regular students. Then run SD and find (partial) matching $Y^{S'}$ for S'.

2: Allocate $R' \subseteq R$ to colleges such that $Y^{S'}$ is feasible and R' is minimal: no $R'' \subsetneq R'$ makes $Y^{S'}$ feasible. Then allocate $R \setminus R'$ based on the preferences of S'.

3: Run DA for $S \setminus S'$. The capacity of each college c is $q_\mu(c) - |Y_c^{S'}|$, where μ is the resource allocation determined in 2.

is endogenously determined as the sum of the capacities of allocated resources. We assume $f(\nu) = 0$ if μ exists such that $\nu_i \leq q_\mu(c_i)$ holds for all $i \in M$. The formal definition of SDA is presented in Mechansm 6.

The basic idea of SDA is to combine SD and ACDA. One major limitation of ACDA is that we need to determine the maximal feasible vector ν^* (which determines the maximum quota of each college) independently from students' preferences. As a result, the maximum quotas of popular colleges can be low, while those of unpopular colleges can be high. In the standard SDA, first, we choose a subset of students $S' \subseteq S$, where $|S'| = k$. We call S' *sampled* students, and $S \setminus S'$ *regular* students. We assign sampled students using SD. Assume the obtained matching for sampled students is Y'. Then, we choose a maximal feasible vector ν^* based on the preferences of sampled students. Liu et al. [51] presented various methods for choosing ν^*. In this thesis, as elaborated later, we apply a simulation-based method utilizing duplicates of sampled students, proven to be highly effective. For regular students, we apply ACDA, where maximum quota q_{c_i} for each college c_i is given as $\nu_i^* - |Y'_{c_i}|$.

However, if the preferences of sampled students are significantly differ from those of regular students, obtained ν^* can be detrimental for regular students. Consequently, not even the no vacant college property is satisfied. We can assume SDA satisfies no empty matching property. This property is violated only in an exceptional case where all sampled students find all colleges unacceptable. In such a case, we can choose additional sampled students until at least one student is assigned to some college.

Table 3.2 summarizes the properties of the strategyproof mechanisms under hereditary constraints.

3.5 Impossibility Theorems

Since Theorem 1 shows that no mechanism exists that is fair, strategyproof, and cut-off nonwasteful under hereditary M$^\natural$-convex set constraints, this result extends to hereditary constraints. The next question pertains to the potential existence of a fair and strategyproof mechanism when cut-off nonwastefulness is relaxed to weak

Table 3.2: Mechanism Properties

	Strategyproof	Fair	Pareto efficient	Nonwasteful
SD [30]	yes	no	yes	yes
ACDA [30]	yes	yes	no	no
ADA [30]	yes	no	no	yes
SDA [51]	yes	no	no	no

Table 3.3: Possible matchings for preference profiles (Theorem 3)

Preference profile	s_1	s_2	Possible matchings
$\succ_S^1$	c_2	c_1	$[c_2, c_1]$
$\succ_S^2$	c_2	c_1c_3	$[c_2, c_1]$, $[\emptyset, c_3]$
$\succ_S^3$	$c_1c_2c_3$	c_1c_3	$[c_1, \emptyset]$, $[\emptyset, c_3]$
$\succ_S^4$	$c_1c_2c_3$	$c_1c_3c_4$	$[c_1, \emptyset]$, $[\emptyset, c_3]$
$\succ_S^5$	c_3c_1	$c_1c_3c_4$	$[c_1, \emptyset]$, $[\emptyset, c_3]$
$\succ_S^6$	c_3c_1	c_4	$[c_1, \emptyset]$, $[c_3, c_4]$
$\succ_S^7$	c_3	c_4	$[c_3, c_4]$

nonwastefulness, characterized as an open question [42]. We prove that no mechanism exists that is fair, strategyproof, and weak nonwastefulness by Theorem 2. We delegate its proof to the following Theorem 3, which is a stronger result than Theorem 2.

Theorem 2 No mechanism can simultaneously satisfy fairness, strategyproofness, and weak nonwastefulness under hereditary constraints.

Building upon Theorem 2, the ensuing question arises: can a strategyproof and fair mechanism satisfy any property weaker than weak nonwastefulness? Theorem 3 shows that no mechanism simultaneously satisfies strategyproofness, fairness, and no vacant college property. Following this, we present a simple mechanism that satisfies strategyproofness, fairness, and no empty matching property (Theorem 4). In essence, we establish tight boundaries, at least within the efficiency properties considered in this thesis, regarding whether a strategyproof and fair mechanism can satisfy certain efficiency properties for each class of constraints.

Theorem 3 No mechanism can simultaneously satisfy fairness, strategyproofness, and no vacant college property under hereditary constraints.

Proof Consider a matching market with two students $S = \{s_1, s_2\}$ and four colleges

$C = \{c_1, c_2, c_3, c_4\}$. The colleges' preferences $\succ_C$ are as follows:

$$\begin{aligned} c_1 &: s_1 \succ_{c_1} s_2 \\ c_2 &: s_1 \succ_{c_2} s_2 \\ c_3 &: s_2 \succ_{c_3} s_1 \\ c_4 &: s_2 \succ_{c_4} s_1 \end{aligned}$$

Suppose $f(\nu) = 0$ if and only if $\nu \leq \nu'$ for some $\nu' \in \{(1,1,0,0), (0,0,1,1)\}$.

Assume, for the sake of contradiction, that a mechanism exists that is fair, strategyproof, and satisfies no vacant college property.

We examine seven possible students' profiles $\succ_S^1, \dots, \succ_S^7$ described in Table 3.3. For each students' profile, we also enumerate all matchings that are fair and satisfy no vacant college property. For $\succ_S^1$, due to no vacant college property, both students must be assigned to their first choice colleges. Thus, the only possible matching is $[c_2, c_1]$. For $\succ_S^2$, another matching, $[\emptyset, c_3]$ is also possible. However, if the mechanism chooses $[\emptyset, c_3]$, student s_2 has an incentive to manipulate (to modify the profile to $\succ_S^1$) so that she is assigned to c_1. Thus, the mechanism must choose $[c_2, c_1]$. For $\succ_S^3$, both students consider c_1 and c_3 acceptable. Due to fairness, only s_1 can be assigned to c_1, and only s_2 can be assigned to c_3. Also, if we assign s_1 to c_2, due to no vacant college property, we need to assign s_2 to c_1, However, this violates fairness. Thus, possible matchings are either $[c_1, \emptyset]$ or $[\emptyset, c_3]$. However, if the mechanism chooses $[\emptyset, c_3]$, student s_1 has an incentive to manipulate (to modify the profile to $\succ_S^2$) so that she is assigned to c_2. Continuing a similar argument, we obtain that the mechanism must choose the matching colored in blue in Table 3.3. In particular, for $\succ_S^6$, the mechanism must choose $[c_1, \emptyset]$. For $\succ_S^7$, the only matching that satisfies no vacant college property is $[c_3, c_4]$. This implies that when the profile is $\succ_S^6$, student s_1 has an incentive to manipulate (to modify the profile to $\succ_S^7$) so that she is assigned to c_3. This violates our assumption that the mechanism is strategyproof. □

Next, we show that a mechanism exists that satisfies fairness, strategyproofness, and no empty matching property under hereditary constraints. This mechanism utilizes GDA. For given f, which is hereditary, we construct a set of vectors F' such that $\forall \nu \in F'$, $f(\nu) = 0$ holds (i.e., F' is a subset of vectors induced by f), and F' is a hereditary M$^\natural$-convex set. Then, we apply GDA by using f' (where $f'(\nu) = 0$ iff $\nu \in F'$) instead of f. F' is constructed as follows. We initialize $F' \leftarrow \{e_0\}$. Then, for each $i \in M$, if $f(e_i) = 0$, we add e_i to F'. Clearly, F' is an M$^\natural$-convex set; it contains only e_0 and e_i $(i \in M)$.

Theorem 4 Under hereditary constraints, GDA using f' is fair, strategyproof, and satisfies no empty matching property.

Proof For the obtained matching Y by GDA, $f'(\nu(Y)) = 0$ holds. Then, by way of constructing F', $f(\nu(Y)) = 0$ holds, i.e., Y is feasible. Since f' induces a hereditary M$^\natural$-convex set, GDA is strategyproof and fair [48]. Also, as long as there exists

$(s, c) \in X$ such that $c \succ_s \emptyset$ and $f'(\nu(\{(s, c)\}) = 0$ hold, $Y \neq \emptyset$ holds. This is because if $Y = \emptyset$, then $(s, c) \in Ch_S(Y \cup \{(s, c)\})$ and $(s, c) \in Ch_C(Y \cup \{(s, c)\})$ hold, which violates the fact that GDA obtains an HM-stable matching. □

3.6 New Fairness Concept: Envy-free up to k Peers

In this section, we introduce a weaker fairness concept called envy-free up to k peers (EF-k). Most existing works in two-sided matching require that the obtained matching be fair. However, just requiring fairness is not sufficient since the matching that no student is assigned to any college is fair; we should achieve some requirement on students' welfare in conjunction with fairness.

Let $Ev(Y, s)$ denote $\{s' \mid s' \in S, s \text{ has justified envy toward } s' \text{ in } Y\}$ for matching Y and student s.

Definition 17 (Envy-free up to k Peers) Matching Y is envy-free up to k peers (EF-k) if $\forall s \in S$, $|Ev(Y, s)| \leq k$ holds.

EF-0 is equivalent to fairness and Any matching is EF-$(n-1)$.

Several approaches can be considered to relax fairness. For instance, instead of counting the number of students to whom each student has envy, we can count the colleges at which each student has envy. Additionally, we can count the number of students by whom each student is envied. The acceptability of these concepts depends on the situation. This work represents an initial step, introducing new research directions in constrained two-sided matching, i.e., how to relax the fairness concept in a socially acceptable way.

We use the following example to show that nonwastefulness and EF-k are incompatible for any $k < n - 1$ even under hereditary M$^\natural$-convex set constraints.

Example 1 There are n students and n colleges. For each student s_i, her preference is: $c_{i+1} \succ_{s_i} c_{i+2} \succ_{s_i} \ldots \succ_{s_i} c_n \succ_{s_i} c_1 \succ_{s_i} \ldots \succ_{s_i} c_i$. For each college c_i, its preference is: $s_i \succ_{c_i} s_{i+1} \succ_{c_i} \ldots \succ_{c_i} s_n \succ_{c_i} s_1 \succ_{c_i} \ldots \succ_{c_i} s_{i-1}$. In short, for each student s_i, her most preferred college c_{i+1} considers her as the least preferred student, and her least preferred college c_i considers her as the most preferred student. Distributional constraints f is defined as: $f(\nu) = 0$ iff $\forall i \in M$, $|\nu_i| \leq 1$ and $\sum_{i \in M} |\nu_i| \leq n - 1$ hold, i.e., each college can accept at most one student, and the total number of students accepted to all colleges is at most $n - 1$. Clearly, f induces a hereditary M$^\natural$-convex set.

Theorem 5 Under hereditary M$^\natural$-convex set constraints, a case exists that no matching is nonwasteful and EF-k for any $k < n - 1$.

Proof Consider the setting in Example 1. The total number of accepted students is at most $n-1$. Also, due to nonwastefulness, exactly one student is unassigned to any college. By symmetry, without loss of generality, we assume s_1 is unassigned. Then, there exists exactly one vacant college, i.e., a college to which no student is assigned. The vacant college must be c_2, since if c_i $(i \neq 2)$ is vacant, student s_{i-1} claims an empty seat of c_i. Also, s_n must be assigned to c_1. Otherwise, she is assigned to c_i where $3 \leq i \leq n$; she claims an empty seat of c_2. Then, s_{n-1} must be assigned to c_n. Otherwise, she is assigned to c_i where $3 \leq i \leq n-1$; she claims an empty seat of c_2. Next, s_{n-2} must be assigned to c_{n-1}. Otherwise, she is assigned to c_i where $3 \leq i \leq n-2$; she claims an empty seat of c_2. By repeating a similar argument, we obtain that each student s_i $(i \neq 1)$ is assigned to her most preferred college c_{i+1}. Then, s_1 has justified envy toward $s_2, \ldots, s_n$. Thus, $|Ev(Y, s_1)| = n-1$ holds. □

Given Theorem 5, a natural question is the complexity of checking the existence of a nonwasteful and EF-k matching (for $k < n-1$). We assume f can be computed in a constant time. To examine this complexity, we utilize the following lemma.

Lemma 1 Checking whether a fair and nonwasteful matching exists or not is NP-complete, even when distributional constraints form a hereditary M$^\natural$-convex set.

Proof Aziz et al. [7] showed that checking the existence of a *strongly stable* matching is NP-complete for *REG* constraints. Strong stability is equivalent to fairness and nonwastefulness. *REG* constraints mean regional maximum quotas for mutually disjoint regions, which is a special case of hereditary M$^\natural$-convex set constraints. Thus, this complexity result carries over to hereditary M$^\natural$-convex set constraints, which is more general than *REG*. □

Theorem 6 Checking whether an EF-k $(k < n-1)$ and nonwasteful matching exists or not is NP-complete, even when distributional constraints form a hereditary M$^\natural$-convex set.

Proof First, for given matching Y, we can check whether Y is EF-k and nonwasteful in polynomial time, so the problem is in NP. Next, we show a reduction from the problem of checking whether a fair and nonwasteful matching exists or not. Consider an original matching problem instance I, where distributional constraints form a hereditary M$^\natural$-convex set. We create an instance of an extended market I' as follows.

- For each college in I, we create a corresponding college c' in I'. Let C' denote the set of these colleges in I'. The distributional constraints over C' are the same as the original instance I.
- For each student s_i in I, we create $k+1$ students $s_{i,1}, \ldots, s_{i,k+1}$, as well as $k+1$ additional colleges $c_{i,1}, \ldots, c_{i,k+1}$. These additional colleges for s_i form a region with regional maximum quota k. Each student in $s_{i,1}, \ldots, s_{i,k+1}$ is a copy of student s_i in the original instance I.

- The preference of additional college $c_{i,j}$ is defined in the same way as Example 1. More specifically, each additional college $c_{i,j}$ can accept only corresponding (copied) students $s_{i,1}, \dots, s_{i,k+1}$, and $c_{i,j}$ most prefers $s_{i,j}$ and least prefers $s_{i,j-1}$.

- Each student $s_{i,j}$ prefers any of its additional colleges over any original college. The order of original colleges is the same as the original instance I. The order of her additional colleges is defined in the same way as Example 1, i.e., $s_{i,j}$ most prefers $c_{i,j+1}$.

- The preference of each college $c' \in C'$ is defined as follows. If $s_i \succ_c s_j$ holds in the original instance, $s_{i,t} \succ_{c'} s_{j,t'}$ holds for any $t, t' \in \{1, \dots, k+1\}$. The preference over $s_{i,1}, \dots, s_{i,k+1}$, i.e., the copied students of the same original student, can be decided arbitrarily.

We can observe the following facts. Matching Y in the extended instance I' is nonwasteful only when for each $i \in N$ and copied students $s_{i,1}, \dots, s_{i,k+1}$, exactly k students are assigned to their additional colleges $c_{i,1}, \dots, c_{i,k+1}$. Also, these k students must be assigned to their first-choice colleges. Thus, the only student who is not assigned to her additional colleges has justified envy toward other k copied students. Let S' denote the set of students who are not assigned to their additional colleges. S' will be assigned to C'. Assume Y is EF-k and nonwasteful, then the matching between S' and C' within Y must be nonwasteful and fair; otherwise, at least one student in S' has justified envy toward more than k students or Y is wasteful for the original instance (to obtain a matching in the original instance from Y, we replacing $s_{i,j}$ to s_i and c' to c). Also, if there exists a fair and nonwasteful matching in the original instance I, then there exists an EF-k and nonwasteful matching in I'; the assignment of $s_{i,1}$ is the same as s_i, and the rest of the students are assigned to their favorite additional colleges. □

Intuitively, we can use the reduction from the problem of Lemma 1 by utilizing Example 1 as a gadget.

3.7 New Strategyproof Mechanisms

In this section, we introduce two contrasting strategyproof mechanisms that work for general hereditary constraints. The first one (called SD*) satisfies the strongest efficiency property, i.e., Pareto efficiency, while it cannot guarantee EF-k for any fixed $k < n-1$. The second one (called SDA with reserved quotas) satisfies EF-k for any fixed $k < n-1$, while it can only guarantee a rather weak efficiency property. In the next section, we experimentally show that SD* can guarantee EF-k where k is much smaller than $n-1$ when colleges' preferences are similar. Furthermore, we experimentally show that SDA with reserved quotas can significantly improve students' welfare compared to a fair (EF-0) mechanism even when k is very small.

3.7.1 Pareto Efficient Mechanism

First, we develop a strategyproof and Pareto efficient mechanism based on SD. For master-list L, a pair of students (s, s'), and college c, we say c disagrees with L for (s, s') if $s' \succ_L s$ and $s \succ_c s' \succ_c \emptyset$ holds. Otherwise, we say c agrees with L for (s, s'). In short, c disagrees with L for (s, s'), when s' is ranked higher than s in L, both s and s' are acceptable for college c, and c prefers s over s'. Assume we use SD based on L. Then, in obtained matching Y, if c disagrees with L for (s, s'), s has a chance to have justified envy toward s' in c, since s' is chosen before s and can be allocated to c, while s might not be allocated to c. On the other hand, if c agrees with L for (s, s'), then s never has justified envy toward s' in c. This is because, the fact that c agrees with L for (s, s') means: (i) s is ranked higher than s' in L, (ii) s is ranked lower than s' in c, or (iii) either s or s' is unacceptable for c. In each of the above three cases, s cannot have justified envy toward s' in c.

Let $d(L, s)$ denote $|\{s' \mid s' \in S \setminus \{s\}, c \in C, c \text{ disagrees with } L \text{ for } (s, s')\}|$, i.e., $d(L, s)$ counts the number of students such that for some college c, a disagreement related to s occurs.

The following theorem holds.

Theorem 7 Assume for master-list L, $\forall s \in S$, $d(L, s) \leq k$ holds. Then, SD using L is EF-k.

Proof Assume, for the sake of contradiction, that in obtained matching Y, there exists student s s.t. $|Ev(Y, s)| > k$ holds. Then, for each $s' \in Ev(Y, s)$, the following conditions must hold: (i) $s' \succ_L s$, and (ii) for $(s', c) \in Y$, $s \succ_c s' \succ_c \emptyset$. Thus, c disagrees L for (s, s'). This is true for each $s' \in Ev(Y, s)$. Then, $d(L, s) > k$ holds. This violates our assumption that $d(L, s) \leq k$. □

Theorem 7 means that if we can choose a good master-list L, such that $\max_{s \in S} d(L, s)$ is small, e.g. at most k, the obtained matching is guaranteed to be EF-k. Note that this guarantee holds independently from the actual distributional constraints and students' preferences; k can be computed using colleges' preference profile $\succ_C$ only.

We examine the problem of finding an optimal master-list (in terms of minimizing $\max_{s \in S} d(L, s)$) for given colleges' preference profile $\succ_C$.

Theorem 8 For given colleges' preference profile $\succ_C$, computing master-list L, which minimizes $\max_{s \in S} d(L, s)$ can be done in polynomial time.

Proof We first introduce a graphical representation of the above optimization problem. Consider a directed graph $G = (S, E)$, where each student is a vertex. For a pair of students s and s', if there exists college c s.t. $s \succ_c s' \succ_c \emptyset$ holds, we add a directed edge (s, s'). This means that to make c agree with the obtained master-list for (s, s'), the master-list must rank s higher than s'. Then, for $G = (S, E)$ and master-list L, $d(L, s)$ is equal to the number of outgoing edges from s toward any of higher-ranked students than s in L. For s, let O_s denote the set of outgoing edges from s. Clearly, for any L, $d(L, s) \leq |O_s|$ holds. Also, $d(L, s) = |O_s|$ holds when s is ranked lowest in L. This implies $\max_{s\in S} d(L, s) \geq \min_{s\in S} |O_s|$ holds, i.e., the optimal k cannot be smaller than $\min_{s\in S} |O_s|$. This is because some student s must be ranked lowest in L, and $d(L, s) = |O_s|$ holds. Then, when choosing the student who should be ranked lowest in L, we can safely choose s with the smallest $|O_s|$ to guarantee L's optimality. Thus, the following greedy algorithm obtains an optimal master-list L (as well as $\max_{s\in S} d(L, s)$).

1. For given graph $G = (V, E)$ (where $V = S$), set $k \leftarrow 0$, and L to an empty list.
2. If $V = \emptyset$, return L and k.
3. Choose $s = \arg\min_{s\in V} |O_s|$. Add s to the top of L. $k \leftarrow \max(k, |O_s|)$. Remove s and all edges related to s from G. Go to (2).

Clearly, the complexity of this greedy algorithm is $O(|V||E|)$. □

We call SD mechanism using optimal L as SD*. SD* is strategyproof and Pareto efficient. When we apply SD* to the matching instance presented in Example 1, the above algorithm returns L with $\max_{s\in S} d(L, s) = n - 1$ and the obtained matching cannot be EF-k for any $k < n - 1$. In the next section, we show that SD* can be EF-k for smaller k when colleges' preferences are similar.

3.7.2 EF-k Mechanism

Next, we develop a strategyproof and EF-k mechanism for any given $k \leq n-1$ based on SDA. SDA is strategyproof and is also EF-k, since for each sampled student s, she has justified envy only toward another sampled student assigned before her. Thus, $|Ev(Y, s)| \leq k - 1$ holds. Since DA is fair, for regular student s, she has justified envy only toward sampled students. Thus, $|Ev(Y, s)| \leq k$ holds.

We propose a generalized version of SDA, such that no vacant college property is satisfied under a mild assumption. The basic idea is that, since there exists a chance that the preferences of sampled students are completely different from those of regular students, we reserve some seats for each college even if the college seems unpopular based on the preferences of sampled students. Let $\widehat{\nu} = (\widehat{\nu}_1, \ldots, \widehat{\nu}_m)$ be *reserved quotas*, where $\widehat{\nu}_i \geq 0$ for each $i \in M$, and $f(\widehat{\nu}) = 0$ holds. The goal of the reserved quotas $\widehat{\nu}$ is to guarantee that each college c_i is guaranteed to accept at least $\widehat{\nu}_i$ students, as long as enough students hope to be assigned to c_i, even if c_i seems unpopular among sampled students.

For two m-element vectors ν and ν', let $\nu \vee \nu'$ denote the element-wise maximum, i.e., $\nu \vee \nu' = (\max(\nu_1, \nu'_1), \ldots, \max(\nu_m, \nu'_m))$.

First, we define SD with reserved quotas $\widehat{\nu}$. As standard SD, we assign students sequentially based on master-list L. Let Y denote the assignment obtained so far. The current student can be assigned to c_i, as long as $f((\nu(Y) + e_i) \vee \widehat{\nu}) = 0$ holds.

In short, the current student s can be assigned to c_i, if c_i can still accept one more student, assuming each college c_j will be assigned at least $\widehat{\nu}_j$ students.

Then, SDA with reserved quotas $\widehat{\nu}$ is defined as follows. Choose k sampled students (the remaining students are regular students). They are assigned by SD with reserved quotas $\widehat{\nu}$. Let Y' denote the matching for sampled students. Then, obtain a matching Y'', by further assigning multiple virtual students, each of which is a copy of sampled students by SD with reserved quotas, until no more student can be assigned. More specifically, we assume sampled students are $s_1, \ldots, s_k$. We create virtual students $s_{i,1}, s_{i,2}, \ldots$, which are copies of each sampled student s_i. Then, after sampled students are assigned. We assign these virtual students in a round-robin order, i.e., $s_{1,1}, s_{2,1}, \ldots, s_{k,1}, s_{1,2}, s_{2,2}, \ldots, s_{k,2}, s_{1,3}, s_{2,3}, \ldots, s_{k,3}, \ldots$. Note that this procedure is just for choosing appropriate ν^*; in reality, these virtual students are not allocated to any college. Then, we choose maximal feasible vector ν^* such that $\nu^* \geq \nu(Y'') \vee \widehat{\nu}$ holds. For each college c_i, we set its maximum quota q_{c_i} as $\nu^*_i - |Y'_{c_i}|$, and run ACDA for regular students.

Theorem 9 Assume for $\widehat{\nu}$, $f(\widehat{\nu}) = 0$ holds, and $\forall i \in M$, such that $f(e_i) = 0$ holds, $\widehat{\nu}_i \geq 1$ also holds. SDA with reserved quotas $\widehat{\nu}$ is strategyproof, EF-k, and satisfies no vacant college property.

Proof It is clear even after the above modifications, SDA with reserved quotas $\widehat{\nu}$ is still strategyproof and EF-k.

We show that it also satisfies no vacant college property. Assume, for the sake of contradiction, that obtained matching Y does not satisfy no vacant college property, i.e., student s strongly claims an empty seat of c_i, while $Y_s = \emptyset$ and $Y_{c_i} = \emptyset$. Since Y is obtained by SDA with reserved quotas ν^*, $\nu(Y) \leq \nu^*$ holds. Also, $Y_{c_i} = \emptyset$ and $\nu^*_i \geq \widehat{\nu}_i \geq 1$ holds. However, this fact means that if s applies to c_i, she must be accepted to c_i (either in SD with reserved quotas or ACDA). This violates the fact that $Y_s = \emptyset$. □

3.8 Empirical Evaluation

In this section, we empirically evaluate SD* and SDA with reserved quotas under hereditary constraints in terms of fairness and efficiency.

First, we show the level of k that SD* can be guaranteed by using an optimal master-list. We set the number of students n to 200 and the number of colleges m to 20. We generate the preference of each college c using the Mallows model [16, 52, 53]; college preference $\succ_c$ is drawn with probability:

$$\Pr(\succ_c) = \frac{\exp(-\theta_C \cdot \delta(\succ_c, \succ_{\widehat{c}}))}{\sum_{\succ'_c} \exp(-\theta_c \cdot \delta(\succ'_c, \succ_{\widehat{c}}))}.$$

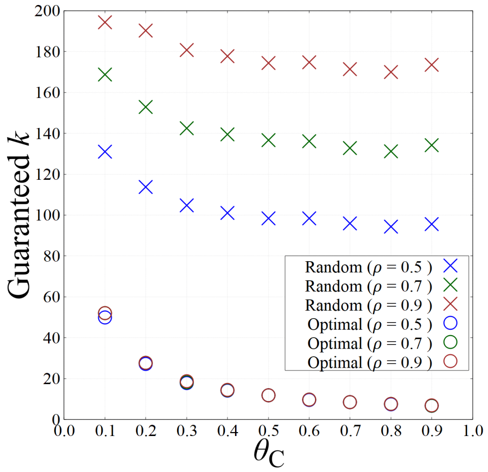
Here $\theta_C \in \mathbf{R}_+$ denotes the spread parameter for colleges, $\succ_{\widehat{c}}$ is a central preference uniformly randomly chosen from all possible preferences, and $\delta(\succ_c, \succ_{\widehat{c}})$ represents the Kendall tau distance, which is the number of pairwise inversions between $\succ_c$ and $\succ_{\widehat{c}}$. Intuitively, colleges' preferences are distributed around a central preference with spread parameter θ_C. When $\theta_C = 0$, the Mallows model becomes identical to the uniform distribution, while increasing θ_C leads to convergence towards a constant distribution, yielding $\succ_{\widehat{c}}$. Initially, each $\succ_c$ does not include $\emptyset$. We insert $\emptyset$ at the position $\lfloor \rho \cdot n \rfloor$ (where $0 < \rho < 1$).

Figure 3.1 (a) shows the guaranteed k when using an optimal master-list by varying the spread parameter θ_C and ρ. Each data point is an average of 10 instances. We also show the result when the master-list is randomly chosen. We can see that when the spread parameter becomes larger (colleges' preferences become more similar), SD* can guarantee EF-k for smaller k. For example, k becomes less than 5% of n when θ_C is 0.6. We can see ρ has almost no effect on SD*, while it significantly affects randomly selected master-lists.

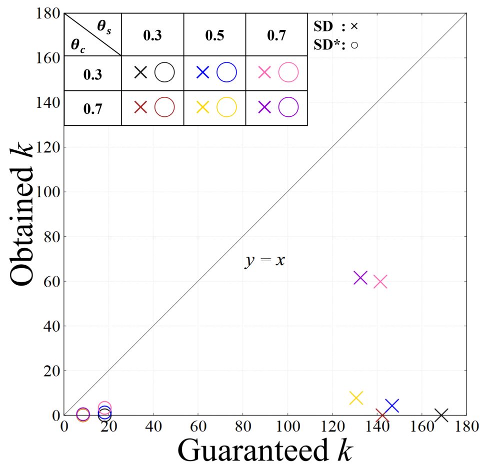
Next, we apply SD* to each matching market and measure the obtained level of k that SD* achieves under SPR [51] (Recall Section 3.4). Each market has $|R| = 100$ resources. For each resource r, we generate T_r such that each college c is included in T_r with probability 0.3. There are 40, 20, and 40 resources with capacity 1, 2, and 3, respectively; thus the total capacity of colleges is equal to n. We generate each student's preference in a similar way as a college's preference, i.e., we utilize the Mallows model with spread parameter θ_S. We do not apply ρ for students; each student considers all colleges acceptable.

Figure 3.1 (b) shows the average of 10 instances. The x-axis shows the guaranteed k and the y-axis shows the actually obtained k. We set colleges' spread parameter θ_C to 0.3 and 0.7, and students' spread parameter θ_S to 0.3, 0.5, and 0.7. ρ is set to 0.7. By definition, each data point must be located in the lower-right half. The result shows the actually obtained k is much smaller than the guaranteed k. In particular, for SD*, it is between 0 and 4. For SD, we can see that when θ_S becomes larger, the competition among students becomes more intense. As a result, more students tend to have justified envy.

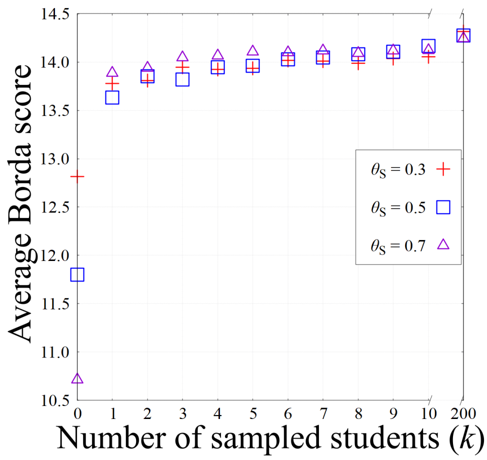
Next, we evaluate SDA with reserved quotas. By varying k, it can be identical to ACDA (when $k = 0$) and SD* (when $k = n$), assuming we use the same master-list as SD* and the same reserved quotas. Figure 3.1 (c) shows the average Borda score of the students varying k and the students' spread parameter θ_S. If a student is assigned to her i-th choice college, her Borda score is $m - i + 1$. We fix the colleges' spread parameter θ_C to 0.7 and ρ to 0.7. We set reserved quotas $\widehat{\nu}$ to $(1, 1, \ldots, 1)$. Each data point represents an average of 10 instances. In this setting, SDA with n sampled students (which is identical to SD*) guarantees EF-k for $k = 9$ in average. The average Borda score significantly improves as k increases from the case where $k = 0$. Note that increasing the average Borda score by one is significant; each student must be assigned to a strictly better college. The difference between $k = 0$ (where SDA is identical to ACDA) and $k = 1$ becomes larger when θ_S becomes larger, i.e., when students' preferences are similar. We can see that SDA achieves a high degree of fairness and efficiency with a few sampled students.



(a) Guaranteed k for optimal/random master-list



(b) Comparison between obtained/guaranteed k for SD*/SD



(c) Average Borda score for SDA with reserved quotas

Figure 3.1: Trade-off between efficiency and fairness under hereditary constraints

Chapter 4

Union of Symmetric M-convex Set Constraints

In this chapter, we explore constraints related to M-convexity and introduce a novel strategyproof mechanism.

4.1 Model

In this section, we define and illustrate the set of constraints related to M-convexity. We posit that all colleges are universally acceptable to all students, and vice versa. In other words, the outside option $\emptyset$ is considered the least preferable position in both student ($\succ_S$) and college ($\succ_C$) preferences. This strong assumption is necessary to guarantee the existence of a feasible matching, a common practice in previous works [22, 29, 30, 76].

4.1.1 M-convex Set

We first present the well-studied class of M-convex constraints. A set of distributional constraints is said to be M-convex if it can be defined as an *M-convex set*.

Definition 18 (M-convex Set) A family of vectors $F \subseteq \mathbf{Z}_+^m$ forms an *M-convex set*, if for all $\nu, \nu' \in F$, for all i such that $\nu_i > \nu'_i$, there exists $j \in \{k \in M \mid \nu_k < \nu'_k\}$ such that $\nu - e_i + e_j \in F$ and $\nu' + e_i - e_j \in F$ hold. Function f, representing distributional constraints, satisfies M-convexity if its induced vectors form an M-convex set.

The concept of an M-convex set closely relates to an M$^\natural$-convex set (Definition 13). The only difference between them is whether allowing 0 as j, indicating that the sum of elements of $\nu - e_i + e_j$ and $\nu' + e_i - e_j$ is preserved as n when $j \neq 0$. An M-convex set represents a set of maximum elements within an M$^\natural$-convex set. Assuming that all students are assigned to colleges, M$^\natural$-convex becomes equivalent to an M-convex set.

Note that the condition to apply GDA in Footnote 1 can be restated as follows: the set of feasible matchings (specifically, the set of feasible contracts and their subsets) forming a matroid structure is equivalent to the set of feasible vectors forming an M-convex set [48].

Definition 18 asserts that for any two vectors, $\nu, \nu' \in F$, we can find another element of F by moving one step from ν towards ν' (i.e., $\nu - e_i + e_j \in F$), as well as by moving one step from ν' toward ν (i.e., $\nu' + e_i - e_j \in F$). As an example, consider a standard school choice market comprising two schools and five students, with distributional constraints forming a feasible set denoted as $F = \{(1,4),(4,1),(2,3)\}$. Upon examining two feasible vectors $\nu^1 = (1,4)$ and $\nu^2 = (4,1)$, we observe that F does not satisfy the definition of M-convexity, since $\nu_1^1 < \nu_1^2$ and $\nu_2^1 > \nu_2^2$, but $(\nu_1^2 - 1, \nu_2^2 + 1) = (3,2)$ is not in F. However, by adding $(3,2)$ to F, we obtain a feasible set $\{(1,4),(4,1),(2,3),(3,2)\}$ which is M-convex. The new feasible vector set corresponds to the distributional constraints where each school's maximum and minimum quotas are four and one.

4.1.2 Symmetric M-convex Set

Before introducing the several families of M-convex set constraints, we first present symmetry. The concept of symmetry is motivated by settings where colleges exhibit comparable sizes, resulting in similar constraints for all colleges. Thus, the set of distributional constraints is symmetric over the set of colleges.

Definition 19 (Symmetry) Set $F \subseteq \mathbb{N}_0^m$ is symmetric if for all $\nu = (\nu_1, \ldots, \nu_m) \in F$, any permutation of $(\nu_1, \ldots, \nu_m)$ also belongs to F.

We identify several distributional constraints that can be represented by a *symmetric M-convex set*. Uniform min/max quotas constraints impose minimum quotas on colleges, which guarantee a minimum number of students in each college, as well as the capacity limits of colleges. that are generally imposed in traditional matching theory.

Definition 20 (Uniform Min/Max Quotas Constraints) Let p and q respectively be the minimum and maximum quotas, i.e., the number of students assigned in each college must be between p and q. Then the feasible set is given as $F = \{\nu \in \mathbb{N}_0^m \mid \forall i \in M, p \le \nu_i \le q, \text{ and } \sum_{i \in M} \nu_i = n\}$.

It is easy to verify that the general individual min/max quotas constraints can be represented by an M-convex set. When all the colleges have identical min/max quotas, F becomes symmetric.

Introducing another class of constraints that can be represented by a symmetric M-convex set, namely, *symmetric distance constraints*, requires us to define the *most balanced vectors*.

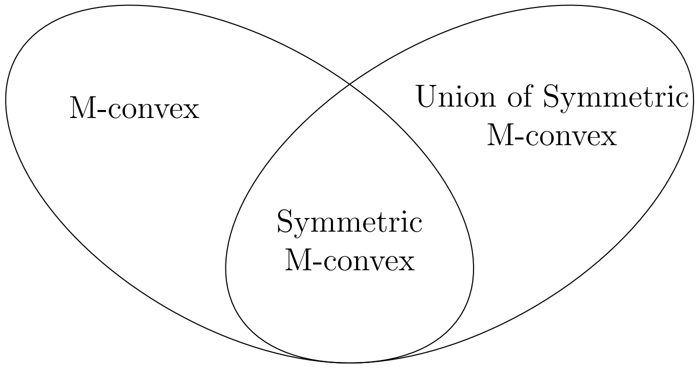


Figure 4.1: Inclusion of notions in discrete convex analysis

Definition 21 (Most Balanced Vectors) Vector $\nu \in F$ is most balanced if for each $i \in M$, ν_i is either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$. Let F^* denote the set of all the most balanced vectors.

Definition 22 (Symmetric Distance Constraints) Let F^* be the set of the most balanced vectors. Given a distance threshold $d \in \mathbb{R}$, the feasible set F of symmetric distance constraints is given as $F = \{\nu \in \mathbb{N}_0^m \mid \delta(\nu) \leq d, \sum_{i \in M} \nu_i = n\}$, where $\delta(\nu)$ is given as either (i) Manhattan distance (L^1 norm), i.e., $\min_{\nu^* \in F^*}(\sum_{i \in M} |\nu_i - \nu_i^*|)$, or (ii) Chebyshev distance (L^∞ norm), i.e., $\min_{\nu^* \in F^*} \max_{i \in M} |\nu_i - \nu_i^*|$.

It is easy to verify that for a single vector ν, a set of vectors whose distance from ν is within a given threshold can be represented by an M-convex set. Since we consider distance constraints from the set of all the most balanced vectors (which is symmetric), F is also symmetric. This class of constraints is appropriate when a policymaker believes that the most balanced vectors are ideal, although she can accept any matching that is not too far from them.

4.1.3 Union of Symmetric M-convex Set

Based on these notions, we present a formal definition of a union of symmetric M-convex set.

Definition 23 (Union of Symmetric M-convex Set) Set $F \subseteq \mathbb{N}_0^m$ is a union of symmetric M-convex set if it is represented by $F_1 \cup \ldots \cup F_\ell$, where for each $i \in \{1, \ldots, \ell\}$, set F_i is a symmetric M-convex set.

By its definition, the set of the unions of symmetric M-convex sets is symmetric. Thus, its intersection with the set of M-convex sets corresponds to the set of symmetric M-convex sets (Figure 4.1).

Several classes of constraints beyond M-convex sets can be represented by a union of symmetric M-convex sets. We first present two classes of real-life distributional constraints that can be represented by a union of symmetric M-convex sets: *ratio* and *difference* constraints. Ratio constraints specify the acceptable minimum ratio between the least and the most popular colleges.

Definition 24 (Ratio Constraints [80]) For a given parameter $\alpha \in [0, 1]$, F is given as $\{\nu \in \mathbb{N}_0^m \mid r(\nu) \geq \alpha, \sum_{i \in M} \nu_i = n\}$, where $r(\nu)$ is given as $\frac{\min_{i \in M} \nu_i}{\max_{i \in M} \nu_i}$, i.e., $r(\nu)$ is the ratio of students between the least and the most popular colleges.

Ratio constraints can be clearly decomposed into several uniform min/max quotas constraints, as Example 2 illustrates.

Example 2 Assume $n = 21$, $m = 4$, and $\alpha = 0.5$. Feasible set F includes $(3, 6, 6, 6)$, $(4, 4, 5, 8)$, $(4, 4, 6, 7)$, $(4, 5, 5, 7)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations. It can be represented by a union of F_1 and F_2, where F_1 includes $(3, 6, 6, 6)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations, and F_2 includes $(4, 4, 5, 8)$, $(4, 4, 6, 7)$, $(4, 5, 5, 7)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations. Here F_1 corresponds to the feasible set for uniform min/max quotas constraints where the minimum quota is 3 and the maximum quota is 6, and F_2 corresponds to the feasible set for uniform min/max quotas constraints where the minimum quota is 4 and the maximum quota is 8.

Difference constraints, which specify the acceptable maximum difference between the least and the most popular colleges, can be interpreted as an absolute version of ratio constraints.

Definition 25 (Difference Constraints) For a given parameter $\beta \in \mathbb{R}$, $F = \{\nu \in \mathbb{N}_0^m \mid \gamma(\nu) \leq \beta, \sum_{i \in M} \nu_i = n\}$, where $\gamma(\nu)$ is given as $\max_{i \in M} \nu_i - \min_{i \in M} \nu_i$, i.e., $\gamma(\nu)$ is the difference between the numbers of students allocated to the most and the least popular colleges.

It is also easy to see that difference constraints can be decomposed into several uniform min/max quotas constraints, as Example 3 illustrates.

Example 3 Assume $n = 21$, $m = 4$, and $\beta = 4$. Feasible set F includes $(3, 4, 7, 7)$, $(3, 5, 6, 7)$, $(3, 6, 6, 6)$, $(4, 4, 5, 8)$, $(4, 4, 6, 7)$, $(4, 5, 5, 7)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations. It can be represented by a union of F_1 and F_2, where V_1 includes $(3, 4, 7, 7)$, $(3, 5, 6, 7)$, $(3, 6, 6, 6)$, $(4, 4, 6, 7)$, $(4, 5, 5, 7)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations, and V_2 includes $(4, 4, 5, 8)$, $(4, 4, 6, 7)$, $(4, 5, 5, 7)$, $(4, 5, 6, 6)$, $(5, 5, 5, 6)$, and their permutations. Here V_1 corresponds to the feasible set for uniform min/max quotas constraints where the minimum quota is 3 and the maximum quota is 7, and V_2 corresponds to the feasible set for uniform min/max quotas constraints where the minimum quota is 4 and the maximum quota is 8.

Note that even if both the ratio and difference constraints can be represented by the union of uniform min/max quotas constraints, they are not interchangeable, i.e., ratio constraints cannot be represented by difference constraints and conversely, as Proposition 1 states.

Proposition 1 Ratio and difference constraints are not interchangeable.

Proof Assume $n = 10$ and $m = 4$.

First, we show that ratio constraints cannot be represented by difference constraints. Suppose $\alpha = 0.5$. Feasible set F_α contains $(2, 2, 2, 4)$ and all its permutations. To represent it as difference constraints, β must be greater than or equal to 2, but then $(1, 3, 3, 3)$, which is not in F_α, must be feasible.

Next we show that difference constraints cannot be represented by ratio constraints. Suppose $\beta = 4$. Feasible set F_β includes $(0, 3, 3, 4)$ and all its permutations. To represent it as ratio constraints, α must be 0, but then $(0, 0, 0, 10)$, which is not in F_β, must be feasible. □

Ratio and difference constraints constitute alternative ways to specify a well-balanced matching. Ratio constraints ensure that the ratio between the least and the most popular colleges must be within interval $[\alpha, 1]$, and the difference constraints ensure that the difference between them remains within β. The appropriateness of each approach depends on the policymakers' preferences and the specific applications under consideration.

We illustrate that M-convexity is not closed under the union by the following example. In particular, this example shows the union of symmetric M-convex sets is not necessarily an M-convex set.

Example 4 Consider the difference constraints with $n = 10$, $m = 4$, and $\beta = 2$. That is, the difference in the student numbers between the most popular college and the least popular one is at most two. The feasible vectors can then be represented by the union of multiple symmetric M-convex sets, i.e., $F_1 \cup F_2$, where F_1 consists of $(1, 3, 3, 3)$, $(2, 2, 3, 3)$ and all their permutations, and F_2 consists of $(2, 2, 2, 4)$, $(2, 2, 3, 3)$, and all their permutations. We consider the following two vectors: $\nu = (1, 3, 3, 3) \in F_1$ and $\nu' = (2, 2, 2, 4) \in F_2$. $\nu_2 > \nu'_2$ holds. For both $j = 1$ and $j = 4$ where $\nu_j < \nu'_j$ holds, the condition of M-convexity is not satisfied by this set of feasible vectors. For $j = 1$, $\nu' + e_2 - e_1 = (1, 3, 2, 4)$ is not feasible, and for $j = 4$, $\nu - e_2 + e_4 = (1, 2, 3, 4)$ is not feasible, since the required difference excesses β.

As described earlier, according to Kojima et al. [48], if the set of feasible vectors forms an M-convex set, then GDA is strategyproof and fair. [1] However, we cannot apply it to a union of symmetric M-convex sets since it is not an M-convex set as illustrated in Example 4. Therefore, beyond M-convexity, identifying constraints, for which we can develop a strategyproof and fair mechanism, is a challenging question. In this chapter, we identify a general class of constraints beyond M-convex sets,

[1] To be precise, they use a condition called M$^\natural$-convex set, which is a generalization of an M-convex set. When all students are assigned to colleges, it becomes equivalent to an M-convex set.

namely, the union of symmetric M-convex sets, for which a strategyproof and fair mechanism can be developed.

Next we present another type of union of symmetric M-convex set constraints obtained by combining several symmetric M-convex set constraints. If a policymaker believes that several symmetric M-convex set constraints are equally appropriate, she can define the union of these vectors as feasible. This situation assumes that all constraints are equally justifiable and that satisfying one is necessary for feasibility. For illustration, by combining uniform min/max and distance constraints, we obtain the following class of constraints, which we call *flexible uniform min/max quotas constraints.*

Definition 26 (Flexible Uniform Min/Max Quotas Constraints) Let p and q respectively be the minimum and maximum quotas. Let $d \in \mathbb{R}$ be the distance threshold, and let F^* be the set of the most balanced vectors. Then $F = \{\nu \in \mathbb{N}_0^m \mid \forall i \in M, p \leq \nu_i \leq q \text{ and } \sum_{i \in M} \nu_i = n\} \bigcup \{\nu \in \mathbb{N}_0^m \mid \delta(\nu) \leq d \text{ and } \sum_{i \in M} \nu_i = n\}$, where $\delta(\nu)$ is given as either (i) Manhattan distance (L^1 norm), $\min_{\nu^* \in F^*}(\sum_{i \in M} |\nu_i - \nu_i^*|)$, or (ii) Chebyshev distance (L^∞ norm), $\min_{\nu^* \in F^*} \max_{i \in M} |\nu_i - \nu_i^*|$.

Notice that, as illustrated in Figure 4.1, components of the unions of symmetric M-convex sets, i.e., the class of symmetric M-convex sets, is a subclass of M-convex sets, while the two classes, the union of symmetric M-convex sets and M-convex set, are not included by each other. We use the following example to show the intuition.

Example 5 (Example 2 continued) We show by reusing Example 2, in which the ratio-constraint feasible set F is the union of symmetric M-convex sets F_1 and F_2. Taking F_1 (uniform min/max quotas of 3/6) as an instance, we can easily extend it by re-specifying the min/max quotas of c_1 (the first element of feasible vectors) as 3/7 instead of 3/6. Then we obtain an asymmetric M-convex set. Moreover, notice that F, a union of two symmetric M-convex sets, is not M-convex. Observe that $\nu = (3, 6, 6, 6)$ and $\nu' = (4, 4, 5, 8)$ are in F, and $\nu'_4 > \nu_4$ and $\nu'_2 < \nu_2$. However, by subtracting 1 from ν_2 and adding 1 to ν_4, we obtain $(3, 5, 6, 7)$, which is not in F.

Note that Definition 23 includes the case that F is represented by just one symmetric M-convex set. Thus, all results in this chapter also hold for a symmetric M-convex set.

4.2 Impossibility Theorems

Under distributional constraints, fairness and nonwastefulness are generally incompatible, especially under the union of symmetric M-convex sets. Yahiro et al. [80] showed that these two notions are incompatible under ratio constraints, which is a special case of the union of symmetric M-convex set constraints. We strengthen this result by showing that fairness and nonwastefulness are incompatible under difference constraints.

Proposition 2 Under difference constraints, fairness and nonwastefulness are incompatible.

Proof Consider the following example (Example 6), to which the example used in Yahiro et al. [80] is trivially extended in the context of difference constraints.

Example 6 (Adapted from [80]) Consider $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, $\beta = 1$. The following are the preferences of the students and the colleges:

$$\begin{array}{ll} s_1, s_2 : c_2 \succ c_3 \succ c_1 \\ s_3, s_4 : c_3 \succ c_2 \succ c_1 \end{array} \qquad \begin{array}{r} c_1 : s_1 \succ s_3 \succ s_2 \succ s_4 \\ c_2 : s_2 \succ s_3 \succ s_1 \succ s_4 \\ c_3 : s_4 \succ s_1 \succ s_3 \succ s_2. \end{array}$$

Since $\beta = 1$, the difference in the number of students between the most and the least popular colleges should be smaller or equal to 1. Hence, the matchings that satisfy these constraints are such that one college is assigned two students and each of the other two colleges is assigned one student. If a feasible matching is fair, it must contain (s_2, c_2) and (s_4, c_3); otherwise, either s_2 or s_4 has justified envy. Here, although c_1 is the least preferred college for each student, at least one student must be assigned to it. Assigning both s_1 and s_3 to c_1 is wasteful. Assume we assign s_1 to c_1. If we assign s_3 to c_2, s_3 claims an empty seat in c_3. If we assign s_3 to c_3, s_1 has justified envy toward s_3. Thus assume we assign s_3 to c_1. If we assign s_1 to c_3, s_1 claims an empty seat in c_2. If we assign s_1 to c_2, s_3 has justified envy toward s_1. □

Example 6 also shows that fairness and nonwastefulness are incompatible under a symmetric M-convex set constraints since it includes only $(1, 1, 2)$ and its permutations, which can be represented by a single symmetric M-convex set. Since fairness and nonwastefulness are incompatible under the union of symmetric M-convex set constraints, we focus on *finding a fair outcome while reducing as much wastefulness as possible*. Observe also that nonwastefulness is weaker than Pareto efficiency (recall Definition 3), and therefore, fairness is also incompatible with Pareto efficiency under the union of symmetric M-convex set constraints.

Now we present two properties that are satisfied by a union of symmetric M-convex set constraints. They are crucial for analyzing our mechanism.

Lemma 2 If $\nu = (\nu_1, \ldots, \nu_i, \ldots, \nu_j, \ldots, \nu_m) \in F$ where F is a union of symmetric M-convex sets and $\nu_i > \nu_j$ holds, then vector $\nu' = (\nu_1, \ldots, \nu_i - 1, \ldots, \nu_j + 1, \ldots, \nu_m)$ and all of its permutations also belong to F.

Proof From symmetry, vector ν'', obtained by exchanging the i-th and j-th elements of ν, is also feasible. Then by the definition of an M-convex set, since $\nu_i > \nu''_i$, $\nu_j < \nu''_j$, and $\nu_k = \nu''_k$ for all $k \neq i, j$, vector $\nu - e_i + e_j = \nu'$ must be in F. □

Intuitively, Lemma 2 states that when transforming a feasible vector ν into a more balanced vector ν', feasibility is preserved. This idea leads to the following lemma about most balanced vectors.

Lemma 3 If F, where $F \neq \emptyset$, is a union of symmetric M-convex set, it contains all of the most balanced vectors.

Proof Assume first that F is a symmetric M-convex set, and let ν be a vector in F. If ν is most balanced, then from symmetry, all of its permutations must belong to F. Thus, F contains all of the most balanced vectors. If ν is not most balanced, from Lemma 2, we can move one student from a popular college to a less popular college in ν and obtain another vector included in F. By repeatedly applying such modification, we eventually obtain a most balanced vector that belongs to F.

Therefore, if F is a union of symmetric M-convex sets, it includes all the most balanced vectors. □

4.3 New Strategyproof Mechanism: Quota Reduction Deferred Acceptance

In this subsection, we introduce the Quota Reduction Deferred Acceptance mechanism (QRDA) and present its axiomatic properties. Furthermore, we provide a theoretical comparison with a baseline mechanism.

To develop a DA-based mechanism considering distributional constraints of the union of symmetric M-convex sets, the crucial aspect is determining appropriate maximum quotas through DA, ensuring feasible matchings. We explore this idea by employing flexible maximum quotas, aiming to enhance efficiency while guaranteeing feasibility and other desired properties. Later in Section 4.5, we compare our proposed mechanism with another that uses a fixed method to determine maximum quotas.

Informally, QRDA produces an initial standard market from a market with distributional constraints. At each stage, it iteratively (i) restricts the constraints on this market (i.e., reduces the maximum quotas) and (ii) applies DA to it until the matching returned by DA is also feasible with respect to the distributional constraints.

Which college's maximum quota is reduced at the beginning of each stage is defined by σ, a sequence of colleges based on the round-robin order $c_1, c_2, \ldots, c_m$. Let $\sigma(k)$ denote the k-th college in σ, i.e., $\sigma(k) = c_j$, where $j = 1 + (k - 1 \mod m)$. For simplicity, although we assume σ is based on a fixed round-robin order, all our results hold for any *balanced* sequence σ. A sequence σ is said to be *balanced* if for each $\ell \in \mathbb{N}_0$, $\sigma(m\ell + 1), \sigma(m\ell + 2), \ldots, \sigma(m\ell + m)$ is a permutation of $c_1, c_2, \ldots, c_m$. The balanced sequence requirement is crucial to guarantee the strategyproofness of QRDA, as Example 8 later demonstrates.

QRDA is defined with respect to a specific quota reduction sequence σ. However, in the following, for simplicity, we assume that σ is based on the round-robin order $c_1, c_2, \ldots, c_m$. When necessary, we specify σ and denote QRDA based on σ by QRDA$^\sigma$.

Let ν^{max} be a value that satisfies $\forall \nu \in F, \forall i \in M, \nu^{max} \geq \nu_i$. Also, let q_c^k denote the quota of college c at stage k of QRDA. Given σ and ν^{max}, QRDA is defined.

Mechanism 7 Quota Reduction Deferred Acceptance (QRDA)

Input: $S, C, X, \succ_S, \succ_C, f$, order σ
Output: matching Y
Initialization: For all $c \in C$, $q_c^1 \leftarrow \nu^{max}$.
Initial Stage:
(i) Run DA where colleges' quotas are q_C^1 and obtain matching Y^1.
(ii) If Y^1 is feasible, then return Y^1. Otherwise, go to **Stage** 2.
Stage k **(**≥ 2**)**:
(i) For college $c' = \sigma(k-1)$, $q_{c'}^k \leftarrow q_{c'}^{k-1} - 1$, and for $c \neq c'$, $q_c^k \leftarrow q_c^{k-1}$.
(ii) Run DA where colleges' quotas are q_C^k and obtain matching Y^k.
(iii) If Y^k is feasible, then return Y^k. Otherwise, go to **Stage** $k+1$.

Note that in the following arguments, especially in Theorem 16, we assume that it only costs constant time to verify whether a matching is feasible or not. Also, note that the complexity/cost for finding $\max_{\nu \in F, i \in M} \nu_i$ depends on the representation of feasible vectors F, which can be high. However, we only require that ν^{max} is not smaller than $\max_{\nu \in F, i \in M} \nu_i$. Thus, we can always choose $\nu^{max} = n$. The choice of ν^{max} does not influence the complexity result in Theorem 16.

4.3.1 Mechanism Properties

First we show that QRDA always returns a feasible and fair matching by Theorem 10.

Theorem 10 QRDA returns a feasible and fair matching when F is a union of symmetric M-convex sets.

Proof If QRDA returns a matching, it is clearly feasible. According to Lemma 3, all the most balanced vectors must be included in F. Eventually, at some stage k', the quota of each college equals either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$, and $\sum_{i \in M} q_{c_i}^{k'} = n$. At this stage, standard DA returns a feasible matching. Thus, QRDA must terminate at an earlier stage k ($\leq k'$) by returning a feasible matching. The matching returned at stage k is identical to the one returned by DA where colleges' quotas are q_C^k. Since DA is fair [27], QRDA is also fair. □

This theorem holds under a much weaker condition; the fact that most balanced vectors are included in F is sufficient.

Now we show that QRDA is strategyproof. Although using DA iteratively, QRDA does not trivially inherit strategyproofness from DA. Since the quotas of colleges are decreasing, a student might have an incentive to terminate the mechanism early to secure the college seat, which might not be available at later stages. Therefore, the quotas have to be reduced carefully to preserve strategyproofness, i.e., the quota reduction sequence has to be balanced.

Moreover, under general non-M-convex constraints, a balanced quota reduction sequence may not be sufficient to preserve the strategyproofness of iterative DA mechanisms. Yahiro et al. [80] provided an example illustrating this fact, which we reproduce with Example 7 for the sake of completeness. In this example, initial quotas are equal to the largest number of students in a college in any feasible matching and the quota reduction sequence is balanced.

Example 7 ([80]) Consider $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $C = \{c_1, c_2, c_3\}$, $\sigma : c_1, c_2, c_3$, and $F = \{(3, 1, 2), (2, 2, 2)\}$. The following are the preferences of the students and the colleges:

$$\begin{aligned} s_1, s_2 &: c_1 \succ c_2 \succ c_3 \\ s_3 &: c_2 \succ c_3 \succ c_1 \qquad c_1, c_2, c_3 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_6. \\ s_4, s_5, s_6 &: c_3 \succ c_1 \succ c_2 \end{aligned}$$

The initial maximum quotas are $q^1_C = (3, 3, 3)$. At stage 1, all the students are assigned to their favorite college, and the matching is not feasible. The mechanism proceeds by reducing the quota of colleges c_1 and c_2 by one at stages 2 and 3, and the matching remains the same. At stage 4, the quota of college c_3 is decreased by one. Student s_6 is rejected and then applies to c_1, which also rejects her. Hence, s_6 is assigned to c_2, and the matching becomes feasible. However, if s_6 misreports her preference with $\succ'_{s_6}$ such that c_1 is her favorite college, she is assigned to c_1 at stage 1 and the matching is feasible. Thus, s_6 can successfully manipulate the mechanism.

Hence, QRDA's strategyproofness is not trivial, and to prove it, we develop novel proof techniques, which require several lemmas.

The first is called the *Scenario lemma* and it requires two definitions. A scenario is a sequence of colleges to which a student plans to apply. For example, suppose that student s has scenario $\mathcal{C}_s$, defined as c_1, c_2, c_3. This scenario $\mathcal{C}_s$ means that student s first applies to college c_1, and if she is rejected, then she applies to college c_2, and then to c_3 if she is rejected again. A scenario is not necessarily exhaustive. A *rejection chain* $\mathcal{R}(\mathcal{C}_s)$, based on scenario $\mathcal{C}_s$, is the sequence of all the students and the colleges' actions (applications and rejections) that follow when student s enters the market with scenario $\mathcal{C}_s$, starting when she applies to the first college of this scenario. A rejection chain ends when (i) student s is rejected by the last college in $\mathcal{C}_s$ or (ii) the mechanism terminates. A simple example of a rejection chain is presented in Table 4.1.

Given these definitions, we can define the Scenario lemma, which is inspired by the original Scenario lemma [17]. It was proven in the context of ratio constraints [80], and the proof is trivially extended to constraints represented by a union of symmetric M-convex sets.

Lemma 4 (Scenario Lemma) Consider two scenarios, $\mathcal{C}_s$ and $\mathcal{C}'_s$, of student s starting from the same stage of QRDA. If (1) each college that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$ (the order is irrelevant), (2) student s applies to all the colleges in $\mathcal{C}_s$,

Stage	#	Action
k	1	Student s applies to college c_1.
	2	College c_1 rejects student s_1.
	3	Student s_1 applies to college c_2 (and is accepted).
$k+1$	1	College c_3 rejects student s_2 (due to its quota reduction).
	2	Student s_2 applies to college c_4.
		...

Table 4.1: Example of a rejection chain

and (3) all the actions of $\mathcal{R}(\mathcal{C}'_s)$ happen at the same stage, then all the actions in $\mathcal{R}(\mathcal{C}'_s)$ also happen in $\mathcal{R}(\mathcal{C}_s)$.

Proof The first action in $\mathcal{R}(\mathcal{C}'_s)$ is "student s applies to college c," where c is the first college that appears in $\mathcal{C}'_s$. Since c also appears in $\mathcal{C}_s$, and s applies to all the colleges in $\mathcal{C}_s$, $\mathcal{R}(\mathcal{C}_s)$ also includes this action. For an inductive step, assume the first $i-1$ actions in $\mathcal{R}(\mathcal{C}'_s)$ also happen in $\mathcal{R}(\mathcal{C}_s)$, and consider the i-th action of $\mathcal{R}(\mathcal{C}'_s)$. The i-th action in $\mathcal{R}(\mathcal{C}'_s)$ must be either (i) "student s' applies to college c'" or (ii) "college c' rejects student s'."

In case (i) with $s' = s$, since college c' must appear in $\mathcal{C}_s$ and s applies to all the colleges in $\mathcal{C}_s$, $\mathcal{R}(\mathcal{C}_s)$ also includes this action. In case (i) with $s' \neq s$, there must be a previous action, "college c'' rejects student s'," in $\mathcal{R}(\mathcal{C}'_s)$. From the inductive assumption, this action also happens in $\mathcal{R}(\mathcal{C}_s)$. Thus, the action "student s' applies to college c'" also happens in $\mathcal{R}(\mathcal{C}_s)$.

In case (ii), let $S'_{c'}$ be the set of students who applied to c' before the i-th action in $\mathcal{R}(\mathcal{C}'_s)$, and let $S_{c'}$ be the set of all the students applying to c' until all actions in $\mathcal{R}(\mathcal{C}_s)$ are executed. Here, $S'_{c'} \subseteq S_{c'}$ holds since every application before the i-th action in $\mathcal{R}(\mathcal{C}'_s)$ also appears in $\mathcal{R}(\mathcal{C}_s)$. Since in the i-th action of $\mathcal{R}(\mathcal{C}'_s)$, s' is rejected by college c', she is not among c''s favorite $q^k_{c'}$ students in set $S'_{c'}$. Since the quotas of colleges are non-increasing as QRDA continues, in some stage k' ($\geq k$), student s' must not be among the favorite $q^{k'}_{c'}$ students in $S_{c'}$. Thus, the action "college c' rejects student s'" eventually occurs in $\mathcal{R}(\mathcal{C}_s)$. $\square$

Another essential lemma for QRDA's strategyproofness in our setting is Lemma 5. Before introducing it, we introduce the following notation and concepts. Given a matching Y, consider that some college c_i's student number is decreased by one and another college c_j's student number is increased by one. We denote by η the corresponding *operation*, i.e., the operation that transforms vector $(|Y_{c_1}|, \ldots, |Y_{c_i}|, |Y_{c_j}|, \ldots, |Y_{c_m}|)$ into vector $(|Y_{c_1}|, \ldots, |Y_{c_i}|-1, |Y_{c_j}|+1, \ldots, |Y_{c_m}|)$. Moreover, at stage k of QRDA, we denote by η_k the operation that transforms vector $\nu(Y^{k-1})$ into vector $\nu(Y^k)$ since at most one student number changes in a QRDA's stage. Finally, at stage k of QRDA, a college c is said to be *full* if $|Y^k_c| = q^k_c$, and *maximum* if $|Y^k_c| = \max_{c' \in C} |Y^k_{c'}|$. Note that c_i is always full and maximum, and c_j is neither full nor maximum in QRDA since σ is a balanced sequence.

Intuitively, Lemma 5 means that, at some stage of QRDA, if operation η exists that makes the current QRDA's matching feasible, at any subsequent stage of QRDA, this operation (more specifically, any operation that resembles η, as defined in Lemma 5) also makes the current QRDA's matching feasible.

Lemma 5 Consider an infeasible matching Y^k obtained at stage k of QRDA and an operation η that leads to a feasible matching when applied to Y^{k-1}. Assume that operation η increases college c_j's student number, and let k' denote the first stage of QRDA after k $(< k')$ where c_j's student number increases. Then at any stage $\hat{k}$ such that $k-1 \leq \hat{k} < k'$ and for any maximum college c_{max} at stage $\hat{k}$, applying an operation that reduces c_{max}'s student number and increases c_j's student number to matching $Y^{\hat{k}}$ leads to a feasible matching.

Proof Consider an infeasible matching Y^k obtained at stage k of QRDA and an operation η such that imposing η on $\nu(Y^{k-1})$ leads to a feasible vector. Assume that operation η reduces college c_i's student number by one and increases college c_j's student number by one. The proof is done by induction.

For $\hat{k} = k-1$, vector $(\ldots, |Y^{k-1}_{c_i}| - 1, |Y^{k-1}_{c_j}| + 1, \ldots)$ is feasible by assumption. If college c_i is the unique maximum college at stage $k-1$, then the claim is satisfied for $\hat{k} = k-1$. Hence, consider a maximum college $c_{max} \neq c_i$ at stage $k-1$. By definition, $|Y^{k-1}_{c_{max}}| \geq |Y^{k-1}_{c_i}|$, which implies $|Y^{k-1}_{c_{max}}| > |Y^{k-1}_{c_i}| - 1$, and thus with Lemma 2 on $(\ldots, |Y^{k-1}_{c_i}| - 1, |Y^{k-1}_{c_j}| + 1, \ldots)$, vector $(\ldots, |Y^{k-1}_{c_{max}}| - 1, |Y^{k-1}_{c_j}| + 1, \ldots)$ is feasible.

Consider some stage $\hat{k}$ such that $k-1 < \hat{k} < k'$ and assume that the claim is true at stage $\hat{k}-1$. We show that the claim is also true for $\hat{k}$. Assume that operation $\eta_{\hat{k}}$ reduces college c_a's student number and increases college c_b's student number. Since college c_a is maximum at stage $\hat{k}-1$ and the claim is true at stage $\hat{k}-1$, vector $(\ldots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_j}| + 1, \ldots)$ is feasible. Note that if $c_b = c_j$ then $\nu(Y^{\hat{k}})$ is feasible and QRDA terminates strictly before k', which is a contradiction. Hence, in the following, we assume $c_b \neq c_j$. Two cases are possible.

If $|Y^{\hat{k}-1}_{c_a}| - 1 = |Y^{\hat{k}-1}_{c_b}|$, then vector $\nu(Y^{\hat{k}})$ is a permutation of vector $\nu(Y^{\hat{k}-1})$. Hence, since $c_b \neq c_j$, for any maximum college c_{max} at $\hat{k}$, vector $(\ldots, |Y^{\hat{k}}_{c_{max}}| - 1, |Y^{\hat{k}}_{c_j}| + 1, \ldots)$ is a permutation of vector $(\ldots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_j}| + 1, \ldots)$. Hence, by symmetry, vector $(\ldots, |Y^{\hat{k}}_{c_{max}}| - 1, |Y^{\hat{k}}_{c_j}| + 1, \ldots)$ is feasible.

Otherwise, $|Y^{\hat{k}-1}_{c_a}| - 1 > |Y^{\hat{k}-1}_{c_b}|$. By Lemma 2 on vector $(\ldots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_j}| + 1, \ldots)$, we obtain that vector $(\ldots, |Y^{\hat{k}-1}_{c_a}| - 2, |Y^{\hat{k}-1}_{c_j}| + 1, |Y^{\hat{k}-1}_{c_b}| + 1 \ldots)$ is feasible, i.e., vector $(\ldots, |Y^{\hat{k}}_{c_a}| - 1, |Y^{\hat{k}}_{c_j}| + 1, |Y^{\hat{k}}_{c_b}| \ldots)$ is feasible. If college c_a is the unique maximum college at stage $\hat{k}$, then the claim is satisfied at $\hat{k}$. Hence, consider a maximum college $c_{max} \neq c_a$ at $\hat{k}$. By definition, $|Y^{\hat{k}}_{c_{max}}| \geq |Y^{\hat{k}}_{c_a}|$, which implies $|Y^{\hat{k}}_{c_{max}}| > |Y^{\hat{k}}_{c_a}| - 1$, and thus with Lemma 2 on $(\ldots, |Y^{\hat{k}}_{c_a}| - 1, |Y^{\hat{k}}_{c_j}| + 1, \ldots)$, vector $(\ldots, |Y^{\hat{k}}_{c_{max}}| - 1, |Y^{\hat{k}}_{c_j}| + 1, \ldots)$

is feasible, which concludes the proof. □

Now we are ready to show that QRDA is strategyproof under constraints represented by a union of symmetric M-convex sets.

Theorem 11 QRDA is strategyproof when F is a union of symmetric M-convex sets.

Proof By contradiction, assume student s is assigned to a better college when she misreports. Without loss of generality, we assume that her true preference is $c_1 \succ_s c_2 \succ_s \ldots \succ_s c_m$, and that s is assigned to college c_j in stage k when misreporting, but s is assigned to c_i at stage k' under s's true preference, where $j < i$.

First, we show that if $k' \leq k$, student s cannot benefit from misreporting. DA satisfies a property called resource monotonicity, i.e., DA's outcome is weakly less preferred by each student if the quotas decrease [18]. This property implies that when student s truthfully reports her preference at both stages k and k', her assignment is (weakly) worse in k than in k'. Furthermore, DA is strategyproof [17, 65]. Hence, at stage k, student s's assignment is worse when she misreports than when she truthfully reports. Therefore, s's assignment is (weakly) worse when she misreports in k than when she truthfully reports in k', and thus, s cannot benefit from misreporting if $k' \leq k$. Hence, in the following, $k < k'$ holds.

At stage k, consider an alternative (and equivalent) DA mechanism where we run DA for $S \setminus \{s\}$ under the quotas of stage k and add s to the market. The matching obtained in this way is identical to the matching obtained by applying DA when all the students enter the market simultaneously [17]. When s is added to the market with a misreported preference, s is assigned to college c_j and a feasible matching $\dot{Y}^k$ is obtained; when s is added with a true preference, s is assigned to college c_o, and an infeasible matching Y^k is obtained.

Now we define two scenarios: (a) $\mathcal{C}_s = (c_1, c_2, \ldots, c_{i-1})$, based on the true preference of s, and so the last action in $\mathcal{R}(\mathcal{C}_s)$ must be "college c_{i-1} rejects student s," and (b) $\mathcal{C}'_s$, based on the misreported preferences in which the last college is c_j. For $\mathcal{C}'_s$, the following two cases are possible: (i) c_j is truly the least preferred college for s within $\mathcal{C}'_s$ or (ii) $\mathcal{C}'_s$ contains at least one college that is less preferred by s than c_j. We start by proving case (i); case (ii) is proven in the last paragraph of the proof. A key point in case (i) is that each college c that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$. Thus by Lemma 4, it implies that action "student s applies to college c_j" appears in rejection chain $\mathcal{R}(\mathcal{C}_s)$.

At stage k, when s enters the market (after all the other students) with true preferences, assume that college c_e gains one student, and when s misreports, another college c_d gains one student. We show the proof for a particular case when $c_e = c_o$ and $c_d = c_j$; a similar argument can be developed when $c_e \neq c_o$ or $c_d \neq c_j$ by replacing c_o to c_e, and c_j to c_d, respectively, in the following argument.

Note that $|\dot{Y}^k_{c_o}| = |Y^k_{c_o}| - 1$, $|\dot{Y}^k_{c_j}| = |Y^k_{c_j}| + 1$, and for all other colleges c ($\neq c_o, c_j$), $|\dot{Y}^k_c| = |Y^k_c|$. Therefore, the vector of feasible matching $\dot{Y}^k$ verifies $\nu(\dot{Y}^k) =$

$(|Y^k_{c_1}|, \dots, |Y^k_{c_o}| - 1, \dots, |Y^k_{c_j}| + 1, \dots, |Y^k_{c_m}|)$.

It follows that if $|Y^k_{c_o}| - 1 < |Y^k_{c_j}| + 1$ holds, then by Lemma 2 on $\nu(\dot{Y}^k)$, Y^k is also feasible, which is a contradiction. Thus we obtain the following equation:

$$|Y^k_{c_o}| \geq |Y^k_{c_j}| + 2. \tag{4.1}$$

The rest of the proof for case (i) is composed of three arguments: (A), (B), and (C).

(A) First, we show that under scenario $\mathcal{C}_s$, college c_j's student number does not increase strictly between k and k'. By contradiction, assume that c_j's student number increases at some stage $\tilde{k}$ ($k < \tilde{k} < k'$). Let $c_\star$ denote a college whose student number is decreased by operation $\eta_{\tilde{k}}$. Since college $c_\star$ is maximum at stage $\tilde{k} - 1$, by Lemma 5 with $\hat{k} \longleftarrow \tilde{k} - 1$, vector $(\dots, |Y^{\tilde{k}-1}_{c_\star}| - 1, |Y^{\tilde{k}-1}_{c_j}| + 1, \dots)$ is feasible, i.e., vector $\nu(Y^{\tilde{k}})$ is feasible. Hence, QRDA terminates strictly before k', which is a contradiction.

(B) Recall that under scenario $\mathcal{C}_s$, action "student s applies to college c_j" must occur; however, in this scenario student s's final assignment is c_i ($\neq c_j$). Thus, college c_j rejects student s at some stage of QRDA. Notice that if college c_j is not full at stage $k' - 1$, then c_j cannot reject any students at stage k' or at any previous stage. Hence, college c_j is full at stage $k' - 1$, i.e., $q^{k'-1}_{c_j} = |Y^{k'-1}_{c_j}|$, which implies that for any college c, $|Y^{k'-1}_{c_j}| + 1 \geq |Y^{k'-1}_{c}|$. Since college c_j's student number does not increase strictly between k and k', the following equation holds for any college c:

$$|Y^k_{c_j}| + 1 \geq |Y^{k'-1}_{c}|. \tag{4.2}$$

Let $\hat{k}$ denote the first stage after k such that $|Y^k_{c_j}| + 1 \geq |Y^{\hat{k}}_c|$ for any college c. With Eqs. 4.1 and 4.2, we know that $k < \hat{k} \leq k' - 1$. Assume that operation $\eta_{\hat{k}}$ decreases c_a's student number and increases c_b. By the definition of stage $\hat{k}$, $|Y^{\hat{k}-1}_{c_a}| = |Y^k_{c_j}| + 2$ and $|Y^{\hat{k}-1}_{c_b}| < |Y^k_{c_j}| + 1$ hold.

(C) Finally, we show that vector $\nu(Y^{\hat{k}})$ is feasible. Since college c_a is maximum at stage $\hat{k} - 1$ and $\hat{k} - 1 \geq k$, by Lemma 5, vector $(\dots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_j}| + 1, \dots)$ is feasible. Recall that college c_j's student number does not increase strictly between k and k'. Moreover, since $|Y^{\hat{k}-1}_{c_a}| = |Y^k_{c_j}| + 2 \geq |Y^{\hat{k}-1}_{c_j}| + 2$, college c_j cannot be full at stage $\hat{k}$, which implies that college c_j's student number does not decrease between k and $\hat{k}$. Therefore, we obtain $|Y^{\hat{k}-1}_{c_j}| = |Y^k_{c_j}|$. Since $|Y^{\hat{k}-1}_{c_b}| < |Y^k_{c_j}| + 1$, it leads to $|Y^{\hat{k}-1}_{c_b}| < |Y^{\hat{k}-1}_{c_j}| + 1$. Thus, by Lemma 2 on vector $(\dots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_j}| + 1, \dots)$, vector $(\dots, |Y^{\hat{k}-1}_{c_a}| - 1, |Y^{\hat{k}-1}_{c_b}| + 1, \dots)$ is also feasible, i.e., vector $\nu(Y^{\hat{k}})$ is feasible. Hence, QRDA terminates strictly before k', which is a contradiction.

Last, for case (ii), we can create a new scenario, $\mathcal{C}''_s$, by removing all the colleges that are less desired than c_j based on $\succ_s$ from $\mathcal{C}'_s$. If s is assigned to c_j in $\mathcal{R}(\mathcal{C}''_s)$, we obtain the same contradiction as for case (i) by comparing $\mathcal{R}(\mathcal{C}''_s)$ and $\mathcal{R}(\mathcal{C}_s)$. Thus, action "college c_j rejects student s" must appear in $\mathcal{R}(\mathcal{C}''_s)$. By Lemma 4, this action also appears in $\mathcal{R}(\mathcal{C}'_s)$, but this also leads to a contradiction. $\square$

QRDA's strategyproofness is heavily based on the fact that σ is balanced. Under ratio constraints, Yahiro et al. [80] provided an example to show that QRDA is not strategyproof if σ is not balanced. To be self-contained, we reproduce this example with Example 8.

Example 8 Consider $S = \{s_1, \ldots, s_9\}$, $C = \{c_1, c_2, c_3\}$, $\alpha = 1/4$ (the ratio constraints), and σ starting with c_2, c_2, c_1. The following are the preferences of the students and the colleges:

$$\begin{aligned} s_1, \ldots, s_5 &: c_1 \succ c_2 \succ c_3 \\ s_6, s_7, s_8 &: c_2 \succ c_3 \succ c_1 \qquad c_1, c_2, c_3 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_6 \succ s_7 \succ s_8 \succ s_9 \succ s_5. \\ s_9 &: c_3 \succ c_1 \succ c_2 \end{aligned}$$

First, QRDA$^\sigma$ sets $q_c^1 = 5$, for all $c \in C$. At stage 1, all students are assigned to their favorite colleges, but this is not feasible. At both stages 2 and 3, the quota of c_2 is decreased by one, although the matching remains the same. Then at stage 4, the quota of c_1 is decreased by one and s_5 is rejected from c_1. Student s_5 applies to c_2, which also rejects her. She is finally assigned to c_3 and the corresponding matching is feasible. However, if student s_5 misreports by declaring that c_2 is her most preferred college, QRDA stops at stage 1 by returning a feasible matching where she is assigned to college c_2. Hence, student s_5 manipulated the mechanism to get a better result (s_5, c_2).

Under ratio constraints, QRDA has been proven to hold additional gratifying properties [80], most of which can be trivially extended to our setting. We define these properties and notions.

Definition 27 (Weak Pareto Efficient and Domination) Matching Y strongly Pareto dominates matching $\dot{Y}$ if $Y_s \succ_s \dot{Y}_s$ holds for every $s \in S$. Feasible matching Y is weakly Pareto efficient if no feasible matching $\dot{Y}$ exists that strongly dominates Y. Mechanism φ dominates mechanism ψ if for each preference profile of students $\succ_S$, either $\varphi(\succ_S)$ Pareto dominates $\psi(\succ_S)$ or $\varphi(\succ_S) = \psi(\succ_S)$ holds, and profile $\succ_S$ exists such that $\varphi(\succ_S)$ Pareto dominates $\psi(\succ_S)$.

As its name shows, weak Pareto efficiency is a lenient property of Pareto efficiency (Definition 3). In other words, Pareto efficiency implies weak Pareto efficiency while it does not imply the other efficiency properties that are presented in this thesis.

Next, we define *weak non-bossiness* and *weak Maskin monotonicity*, which are related to weak group strategyproofness. First, we present some definitions to describe these properties.

The *strict upper contour* set of college c at preference $\succ_s$, denoted as $U(\succ_s, c)$, is the set of colleges that student s strictly prefers to college c:

$$U(\succ_s, c) = \{c' \in C \mid c' \succ_s c\}.$$

Preference $\succ'_s$ is a *monotonic transformation* of preference $\succ_s$ at college c (or equivalently at contract (s, c)) if $U(\succ'_s, c) \subseteq U(\succ_s, c)$. In other words, the set of colleges preferred to c in $\succ'_s$ is a subset of the colleges preferred to c in $\succ_s$. Preference $\succ'_s$ is an *upper-contour-set preserving transformation* of $\succ_s$ at college c (or equivalently at contract (s, c)) if $U(\succ'_s, c) = U(\succ_s, c)$. Informally, the set of colleges preferred to c in $\succ'_s$ is exactly the set of colleges preferred to c in $\succ_s$.

These notions naturally extend to the preference profiles. Profile $\succ'_S$ is a monotonic transformation (respectively, upper-contour-set preserving transformation) of profile $\succ_S$ at matching Y if for each student s, preference $\succ'_s$ is a monotonic transformation (respectively, upper-contour-set preserving transformation) of preference $\succ_s$ at contract Y_s.

Definition 28 (Weak Non-bossiness) Mechanism φ is weakly non-bossy if for any preference profile $\succ_S$, student s and preference $\succ'_s$ such that $\succ'_s$ is an upper-contour-set preserving transformation of $\succ_s$ at $\varphi_s(\succ_S)$, the following holds:

$$[\varphi_s(\succ'_s, \succ_{S\setminus\{s\}}) = \varphi_s(\succ_S)] \Rightarrow [\varphi(\succ'_s, \succ_{S\setminus\{s\}}) = \varphi(\succ_S)].$$

This is a weaker version of a property called non-bossiness, which was first introduced by Satterthwaite and Sonnenschein [69]. Non-bossiness demands that a student cannot alter the assignments of other students by modifying her stated preference unless her assignment changes. We call the above property weak non-bossiness since we restrict the possible modification to the upper-contour-set preserving transformation.

Definition 29 (Weak Maskin Monotonicity) Mechanism φ satisfies weak Maskin monotonicity if for all profiles $\succ_S$ and $\succ'_S$ such that $\succ'_S$ is a monotonic transformation of $\succ_S$ at $\varphi(\succ_S)$, $\varphi_s(\succ'_S) \succeq'_s \varphi_s(\succ_S)$ holds for each student s.

This is a weaker version of a property called Maskin monotonicity [54], which requires that for all $s \in S$, $\varphi_s(\succ_S) = \varphi_s(\succ'_S)$ when $\succ'_s$ is a monotonic transformation of $\succ_s$ at $\varphi_s(\succ_S)$. Intuitively, Maskin monotonicity requires that the mechanism's outcome be identical if each student s ranks her assigned college $\varphi_s(\succ_S)$ weakly higher in $\succ'_s$ compared to $\succ_s$. We call the above property weak Maskin monotonicity since we assume $\varphi(\succ'_S)$ is not necessarily identical to $\varphi(\succ_S)$, although each $\varphi_s(\succ'_S)$ must be at least as good as $\varphi_s(\succ_S)$, according to $\succ'_s$. A similar concept called the inconspicuous manipulation has been studied [38, 77, 37], where a student is allowed to promote exactly one college.

Their proofs for weak non-bossiness, weak Maskin monotonicity, weak group strategyproofness, and non-domination rely only on QRDA's strategyproofness under ratio constraints and DA's properties. Since Theorem 11 shows that QRDA is strategyproof in our setting, Theorems 12 and 13 trivially hold.

Theorem 12 QRDA is weakly non-bossy, weakly Maskin monotone, and weakly group strategyproof.

Theorem 13 Given a balanced σ, no strategyproof mechanism exists that dominates QRDA$^\sigma$.

However, weak Pareto efficiency does not trivially extend to a union of symmetric M-convex sets since its proof under ratio constraints strongly relies on the structure of the ratio constraints. With Theorem 14, we show that QRDA remains weakly Pareto efficient in our setting.

Theorem 14 QRDA is weakly Pareto efficient.

Proof Toward a contradiction, assume a matching Y that strongly Pareto dominates the matching returned by QRDA, denoted by $\dot{Y}$, i.e., $Y_s \succ_s \dot{Y}_s$ holds for all $s \in S$. Let k denote the last stage of QRDA, i.e., $\dot{Y}^k = \dot{Y}$. The proof is conducted by showing that matching $\dot{Y}^{k-1}$ is feasible, i.e., the matching returned by QRDA at stage $k-1$ is feasible, which contradicts that QRDA terminates at stage k. We develop the proof through arguments (A), (B), and (C).

(A) This argument shows that for all $c \in C$ such that $|\dot{Y}_c^k| = 0$, it holds $|Y_c| = 0$.

First, we show that for all $c \in C$, $q_c^k \geq 1$. Assume $c' \in C$ exists such that $q_{c'}^k = 0$. The definition of QRDA's quota reduction sequence implies that for all $c \in C$, both $q_c^k \leq 1$ and $q_c^{k-1} \leq 1$ hold. Therefore, for any college $c \in C$, both $|\dot{Y}_c^k| \leq 1$ and $|\dot{Y}_c^{k-1}| \leq 1$ hold. Hence, vector $\nu(\dot{Y}^{k-1})$ is a permutation of vector $\nu(\dot{Y}^k)$ and thus, by symmetry, $\dot{Y}^{k-1}$ is feasible, which contradicts that QRDA terminates at stage k.

Then consider a college $c' \in C$ such that $|\dot{Y}_{c'}^k| = 0$. Since for all $c \in C$, $q_c^k \geq 1$, no student applies to c' during QRDA, implying that for all $s \in S$, $\dot{Y}_s^k \succ_s c'$. If $|Y_{c'}| \neq 0$, then for any student s assigned to c' in Y, it holds that $c' \succ_s \dot{Y}_s^k$, which is a contradiction.

(B) The next argument shows that any college c such that $|Y_c| \neq 0$ is full at stage $k-1$ of QRDA, i.e., $|\dot{Y}_c^{k-1}| = q_c^{k-1}$.

Consider a college c', which is not full at stage $k-1$, implying that it rejects no student during QRDA. Toward a contradiction, assume that $|Y_{c'}| \neq 0$, and consider a student s' who is assigned to c' in Y. Since Y strongly dominates $\dot{Y}$, $Y_{s'} \succ_{s'} \dot{Y}_{s'}$ holds. Hence, during QRDA, student s' has to be rejected by college c' before being allowed to apply to $\dot{Y}_{s'}$, which contradicts that c' rejects no student during QRDA.

(C) Finally, we show that we can transform matching Y into matching $\dot{Y}^{k-1}$ by applying permutations and operations that satisfy the conditions of Lemma 2.

Denote $C' = \{c \in C \mid |Y_c| \neq 0\}$ and thus $\sum_{c \in C'} |Y_c| = n$. Let matching $\tilde{Y}$ be one of the matchings such that (i) for all $c \in C \setminus C'$, $|\tilde{Y}_c| = 0$, and (ii) for all $c \in C'$, $|\tilde{Y}_c| = \lceil n/|C'| \rceil$ or $\lfloor n/|C'| \rfloor$, i.e., $\tilde{Y}$ is one of the most balanced matchings when we restrict the setting to colleges in C'. With a similar argument as proof of Lemma 3, we can show that any matching can be transformed into a most balanced matching only by permutations and operations satisfying the conditions of Lemma 2. Hence, since $\tilde{Y}$ is one of the most balanced vectors when we restrict colleges to C', we can transform matching Y into $\tilde{Y}$ only by permutations and operations satisfying the conditions of Lemma 2. Therefore, since Y is feasible when considering all colleges C, $\tilde{Y}$ is also feasible when considering all colleges C.

By definition of $\tilde{Y}$, $\sum_{c\in C'} |\tilde{Y}_c| = n$ holds, and thus $\sum_{c\in C'} |\tilde{Y}_c| \geq \sum_{c\in C'} |\dot{Y}_c^{k-1}|$ also holds. Furthermore, by argument (B), all colleges $c \in C'$ are full at stage $k-1$, i.e., $|\dot{Y}_c^{k-1}| = q_c^{k-1}$. With QRDA's quota reduction sequence, it implies that for all $c \in C$, $q_c^{k-1} \leq \lceil n/|C'| \rceil$. Therefore, in addition to satisfying (i) and (ii), matching $\tilde{Y}$ can be chosen such that it also satisfies (iii) for all $c \in C'$, $|\tilde{Y}_c| \geq |\dot{Y}_c^{k-1}|$.

Now note that $\sum_{c\in C'} |\tilde{Y}_c| = n = \sum_{c\in C} |\dot{Y}_c^{k-1}|$ holds, and, with condition (iii), $\sum_{c\in C'}(|\tilde{Y}_c| - |\dot{Y}_c^{k-1}|) = \sum_{c\in C\setminus C'} |\dot{Y}_c^{k-1}|$ holds. It intuitively means that the surplus of students in the colleges of C' when comparing $\tilde{Y}$ and $\dot{Y}^{k-1}$ equals the student number in the colleges of $C \setminus C'$ in $\dot{Y}^{k-1}$. To transform $\tilde{Y}$ into $\dot{Y}^{k-1}$, we can transfer the surplus of students from a college in C' to a college in $C \setminus C'$ with operations that satisfy the conditions of Lemma 2 and with permutations.

$\dot{Y}^{k-1}$ is feasible by Lemma 2 and symmetry, which is a contradiction. Hence, no matching exists that strongly Pareto dominates $\dot{Y}$. □

As Proposition 2 shows, fairness and nonwastefulness are incompatible. Since QRDA is fair, it cannot be nonwasteful. Nevertheless, QRDA satisfies the following weaker version of nonwastefulness.

Definition 30 (Nonwastefulness Based on Balance) In matching Y, where $(s, c) \in Y$, student s claims an empty seat in c' based on balance if $(s, c') \succ_s (s, c)$, $(Y \setminus \{(s, c)\}) \cup \{(s, c')\}$ is feasible, and $|Y_{c'}| + 2 \leq |Y_c|$. Matching Y is weakly nonwasteful if no student claims an empty seat in Y based on balance.

Intuitively, a student claims an empty seat based on balance if, by unilaterally moving the student to her more preferred college, the result allocation vector is not only feasible but also more balanced than the original allocation vector; if we move s from college c to college c', the student number in c' is still fewer than or equal to that in c.

We have the following result in terms of this weaker version of nonwastefulness.

Theorem 15 QRDA is nonwasteful based on balance when F is a union of symmetric M-convex set.

Proof We prove this theorem towards a contradiction. Assume that QRDA terminates at stage k, outputting feasible matching Y^k. Student s exists such that $(s, c) \in Y^k$ and s claims an empty seat in college c' in matching Y^k based on balance. According to Definition 30, s prefers c' over c and $|Y_{c'}^k| + 2 \leq |Y_c^k|$ holds. Since s prefers c' over c, s must have applied to c' and been rejected. Thus, c' must be full, i.e., $|Y_{c'}^k| = q_{c'}^k$ holds. Also, $|Y_c^k| \leq q_c^k$ must hold, since college c's student number cannot exceed its quota. By the fact that $|Y_{c'}^k| + 2 \leq |Y_c^k|$ holds, we obtain $q_{c'}^k + 2 \leq q_c^k$. However, this contradicts the fact that the quota reduction sequence is balanced, that is, the difference between the maximum quotas of any pair of colleges is at most one. □

Mechanism 8 ACDA (based on a most balanced vector)

Input: $S, C, X, \succ_S, \succ_C$

Output: matching Y

1: For each i where $i \leq (n \bmod m)$, $q_{c_i} \leftarrow \lfloor n/m \rfloor$, and for each i where $i > (n \bmod m)$, $q_{c_i} \leftarrow \lceil n/m \rceil$.

2: Run DA with q_C.

Lastly, we briefly delve into the time complexity of QRDA. At each stage k of QRDA, we do not have to run DA from the beginning (that is, each student starts by applying to the first-ranked college). Instead, at stage k of QRDA, each student first applies to her most preferred college among those that have not rejected her at all previous stages. This modification ensures that the resulting matching remains unaffected, as the execution order in DA can be flexible [17].

Theorem 16 The time complexity of QRDA is $O(mn)$, assuming that verifying the feasibility of a vector can be done in constant time.

Proof QRDA repeatedly applies DA. By the above modification, a student is rejected by each college at most once during the whole QRDA's execution. Thus, each step in DA is executed at most mn times in total. Hence, the time complexity of QRDA is $O(mn)$. □

4.3.2 Theoretical Comparison with Baseline Mechanism

To the best of our knowledge, no fair and strategyproof mechanism exists in the literature that can handle the union of symmetric M-convex set constraints except QRDA.

As a baseline fair mechanism, we utilize ACDA. If prior information were available regarding the popularity of colleges, it might be possible to set q_C to meet distributional constraints and maximize student welfare. Otherwise, a pragmatic and rational approach to determine suitable q_C is by utilizing a most balanced vector. Similar to QRDA, ACDA is defined with respect to a specific sequence σ, denoted as ACDA$^\sigma$. We assume that σ is the round-robin order $c_1, c_2, \ldots, c_m$, unless otherwise specified. ACDA (based on a most balanced vector) is defined as Mechanism 8.

ACDA inherits the properties of DA like fairness and strategyproofness.

Theorem 17 ACDA is strategyproof and returns a feasible and fair matching.

Proof Since all the most balanced vectors are included in F, the matching obtained under q_C is clearly feasible. Since standard DA is fair [27], the matching obtained by ACDA is also fair. Furthermore, since q_C is determined independently from $\succ_S$ and DA is strategyproof, ACDA is strategyproof. □

The next result shows that QRDA is overwhelmingly favored by stating that all students weakly prefer the college that they obtain in QRDA to the one in ACDA.

Theorem 18 Given a balanced sequence σ, all students weakly prefer the matching obtained by QRDA$^\sigma$ over that by ACDA$^\sigma$.

Proof Let k' denote the stage of QRDA$^\sigma$ where $q_C^{k'}$ becomes identical to the q_C used in ACDA$^\sigma$. This situation can happen since ACDA$^\sigma$ generates the most balanced vector based on the quota reduction sequence σ. As shown in the proof of Theorem 10, QRDA$^\sigma$ terminates at stage k where $k \leq k'$. Since $q_c^k \geq q_c^{k'}$ holds for all $c \in C$ and DA satisfies resource monotonicity (described in the proof of Theorem 11), each student weakly prefers the matching obtained by QRDA$^\sigma$ over that by ACDA$^\sigma$. □

Since QRDA always obtains a (weakly) better matching for students than ACDA, one may assume that QRDA is always less wasteful than ACDA, i.e., more students claim empty seats in ACDA compared to QRDA. However, we cannot guarantee that QRDA is less wasteful than ACDA. Yahiro et al. [80] identified a case where the number of students who claim empty seats in QRDA exceeds that of ACDA under ratio constraints.

Theorem 19 (based on Theorem 11 by Yahiro et al. [80]) Given a balanced σ, a case exists where the number of students who claim empty seats in QRDA$^\sigma$ exceeds that of ACDA$^\sigma$.

Theorem 19 implies that there are cases where QRDA still has room for amelioration by mitigating students' claims compared to ACDA. Under ratio constraints, this QRDA's drawback is compensated by the fact that when ACDA returns a nonwasteful matching, QRDA returns the same nonwasteful matching [80]. Theorem 20 shows that this fact extends to a union of symmetric M-convex sets.

Theorem 20 Given a balanced σ, when ACDA$^\sigma$ returns a nonwasteful matching, QRDA$^\sigma$ and ACDA$^\sigma$ return the same matching.

Proof Assume that QRDA$^\sigma$ returns matching $\dot{Y}$ and that ACDA$^\sigma$ returns matching Y that differs from $\dot{Y}$. By contradiction assume also that Y is nonwasteful.

Consider a procedure that starts with matching $\dot{Y}$ and keeps applying QRDA's stages, i.e., reducing quotas and applying DA (even though $\dot{Y}$ is feasible) until the quotas reach the same quotas as in ACDA$^\sigma$, and thus the procedure returns Y. During this procedure, since Y differs from $\dot{Y}$, some rejection chains must occur and let r denote the last of these rejection chains. Assume rejection chain r decreases the student number in college c' and increases c''. Denote by Y^{-r} the matching returned by this procedure before rejection chain r happens, i.e., $\nu(Y^{-r}) = (|Y_{c_1}|, \ldots, |Y_{c'}| + 1, |Y_{c''}| - 1, \ldots, |Y_{c_m}|)$. Since $\dot{Y}$ is feasible, by Lemma 2, any matching returned during this procedure is feasible, in particular Y^{-r} is feasible. In the following, we show that one student involved in the rejection chain r is a claiming student in Y.

Notice that each college that rejects a student during rejection chain r has to be full in matching Y^{-r}, and can be either maximum or not in Y^{-r}, i.e., its student number can be $|Y^{-r}_{c'}|$ or $|Y^{-r}_{c'}| - 1$.

First, we assume that rejection chain r involves a rejection from a college that is maximum in Y^{-r}. Consider the first maximum college in Y^{-r} that rejects a student in r, denoted as c_b, and then $|Y^{-r}_{c_b}| = |Y^{-r}_{c'}|$. Assume that college c_b accepts student s_b (and then rejects another one), and that s_b was initially rejected by college c_a. We have two cases to consider: either $c_a = c'$ or $c_a \neq c'$.

If $c_a = c'$, consider matching $(Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c')\}$. Then note that $\nu((Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c')\}) = (|Y_{c_1}|, \dots, |Y_{c'}| + 1, |Y_{c_b}| - 1, \dots, |Y_{c_m}|)$. By assumption, $|Y^{-r}_{c_b}| = |Y^{-r}_{c'}|$, and since $|Y^{-r}_{c'}| = |Y_{c'}| + 1$ and $|Y^{-r}_{c_b}| = |Y_{c_b}|$, vector $\nu((Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c')\})$ is a permutation of Y. By symmetry, matching $(Y \setminus \{(s', c_b)\}) \cup \{(s', c')\}$ is feasible, and thus student s_b is a claiming student in Y.

Otherwise, $c_a \neq c'$, and consider matching $(Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c_a)\}$. Note that $\nu((Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c_a)\}) = (|Y_{c_1}|, \dots, |Y_{c_a}| + 1, |Y_{c_b}| - 1, \dots, |Y_{c_m}|)$. By definition of c_b, college c_a is not maximum in Y^{-r}, and thus $|Y^{-r}_{c_a}| = |Y^{-r}_{c'}| - 1$ holds. In addition with $|Y^{-r}_{c_b}| = |Y_{c_b}|$, it implies that vector $\nu((Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c_a)\})$ is a permutation of Y. By symmetry, matching $(Y \setminus \{(s_b, c_b)\}) \cup \{(s_b, c_a)\}$ is feasible, and thus student s_b is a claiming student in Y.

Therefore, assume no rejection exists in r that involves a college that is maximum in Y^{-r}. Consider the *last* rejection in rejection chain r, and assume that college c_d rejects student s_d who is then accepted by college c'' (and rejection chain r stops). Note that $\nu((Y \setminus \{(s_d, c'')\}) \cup \{(s_d, c_d)\}) = (|Y_{c_1}|, \dots, |Y_{c_d}| + 1, |Y_{c''}| - 1, \dots, |Y_{c_m}|)$. By assumption, since college c_d is not maximum in Y^{-r}, $|Y^{-r}_{c_d}| = |Y^{-r}_{c'}| - 1$. Then vector $\nu((Y \setminus \{(s_d, c'')\}) \cup \{(s_d, c_d)\})$ is a permutation of Y^{-r}. By symmetry, matching $(Y \setminus \{(s_d, c'')\}) \cup \{(s_d, c_d)\}$ is feasible, and thus student s_d is a claiming student in Y.
□

4.4 Generalized Asymmetric Constraints: Union of Symmetric M-convex Sets with an Offset

In Section 4.1.3, we introduced the union of symmetric M-convex set constraints, each component of which is symmetric. Symmetry assumes that all colleges have approximately equal size, which might not hold true in real-world scenarios where colleges have different sizes and enrollment capacities. To address such cases, we extend the symmetric M-convex set constraints by introducing an asymmetric factor called an *offset*.

4.4.1 Union of Symmetric M-convex Set Constraints with an Offset

A union of symmetric M-convex set constraints with an offset is defined as follows.

Definition 31 (Union of Symmetric M-convex Sets with an Offset) Set $F \subseteq \mathbb{N}_0^m$ is a union of symmetric M-convex sets with an offset if offset vector $\nu^* \in \mathbb{Z}^m$ and $F' \subseteq \mathbb{N}_0^m$ exist such that F' is a union of symmetric M-convex sets and F is given as $\{\nu \mid \nu' \in F', \nu = \nu' + \nu^*, \sum_{c \in C} \nu_c = n\}$.

In short, a union of symmetric M-convex sets with an offset is derived from a union of symmetric M-convex sets F' by adding offset vector ν^* to each element in F'. Note that offset vector ν^* may contain negative elements, as long as each element in $\nu \in F$ is non-negative.

One practical example of a union of symmetric M-convex sets with an offset is when each college requires a specific number of students to operate, represented as its minimum quota. More specifically, each college c has its own minimum quota p_c, which is defined analogously to Definition 20. However, these minimum quotas may vary based on the size of the colleges. We apply the union of symmetric M-convex constraints to the number of surplus students beyond p_c for all $c \in C$, rather than the number of overall students. In this market, the constraints are represented as follows.

1. The individual minimum quotas, $p_C = (p_{c_1}, \dots, p_{c_m})$, which is an m-dimensional vector and each p_c is the minimum quota of college c. We require $\sum_{c \in C} p_c < n$.
2. The constraints on the number of surplus students for each college beyond p_C are given as a union of m-dimensional symmetric M-convex sets $F \subseteq \mathbb{N}_0^m$.
3. Matching Y is feasible if $\nu(Y) - p_C \in F$ holds.

4.4.2 QRDA for Union of Symmetric M-convex Set Constraints with an Offset

To handle this generalized class of constraints, we slightly modify our proposed mechanism, QRDA, with different initial individual maximum quotas to handle the asymmetry introduced by the offset vector. Recall that under the constraints in Section 4.1.3, QRDA commences with uniform maximum quota ν^{max}, which satisfies $\nu^{max} \geq \max_{\nu \in F, i \in M} \nu_i$ for each college, and reduces the colleges' quotas based on a given sequence σ. On the other hand, for any union of symmetric M-convex sets with an offset F, we define $\tilde{\nu}^{max}$ as a value that satisfies $\tilde{\nu}^{max} \geq \max_{\nu' \in F', i \in M} \nu'_i$. For each college c, ν_c^{max} is defined as $\tilde{\nu}^{max} + \nu_c^*$. In short, ν_c^{max} is the sum of the common upper-bound $\tilde{\nu}^{max}$ and the offset of c. The modified QRDA sets q_c^1 to ν_c^{max}, signifying that the initial maximum quota for each college is an upper-bound for any possible value in F. Intuitively, in the generalized constraints handled in this section, a set of "most balanced vectors" still exists, although they have been shifted by the offset vector. The constraints are therefore symmetric and M-convex around the most balanced vectors. Therefore, the modified QRDA ensures that any returned matching still satisfies all the properties described above in this section. Applying similar arguments, we deduce the following theorem.

Theorem 21 In any market under a union of symmetric M-convex set constraints with an offset, the modified QRDA is feasible, fair, and strategyproof.

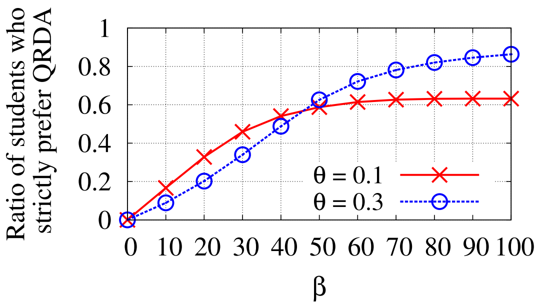
4.5 Empirical Evaluation

Concerning student welfare, Theorem 18 guarantees that students weakly prefer QRDA over ACDA. To delve deeper into the quantitative disparities in student welfare between QRDA and ACDA, we rely on computer simulations for a comprehensive analysis. On the other hand, Theorem 19 shows that we cannot guarantee that QRDA is always better than ACDA in terms of nonwastefulness. Neverthless, our conjecture remains that situations in which QRDA is more wasteful than ACDA are likely to be rare and atypical; in the majority of cases, QRDA should prove superior. This conjecture is substantiated by our computer simulations. We undertake extensive investigations into these inquiries concerning both difference constraints and flexible uniform min/max quotas constraints.

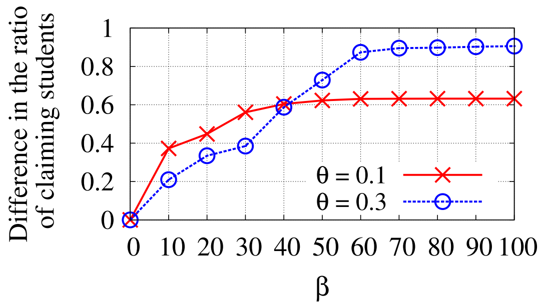
Additionally, for difference constraints, we explore an alternative indirect approach, whereby the union of symmetric M-convex set constraints is transformed into individual min/max quotas constraints. Within this method, each college c has its minimum quota p_c as well as its maximum quota q_c. At least p_c and at most q_c students have to be assigned to college c. Fragiadakis et al. [22] presented a strategyproof and fair mechanism called Extended Seat Deferred Acceptance (ESDA) to handle individual min/max quotas constraints. As previously mentioned in Section 4.1.3, difference constraints with parameter β can be represented by a union of uniform min/max quotas constraints. To adapt ESDA to handle difference constraints, we calculate uniform min/max quotas $\tilde{p}$ and $\tilde{q}$: choose the maximum value of minimum quota $\tilde{p}$, with corresponding maximum quota $\tilde{q} = \tilde{p} + \beta$, such that $\tilde{p}$ and $\tilde{q}$ satisfy $\tilde{q} + (m-1)\tilde{p} \leq n \leq (m-1)\tilde{q} + \tilde{p}$. In simpler terms, we choose the largest value of $\tilde{p}$ among the union of uniform min/max quotas constraints. The minimum quota $\tilde{p}$ signifies the minimum number of students to be allocated to each college, while the maximum quota $\tilde{q}$ allows for more students to attend popular colleges. We implemented alternative methods for choosing $\tilde{p}$ and $\tilde{q}$, such as selecting the smallest $\tilde{p}$ from the union of uniform min/max quotas constraints [2].

We consider a market with $n = 800$ students and $m = 20$ colleges. We generate student preferences using the Mallows model. To ensure reliable findings, we

[2]Remarkably, the results obtained from these alternative methods were similar.



(a) Ratio of students who strictly prefer QRDA to ACDA



(b) Difference between ratio of claiming students in QRDA and in ACDA

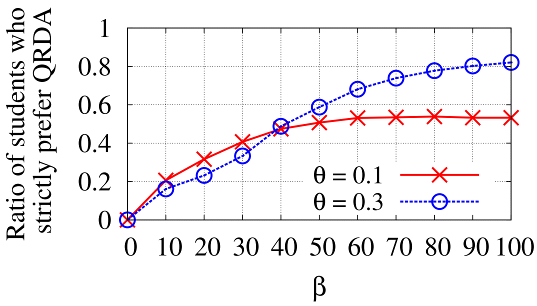
Figure 4.2: Comparisons between QRDA and ACDA under difference constraints

choose two values for θ_S, 0.1 and 0.3, reflecting real-world diversity in preferences and some colleges being more preferred. The priority ranking of each college c is drawn uniformly at random. We create 100 problem instances for each parameter setting.

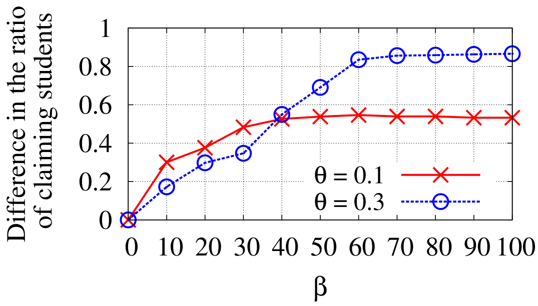
We first conduct a computer simulation with *difference constraints* (Definition 25) to quantitatively measure the weak domination of QRDA over ACDA and to show that QRDA outperforms ACDA in terms of nonwastefulness.

In Figure 4.2 (a), we show the proportion of students who strictly prefer QRDA over ACDA depending on the allowed difference β in Definition 25. When $\beta = 10$ and $\theta_S = 0.1$, approximately 18% of the students strictly prefer QRDA's outcome; this number increases before plateauing at 60% when $\beta = 50$. As β increases, since the set of feasible sets expands, the potential of amelioration with QRDA increases as well. We expect policymakers to prefer QRDA over ACDA since it outperforms ACDA and a significant amount of students strictly prefer it. When $\theta_S = 0.3$, the competition among students rises since their preferences tend to be more similar. Compared to $\theta_S = 0.1$, the improvement by QRDA is smaller when β is small ($\beta \leq 45$) and larger when $\beta \geq 45$. A reason might be that as the competition becomes more severe ($\theta_S = 0.3$), it is more difficult to enhance students' welfare since the resources in popular colleges are limited even though QRDA has more flexibility.

To compare the nonwastefulness of QRDA and ACDA, we measure the proportion of claiming students, i.e., students who claim an empty seat, in both mechanisms and show the difference in Figure 4.2 (b), i.e., by plotting $(|S_{\text{ACDA}}| - |S_{\text{QRDA}}|)/n$, where S_{ACDA} (respectively, S_{QRDA}) denotes the set of claiming students in ACDA (respectively, QRDA). This value is always positive for both $\theta_S = 0.1$ and $\theta_S = 0.3$, which means that on average more students claim empty seats in ACDA than in QRDA. When $\theta_S = 0.1$, this number is 40% for $\beta = 10$ and increases to approximately 60% for $\beta = 40$ before plateauing. Similarly, this value is lower (less wasteful) for $\theta_S = 0.3$ when $\beta < 40$ and higher (more wasteful) when $\beta > 40$. We conclude that QRDA is less wasteful than ACDA in a difference constraints setting. Similar to Figure 4.2 (a), for both cases of $\theta_S = 0.1$ and $\theta_S = 0.3$, QRDA's amelioration becomes greater as β increases.



(a) Ratio of students who strictly prefer QRDA to ESDA



(b) Difference between ratio of claiming students in QRDA and in ESDA

Figure 4.3: Comparisons between QRDA and ESDA under difference constraints

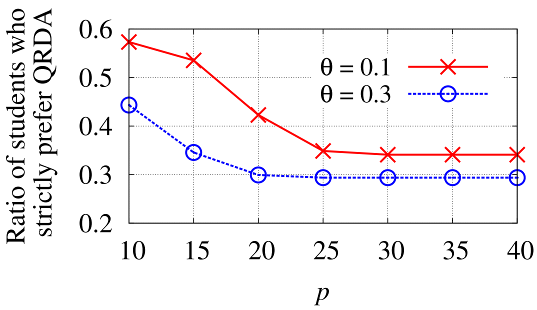
We next compare QRDA and ESDA through similar simulations. In Figure 4.3 (a), we plot the ratio of students who strictly prefer QRDA over ESDA depending on β. The trend closely resembles Figure 4.2 (a). This is reasonable since, similar to ACDA, many students have to be assigned to less preferred colleges to meet the minimum quotas in ESDA. The difference at the plateau is slightly smaller than that of Figure 4.2 when both $\theta_S = 0.1$ and 0.3.

Figure 4.3 (b) presents the difference in the ratio of the claiming students between QRDA and ESDA. The graph resembles Figure 4.2. Overall, we confirm that QRDA experimentally outperforms ACDA and ESDA under difference constraints.

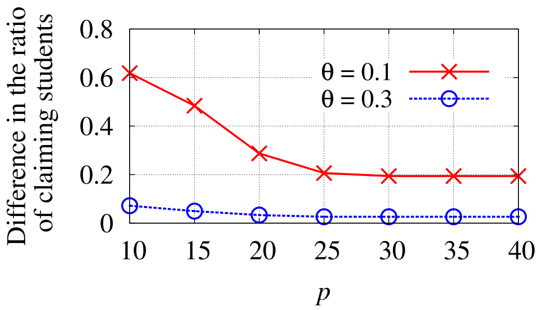
Finally, we conduct computer simulations with *flexible uniform min/max quotas constraints* (Definition 26). Among the three variables, p, q and d, in these constraints, we choose to fix $q = 60$ and vary p and d since they have the greatest impact on feasibility. When varying q, the results were almost constant since maximum quotas only impact the most popular colleges.

Figure 4.4 (a) shows the ratio of students who strictly prefer QRDA over ACDA, and Figure 4.4 (b) shows the difference in the ratio of claiming students between QRDA and ACDA when minimum quota p is varied. We set $d = 100$. Both graphs gradually decrease as p increases until reaching a stabilized value. This result confirms that QRDA outperforms ACDA in this setting. It further shows that the improvement obtained by QRDA is limited when the minimum quotas are large. This is because a larger minimum quota can significantly shrink the constraint set, which leads to much less flexibility. Therefore, it limits the QRDA performance to a large extent.

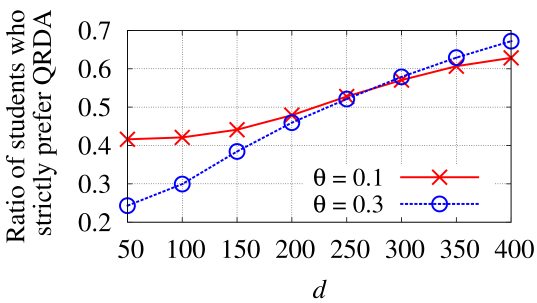
We present similar simulations when varying distance parameter d in Figures 4.4 (c) and (d). We set $p = 20$. Figure 4.4 (c) shows that, as d increases, the ratio of students who strictly prefer QRDA over ACDA increases. Even with higher competition ($\theta_S = 0.3$), QRDA significantly outperforms ACDA when d is large. In Figure 4.4 (d), the trend is similar for $\theta_S = 0.1$, but for the averages of $\theta_S = 0.3$, the curve only increases slightly.



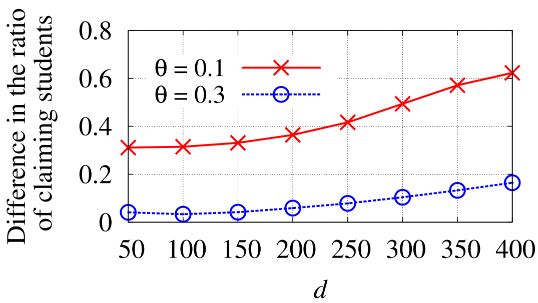
(a) Ratio of students who strictly prefer QRDA to ACDA



(b) Difference between ratio of claiming students in QRDA and ACDA



(c) Ratio of students who strictly prefer QRDA to ACDA



(d) Difference between ratio of claiming students in QRDA and in ACDA

Figure 4.4: Comparisons between QRDA and ACDA under flexible uniform min/max quotas constraints

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Chapter 5

Initial Endowment Settings

In this chapter, we model the problem where each student is endowed with an initial college. Essentially, we assume $\omega(s) \neq \emptyset$ for every $s \in S$.

5.1 Model

Let $\tilde{Y}$ denote the initial endowment, i.e., for all $s \in S$, $\tilde{Y}(s) = \omega(s)$. Matching Y is *individually rational* if $Y_s \succeq_s \omega(s)$ holds for all $s \in S$. In other words, individual rationality requires that each student is matched to a college that is at least as good as her initial endowment college. Additionally, for any student s, any college, which is ranked weakly higher than $\omega(s)$, is *acceptable* to s. We include individual rationality condition in the feasibility with distributional constraints (Definition 1). We assume that initial endowment matching $\tilde{Y}$ is feasible. We also posit that all students and colleges regard $\emptyset$ as the worst-ranked on $\succ_S$ and $\succ_C$.

5.2 Impossibility Theorems

Fairness and individual rationality are generally incompatible under the initial endowment settings.

Theorem 22 Fairness and individual rationality are incompatible.

Proof Consider a market, in which $S = \{s_1, s_2\}$, $C = \{c_1, c_2\}$, $\omega(s_1) = c_1$, $\omega(s_2) = c_2$, and $F = \{(1, 1)\}$. Both students prefer c_1 over c_2, and both colleges prefer s_2 over s_1. Since a fair matching must assign s_2 to c_1, s_1 must be assigned to c_2. However, this violates individual rationality. □

Thus, we introduce a weaker version of fairness, which we call fairness among non-initial endowment students. Intuitively, any justified envy toward student s, who is assigned to college c, is not deemed legitimate if s is an initial endowment student of c.

Definition 32 (Fairness among Non-Initial Endowment Students (NIE-fairness)) In matching Y, where $(s, c) \in Y$, student s has justified envy respecting initial endowments (RIE-envy) toward another student s' if $\exists c' \in C$, such that $(s, c') \in X \setminus Y$, $c' \succ_s c$, $(s', c') \notin \tilde{Y}$, and $s \succ_{c'} s'$ hold. Matching Y is fair among non-initial endowment students if no student has justified RIE-envy in Y.

Unfortunately, NIE-fairness and nonwastefulness are generally incompatible.

Theorem 23 NIE-fairness and nonwastefulness are incompatible.

Proof Consider $S = \{s_1, s_2\}$, $C = \{c_1, c_2, c_3\}$, and $\omega(s_1) = \omega(s_2) = c_1$, $\omega(s_1) = \omega(s_2) = c_1$. The following are the preferences of the students and colleges.

$$\begin{array}{l} s_1 : c_2 \succ_{s_1} c_3 \succ_{s_1} c_1 \\ s_2 : c_3 \succ_{s_2} c_2 \succ_{s_2} c_1 \end{array} \qquad \begin{array}{l} c_1 : s_1 \succ_{c_1} s_2 \\ c_2 : s_2 \succ_{c_2} s_1 \\ c_3 : s_1 \succ_{c_3} s_2 \end{array}$$

The set of feasible vectors F is given as follows.

$$F = \left\{ \nu \in \mathbb{Z}^3_{\geq 0} \;\middle|\; \begin{array}{c} \sum_{i \in M} \nu_i = 2, \\ 1 \leq \nu_1 \leq 2, \\ 0 \leq \nu_2, \nu_3 \leq 1. \end{array} \right\}$$

Set F represents a situation where the minimum quota of c_1 is one, i.e., at least one student must be assigned to c_1 (while the other colleges do not have any minimum quota). To satisfy this minimum quota, c_2 and c_3 can accept at most one student.

By the student preference profile, even though c_1 is the least popular college for both s_1 and s_2, at least one student must be assigned to it since $1 \leq \nu_1 \leq 2$. Assume s_1 is assigned to c_1. Then s_2 must be assigned to her favorite college c_3, or otherwise s_2 claims an empty seat in c_3. However, s_1 then has justified RIE-envy toward s_2 since $s_1 \succ_{c_3} s_2$. Similarly, assume that s_2 is assigned to c_1. Then s_1 must be assigned to her favorite college c_2, or otherwise s_1 claims an empty seat in c_2. However, s_2 then has justified RIE-envy toward s_1 since $s_2 \succ_{c_2} s_1$. $\square$

Since NIE-fairness and nonwastefulness are generally incompatible, we introduce a weaker version of nonwastefulness called nonwastefulness based on ranks. To define it, we introduce a concept called the *rank* of a student for each college.

Definition 33 (Rank) For any contract $(s, c) \in X$, $rank((s, c))$ is defined as follows:

$$rank((s,c)) = \begin{cases} 0 & \text{if } \omega(s) = c, \\ |\{s' \in S \mid \omega(s') \neq c, s' \succ_c s\}| + 1 & \text{otherwise.} \end{cases}$$

In other words, $rank((s, c))$ denotes the ranking of student s by college c based on $\succ_c$, except for c's initial endowment students. If $\omega(s) = c$, then $rank((s, c)) = 0$. If student s is ranked highest by c among all the students in $\{s \in S \mid \omega(s) \neq c\}$, $rank((s, c)) = 1$, and if s is ranked second, $rank((s, c)) = 2$, and so on. Nonwastefulness based on ranks is defined as follows.

Definition 34 (Nonwastefulness Based on Ranks) Given matching Y, assume that contract $(s, c) \in Y$. Student s claims an empty seat in c' based on ranks if all of the following conditions hold:
(i) $(s, c') \in X \setminus Y$,
(ii) $c' \succ_s c$,
(iii) $(Y \setminus \{(s, c)\}) \cup \{(s, c')\}$ is feasible, and
(iv) $rank((s, c')) < rank((s, c))$.
Matching Y is nonwasteful based on ranks if no student claims an empty seat based on ranks.

Conditions (i) – (iii) are equivalent to the conditions that a student claims an empty seat in Definition 4. Intuitively, a student claims an empty seat in a more preferred college c' *based on ranks* if she is ranked higher in c' than in her current allocated college c. Obviously, if a mechanism is nonwasteful, it is also nonwasteful based on ranks.

Next we introduce another property called *core*.

Definition 35 (Core) Feasible matching Y is in the *strong* core if no feasible matching Y' and group of students $S' \subseteq S$ exist such that (i) for any $c \in C$, $|\{(s, c) \in Y' \mid s \in S'\}| = |\{(s, c) \in \tilde{Y} \mid s \in S'\}|$ and (ii) $\forall s \in S'$, $Y'_s \succeq_s Y_s$, and $\exists s \in S'$, $Y'_s \succ_s Y_s$. Feasible matching Y is in the *weak* core if no feasible matching Y' and group of students $S' \subseteq S$ exist such that (i) for any $c \in C$, $|\{(s, c) \in Y' \mid s \in S'\}| = |\{(s, c) \in \tilde{Y} \mid s \in S'\}|$ and (ii) $\forall s \in S', Y'_s \succ_s Y_s$ holds.

In other words, in any matching in the strong core (resp. weak core), no group of students can be (resp. strictly) better off by swapping their initial endowment colleges among themselves (if such a group exists, it is called a *blocking coalition*). A strong core is a subset of the weak core.

Roth and Postlewaite [67] showed that the matching obtained by the mechanism called Top Trading Cycles is the unique element in the strong core when the objects assigned to students are completely distinct (e.g., a housing market). In our setting, we allocate college seats to students. college seats are not completely distinct: all seats in one college are identical. The following example shows that the *strong* core can be empty in our setting.

Theorem 24 A strong core may be empty.

Proof Consider $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2\}$, and $\omega(s_1) = \omega(s_2) = c_1$, $\omega(s_3) = c_2$. The following are the preferences of students.

$$s_1 : c_2 \succ_{s_1} c_1$$
$$s_2 : c_2 \succ_{s_2} c_1$$
$$s_3 : c_1 \succ_{s_3} c_2$$

The set of feasible vectors F contains only one element: $(2, 1)$. Each element in the strong core must be Pareto efficient; otherwise, the grand coalition (the coalition of all students) is a blocking coalition. There are two Pareto efficient matchings: $Y = \{(s_1, c_2), (s_2, c_1), (s_3, c_1)\}$ and $Y' = \{(s_1, c_1), (s_2, c_2), (s_3, c_1)\}$.

For Y, s_2 and s_3 form a blocking coalition since by swapping their initial endowment colleges, s_2 becomes strictly better while s_3 is indifferent. Similarly, for Y', s_1 and s_3 form a blocking coalition since by swapping their initial endowment colleges, s_1 strictly improves while s_3 is indifferent. However these two matchings are both in the weak core. □

In the next theorem, we show that feasibility, Pareto efficiency, and strategyproofness are incompatible when the set of feasible vectors is *not* an M-convex set.

Theorem 25 A market exists where the set of feasible vectors is not M-convex, such that no mechanism simultaneously satisfies feasibility, Pareto efficiency, and strategyproofness.

Proof The theorem can be proved even with a simple market with two students, three colleges, and the set of feasible vectors becomes an M-convex set by just adding one vector. Consider $S = \{s_1, s_2\}$, $C = \{c_1, c_2, c_3\}$, and $\omega(s_1) = \omega(s_2) = c_1$. The following are the preferences of students.

$$s_1 : c_3 \succ_{s_1} c_2 \succ_{s_1} c_1$$
$$s_2 : c_3 \succ_{s_2} c_2 \succ_{s_2} c_1$$

The set of feasible vectors F is given as $\{(2, 0, 0), (1, 1, 0), (0, 1, 1)\}$.

This market can be interpreted as follows. Within a college district, there are three colleges, c_1, c_2, and c_3, where c_1 has a larger capacity than the others. Due to the constraints on the district's logistic resources, at most two colleges can be simultaneously operated only if they are close to each other (i.e., c_1 and c_2, or c_2 and c_3), otherwise only c_1 is open. Note that F is not an M-convex set by the following argument. For $\nu = (2, 0, 0)$ and $\nu' = (0, 1, 1)$, consider $\nu_3 < \nu'_3$, and then $\nu_j > \nu'_j$ holds only for $j = 1$, but $\nu + e_3 - e_1 = (1, 0, 1)$ is not in F. However, $F \cup \{(1, 0, 1)\}$ is an M-convex set.

In this market, there are two feasible and Pareto effieicnt matchings, $\{(s_1, c_3), (s_2, c_2)\}$ and $\{(s_1, c_2), (s_2, c_3)\}$. Assume that Pareto efficient mechanism φ chooses $\varphi(\succ_S) = \{(s_1, c_3), (s_2, c_2)\}$. Then s_2 can misreport $\succ'_{s_2}$: $c_3 \succ'_{s_2} c_1 \succ'_{s_2} c_2$. With this misreport, the only feasible and Pareto effieicnt matching is $\varphi(\succ_{s_1}, \succ'_{s_2}) = \{(s_1, c_2), (s_2, c_3)\}$ and $\varphi_{s_2}(\succ_{s_1}, \succ'_{s_2}) \succ_{s_2} \varphi_{s_2}(\succ_{s_1}, \succ_{s_2})$. Similarly, s_1 has an incentive to misreport if $\varphi(\succ_S) = \{(s_1, c_2), (s_2, c_3)\}$. In both cases, since a student can benefit by misreporting, φ cannot satisfy strategyproofness. □

This theorem implies that the violation of M-convexity easily leads to the nonexistence of any feasible, strategyproof, and Pareto efficient mechanism. In Section 5.4, we show that if the distributional constraints of a market can be represented as an M-convex set, a mechanism exists that satisfies strategyproofness and Pareto efficiency. Our conjecture is that M-convexity would be the most general classes of distributional constraints under which we can still have a mechanism with such desirable properties.

5.3 Properties of M-convex Set

In this section, we present several properties associated with M-convexity, crucial for subsequent sections. These properties are either well-documented in the existing literature or possess relatively straightforward proofs. For the sake of completeness, we provide proofs.

Lemma 6 (Fujishige [23, Lemma 4.5], Murota [55, Lemma 9.23]) Let F be an M-convex set. For some $\nu \in F$, there exist $i_1, j_1, \dots, i_r, j_r$, all of which are in M and distinct, such that

$$\begin{cases} \nu + e_{i_h} - e_{j_k} \in F & \text{if } h = k \\ \nu + e_{i_h} - e_{j_k} \notin F & \text{if } h > k \end{cases} \qquad (h, k \in \{1, \dots, r\}). \tag{5.1}$$

Then it holds that $\nu + \sum_{k \in \{1,\dots,r\}} (e_{i_k} - e_{j_k}) \in F$.

Proof The proof is conducted by induction on r. When $r = 1$, it obviously holds. Assume the supposition is true up to $r = \ell$ and consider a case where $r = \ell + 1$. Take two vectors, $a = \nu + (e_{i_1} - e_{j_1})$ and $b = \nu + \sum_{k=2}^{r} (e_{i_k} - e_{j_k})$. It holds that $a \in F$ by assumption. It also holds that $b \in F$ from induction, because $i_2, j_2, \dots, i_r, j_r$ are 2ℓ (which equals $2(r-1)$) distinct elements in M, which satisfy Equation (5.1). Since $i_1, j_1, \dots, i_r, j_r$ are distinct, it holds that $a_{i_1} > b_{i_1}$. It also holds that $\{k \in M \mid a_k < b_k\} = \{j_1, i_2, i_3, \dots, i_r\}$. From the M-convexity of F, $j \in \{j_1, i_2, i_3, \dots, i_r\}$ must exist such that $a + (e_j - e_{i_1}) = \nu + (e_j - e_{j_1}) \in F$. It follows that j_1 is the only candidate, since $\nu + e_{i_h} - e_{j_1} \notin F$ for any i_h, $h > 1$. It then follows that $b - (e_{j_1} - e_{i_1}) = b + (e_{i_1} - e_{j_1}) = \nu + \sum_{k \in \{1,\dots,r\}} (e_{i_k} - e_{j_k}) \in F$. □

In other words, Lemma 6 means that we can unilaterally apply feasible moves (i.e., to reduce an element by one and simultaneously increase another by one) simultaneously if they are *sorted properly* in some sense.

Lemma 7 Let F be an M-convex set and assume vector $\nu \in F$. Let $J = \{1, \dots, r\}$, and for a fixed $q \in J$, if elements $i_1, j_1, \dots, i_r, j_r \in M$ are given such that $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_r\} = \emptyset$ and

$$\begin{cases} \nu + e_{i_h} - e_{j_k} \in F & \text{if } h = k \neq q \\ \nu + e_{i_h} - e_{j_k} \notin F & \text{if } h = k = q \\ \nu + e_{i_h} - e_{j_k} \notin F & \text{if } h > k \end{cases} \qquad (h, k \in J) \tag{5.2}$$

hold, then we have

$$\nu + \sum_{\ell \in J} (e_{i_\ell} - e_{j_\ell}) \notin F.$$

Proof Assume to the contrary that $\nu' = \nu + \sum_{\ell \in J} (e_{i_\ell} - e_{j_\ell})$ is in F. If $r > q$, by the exchange property for ν', ν and i_r with $\nu'_{i_r} > \nu_{i_r}$, by $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_r\} = \emptyset$, k exists such that $\nu'_{j_k} < \nu_{j_k}$ and

$$\nu' - e_{i_r} + e_{j_k},\ \nu + e_{i_r} - e_{j_k} \in F.$$

Furthermore, from Equation (5.2), k must equal r. Then we have $\nu + \sum_{\ell \in J \setminus \{r\}} (e_{i_\ell} - e_{j_\ell}) \in F$. By repeating the same argument, we can assume that $q = r$. Since $\nu'_{i_q} > \nu_{i_q}$ holds, by the exchange property for ν', ν and i_q, k exists such that $\nu'_{j_k} < \nu_{j_k}$ and

$$\nu' - e_{i_q} + e_{j_k},\ \nu + e_{i_q} - e_{j_k} \in F.$$

However, all of the elements of J are less than or equal to q, which contradicts Equation (5.2). □

In other words, Lemma 7 states that if a move is infeasible, it remains infeasible even after applying a set of feasible moves.

Lemma 8 Assume that for $\nu \in F$, $I, J \subset M$ exist, $I \cap J = \emptyset$ such that $\forall i \in I$, $\forall j \in J$, the following condition holds:

$$\nu + e_i - e_j \notin F. \tag{5.3}$$

Then no $\nu' \in F$ exists such that the following condition holds:

$$\forall i \in M \setminus J,\ \nu'_i \geq \nu_i \text{ and } \exists j \in I,\ \nu'_j > \nu_j. \tag{5.4}$$

Proof Assume to the contrary that ν' exists and satisfies Equation (5.4). Since F is an M-convex set, for $\nu', \nu \in F$ and $i \in I$ with $\nu'_i > \nu_i$, j exists such that $\nu'_j < \nu_j$ and

$$\nu' - e_i + e_j,\ \nu + e_i - e_j \in F.$$

By Equation (5.4), j must belong to J, which contradicts Equation (5.3). □

In other words, Lemma 8 means that if we cannot move one student from J to I, then we cannot increase the number of students in I without decreasing the number of students in $M \setminus J$.

5.4 Top Trading Cycles under M-convex Constraints

In this section, we introduce a strategyproof mechanism, Top Trading Cycles under M-convex constraints (TTC-M), which achieves a feasible and Pareto efficient outcome in our settings.

We introduce some notions with which we describe our mechanism. A directed graph is a pair (V, E), where V is a set of vertices and $E \subseteq V \times V$ is a collection of directed edges, i.e., any $(i, j) \in E$ is an ordered pair of vertices. A sequence of distinct vertices, $(i_1, \ldots, i_k)$, with $k \geq 2$, is a directed path in (V, E) from i_1 to i_k if $(i_h, i_{h+1}) \in E$ for all $h \in \{1, \ldots, k-1\}$. A directed path is a *cycle* if the first and last vertices coincide.

First we outline TTC-M. It repeats several rounds. At the beginning of round k, let Y^{k-1} denote the matching of students who have already left the market, and let $\tilde{Y}^{k-1}$ denote the initial endowment matching of the remaining students. Then $\widehat{Y}^{k-1} = Y^{k-1} \cup \tilde{Y}^{k-1}$ denotes the provisional matching at the beginning of round k.

Let S^k denote $\{s \mid (s, c) \in \tilde{Y}^{k-1}\}$, i.e., the set of remaining students at round k. Student $s \in S^k$, whose initial endowment college is c_j, is *admissible* to college c_i at round k if $\nu(\widehat{Y}^{k-1}) + e_i - e_j \in F$ holds, i.e., we can move s to c_i from her initial endowment college c_j without violating distributional constraints. If there is no student admissible to college c in S^k, the college leaves the market at the beginning of round k. Let C^k denote the set of remaining colleges at the beginning of round k. Note that by this definition, if $\omega(s) = c$, s is always admissible to c at any round k as long as $\nu(\widehat{Y}^{k-1}) \in F$ holds.

The mechanism uses a college preference profile $\succ^*_C$, which is artificially constructed based on a common serial order over students $\succ$.[1] We assume $\succ$ is complete and strict over S. Based on $\succ$, the (artificially constructed) preference of each college $c \in C$, $\succ^*_c$, is constructed, which is basically identical to $\succ$, but all initial endowment students of c are prioritized. More specifically, $s \succ^*_c s'$ holds if and only if one of the following conditions holds:
(i) $\omega(s) = \omega(s') = c$ and $s \succ s'$,
(ii) $\omega(s) \neq c$, $\omega(s') \neq c$, and $s \succ s'$, or
(iii) $\omega(s) = c$ and $\omega(s') \neq c$.
Without loss of generality, we assume $s_1 \succ s_2 \succ \cdots \succ s_n$ holds.

Now we are ready to introduce TTC-M, which is described as in Mechanism 9. Intuitively, we can assume that in TTC-M at each round k, each college selects one student and gives her the right to obtain its seat. Then students with such rights can trade seats among themselves by constructing trading cycles in G^k by the standard TTC mechanism. Therefore, a student can only participate in a trade when she is selected by some college. By definition of $\succ^*_c$, the priority right of college c is first given to its initial endowment students, where ties are broken by the common serial order $\succ$. When all of its initial endowment students have left the market, college c gives the right to a remaining student who is admissible (i.e., unilaterally moving her to c from her initial endowment college does not violate the distributional constraints), where ties are broken by $\succ$.

Since a student considers her initial endowment college acceptable, and the college deems her admissible at any round as long as she remains in the market, she is guaranteed a seat in her initial endowment college. Thus, every student is included in a cycle at some round before TTC-M terminates.

One particular feature of TTC-M is how it deals with the underlying distributional constraints. At round k of TTC-M, the priority right of a college is given to a student based on $\succ^*_C$, $\nu(\widehat{Y}^{k-1})$, and F. For instance, if the distributional constraints are individual maximum quotas, a college gives its priority right to a student if the

[1] In specific applications, colleges (as well as students) may come to a consensus on a preferred order, possibly based on measures like GPA. This order is also employed in SD, which is feasible, strategyproof, and Pareto efficient, albeit lacking individual rationality. If no student has an initial endowment college, TTC-M becomes identical to SD.

Mechanism 9 Top Trading Cycles for M-convex Constraints (TTC-M)

Input: $S, C, X, \succ_S, \succ_C, \succ_C^*, \omega, f$
Output: matching Y
Initialization: $\tilde{Y}^0 \leftarrow \tilde{Y}, Y^0 \leftarrow \emptyset, k \leftarrow 1$.
Round k:
Step 1: Construct directed graph G^k as follows.
- Each college c leaves the market if it has no admissible student in S^k. Otherwise, c points to an admissible student who is ranked the highest according to $\succ_c^*$ in S^k.
- Each student in S^k points to her favorite college in C^k.
- This creates directed graph $G^k = (V^k, E^k)$, where $V^k = S^k \cup C^k$, $(s, c) \in E^k$ (resp. $(c, s) \in E^k$) represents the fact that student s (resp. college c) points to college c (resp. student s).

Step 2: Let $\mathscr{C}^k$ be the set of all directed edges that form cycles in G^k. Since V^k is finite, at least one cycle exists in G^k and thus $\mathscr{C}^k$ is nonempty.
Step 3: $\tilde{Y}^k \leftarrow \tilde{Y}^{k-1} \setminus \{(s, \omega(s)) \mid (s, c) \in \mathscr{C}^k\}$ and $Y^k \leftarrow Y^{k-1} \cup \{(s, c) \mid (s, c) \in \mathscr{C}^k\}$.
Each student s such that $(s, c) \in \mathscr{C}^k$ leaves the market.
Step 4: If $\tilde{Y}^k = \emptyset$, then return Y^k. Otherwise, $k \leftarrow k + 1$ and go to the next round.

number of allocated students is fewer than the quota. Under more complex distributional constraints, such as minimum quotas and/or regional quotas, just looking at the current status and its own quota is insufficient for a college to determine which student it should prioritize. For example, allocating a student to college c', who was initially endowed a seat in another college c, decreases the number of students allocated to c, i.e., $\nu_c(\widehat{Y}^{k-1})$. This decision might violate distributional constraints, even though c' can accept one more student. The next example describes how TTC-M works with regional quotas constraints. In the example, for each student $s \in S$, we only list in $\succ_s$ her acceptable colleges.

Example 9 Consider $S = \{s_1, s_2, s_3, s_4, s_5\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $\succ$: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5$. The following are the preferences of students.

$$
\begin{aligned}
s_1 &: c_2 \succ_{s_1} c_1 = \omega(s_1) \\
s_2 &: c_3 \succ_{s_2} c_2 = \omega(s_2) \\
s_3 &: c_2 \succ_{s_3} c_3 = \omega(s_3) \\
s_4 &: c_3 \succ_{s_4} c_4 = \omega(s_4) \\
s_5 &: c_2 \succ_{s_5} c_4 = \omega(s_5)
\end{aligned}
$$

The set of feasible vectors F is given as follows.

$$
F = \left\{ \nu \in \mathbb{Z}_{\geq 0}^4 \;\middle|\; \begin{array}{c} \sum_{i \in M} \nu_i = 5, \\ 0 \leq \nu_i \leq 2 \quad \forall i \in M, \\ 2 \leq \nu_3 + \nu_4 \leq 3. \end{array} \right\}
$$

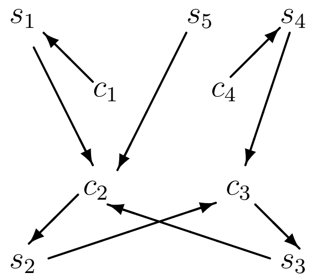


Figure 5.1: G^1 obtained in Round 1 by TTC-M in Example 9

Set F represents the situation where the colleges in the same region (c_3 and c_4) are jointly subjected to the regional minimum and maximum quotas in addition to the individual maximum quota of 2.

First, $\tilde{Y}^0$ is determined: $\tilde{Y}^0 = \tilde{Y} = \{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_4), (s_5, c_4)\}$. Note that $\nu(\tilde{Y}^0) = (1, 1, 1, 2) \in F$.

At Step 1 of Round 1, since every college still has its initial endowment student in the market, all the colleges remain in the market. Each college c points to a student based on $\succ^*_c$, $\nu(\widehat{Y}^0)$, and F. In this round, each college points to its initial endowment student who is ranked highest according to $\succ$. Each student points to her favorite college that remains in the market. This results in G^1, as shown in Figure 5.1. There is one cycle: $(c_2, s_2, c_3, s_3, c_2)$. At Step 2, $\mathscr{C}^1$ is $\{(c_2, s_2), (s_2, c_3), (c_3, s_3), (s_3, c_2)\}$. At Step 3, (s_2, c_2) and (s_3, c_3) are removed from $\tilde{Y}^0$, and (s_2, c_3) and (s_3, c_2) are added to Y^0. $\tilde{Y}^1$ and Y^1 are determined:

$$\begin{aligned}\tilde{Y}^1 &= \{(s_1, c_1), (s_4, c_4), (s_5, c_4)\},\\ Y^1 &= \{(s_2, c_3), (s_3, c_2)\}.\end{aligned}$$

Note that $\nu(\widehat{Y}^1) = \nu(\widehat{Y}^0) = (1, 1, 1, 2)$, since at this round s_2 and s_3 *exchange* the seats of their initial endowment colleges and thus the distributional vector does not change. At Step 4, TTC-M goes to Round 2 because $\tilde{Y}^1 \neq \emptyset$.

At Step 1 of Round 2, colleges c_2 and c_3 do not have their initial endowment students. college c_2 points to s_1 because based on $\succ^*_{c_2}$, she is the highest among the remaining students and allocating her to c_2 from her initial endowment college c_1 does not violate distributional constraints ($(1, 1, 1, 2) + e_{c_2} - e_{c_1} = (0, 2, 1, 2) \in F$); that is, she is admissible to c_2 at Round 2. To college c_3, however, s_1 is not admissible because $(1, 1, 1, 2) + e_{c_3} - e_{c_1} = (0, 1, 2, 2) \notin F$ due to the maximum quota of the region containing c_3 and c_4. On the other hand, moving a student from c_4 to c_3 is feasible. Thus, c_3 points to s_4 according to $\succ^*_{c_3}$. Therefore, G^2 is determined (Figure 5.2). There are two cycles: (c_2, s_1, c_2) and (c_3, s_4, c_3). At Step 2, $\mathscr{C}^2$ is $\{(c_2, s_1), (s_1, c_2), (c_3, s_4), (s_4, c_3)\}$. $\tilde{Y}^2$ and Y^2 are determined:

$$\begin{aligned}\tilde{Y}^2 &= \{(s_5, c_4)\},\\ Y^2 &= \{(s_2, c_3), (s_3, c_2), (s_1, c_2), (s_4, c_3)\}.\end{aligned}$$

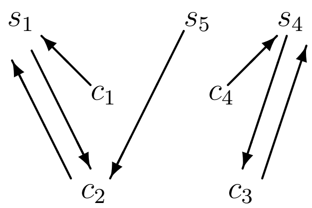


Figure 5.2: G^2 obtained in Round 2 by TTC-M in Example 9

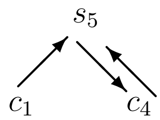


Figure 5.3: G^3 obtained in Round 3 by TTC-M in Example 9

Observe that $\nu(\widehat{Y}^2) = (0,2,2,1) \in F$. Note also that c_2's decision to give its right to student s_1 is based on the fact that moving s_1 from c_1 to c_2 is feasible, i.e., $(0,2,1,2)$ is in F. Similarly, c_3's decision to give its right to student s_4 is based on the fact that it is feasible to move s_4 from c_4 to c_3, i.e., $(1,1,2,1)$ is in F. The fact that implementing both moves still gives a feasible vector is guaranteed by M-convexity, as shown in the proof of Theorem 26. At Step 4, TTC-M goes to Round 3 because $\tilde{Y}^2 \neq \emptyset$.

At Step 1 of Round 3, G^3 is determined (Figure 5.3). Since no student is admissible to c_2 or c_3, these colleges leave the market. There is one cycle: (c_4, s_5, c_4). At Step 2, $\mathscr{C}^3$ is $\{(c_4, s_5), (s_5, c_4)\}$. Therefore, $\tilde{Y}^3$ and Y^3 are determined:

$$\begin{aligned} \tilde{Y}^3 &= \emptyset, \\ Y^3 &= \{(s_2,c_3),(s_3,c_2),(s_1,c_2),(s_4,c_3),(s_5,c_4)\}. \end{aligned}$$

At Step 4, TTC-M returns Y^3 because $\tilde{Y}^3 = \emptyset$.

5.4.1 Mechanism Properties

We introduce a property, which guarantees that TTC-M is feasible even though each college independently makes its decision. Before we show the property, we prove the following lemma.

Lemma 9 In TTC-M, for k (≥ 1), if $\nu(\widehat{Y}^k) \neq \nu(\widehat{Y}^{k-1})$, r (≥ 1) exists and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ such that

$$\nu(\widehat{Y}^k) = \nu(\widehat{Y}^{k-1}) + \sum_{\ell \in \{1,\ldots,r\}} (e_{\tilde{c}_\ell} - e_{\omega(\tilde{s}_\ell)}), \tag{5.5}$$

where $\tilde{s}_1, \ldots, \tilde{s}_r$ are ordered such that $\tilde{s}_1 \succ \cdots \succ \tilde{s}_r$ and $\tilde{c}_1, \ldots, \tilde{c}_r$ are ordered such that $(\tilde{c}_\ell, \tilde{s}_\ell) \in E^k$ for all $\ell \in \{1, \ldots, r\}$. Furthermore, the colleges in $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct, no college in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ has any initial endowment student in S^k, and

$$\begin{cases} \nu(\widehat{Y}^{k-1}) + e_{\tilde{c}_h} - e_{\omega(\tilde{s}_\ell)} \in F & \text{if } h = \ell \\ \nu(\widehat{Y}^{k-1}) + e_{\tilde{c}_h} - e_{\omega(\tilde{s}_\ell)} \notin F & \text{if } h > \ell \end{cases} \qquad (h, \ell \in \{1, \ldots, r\}). \tag{5.6}$$

Proof In TTC-M, the fact that $(c, s) \in \mathscr{C}^k$ means that "college c accepts a student (who is pointing to c), while student s moves from her initial endowment college $\omega(s)$ to a college (to which s is pointing)." Therefore, $\nu(\widehat{Y}^k)$ can be expressed:

$$\nu(\widehat{Y}^k) = \nu(\widehat{Y}^{k-1}) + \sum_{(c,s) \in \mathscr{C}^k} (e_c - e_{\omega(s)}).$$

From this expression, it is clear that moving s from $\omega(s)$ to $\widehat{Y}^k(s)$ does not affect the resulting distributional vector if $(c, s) \in \mathscr{C}^k$ and $\omega(s) = c$. When $\nu(\widehat{Y}^k) \neq \nu(\widehat{Y}^{k-1})$, $r \geq 1$ exists and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ such that

$$\begin{aligned} \nu(\widehat{Y}^k) =& \nu(\widehat{Y}^{k-1}) + \sum_{(c,s) \in \mathscr{C}^k} (e_c - e_{\omega(s)}) \\ =& \nu(\widehat{Y}^{k-1}) + \sum_{(c,s) \in \mathscr{C}^k,\ \omega(s) \neq c} (e_c - e_{\omega(s)}) \\ =& \nu(\widehat{Y}^{k-1}) + \sum_{\ell \in [r]} (e_{\tilde{c}_\ell} - e_{\omega(\tilde{s}_\ell)}), \end{aligned}$$

where $\tilde{s}_1, \ldots, \tilde{s}_r$ are ordered such that $\tilde{s}_1 \succ \cdots \succ \tilde{s}_r$ and $\tilde{c}_1, \ldots, \tilde{c}_r$ are ordered such that $(\tilde{c}_\ell, \tilde{s}_\ell) \in \mathscr{C}^k$ for all $\ell \in \{1, \ldots, r\}$.

Next we show that the elements in $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct. Since each college can receive at most one student in a round, the elements in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ are clearly distinct. The elements in $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r)\}$ are also clearly distinct since if $\omega(\tilde{s}_i) = \omega(\tilde{s}_h)$ holds ($i < h$), $\tilde{c}_h$ should have pointed to $\tilde{s}_i$ (not $\tilde{s}_h$) according to $\succ^*_{\tilde{c}_h}$. Furthermore, in G^k, each college in $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ points to a student who is not among its initial endowment students, while each college in set $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r)\}$ points to its initial endowment student. Therefore, all the colleges in set $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ are distinct. Besides, each college in set $\{\tilde{c}_1, \ldots, \tilde{c}_r\}$ points to its top-ranked admissible student based on the common order $\succ$. Thus Equation (5.6) holds. □

Now we are ready to prove that TTC-M is feasible.

Theorem 26 TTC-M is feasible.

Proof We show that TTC-M always obtains a feasible and individually rational outcome. First, we show by induction on k that $\widehat{Y}^k$ is feasible. For $k = 0$, it is clear by assumption that $\widehat{Y}^0 = \tilde{Y}$ is feasible. Then by assuming that $\nu(\widehat{Y}^{k-1}) \in F$ is true for any $k \geq 1$, the induction is completed by showing $\nu(\widehat{Y}^k) \in F$. Since $\nu(\widehat{Y}^k)$ is represented as Equation (5.5), by Equation (5.6) and Lemma 6, $\nu(\widehat{Y}^k) = \nu(\widehat{Y}^{k-1}) + \sum_{\ell \in \{1,\ldots,r\}} (e_{\tilde{c}_\ell} - e_{\omega(\tilde{s}_\ell)})$ must be in F.

We then show that the outcome of TTC-M is individually rational by the following two claims: (i) each student is contained in a cycle exactly once and assigned to the college she points to in this formed cycle, (ii) a student never points to a college that is worse than her initial endowment college based on her preference. Claim (i) is clear from the definition of TTC-M. Claim (ii) is true because $\nu(\widehat{Y}^k) + e_i - e_j \in F$ is satisfied for any k if $j = i$; that is, as long as a student is in the market, her initial endowment college remains in the market and deems her admissible. □

To show that TTC-M is strategyproof and Pareto efficient, we prove the following lemma, which illustrates that if student s is inadmissible to college c at round k, then s remains inadmissible to c at any round after k. This is a key property for proving strategyproofness and Pareto effieicncy.

Lemma 10 Let s' be a student in S^{k+1} and $c' = \omega(s')$. For college $c \in C$, which has no initial endowment student in S^k (c is not required to remain in the market at round k), if $\nu(\widehat{Y}^{k-1}) + e_c - e_{\omega(s)} \notin F$ for all students $s \in S^k$ with $s \succeq s'$, then $\nu(\widehat{Y}^k) + e_c - e_{c'} \notin F$.

Proof If $\nu(\widehat{Y}^k) = \nu(\widehat{Y}^{k-1})$, then the lemma obviously holds.

We assume $\nu(\widehat{Y}^k) \neq \nu(\widehat{Y}^{k-1})$. By Lemma 9, $r \geq 1$ and $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\} \subseteq C^k$ exist, which satisfy Equations (5.5) and (5.6). Note that $s' \in S^{k+1}$, and no student in $\{\tilde{s}_1, \ldots, \tilde{s}_r\}$ is included in S^{k+1}. We assume that, based on $\succ$, the strict order on $\{\tilde{s}_1, \ldots, \tilde{s}_r, s'\}$ is $\tilde{s}_1 \succ \cdots \succ \tilde{s}_p \succ s' \succ \tilde{s}_{p+1} \succ \cdots \succ \tilde{s}_r$. By this order, we consider the order on $\{\tilde{c}_1, \ldots, \tilde{c}_r, c\}$ and $\{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r), \omega(s')\}$ as: $\tilde{c}_1, \ldots, \tilde{c}_p, c, \tilde{c}_{p+1}, \ldots, \tilde{c}_r$, and $\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_p), \omega(s'), \omega(\tilde{s}_{p+1}), \ldots, \omega(\tilde{s}_r)$.

We apply Lemma 7 to the above colleges and have the following facts:

- $\nu(\widehat{Y}^{k-1}) \in F$ by Theorem 26,
- $\{\tilde{c}_1, \ldots, \tilde{c}_r, c\} \cap \{\omega(\tilde{s}_1), \ldots, \omega(\tilde{s}_r), \omega(s')\} = \emptyset$ by Lemma 9,
- $\{\tilde{c}_1, \omega(\tilde{s}_1), \ldots, \tilde{c}_r, \omega(\tilde{s}_r)\}$ satisfies Equation (5.6) by Lemma 9,
- $\nu(\widehat{Y}^{k-1}) + e_c - e_{\omega(s')} \notin F$ by the assumption,
- $\nu(\widehat{Y}^{k-1}) + e_c - e_{\omega(\tilde{s}_\ell)} \notin F$ for $\ell \in \{1, \ldots, p\}$ by the assumption, and
- $\nu(\widehat{Y}^{k-1}) + e_{\tilde{c}_h} - e_{\omega(s')} \notin F$ for $h \in \{p+1, \ldots, r\}$; otherwise, $\tilde{c}_h$ should have pointed to s' (not $\tilde{s}_h$).

From Lemma 7 and the above facts, $\nu(\widehat{Y}^k) + e_c - e_{c'} = \nu(\widehat{Y}^{k-1}) + \sum_{\ell\in\{1,\ldots,r\}}(e_{\tilde{c}_\ell} - e_{\omega(\tilde{s}_\ell)}) + (e_c - e_{\omega(s')})$ is not contained in F. $\square$

Next we show that TTC-M satisfies strategyproofness and Pareto efficiency. To show the strategyproofness, we use the following lemma.

Lemma 11 For any directed path $(c^*, \ldots, s^*)$, if $(c^*, \ldots, s^*)$ is in G^k and $s^* \in V^{k+1}$ in the execution of TTC-M, then $(c^*, \ldots, s^*)$ is also in G^{k+1}.

Proof We first show the following:

(i) For any $c \in C$ and $s \in S$, if $(s, c) \in E^k$ and $s, c \in V^{k+1}$, then $(s, c) \in E^{k+1}$ holds.

(ii) For any $c \in C$ and $s \in S$, if $(c, s) \in E^k$ and $s \in V^{k+1}$, then $c \in V^{k+1}$ and $(c, s) \in E^{k+1}$ hold.

(iii) For any $s, s' \in S$ and $c \in C$, if $(s, c) \in E^k$, $(c, s') \in E^k$, and $s' \in V^{k+1}$, then $s \in V^{k+1}$ holds.

(i) means that, if $(s, c) \in E^k$ holds, i.e., c is the best remaining college for s at round k, and s and c remain in the market at round $k+1$, then c is still the best choice for s at round $k+1$. This is true because the colleges that left the market will never return in a later round in TTC-M.

(ii) means that if s is the highest-ranked admissible student for c at round k, then c still ranks s as the highest-ranked admissible student in the next round, given that s still remains in the market. To prove the claim, it is sufficient to show that at round $k+1$, s is still admissible to c, and s is still c's top-ranked admissible student. Let us first prove that s is admissible to c at round $k+1$, i.e., $\nu(\widehat{Y}^k) + e_c - e_{\omega(s)} \in F$ holds. If $\omega(s) = c$, it obviously holds. If $\omega(s) \neq c$, then consider a hypothetical case, where c is the favorite college of s at round k. This case could have happened, since the preference of each student is arbitrary. Note that in this case, since F and ω do not change, s is still admissible to c at round k. Therefore, in this hypothetical case, in addition to the cycles formed at round k in the original market, exactly one cycle, (c, s, c), is formed at round k. From Theorem 26, the resulting vector at the end of k in this hypothetical setting should be feasible: $\nu(\widehat{Y}^k) + e_c - e_{\omega(s)} \in F$. This fact implies that s is admissible to c at round $k+1$ in the original market (where c is not the favorite college of s at round k and s remains in the market at round $k+1$). Since c has an admissible student, c remains in the market, i.e., $c \in V^{k+1}$. Next we show that s remains the highest-ranked admissible student for c at $k+1$. If $\omega(s) = c$, then it clearly holds. Assume $\omega(s) \neq c$, and let s' be a student in V^{k+1} and $s' \succ s$. The fact that $(c, s') \notin E^k$ implies that $\nu(\widehat{Y}^{k-1}) + e_c - e_{\omega(s')} \notin F$. Then from Lemma 10 and $\nu(\widehat{Y}^{k-1}) + e_c - e_{\omega(s)} \in F$, it holds that $\nu(\widehat{Y}^k) + e_c - e_{\omega(s')} \notin F$, and thus s' is inadmissible to c at round $k+1$. Therefore, we have $(c, s) \in E^{k+1}$.

(iii) is an elementary property, inherited from the standard TTC mechanism. Assume $(s, c) \in E^k$, $(c, s') \in E^k$, and student s leaves the market at round k. She leaves the market only when (s, c) is included in a cycle. If (s, c) is included in a cycle, then (c, s') must also be included in the same cycle. Then s' must leave the market at round k. Thus, the fact that s' remains in the market implies that s also remains in it.

From (i), (ii), and (iii), for any directed path $(c, \ldots, s)$, if $(c, \ldots, s)$ is in G^k and $s \in V^{k+1}$, then $(c, \ldots, s)$ is also in G^{k+1}. □

Theorem 27 TTC-M is strategyproof.

Proof From Lemma 11, for any directed path $(c, \ldots, s)$, if $(c, \ldots, s)$ is in G^k and $s \in V^{k+1}$, then $(c, \ldots, s)$ is also in G^{k+1}. By repeatedly applying this lemma, once there is a directed path to a student in a round, it remains in any later round as long as the student also remains in the market. By the construction of TTC-M, a student can only obtain a college seat if there is a directed path from the college to her. Fix s and $\succ_{-s}$. We call a college obtainable at k if s can obtain its seat by pointing to it at k. Observe that any college on any directed path to s in G^k is obtainable for s. It is now clear that the set of obtainable colleges for s is increasing on k, and the set's growth depends only on $\succ_{-s}$. Since s can only obtain a college seat from her obtainable colleges at any round, her best strategy is to choose her most preferred college among all the obtainable colleges at the round. Choosing based on her true preference $\succ_s$ gives her the best assignment, and therefore TTC-M is strategyproof. □

Theorem 28 TTC-M is Pareto efficient.

Proof Suppose we run TTC-M and obtain feasible matching Y. Consider a student who is matched at round r in TTC-M. We show that if in another matching Y', the student is assigned to a more preferred college than her allocation in Y, another student must exist who is (assigned before r) worse off in Y'. The proof is done by induction on r.

When $r = 1$, the statement is trivially true because any student being assigned at round 1 is allocated to her top choice.

Assume the supposition is true up to $r = k - 1$ with $k \geq 2$. We consider $r = k$ and define the following notations:

> $I^k = C \setminus C^k$: colleges that are not in the market at round k.
>
> $J^k = \{c \mid \tilde{Y}^{k-1}(c) \neq \emptyset\}$: a set of colleges, each of which has at least one remaining initial endowment student in the market at round k. Note that $J^k \subseteq C^k$ holds.

Each $c_i \in C \setminus J^k$ is filled with $\nu_i(\widehat{Y}^{k-1}) = \nu_i(Y^{k-1})$ students at the beginning of round k. Without loss of generality, assume $I^k \neq \emptyset$ (if $I^k = \emptyset$, every student allocated at k goes to her top choice). Consider student s, who is allocated to college c at round

k, and assume c is not her top choice. Then all the colleges that she prefers over c are in I^k. From Lemma 10 and the definitions of I^k and J^k, $\forall c_i \in I^k$ and $\forall c_j \in J^k$, $\nu(\widehat{Y}^{k-1}) + e_i - e_j \notin F$ holds. Lemma 8 then implies that there is no feasible matching Y' such that

$$\forall c_i \in C \setminus J^k,\ \nu_i(Y') \geq \nu_i(Y^{k-1}) \text{ and } \exists c_i \in I^k,\ \nu_i(Y') > \nu_i(Y^{k-1}).$$

Put differently, for any feasible matching Y' with $Y'(s) \in I^k$ (which covers all the feasible matchings where the allocation of s is better than c), at least

$$\exists c_i \in C \setminus J^k,\ \nu_i(Y') < \nu_i(Y^{k-1}) \text{ or } \nu_{Y'(s)}(Y') \leq \nu_{Y'(s)}(Y^{k-1})$$

holds. Whichever is the case, a student exists who is matched before k in Y and has a different allocation in Y'. From the induction argument, however, such a change necessarily makes someone who is matched before k worse off. □

Next we prove that the TTC-M is strongly group strategyproof. We first introduce a concept called *non-bossiness* [62].

Definition 36 (Non-bossiness) Mechanism φ is non-bossy if for any $s \in S$, $\succ_{-s}$, $\succ_s$, and $\succ'_s$, as long as $\varphi_s(\succ'_s, \succ_{-s}) = \varphi_s(\succ_s, \succ_{-s})$ holds, $\varphi(\succ'_s, \succ_{-s}) = \varphi(\succ_s, \succ_{-s})$ holds.

In other words, a mechanism is non-bossy if a student can only change the assignments of other students when her own assignment changes.

Theorem 29 TTC-M is non-bossy.

Proof Consider two markets, which are basically identical except for the preference of student s: denoted as $\succ_s$ in the first market and $\succ'_s$ in the second. Let Y and Y' denote matchings obtained by TTC-M in the first and second markets, respectively. Towards a contradiction, assume $Y_s = Y'_s$ holds, but Y and Y' are different.

Assume that s is contained in a cycle formed at round k in the first market and at round k' in the second market. We examine the following two cases: (i) $k = k'$, and (ii) $k \neq k'$. For case (ii), without loss of generality, we assume $k < k'$ holds.

Case (i) $k = k'$: Since s is not contained in any cycle at any round before k, the cycles formed at each round before k are identical in both markets. Then at round k, since $Y_s = Y'_s$, in both markets, s points to Y_s. Furthermore, for all $s' \in S \setminus \{s\}$ and all $c \in C$, the preferences are identical in both markets. Thus, the cycles formed at round k are identical in both markets, and so are the cycles at rounds after k. This implies that $Y = Y'$.

Case (ii) $k < k'$: It is clear that all the cycles formed before round k must be identical in both markets. Let us first examine why any cycle formed at round k in the first market is also formed in the second market. Let $\mathscr{C}$ denote the cycle containing s at round k in the first market. We show that $\mathscr{C}$ is also formed at round k' in the second market. At round k, since s is assigned to Y_s, s points to Y_s, and there is a directed path from Y_s to s in the first market. This path also exists in the second market at round k. By Lemma 11, this path remains until round k' in the second market. Since $Y'_s = Y_s$, s points to Y_s at round k' in the second market, which implies that $\mathscr{C}$ is formed. Besides $\mathscr{C}$, the other cycles at round k in the first market are also formed at round k in the second market since the preferences of the other students are identical in both markets.

Next we show that all the cycles formed after round k in the first market are also formed in the second market. We first show that all the cycles formed at round $k+1$ in the first market are also formed in the second market. Let C' denote the set of colleges contained in $\mathscr{C}$, i.e., $C' = \{c \in C \mid \exists s' \in S, (s', c) \in \mathscr{C}\}$. For a cycle formed at round $k+1$ in the first market, if it does not include any college in C', it is also formed at round $k+1$ in the second market, since it is not influenced by $\mathscr{C}$. Thus, we concentrate on a cycle that includes at least one college in C'. Let $\mathscr{C}'$ denote such a cycle. We divide the students in $\mathscr{C}'$ into two groups (1 and 2). Each group 1 student points to a college in C' at round k in the first market, but she points to a different college at round $k+1$ according to $\mathscr{C}'$, since $\mathscr{C}$ is formed and the college in C' (to which she pointed) leaves the market after round k. Group 2 students are the complement of group 1 students. At round $k+1$ in the second market, a group 1 student continues to point to the same college in C' since $\mathscr{C}$ is not formed. On the other hand, each group 2 student points to a college according to $\mathscr{C}'$ at round $k+1$ in the second market. Using a similar argument to the proof of Lemma 11, we can show that all of the group 1 and group 2 students continue to point to the same college until $\mathscr{C}$ is formed, i.e., until round k' in the second market. At round $k'+1$ in the second market, group 1 students point to colleges according to $\mathscr{C}'$, and group 2 students continue to point to colleges according to $\mathscr{C}'$. Therefore, $\mathscr{C}'$ is formed at round $k'+1$ in the second market.

We can construct a similar argument for the rounds after $k+1$ to complete the proof. □

With Theorem 29, we can immediately show that TTC-M is strongly group strategyproof.

Theorem 30 TTC-M is strongly group strategyproof.

Proof Pápai [62] showed that a mechanism is strongly group strategyproof if and only if it is strategyproof and non-bossy. Since TTC-M is strategyproof (Theorem 27) and non-bossy (Theorem 29), TTC-M is strongly group strategyproof. □

Recall that the strong core can be empty in our setting (Example 5.2). We show that the matching obtained by TTC-M is always in the weak core.

Theorem 31 TTC-M always yields a matching in the weak core.

Proof Towards a contradiction, we assume that TTC-M returns matching Y that is not in the weak core: i.e., another matching $Y' \in X$ and a group of students $S' \subseteq S$

exist such that (i) for each $c \in C$, $|\{(s,c) \in Y' \mid s \in S'\}| = |\{(s,c) \in \tilde{Y} \mid s \in S'\}|$ and (ii) for every $s \in S', Y'_s \succ_s Y_s$. Since TTC-M is individually rational, $Y_s \succeq_s \tilde{Y}_s$ for each $s \in S'$. By condition (ii), $Y'_s \neq \tilde{Y}_s$ for every $s \in S'$. Moreover, condition (i) guarantees that permutation σ of S' exists such that $Y'_s = \tilde{Y}_{\sigma(s)}$ and $s \neq \sigma(s)$ for each $s \in S'$. It implies that a set of students $\hat{S} = \{\hat{s}_1, \ldots, \hat{s}_{|\hat{S}|}\} \subseteq S'$ exists such that $Y'_{\hat{s}_i} = \{(\hat{s}_i, \omega(\hat{s}_{i+1}))\}$ for each $i \in \{1, \ldots, |\hat{S}| - 1\}$ and $Y'_{\hat{s}_{|\hat{S}|}} = \{(\hat{s}_{|\hat{S}|}, \omega(\hat{s}_1))\}$. Let $\hat{C}$ denote the set of initial endowment colleges of students in $\hat{S}$.

Consider student $\hat{s}_i \in \hat{S}$ who is the first assigned by TTC-M among the students in $\hat{S}$. Let k_i denote the round at which $\hat{s}_i$ is assigned by TTC-M. Students $\hat{S}$ and colleges $\hat{C}$ are in V^{k_i} since $\hat{s}_i$ is the first student assigned among $\hat{S}$. At round k_i, student $\hat{s}_i$ points to $Y_{\hat{s}_i}$ and is accepted, even though $\omega(\hat{s}_{i+1}) = Y'_{\hat{s}_i} \succ_{\hat{s}_i} Y_{\hat{s}_i}$ holds (when $i = |\hat{S}|$, $\omega(\hat{s}_1) = Y'_{\hat{s}_i} \succ_{\hat{s}_i} Y_{\hat{s}_i}$ holds). This is possible only when $\omega(\hat{s}_{i+1})$ (or $\omega(\hat{s}_1)$) has already quit the market. However, this contradicts the fact that $\omega(\hat{s}_{i+1})$ (or $\omega(\hat{s}_1)$) remains at round k_i. □

Finally, we examine the time complexity of TTC-M.

Theorem 32 The time complexity of TTC-M is $O(mn)$ under the assumption that we can check in $O(1)$ time whether $\nu \in F$ for an M-convex set F and a vector ν on C.

Proof Since at least one cycle exists in each round, at least one student in S leaves the market with her allocation. Therefore, the number of rounds required for TTC-M is at most n. At each round, there are at most m students who can be part of the cycles since each remaining college points to exactly one student, and finding the cycles can be done in $O(m)$. Furthermore, a college needs to check whether a student is admissible. By Lemma 5, when a student becomes inadmissible (or she is inadmissible in the initial round), she remains inadmissible in all future rounds. Thus for each college, the cost required for this check (until it leaves the market) is $O(n)$. To be more precise, in the first round, each college goes through $\succ^*_c$ from the top until it finds the first admissible student. In the next round, it checks whether the student, to whom it points in the previous round, remains in the market and is admissible. If she is no longer admissible (or left the market), it again goes through $\succ^*_c$. Thus, the total number of checks for each college is $O(n)$. Hence, the overall time complexity is $O(mn)$. □

5.4.2 TTC among Representatives

In this subsection, we study a special case of TTC-M, which restricts the set of feasible vectors F to $\{\nu(\tilde{Y})\}$, i.e., the mechanism only outputs matchings whose distribution vector is identical to that of the initial endowment. This mechanism is identical to a mechanism called the Top Trading Cycles among Representatives (TTC-R) [33]. In TTC-R, the number of students assigned to each college is fixed; if a college accepts a student from another college, one of its initial endowment students must leave. At each round, each college points to its top-ranked initial endowment students, each student points to her favorite college, and all the students caught in cycles are allocated to their favorite colleges. That is, these top-ranked students (representatives) exchange seats among themselves. When the top-ranked student of college c leaves the market, the second-ranked student becomes the representative of college c, and so forth.

Since TTC-R is equivalent to TTC-M, where we artificially restrict the set of feasible vectors to a set that contains only the initial endowment vector, TTC-R remains feasible, individually rational, and strategyproof. However it fails to satisfy the efficiency requirements. In addition, since we can show that it is non-bossy using a similar argument from the proof of Theorem 29, it is also strongly group strategyproof. We can also show that the matching obtained by TTC-R is in the weak core by using a similar argument from the proof of Theorem 31.

Theorem 33 TTC-R is strongly group strategyproof and always yields a matching in the weak core.

However, since TTC-R artificially restricts the set of feasible vectors, the following example easily shows that TTC-R is not nonwasteful based on ranks.

Example 10 Consider $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, and $\omega(s_1) = c_1, \omega(s_2) = c_1, \omega(s_3) = c_2$. The following are the preferences of the students and colleges.

$$\begin{array}{ll} s_1 : c_1 \succ_{s_1} c_2 \succ_{s_1} c_3 & \quad c_1 : s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \\ s_2 : c_3 \succ_{s_2} c_2 \succ_{s_2} c_1 & \quad c_2 : s_3 \succ_{c_2} s_1 \succ_{c_2} s_2 \\ s_3 : c_1 \succ_{s_3} c_2 \succ_{s_3} c_3 & \quad c_3 : s_2 \succ_{c_3} s_3 \succ_{c_3} s_1 \end{array}$$

The set of feasible vectors is given as follows:

$$F = \left\{ \nu \in \mathbb{Z}^3_{\geq 0} \;\middle|\; \begin{array}{c} \sum_{i \in M} \nu_i = 3, \\ 0 \leq \nu_1, \nu_2, \nu_3 \leq 3. \end{array} \right\}$$

Under common serial order $s_1 \succ s_2 \succ s_3$, artificially modified colleges' preferences $\succ^*_C$ are given as follows:

$$\begin{array}{l} c_1 : s_1 \succ^*_{c_1} s_2 \succ^*_{c_1} s_3 \\ c_2 : s_3 \succ^*_{c_2} s_1 \succ^*_{c_2} s_2 \\ c_3 : s_1 \succ^*_{c_3} s_2 \succ^*_{c_3} s_3 \end{array}$$

In TTC-R, the set of feasible vectors is artificially restricted to $F = \{(2, 1, 0)\}$. Then c_3 immediately leaves the market since it has no admissible student. In the first round, each student points to her favorite remaining college: s_1 to c_1, s_2 to c_2, and s_3 to c_1. According to $\succ^*_C$, c_1 points to s_1 and c_2 points to s_3, forming cycle (s_1, c_1, s_1). In the second round, cycle $(s_2, c_2, s_3, c_1, s_2)$ is formed. Therefore TTC-R outputs matching: $\{(s_1, c_1), (s_2, c_2), (s_3, c_1)\}$. Student s_2 claims an empty seat in c_3 based on ranks since $rank((s_2, c_3)) = 1 < rank((s_2, c_2)) = 2$. Hence, TTC-R is not nonwasteful based on ranks.

Example 10 also implies that TTC-R is neither nonwasteful nor Pareto efficient. Although TTC-R is not nonwasteful, NIE-fairness does not hold, as shown in the following example.

Example 11 Consider $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, and $\omega(s_1) = c_1, \omega(s_2) = c_3, \omega(s_3) = c_2$. The following are the preferences of the students and colleges.

$$\begin{array}{ll} s_1 : c_2 \succ_{s_1} c_1 \succ_{s_1} c_3 & c_1 : s_1 \succ_{c_1} s_2 \succ_{c_1} s_3 \\ s_2 : c_2 \succ_{s_2} c_3 \succ_{s_2} c_1 & c_2 : s_3 \succ_{c_2} s_1 \succ_{c_2} s_2 \\ s_3 : c_3 \succ_{s_3} c_2 \succ_{s_3} c_1 & c_3 : s_2 \succ_{c_3} s_3 \succ_{c_3} s_1 \end{array}$$

Set of feasible vectors F is given as $\{(1, 1, 1)\}$. That is, each college, which has exactly one initial endowment student, leaves the market after it is included in a cycle. Thus, the choice of a common serial order does not affect the behavior of TTC-R. In the first round, cycle $(s_2, c_2, s_3, c_3, s_2)$ is formed. In the second round, colleges c_2 and c_3 leave the market; cycle (s_1, c_1, s_1) is formed. Thus the obtained matching is: $\{(s_1, c_1), (s_2, c_2), (s_3, c_3)\}$.

Observe that student s_1 has justified RIE-envy towards s_2 since $s_1 \succ_{c_2} s_2$ and $\omega(s_2) \neq c_2$, which implies that TTC-R is not NIE-fair.

5.5 Deferred Acceptance Based Mechanisms

In this section, we study DA-based mechanisms. First we describe a simple extension of DA called the Artificial Cap Deferred Acceptance (ACDA) mechanism. Then we introduce our second main mechanism called DA based on Ranks (DA-R).

5.5.1 Artificial Cap Deferred Acceptance

In this subsection, we introduce Artificial Cap Deferred Acceptance (ACDA) for initial endowment settings.

For each college c, we set artificial maximum quota q_c to $|\tilde{Y}_c|$, i.e., the number of its initial endowment students. Therefore, the distribution vector of output matching by ACDA is $\nu(\tilde{Y})$, which must be feasible. The preference of each college, $\succ_c$, is modified as $\tilde{\succ}_c$ such that each of its initial endowment students is preferred over any non-initial endowment students. More specifically, for each college c, for each pair of students s and s', $s \tilde{\succ}_c s'$ holds if one of the following conditions holds:

1. $\omega(s) = c$ and $\omega(s') \neq c$,
2. $s \succ_c s'$ and $\omega(s') \neq c$, $\omega(s') \neq c$, or
3. $s \succ_c s'$ and $\omega(s') = \omega(s') = c$.

ACDA uses DA as subroutine with $(\tilde{\succ}_c)_{c \in C}$ and $(q_c)_{c \in C}$. We show an example to illustrate how ACDA works.

Example 12 Consider $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3\}$, and $\omega(s_1) = c_1$, $\omega(s_2) = \omega(s_3) = c_2$, $\omega(s_4) = c_3$. The following are the preferences of the students and colleges.

$$\begin{array}{rl} s_1 : & c_2 \succ_{s_1} c_3 \succ_{s_1} c_1 \\ s_2, s_3, s_4 : & c_1 \succ_s c_2 \succ_s c_3 \end{array} \qquad \begin{array}{l} c_1 : s_1 \tilde{\succ}_{c_1} s_2 \tilde{\succ}_{c_1} s_4 \tilde{\succ}_{c_1} s_3 \\ c_2 : s_2 \tilde{\succ}_{c_2} s_3 \tilde{\succ}_{c_2} s_4 \tilde{\succ}_{c_2} s_1 \\ c_3 : s_4 \tilde{\succ}_{c_3} s_1 \tilde{\succ}_{c_3} s_2 \tilde{\succ}_{c_3} s_3 \end{array}$$

The set of feasible vectors F is given as follows:

$$F = \left\{ \nu \in \mathbb{Z}^3_{\geq 0} \left| \begin{array}{c} \sum_{i \in M} \nu_i = 4, \\ 0 \leq \nu_1, \nu_2 \leq 3, \\ 1 \leq \nu_3 \leq 4. \end{array} \right. \right\}$$

Set F represents a situation where at least one student must be assigned to c_3. Since ACDA sets artificial maximum quotas according to the initial endowment ω, quotas q_1, q_2, and q_3 are set to 1, 2, and 1, respectively.

Each student applies to her favorite college, specifically, s_2, s_3, and s_4 apply to c_1, and s_1 applies to c_2. Since three students are applying to c_1, which exceeds maximum quota $q_{c_1} = 1$, c_1 accepts s_2, who is more preferred according to $\tilde{\succ}_{c_1}$, and rejects s_3 and s_4. Similarly, c_2 accepts s_1. Students who were rejected in the previous step, s_3 and s_4, apply to c_2. Then three students, s_1, s_3, and s_4, apply to c_2, which exceeds maximum quota $q_{c_2} = 2$. Hence, c_2 accepts s_3 and s_4 and rejects s_1. Finally, s_1 applies to c_3 and is accepted. All students are accepted, and the mechanism terminates.

The following is the obtained matching: $\{(s_1, c_3), (s_2, c_1), (s_3, c_2), (s_4, c_2)\}$.

We show ACDA satisfies several fundamental desiderata although it can be very inefficient.

Theorem 34 ACDA is feasible, individually rational, NIE-fair, and strategyproof, but is not nonwasteful based on ranks.

Proof Since $q_c = |\tilde{Y}_c|$ and $\tilde{\succ}_c$ is constructed such that each initial endowment student is preferred more than any non-initial endowment student, for any $S' \subseteq S$, student s, who is an initial endowment student of c, is within top q_c students according to $\tilde{\succ}_c$. This means that s is never rejected from c in ACDA if she applies to it. Thus ACDA is individually rational.

Furthermore, DA satisfies fairness according to the given colleges' preferences $\tilde{\succ}_C$ [27]; if s prefers c over her assigned college, then each student s' assigned to c is more preferred than s according to $\tilde{\succ}_c$. This means that either $s' \succ_c s$ or $\omega(s') = c$ holds. Thus, ACDA satisfies NIE-fairness.

Since DA is strategyproof [17, 65], and the artificial maximum quotas are determined independently from the students' preferences, ACDA is also strategyproof. Furthermore, since DA is feasible for given maximum quotas [27], ACDA is feasible.

We show that ACDA fails to satisfy nonwastefulness based on ranks by Example 10. It is clear that $\tilde{\succ}_C$ is consistent with $\succ_C$ in this example. Since $\nu(\tilde{Y}) =$

$(2, 1, 0)$, the artificial cap of ACDA is $(2, 1, 0)$. Then each student applies to her favorite college, that is, s_1 and s_3 apply to c_1, and s_2 applies to c_3. However, since $(2, 0, 1)$ violates the artificial cap, s_2 is rejected by c_3 and then applies to c_2. The following matching is obtained: $\{(s_1, c_1), (s_2, c_2), (s_3, c_1)\}$.

In the matching, student s_2 claims an empty seat in c_3 based on ranks since $rank((s_2, c_3)) = 1 < rank((s_2, c_2)) = 2$. Thus, ACDA fails to satisfy nonwastefulness based on ranks. □

5.5.2 Deferred Acceptance Mechanism Based on Ranks

In this subsection, we introduce the Deferred Acceptance mechanism based on Ranks (DA-R) and describe its properties. DA-R is one particular, concrete instance of a class of GDA (Mechanism 2). We construct DA-R such that it also satisfies individual rationality.

Definition 37 (Students' Choice Function w.r.t. Init. Endowments) For each student s, let Ch_s denote her choice function, which is defined as follows. Given $Y \subseteq X$, let $\hat{Y}_s$ denote $\{(s, c) \in Y_s \mid c \succeq_s \omega(s)\}$. Then $Ch_s(Y)$ returns $\{x\}$ such that $x \in \hat{Y}_s$ and x is the most preferred contract in $\hat{Y}_s$ for s. The choice function of all students Ch_S is defined as: $Ch_S(Y) = \bigcup_{s \in S} Ch_s(Y)$.

Definition 38 (Colleges' Choice Function w.r.t. Init. Endowments) Given $Y \subseteq X$, the choice function of the schools returns set of contracts $Ch_C(Y)$, which is defined as:

$$Ch_C(Y) = \arg\max_{Y' \subseteq Y} f(\nu(Y')) + w(Y').$$

where $w((s_i, c_j))$ is defined as: $w((s_i, c_j)) = C_1(n - rank((s_i, c_j))) + C_2(m - j) + (n - i)$.

The values C_1, C_2 are constants that satisfy $C_1 \gg C_2 \gg n$, and $rank((s_i, c_j))$ is the ranking of student s_i by college c_j (Definition 33). From this definition, it is easy to show that w satisfies the following properties:

1. For two contracts, (s, c) and (s', c'), such that $rank((s, c)) < rank((s', c'))$, $w((s, c)) > w((s', c'))$ holds.

2. For two contracts, (s, c_j) and (s', c_k), such that $rank((s, c_j)) = rank((s', c_k))$ and $j < k$, $w((s, c_j)) > w((s', c_k))$ holds.

3. For two contracts, (s_i, c) and (s_k, c), where $rank((s_i, c)) = rank((s_k, c))$ and $i < k$, $w(s_i, c) > w(s_k, c)$ holds.

The first property denotes that a higher ranked contract is more valuable. In particular, a contract that corresponds to an initial endowment is more valuable than any non-initial endowment contracts. The second property is used for tie-breaking among contracts with the same ranking by different colleges. The third property guarantees that, for any pair of contracts x and x', $x \neq x'$ implies $w(x) \neq w(x')$. More specifically, $rank((s_i, c)) = rank((s_k, c))$ holds only if $\omega(s_i) = \omega(s_k) = c$ holds, i.e., both s_i and s_k are c's initial endowment students. Neither contract is ever rejected; their relative ordering does not affect the Ch_C output.

DA-R utilizes DA with these Ch_S and Ch_C and it is one particular and concrete instance of GDA. We show the execution of DA-R with Example 13.

Example 13 We consider the same instance as in Example 12. First, each student chooses her favorite acceptable contract. Thus, $Y = \{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_1)\}$. Based on the greedy procedure (Recall the description after Definition 15), (s_2, c_1), (s_4, c_1), and (s_1, c_2) are added to Y'. Next (s_3, c_1) is selected, but no $\nu \in F$ exists such that $(3, 1, 0) \leq \nu$ holds. Thus, (s_3, c_1) is not included in Y'. Then $Y' = \{(s_1, c_2), (s_2, c_1), (s_4, c_1)\}$, and $Y \setminus Y' = \{(s_3, c_1)\}$ is rejected.

Student s_3 next chooses her second favorite contract (s_3, c_2), and other students choose the same colleges as before. Thus $Y = \{(s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_1)\}$. Based on the greedy procedure, (s_3, c_2), (s_2, c_1), and (s_4, c_1) are added to Y'. Next (s_1, c_2) is selected[2], but no $\nu \in F$ exists such that $(2, 2, 0) \leq \nu$ holds. Thus, (s_1, c_2) is not included in Y'. Then $Y' = \{(s_2, c_1), (s_3, c_2), (s_4, c_1)\}$, and $Y \setminus Y' = \{(s_1, c_2)\}$ is rejected.

Finally, s_1 chooses her second favorite contract (s_1, c_3). Thus, $Y = \{(s_1, c_3), (s_2, c_2), (s_3, c_2), (s_4, c_1)\}$, which is feasible since $\nu(Y) = (2, 1, 1) \in F$. No contract is rejected, and the mechanism terminates.

The following is the obtained matching: $\{(s_1, c_3), (s_2, c_1), (s_3, c_2), (s_4, c_1)\}$, which is individually rational and NIE-fair.

ACDA is a special case of DA-R, if the set of feasible vectors F is artificially restricted to $\{\nu(\tilde{Y})\}$, i.e., it contains exactly one vector that is identical to the distribution vector of the initial endowment. ACDA is much less flexible than DA-R.

Next we describe the theoretical properties of DA-R.

Theorem 35 DA-R is feasible, strategyproof, and NIE-fair.

Proof Since DA-R is one particular and concrete instance of GDA, it is guaranteed to be feasible and strategyproof. Next we show that DA-R is individually rational. Observe that student s is never rejected by her initial endowment college $\omega(s)$ because, for any Y and initial endowment $\tilde{Y}$, and two contracts $x \in Y \cap \tilde{Y}$ and $x' \in Y \setminus \tilde{Y}$, $w(x) > w(x')$ holds. Thus, all the contracts in $Y \cap \tilde{Y}$ are chosen before any contract in $Y \setminus \tilde{Y}$. Furthermore, $\widehat{f}(Y \cap \tilde{Y}) = 0$ since $\nu(\tilde{Y}) \in F$ holds. Hence, DA-R satisfies individual rationality.

We next show that DA-R is NIE-fair. Assume that student s prefers c over her assigned college, then for each student s' assigned to c, $v((s', c)) > v((s, c))$ holds. This means that either s' is c's initial endowment student or $s' \succ_c s$ holds. □

[2] Here $rank((s_1, c_2)) = rank((s_4, c_1)) = 2$. $w((s_1, c_2)) < w((s_4, c_1))$ is derived by the second property of w.

Theorem 36 DA-R is nonwasteful based on ranks.

Proof Toward a contradiction, we assume for matching Y obtained by DA-R, s claims an empty seat in c' based on ranks, i.e., $(s,c) \in Y$, $(s,c') \in X \setminus Y$, $c' \succ_s c$ and $rank((s,c')) < rank((s,c))$ holds. Furthermore, $(Y \setminus \{(s,c)\}) \cup \{(s,c')\}$ is feasible.

Since $c' \succ_s c$, it is clear that $(s,c') \in Ch_S(Y \cup \{(s,c')\})$ holds. We next examine $Ch_C(Y \cup \{(s,c')\})$. In the greedy procedure, since $rank((s,c')) < rank((s,c))$, $w((s,c')) > w((s,c))$ holds. Thus, (s,c') is selected before (s,c). Since Y is feasible, all contracts selected before (s,c') are added to Y'. Since $(Y \setminus \{(s,c)\}) \cup \{(s,c')\}$ is feasible, its subset $Y' \cup \{(s,c')\}$ is also feasible. Thus, $(s,c) \in Ch_C(Y \cup \{(s,c')\})$ holds, violating the fact that DA-R (GDA) is HM-stable. □

Note that DA-R fails to satisfy nonwastefulness (or Pareto efficiency), since it is NIE-fair, and NIE-fairness is incompatible with nonwastefulness as Example 5.2 shows.

Next we show that DA-R does not satisfy strong group strategyproofness, and the obtained matching is not in the weak core by the following example.

Example 14 We use the same instance as in Example 11. Notice that in this example, $\tilde{\succ}_C$ is identical to $\succ_C$. First, each student chooses her favorite acceptable contract. Thus, $Y = \{(s_1,c_2),(s_2,c_2),(s_3,c_3)\}$. Based on the greedy procedure, (s_1,c_2) and (s_3,c_3) are added to Y'. Next (s_2,c_2) is selected, but no $\nu \in F$ exists such that $(0,2,1) \leq \nu$ holds. The colleges choose $Y' = \{(s_1,c_2),(s_3,c_1)\}$, and $Y \setminus Y' = \{(s_2,c_2)\}$ is rejected.

Student s_2 next chooses her second favorite contract (s_2,c_3), and the other students choose the same colleges as before. Thus, $Y = \{(s_1,c_2),(s_2,c_3),(s_3,c_3)\}$. By the greedy procedure, (s_2,c_3) and (s_1,c_2) are added to Y'. Next (s_3,c_3) is selected, but no $\nu \in F$ exists such that $(0,1,2) \leq \nu$ holds. The colleges choose $Y' = \{(s_1,c_2),(s_2,c_3)\}$, and $Y \setminus Y' = \{(s_3,c_3)\}$ is rejected.

Student s_3 next chooses her second favorite contract (s_3,c_2), and the other students keep their previous choices. Thus, $Y = \{(s_1,c_2),(s_3,c_2),(s_2,c_3)\}$. Based on the greedy procedure, (s_3,c_2) and (s_2,c_3) are added to Y'. Next (s_1,c_2) is selected, but no $\nu \in F$ exists such that $(0,2,1) \leq \nu$ holds. The colleges choose $Y' = \{(s_2,c_3),(s_3,c_2)\}$, and $Y \setminus Y' = \{(s_1,c_2)\}$ is rejected.

Finally, s_1 chooses her second favorite contract (s_1,c_1). Thus, $Y = \{(s_1,c_1),(s_3,c_2),(s_2,c_3)\}$, which is feasible since $\nu(Y) = (1,1,1) \in F$. No contract is rejected, and the mechanism terminates. The obtained matching is identical to the initial endowment matching $\tilde{Y}$ described as follows: $\{(s_1,c_1),(s_2,c_3),(s_3,c_2)\}$. We assume s_1 misreports her preference as $c_1 \succ_{s_1} c_2 \succ_{s_1} c_3$, and s_2 and s_3 report their true preferences. Then DA-R obtains the following Pareto efficient matching: $\{(s_1,c_1),(s_2,c_2),(s_3,c_3)\}$. This implies that DA-R is not strong group strageyproof: for $S' = \{s_1,s_2,s_3\}$ and the above manipulation, s_1's assignment is unchanged (she is weakly better off), and s_2 and s_3 are strictly better off. Furthermore, if s_2 and s_3 exchange their initial endowments, both students are strictly better off than $\tilde{Y}$. Hence, the matching obtained by DA-R is not in the weak core. In this example, ACDA also obtains $\tilde{Y}$ since

$F = \{\nu(\tilde{Y})\}$. Thus, ACDA fails to satisfy strong group strategyproofness, and the matching obtained by ACDA is not in the weak core.

Finally, we discuss DA-R's time complexity with the following theorem.

Theorem 37 The time complexity of DA-R is $O(T(f) \times |X|^2)$, where $T(f)$ denotes the time required to calculate f.

Proof According to Theorem 12 in a previous work [48], the required stages of DA-R are $O(|X|^2)$, and the time complexity of DA-R is $O(T(f) \cdot |X|^2)$. □

5.6 Empirical Evaluation

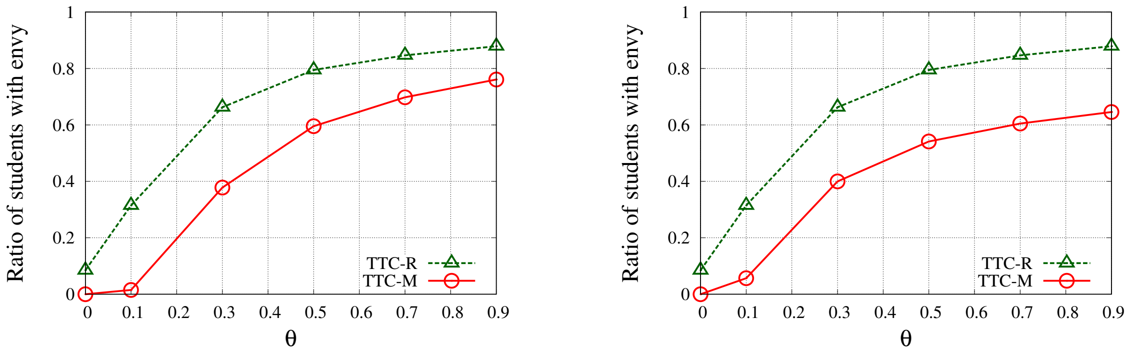
In this section, we conduct quantitative evaluations with several mechanisms, including TTC-based and DA-based mechanisms. We theoretically showed that TTC-M always outputs Pareto efficient matchings and DA-R and ACDA always yield NIE-fair matchings. Although efficiency and fairness are both desirable properties in matching markets, they are incompatible in general. Therefore, we conduct computer simulations to evaluate the degree of efficiency/fairness that these mechanisms can achieve in random markets.

In previous sections, we compared the proposed mechanisms: TTC-M, TTC-R, ACDA, and DA-R. As a benchmark, we show the performance of another trivial mechanism called Initial, which always returns the initial endowment matching $\tilde{Y}$. We evaluate the performance of these mechanisms by comparing several measurements, which can reflect the efficiency or the fairness (e.g., the ratios of students who claim empty seats and who have justified RIE-envy) of the matchings obtained by these mechanisms.

In the experiments, we use instances with 800 students and 20 colleges. Instances are randomly generated in two types of markets: A and B. In Market A, each college has individual minimum/maximum quotas: $p_c = 10$ and $q_c = 80$. In Market B, distance constraints are implemented, where matching Y is feasible if the Manhattan distance between $\nu(Y)$ and the ideal distribution $\nu^* = (40, \ldots, 40)$ is smaller than or equal to 300. As described in Section 3.3, both constraints can be represented as M-convex sets. For ACDA, the artificially restricted maximum quotas of each college is set to 40 in both markets. In all instances, the student preferences are generated by the Mallows model.

In each of the following evaluations, we randomly constructed 100 instances for each parameter setting and compared the average performances of the five mechanisms (TTC-R, TTC-M, ACDA, DA-R, and Initial) over 100 instances except for Figure 5.8, which plots for each problem instance.

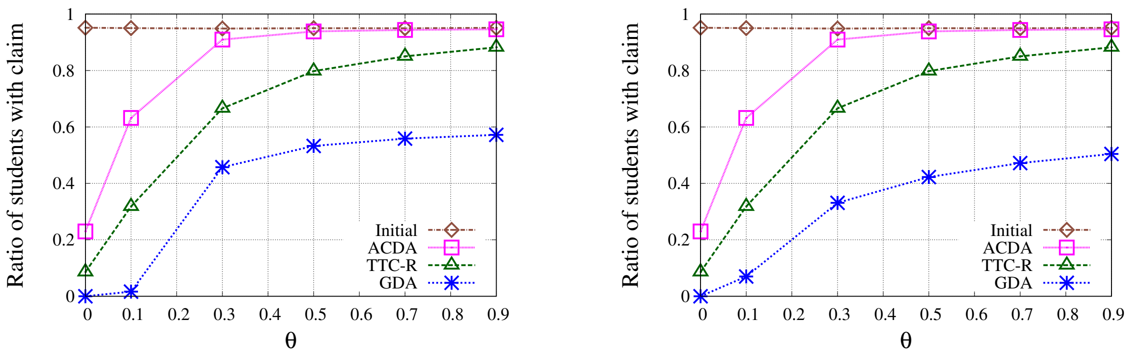
To evaluate the degree of fairness, we examine the ratio of students who have justified RIE-envy in the matchings obtained by the mechanisms. Since the DA-based mechanisms (i.e., ACDA and DA-R) as well as Initial are NIE-fair, it is guaranteed



(a) Market A

(b) Market B

Figure 5.4: Ratio of students with justified RIE-envy



(a) Market A

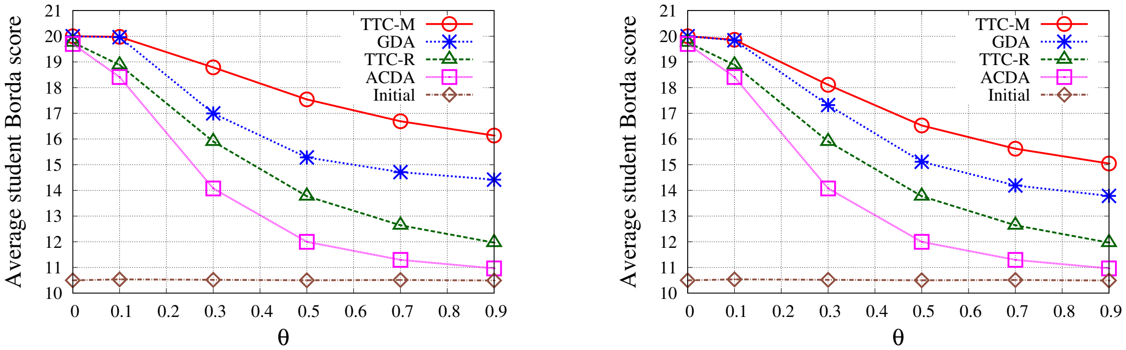
(b) Market B

Figure 5.5: Ratios of students claiming empty seats

that the ratio of students with justified RIE-envy is zero. Thus, we only show the TTC-R and TTC-M results. Figure 5.4 shows the average ratio of students with justified RIE-envy with varying spread parameter θ_S for the students' preferences. Recall that when $\theta_S = 0$, the students' preferences are random, and as θ_S increases, they quickly become similar. Thus, as θ_S increases, the competition among students deepens, and fewer students can be allocated to their first (or second) choice colleges. Since TTC-based mechanisms do not use colleges' preferences, more students have justified RIE-envy. In both Markets A and B, under each θ_S setting, the ratio of TTC-M is smaller than that of TTC-R, which implies that TTC-M outperforms TTC-R in terms of fairness. This is because TTC-M is more flexible than TTC-R, where the number of students allocated to each college is fixed.

To compare the efficiency of the mechanisms, we measured the ratio of students who claim empty seats, the Borda scores, and the *Cumulative Distribution Functions*.

Figure 5.5 shows the average ratios of students who claim empty seats with varying θ_S. Since TTC-M is Pareto efficient, it is also nonwasteful, i.e., no student claims an



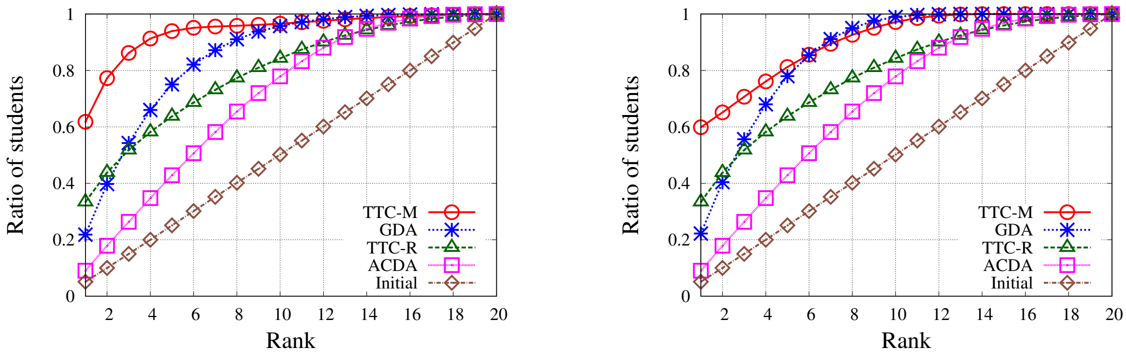
(a) Market A (b) Market B

Figure 5.6: Average Borda scores of students

empty seat. Thus, we compared four mechanisms: TTC-R, DA-R, ACDA, and Initial. In terms of the ratio of students who claim empty seats, these mechanisms are ranked from lowest/best to highest/worst as follows: DA-R, TTC-R, ACDA, and Initial. That is, a more flexible mechanism performs better in terms of nonwastefulness. Similar to Figure 5.4, as θ_S increases, the competition among students becomes more severe, and more students claim empty seats, except for Initial.

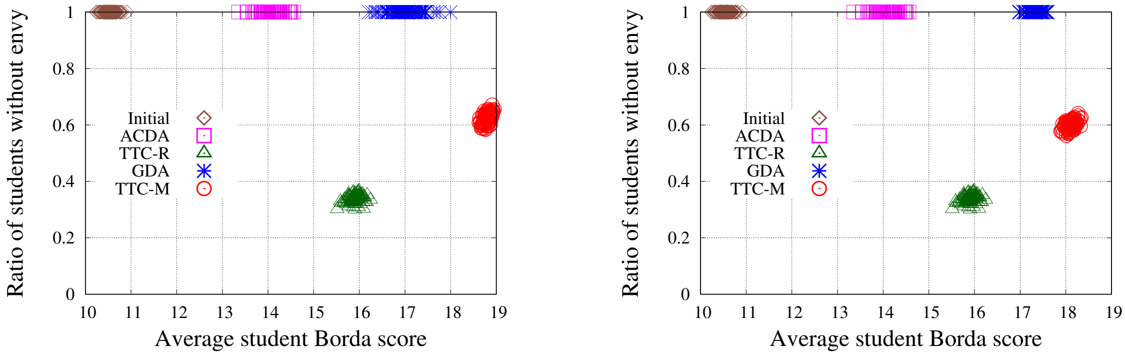
Next we evaluate the students' welfare by the average Borda scores. Figure 5.6 shows the average Borda scores of all the students in five mechanisms with varying θ_S. In terms of the average Borda scores, these mechanisms are ranked from largest/best to smallest/worst as follows: TTC-M, DA-R, TTC-R, ACDA, and Initial. This result implies that a more flexible mechanism performs better in terms of students' welfare. ACDA/TTC-R is equivalent to DA-R/TTC-M when F is restricted to one feasible distribution vector $\nu(\tilde{Y})$, although in TTC-M and DA-R, F is expanded to fully utilize the flexibility allowed by the given constraints. Similar to Figures 5.4 and 5.5, as θ_S increases, the competition among students becomes more severe, and their welfare falls, except for Initial.

Another way to compare students' welfare is to check the Cumulative Distribution Functions (CDFs) of their college rankings. Figure 5.7 shows the average ratios of the students, who are assigned to their i-th choice or a better college, with varying i when $\theta_S = 0.3$ for the five mechanisms. If the curve of mechanism α is always above that of another mechanism, β, we can conclude that mechanism α outperforms β in terms of students' welfare. Roughly speaking, these mechanisms are ranked in terms of CDFs from best to worst as follows: TTC-M, DA-R, TTC-R, ACDA, and Initial. Note that TTC-R is slightly better than DA-R for the first and second rankings in both markets. In addition, DA-R is only slightly better than TTC-M when $i \geq 7$ in Market B. Perhaps in TTC-based mechanisms, the welfare of the students can be slightly uneven; the welfare of a student who is ranked higher in the common serial order is better, although the welfare of a student who is ranked lower can be rather degraded. Compared to TTC-based mechanisms, DA-R is more balanced.



(a) Market A (b) Market B

Figure 5.7: Cumulative distribution functions when $\theta_S = 0.3$



(a) Market A (b) Market B

Figure 5.8: Trade-off between fairness and efficiency when $\theta_S = 0.3$

Finally, to illustrate a trade-off between efficiency and fairness, we plotted the results of the obtained matchings for each problem instance in a two-dimensional space in Figure 5.8, where the x-axis denotes the average student Borda score (which is a measurement of efficiency/students' welfare), and the y-axis denotes the ratio of students *without* justified RIE-envy (which is positively correlated to fairness). Thus, the points located north-east are preferable. Since Initial, ACDA, and DA-R are NIE-fair mechanisms, their ratios are 1. In terms of efficiency, DA-R is clearly optimal in both markets among these three. On the other hand, TTC-R is dominated by TTC-M. Therefore, among these five mechanisms, TTC-M and DA-R are clear winners in terms of these two criteria. TTC-M is the best choice for efficiency; DA-R is the best choice when we emphasize fairness.

Chapter 6

Conclusion and Future Work

When distributional constraints are imposed in two-sided matching, a trade-off between fairness and efficiency surfaces. We initially scrutinized the existence of strategyproof mechanisms, uncovering compelling impossibility results. This clarification precisely delineates the stringent boundaries dictating whether a strategyproof mechanism can satisfy particular fairness and efficiency properties for each class of constraints. Based on these seminal findings, we devised strategyproof mechanisms tailored to hereditary, hereditary $\mathrm{M}^\natural$-convex set, and the union of symmetric M-convex set. We also introduced two strategyproof mechanisms that respect individual rationality under M-convex set constraints. Beyond theoretical contributions, our work extended to a comprehensive evaluation of the performance of strategyproof mechanisms through computer simulations. Following this, we provide a summary of each model in this thesis and outline potential directions for future research.

Hereditary Constraints We clarified the tight boundaries on whether a strategyproof and fair mechanism can satisfy certain efficiency properties under hereditary ($\mathrm{M}^\natural$-convex set) constraints. Specifically, we showed two new impossibility theorems: no mechanism can simultaneously satisfy fairness, strategyproofness, and cut-off nonwastefulness under hereditary $\mathrm{M}^\natural$-convex set constraints (Theorem 1), and no mechanism can simultaneously satisfy fairness, strategyproofness, and no vacant college property under hereditary constraints (Theorem 3). We addressed an open question in Kamada and Kojima [42] by Theorem 3. Building on these results, we introduced a novel relaxed fairness concept called EF-k. This thesis is groundbreaking as it introduces a relaxed notion of fairness in constrained two-sided matching. We also proposed two new strategyproof and EF-k mechanisms: SD* and SDA with reserved quotas. While SD* is Pareto efficient, it cannot guarantee EF-k for any fixed $k < n-1$. Conversely, SDA with reserved quotas satisfies EF-k for any fixed $k < n-1$, although it guarantees a relatively weak efficiency property, no vacant college. Lastly, we empirically assessed their performance through computer simulations, demonstrating that they strike a well-balanced compromise between fairness and efficiency.

The significance of EF-k is emphasized as it introduces a plethora of new research possibilities in constrained matching literature, raising several open questions. For instance, can any strategyproof mechanism ensure EF-k for a fixed k along with a specific efficiency property (stronger than the no vacant college property, such as weak nonwastefulness)? Additionally, a comparative analysis between our proposed mechanisms and ADA is worth exploring. If we use the same optimal master-list as SD*, ADA can also guarantee EF-k. However, in theory, SD* dominates ADA; the guaranteed level of k is identical, and SD* is Pareto efficient, while ADA can only guarantee nonwastefulness. Investigating this in detail constitutes our immediate future work.

Union of Symmetric M-convex Set Constraints We identified a groundbreaking class of distributional constraints, defined as a union of symmetric M-convex sets. Since M-convexity is not closed under union, a union of symmetric M-convex sets is not an M-convex set. We devised a fair and strategyproof mechanism named QRDA based on DA. Additionally, we demonstrated that QRDA inherits several significant axiomatic properties from DA such as weak group strategyproofness, weak Pareto efficiency, or weak non-bossiness. Importantly, these theoretical results have been further verified in a generalized class of distributional constraints: union of symmetric M-convex set constraints with an offset. In terms of student welfare, we proved that QRDA theoretically outperforms ACDA. Moreover, through rigorous experimentation, we showed that QRDA outperforms both ACDA and ESDA in terms of student welfare and nonwastefulness.

We seek to clarify whether any class of constraints exists that is broader than the union of symmetric M-convex sets, where a non-trivial fair and strategyproof mechanism exists. Another direction involves developing an efficient mechanism capable of handling the union of symmetric M-convex set constraints.

Initial Endowment Settings We investigated an allocation problem concerning multiple types of objects assigned to agents, where each type of an object has multiple copies. Each agent is endowed with an object, subject to distributional constraints are imposed on the allocation, reflecting scenarios like the school choice problem. In this context, students can attend their nearby college and transfer based on preferences, adhering to constraints like minimum/maximum quotas in regions. Two mechanisms, TTC-M and DA-R, were developed to be both feasible and strategyproof under M-convex set constraints. TTC-M prioritizes Pareto efficiency, while DA-R emphasizes NIE-fairness. The performance of these mechanisms, balancing fairness and efficiency, was rigorously evaluated through computer simulations.

Future research should focus on developing mechanisms capable of handling constraints beyond M-convex sets, taking into account weaker efficiency and fairness properties.

References

[1] Atila Abdulkadiroğlu and Tayfun Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003. doi: 10.1257/000282803322157061.

[2] Atila Abdulkadiroglu, Yeon-Koo Che, Parag Pathak, Alvin Roth, and Olivier Tercieux. Minimizing justified envy in school choice: The design of new orleans' oneapp. Technical Report 23265, National Bureau of Economic Research, 2017. Working Paper.

[3] Atila Abdulkadiroğlu. College admissions with affirmative action. *International Journal of Game Theory*, 33(4):535–549, 2005. doi: https://doi.org/10.1007/s00182-005-0215-7.

[4] Atila Abdulkadiroğlu and Tayfun Sönmez. House allocation with existing tenants. *Journal of Economic Theory*, 88:233–260, 1999.

[5] Haris Aziz and Bettina Klaus. Random matching under priorities: stability and no envy concepts. *Social Choice and Welfare*, 53(2):213–259, 2019.

[6] Haris Aziz, Serge Gaspers, Zhaohong Sun, and Toby Walsh. From matching with diversity constraints to matching with regional quotas. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS-2019)*, pages 377–385, 2019.

[7] Haris Aziz, Anton Baychkov, and Peter Biró. Cutoff stability under distributional constraints with an application to summer internship matching. *Mathematical Programming*, 2022. doi: https://doi.org/10.1007/s10107-022-01917-1.

[8] Haris Aziz, Péter Biró, Tamás Fleiner, Serge Gaspers, Ronald de Haan, Nicholas Mattei, and Baharak Rastegari. Stable matching with uncertain pairwise preferences. *Theoretical Computer Science*, 909:1–11, 2022. doi: 10.1016/j.tcs.2022.01.028.

[9] Haris Aziz, Peter Biró, and Makoto Yokoo. Matching market design with constraints. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI-2022)*, pages 12308–12316, 2022. doi: 10.1609/aaai.v36i11.21495.

[10] Péter Biró, Tamás Fleiner, Robert Irving, and David Manlove. The college admissions problem with lower and common quotas. *Theoretical Computer Science*, 411(34-36):3136–3153, 2010.

[11] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011. doi: 10.1086/664613.

[12] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum nash welfare. *ACM Transactions on Economics and Computation*, 7:12:1–12:32, 2019. doi: 10.1145/3355902.

[13] Katarína Cechlárová and Tamás Fleiner. Pareto optimal matchings with lower quotas. *Mathematical Social Sciences*, 88:3–10, 2017.

[14] Katarína Cechlárová, Bettina Klaus, and David Manlove. Pareto optimal matchings of students to courses in the presence of prerequisites. *Discrete Optimization*, 29:174–195, 2018.

[15] Vladimir Danilov, Gleb Koshevoy, and Kazuo Murota. Discrete convexity and equilibria in economies with indivisible goods and money. *Mathematical Social Sciences*, 41:251–273, 2001. doi: https://doi.org/10.1016/S0165-4896(00)00071-8.

[16] Joanna Drummond and Craig Boutilier. Elicitation and approximately stable matching with partial preferences. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI-2013)*, pages 97–105, 2013.

[17] Lester Dubins and David Freedman. Machiavelli and the Gale-Shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494, 1981. doi: 10.2307/2321753.

[18] Lars Ehlers and Bettina Klaus. Object allocation via deferred-acceptance: Strategy-proofness and comparative statics. *Games and Economic Behavior*, 97:128–146, 2016.

[19] Lars Ehlers, Isa Hafalir, M. Bumin Yenmez, and Muhammed Yildirim. School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic Theory*, 153:648–683, 2014. doi: 10.1016/j.jet.2014.03.004.

[20] Tamás Fleiner. A matroid generalization of the stable matching polytope. In *Proceedings of the 8th International Conference on Integer Programming and Combinatorial Optimization (IPCO-01)*, pages 105–114. Springer, 2001.

[21] Daniel Fragiadakis and Peter Troyan. Improving matching under hard distributional constraints. *Theoretical Economics.*, 12(2):863–908, 2017. doi: 10.3982/TE2195. URL `https://ideas.repec.org/a/the/publsh/2195.html`.

[22] Daniel Fragiadakis, Atsushi Iwasaki, Peter Troyan, Suguru Ueda, and Makoto Yokoo. Strategyproof matching with minimum quotas. *ACM Transactions on Economics and Computation*, 4(1):6:1–6:40, 2015. doi: 10.1145/2841226. URL `https://doi.org/10.1145/2841226`.

[23] Satoru Fujishige. *Submodular Functions and Optimizations*, volume 58 of *Annals of Discrete Mathematics*. Elsevier, Amsterdam, 2nd edition, 2005.

[24] Satoru Fujishige and Akihisa Tamura. A general two-sided matching market with discrete concave utility functions. *Discrete Applied Mathematics*, 154(6): 950–970, 2006. doi: https://doi.org/10.1016/j.dam.2005.10.006.

[25] Satoru Fujishige and Akihisa Tamura. A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete convex analysis. *Mathematics of Operations Research*, 32(1):136–155, 2007. doi: 10.1287/moor.1070.0227.

[26] Etsushi Fujita, Julien Lesca, Akihisa Sonoda, Taiki Todo, and Makoto Yokoo. A complexity approach for core-selecting exchange with multiple indivisible goods under lexicographic preferences. *Journal of Artificial Intelligence Research*, 63: 515–555, 2018.

[27] David Gale and Lloyd Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962. doi: 10.2307/2312726.

[28] Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601, 1973. doi: 10.2307/1914083.

[29] Masahiro Goto, Atsushi Iwasaki, Yujiro Kawasaki, Ryoji Kurata, Yosuke Yasuda, and Makoto Yokoo. Strategyproof matching with regional minimum and maximum quotas. *Artificial Intelligence*, 235:40–73, 2016. doi: https://doi.org/10.1016/j.artint.2016.02.002.

[30] Masahiro Goto, Fuhito Kojima, Ryoji Kurata, Akihisa Tamura, and Makoto Yokoo. Designing matching mechanisms under general distributional constraints. *American Economic Journal: Microeconomics*, 9(2):226–262, 2017. doi: 10.1257/mic.20160124.

[31] Pablo Guillen and Onur Kesten. Matching markets with mixed ownership: the case for a real-life assignment mechanism. *International Economic Review*, 53: 1027–1046, 2012. doi: 10.1111/j.1468-2354.2012.00710.x.

[32] Isa Hafalir, M. Bumin Yenmez, and Muhammed Yildirim. Effective affirmative action in school choice. *Theoretical Economics*, 8(2):325–363, 2013.

[33] Naoto Hamada, Chia-Ling Hsu, Ryoji Kurata, Takamasa Suzuki, Suguru Ueda, and Makoto Yokoo. Strategy-proof school choice mechanisms with minimum quotas and initial endowments. *Artificial Intelligence*, 249:47 – 71, 2017. doi: https://doi.org/10.1016/j.artint.2017.04.006.

[34] John Hatfield and Paul Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, 2005. doi: 10.1257/0002828054825466.

[35] Hadi Hosseini and Kate Larson. Multiple assignment problems under lexicographic preferences. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS-19)*, pages 837–845, 2019.

[36] Hadi Hosseini, Kate Larson, and Robin Cohen. On manipulablity of random serial dictatorship in sequential matching with dynamic preferences. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI-2015)*, pages 4168–4169, 2015. doi: 10.1609/aaai.v29i1.9744.

[37] Hadi Hosseini, Fatima Umar, and Rohit Vaish. Accomplice manipulation of the deferred acceptance algorithm. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI-21)*, pages 231–237, 2021.

[38] Chien-Chung Huang. Cheating by men in the Gale-Shapley stable matching algorithm. In *Proceedings of the 14th Annual European Symposium on Algorithms (ESA-06)*, volume 4168 of *Lecture Notes in Computer Science*, pages 418–431. Springer, 2006.

[39] Woonghee Tim Huh and Ganesh Janakiraman. On the optimal policy structure in serial inventory systems with lost sales. *Operations Research*, 58:486–491, 2010. doi: 10.1287/opre.1090.0716.

[40] Anisse Ismaili, Naoto Hamada, Yuzhe Zhang, Takamasa Suzuki, and Makoto Yokoo. Weighted matching markets with budget constraints. *Journal of Artificial Intelligence Research*, 65:393–421, 2019. doi: 10.1613/jair.1.11582.

[41] Yuichiro Kamada and Fuhito Kojima. Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105(1):67–99, 2015. doi: 10.1257/aer.20101552.

[42] Yuichiro Kamada and Fuhito Kojima. Stability concepts in matching under distributional constraints. *Journal of Economic Theory*, 168:107–142, 2017. doi: https://doi.org/10.1016/j.jet.2016.12.006.

[43] Yuichiro Kamada and Fuhito Kojima. Fair matching under constraints: Theory and applications. *Review of Economic Studies*, page rdad046, 2023.

[44] Takehiro Kawasaki, Ryoji Wada, Taiki Todo, and Makoto Yokoo. Mechanism design for housing markets over social networks. In Frank Dignum, Alessio Lomuscio, Ulle Endriss, and Ann Nowé, editors, *Proceeding of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS-21)*, pages 692–700, 2021.

[45] Yasushi Kawase and Atsushi Iwasaki. Near-feasible stable matchings with budget constraints. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI-2017)*, pages 242–248, 2017. doi: 10.24963/ijcai.2017/35.

[46] Kei Kimura, Kwei-guu Liu, Zhaohong Sun, Kentaro Yahiro, and Makoto Yokoo. Multi-stage generalized deferred acceptance mechanism: Strategyproof mechanism for handling general hereditary constraints. *arXiv preprint arXiv: 2309.10968*, 2023.

[47] Fuhito Kojima. School choice: Impossibilities for affirmative action. *Games and Economic Behavior*, 75(2):685–693, 2012.

[48] Fuhito Kojima, Akihisa Tamura, and Makoto Yokoo. Designing matching mechanisms under constraints: An approach from discrete convex analysis. *Journal of Economic Theory*, 176:803–833, 2018. doi: 10.1016/j.jet.2018.05.004.

[49] Bernhard Korte and Jens Vygen. *Combinatorial Optimization, Theory and Algorithms, Fifth Edition.* Springer, 2012.

[50] Ryoji Kurata, Naoto Hamada, Atsushi Iwasaki, and Makoto Yokoo. Controlled school choice with soft bounds and overlapping types. *Journal of Artificial Intelligence Research*, 58:153–184, 2017. doi: 10.1613/jair.5297.

[51] Kwei-guu Liu, Kentaro Yahiro, and Makoto Yokoo. Strategyproof mechanism for two-sided matching with resource allocation. *Artificial Intelligence*, 316:103855, 2023. doi: 10.1016/j.artint.2023.103855.

[52] Tyler Lu and Craig Boutilier. Effective sampling and learning for mallows models with pairwise-preference data. *Journal of Machine Learning Research*, 15:3963–4009, 2014.

[53] Colin Mallows. Non-null ranking models. I. *Biometrika*, 44(1-2):114–130, 1957. doi: 10.2307/2333244.

[54] Eric Maskin. Nash Equilibrium and Welfare Optimality. *The Review of Economic Studies*, 66(1):23–38, 1999.

[55] Kazuo Murota. *Discrete Convex Analysis*, volume 10 of *SIAM Monographs on Discrete Mathematics and Applications.* Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[56] Kazuo Murota. *Matrices and matroids for systems analysis*, volume 20. Springer Science & Business Media, 2009.

[57] Kazuo Murota. Discrete convex analysis: A tool for economics and game theory. *Journal of Mechanism and Institution Design*, 1:151–273, 2016. doi: 10.22574/jmid.2016.12.005.

[58] Kazuo Murota and Akiyoshi Shioura. M-convex function on generalized polymatroid. *Mathematics of Operations Research*, 24(1):95–105, 1999. doi: 10.1287/moor.24.1.95.

[59] Kazuo Murota and Akihisa Tamura. Application of M-convex submodular flow problem to mathematical economics. *Japan Journal of Industrial and Applied Mathematics*, 20:257–277, 2003. doi: https://doi.org/10.1007/BF03167422.

[60] Kazuo Murota and Yu Yokoi. On the lattice structure of stable allocations in a two-sided discrete-concave market. *Mathematics of Operations Research*, 40(2): 460–473, 2015. doi: 10.1287/moor.2014.0679.

[61] Kazuo Murota, Akiyoshi Shioura, and Zaifu Yang. Computing a walrasian equilibrium in iterative auctions with multiple differentiated items. In *Proceedings of the 24th International Symposium on Algorithms and Computation (ISAAC-13)*, volume 8283 of *Lecture Notes in Computer Science*, pages 468–478. Springer, 2013.

[62] Szilvia Pápai. Strategyproof assignment by hierarchical exchange. *Econometrica*, 68(6):1403–1433, 2000.

[63] Szilvia Pápai. Strategyproof and nonbossy multiple assignments. *Journal of Public Economic Theory*, 3(3):257–271, 2001.

[64] Marek Pycia and M. Utku Ünver. Incentive compatible allocation and exchange of discrete resources. *Theoretical Economics*, 12:287–329, 2017. doi: 10.3982/TE2201.

[65] Alvin Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4):617–628, 1982. doi: 10.1287/moor.7.4.617.

[66] Alvin Roth. The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory*, 36(2):277–288, 1985. doi: 10.1016/0022-0531(85)90106-1.

[67] Alvin Roth and Andrew Postlewaite. Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics*, 4(2):131–137, 1977.

[68] Alvin Roth and Marilda Oliveira Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis (Econometric Society Monographs)*. Cambridge University Press, 1990. doi: 10.1017/CCOL052139015X.

[69] Mark Satterthwaite and Hugo Sonnenschein. Strategy-proof allocation mechanisms at differentiable points. *Review of Economic Studies*, 48(4):587–597, 1981.

[70] Alexander Schrijver. *Combinatorial Optimization – Polyhedra and Efficiency.* Springer, 2003.

[71] Lloyd Shapley and Herbert Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1:23 – 37, 1974.

[72] Sujoy Sikdar, Sibel Adali, and Lirong Xia. Mechanism design for multi-type housing markets. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI-17)*, pages 684–690, 2017.

[73] Tayfun Sönmez and Tobias Switzer. Matching with (branch-of-choice) contracts at the United States military academy. *Econometrica*, 81(2):451–488, 2013. ISSN 1468-0262.

[74] Ning Sun and Zaifu Yang. Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica*, 74:1385–1402, 2006.

[75] Takamasa Suzuki, Akihisa Tamura, Kentaro Yahiro, Makoto Yokoo, and Yuzhe Zhang. Strategyproof allocation mechanisms with endowments and M-convex distributional constraints. *Artificial Intelligence*, 315:103825, 2023. doi: 10.1016/j.artint.2022.103825.

[76] Kentaro Tomoeda. Finding a stable matching under type-specific minimum quotas. *Journal of Economic Theory*, 176:81–117, 2018.

[77] Rohit Vaish and Dinesh Garg. Manipulating Gale-Shapley algorithm: Preserving stability and remaining inconspicuous. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI-17)*, pages 437–443, 2017.

[78] Robert Wilson. *Game-theoretic analyses of trading processes.* Institute for Mathematical Studies in the Social Sciences, Stanford University, 1985.

[79] Qingyun Wu and Alvin Roth. The lattice of envy-free matchings. *Games and Economic Behavior*, 109:201–211, 2018.

[80] Kentaro Yahiro, Yuzhe Zhang, Nathanaël Barrot, and Makoto Yokoo. Strategyproof and fair matching mechanism for ratio constraints. *Autonomous Agents and Multi-Agent Systems*, 34:1–29, 2020. doi: 10.1007/s10458-020-09448-9.

[81] Yuzhe Zhang, Kentaro Yahiro, Nathanaël Barrot, and Makoto Yokoo. Strategyproof and fair matching mechanism for union of symmetric M-convex constraints. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI-18)*, pages 590–596, 2018.

[82] Paul Zipkin. On the structure of lost-sales inventory models. *Operational Research*, 56:937–944, 2008. doi: 10.1287/opre.1070.0482.