Classifications of singularities of plane curves and geometric invariants of flat fronts in the unit 3-sphere

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Classifications of singularities of plane curves and geometric invariants of flat fronts in the unit 3-sphere

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1. **Introduction**

Let f be a C^∞ map from a smooth manifold into a smooth manifold. A point p is called a singular point of f if f is not an immersion at p. Singularities are unavoidable and important subjects in differential geometry, since they naturally appear on parallel plane curves, evolutes, caustics and surfaces with limited curvatures. However, there have been many studies of regular curves and regular surfaces which do not have singularities. This is because the unit normal vector field which is defined on regular parts may not necessarily be extended smoothly to singular points and curvatures may diverge at singular points. Moreover, treatments of singularities are not straightforward even for local behavious. For example, as is well known in the general theory of manifolds, there are essentially only the two types of behaviors at regular points, but there are uncountable types of behaviors at singular points, even in the cases of simple functions ([51]). Thus, singularities have difficulties in various aspects such as geometrical considerations and classifications of local behaviors. In recent years, studies of wave fronts which originate from the Huygens' principle are remarkable ([21–23, 25, 29, 33, 42–44, 46]).

In this thesis, we consider flat fronts in the unit 3-sphere S^3 and classifications of cusps appearing on wave fronts in the Euclidean plane $\mathbb{R}^2$.

In Chapter 2, we consider geometrical invariants of flat front in S^3 at non-degenerate singular points. Hartman and Nirenberg proved that a complete flat regular surface in the Euclidean space $\mathbb{R}^3$ is a cylinder over a regular plane curve ([11]). Thus, it is important to consider flat surfaces with singular points in studying global properties of such surfaces. Recently, in addition to flat fronts in $\mathbb{R}^3$, there has been studies on flat fronts in S^3 and the hyperbolic 3-space H^3 ([20–23, 29, 33, 41, 43, 46]). In particular, Saji, Umehara and Yamada studied a relationship between the *extrinsic Gaussian curvature* K_{E} (i.e. the product of the principal curvatures) and the *singular curvature* ([46, Theorem 3.1]). For geometrical meaning of the singular curvature, see ([46, Corollary 1.18] or Figures 9 and 10)

Fact 1.0.1. *([46, Theorem 3.1]). Let M^2 be an oriented 2-dimensional manifold, N^3 an oriented Riemannian 3-dimensional manifold and $f\colon M^2 \to N^3$ a flat front (see Definition 2.2.7). Let p be a singular point of f and γ a singular curve consisting of non-degenerate singular points with $\gamma(0) = p$ defined on an open interval $I \subset \mathbb{R}$.*

(1) *If the extrinsic Gaussian curvature of f is nonnegative on $V^2 \setminus \gamma(I)$ for a neighborhood V^2 of p, then its singular curvature is nonpositive at p. Moreover, if the extrinsic Gaussian curvature of f is a positive constant on $V^2 \setminus \gamma(I)$, then its singular curvature is strictly negative at p.*
(2) *In particular, assume that $N^3 = \mathbb{R}^3$. If the Guassian curvature of f is nonnegative on $V^2 \setminus \gamma(I)$, then its singular curvature is nonpositive.*

We note that even if the Gaussian curvature is strictly negative at p, the singular curvature is not necessarily positive (for example, see [46], [31, Theorem 3.1]). In [48, Theorem A], Teramoto investigated the inverse of the statement of Fact 1.0.1 (2). By Fact 1.0.1 and $K_{\mathrm{I}} = K_{\mathrm{E}} + K_{N^3}$, we notice that the sign of the singular curvature of flat fronts in H^3 is strictly negative, where K_{I} is the *intrinsic Gaussian curvature* of the considered front and K_{N^3} is the sectional curvature of N^3. However, it is difficult to investigate the singular curvature of flat fronts in S^3 using only Fact 1.0.1. In this chapter, we use a solution of the wave equation to give an expression for the singular curvature (Theorem 2.4.3). Moreover, we study the mean curvature of flat fronts in S^3 (Propositions 2.5.1, 2.5.2 and Theorem 2.6.7).

In Chapter 3, we study relationships between $(n, n+1)$-cusps and evolutes of wave fronts in $\mathbb{R}^2$. We note that singularities of $(n, n+1)$-cusps appear on wave fronts. If a plane curve has singular points, we cannot define the evolute of the plane curve because the curvature may diverge at singular points. However, Fukunaga and Takahashi defined evolutes of wave fronts and gave their properties ([8]). As applications of their properties and invariants of $(n, n+1)$-cusps, we give characterizations of $(n, n+1)$-cusps and negative judgments for $(n, n+1)$-cusps in the sense of $\mathcal{A}$-equivalent (Theorem 3.3.2 and Corollary 3.3.7). Moreover, we prove that the invariants correspond to rough criteria for $(n, n+1)$-cusps (Theorem 3.3.1).

In Chapter 4, we give a criterion for $(4, 5)$-cusps. Bruce and Gaffney classified cusps in the sense of complex C^ω-equivalence and gave important types of cusps ([4]). In general, it is difficult to examine such types of cusps from definition, directly. Therefore, investigating simpler criteria for their singularities is an important subject. In particular, it is important to give a criterion for $(4, 5)$-cusps, because they relate not only the evolutes of fronts but also classifications in the sense of complex C^ω-equivalence. However, local behaviors of $(4, 5)$-cusps are complex and have not been well analyzed. To analyze the behaviors, we introduce a new invariant of $(4, 5)$-cusps. This new invariant is called the $(4, 5; \pm 7)$-*cuspidal curvature*. Using a property of the invariant, we give complete classifications with respect to $(4, 5)$-cusps (Theorem 4.5.2). As a consequence of this result, we notice a new type singularity.

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2. Flat fronts and their caustics in the unit 3-sphere

2.1. **Introduction.** A flat surface in the unit 3-sphere $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ is an immersion with vanishing (intrinsic) Gaussian curvature. It is known that a flat surface in S^3 corresponds to a solution of the (linear) *wave equation*:

$$\theta_{uu} - \theta_{vv} = 0, \tag{2.1}$$

where $\theta \colon \Sigma \to (0, \pi)$ is a C^∞ function defined on a simply connected domain $\Sigma \subset \mathbb{R}^2$ (cf. [6, 35, 47]). In particular, the first and the second fundamental forms I^f, II^f of the surface can be written as

$$I^f = \cos^2 \frac{\theta}{2} du^2 + \sin^2 \frac{\theta}{2} dv^2, \quad II^f = \frac{1}{2} \sin \theta (du^2 - dv^2), \tag{2.2}$$

where θ is a solution to the equation (2.1) ([35]). If we extend the target of θ to $\mathbb{R}$, then the corresponding flat surface in S^3 may have singular points. In fact, if $p \in \theta^{-1}(\pi\mathbb{Z})$, then the first fundamental form I^f as in (2.2) degenerates. This implies that the flat surface associated to the solution θ is not an immersion at $p \in \theta^{-1}(\pi\mathbb{Z})$. On the other hand, it is known that a unit normal vector (or the *dual*) of the flat surface can be defined smoothly even at a singular point ([43, Theorem 2.11 and Corollary 2.12]). Thus one may extend a flat surface to a *flat front* in S^3.

In this chapter, we investigate singularities and geometric properties of a flat front associated to the solution θ of (2.1) and its dual front. To observe these properties, we use curvature line moving frame along the flat front. We give characterizations of singularities of a flat front and its dual by the solution θ to (2.1). After that, we investigate geometric invariants defined at a cuspidal edge on a flat front and its dual in Section 2.4. In particular, we show explicit representations for some geometric invariants by using the solution θ of the corresponding wave equation. Namely,

Theorem 2.1.1. *([32]) Let $f : \Sigma \to S^3$ be a flat front corresponding to the solution θ of (2.1). Let ν be its dual flat front. Suppose that $p \in \theta^{-1}(\pi\mathbb{Z})$ is a cuspidal edge of both f and ν, i.e., $\theta_u \neq 0$ and $\theta_v \neq 0$ at p.*

(1) *If $p \in \theta^{-1}(Z_1)$, then the invariants for f along a singular curve $\gamma(t)$ through p defined in Definition 2.4.1 are*

$$\kappa_s(t) = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}, \quad \kappa_c(t) = -\mathrm{sgn}(\theta_u) \frac{2\sqrt{2}}{|\theta_u|^{1/2}}, \quad \kappa_t(t) = \frac{\theta_v}{\theta_u} \tag{2.3}$$

and the invariants for ν along $\gamma(t)$ are

$$\kappa_s^\nu(t) = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}, \quad \kappa_c^\nu(t) = \mathrm{sgn}(\theta_v) \frac{2\sqrt{2}}{|\theta_v|^{1/2}}, \quad \kappa_t^\nu(t) = \frac{\theta_u}{\theta_v}. \tag{2.4}$$

(2) *If $p \in \theta^{-1}(Z_2)$, then the invariants for f along a singular curve $\gamma(t)$ through p defined in Definition 2.4.1 are*

$$\kappa_s(t) = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}, \quad \kappa_c(t) = \mathrm{sgn}(\theta_v) \frac{2\sqrt{2}}{|\theta_v|^{1/2}} \quad \kappa_t(t) = \frac{\theta_u}{\theta_v} \tag{2.5}$$

and the invariants for ν along $\gamma(t)$ are

$$\kappa_s^\nu(t) = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}, \quad \kappa_c^\nu(t) = -\,\mathrm{sgn}(\theta_u) \frac{2\sqrt{2}}{|\theta_u|^{1/2}}, \quad \kappa_t^\nu(t) = \frac{\theta_v}{\theta_u}. \tag{2.6}$$

By this result, we notice some dual relations between a flat front and its dual front via geometric invariants. In Section 2.6, we investigate caustics of flat fronts in S^3. Considering singularities of caustics, we clarify relationships between types of singularities of the initial flat front and its caustic. Moreover, we observe the mean curvatures of caustics. Namely,

Theorem 2.1.2. *([32]) Let $f\colon \Sigma \to S^3$ be a flat front corresponding to the function θ satisfying (2.1). Then the mean curvature of the caustic C_1 (resp. C_2) as in (2.34) is represented as*

$$H^{C_1} = -\frac{\varepsilon_1}{\theta_u}\left(\frac{\theta_u^2}{4} - \frac{\theta_v^2}{4} - 1\right) \quad \left(\textit{resp. } H^{C_2} = \frac{\varepsilon_2}{\theta_v}\left(\frac{\theta_v^2}{4} - \frac{\theta_u^2}{4} - 1\right)\right) \tag{2.7}$$

on the set of regular points, where $\varepsilon_1 = \operatorname{sgn}(\csc\frac{\theta}{2})$ (resp. $\varepsilon_2 = \operatorname{sgn}(\sec\frac{\theta}{2})$). If p is a singular point of C_1 (resp. C_2), then H^{C_1} (resp. H^{C_2}) is unbounded near p.

We note that the Gaussian curvatures of caustics of flat fronts also vanishes ([20]). Furthermore, investigating the dual of the caustic, we clarify relation between mean curvatures of the caustic and its dual (Proposition 2.6.10). Finally, we give an explicit representation of the solution θ to (2.1) corresponding to a flat front whose caustic is a minimal surface (Proposition 2.7.1).

2.2. **Preliminaries.**

2.2.1. **Fronts.** Let $f\colon \Sigma \to S^3$ be a C^∞ map, where Σ is an open domain in $(\mathbb{R}^2; u, v)$. Then a point $p \in \Sigma$ is called a *singular point* of f if f is not an immersion at p, that is, $\operatorname{rank} df_p < 2$ holds. The map f is said to be a *frontal* if there exists a C^∞ map $\nu\colon \Sigma \to S^3$ such that $\langle f, \nu\rangle = \langle f_u, \nu\rangle = \langle f_v, \nu\rangle = 0$ on Σ, where $\langle\cdot,\cdot\rangle$ is the canonical inner product of $\mathbb{R}^4$, $(\cdot)_u = \partial(\cdot)/\partial u$ and $(\cdot)_v = \partial(\cdot)/\partial v$. Moreover, a frontal f is called a *front* if the mapping $(f, \nu)\colon \Sigma \to \Delta \subset S^3 \times S^3$ gives an immersion, where $\Delta = \{(\boldsymbol{x}, \boldsymbol{y}) \in S^3 \times S^3 \mid \langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0\}$. We call ν a *unit normal vector field* or (Δ-)*dual* of f, because ν is also a front in S^3 with unit normal vector $-f$ ([1,37,38]). Thus f and ν are in *dual relation* ([16,34]).

Let $f\colon \Sigma \to S^3$ be a frontal with the dual ν. We set a function $\lambda\colon \Sigma \to \mathbb{R}$ by

$$\lambda(u, v) = \det(f, f_u, f_v, \nu)(u, v), \tag{2.8}$$

where $\det$ is the determinant of 4×4 matrices. We call the function λ the *signed area density function* of f. A function $\tilde\lambda$ which is nowhere-vanishing functional multiple of λ is called an *identifier of singularities* of f. Let us denote by $S(f)$ the set of singular points of f. Then we see that $S(f) = \lambda^{-1}(0) = \tilde\lambda^{-1}(0)$ holds.

Definition 2.2.1. *Let $f\colon \Sigma \to S^3$ be a frontal. A singular point $p \in S(f)$ is said to be non-degenerate if $(\tilde\lambda_u, \tilde\lambda_v) \neq (0, 0)$ at p.*

Assume that p is a non-degenerate singular point of f. By the implicit function theorem, there exist an open neighborhood $U(\subset \Sigma)$ of p and a C^∞ regular curve $\gamma\colon (-\varepsilon, \varepsilon) \ni t \mapsto \gamma(t) \in U$ ($\varepsilon > 0$) such that $\gamma(0) = p$ and $\tilde\lambda(\gamma(t)) = 0$ on U. We call γ a *singular curve* of f. This implies that non-degenerate singular points are parametrized by a regular plane curve. Since $\operatorname{rank} df_p = 1$ at the non-degenerate singular point p, there exists a non-vanishing vector field η on U such that $df_q(\eta) = 0$ holds for any $q \in S(f) \cap U$. We call η a *null vector field* of f.

Definition 2.2.2. *Let $f\colon \Sigma \to S^3$ be a frontal and p be a non-degenerate singular point of f. Then p is said to be of the first kind if the tangent vector $\gamma' = d\gamma/dt$ to γ and a null vector field η are linearly independent at p. Otherwise, it is said to be of the second kind ([29]).*

Definition 2.2.3. *Let M^3 be a 3-dimensional manifold, $f\colon \Sigma \to M^3$ be a C^∞ map and p be a singular point of f.*

(1) *A point p is called a cuspidal edge if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ at the origin.*
(2) *A point p is called a swallowtail if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at the origin.*
(3) *A point p is called a cuspidal cross cap if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, uv^3)$ at the origin.*
(4) *A point p is called a cuspidal butterfly if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 5v^4 + 2uv, uv^2 + 4v^5)$ at the origin.*
(5) *A point p is called a cuspidal beaks if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^3 - u^2v, 3v^4 - 2u^2v^2)$ at the origin.*
(6) *A point p is called a cuspidal lips if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, vu^2 + v^3, 2u^2v^2 + 3v^4)$ at the origin.*
(7) *A point p is called a $D_4^\pm$ singularity if the map germ f at p is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (2uv, \pm u^2 + 3v^2, \pm 2u^2v + 2v^3)$ at the origin.*

Here we consider f as a map germ $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ by taking local coordinate systems of Σ and M^3 at p and $f(p)$, respectively, and two map germs $f_1, f_2\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\varphi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\Phi\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\Phi \circ f_1 \circ \varphi^{-1} = f_2$ holds.

Cuspidal edges are singular points of the first kind and swallowtails and cuspidal butterflies are singular points of the second kind. In particular it is known that cuspidal edges and swallowtails are the generic singularities of fronts. Cuspidal cross caps are singular points of the first kind of frontals. Cuspidal beaks and cuspidal lips are corank one degenerate singular points and $D_4^\pm$ singularities are corank two degenerate singular points. It is known that cuspidal butterflies, cuspidal beaks, cuspidal lips and $D_4^\pm$ singularities are singularities of the bifurcations in generic one parameter families of fronts (cf. [2, 3]). In this chapter, we mainly deal with cuspidal edges and swallowtails. Criteria for these singularities are known ([21, Proposition 1.3] and [42, Corollary 2.5]).

Fact 2.2.4. *Let $f\colon \Sigma \to S^3$ be a front. Let p be a non-degenerate singular point of f. Take a null vector field η defined near p. Then*

(1) *f is a cuspidal edge at p if and only if $\eta\lambda \neq 0$ at p. This is equivalent to $\eta\tilde{\lambda}(p) \neq 0$.*
(2) *f is a swallowtail at p if and only if $\eta\lambda = 0$ and $\eta\eta\lambda \neq 0$ at p. This is equivalent to $\eta\tilde{\lambda}(p) = 0$ and $\eta\eta\tilde{\lambda}(p) \neq 0$,*

where λ is the signed area density function of f, $\tilde{\lambda}$ an identifier of singularities of f and $\eta\lambda$ means the directional derivative of λ in the direction η.

It is known that criteria for cuspidal butterflies, cuspidal beaks, cuspidal lips and $D_4^\pm$ singularities in $\mathbb{R}^3$ ([17, 18, 40, 42]) .

Fact 2.2.5. *Let $f\colon \Sigma \to \mathbb{R}^3$ be a front. Assume that p is a singular point of f. Then the followings hold.*

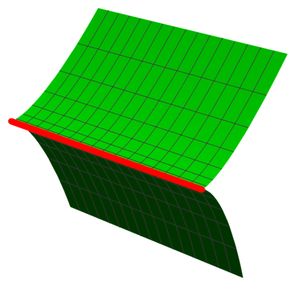

FIGURE 1.
cuspidal edge

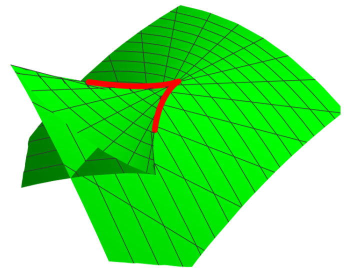

FIGURE 2.
swallowtail

- *f is a cuspidal butterfly at p if and only if $d\lambda \neq 0$, $\eta\lambda = \eta\eta\lambda = 0$ and $\eta\eta\eta\lambda \neq 0$ at p.*
- *f is a cuspidal beaks at p if and only if $d\lambda = 0$, $\eta\eta\lambda \neq 0$ and $\det Hess(\lambda) < 0$ at p.*
- *f is a cuspidal lips at p if and only if $d\lambda = 0$ and $\det Hess(\lambda) > 0$ at p.*
- *f is a D_4^+ singularity (resp. D_4^- singularity) at p if and only if $\operatorname{rank} df = 0$ and $\det Hess(\lambda) < 0$ (resp. $\det Hess(\lambda) < 0$) at p.*

Where $Hess(\lambda)$ is defined as follow :

$$\begin{pmatrix} \lambda_{uu} & \lambda_{uv} \\ \lambda_{uv} & \lambda_{vv} \end{pmatrix}.$$

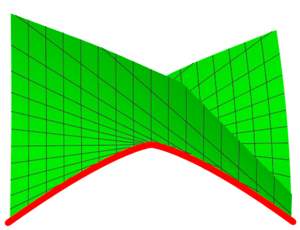

FIGURE 3.
cuspidal butterfly

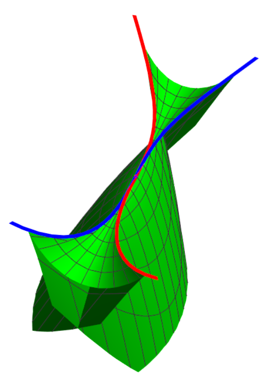

FIGURE 4.
cuspidal beaks

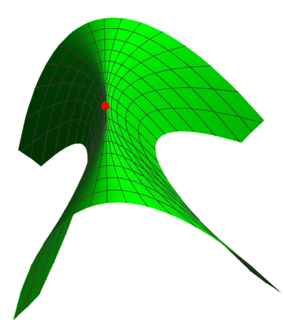

FIGURE 5.
cuspidal lips

Moreover, a criterion for cuspidal cross caps are known. Let $f : U \to \mathbb{R}^3$ be a frontal defined on an open neighborhood of the origin in $\mathbb{R}^2$ and $\hat{\nu}$ be a (not neccessarily unit) normal vector field of f. Assume that the origin in $\mathbb{R}^2$ is a non-degenerate singular point of f. Let $\gamma(t) = (u(t), v(t))$ be a singular curve such that $\gamma(0) = (0, 0)$ and $\eta(t) = (a(t), b(t))$ be a null vector field along γ. Then we set

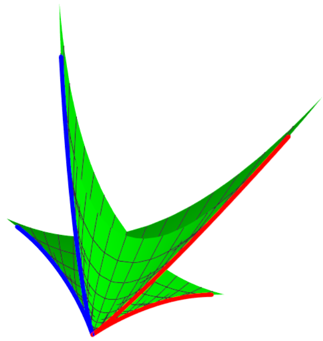

FIGURE 6. D_4^+ singularity

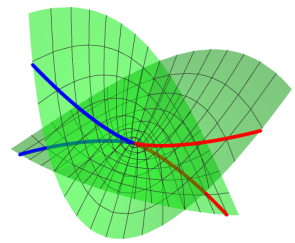

FIGURE 7. D_4^- singularity

$$\tau(t) = \det\left(\hat{\gamma}'(t), \hat{\nu}(u(t), v(t)), \hat{\nu}_\eta(t)\right),$$
$$\hat{\nu}_\eta(t) = a(t)\hat{\nu}_u(u(t), v(t)) + b(t)\hat{\nu}_v(u(t), v(t)),$$

where $\hat{\gamma} = f \circ \gamma$ and $\hat{\gamma}'(t)$ is the differentiation of $\hat{\gamma}$ with respect to t.

Fact 2.2.6. *([7]) Let $f : \Sigma \to \mathbb{R}^3$ be a C^∞ map defined on an open neighborhood of the origin in $\mathbb{R}^2$. Then f is a cuspidal cross cap at $(0,0)$ if and only if the origin is a singular point of the first kind, $\tau(0) = 0$ and $\tau'(0) \neq 0$.*

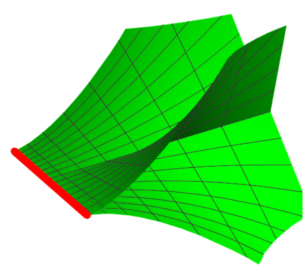

FIGURE 8.
cuspidal cross cap

2.2.2. **Flat fronts in S^3.** Let $f : \Sigma \to S^3$ be a *flat* surface, i.e., an immersed surface with vanishing (*intrinsic*) *Gaussian curvature* K_{I} in S^3. Since the *extrinsic Gaussian curvature* (or the *Gauss-Kronecker curvature*) K_{E} of f satisfies $K_{\mathrm{I}} = K_{\mathrm{E}} + 1$, we have $K_{\mathrm{E}} \equiv -1$.

To extend this notion, we give the following definition.

Definition 2.2.7. *Let $f : \Sigma \to S^3$ be a front. Then f is called a flat front if there exists an open dense subset O in Σ such that $f|_O : O \to S^3$ gives a flat surface.*

As mentioned in Section 2.1, for a flat front, we can take a local coordinate system (u, v) and a function $\theta\colon \Sigma \to \mathbb{R}$ satisfying (2.1) so that the first and second fundamental forms can be written as (2.2). Thus there exist C^∞ maps $\boldsymbol{e}_1, \boldsymbol{e}_2\colon \Sigma \to S^3$ such that

$$f_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad f_v = \sin\frac{\theta}{2}\boldsymbol{e}_2. \tag{2.9}$$

We consider derivatives of the frame $\{f, \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\}$ along f.

Proposition 2.2.8. *The Christoffel symbols Γ^i_{jk} $(i, j, k = 1, 2)$ are given by*

$$\begin{aligned}
&\Gamma^1_{11} = -\frac{\theta_u}{2}\tan\frac{\theta}{2}, \quad \Gamma^2_{11} = \frac{\theta_v}{2}\cot\frac{\theta}{2}, \quad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{\theta_v}{2}\tan\frac{\theta}{2},\\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{\theta_u}{2}\cot\frac{\theta}{2}, \quad \Gamma^1_{22} = -\frac{\theta_u}{2}\tan\frac{\theta}{2}, \quad \Gamma^2_{22} = \frac{\theta_v}{2}\cot\frac{\theta}{2}
\end{aligned} \tag{2.10}$$

on the set of regular points $\Sigma \setminus \theta^{-1}(\pi\mathbb{Z})$ of f.

Proof. Since the first fundamental form of f is given by (2.2), we have the assertion by direct calculations. $\square$

We note that the functions Γ^i_{jk} in (2.10) might be unbounded near a singular point $p \in \theta^{-1}(\pi\mathbb{Z})$. Moreover, we have the following by direct calculations.

Proposition 2.2.9.

$$\begin{aligned}
f_{uu} &= -\cos^2\frac{\theta}{2}f + \Gamma^1_{11}f_u + \Gamma^2_{11}f_v + \frac{1}{2}\sin\theta\nu\\
&= -\cos^2\frac{\theta}{2}f - \frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{1}{2}\sin\theta\nu,\\
f_{uv} = f_{vu} &= \Gamma^1_{12}f_u + \Gamma^2_{12}f_v = -\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2,\\
f_{vv} &= -\sin^2\frac{\theta}{2}f + \Gamma^1_{22}f_u + \Gamma^2_{22}f_v - \frac{1}{2}\sin\theta\nu\\
&= -\sin^2\frac{\theta}{2}f - \frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\sin\theta\nu.
\end{aligned} \tag{2.11}$$

These can be extended to the set of singular points $\theta^{-1}(\pi\mathbb{Z})$.

On the other hand, by differentiating (2.9), we have

$$\begin{aligned}
&f_{uu} = -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \cos\frac{\theta}{2}(\boldsymbol{e}_1)_u,\\
&f_{uv} = -\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \cos\frac{\theta}{2}(\boldsymbol{e}_1)_v, \quad f_{vu} = \frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}(\boldsymbol{e}_2)_u,\\
&f_{vv} = \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}(\boldsymbol{e}_2)_v.
\end{aligned} \tag{2.12}$$

Comparing (2.11) and (2.12), we have

$$\begin{aligned}
&(\boldsymbol{e}_1)_u = -\cos\frac{\theta}{2}f + \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\nu, \quad (\boldsymbol{e}_1)_v = \frac{\theta_u}{2}\boldsymbol{e}_2,\\
&(\boldsymbol{e}_2)_u = -\frac{\theta_v}{2}\boldsymbol{e}_1, \quad (\boldsymbol{e}_2)_v = -\sin\frac{\theta}{2}f - \frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu.
\end{aligned} \tag{2.13}$$

Moreover, since the second fundamental form of f is given by (2.2), we see that

$$\nu_u = -\sin\frac{\theta}{2}\boldsymbol{e}_1, \quad \nu_v = \cos\frac{\theta}{2}\boldsymbol{e}_2. \tag{2.14}$$

By (2.13) and (2.14), we have the following equations for the frame $\{f, \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\}$:

$$\begin{aligned}
\begin{pmatrix} f \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \nu \end{pmatrix}_u &= \begin{pmatrix} 0 & \cos\frac{\theta}{2} & 0 & 0 \\ -\cos\frac{\theta}{2} & 0 & \frac{\theta_v}{2} & \sin\frac{\theta}{2} \\ 0 & -\frac{\theta_v}{2} & 0 & 0 \\ 0 & -\sin\frac{\theta}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \nu \end{pmatrix}, \\
\begin{pmatrix} f \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \nu \end{pmatrix}_v &= \begin{pmatrix} 0 & 0 & \sin\frac{\theta}{2} & 0 \\ 0 & 0 & \frac{\theta_u}{2} & 0 \\ -\sin\frac{\theta}{2} & -\frac{\theta_u}{2} & 0 & -\cos\frac{\theta}{2} \\ 0 & 0 & \cos\frac{\theta}{2} & 0 \end{pmatrix} \begin{pmatrix} f \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \nu \end{pmatrix}.
\end{aligned} \tag{2.15}$$

We set $\mathcal{F} = (f, \boldsymbol{e}_1, \boldsymbol{e}_2, \nu)^t$, where ^{t} is the transpose. By (2.15), $\mathcal{F}_u = \Omega\mathcal{F}$ and $\mathcal{F}_v = \Lambda\mathcal{F}$ hold, where Ω and Λ are 4×4 matrices as in (2.15). We consider the integrability condition for these frame equations. By a direct calculation, $(\mathcal{F}_u)_v = (\mathcal{F}_v)_u$ holds if and only if $\Lambda_u - \Omega_v = \Omega\Lambda - \Lambda\Omega$. This is equivalent to (2.1).

2.3. **Singularities of flat fronts and their dual fronts.** In this section, we consider singularities of flat fronts in S^3. To give characterizations, we use criteria for a cuspidal edge and swallowtail (see Fact 2.2.4).

Proposition 2.3.1. *Let $f : \Sigma \to S^3$ be a flat front associated to the solution θ to (2.1). Then we have the following.*

(1) *Suppose that $p \in \theta^{-1}(Z_1)$. Then*
   - *f at p is a cuspidal edge if and only if $\theta_u \neq 0$ at p;*
   - *f at p is a swallowtail if and only if $\theta_u = 0$, $\theta_v \neq 0$ and $\theta_{uu} \neq 0$ at p.*

(2) *Suppose that $p \in \theta^{-1}(Z_2)$. Then*
   - *f at p is a cuspidal edge if and only if $\theta_v \neq 0$ at p;*
   - *f at p is a swallowtail if and only if $\theta_v = 0$, $\theta_u \neq 0$ and $\theta_{vv} \neq 0$ at p.*

Proof. We assume that $p \in \theta^{-1}(Z_1)$. By (2.9), p is a rank one singular point of f. Thus there exists a null vector field η_1 for f near p. Then we can take $\eta_1 = \partial_u$. In fact, by $f_u(p) = \mathbf{0}$ and $f_v(p) \neq \mathbf{0}$, we see that

$$\mathbf{0} = (df)_p(\eta_1) = f_v(p)dv(\eta_1).$$

By (2.9), the signed area density function λ of f can be calculated as $\lambda = (1/2)\sin\theta$. The gradient vector of λ is

$$(\lambda_u, \lambda_v) = \frac{1}{2}\cos\theta(\theta_u, \theta_v).$$

When $p \in \theta^{-1}(Z_1)$, $\cos\theta = -1$ at p and hence $(\lambda_u, \lambda_v) \neq (0,0)$ at p if and only if $(\theta_u, \theta_v) \neq (0,0)$ at p. This implies that p is a non-degenerate singular point of f if and only if $(\theta_u, \theta_v) \neq (0,0)$ at p.

In these preparations, we prove (1). Calculating $\eta_1\lambda$ and $\eta_1\eta_1\lambda$, we see that

$$\eta_1\lambda = \frac{\theta_u}{2}\cos\theta, \quad \eta_1\eta_1\lambda = \frac{\theta_{uu}}{2}\cos\theta - \frac{\theta_u^2}{2}\sin\theta.$$

By Fact 2.2.4 (1), p is a cuspidal edge if and only if $\theta_u \neq 0$ at p. Moreover, by Fact 2.2.4 (2), p is a swallowtail if and only if $\theta_u = 0$ and $\theta_{uu} \neq 0$ at p. Considering the non-degeneracy, we have the assertion.

For the case of $p \in \theta^{-1}(Z_2)$, we can show in a similar way by using $\eta_2 = \partial_v$. □

Next we consider singularities of dual surface ν. Assume that $S(f) = \theta^{-1}(\pi\mathbb{Z})$ consists of non-degenerate singular points of f. Then $S(\nu) = S(f)$ and $S(\nu)$ also consists of non-degenerate singular points of ν. By (2.14), when $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$), $\nu_u \neq 0$ and $\nu_v = 0$ (resp. $\nu_u = 0$ and $\nu_v \neq 0$) hold at p. Thus p is a rank one singular point of ν. By the same argument as the case of f, we can take $\eta_1^\nu = \partial_v$ (resp. $\eta_2^\nu = \partial_u$) as a null vector field of ν around $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$).

Proposition 2.3.2. *Let $f\colon \Sigma \to S^3$ be a flat front associated to the solution θ of (2.1). Let ν be the dual of f. Then we have the following.*

(1) *Suppose $p \in \theta^{-1}(Z_1)$. Then*
- *ν at p is a cuspidal edge if and only if $\theta_v \neq 0$ at p;*
- *ν at p is a swallowtail if and only if $\theta_v = 0$, $\theta_u \neq 0$ and $\theta_{vv} \neq 0$ at p.*

(2) *Suppose $p \in \theta^{-1}(Z_2)$. Then*
- *ν at p is a cuspidal edge if and only if $\theta_u \neq 0$ at p;*
- *ν at p is a swallowtail if and only if $\theta_u = 0$, $\theta_v \neq 0$ and $\theta_{uu} \neq 0$ at p.*

Proof. For ν, $-f$ can be taken as a unit normal vector to ν. Thus the signed area density function λ^ν of ν can be calculated as

$$\lambda^\nu = \det(\nu, \nu_u, \nu_v, -f) = -\frac{1}{2}\sin\theta = -\lambda,$$

where λ is the signed area density function of f. Therefore, we have the results by similar calculations for the case of f. □

Corollary 2.3.3. *Let $f\colon \Sigma \to S^3$ be a flat front associated to the solution θ of (2.1) and ν its dual. If $p \in \theta^{-1}(\pi\mathbb{Z})$ is a swallowtail of f (resp. ν), then p is a cuspidal edge of ν (resp. f).*

Proof. By Propositions 2.3.1 and 2.3.2, we have the assertion. □

2.4. **Geometric invariants.** In this section, we investigate geometric invariants of a flat front and its dual defined on cuspidal edges. First we recall definitions of invariants. Let $f\colon \Sigma \to S^3$ be a front and ν its dual front. Suppose that $p \in \Sigma$ is a cuspidal edge of f. Then there exist a neighborhood U around p and the pair of vector fields (ξ, η) on U such that

- ξ is tangent to the singular set $S(f) \cap U$;
- η is a null vector field of f.

This pair (ξ, η) is called an *adapted pair*.

Definition 2.4.1. *([25, 29, 46]) Using an adapted pair of vector fields, we set the following :*

$$
\begin{aligned}
&\kappa_s = \varepsilon_\gamma \frac{\det(f, \xi f, D_\xi \xi f, \nu)}{|\xi f|^3}, \quad \kappa_\nu = \frac{\langle D_\xi \xi f, \nu \rangle}{|\xi f|^2},\\
&\kappa_c = \frac{|\xi f|^{3/2} \det(f, \xi f, D_\eta \eta f, D_\eta D_\eta \eta f)}{|f \wedge \xi f \wedge D_\eta \eta f|^{5/2}} \qquad (2.16)\\
&\kappa_t = \frac{\det(f, \xi f, D_\eta \eta f, D_\xi D_\eta \eta f)}{|f \wedge \xi f \wedge D_\eta \eta f|^2} - \frac{\langle \xi f, D_\eta \eta f \rangle \det(f, \xi f, D_\eta \eta f, D_\xi \xi f)}{|\xi f|^2 |f \wedge \xi f \wedge D_\eta \eta f|^2}
\end{aligned}
$$

along a singular curve γ of f, where $\varepsilon_\gamma = \operatorname{sgn}(\det(\xi, \eta) \cdot \eta\lambda)$ along γ and D is the metric connection defined for a vector $\zeta \in T_{f(p)}S^3$ (resp. $\zeta \in T_{\nu(p)}S^3$),

$$D_\zeta k = \zeta k - \langle \zeta k, f \rangle f \quad (\textit{resp. } D_\zeta k = \zeta k - \langle \zeta k, \nu \rangle \nu).$$

These geometric invariants κ_s, κ_ν, κ_c and κ_t are called *the singular curvature, the limiting normal curvature, cuspidal curvature and the cusp-directional torsion*, respectively. It is known that the singular curvature κ_s is an intrinsic invariant of frontals ([46, Theorem 1.6]). The sign of this invariant relates to concavities and convexities of cuspidal edges ([46, Theorem 1.17 and Corollary 1.18], Figures 9 and 10). The limiting normal curvature relates to the boundedness of the Gaussian curvature around cuspidal edges ([29, 46], Figures 11 and 12). If a point p is a non-degenerate singular point of first kind of a frontal f, then f is a front at p if and only if the cuspidal curvature κ_c does not vanish at p ([29, Proposition 3.11]). The cusp-directional torsion κ_t measures the rotation of cusp-direction along the singular curve consisting of cuspidal edges ([25, Proposition 5.3]).

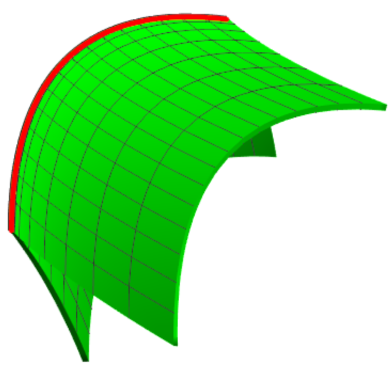

FIGURE 9. $\kappa_s > 0$

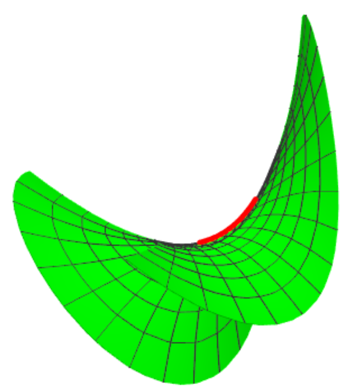

FIGURE 10. $\kappa_s < 0$

We seek an adapted pair of vector fields.

Lemma 2.4.2. *Let $f : \Sigma \to S^3$ be a flat front associated to the solution θ of the wave equation* (2.1). *We assume that $p \in \theta^{-1}(\pi\mathbb{Z})$ is a cuspidal edge of f.*

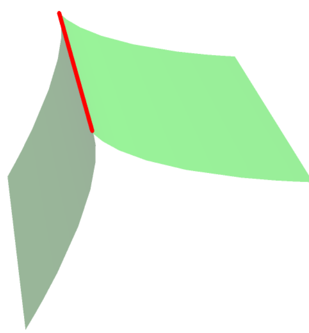

FIGURE 11. $\kappa_\nu = 0$.

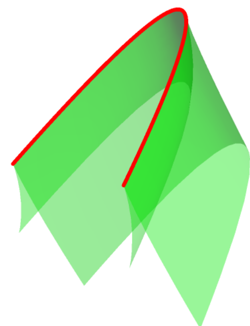

FIGURE 12. $\kappa_\nu \neq 0$.

(1) *If* $p \in \theta^{-1}(Z_1)$*, we can take* $\eta_1 = \partial_u$ *as a null vector field of* f *on an open neighborhood* U *of* p *and* $\xi_1 = -\theta_v \partial_u + \theta_u \partial_v$ *as a tangent to* γ*.*
(2) *If* $p \in \theta^{-1}(Z_2)$*, we can take an adapted pair* (ξ_2, η_2) *as* $\xi_2 = \xi_1$ *and* $\eta_2 = \partial_v$ *on a neighborhood* U *of* p*.*

Proof. Let $\gamma(t) = (u(t), v(t))$ $(|t| < \varepsilon)$ be a singular curve passing through p. By the proof of Proposition 2.3.1, we see that $\eta_1 = \partial_u$ and $\lambda(\gamma(t)) = \dfrac{1}{2}\sin\theta(t) = 0$, where $\theta(t) = \theta(u(t), v(t))$. By differentiating $\lambda(\gamma(t))$, we have

$$0 = u'(t)\lambda_u(\gamma(t)) + v'(t)\lambda_v(\gamma(t)) = \frac{\theta'(t)}{2}\cos\theta(t).$$

We notice that $\theta'(t) = 0$ by $\cos\theta(t) = -1$ on $\gamma \subset \theta^{-1}(Z_1)$. Since $\theta'(t) = u'(t)\theta_u(t) + v'(t)\theta_v(t)$, we can take $u'(t) = -\theta_v(t)$ and $v'(t) = \theta_u(t)$. By a similar observation, for $p \in \theta^{-1}(Z_2)$, we can take an adapted pair (ξ_2, η_2) as $\xi_2 = \xi_1$ and $\eta_2 = \partial_v$ on a neighborhood U of p. Therefore we have the assertion. □

For the dual front ν of f, since $S(f) = S(\nu)$, we can take a vector field tangent ξ_1^ν (resp. ξ_2^ν) to $S(\nu)$ around $p \in \theta^{-1}(Z_1)$ (resp. $p \in \theta^{-1}(Z_2)$) as $\xi_1^\nu = \xi_1$ (resp. $\xi_2^\nu = \xi_1$). Moreover, by (2.14), we see that $\eta_1^\nu = \partial_v$ is a null vector field for ν around $p \in \theta^{-1}(Z_1)$ and $\eta_2^\nu = \partial_u$ around $p \in \theta^{-1}(Z_2)$. In both cases, pairs (ξ_1^ν, η_1^ν) and (ξ_2^ν, η_2^ν) are adapted pair for ν.

2.4.1. **Geometric invariants for flat fronts and their dual.** Under the above settings, we shall show the following:

Theorem 2.4.3. *Let* $f \colon \Sigma \to S^3$ *be a flat front corresponding to the solution* θ *of* (2.1)*. Let* ν *be its dual flat front. Suppose that* $p \in \theta^{-1}(\pi\mathbb{Z})$ *is a cuspidal edge of both* f *and* ν*, i.e.,* $\theta_u \neq 0$ *and* $\theta_v \neq 0$ *at* p *.*

(1) *If* $p \in \theta^{-1}(Z_1)$*, then invariants for* f *along a singular curve* $\gamma(t)$ *through* p *are*

$$\kappa_s(t) = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}, \quad \kappa_c(t) = -\mathrm{sgn}(\theta_u)\frac{2\sqrt{2}}{|\theta_u|^{1/2}}, \quad \kappa_t(t) = \frac{\theta_v}{\theta_u} \tag{2.17}$$

and invariants for ν along $\gamma(t)$ are

$$\kappa_s^\nu(t) = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}, \quad \kappa_c^\nu(t) = \operatorname{sgn}(\theta_v)\frac{2\sqrt{2}}{|\theta_v|^{1/2}}, \quad \kappa_t^\nu(t) = \frac{\theta_u}{\theta_v}. \tag{2.18}$$

(2) *If $p \in \theta^{-1}(Z_2)$, then invariants for f along a singular curve $\gamma(t)$ through p are*

$$\kappa_s(t) = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}, \quad \kappa_c(t) = \operatorname{sgn}(\theta_v)\frac{2\sqrt{2}}{|\theta_v|^{1/2}} \quad \kappa_t(t) = \frac{\theta_u}{\theta_v} \tag{2.19}$$

and invariants for ν along $\gamma(t)$ are

$$\kappa_s^\nu(t) = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}, \quad \kappa_c^\nu(t) = -\operatorname{sgn}(\theta_u)\frac{2\sqrt{2}}{|\theta_u|^{1/2}}, \quad \kappa_t^\nu(t) = \frac{\theta_v}{\theta_u}. \tag{2.20}$$

For both cases, the limiting normal curvatures vanish along γ.

Remark 2.4.4. *We notice that ν is also flat. In fact, the first and the second fundamental form I^ν, II^ν of ν can be written as*

$$I^\nu = \sin^2\frac{\theta}{2}du^2 + \cos^2\frac{\theta}{2}dv^2, \quad II^\nu = -\langle d\nu, d(-f)\rangle = \langle d\nu, df\rangle = -\frac{1}{2}\sin\theta(du^2 - dv^2).$$

Thus we have $K_{\mathrm{E}}^\nu \equiv -1$ by a direct calculation.

To prove this theorem, we prove the following lemma.

Lemma 2.4.5. *Under the setting of Theorem 2.4.3, we have the following.*

(1) *If $p \in \theta^{-1}(Z_1)$, then derivatives of f by an adapted pair are given as follows :*

$$\begin{aligned} \xi_1 f = \varepsilon_s\theta_u\boldsymbol{e}_2, \quad D_{\xi_1}\xi_1 f &\equiv -\varepsilon_s\frac{\theta_u}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_1 \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}, \\ D_{\eta_1}\eta_1 f = -\varepsilon_s\frac{\theta_u}{2}\boldsymbol{e}_1, \quad D_{\eta_1}D_{\eta_1}\eta_1 f &\equiv -\theta_u\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)} \\ D_{\xi_1}D_{\eta_1}\eta_1 f &\equiv \frac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)} \end{aligned} \tag{2.21}$$

along γ. Moreover, derivatives of ν by an adapted pair are given as follows :

$$\begin{aligned} \xi_1^\nu\nu = \varepsilon_s\theta_v\boldsymbol{e}_1, \quad D_{\xi_1^\nu}\xi_1^\nu\nu &\equiv \varepsilon_s\frac{\theta_v}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_2 \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}, \\ D_{\eta_1^\nu}\eta_1^\nu\nu = -\varepsilon_s\frac{\theta_v}{2}\boldsymbol{e}_2, \quad D_{\eta_1^\nu}D_{\eta_1^\nu}\eta_1^\nu\nu &\equiv \theta_v f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}, \\ D_{\xi_1^\nu}D_{\eta_1^\nu}\eta_1^\nu\nu &\equiv \frac{\theta_u\theta_v}{2}f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)} \end{aligned} \tag{2.22}$$

along γ.

(2) *If $p \in \theta^{-1}(Z_2)$, then derivatives of f by an adapted pair are given as follows :*

$$\xi_2 f = -\varepsilon_c\theta_v \boldsymbol{e}_1, \quad D_{\xi_2}\xi_2 f \equiv -\varepsilon_c \frac{\theta_v}{2}(\theta_u^2-\theta_v^2)\boldsymbol{e}_2 \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)},$$

$$D_{\eta_2}\eta_2 f = \varepsilon_c\frac{\theta_v}{2}\boldsymbol{e}_2, \quad D_{\eta_2}D_{\eta_2}\eta_2 f \equiv -\theta_v \nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}, \tag{2.23}$$

$$D_{\xi_2}D_{\eta_2}\eta_2 f \equiv -\frac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$$

along γ. Moreover, derivatives of ν by an adapted pair are given as follows :

$$\xi_2^\nu \nu = \varepsilon_c\theta_u \boldsymbol{e}_2, \quad D_{\xi_2^\nu}\xi_2^\nu \nu \equiv -\varepsilon_c \frac{\theta_u}{2}(\theta_u^2-\theta_v^2)\boldsymbol{e}_1 \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)},$$

$$D_{\eta_2^\nu}\eta_2^\nu \nu = -\varepsilon_c\frac{\theta_u}{2}\boldsymbol{e}_1, \quad D_{\eta_2^\nu}D_{\eta_2^\nu}\eta_2^\nu \nu \equiv \theta_u f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}, \tag{2.24}$$

$$D_{\xi_2^\nu}D_{\eta_2^\nu}\eta_2^\nu \nu \equiv -\frac{\theta_u\theta_v}{2} f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}$$

along γ.

Where $\varepsilon_s = \operatorname{sgn}(\sin\frac{\theta}{2})$ and $\varepsilon_c = \operatorname{sgn}(\cos\frac{\theta}{2})$ and $\mathcal{E}(U) = \{h\colon U \to \mathbb{R}\}$ is the local ring consisting of function germs, $\langle k_1, k_2, \dots, k_l\rangle_{\mathcal{E}(U)} = \{a_1k_1+a_2k_2+\cdots+a_lk_l \mid a_1, a_2, \dots, a_l \in \mathcal{E}(U)\}$, and $f_1 \equiv f_2 \mod A$ means that $f_1 - f_2 \in A$.

Proof. We prove (1). We note that $f_u = \boldsymbol{0}$ at p. By $\xi_1 = -\theta_v\partial_u + \theta_u\partial_v$ and (2.9), we have

$$\xi_1 f = -\theta_v f_u + \theta_u f_v = \theta_u \sin\frac{\theta}{2}\boldsymbol{e}_2 = \varepsilon_s\theta_u\boldsymbol{e}_2,$$

along γ. We calculate $D_{\xi_1}\xi_1 f$ along γ. We see that

$$\begin{aligned} D_{\xi_1}\xi_1 f &= D_{\xi_1}(-\theta_v f_u + \theta_u f_v) = \xi_1(-\theta_v f_u + \theta_u f_v) - \langle \xi_1(-\theta_v f_u + \theta_u f_v), f\rangle f \\ &\equiv \xi_1(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1) + \xi_1(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2) \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}. \end{aligned}$$

By $\cos\frac{\theta}{2} = 0$ on $\gamma \subset \theta^{-1}(Z_1)$, we have

$$\begin{aligned} &\xi_1(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1) = (-\theta_v\partial_u + \theta_u\partial_v)(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1) \\ &= -\theta_v(\frac{\theta_v\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1) + \theta_u(\frac{\theta_v^2}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1) = -\frac{\theta_v^2\theta_u}{2}\varepsilon_s\boldsymbol{e}_1 + \frac{\theta_v^2\theta_u}{2}\varepsilon_s\boldsymbol{e}_1 = \boldsymbol{0} \end{aligned}$$

and

$$\begin{aligned} &\xi_1(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2) = (-\theta_v\partial_u + \theta_u\partial_v)(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2) \\ &= -\theta_v(\theta_{uu}\sin\frac{\theta}{2}\boldsymbol{e}_2 + \theta_u\sin\frac{\theta}{2}(\boldsymbol{e}_2)_u) + \theta_u(\theta_{uv}\sin\frac{\theta}{2}\boldsymbol{e}_2 + \theta_u\sin\frac{\theta}{2}(\boldsymbol{e}_2)_v) \\ &\equiv -\theta_v\theta_u\varepsilon_s(\boldsymbol{e}_2)_u + \theta_u^2\varepsilon_s(\boldsymbol{e}_2)_v \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)} \end{aligned}$$

along γ. By (2.13), we have

$$
\begin{aligned}
&-\theta_v\theta_u\varepsilon_s(\boldsymbol{e}_2)_u+\theta_u^2\varepsilon_s(\boldsymbol{e}_2)_v=-\theta_v\theta_u\varepsilon_s(-\frac{\theta_v}{2}\boldsymbol{e}_1)+\theta_u^2\varepsilon_s(-\sin\frac{\theta}{2}f-\frac{\theta_u}{2}\boldsymbol{e}_1-\cos\frac{\theta}{2}\nu)\\
&\equiv-\varepsilon_s\frac{\theta_u}{2}(\theta_u^2-\theta_v^2)\boldsymbol{e}_1 \mod \langle f,\boldsymbol{e}_2,\nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

Hence we see that

$$
D_{\xi_1}\xi_1 f\equiv-\varepsilon_s\frac{\theta_u}{2}(\theta_u^2-\theta_v^2)\boldsymbol{e}_1 \mod \langle f,\boldsymbol{e}_2,\nu\rangle_{\mathcal{E}(U)}
$$

along γ. We calculate $D_{\eta_1}\eta_1 f$ along γ. By $\cos\frac{\theta}{2}=0$, $\sin\theta=0$, $\eta_1=\partial_u$ and $f_{uu}=-\cos^2\frac{\theta}{2}f-\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1+\frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2+\frac{1}{2}\sin\theta\nu$, we have

$$
\begin{aligned}
&D_{\eta_1}\eta_1 f=D_{\partial_u}f_u=f_{uu}-\langle f_{uu},f\rangle f\\
&=-\cos^2\frac{\theta}{2}f-\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1+\frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2+\frac{1}{2}\sin\theta\nu+\cos^2\frac{\theta}{2}f=-\varepsilon_s\frac{\theta_u}{2}\boldsymbol{e}_1
\end{aligned}
$$

along γ. We show $D_{\eta_1}D_{\eta_1}\eta_1 f\equiv-\theta_u\nu \mod \langle f,\boldsymbol{e}_1,\boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$ along γ. We note that

$$
D_{\eta_1}\eta_1 f=-\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1+\frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2+\frac{1}{2}\sin\theta\nu. \tag{2.25}
$$

From the definition of D, we only need to calculate $\partial_u(D_{\eta_1}\eta_1 f)$. By $\cos\frac{\theta}{2}=0$, $\nu_u=-\sin\frac{\theta}{2}\boldsymbol{e}_1$, $(\boldsymbol{e}_1)_u=-\cos\frac{\theta}{2}f+\frac{\theta_v}{2}\boldsymbol{e}_2+\sin\frac{\theta}{2}\nu$ and $(\boldsymbol{e}_2)_u=-\frac{\theta_v}{2}\boldsymbol{e}_1$, we see that

$$
\begin{aligned}
&\partial_u(D_{\eta_1}\eta_1 f)=-\frac{\theta_{uu}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1-\frac{\theta_u}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_1)_u-\frac{\theta_v\theta_u}{4}\sin\frac{\theta}{2}\boldsymbol{e}_2+\frac{\theta_u}{2}\cos\theta\nu+\frac{1}{2}\sin\theta\nu_u\\
&\equiv-\frac{\theta_u}{2}\sin^2\frac{\theta}{2}\nu+\frac{\theta_u}{2}\cos\theta\nu=-\frac{\theta_u}{2}\nu-\frac{\theta_u}{2}\nu=-\theta_u\nu \mod \langle f,\boldsymbol{e}_1,\boldsymbol{e}_2\rangle_{\mathcal{E}(U)}
\end{aligned}
$$

along γ. Next we show $D_{\xi_1}D_{\eta_1}\eta_1 f\equiv\frac{\theta_u\theta_v}{2}\nu \mod \langle f,\boldsymbol{e}_1,\boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$ along γ. From the definition of D, we only need to calculate $\xi_1(D_{\eta_1}\eta_1 f)$. By (2.25), $\cos\frac{\theta}{2}=0$, $\sin\theta=0$, $(\boldsymbol{e}_1)_u=-\cos\frac{\theta}{2}f+\frac{\theta_v}{2}\boldsymbol{e}_2+\sin\frac{\theta}{2}\nu$ and $(\boldsymbol{e}_1)_v=\frac{\theta_u}{2}\boldsymbol{e}_2$, we have

$$
\begin{aligned}
&\xi_1(D_{\eta_1}\eta_1 f) = \xi_1(-\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{1}{2}\sin\theta\nu)\\
&= -\theta_v(-\frac{\theta_{uu}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_u}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_1)_u - \frac{\theta_v\theta_u}{4}\sin\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_u}{2}\cos\theta\nu)\\
&+ \theta_u(-\frac{\theta_{uv}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_u}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_1)_v - \frac{\theta_v^2}{4}\sin\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_v}{2}\cos\theta\nu)\\
&= -\theta_v(-\frac{\theta_{uu}}{2}\varepsilon_s\boldsymbol{e}_1 - \frac{\theta_u}{2}\varepsilon_s(\boldsymbol{e}_1)_u - \frac{\theta_v\theta_u}{4}\varepsilon_s\boldsymbol{e}_2 - \frac{\theta_u}{2}\nu)\\
&+ \theta_u(-\frac{\theta_{uv}}{2}\varepsilon_s\boldsymbol{e}_1 - \frac{\theta_u}{2}\varepsilon_s(\boldsymbol{e}_1)_v - \frac{\theta_v^2}{4}\varepsilon_s\boldsymbol{e}_2 - \frac{\theta_v}{2}\nu)\\
&\equiv -\theta_v(-\frac{\theta_u}{2}\nu - \frac{\theta_u}{2}\nu) - \frac{\theta_u\theta_v}{2}\nu = \frac{\theta_u\theta_v}{2} \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}
\end{aligned}
$$

along γ. Hence we see that

$$
D_{\xi_1}D_{\eta_1}\eta_1 f \equiv \frac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}
$$

along γ.

We calculate derivatives of ν by an adapted. We note that $\nu_v = \boldsymbol{0}$ at p. By $\xi_1^\nu = -\theta_v\partial_u + \theta_u\partial_v$, we have

$$
\xi_1^\nu\nu = -\theta_v\nu_u = -\theta_v(-\sin\frac{\theta}{2}\boldsymbol{e}_1) = \varepsilon_s\theta_v\boldsymbol{e}_1
$$

along γ. We calculate $D_{\xi_1^\nu}\xi_1^\nu\nu$ along γ. We see that

$$
\begin{aligned}
&D_{\xi_1^\nu}\xi_1^\nu\nu = D_{\xi_1^\nu}(-\theta_v\nu_u + \theta_u\nu_v) = \xi_1^\nu(-\theta_v\nu_u + \theta_u\nu_v) - \left\langle \xi_1^\nu(-\theta_v\nu_u + \theta_u\nu_v), \nu\right\rangle \nu\\
&\equiv \xi_1^\nu(\theta_v\sin\frac{\theta}{2}\boldsymbol{e}_1) + \xi_1^\nu(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2) \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

By $\cos\dfrac{\theta}{2} = 0$ on $\gamma \subset \theta^{-1}(Z_1)$, we have

$$
\begin{aligned}
&\xi_1^\nu(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2) = (-\theta_v\partial_u + \theta_u\partial_v)(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2)\\
&= -\theta_v(-\frac{\theta_u^2}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2) + \theta_u(-\frac{\theta_u\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\xi_1^\nu(\theta_v\sin\frac{\theta}{2}\boldsymbol{e}_1) = (-\theta_v\partial_u + \theta_u\partial_v)(\theta_v\sin\frac{\theta}{2}\boldsymbol{e}_1)\\
&= -\theta_v(\theta_{vu}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \theta_v\sin\frac{\theta}{2}(\boldsymbol{e}_1)_u) + \theta_u(\theta_{vv}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \theta_v\sin\frac{\theta}{2}(\boldsymbol{e}_1)_v)\\
&\equiv -\theta_v^2\varepsilon_s(\boldsymbol{e}_1)_u + \theta_u\theta_v\varepsilon_s(\boldsymbol{e}_1)_v \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

along γ. By (2.13), we have

$$
\begin{aligned}
&-\theta_v^2\varepsilon_s(\boldsymbol{e}_1)_u+\theta_u\theta_v\varepsilon_s(\boldsymbol{e}_1)_v=-\theta_v^2\varepsilon_s(-\cos\frac{\theta}{2}f+\frac{\theta_v}{2}\boldsymbol{e}_2+\sin\frac{\theta}{2}\nu)+\theta_u\theta_v\varepsilon_s\frac{\theta_u}{2}\boldsymbol{e}_2\\
&\equiv-\varepsilon_s\frac{\theta_v^3}{2}\boldsymbol{e}_2+\varepsilon_s\frac{\theta_u^2\theta_v}{2}\boldsymbol{e}_2=\varepsilon_s\frac{\theta_v}{2}(\theta_u^2-\theta_v^2)\mod\langle f,\boldsymbol{e}_1,\nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

Hence we see that

$$
D_{\xi_1^\nu}\xi_1^\nu\nu\equiv\varepsilon_s\frac{\theta_v}{2}(\theta_u^2-\theta_v^2)\boldsymbol{e}_2\mod\langle f,\boldsymbol{e}_1,\nu\rangle_{\mathcal{E}(U)}
$$

along γ. We calculate $D_{\eta_1^\nu}\eta_1^\nu\nu$ along γ. By $\nu_v=\cos\frac{\theta}{2}\boldsymbol{e}_2$, we see that

$$
\nu_{vv}=-\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2+\cos\frac{\theta}{2}(\boldsymbol{e}_2)_v=-\varepsilon_s\frac{\theta_v}{2}\boldsymbol{e}_2 \tag{2.26}
$$

along γ. Thus we have

$$
D_{\eta_1^\nu}\eta_1^\nu\nu=D_{\eta_1^\nu}\nu_v=\nu_{vv}-\langle\nu_{vv},\nu\rangle\,\nu=-\varepsilon_s\frac{\theta_v}{2}\boldsymbol{e}_2
$$

along γ. We show $D_{\eta_1^\nu}D_{\eta_1^\nu}\eta_1^\nu\nu\equiv\theta_v f\mod\langle\boldsymbol{e}_1,\boldsymbol{e}_2,\nu\rangle_{\mathcal{E}(U)}$ along γ. By (2.26) and $(\boldsymbol{e}_2)_v=-\sin\frac{\theta}{2}f-\frac{\theta_u}{2}\boldsymbol{e}_1-\cos\frac{\theta}{2}\nu$, we see that

$$
\begin{aligned}
D_{\eta_1^\nu}\eta_1^\nu\nu&=\nu_{vv}-\langle\nu_{vv},\nu\rangle\,\nu\\
&=-\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2+\cos\frac{\theta}{2}(-\sin\frac{\theta}{2}f-\frac{\theta_u}{2}\boldsymbol{e}_1-\cos\frac{\theta}{2}\nu)+\cos^2\frac{\theta}{2}\nu\\
&=-\frac{1}{2}\sin\theta f-\frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1-\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2.
\end{aligned}\tag{2.27}
$$

From the definition of D, we only need to calculate $\partial_v(D_{\eta_1^\nu}\eta_1^\nu\nu)$. By $\cos\frac{\theta}{2}=0$, $\sin\theta=0$, $\cos\theta=-1$ and $(\boldsymbol{e}_2)_v=-\sin\frac{\theta}{2}f-\frac{\theta_u}{2}\boldsymbol{e}_1-\cos\frac{\theta}{2}\nu$, we have

$$
\begin{aligned}
&\partial_v(D_{\eta_1^\nu}\eta_1^\nu\nu)=\partial_v(-\frac{1}{2}\sin\theta f-\frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1-\frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2)\\
&=-\frac{\theta_v}{2}\cos\theta f+\frac{\theta_u\theta_v}{4}\sin\frac{\theta}{2}\boldsymbol{e}_1-\frac{\theta_{vv}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2-\frac{\theta_v}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_2)_v\\
&=\frac{\theta_v}{2}f+\frac{\theta_u\theta_v}{4}\varepsilon_s\boldsymbol{e}_1-\frac{\theta_{vv}}{2}\varepsilon_s\boldsymbol{e}_2-\frac{\theta_v}{2}\varepsilon_s(\boldsymbol{e}_2)_v\equiv\theta_v f\mod\langle\boldsymbol{e}_1,\boldsymbol{e}_2,\nu\rangle_{\mathcal{E}(U)}
\end{aligned}
$$

along γ. Next we show $D_{\xi_1^\nu}D_{\eta_1^\nu}\eta_1^\nu\nu\equiv\frac{\theta_u\theta_v}{2}f\mod\langle\boldsymbol{e}_1,\boldsymbol{e}_2,\nu\rangle_{\mathcal{E}(U)}$ along γ. By (2.27), $\cos\frac{\theta}{2}=0$, $\sin\theta=0$, $\cos\theta=-1$, $(\boldsymbol{e}_2)_u=-\frac{\theta_v}{2}\boldsymbol{e}_1$ and $(\boldsymbol{e}_2)_v=-\sin\frac{\theta}{2}f-\frac{\theta_u}{2}\boldsymbol{e}_1-\cos\frac{\theta}{2}\nu$, we have

$$
\begin{aligned}
&\xi_1^{\nu}(D_{\eta_1^{\nu}}\eta_1^{\nu}\nu) = (-\theta_v\partial_u + \theta_u\partial_v)(-\frac{1}{2}\sin\theta f - \frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_v}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2)\\
&= -\theta_v(\frac{\theta_u}{2}\cos\theta f + \frac{\theta_u^2}{4}\sin\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_{vu}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_2)_u)\\
&+ \theta_u(\frac{\theta_v}{2}\cos\theta f + \frac{\theta_u\theta_v}{4}\sin\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_{vv}}{2}\sin\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\sin\frac{\theta}{2}(\boldsymbol{e}_2)_v)\\
&= -\theta_v(-\frac{\theta_u}{2}f + \frac{\theta_u^2}{4}\varepsilon_s\boldsymbol{e}_1 - \frac{\theta_{vu}}{2}\varepsilon_s\boldsymbol{e}_2 - \frac{\theta_v}{2}\varepsilon_s(\boldsymbol{e}_2)_u)\\
&+ \theta_u(-\frac{\theta_v}{2}f + \frac{\theta_u\theta_v}{4}\varepsilon_s\boldsymbol{e}_1 - \frac{\theta_{vv}}{2}\varepsilon_s\boldsymbol{e}_2 - \frac{\theta_v}{2}\varepsilon_s(\boldsymbol{e}_2)_v)\\
&\equiv \frac{\theta_u\theta_v}{2}f - \frac{\theta_u\theta_v}{2}f + \frac{\theta_u\theta_v}{2}f = \frac{\theta_u\theta_v}{2}f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

along γ. This completes the proof for the assertion (1).

We prove (2). We note that $f_v = \boldsymbol{0}$ at p. By $\xi_2 = -\theta_v\partial_u + \theta_u\partial_v$ and (2.9), we have

$$
\xi_2 f = -\theta_v f_u + \theta_u f_v = -\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1 = -\varepsilon_c\theta_v\boldsymbol{e}_1,
$$

along γ. We calculate $D_{\xi_2}\xi_2 f$ along γ. We see that

$$
\begin{aligned}
D_{\xi_2}\xi_2 f &= D_{\xi_2}(-\theta_v f_u + \theta_u f_v) = \xi_2(-\theta_v f_u + \theta_u f_v) - \langle \xi_2(-\theta_v f_u + \theta_u f_v), f\rangle f\\
&\equiv \xi_2(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1) + \xi_2(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2) \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}
$$

By $\sin\dfrac{\theta}{2} = 0$ on $\gamma \subset \theta^{-1}(Z_2)$, we have

$$
\begin{aligned}
&\xi_2(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1) = (-\theta_v\partial_u + \theta_u\partial_v)(-\theta_v\cos\frac{\theta}{2}\boldsymbol{e}_1)\\
&= -\theta_v(-\theta_{vu}\cos\frac{\theta}{2}\boldsymbol{e}_1 - \theta_v\cos\frac{\theta}{2}(\boldsymbol{e}_1)_u) + \theta_u(-\theta_{vv}\cos\frac{\theta}{2}\boldsymbol{e}_1 - \theta_v\cos\frac{\theta}{2}(\boldsymbol{e}_1)_v)\\
&= \theta_v\theta_{vu}\varepsilon_c\boldsymbol{e}_1 + \theta_v^2\varepsilon_c(\boldsymbol{e}_1)_u - \theta_u\theta_{vv}\varepsilon_c\boldsymbol{e}_1 - \theta_u\theta_v\varepsilon_c(\boldsymbol{e}_1)_v\\
&= \theta_v\theta_{vu}\varepsilon_c\boldsymbol{e}_1 + \theta_v^2\varepsilon_c(-\varepsilon_c f + \frac{\theta_v}{2}\boldsymbol{e}_2) - \theta_u\theta_{vv}\varepsilon_c\boldsymbol{e}_1 - \theta_u\theta_v\varepsilon_c(\frac{\theta_u}{2}\boldsymbol{e}_2)\\
&\equiv \varepsilon_c\frac{\theta_v^3}{2}\boldsymbol{e}_2 - \varepsilon_c\frac{\theta_v\theta_u^2}{2}\boldsymbol{e}_2 = -\varepsilon_c\frac{\theta_v}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_2 \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_2(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2) &= (-\theta_v\partial_u + \theta_u\partial_v)(\theta_u\sin\frac{\theta}{2}\boldsymbol{e}_2)\\
&= -\theta_v(\frac{\theta_u^2}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2) + \theta_u(\frac{\theta_u\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2) = \boldsymbol{0}
\end{aligned}
$$

along γ.

Hence we see that

$$D_{\xi_2}\xi_2 f \equiv -\varepsilon_c \frac{\theta_v}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_1 \mod \langle f, \boldsymbol{e}_1, \nu\rangle_{\mathcal{E}(U)}$$

along γ. We calculate $D_{\eta_2}\eta_2 f$ along γ. By $\sin\dfrac{\theta}{2} = 0$, $\sin\theta = 0$, $\eta_2 = \partial_v$ and $f_{vv} = -\sin^2\dfrac{\theta}{2}f - \dfrac{\theta_u}{2}\sin\dfrac{\theta}{2}\boldsymbol{e}_1 + \dfrac{\theta_v}{2}\cos\dfrac{\theta}{2}\boldsymbol{e}_2 - \dfrac{1}{2}\sin\theta\nu$, we have

$$\begin{aligned}
&D_{\eta_2}\eta_2 f = D_{\partial_v} f_v = f_{vv} - \langle f_{vv}, f\rangle f\\
&= -\sin^2\frac{\theta}{2}f - \frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\sin\theta\nu + \sin^2\frac{\theta}{2}f = \varepsilon_c\frac{\theta_v}{2}\boldsymbol{e}_2
\end{aligned}$$

along γ. We show $D_{\eta_2}D_{\eta_2}\eta_2 f \equiv -\theta_v\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$ along γ. We note that

$$D_{\eta_2}\eta_2 f = -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\sin\theta\nu. \tag{2.28}$$

From the definition of D, we only need to calculate $\partial_v(D_{\eta_2}\eta_2 f)$. By $\sin\dfrac{\theta}{2} = 0$, $\sin\theta = 0$, $\cos\theta = 1$, $\nu_v = \cos\dfrac{\theta}{2}\boldsymbol{e}_1$ and $(\boldsymbol{e}_2)_v = -\sin\dfrac{\theta}{2}f - \dfrac{\theta_u}{2}\boldsymbol{e}_1 - \cos\dfrac{\theta}{2}\nu$, we see that

$$\begin{aligned}
&\partial_v(D_{\eta_2}\eta_2 f) = -\frac{\theta_u\theta_v}{4}\cos\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_{vv}}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_v}{2}\cos\frac{\theta}{2}(\boldsymbol{e}_2)_v - \frac{\theta_v}{2}\cos\theta\nu\\
&\equiv -\frac{\theta_v}{2}\nu - \frac{\theta_u}{2}\nu = -\theta_v\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}
\end{aligned}$$

along γ. Next we show $D_{\xi_2}D_{\eta_2}\eta_2 f \equiv -\dfrac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$ along γ. From the definition of D, we only need to calculate $\xi_2(D_{\eta_2}\eta_2 f)$. By (2.28), $\sin\dfrac{\theta}{2} = 0$, $\sin\theta = 0$, $\cos\theta = 1$, $(\boldsymbol{e}_2)_u = -\dfrac{\theta_v}{2}\boldsymbol{e}_1$ and $(\boldsymbol{e}_2)_v = -\sin\dfrac{\theta}{2}f - \dfrac{\theta_u}{2}\boldsymbol{e}_1 - \cos\dfrac{\theta}{2}\nu$, we have

$$\begin{aligned}
&\xi_2(D_{\eta_2}\eta_2 f) = \xi_2(-\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 - \frac{1}{2}\sin\theta\nu)\\
&= -\theta_v(-\frac{\theta_u^2}{4}\cos\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_{vu}}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_v}{2}\cos\frac{\theta}{2}(\boldsymbol{e}_2)_u - \frac{\theta_u}{2}\cos\theta\nu)\\
&+ \theta_u(-\frac{\theta_u\theta_v}{4}\cos\frac{\theta}{2}\boldsymbol{e}_1 + \frac{\theta_{vv}}{2}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \frac{\theta_v}{2}\cos\frac{\theta}{2}(\boldsymbol{e}_2)_v - \frac{\theta_v}{2}\cos\theta\nu)\\
&= -\theta_v(-\frac{\theta_u^2}{4}\varepsilon_c\boldsymbol{e}_1 + \frac{\theta_{vu}}{2}\varepsilon_c\boldsymbol{e}_2 + \frac{\theta_v}{2}\varepsilon_c(\boldsymbol{e}_2)_u - \frac{\theta_u}{2}\nu)\\
&+ \theta_u(-\frac{\theta_u\theta_v}{4}\varepsilon_c\boldsymbol{e}_1 + \frac{\theta_{vv}}{2}\varepsilon_c\boldsymbol{e}_2 + \frac{\theta_v}{2}\varepsilon_c(\boldsymbol{e}_2)_v - \frac{\theta_v}{2}\nu)\\
&\equiv -\theta_v(-\frac{\theta_u}{2}\nu) + \theta_u(-\frac{\theta_v}{2}\nu - \frac{\theta_v}{2}\nu) = -\frac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}
\end{aligned}$$

along γ. Hence we see that

$$D_{\xi_2}D_{\eta_2}\eta_2 f \equiv -\frac{\theta_u\theta_v}{2}\nu \mod \langle f, \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathcal{E}(U)}$$

along γ.

We calculate derivatives of ν by an adapted. We note that $\nu_u = \mathbf{0}$ at p. By $\xi_2^\nu = -\theta_v\partial_u + \theta_u\partial_v$, we have

$$\xi_2^\nu \nu = \theta_u \nu_v = \theta_u(\cos\frac{\theta}{2}\boldsymbol{e}_2) = \varepsilon_c\theta_u\boldsymbol{e}_2$$

along γ. We calculate $D_{\xi_2^\nu}\xi_2^\nu\nu$ along γ. We see that

$$\begin{aligned}
&D_{\xi_2^\nu}\xi_2^\nu\nu = D_{\xi_2^\nu}(-\theta_v\nu_u + \theta_u\nu_v) = \xi_2^\nu(-\theta_v\nu_u + \theta_u\nu_v) - \left\langle \xi_2^\nu(-\theta_v\nu_u + \theta_u\nu_v), \nu\right\rangle \nu \\
&\equiv \xi_2^\nu(\theta_v \sin\frac{\theta}{2}\boldsymbol{e}_1) + \xi_2^\nu(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2) \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}$$

By $\sin\dfrac{\theta}{2} = 0$ on $\gamma \subset \theta^{-1}(Z_2)$, we have

$$\begin{aligned}
\xi_2^\nu(\theta_v \sin\frac{\theta}{2}\boldsymbol{e}_1) &= (-\theta_v\partial_u + \theta_u\partial_v)(\theta_v\sin\frac{\theta}{2}\boldsymbol{e}_1) \\
&= -\theta_v(\frac{\theta_u\theta_v}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1) + \theta_u(\frac{\theta_v^2}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1) = \mathbf{0}
\end{aligned}$$

and

$$\begin{aligned}
&\xi_2^\nu(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2) = (-\theta_v\partial_u + \theta_u\partial_v)(\theta_u\cos\frac{\theta}{2}\boldsymbol{e}_2) \\
&= -\theta_v(\theta_{uu}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \theta_u\cos\frac{\theta}{2}(\boldsymbol{e}_2)_u) + \theta_u(\theta_{uv}\cos\frac{\theta}{2}\boldsymbol{e}_2 + \theta_u\varepsilon_c(\boldsymbol{e}_2)_v) \\
&\equiv -\varepsilon_c\theta_v\theta_u(\boldsymbol{e}_2)_u + \varepsilon_c\theta_u^2(\boldsymbol{e}_2)_v \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}$$

along γ. By (2.13), we have

$$\begin{aligned}
&-\varepsilon_c\theta_v\theta_u(\boldsymbol{e}_2)_u + \varepsilon_c\theta_u^2(\boldsymbol{e}_2)_v = -\varepsilon_c\theta_v\theta_u(-\frac{\theta_v}{2}\boldsymbol{e}_1) + \varepsilon_c\theta_u^2(-\frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu) \\
&\equiv \varepsilon_c\frac{\theta_v^2\theta_u}{2}\boldsymbol{e}_1 - \varepsilon_c\frac{\theta_u^3}{2}\boldsymbol{e}_1 = -\varepsilon_c\frac{\theta_u}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_1 \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}.
\end{aligned}$$

Hence we see that

$$D_{\xi_2^\nu}\xi_2^\nu\nu \equiv -\varepsilon_c\frac{\theta_u}{2}(\theta_u^2 - \theta_v^2)\boldsymbol{e}_1 \mod \langle f, \boldsymbol{e}_2, \nu\rangle_{\mathcal{E}(U)}$$

along γ. We calculate $D_{\eta_2^\nu}\eta_2^\nu\nu$ along γ. By $\nu_u = -\sin\dfrac{\theta}{2}\boldsymbol{e}_1$ and $\sin\dfrac{\theta}{2} = 0$ on $\gamma \subset \theta^{-1}(Z_2)$, we see that

$$\nu_{uu} = -\frac{\theta_u}{2}\cos\frac{\theta}{2}\boldsymbol{e}_1 - \sin\frac{\theta}{2}(\boldsymbol{e}_1)_u = -\varepsilon_c\frac{\theta_u}{2}\boldsymbol{e}_1 \tag{2.29}$$

along γ. Thus we have

$$D_{\eta_2^\nu}\eta_2^\nu\nu = D_{\eta_2^\nu}\nu_u = \nu_{uu} - \langle \nu_{uu}, \nu\rangle\, \nu = -\varepsilon_c\frac{\theta_u}{2}\boldsymbol{e}_1$$

along γ. We show $D_{\eta_2^\nu} D_{\eta_2^\nu} \eta_2^\nu \nu \equiv \theta_u f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu \rangle_{\mathcal{E}(U)}$ along γ. By (2.29), we see that

$$
\begin{aligned}
D_{\eta_2^\nu} \eta_2^\nu \nu &= \nu_{uu} - \langle \nu_{uu}, \nu \rangle \nu \\
&= -\frac{\theta_u}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \sin \frac{\theta}{2} (-\cos \frac{\theta}{2} f + \frac{\theta_v}{2} \boldsymbol{e}_2 + \sin \frac{\theta}{2} \nu) + \sin^2 \frac{\theta}{2} \nu \\
&= \frac{1}{2} \sin \theta f - \frac{\theta_u}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_v}{2} \sin \frac{\theta}{2} \boldsymbol{e}_2. \qquad (2.30)
\end{aligned}
$$

From the definition of D, we only need to calculate $\partial_u (D_{\eta_2^\nu} \eta_2^\nu \nu)$. By $\cos \theta = 1$, $\sin \theta = 0$, $\sin \frac{\theta}{2} = 0$, $(\boldsymbol{e}_1)_u = -\cos \frac{\theta}{2} f + \frac{\theta_v}{2} \boldsymbol{e}_2 + \sin \frac{\theta}{2} \nu$ and (2.30), we have

$$
\begin{aligned}
\partial_u (D_{\eta_2^\nu} \eta_2^\nu \nu) &= \partial_u (\frac{1}{2} \sin \theta f - \frac{\theta_u}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_v}{2} \sin \frac{\theta}{2} \boldsymbol{e}_2) \\
&= \frac{\theta_u}{2} \cos \theta f - \frac{\theta_{uu}}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_u}{2} \cos \frac{\theta}{2} (\boldsymbol{e}_1)_u - \frac{\theta_v \theta_u}{4} \cos \frac{\theta}{2} \boldsymbol{e}_2 \\
&= \frac{\theta_u}{2} f - \frac{\theta_{uu}}{2} \varepsilon_c \boldsymbol{e}_1 - \frac{\theta_u}{2} \varepsilon_c (\boldsymbol{e}_1)_u - \frac{\theta_v \theta_u}{4} \varepsilon_c \boldsymbol{e}_2 \equiv \frac{\theta_u}{2} f + \frac{\theta_u}{2} f = \theta_u f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu \rangle_{\mathcal{E}(U)}
\end{aligned}
$$

along γ. Next we show $D_{\xi_2^\nu} D_{\eta_2^\nu} \eta_2^\nu \nu \equiv -\frac{\theta_u \theta_v}{2} f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu \rangle_{\mathcal{E}(U)}$ along γ. By $\cos \theta = 1$, $\sin \theta = 0$, $\sin \frac{\theta}{2} = 0$, $(\boldsymbol{e}_1)_u = -\cos \frac{\theta}{2} f + \frac{\theta_v}{2} \boldsymbol{e}_2 + \sin \frac{\theta}{2} \nu$, $(\boldsymbol{e}_1)_v = \frac{\theta_u}{2} \boldsymbol{e}_2$ and (2.30), we have

$$
\begin{aligned}
\xi_2^\nu (D_{\eta_2^\nu} \eta_2^\nu \nu) &= (-\theta_v \partial_u + \theta_u \partial_v)(\frac{1}{2} \sin \theta f - \frac{\theta_u}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_v}{2} \sin \frac{\theta}{2} \boldsymbol{e}_2) \\
&= -\theta_v (\frac{\theta_u}{2} \cos \theta f - \frac{\theta_{uu}}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_u}{2} \cos \frac{\theta}{2} (\boldsymbol{e}_1)_u - \frac{\theta_{vu}}{4} \cos \frac{\theta}{2} \boldsymbol{e}_2) \\
&+ \theta_u (\frac{\theta_v}{2} \cos \theta f - \frac{\theta_{uv}}{2} \cos \frac{\theta}{2} \boldsymbol{e}_1 - \frac{\theta_u}{2} \cos \frac{\theta}{2} (\boldsymbol{e}_1)_v - \frac{\theta_v^2}{4} \cos \frac{\theta}{2} \boldsymbol{e}_2) \\
&= -\theta_v (\frac{\theta_u}{2} f - \frac{\theta_{uu}}{2} \varepsilon_c \boldsymbol{e}_1 - \frac{\theta_u}{2} \varepsilon_c (\boldsymbol{e}_1)_u - \frac{\theta_{vu}}{4} \varepsilon_c \boldsymbol{e}_2) \\
&+ \theta_u (\frac{\theta_v}{2} f - \frac{\theta_{uv}}{2} \varepsilon_c \boldsymbol{e}_1 - \frac{\theta_u}{2} \varepsilon_c (\boldsymbol{e}_1)_v - \frac{\theta_v^2}{4} \varepsilon_c \boldsymbol{e}_2) \\
&\equiv -\frac{\theta_u \theta_v}{2} f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu \rangle_{\mathcal{E}(U)}
\end{aligned}
$$

along γ. Hence we see that

$$
D_{\xi_2^\nu} D_{\eta_2^\nu} \eta_2^\nu \nu \equiv -\frac{\theta_u \theta_v}{2} f \mod \langle \boldsymbol{e}_1, \boldsymbol{e}_2, \nu \rangle_{\mathcal{E}(U)}.
$$

along γ. This completes the proof for the assertion (2). □

Proof of Theorem 2.4.3. We show (1). We calculate the singular curvature κ_s and κ_s^ν, respectively. By (2.21), we have

$$
|\xi_1 f| = |\theta_u|, \quad \det(f, \xi_1 f, D_{\xi_1} \xi_1 f, \nu) = \frac{\theta_u^2}{2} (\theta_u^2 - \theta_v^2)
$$

along γ. Since $\lambda = \frac{1}{2} \sin \theta$, $\xi_1 = -\theta_v \partial_u + \theta_u \partial_v$ and $\eta_1 = \partial_u$, we see that

$$\eta_1\lambda = \frac{\theta_u}{2}\cos\theta, \quad \det(\xi_1, \eta_1) = -\theta_u$$

hold along γ. Thus have $\varepsilon_\gamma = \operatorname{sgn}(\det(\xi_1, \eta_1) \cdot \eta_1\lambda) = 1$ by $\cos\theta = -1$. Therefore we have

$$\kappa_s = \varepsilon_\gamma \frac{\det(f, \xi_1 f, D_{\xi_1}\xi_1 f, \nu)}{|\xi_1 f|^3} = \frac{\theta_u^2}{2}(\theta_u^2 - \theta_v^2)\frac{1}{|\theta_u|^3} = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}$$

along γ. For ν, by (2.22), we have

$$|\xi_1^\nu \nu| = |\theta_v|, \quad \det(\nu, \xi_1^\nu \nu, D_{\xi_1^\nu}\xi_1^\nu \nu, -f) = \frac{\theta_v^2}{2}(\theta_u^2 - \theta_v^2)$$

along γ. Since $\lambda^\nu = -\lambda = -\dfrac{1}{2}\sin\theta$, $\xi_1^\nu = -\theta_v\partial_u + \theta_u\partial_v$ and $\eta_1^\nu = \partial_v$, we see that

$$\eta_1^\nu\lambda^\nu = -\frac{\theta_v}{2}\cos\theta, \quad \det(\xi_1^\nu, \eta_1^\nu) = -\theta_v$$

hold along γ. Thus have $\varepsilon_\gamma^\nu = \operatorname{sgn}(\det(\xi_1^\nu, \eta_1^\nu) \cdot \eta_1^\nu\lambda^\nu) = -1$ by $\cos\theta = -1$. Therefore we have

$$\kappa_s^\nu = \varepsilon_\gamma^\nu \frac{\det(\nu, \xi_1^\nu \nu, D_{\xi_1^\nu}\xi_1^\nu \nu, -f)}{|\xi_1^\nu \nu|^3} = -\frac{\theta_v^2}{2}(\theta_u^2 - \theta_v^2)\frac{1}{|\theta_v|^3} = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}$$

along γ.

Next we calculate the cuspidal curvature κ_c and κ_c^ν. Since $|\xi_1 f| = |\theta_u|$ and

$$f \wedge \xi_1 f \wedge D_{\eta_1}\eta_1 f = \frac{\theta_u^2}{2} f \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2,$$

we have

$$|\xi_1 f|^{3/2} = |\theta_u|^{3/2}, \quad |f \wedge \xi_1 f \wedge D_{\eta_1}\eta_1 f|^{5/2} = \frac{\theta_u^4|\theta_u|}{2^{5/2}}$$

along γ. Moreover, by (2.21), we see that

$$\det(f, \xi_1 f, D_{\eta_1}\eta_1 f, D_{\eta_1}D_{\eta_1}\eta_1 f) = -\frac{\theta_u^3}{2}$$

holds along γ. Therefore we have

$$\begin{aligned}
\kappa_c &= \frac{|\xi_1 f|^{3/2}\det(f, \xi_1 f, D_{\eta_1}\eta_1 f, D_{\eta_1}D_{\eta_1}\eta_1 f)}{|f \wedge \xi_1 f \wedge D_{\eta_1}\eta_1 f|^{5/2}} = -\frac{\theta_u^3}{2}|\theta_u|^{3/2}\frac{2^{5/2}}{\theta_u^4|\theta_u|} \\
&= -2\sqrt{2}\frac{|\theta_u|^{3/2}\theta_u^3}{\theta_u^4|\theta_u|} = -\operatorname{sgn}(\theta_u)\frac{2\sqrt{2}}{|\theta_u|^{1/2}}
\end{aligned}$$

along γ. For ν, since $|\xi_1^\nu \nu| = |\theta_v|$ and

$$\nu \wedge \xi_1^\nu \nu \wedge D_{\eta_1^\nu}\eta_1^\nu \nu = -\frac{\theta_v^2}{2}\nu \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2,$$

we have

$$|\xi_1^\nu \nu|^{3/2} = |\theta_v|^{3/2}, \quad |\nu \wedge \xi_1^\nu \nu \wedge D_{\eta_1^\nu}\eta_1^\nu \nu|^{5/2} = \frac{\theta_v^4|\theta_v|}{2^{5/2}}$$

along γ. Moreover, by (2.22), we see that

$$\det(\nu, \xi_1^\nu \nu, D_{\eta_1^\nu}\eta_1^\nu \nu, D_{\eta_1^\nu}D_{\eta_1^\nu}\eta_1^\nu \nu) = \frac{\theta_v^3}{2}$$

holds along γ. Therefore we have

$$\begin{aligned}\kappa_c^\nu &= \frac{|\xi_1^\nu \nu|^{3/2}\det(\nu, \xi_1^\nu \nu, D_{\eta_1^\nu}\eta_1^\nu \nu, D_{\eta_1^\nu}D_{\eta_1^\nu}\eta_1^\nu \nu)}{|\nu \wedge \xi_1^\nu \nu \wedge D_{\eta_1^\nu}\eta_1^\nu \nu|^{5/2}} = \frac{\theta_v^3}{2}|\theta_v|^{3/2}\frac{2^{5/2}}{\theta_v^4|\theta_v|}\\ &= 2\sqrt{2}\frac{|\theta_v|^{3/2}\theta_v^3}{\theta_v^4|\theta_v|} = \operatorname{sgn}(\theta_v)\frac{2\sqrt{2}}{|\theta_v|^{1/2}}\end{aligned}$$

along γ.

Finally, we consider the cuspidal torsion κ_t and κ_t^ν. We note that

$$\left\langle \xi_1 f, D_{\eta_1}\eta_1 f\right\rangle = 0$$

holds along γ by (2.21). Hence we only need to calculate $\det(f, \xi_1 f, D_{\eta_1}\eta_1 f, D_{\xi_1}D_{\eta_1}\eta_1 f)$ along γ. By (2.21), we have

$$\det(f, \xi_1 f, D_{\eta_1}\eta_1 f, D_{\xi_1}D_{\eta_1}\eta_1 f) = \frac{\theta_u^3\theta_v}{4}$$

along γ. Since $|f \wedge \xi_1 f \wedge D_{\eta_1}\eta_1 f|^2 = \dfrac{\theta_u^4}{4}$, κ_t can be expressed as

$$\kappa_t = \frac{\det(f, \xi_1 f, D_{\eta_1}\eta_1 f, D_{\xi_1}D_{\eta_1}\eta_1 f)}{|f \wedge \xi_1 f \wedge D_{\eta_1}\eta_1 f|^2} = \frac{4}{\theta_u^4}\frac{\theta_u^3\theta_v}{4} = \frac{\theta_v}{\theta_u}$$

along γ. For ν, we see that

$$\left\langle \xi_1^\nu \nu, D_{\eta_1^\nu}\eta_1^\nu \nu\right\rangle = 0$$

holds along γ. By (2.22), we have

$$\det(\nu, \xi_1^\nu \nu, D_{\eta_1^\nu}\eta_1^\nu \nu, D_{\xi_1^\nu}D_{\eta_1^\nu}\eta_1^\nu \nu) = \frac{\theta_u\theta_v^3}{4}$$

along γ. Since $|\nu \wedge \xi_1^\nu \nu \wedge D_{\eta_1^\nu}\eta_1^\nu \nu|^2 = \dfrac{\theta_v^4}{4}$, we see that

$$\kappa_t^\nu = \frac{\det(\nu, \xi_1^\nu \nu, D_{\eta_1^\nu}\eta_1^\nu \nu, D_{\xi_1^\nu}D_{\eta_1^\nu}\eta_1^\nu \nu)}{|\nu \wedge \xi_1^\nu \nu \wedge D_{\eta_1^\nu}\eta_1^\nu \nu|^2} = \frac{4}{\theta_v^4}\frac{\theta_u\theta_v^3}{4} = \frac{\theta_u}{\theta_v}$$

holds along γ. This completes the proof for the assertion (1).

We show (2). We calculate the singular curvature κ_s and κ_s^ν, repectively. By (2.23), we have

$$|\xi_2 f| = |\theta_v|, \quad \det(f, \xi_2 f, D_{\xi_2}\xi_2 f, \nu) = \frac{\theta_v^2}{2}(\theta_u^2 - \theta_v^2)$$

along $\gamma \subset \theta^{-1}(Z_2)$. Since $\lambda = \dfrac{1}{2}\sin\theta$, $\xi_2 = -\theta_v\partial_u + \theta_u\partial_v$ and $\eta_2 = \partial_v$, we see that

$$\eta_2\lambda = \frac{\theta_v}{2}\cos\theta, \quad \det(\xi_1, \eta_1) = -\theta_v$$

hold along γ. Thus have $\varepsilon_\gamma = \mathrm{sgn}(\det(\xi_2, \eta_2) \cdot \eta_2\lambda) = -1$ by $\cos\theta = 1$. Therefore we have

$$\kappa_s = \varepsilon_\gamma \frac{\det(f, \xi_2 f, D_{\xi_2}\xi_2 f, \nu)}{|\xi_2 f|^3} = -\frac{\theta_v^2}{2}(\theta_u^2 - \theta_v^2)\frac{1}{|\theta_v|^3} = \frac{\theta_v^2 - \theta_u^2}{2|\theta_v|}$$

along γ. For ν, by (2.24), we have

$$|\xi_2^\nu \nu| = |\theta_u|, \quad \det(\nu, \xi_2^\nu\nu, D_{\xi_2^\nu}\xi_2^\nu\nu, -f) = \frac{\theta_u^2}{2}(\theta_u^2 - \theta_v^2)$$

along γ. Since $\lambda^\nu = -\lambda = -\dfrac{1}{2}\sin\theta$, $\xi_2^\nu = -\theta_v\partial_u + \theta_u\partial_v$ and $\eta_2^\nu = \partial_u$, we see that

$$\eta_2^\nu\lambda^\nu = -\frac{\theta_u}{2}\cos\theta, \quad \det(\xi_2^\nu, \eta_2^\nu) = -\theta_u$$

hold along γ. Thus have $\varepsilon_\gamma^\nu = \mathrm{sgn}(\det(\xi_2^\nu, \eta_2^\nu) \cdot \eta_2^\nu\lambda^\nu) = 1$ by $\cos\theta = 1$. Therefore we have

$$\kappa_s^\nu = \varepsilon_\gamma^\nu \frac{\det(\nu, \xi_2^\nu\nu, D_{\xi_2^\nu}\xi_2^\nu\nu, -f)}{|\xi_2^\nu\nu|^3} = \frac{\theta_u^2}{2}(\theta_u^2 - \theta_v^2)\frac{1}{|\theta_u|^3} = \frac{\theta_u^2 - \theta_v^2}{2|\theta_u|}$$

along γ.

Next we calculate the cuspidal curvature κ_c and κ_c^ν. Since $|\xi_2 f| = |\theta_v|$ and

$$f \wedge \xi_2 f \wedge D_{\eta_2}\eta_2 f = -\frac{\theta_v^2}{2} f \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2,$$

we have

$$|\xi_2 f|^{3/2} = |\theta_v|^{3/2}, \quad |f \wedge \xi_2 f \wedge D_{\eta_2}\eta_2 f|^{5/2} = \frac{\theta_v^4|\theta_v|}{2^{5/2}}$$

along γ. Moreover, by (2.23), we see that

$$\det(f, \xi_2 f, D_{\eta_2}\eta_2 f, D_{\eta_2}D_{\eta_2}\eta_2 f) = \frac{\theta_v^3}{2}$$

holds along γ. Therefore we have

$$\begin{aligned}\kappa_c &= \frac{|\xi_2 f|^{3/2}\det(f, \xi_2 f, D_{\eta_2}\eta_2 f, D_{\eta_2}D_{\eta_2}\eta_2 f)}{|f \wedge \xi_2 f \wedge D_{\eta_2}\eta_2 f|^{5/2}} = \frac{\theta_v^3}{2}|\theta_v|^{3/2}\frac{2^{5/2}}{\theta_v^4|\theta_v|} \\ &= 2\sqrt{2}\frac{|\theta_v|^{3/2}\theta_v^3}{\theta_v^4|\theta_v|} = \mathrm{sgn}(\theta_v)\frac{2\sqrt{2}}{|\theta_v|^{1/2}}\end{aligned}$$

along γ. For ν, since $|\xi_2^\nu\nu| = |\theta_u|$ and

$$\nu \wedge \xi_2^\nu\nu \wedge D_{\eta_2^\nu}\eta_2^\nu\nu = \frac{\theta_u^2}{2}\nu \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2,$$

we have

$$|\xi_2^\nu \nu|^{3/2} = |\theta_u|^{3/2}, \quad |\nu \wedge \xi_2^\nu \nu \wedge D_{\eta_2^\nu}\eta_2^\nu \nu|^{5/2} = \frac{\theta_u^4 |\theta_u|}{2^{5/2}}$$

along γ. Moreover, by (2.24), we see that

$$\det(\nu, \xi_2^\nu \nu, D_{\eta_2^\nu}\eta_2^\nu \nu, D_{\eta_2^\nu} D_{\eta_2^\nu}\eta_2^\nu \nu) = -\frac{\theta_u^3}{2}$$

holds along γ. Therefore we have

$$\begin{aligned}
\kappa_c^\nu &= \frac{|\xi_2^\nu \nu|^{3/2} \det(\nu, \xi_2^\nu \nu, D_{\eta_1^\nu}\eta_2^\nu \nu, D_{\eta_2^\nu} D_{\eta_2^\nu}\eta_2^\nu \nu)}{|\nu \wedge \xi_2^\nu \nu \wedge D_{\eta_2^\nu}\eta_2^\nu \nu|^{5/2}} = -\frac{\theta_u^3}{2} |\theta_u|^{3/2} \frac{2^{5/2}}{\theta_u^4 |\theta_u|} \\
&= -2\sqrt{2} \frac{|\theta_u|^{3/2}\theta_u^3}{\theta_u^4 |\theta_u|} = -\mathrm{sgn}(\theta_u) \frac{2\sqrt{2}}{|\theta_u|^{1/2}}
\end{aligned}$$

along γ.

Finally, we consider the cuspidal torsion κ_t and κ_t^ν. We note that

$$\left\langle \xi_2 f, D_{\eta_2}\eta_2 f \right\rangle = 0$$

holds along γ by (2.23). Hence we only need to calculate $\det(f, \xi_2 f, D_{\eta_2}\eta_2 f, D_{\xi_2} D_{\eta_2}\eta_2 f)$ along γ. By (2.23), we have

$$\det(f, \xi_2 f, D_{\eta_2}\eta_2 f, D_{\xi_2} D_{\eta_2}\eta_2 f) = \frac{\theta_v^3 \theta_u}{4}$$

along γ. Since $|f \wedge \xi_2 f \wedge D_{\eta_2}\eta_2 f|^2 = \dfrac{\theta_v^4}{4}$, κ_t can be expressed as

$$\kappa_t = \frac{\det(f, \xi_2 f, D_{\eta_2}\eta_2 f, D_{\xi_2} D_{\eta_2}\eta_2 f)}{|f \wedge \xi_2 f \wedge D_{\eta_2}\eta_2 f|^2} = \frac{4}{\theta_v^4} \frac{\theta_v^3 \theta_u}{4} = \frac{\theta_u}{\theta_v}$$

along γ. For ν, we see that

$$\left\langle \xi_2^\nu \nu, D_{\eta_2^\nu}\eta_2^\nu \nu \right\rangle = 0$$

holds along γ. By (2.24), we have

$$\det(\nu, \xi_2^\nu \nu, D_{\eta_2^\nu}\eta_2^\nu \nu, D_{\xi_2^\nu} D_{\eta_2^\nu}\eta_2^\nu \nu) = \frac{\theta_u^3 \theta_v}{4}$$

along γ. Since $|\nu \wedge \xi_2^\nu \nu \wedge D_{\eta_2^\nu}\eta_2^\nu \nu|^2 = \dfrac{\theta_u^4}{4}$, we see that

$$\kappa_t^\nu = \frac{\det(\nu, \xi_2^\nu \nu, D_{\eta_2^\nu}\eta_2^\nu \nu, D_{\xi_2^\nu} D_{\eta_2^\nu}\eta_2^\nu \nu)}{|\nu \wedge \xi_2^\nu \nu \wedge D_{\eta_2^\nu}\eta_2^\nu \nu|^2} = \frac{4}{\theta_u^4} \frac{\theta_u^3 \theta_v}{4} = \frac{\theta_v}{\theta_u}$$

holds along γ. This completes the proof for the assertion (2).

We note that similar expression of κ_s can be found in [46, Example 1.20]. Moreover, it is known that duality between geometric invariants on cuspidal edges of flat fronts in the hyperbolic 3 -space H^3 and their Δ_1-dual flat fronts in the de Sitter 3-space S_1^3 are studied ([41, Theorem 3.4]) .

As a consequence of Theorem 2.4.3, the following assertion holds.

Corollary 2.4.6. *Let $f\colon \Sigma \to S^3$ be a flat front and ν its dual front. Suppose that f and ν are cuspidal edge at $p \in \Sigma$. Then $\operatorname{sgn}(\kappa_s) = -\operatorname{sgn}(\kappa_s^\nu)$ holds at p.*

Remark 2.4.7. *This can be considered as a special case of [44, Lemma 3.25 (f)].*

Corollary 2.4.8. *Let $f\colon \Sigma \to S^3$ be a flat front and ν its dual flat front. Let $p \in \Sigma$ be a cuspidal edge of f (resp. ν). If ν at p (resp. f at p) is not a cuspidal edge, then κ_s (resp. κ_s^ν) is strictly positive at p.*

Proof. By Theorem 2.4.3, Propositions 2.3.1 and 2.3.2, we have the assertion. □

It seems that this property is related to [48, Theorem A].

Proposition 2.4.9. *Let $f\colon \Sigma \to S^3$ be a flat front and ν its dual front. Let $p \in \Sigma$ be a cuspidal edge of f (resp. ν) and γ a singular curve through p. Suppose that the cuspidal torsion κ_t of f (resp. κ_t^ν of ν) vanishes along γ. Then the singular locus $\nu \circ \gamma$ (resp. $\hat{\gamma} = f \circ \gamma$) is a point.*

Proof. We show the case where κ_t vanishes. We assume that $p \in \theta^{-1}(Z_1)$. Let $\gamma(t) = (u(t), v(t))$ be a singular curve passing through $p = \gamma(0)$. Then we consider the singular locus $\nu(\gamma(t))$ of ν. By (2.14), we have

$$\frac{d}{dt}\nu(\gamma(t)) = u'(t)\nu_u(\gamma(t)) + v'(t)\nu_v(\gamma(t)) = -\varepsilon_s u'(t)\boldsymbol{e}_1(\gamma(t)), \tag{2.31}$$

where $\varepsilon_s = \operatorname{sgn}(\sin\dfrac{\theta}{2})$. On the other hand, differentiating $\lambda(\gamma(t)) = 0$ by t, it holds that

$$\begin{aligned}\frac{d}{dt}\lambda(\gamma(t)) &= u'(t)\lambda_u(\gamma(t)) + v'(t)\lambda_v(\gamma(t)) \\ &= -\frac{1}{2}(u'(t)\theta_u(\gamma(t)) + v'(t)\theta_v(\gamma(t))) = 0,\end{aligned}$$

where we used the relation $\cos\theta(\gamma(t)) = -1$. When κ_t vanishes along γ, then $\theta_v(\gamma(t)) = 0$ by Theorem 2.4.3. Thus under this assumption, we have $u'(t)\theta_u(\gamma(t)) = 0$. Moreover, since f at p is a cuspidal edge, $\theta_u(\gamma(t)) \neq 0$. Hence we get $u'(t) = 0$. By (2.31), we have

$$\left(\frac{d}{dt}\right)\nu(\gamma(t)) \equiv 0.$$

This implies that $\nu(\gamma(t))$ is a point. Thus the assertion follows. For the case where $p \in \theta^{-1}(Z_2)$, one can show in a similar way.

We next consider the case that κ_t^ν vanishes along the singular curve. Suppose $p \in \theta^{-1}(Z_1)$. Let $\gamma(t) = (u(t), v(t))$ a singular curve through $p = \gamma(0)$. We then consider $\hat{\gamma} = f(\gamma(t))$. By direct calculations, we have

$$\hat{\gamma}'(t) = \varepsilon_s v'(t)\boldsymbol{e}_2(\gamma(t)),\ \frac{d}{dt}\lambda(\gamma(t)) = -\frac{1}{2}(u'(t)\theta_u(\gamma(t)) + v'(t)\theta_v(\gamma(t))) = 0$$

Since $\kappa_t^\nu = 0$ along γ, we have $\theta_u(\gamma(t)) = 0$ by Theorem 2.4.3. Moreover, since ν at p is a cuspidal edge, $\theta_v(\gamma(t)) \neq 0$. Thus we have $v'(t) = 0$, and hence $\hat{\gamma}'$ vanishes identically. This implies that $\hat{\gamma}$ is a point. For the case of $p \in \theta^{-1}(Z_2)$, one can show similarly. □

This may be related to [41, Corollary 4.3] and [48, Proposition 3.7]. In the situation above, it can be considered as ν at p is a *cone-like singular point* which is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, u\cos v, u\sin v)$ at the origin.

2.5. **Mean curvatures of a flat front and its dual front.** We examine a relationship between the mean curvature of a flat front f and its dual front ν. Moreover, we examine a relationship between mean curvatures of parallel flat fronts and their dual fronts.

Proposition 2.5.1. *Let $f : \Sigma \to S^3$ be a flat front associated to the solution θ to (2.1) and ν be the dual of f. Then on the set of regular points of f (also ν), the mean curvature H^f of f coincides with the mean curvature H^ν of ν.*

Proof. By (2.2), the mean curvature H^f of f is

$$H^f = \frac{\cos^2\dfrac{\theta}{2}\left(-\dfrac{1}{2}\sin\theta\right) + \sin^2\dfrac{\theta}{2}\left(\dfrac{1}{2}\sin\theta\right)}{2\cos^2\dfrac{\theta}{2}\sin^2\dfrac{\theta}{2}} = \frac{-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{2}\frac{\cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2}}{2\cos^2\dfrac{\theta}{2}\sin^2\dfrac{\theta}{2}}$$
$$= -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

On the other hand, the first fundamental form I^ν of ν is

$$I^\nu = \sin^2\frac{\theta}{2}du^2 + \cos^2\frac{\theta}{2}dv^2.$$

Since $-f$ can be taken as a dual of ν, the second fundamental form II^ν of ν can be calculated as

$$II^\nu = -\langle d\nu, d(-f)\rangle = \langle d\nu, df\rangle = -\frac{1}{2}\sin\theta(du^2 - dv^2) = -II^f.$$

Thus we have

$$H^\nu = \frac{\sin^2\dfrac{\theta}{2}\left(\dfrac{1}{2}\sin\dfrac{\theta}{2}\right) + \cos^2\dfrac{\theta}{2}\left(-\dfrac{1}{2}\sin\dfrac{\theta}{2}\right)}{2\sin^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2}} = -\cot\theta.$$

Therefore we have the assertion. □

We next consider mean curvatures of parallel flat fronts and their dual fronts. Let $f : \Sigma \to S^3$ be a flat front associated to the solution θ to (2.1) and ν its dual front. Then a *parallel front* $f^t : \Sigma \to S^3$ of f is defined as

$$f^t = \cos t f + \sin t \nu$$

for some fixed $t \in \mathbb{R}$. The dual front ν^t of f^t is given by

$$\nu^t = -\sin t f + \cos t \nu.$$

For a parallel flat front f^t and its dual front ν^t, we shall show the following.

Proposition 2.5.2. *The mean curvature H^t of a parallel flat front f^t coincides with the mean curvature H^t_ν of ν^t.*

Proof. By (2.9) and (2.14), we see that

$$
\begin{aligned}
f_u^t &= \left(\cos t\cos\frac{\theta}{2} - \sin t\sin\frac{\theta}{2}\right)\boldsymbol{e}_1 = \cos\left(\frac{\theta}{2}+t\right)\boldsymbol{e}_1,\\
f_v^t &= \left(\cos t\sin\frac{\theta}{2} + \sin t\cos\frac{\theta}{2}\right)\boldsymbol{e}_2 = \sin\left(\frac{\theta}{2}+t\right)\boldsymbol{e}_2,\\
\nu_u^t &= -\left(\sin t\cos\frac{\theta}{2} + \cos t\sin\frac{\theta}{2}\right)\boldsymbol{e}_1 = -\sin\left(\frac{\theta}{2}+t\right)\boldsymbol{e}_1,\\
\nu_v^t &= \left(-\sin t\sin\frac{\theta}{2} + \cos t\cos\frac{\theta}{2}\right)\boldsymbol{e}_2 = \cos\left(\frac{\theta}{2}+t\right)\boldsymbol{e}_2
\end{aligned}
$$

hold. Thus the first and second fundamental forms of f^t and ν^t are

$$
\begin{aligned}
I^{f^t} &= \cos^2\left(\frac{\theta}{2}+t\right)du^2 + \sin^2\left(\frac{\theta}{2}+t\right)dv^2,\\
II^{f^t} &= \cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)(du^2-dv^2),\\
I^{\nu^t} &= \sin^2\left(\frac{\theta}{2}+t\right)du^2 + \cos^2\left(\frac{\theta}{2}+t\right)dv^2,\\
II^{\nu^t} &= -\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)(du^2-dv^2),
\end{aligned}
$$

where we take $-f^t$ as a dual front of ν^t. Therefore we get

$$
\begin{aligned}
H^t &= \frac{\cos^2\left(\frac{\theta}{2}+t\right)\left(-\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)\right) + \sin^2\left(\frac{\theta}{2}+t\right)\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)}{2\cos^2\left(\frac{\theta}{2}+t\right)\sin^2\left(\frac{\theta}{2}+t\right)}\\
&= -\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)\frac{\cos(\theta+2t)}{2\cos^2\left(\frac{\theta}{2}+t\right)\sin^2\left(\frac{\theta}{2}+t\right)} = -\frac{\cos(\theta+2t)}{\sin(\theta+2t)} = -\cot(\theta+2t)
\end{aligned}
$$

and

$$
\begin{aligned}
H^\nu &= \frac{\sin^2\left(\frac{\theta}{2}+t\right)\left(\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)\right) + \cos^2\left(\frac{\theta}{2}+t\right)\left(-\cos\left(\frac{\theta}{2}+t\right)\sin\left(\frac{\theta}{2}+t\right)\right)}{2\cos^2\left(\frac{\theta}{2}+t\right)\sin^2\left(\frac{\theta}{2}+t\right)}\\
&= -\cot(\theta+2t).
\end{aligned}
$$

This completes the proof. □

Remark 2.5.3. *We note that a parallel front of a flat front is also flat (cf. [20]). This property also holds for flat fronts in $\mathbb{R}^3$ and $\mathbb{H}^3$ (see [10, 23, 33]).*

In Section 2.6, we shall show similar relation for the caustic and its dual.

2.6. **Caustics of flat fronts.** In this section, we consider singularities and geometric properties of caustics of flat fronts in S^3. To investigate them, we do some preparations.

We assume that a flat front $f\colon \Sigma \to S^3$ is constructed by the solution θ to (2.1). Then the Weingarten matrix W^f for f can be given as

$$W^f = \begin{pmatrix} \tan\dfrac{\theta}{2} & 0 \\ 0 & -\cot\dfrac{\theta}{2} \end{pmatrix}. \tag{2.32}$$

by (2.2). Thus the principal curvatures κ_1 and κ_2 of f are given as

$$\kappa_1 = \tan\frac{\theta}{2}, \quad \kappa_2 = -\cot\frac{\theta}{2} \tag{2.33}$$

defined on the set of regular points $\Sigma \setminus \theta^{-1}(\pi\mathbb{Z})$ of f. We remark that κ_i $(i = 1, 2)$ is unbounded near $p \in \theta^{-1}(Z_i)$. On the other hand, the curvature radius function $\rho_i = 1/\kappa_i$ $(i = 1$ or $2)$ can be extended as a C^∞ function near $p \in \theta^{-1}(Z_i)$.

Definition 2.6.1. *Define maps $C_i\colon \Sigma \setminus \theta^{-1}(\pi\mathbb{Z}) \to S^3$ $(i = 1, 2)$ as follows:*

$$C_i^{\pm} = \pm\frac{1}{\sqrt{1+\rho_i^2}}\,(f + \rho_i \nu)\,, \tag{2.34}$$

where $\rho_i = 1/\kappa_i$ (cf. [14, (4.6)]). These $C_i^{\pm}$ are called the caustics (or focal surfaces) of f.

Setting $C_i = C_i^+$ $(i = 1, 2)$, we shall study singularities and geometrical properties of C_i near a singular point $p \in \theta^{-1}(\pi\mathbb{Z})$ of f. By the above discussion, ρ_i $(i = 1$ or $2)$ is bounded C^∞ function near $p \in \theta^{-1}(Z_i)$. Therefore, we can consider C_i on a sufficiently small neighborhood of $p \in \theta^{-1}(Z_i)$. We mainly consider C_1 defined near $p \in \theta^{-1}(Z_1)$ in the following.

2.6.1. **Singularities of the caustic.** We consider singularities of C_1 of a flat front relating to the solution θ of (2.1). We set $R_1 = (1+\rho_1^2)^{-1/2}$, where $\rho_1 = \cot(\theta/2)$ (see (2.33)). Then C_1 can be written as

$$C_1 = R_1(f + \rho_1 \nu).$$

Proposition 2.6.2. *Under the setting, we obtain the following results on an neighborhood of $p \in \theta^{-1}(Z_1)$.*

$$(C_1)_u = \frac{\theta_u}{2} R_1\,(\rho_1 f - \nu)\,, \quad (C_1)_v = \frac{\theta_v}{2} R_1\,(\rho_1 f - \nu) + R_1 \csc\frac{\theta}{2}\boldsymbol{e}_2. \tag{2.35}$$

Proof. We note that

$$(\rho_1)_u = \partial_u\left(\cot\frac{\theta}{2}\right) = \frac{\theta_u}{2}\frac{-1}{\sin^2\dfrac{\theta}{2}} = -\frac{\theta_u}{2}\csc^2\frac{\theta}{2}. \tag{2.36}$$

By (2.9), (2.14) and (2.36), we have

$$
\begin{aligned}
(C_1)_u &= (R_1)_u(f + \rho_1 \nu) + R_1(f_u + (\rho_1)_u \nu + \rho_1(\nu)_u) \\
&= (R_1)_u(f + \rho_1 \nu) + R_1\left(\cos\frac{\theta}{2}\boldsymbol{e}_1 - \frac{\theta_u}{2}\csc^2\frac{\theta}{2}\nu - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\sin\frac{\theta}{2}\boldsymbol{e}_1\right) \\
&= (R_1)_u(f + \rho_1 \nu) - R_1\frac{\theta_u}{2}\csc^2\frac{\theta}{2}\nu. \qquad (2.37)
\end{aligned}
$$

By the same argument as above, since $(\rho_1)_v = -\frac{\theta_v}{2}\csc^2\frac{\theta}{2}$, we have

$$
\begin{aligned}
(C_1)_v &= (R_1)_v(f + \rho_1 \nu) + R_1(f_v + (\rho_1)_v \nu + \rho_1(\nu)_v) \\
&= (R_1)_v(f + \rho_1 \nu) + R_1(\sin\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc^2\frac{\theta}{2}\nu + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}\boldsymbol{e}_2) \\
&= (R_1)_v(f + \rho_1 \nu) + R_1\csc\frac{\theta}{2}(\sin^2\frac{\theta}{2}\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc\frac{\theta}{2}\nu + \cos^2\frac{\theta}{2}\boldsymbol{e}_2) \\
&= (R_1)_v(f + \rho_1 \nu) + R_1\csc\frac{\theta}{2}(\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc\frac{\theta}{2}\nu). \qquad (2.38)
\end{aligned}
$$

We remark that

$$
\csc^2\frac{\theta}{2}R_1^2 = \frac{1}{\sin^2\frac{\theta}{2}}\frac{1}{1+\cot^2\frac{\theta}{2}} = 1.
$$

Thus we see that

$$
(R_1)_u = -\frac{1}{2}(1+\rho_1^2)^{-\frac{3}{2}}2\rho_1(\rho_1)_u = -R_1^3(-\frac{\theta_u}{2}\csc^2\frac{\theta}{2})\rho_1 = \frac{\theta_u}{2}\rho_1 R_1, \qquad (2.39)
$$

$$
(R_1)_v = -\frac{1}{2}(1+\rho_1^2)^{-\frac{3}{2}}2\rho_1(\rho_1)_v = -R_1^3(-\frac{\theta_v}{2}\csc^2\frac{\theta}{2})\rho_1 = \frac{\theta_v}{2}\rho_1 R_1. \qquad (2.40)
$$

By (2.37), (2.39) and $\rho_1^2 - \csc^2\frac{\theta}{2} = -1$, we have

$$
\begin{aligned}
(C_1)_u &= \frac{\theta_u}{2}\rho_1 R_1(f + \rho_1\nu) - R_1\frac{\theta_u}{2}\csc^2\frac{\theta}{2}\nu = \frac{\theta_u R_1}{2}(\rho_1 f + \rho_1^2\nu - \csc^2\frac{\theta}{2}\nu) \\
&= \frac{\theta_u R_1}{2}(\rho_1 f - \nu).
\end{aligned}
$$

Moreover, by (2.38), (2.40) and $\rho_1^2 - \csc^2\frac{\theta}{2} = -1$, we have

$$\begin{aligned}(C_1)_v &= \frac{\theta_v}{2}\rho_1 R_1(f+\rho_1\nu) + R_1\csc\frac{\theta}{2}(\boldsymbol{e}_2 - \frac{\theta_v}{2}\csc\frac{\theta}{2}\nu)\\ &= \frac{\theta_v}{2}R_1(\rho_1 f + \rho_1^2\nu - \csc^2\frac{\theta}{2}\nu) + R_1\csc\frac{\theta}{2}\boldsymbol{e}_2\\ &= \frac{\theta_v}{2}R_1(\rho_1 f - \nu) + R_1\csc\frac{\theta}{2}\boldsymbol{e}_2.\end{aligned}$$

Therefore we have the assertion. □

Remark 2.6.3. *By the same argument as Proposition 2.6.2, we have*

$$(C_2)_u = \frac{\theta_u}{2}R_2(\rho_2 f - \nu) + R_2\sec\frac{\theta}{2}\boldsymbol{e}_1, \quad (C_2)_v = \frac{\theta_v}{2}R_2(\rho_2 f - \nu), \tag{2.41}$$

where $R_2 = (1+\rho_2^2)^{-1/2}$*. For the proof of (2.41), we use the relation* $R_2^2\sec^2(\theta/2) = 1$*.*

By (2.35), we see that $\boldsymbol{e}_1$ can be taken as a unit normal vector of C_1. Thus $\boldsymbol{e}_2$ can be taken as a unit normal vector to C_2. Moreover, we see that

$$\begin{aligned}(C_1)_u \wedge (C_1)_v \wedge \boldsymbol{e}_1 &= \frac{\theta_u}{2}\csc\frac{\theta}{2}R_1^2(\rho_1 f - \nu)\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_1,\\ (C_2)_u \wedge (C_2)_v \wedge \boldsymbol{e}_2 &= \frac{\theta_v}{2}\sec\frac{\theta}{2}R_2^2\boldsymbol{e}_1\wedge(\rho_2 f - \nu)\wedge\boldsymbol{e}_2,\end{aligned} \tag{2.42}$$

by (2.35) and (2.41). For C_1, since $R_1 \neq 0$ and $\csc(\theta/2) \neq 0$ around $p \in \theta^{-1}(Z_1)$, the set of singular points of C_1 might be considered as $(\theta_u)^{-1}(0)$ near p. Meanwhile, for C_2, $R_2 \neq 0$ and $\sec(\theta/2) \neq 0$ around $p \in \theta^{-1}(Z_2)$, the set of singular points of C_2 is considered as $(\theta_v)^{-1}(0)$ near p.

Lemma 2.6.4. *C_i is a front with its dual front $\boldsymbol{e}_i$ at $p \in \theta^{-1}(Z_i)$ ($i = 1$ or 2).*

Proof. First we prove that C_1 and $\boldsymbol{e}_1$ are fronts at p. Let us calculate the rank of the following matrix:

$$\begin{pmatrix}(C_1)_u & (C_1)_v\\ (\boldsymbol{e}_1)_u & (\boldsymbol{e}_1)_v\end{pmatrix}. \tag{2.43}$$

If p is a regular point of C_1, i.e., $\theta_u(p) \neq 0$, then we see that (2.43) is a rank two matrix at p. Thus C_1 is a front at p. Assume that p is a singular point of C_1, i.e., $\theta_u(p) = 0$. Since $\boldsymbol{e}_2$ and ν are linearly independent, we have

$$(\boldsymbol{e}_1)_u(p) = \frac{\theta_v}{2}\boldsymbol{e}_2(p) + \varepsilon_s\nu(p) \neq \boldsymbol{0}.$$

where $\varepsilon_s = \operatorname{sgn}(\sin\frac{\theta}{2})$. Moreover, by $(C_1)_v = \varepsilon_s \boldsymbol{e}_2 - \frac{\theta_v}{2}\nu \neq \boldsymbol{0}$ and $(\boldsymbol{e}_1)_v = \boldsymbol{0}$ at p, we notice that (2.43) is a rank two matrix at p. Hence C_1 is a front at p. Moreover, by (2.13), we have

$$\begin{aligned}
\langle (\boldsymbol{e}_1)_u, -C_1\rangle &= \left\langle -\cos\frac{\theta}{2} f + \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\nu, -R_1(f+\rho_1\nu)\right\rangle \\
&= R_1\cos\frac{\theta}{2} - R_1\rho_1\sin\frac{\theta}{2} = R_1\cos\frac{\theta}{2} - R_1\cot\frac{\theta}{2}\sin\frac{\theta}{2} = 0, \\
\langle (\boldsymbol{e}_1)_v, -C_1\rangle &= \left\langle \frac{\theta_u}{2}\boldsymbol{e}_2, -R_1(f+\rho_1\nu)\right\rangle = 0.
\end{aligned}$$

Therefore, $\boldsymbol{e}_1$ is also a front and $-C_1$ can be taken as a dual front of $\boldsymbol{e}_1$.

Next we prove that C_2 and $\boldsymbol{e}_2$ are fronts at p. We calculate the rank of the following matrix:

$$\begin{pmatrix} (C_2)_u & (C_2)_v \\ (\boldsymbol{e}_2)_u & (\boldsymbol{e}_2)_v \end{pmatrix}. \tag{2.44}$$

If p is a regular point of C_2, i.e., $\theta_v(p) \neq 0$, then we see that (2.44) is a rank two matrix at p. Thus C_1 is a front at p. Assume that p is a singular point of C_2, i.e., $\theta_v(p) = 0$. Since $\boldsymbol{e}_1$ and ν are linearly independent, we have

$$(\boldsymbol{e}_2)_v(p) = -\frac{\theta_u}{2}\boldsymbol{e}_1 - \varepsilon_c\nu \neq \boldsymbol{0}.$$

where $\varepsilon_c = \operatorname{sgn}(\cos\frac{\theta}{2})$. Moreover, by $(C_2)_u = -\frac{\theta_u}{2}\nu + \varepsilon_c\boldsymbol{e}_1 \neq \boldsymbol{0}$ and $(\boldsymbol{e}_2)_u = \boldsymbol{0}$ at p , we see that (2.44) is a rank two matrix at p. Hence C_2 is a front at p. Moreover, by (2.13), we have

$$\begin{aligned}
\langle (\boldsymbol{e}_2)_u, -C_2\rangle &= \left\langle -\frac{\theta_v}{2}\boldsymbol{e}_1, -R_2(f+\rho_2\nu)\right\rangle = 0, \\
\langle (\boldsymbol{e}_2)_v, -C_2\rangle &= \left\langle -\sin\frac{\theta}{2} f - \frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu, -R_2(f+\rho_2\nu)\right\rangle \\
&= R_2\sin\frac{\theta}{2} + R_2\rho_2\cos\frac{\theta}{2} = R_2\sin\frac{\theta}{2} - R_2\tan\frac{\theta}{2}\cos\frac{\theta}{2} = 0.
\end{aligned}$$

Therefore, $\boldsymbol{e}_2$ is also a front and $-C_2$ can be taken as a dual front of $\boldsymbol{e}_2$ and we have the assertion. □

Remark 2.6.5. *By the proof of Lemma 2.6.4, we notice that the singular sets of C_i and $\boldsymbol{e}_i$ coincide.*

Proposition 2.6.6. *Let $f\colon \Sigma \to S^3$ be a flat front associated to the solution θ to the equation (2.1). Let $p \in \theta^{-1}(\pi\mathbb{Z})$ be a non-degenerate singular point of f, i.e., $(\theta_u, \theta_v) \neq (0,0)$ at p.*

(1) *When $p \in \theta^{-1}(Z_1)$, then*
   - *p is a regular point of C_1 if and only if p is a cuspidal edge of f ;*
   - *p is a cuspidal edge of C_1 if and only if p is a swallowtail of f.*

(2) *When $p \in \theta^{-1}(Z_2)$, then*
   - *p is regular point of C_2 if and only if p is a cuspidal edge of f ;*
   - *p is a cuspidal edge of C_2 if and only if p is a swallowtail of f.*

We note that C_i is a front at $p \in \theta^{-1}(Z_i)$ ($i = 1$ or 2) by Lemma 2.6.4.

Proof. We show (1). By (2.42), p is a regular point of C_1 if and only if $\theta_u(p) \neq 0$. Assume that $\theta_u(p) = 0$. By the proof of Lemma 2.6.4, we see that $(C_1)_u = \mathbf{0}$ and $(C_1)_v \neq \mathbf{0}$ at p. It implies that p is a rank one singular point of C_1. Moreover, a null vector field of C_1 can be taken as $\eta^{C_1} = \partial_u$. Since the zero set of θ_u is the set of singular points of C_1 near p, an identifier of singularities $\tilde{\lambda}^{C_1}$ can be considered as $\tilde{\lambda}^{C_1} = \theta_u$. Thus $\eta^{C_1}\tilde{\lambda}^{C_1} = \theta_{uu}$. By Fact 2.2.4 and Proposition 2.3.1, we have the assertion.

We show (2). By (2.42), p is a regular point of C_2 if and only if $\theta_v(p) \neq 0$. Assume that $\theta_v(p) = 0$. By the proof of Lemma 2.6.4, we see that $(C_2)_u \neq \mathbf{0}$ and $(C_2)_v = \mathbf{0}$ at p. It implies that p is a rank one singular point of C_2. Moreover, a null vector field of C_2 can be taken as $\eta^{C_2} = \partial_v$. Since the zero set of θ_v is the set of singular points of C_2 near p, an identifier of singularities $\tilde{\lambda}^{C_2}$ can be considered as $\tilde{\lambda}^{C_2} = \theta_v$. Thus $\eta^{C_2}\tilde{\lambda}^{C_2} = \theta_{vv}$. By Fact 2.2.4 and Proposition 2.3.1, we have the assertion. This completes the proof. □

2.6.2. **Mean curvature of the caustic and its dual.** In this section, we consider curvatures of caustic. It is well known that caustics of a flat front are also flat [20] (we will see below). On the other hand, we have the following property of the mean curvature.

Theorem 2.6.7. *Let $f\colon \Sigma \to S^3$ be a flat front corresponding to the function θ satisfying (2.1). Then the mean curvature of the caustic C_1 (resp. C_2) as in (2.34) is represented as*

$$H^{C_1} = -\frac{\varepsilon_1}{\theta_u}\left(\frac{\theta_u^2}{4} - \frac{\theta_v^2}{4} - 1\right) \quad \left(\textit{resp. } H^{C_2} = \frac{\varepsilon_2}{\theta_v}\left(\frac{\theta_v^2}{4} - \frac{\theta_u^2}{4} - 1\right)\right) \tag{2.45}$$

on the set of regular points, where $\varepsilon_1 = \operatorname{sgn}(\csc\frac{\theta}{2})$ (resp. $\varepsilon_2 = \operatorname{sgn}(\sec\frac{\theta}{2})$). If p is a singular point of C_1 (resp. C_2), then H^{C_1} (resp. H^{C_2}) is unbounded near p.

Proof. We first deal with C_1 defined near $p \in \theta^{-1}(Z_1)$. By (2.35), we have

$$\begin{aligned}
\langle (C_1)_u, (C_1)_u\rangle &= \left\langle \frac{\theta_u}{2}R_1(\rho_1 f - \nu), \frac{\theta_u}{2}R_1(\rho_1 f - \nu)\right\rangle = \frac{\theta_u^2 R_1^2}{4}\rho_1^2 + \frac{\theta_u^2 R_1^2}{4} \\
&= \frac{\theta_u^2}{4}R_1^2\left(\rho_1^2 + 1\right) = \frac{\theta_u^2}{4}\left(\frac{1}{\rho_1^2+1}\right)\left(\rho_1^2+1\right) = \frac{\theta_u^2}{4}, \\
\langle (C_1)_u, (C_1)_v\rangle &= \left\langle \frac{\theta_u}{2}R_1(\rho_1 f - \nu), \frac{\theta_v}{2}R_1(\rho_1 f - \nu) + R_1\csc\frac{\theta}{2}\boldsymbol{e}_2\right\rangle = \frac{\theta_u\theta_v}{4}R_1^2\rho_1^2 + \frac{\theta_u\theta_v}{4}R_1^2 \\
&= \frac{\theta_u\theta_v}{4}R_1^2\left(1+\rho_1^2\right) = \frac{\theta_u\theta_v}{4}\left(\frac{1}{1+\rho_1^2}\right)\left(1+\rho_1^2\right) = \frac{\theta_u\theta_v}{4},
\end{aligned}$$

and

$$\begin{aligned}
\langle (C_1)_v, (C_1)_v\rangle &= \left\langle \frac{\theta_v}{2} R_1\left(\rho_1 f - \nu\right) + R_1 \csc\frac{\theta}{2}\boldsymbol{e}_2, \frac{\theta_v}{2} R_1\left(\rho_1 f - \nu\right) + R_1 \csc\frac{\theta}{2}\boldsymbol{e}_2\right\rangle \\
&= \frac{\theta_v^2 R_1^2}{4}\rho_1^2 + \frac{\theta_v^2 R_1^2}{4} + R_1^2 \csc^2\frac{\theta}{2} = R_1^2\left(\frac{\theta_v^2}{4}\rho_1^2 + \frac{\theta_v^2}{4}\right) + R_1^2\csc^2\frac{\theta}{2} \\
&= \frac{\theta_v^2}{4}\left(\frac{1}{1+\rho_1^2}\right)\left(\rho_1^2+1\right) + \left(\frac{1}{1+\rho_1^2}\right)\csc^2\frac{\theta}{2} = \frac{\theta_v^2}{4} + \left(\frac{1}{1+\cot^2\frac{\theta}{2}}\right)\frac{1}{\sin^2\frac{\theta}{2}} \\
&= \frac{\theta_v^2}{4} + 1.
\end{aligned}$$

Thus the first fundamental form (i.e., the induced metric) of C_1 can be calculated as

$$I^{C_1} = \frac{\theta_u^2}{4}du^2 + \frac{\theta_u\theta_v}{2}dudv + \left(\frac{\theta_v^2}{4}+1\right)dv^2. \tag{2.46}$$

On the other hand, by (2.13) and (2.35), we have

$$\begin{aligned}
-\langle (C_1)_u, (\boldsymbol{e}_1)_u\rangle &= -\left\langle \frac{\theta_u}{2}R_1\left(\rho_1 f - \nu\right), -\cos\frac{\theta}{2}f + \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\nu\right\rangle \\
&= \frac{\theta_u}{2}R_1\rho_1\cos\frac{\theta}{2} + \frac{\theta_u}{2}R_1\sin\frac{\theta}{2} = \frac{\theta_u}{2}R_1\left(\rho_1\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \\
&= \frac{\theta_u}{2}R_1\left(\cot\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) = \frac{\theta_u}{2}R_1\left(\frac{1}{\sin\frac{\theta}{2}}\right) \\
&= \frac{\theta_u}{2}\left(\frac{1}{\sqrt{1+\cot^2\frac{\theta}{2}}}\right)\left(\frac{1}{\sin\frac{\theta}{2}}\right) = \frac{\theta_u}{2}\frac{|\sin\frac{\theta}{2}|}{\sin\frac{\theta}{2}} = \varepsilon_1\frac{\theta_u}{2},
\end{aligned}$$

$$\begin{aligned}
-\langle (C_1)_u, (\boldsymbol{e}_1)_v\rangle &= -\left\langle \frac{\theta_u}{2}R_1\left(\rho_1 f - \nu\right), \frac{\theta_u}{2}\boldsymbol{e}_2\right\rangle = 0, \\
-\langle (C_1)_v, (\boldsymbol{e}_1)_v\rangle &= -\left\langle \frac{\theta_v}{2}R_1\left(\rho_1 f - \nu\right) + R_1\csc\frac{\theta}{2}\boldsymbol{e}_2, \frac{\theta_u}{2}\boldsymbol{e}_2\right\rangle \\
&= -\frac{\theta_u}{2}R_1\csc\frac{\theta}{2} = -\varepsilon_1\frac{\theta_u}{2}.
\end{aligned}$$

Thus the second fundamental form of C_1 with respect to $\boldsymbol{e}_1$ is given as

$$II^{C_1} = \varepsilon_1\frac{\theta_u}{2}(du^2 - dv^2). \tag{2.47}$$

Thus we see that the Gauss-Kronecker curvature $K_{\mathrm{E}}^{C_1}$ can be calculated as

$$K_{\mathrm{E}}^{C_1} = \frac{\varepsilon_1 \dfrac{\theta_u}{2}\left(-\varepsilon_1 \dfrac{\theta_u}{2}\right)}{\dfrac{\theta_u^2}{4}\left(\dfrac{\theta_v^2}{4}+1\right) - \left(\dfrac{\theta_u\theta_v}{4}\right)^2} = -1.$$

by (2.46) and (2.47). This implies that the intrinsic Gaussian curvature $K_{\mathrm{I}}^{C_1}$ of C_1 vanishes identically ([47, Page 86]). We note that this is a classical fact (cf. [20]).

We investigate the mean curvature H^{C_1} of C_1 with respect to $\boldsymbol{e}_1$. By (2.46) and (2.47), we have

$$\begin{aligned} H^{C_1} &= \frac{\dfrac{\theta_u^2}{4}\left(-\varepsilon_1 \dfrac{\theta_u}{2}\right) + \left(\dfrac{\theta_v^2}{4}+1\right)\varepsilon_1\dfrac{\theta_u}{2}}{2\left(\dfrac{\theta_u^2}{4}\left(\dfrac{\theta_v^2}{4}+1\right) - \left(\dfrac{\theta_u\theta_v}{4}\right)^2\right)} = \frac{-\varepsilon_1 \dfrac{\theta_u}{2}\left(\dfrac{\theta_u^2}{4} - \dfrac{\theta_v^2}{4} - 1\right)}{\dfrac{\theta_u^2}{2}} \\ &= -\frac{\varepsilon_1}{\theta_u}\left(\frac{\theta_u^2}{4} - \frac{\theta_v^2}{4} - 1\right) \end{aligned}$$

on the set of regular points. We note that p is a regular point of C_1 if and only if $\theta_u(p) \neq 0$ by (2.42). Moreover, since C_1 is a front near p, H^{C_1} is unbounded near p by [46, Corollary 3.5] and [26, Proposition 2.4].

For C_2 defined near $p \in \theta^{-1}(Z_2)$, by (2.41), we have

$$\begin{aligned} \langle (C_2)_u, (C_2)_u \rangle &= \left\langle \frac{\theta_u}{2} R_2(\rho_2 f - \nu) + R_2 \sec\frac{\theta}{2}\boldsymbol{e}_1, \frac{\theta_u}{2} R_2(\rho_2 f - \nu) + R_2 \sec\frac{\theta}{2}\boldsymbol{e}_1 \right\rangle \\ &= \frac{\theta_u^2}{4} R_2^2\rho_2^2 + \frac{\theta_u^2}{4} R_2^2 + R_2^2 \sec^2\frac{\theta}{2} = \frac{\theta_u^2}{4} R_2^2\left(\rho_2^2+1\right) + R_2^2\sec^2\frac{\theta}{2} \\ &= \frac{\theta_u^2}{4}\left(\frac{1}{1+\rho_2^2}\right)\left(\rho_2^2+1\right) + \left(\frac{1}{1+\rho_2^2}\right)\sec^2\frac{\theta}{2} = \frac{\theta_u^2}{4} + \left(\frac{1}{1+\rho_2^2}\right)\sec^2\frac{\theta}{2} \\ &= \frac{\theta_u^2}{4} + \left(\frac{1}{1+\tan^2\dfrac{\theta}{2}}\right)\sec^2\frac{\theta}{2} = \frac{\theta_u^2}{4} + 1, \end{aligned}$$

$$\begin{aligned} \langle (C_2)_u, (C_2)_v \rangle &= \left\langle \frac{\theta_u}{2} R_2(\rho_2 f - \nu) + R_2 \sec\frac{\theta}{2}\boldsymbol{e}_1, \frac{\theta_v}{2} R_2(\rho_2 f - \nu) \right\rangle \\ &= \frac{\theta_u\theta_v}{4} R_2^2\rho_2^2 + \frac{\theta_u\theta_v}{4} R_2^2 = \frac{\theta_u\theta_v}{4} R_2^2\left(1+\rho_2^2\right) = \frac{\theta_u\theta_v}{4}, \end{aligned}$$

$$\begin{aligned} \langle (C_2)_v, (C_2)_v \rangle &= \left\langle \frac{\theta_v}{2} R_2(\rho_2 f - \nu), \frac{\theta_v}{2} R_2(\rho_2 f - \nu) \right\rangle = \frac{\theta_v^2}{4} R_2^2\rho_2^2 + \frac{\theta_v^2}{4} R^2 \\ &= \frac{\theta_v^2}{4} R_2^2\left(1+\rho_2^2\right) = \frac{\theta_v^2}{4}, \end{aligned}$$

Thus the first fundamental form of C_2 can be calculated as

$$I^{C_2} = \left(1 + \frac{\theta_u^2}{4}\right) du^2 + \frac{\theta_u\theta_v}{2} dudv \frac{\theta_v^2}{4} dv^2 \tag{2.48}$$

On the oher hand, by (2.13) and (2.41), we have

$$\begin{aligned}
-\langle (C_2)_u, (\boldsymbol{e}_2)_u \rangle &= -\left\langle \frac{\theta_u}{2} R_2(\rho_2 f - \nu) + R_2 \sec\frac{\theta}{2}\boldsymbol{e}_1, -\frac{\theta_v}{2}\boldsymbol{e}_1 \right\rangle = \frac{\theta_v}{2} R_2 \sec\frac{\theta}{2} \\
&= \frac{\theta_v}{2} \frac{1}{\sqrt{1+\tan^2\dfrac{\theta}{2}}} \sec\frac{\theta}{2} = \frac{\theta_v}{2} \frac{|\cos\dfrac{\theta}{2}|}{\cos\dfrac{\theta}{2}} = \varepsilon_2 \frac{\theta_v}{2},
\end{aligned}$$

$$\begin{aligned}
-\langle (C_2)_u, (\boldsymbol{e}_2)_v \rangle &= -\left\langle \frac{\theta_u}{2} R_2(\rho_2 f - \nu) + R_2 \sec\frac{\theta}{2}\boldsymbol{e}_1, -\sin\frac{\theta}{2} f - \frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu \right\rangle \\
&= \frac{\theta_u}{2} R_2\rho_2 \sin\frac{\theta}{2} - \frac{\theta_u}{2} R_2 \cos\frac{\theta}{2} + \frac{\theta_u}{2} R_2 \sec\frac{\theta}{2} \\
&= \frac{\theta_u}{2} R_2 \left(\rho_2 \sin\frac{\theta}{2} - \cos\frac{\theta}{2} + \sec\frac{\theta}{2}\right) = \frac{\theta_u}{2} R_2 \left(-\tan\frac{\theta}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2} + \sec\frac{\theta}{2}\right) \\
&= \frac{\theta_u}{2} R_2 \left(-\frac{\sin^2\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} - \frac{\cos^2\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} + \sec\frac{\theta}{2}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
-\langle (C_2)_v, (\boldsymbol{e}_2)_v \rangle &= -\left\langle \frac{\theta_v}{2} R_2(\rho_2 f - \nu), -\sin\frac{\theta}{2} f - \frac{\theta_u}{2}\boldsymbol{e}_1 - \cos\frac{\theta}{2}\nu \right\rangle \\
&= \frac{\theta_v}{2} R_2 \left(\rho_2 \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) = \frac{\theta_v}{2} R_2 \left(-\frac{\sin^2\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} - \frac{\cos^2\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}\right) \\
&= -\frac{\theta_v}{2} R_2 \frac{1}{\cos\dfrac{\theta}{2}} = -\varepsilon_2 \frac{\theta_v}{2}.
\end{aligned}$$

Hence the second fundamental forms can be calculated as

$$II^{C_2} = \varepsilon_2 \frac{\theta_v}{2}(du^2 - dv^2), \tag{2.49}$$

where we take $\boldsymbol{e}_2$ as a unit normal vector to C_2. Thus we see that the Gauss-Kronecker curvature $K_{\mathrm{E}}^{C_2}$ can be calculated as

$$K_{\mathrm{E}}^{C_2} = \frac{\varepsilon_2 \dfrac{\theta_v}{2}\left(-\varepsilon_2 \dfrac{\theta_v}{2}\right)}{\left(\dfrac{\theta_u^2}{4} + 1\right)\dfrac{\theta_v^2}{4} - \left(\dfrac{\theta_u\theta_v}{4}\right)^2} = -1$$

Moreover, the mean curvature H^{C_2} is

$$H^{C_2} = \frac{\left(\frac{\theta_u^2}{4}+1\right)\left(-\varepsilon_2\frac{\theta_v}{2}\right)+\frac{\theta_v^2}{4}\varepsilon_2\frac{\theta_v}{2}}{\frac{\theta_v^2}{2}} = \frac{\varepsilon_2\frac{\theta_v}{2}\left(\frac{\theta_v^2}{4}-\frac{\theta_u^2}{4}-1\right)}{\frac{\theta_v^2}{2}}$$
$$= \frac{\varepsilon_2}{\theta_v}\left(\frac{\theta_v^2}{4}-\frac{\theta_u^2}{4}-1\right)$$

on the set of regular points. We note that p is a regular point of C_2 if and only if $\theta_v(p) \neq 0$ by (2.42). This completes the proof.

□

Proposition 2.6.8. *If the caustic C_i ($i = 1$ or 2) has constant mean curvature, then C_i is regular.*

Proof. We consider the case of C_1 defined near $p \in \theta^{-1}(Z_1)$. By Theorem 2.6.7, if a singular point q of C_1 exists on the source of C_1, then H^{C_1} is unbounded at q. This implies that C_1 cannot be a constant mean curvature surface. Thus we have the conclusion. □

Remark 2.6.9. *It is known that caustics of flat fronts in the Euclidean 3-space $\mathbb{R}^3$ and hyperbolic 3-space $\mathbb{H}^3$ are also flat (see [22, 33, 39]).*

For the dual front $\boldsymbol{e}_i$ ($i = 1, 2$) of C_i, these can be considered as maps into S^3. In particular, we have the following.

Proposition 2.6.10. *Let $f : \Sigma \to S^3$ be a flat front corresponding to the function θ satisfying (2.1). Let C_i ($i = 1$ or 2) be the caustic of f defined on a neighborhood U_i of $p \in \theta^{-1}(Z_i)$. Then its dual $\boldsymbol{e}_i$ is also a flat front and the mean curvature satisfies $H^{\boldsymbol{e}_i} = H^{C_i}$, where we take $-C_i$ as dual of $\boldsymbol{e}_i$.*

Proof. We show the case for $i = 1$. By Lemma 2.6.4, $\boldsymbol{e}_1$ is a front. Since $(\boldsymbol{e}_1)_u = -\cos\frac{\theta}{2}f + \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\nu$ and $(\boldsymbol{e}_1)_v = \frac{\theta_u}{2}\boldsymbol{e}_2$, we see that

$$\begin{aligned}
\langle(\boldsymbol{e}_1)_u, (\boldsymbol{e}_1)_u\rangle &= \cos^2\frac{\theta}{2} + \frac{\theta_v^2}{4} + \sin^2\frac{\theta}{2} = \frac{\theta_v^2}{4} + 1,\\
\langle(\boldsymbol{e}_1)_u, (\boldsymbol{e}_1)_v\rangle &= \frac{\theta_u\theta_v}{4},\\
\langle(\boldsymbol{e}_1)_v, (\boldsymbol{e}_1)_u\rangle &= \frac{\theta_u^2}{4}.
\end{aligned}$$

Thus the first fundamental form of $\boldsymbol{e}_1$ is given by

$$I^{\boldsymbol{e}_1} = \left(1+\frac{\theta_v^2}{4}\right)du^2 + \frac{\theta_u\theta_v}{2}dudv + \frac{\theta_u^2}{4}dv^2. \tag{2.50}$$

Moreover, we see that

$$
\begin{aligned}
-\langle(\boldsymbol{e}_1)_u,(-C_1)_u\rangle &= \langle(\boldsymbol{e}_1)_u,(C_1)_u\rangle = -\varepsilon_1\frac{\theta_u}{2},\\
-\langle(\boldsymbol{e}_1)_u,(-C_1)_v\rangle &= \langle(\boldsymbol{e}_1)_u,(C_1)_v\rangle = 0,\\
-\langle(\boldsymbol{e}_1)_v,(-C_1)_v\rangle &= \langle(\boldsymbol{e}_1)_v,(C_1)_v\rangle = \varepsilon_1\frac{\theta_u}{2},
\end{aligned}
$$

where we can be taken $-C_1$ as dual of $\boldsymbol{e}_1$. Hence the second fundamental form of $\boldsymbol{e}_1$ can be calculated as

$$
II^{\boldsymbol{e}_1} = -II^{C_1} = -\varepsilon_1\frac{\theta_u}{2}(du^2-dv^2)
$$

Thus Gauss-Kronecker curvature $K_{\mathrm{E}}^{\boldsymbol{e}_1}$ of $\boldsymbol{e}_1$ can be calculated as

$$
K_{\mathrm{E}}^{\boldsymbol{e}_1} = \frac{\dfrac{-\theta_u^2}{4}}{\left(\dfrac{\theta_v^2}{4}+1\right)\dfrac{\theta_u^2}{4}-\dfrac{\theta_u^2\theta_v^2}{16}} = -1
$$

and hence $\boldsymbol{e}_1$ is flat. Moreover, the mean curvature $H^{\boldsymbol{e}_1}$ of $\boldsymbol{e}_1$ with respect to $-C_1$ is

$$
H^{\boldsymbol{e}_1} = \frac{\left(\dfrac{\theta_v^2}{4}+1\right)\varepsilon_1\dfrac{\theta_u}{2}+\dfrac{\theta_u^2}{4}\left(-\varepsilon_1\dfrac{\theta_u}{2}\right)}{\dfrac{\theta_u^2}{2}} = -\frac{\varepsilon_1}{\theta_u}\left(\frac{\theta_u^2}{4}-\frac{\theta_v^2}{4}-1\right) = H^{C_1}.
$$

We next consider $i=2$. Since $(\boldsymbol{e}_2)_u = -\dfrac{\theta_v}{2}\boldsymbol{e}_1$ and $(\boldsymbol{e}_2)_v = -\sin\dfrac{\theta}{2}f - \dfrac{\theta_u}{2}\boldsymbol{e}_1 - \cos\dfrac{\theta}{2}\nu$, we see that

$$
\langle(\boldsymbol{e}_2)_u,(\boldsymbol{e}_2)_u\rangle = \frac{\theta_v^2}{4},\quad \langle(\boldsymbol{e}_2)_u,(\boldsymbol{e}_2)_v\rangle = \frac{\theta_u\theta_v}{4},\quad \langle(\boldsymbol{e}_2)_v,(\boldsymbol{e}_2)_v\rangle = \frac{\theta_u^2}{4}+1.
$$

Thus the first fundamental form of $\boldsymbol{e}_2$ is given by

$$
I^{\boldsymbol{e}_1} = \frac{\theta_v^2}{4}du^2 + \frac{\theta_u\theta_v}{2}dudv + \left(\frac{\theta_u^2}{4}+1\right)dv^2.
$$

Moreover, we see that

$$
\begin{aligned}
-\langle(\boldsymbol{e}_2)_u,(-C_2)_u\rangle &= \langle(\boldsymbol{e}_2)_u,(C_2)_u\rangle = -\varepsilon_2\frac{\theta_v}{2},\\
-\langle(\boldsymbol{e}_2)_u,(-C_2)_v\rangle &= \langle(\boldsymbol{e}_2)_u,(C_2)_v\rangle = 0,\\
-\langle(\boldsymbol{e}_2)_v,(-C_2)_v\rangle &= \langle(\boldsymbol{e}_2)_v,(C_2)_v\rangle = \varepsilon_2\frac{\theta_v}{2},
\end{aligned}
$$

where we can be taken $-C_2$ as dual of $\boldsymbol{e}_2$. Hence the second fundamental form of $\boldsymbol{e}_2$ can be calculated as

$$
II^{\boldsymbol{e}_2} = -\varepsilon_2\frac{\theta_v}{2}(du^2-dv^2).
$$

By the same argument as above, we have

$$K_{\mathrm{E}}^{\boldsymbol{e}_2} = \frac{-\dfrac{\theta_v^2}{4}}{\left(\dfrac{\theta_v^2}{4}\right)\left(\dfrac{\theta_u^2}{4}+1\right)-\dfrac{\theta_u^2\theta_v^2}{16}} = -1,$$

$$H^{\boldsymbol{e}_2} = \frac{\dfrac{\theta_v^2}{4}\left(\varepsilon_2\dfrac{\theta_v}{2}\right)+\left(\dfrac{\theta_u^2}{4}+1\right)\left(-\varepsilon_2\dfrac{\theta_v}{2}\right)}{\dfrac{\theta_v^2}{2}} = \frac{\varepsilon_2}{\theta_v}\left(\frac{\theta_v^2}{4}-\frac{\theta_u^2}{4}-1\right) = H^{C_2}.$$

Therefore we get the conclusions. □

Remark 2.6.11. *If we choose C_i ($i = 1$ or 2) as dual of $\boldsymbol{e}_i$, we have $H^{\boldsymbol{e}_i} = -H^{C_i}$.*

As an application of Propositions 2.6.8 and 2.6.10, we notice the following result.

Corollary 2.6.12. *The caustic C_i ($i = 1$ or 2) has constant mean curvature if and only if the dual front $\boldsymbol{e}_i$ has constant mean curvature. In particular, these surfaces are regular.*

Since C_i and $\boldsymbol{e}_i$ ($i = 1$ or 2) are flat front, we can see dual relations with respect to their singularities by using Corollaries 2.3.3 and 2.4.6.

Corollary 2.6.13. *Let p be a singular point of C_i and $\boldsymbol{e}_i$ ($i = 1$ or 2).*

(1) *If p is swallowtail of C_i (resp. $\boldsymbol{e}_i$), then p is a cuspidal edge of $\boldsymbol{e}_i$ (resp. C_i)*
(2) *Assume that p is a cuspidal edge of C_i and $\boldsymbol{e}_i$. Then $\operatorname{sgn}(\kappa_s^{C_i}) = -\operatorname{sgn}(\kappa_s^{\boldsymbol{e}_i})$ holds at p. Where $\kappa_s^{C_i}$ (resp. $\kappa_s^{\boldsymbol{e}_i}$) is the singular curvature of C_i (resp. $\boldsymbol{e}_i$).*

2.7. **Flat fronts whose caustics are minimal surfaces.** We consider a flat front whose caustic is a minimal surface. In particular, we focus on the caustic C_1 of a flat front $f\colon \Sigma \to S^3$ which is defined on a simply connected neighborhood of $p \in \theta^{-1}(Z_1)$. In order to do this, we investigate the system of partial differential equations:

$$\begin{cases} \theta_{uu} - \theta_{vv} = 0, \\ \theta_u^2 - \theta_v^2 = 4. \end{cases} \tag{2.51}$$

This system arises from (2.1) and (2.45).

Proposition 2.7.1. *If a function θ defined on a simply connected open neighborhood of $p \in \theta^{-1}(Z_1)$ satisfies the system (2.51), then there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$ such that*

$$\theta(u, v) = \left(\beta + \frac{1}{\beta}\right)u + \left(\beta - \frac{1}{\beta}\right)v + \alpha. \tag{2.52}$$

Proof. We first note that $\theta_u \neq 0$ near p by Proposition 2.6.8. Differentiating the second equation of (2.51) by $\partial_u + \partial_v$ and $\partial_u - \partial_v$, we have

$$2(\theta_u - \theta_v)(\theta_{uu} + \theta_{uv}) = 0, \quad 2(\theta_u + \theta_v)(\theta_{uu} - \theta_{uv}) = 0,$$

respectively. Since the second equation of (2.51) holds, both $\theta_u + \theta_v$ and $\theta_u - \theta_v$ do not vanish near p. Thus it follows that

$$\theta_{uu} + \theta_{uv} = \theta_{uu} - \theta_{uv} = 0,$$

and hence we have $\theta_{uu} = \theta_{uv} = 0$. Moreover, we get $\theta_{vv} = 0$ by (2.1). Therefore θ is a linear function of u and v. Since the second equation can be rewritten as

$$(\theta_u + \theta_v)(\theta_u - \theta_v) = 4,$$

there exists a constant $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$\theta_u + \theta_v = 2\beta, \quad \theta_u - \theta_v = \frac{2}{\beta}.$$

Thus we get

$$d\theta = \theta_u du + \theta_v dv = \left(\beta + \frac{1}{\beta}\right) du + \left(\beta - \frac{1}{\beta}\right) dv.$$

Integrating this, we have

$$\theta = \left(\beta + \frac{1}{\beta}\right) u + \left(\beta - \frac{1}{\beta}\right) v + \alpha,$$

where α is a constant of integration. □

For the function θ as in (2.52), if we choose $\beta = 1$ or $\beta = -1$, then it holds that $\theta = 2u + \alpha$ or $\theta = -2u + \alpha$. Thus in these cases, the resulting flat fronts are rotational surfaces in S^3 ([10]).

3. Relations between $(n, n+1)$-cusps and evolutes of fronts

3.1. **Introduction.** It has been known that for regular curves without inflection points, we can define their evolutes. The evolute is the locus of the centers of the osculating circles of a given plane curve $c : I \to \mathbb{R}^2$. The evolutes (or caustics) are important because they relate to applications concerning envelopes, optical systems and general relativity theory ([5, 12]). If c has a singular point, then the evolute of c cannot be defined as the conventional method. However, in the work of Fukunaga-Takahashi, an evolute of a wave front was defined and an useful representation of it was given ([8]). Furthermore, they proved that multiplicities of singularities of fronts were lowered by considering evolutes, and the evolutes of $Cusp_{(2,3)}(t) = (t^2, t^3)$ and $Cusp_{(3,4)}(t) = (t^3, t^4)$ were considered as examples (see Figures 13 and 14).

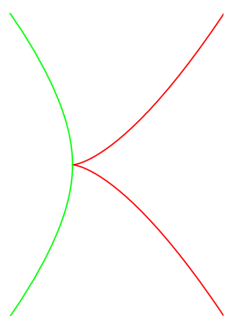

Figure 13. The red curve is the $(2, 3)$-cusp and the green curve is $Ev(c)$.

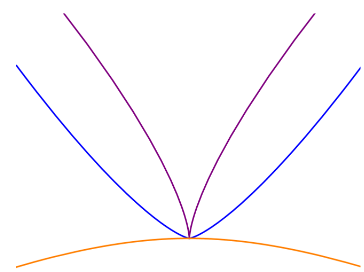

Figure 14. The blue curve is the $(3, 4)$-cusp, the purple curve is $Ev(c)$ and the orange curve is $Ev(Ev(c))$.

It is known that $(2, 3)$-cusps appear on cycloids and asteroids and $(3, 4)$-cusps appear on parallel curves of parabolas (cf.[45]). Moreover, $(2, 3)$-cusp is the generic singularity of fronts. The following simple criteria are known for these singularities ([45, Theorem 1.3.2], [45, Theorem 1.3.4]).

Fact 3.1.1. *Let $c : I \subset \mathbb{R} \to \mathbb{R}^2$ be a C^∞ curve.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(2,3)}(t) = (t^2, t^3)$ at $t = 0$ if and only if $c'(0) = \mathbf{0}$ and* $\det(c''(0), c'''(0)) \neq 0$.

(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(3,4)}(t) = (t^3, t^4)$ at $t = 0$ if and only if $c'(0) = c''(0) = \mathbf{0}$ and* $\det(c'''(0), c^{(4)}(0)) \neq 0$.

For the definition of $\mathcal{A}$-equivalence, see Definition 3.2.1. By using above criteria and properties of the evolutes of fronts, we easily see the following.

Corollary 3.1.2. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(2,3)}(t) = (t^2, t^3)$ at $t = 0$ if and only if the evolute $Ev(c)$ of c is regular at $t = 0$.*

(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(3,4)}(t) = (t^3, t^4)$ at $t = 0$ if and only if the evolute of the evolute $Ev(Ev(c))$ of c is regular at $t = 0$.*

It seems to be natural to expect that the criteria for $(2,3)$-cusps and $(3,4)$-cusps given in Corollary 3.1.2 can be naturally extended to criteria for $(n, n+1)$-cusps. However, this is not true in the case of $(4,5)$-cusps (for details, see Chapter 4). We are also interested in what these conditions correspond to.

In this Chapter we prove the following results.

Theorem 3.1.3. *([30]) Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c is C^1-equivalent to $Cusp_{(n,n+1)}(t) = (t^n, t^{n+1})$ at $t = 0$ if and only if*

$$c'(0) = c''(0) = \cdots = c^{(n-1)}(0) = 0, \quad \det\left(c^{(n)}(0), c^{(n+1)}(0)\right) \neq 0. \tag{3.1}$$

Theorem 3.1.4. *([30]) If c is $\mathcal{A}$-equivalent to $Cusp_{(n,n+1)}(t) = (t^n, t^{n+1})$ at $t = 0$, then $t = 0$ is a singular point of $Ev(c), \cdots Ev^{n-2}(c)$ and a regular point of $Ev^{n-1}(c)$.*

Corollary 3.1.5. *([30]) Let $\gamma : I \to \mathbb{R}^2$ be a front without inflection points. Assume that $t = 0$ is a singular point of γ, that is, $(d\gamma/dt)(0) = \mathbf{0}$. If $t = 0$ is a singular point of $Ev(\gamma)$, $\cdots$, $Ev^{n-1}(\gamma)$, then γ is not $\mathcal{A}$-equivalent to $Cusp_{(n,n+1)}$ at $t = 0$.*

We note that the condition (3.1) does not depend on the choices of coordinates on the source nor on the target (Proposition 3.3.3). As an application of Proposition 3.3.3, we prove Theorem 3.1.3 and Theorem 3.1.4.

3.2. **Preliminaries.**

3.2.1. *Fronts and singularities.* Let I be an open interval containing the origin of $\mathbb{R}$ and $c : I \to \mathbb{R}^2$ be a C^∞ map. A point p is called a *singular point* of c if $c'(p) = \mathbf{0}$. We say that the curve c is a *front* if there exists a C^∞ map $\nu : I \to \mathbb{R}^2$ such that the pair $(c, \nu) : I \to \mathbb{R}^2 \times S^1$ is a *Legendre immersion*, namely, $c'(t) \cdot \nu(t) = 0$ for each $t \in I$ and (c, ν) is an immersion. For properties of fronts, see [8,9,45,46].

Definition 3.2.1. *Let $c_1, c_2 : (\mathbb{R}, 0) \to (\mathbb{R}^2, \mathbf{0})$ be C^∞ map germs. We say that c_1 and c_2 are C^r-equivalent if there exist C^r diffeomorphism germs $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $\Psi : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ such that $\Psi \circ c_1 \circ \psi^{-1} = c_2$ holds, where $r \in \mathbb{N}_{>0} \cup \{\infty\}$. In particular, if $r = \infty$, we call it $\mathcal{A}$-equivalent.*

3.2.2. **Evolutes of fronts.** Let $c : I \to \mathbb{R}^2$ be a regular curve. We set $\mathbf{e}(t) = \dot{c}(t)/\|\dot{c}(t)\|$ and $\mathbf{n}(t) = M(\mathbf{e}(t))$, where $\dot{c}(t) = (dc/dt)(t)$, $\|\dot{c}(t)\| = \sqrt{\dot{c}(t) \cdot \dot{c}(t)}$ and M is the anticlockwise rotation by $\pi/2$. Moreover, $\kappa(t) = \det(\dot{c}(t), \ddot{c}(t))/\|\dot{c}(t)\|^3 = \dot{\mathbf{e}}(t) \cdot \mathbf{n}(t)/\|\dot{c}(t)\|$ is called the *curvature* of c. An *evolute* $Ev(c) : I \to \mathbb{R}^2$ of c is given by

$$Ev(c)(t) = c(t) + \frac{1}{\kappa(t)}\mathbf{n}(t) \tag{3.2}$$

except for the points where $\kappa(t) = 0$ (cf.[8,49]). A point t is called a *inflection point* if $\kappa(t) = 0$. In general, if c has a singular point, then we cannot define the evolute as above, because the curvature $\kappa(t)$ is not defined at a singular point. However, Fukunaga and Takahashi defined an evolute of a front and gave the representation formula of it ([8, Definition 2.10, Theorem 3.3]). Let us see this in the following. Let $(c, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. We call the curve defined by $c_\lambda(t) = c(t) + \lambda\nu(t)$ a *parallel curve* of c, where λ is a real number. Moreover, we denote the curvature of the parallel curve $c_\lambda(t)$ by $\kappa_\lambda(t)$ when c_λ is a regular curve.

Definition 3.2.2. *Let $c : I \to \mathbb{R}^2$ be a front without inflection points, that is, $\kappa(t) \neq 0$. Define an evolute $Ev(c) : I \to \mathbb{R}^2$ of the front γ by*

$$Ev(c)(t) = \begin{cases} c(t) + \dfrac{1}{\kappa(t)}\mathbf{n}(t) & \text{if } t \text{ is a regular point of } c, \\ c_\lambda(t) + \dfrac{1}{\kappa_\lambda(t)}\mathbf{n}_\lambda(t) & \text{if } t \in (t_0 - \delta, t_0 + \delta),\ t = t_0 \text{ is a singular point of } c\ , \end{cases}$$

where δ is sufficiently small positive number, $\mathbf{n}_\lambda(t) = \dfrac{1 - \lambda\kappa(t)}{|1 - \lambda\kappa(t)|}\mathbf{n}(t)$ and $\lambda \in \mathbb{R}$ is satisfied the condition $\lambda \neq 1/\kappa(t)$.

Moreover, they proved that the following fact ([8, Proposition 5.2 (1) and (3)]).

Fact 3.2.3. *Let $c : I \to \mathbb{R}^2$ be a front without inflection points. Assume that t_0 is a singular point of c. Then $(d^i c/dt^i)(t_0) = \mathbf{0}$ for $i = 2, \cdots, n+1$ if and only if t_0 is a singular point of $Ev^i(c)$ for $i = 1, \cdots, n$, where $Ev^i(c)$ is the i-th evolute of c, that is, $Ev^i(c)(t) = Ev(Ev^{i-1}(c))(t)$.*

By Fact 3.2.3, we immediately have the following result.

Corollary 3.2.4. *Let $c : I \to \mathbb{R}^2$ be a front without inflection points. Assume that t_0 is a singular point of c. Then $(d^2c/dt^2)(t_0) = \mathbf{0}$, $\cdots$, $(d^{(n-1)}c/dt^{(n-1)})(t_0) = \mathbf{0}$ and $(d^n c/dt^n)(t_0) \neq \mathbf{0}$ if and only if t_0 is a singular point of $Ev(c), \cdots Ev^{n-2}(c)$ and a regular point of $Ev^{n-1}(c)$.*

As mentioned in Section 3.1, we give properties of $(n, n+1)$ cusps by using Fact 3.2.3 and Corollary 3.2.4 in Section 3.3.

In order to give another representation of the evolute of a front, we recall the Frenet formula of a front. Let $(c, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. We set $\mu(t) = M(\nu(t))$ and $\ell(t) = \nu'(t) \cdot \mu(t)$. The pair $\{\nu(t), \mu(t)\}$ is called a *moving frame along of c*. Then the Frenet formula of c is given by

$$\begin{pmatrix} \nu'(t) \\ \mu'(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}.$$

We set $\beta(t) = c'(t) \cdot \mu(t)$ and we call the pair (ℓ, β) *the curvature of the Legendre immersion.* Note that t_0 is an inflection point of c if and only if $\ell(t_0) = 0$. By using the moving frame of a front, the evolute of a front can be written as follows ([8, Theorem 3.3]).

Fact 3.2.5. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points.*

(1) *The evolute of a front $Ev(\gamma)$ can be expressed by*

$$Ev(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).$$

Moreover, $(Ev(\gamma), \mu)$ is a Legendre immersion with the curvature

$$\left(\ell(t), \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right).$$

(2) *The evolute $Ev(Ev(\gamma))$ of an evolute of a front γ can be expressed by*

$$Ev(Ev(\gamma))(t) = Ev(\gamma)(t) - \frac{\beta'(t)\ell(t) - \beta(t)\ell'(t)}{\ell^3(t)}\mu(t).$$

Moreover, $(Ev(Ev(\gamma)), -\nu)$ is a Legendre immersion with the curvature

$$\left(\ell(t), \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right)\right).$$

Finally, we recall representations of the n-th evolutes of fronts. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. We denote $Ev^0(t) = \gamma(t)$ and $Ev^1(t) = Ev(\gamma)(t)$. We define the followings inductively.

$$Ev^n(t) = Ev(Ev^{n-1}(\gamma))(t), \quad \beta_0(t) = \beta(t), \quad \textit{and} \quad \beta_n(t) = \frac{d}{dt}\Big(\frac{\beta_{n-1}(t)}{\ell(t)}\Big).$$

Fukunaga and Takahashi gave the representations of the n-th evolutes of fronts.

Fact 3.2.6. *([8, Theorem 5.1]) Let $\gamma : I \to \mathbb{R}$ be a front. Then the n-th evolute of γ is given by*

$$Ev^n(\gamma)(t) = Ev^{n-1}(\gamma)(t) - \frac{\beta_{n-1}(t)}{\ell(t)}M^{n-1}(\nu(t)), \tag{3.4}$$

where M^n is n-times operations of M. Moreover, $(Ev^n(\gamma), M^n(\nu))$ is a Legendre immersion with the curvature $(\ell(t), \beta_n(t))$.

3.3. **Classifications of $(n, n+1)$-cusps in the sense of C^1-equivalence and evolutes of fronts.** In this section, we prove the following theorems.

Theorem 3.3.1. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c is C^1-equivalent to $Cusp_{(n,n+1)}(t) = (t^n, t^{n+1})$ at $t = 0$ if and only if*

$$c'(0) = c''(0) = \cdots = c^{(n-1)}(0) = 0, \tag{3.5}$$

$$\det\,(c^{(n)}(0), c^{(n+1)}(0)) \neq 0. \tag{3.6}$$

Theorem 3.3.2. *If c is $\mathcal{A}$-equivalent to $Cusp_{n,n+1}(t) = (t^n, t^{n+1})$ at $t = 0$, then $t = 0$ is a singular point of $Ev(c), \cdots Ev^{n-2}(c)$ and a regular point of $Ev^{n-1}(c)$.*

For the proofs of these theorems, we divide them into several steps.

3.3.1. *Invariants of $(n, n+1)$-cusps.*

Proposition 3.3.3. *The conditions (3.5) and (3.6) do not depend on the choices of coordinates on the source nor on the target.*

For the proof, it is sufficient to prove the following two lemmas.

Lemma 3.3.4. *The condition (3.5) does not depend on the choices of C^∞ local diffeomorphisms $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ nor $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.*

Lemma 3.3.5. *The condition (3.6) does not depend on the choices of C^∞ local diffeomorphisms $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ nor $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.*

We will use the following fact to prove Lemma 3.3.4 and Lemma 3.3.5.

Fact 3.3.6. *(Faà di Bruno's formula). Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^∞ functions and $m \in \mathbb{Z}_{>0}$. Then the following holds.*

$$(3.7) \qquad (f \circ g)^{(m)} = \sum_{k=1}^{m} \sum_{\substack{i_1+\cdots+i_k=m \\ i_1,\cdots,i_k \neq 0}} \frac{m!}{k!} (f^{(k)} \circ g) \prod_{l=1}^{k} \frac{g^{(i_l)}}{i_l!}.$$

Proof of Lemma 3.3.4. First, we show that the condition (3.5) does not depend on the choice of a C^∞ local diffeomorphism Φ. We put $\Phi = (\Phi_1, \Phi_2)$, $\gamma(t) = (x(t), y(t))$ Setting $\hat{c}(t) := \Phi \circ c(t) = \Phi(x(t), y(t))$, we have $\hat{c}'(t) = J(t)c'(t)$, where $J(t) := J(x(t), y(t)) = \begin{pmatrix} (\Phi_1)_x(x(t), y(t)) & (\Phi_1)_y(x(t), y(t)) \\ (\Phi_2)_x(x(t), y(t)) & (\Phi_2)_y(x(t), y(t)) \end{pmatrix}$. Thus we see that

$$(3.8) \qquad \hat{c}^{(n)}(t) = \sum_{k=0}^{n-1} {}_{n-1}C_k J(t)^{(n-k-1)} c^{(k+1)}(t).$$

By $c'(0) = c''(0) = \cdots = c^{(n-1)}(0) = \mathbf{0}$, we get

$$\hat{c}'(0) = \hat{c}''(0) = \cdots = \hat{c}^{(n-1)}(0) = \mathbf{0}.$$

Next, we show that the condition (3.5) does not depend on the choice of a C^∞ local diffeomorphism ψ . We set $\tilde{c}(s) := c \circ \psi(s) = (x(\psi(s)), y(\psi(s)))$. Then, by Fact 3.3.6, we see that

$$\tilde{c}^{(n)}(0) = \sum_{k=1}^{n} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\cdots,i_k \neq 0}} \left(\frac{n!}{k!} (x^{(k)}(0)) \prod_{l=1}^{k} \frac{\psi^{(i_l)}(0)}{i_l!}, \frac{n!}{k!} (y^{(k)}(0)) \prod_{l=1}^{k} \frac{\psi^{(i_l)}(0)}{i_l!} \right).$$

By $c'(0) = c''(0) = \cdots = c^{(n-1)}(0) = \mathbf{0}$, we get

$$\tilde{c}'(0) = \hat{c}''(0) = \cdots = \hat{c}^{(n-1)}(0) = \mathbf{0}.$$

Hence we have the assertion. ◻

Proof of Lemma 3.3.5. First, we show that the condition (3.6) does not depend on the choice of a C^∞ local diffeomorphism Φ. By (3.8), we have

$$(3.9) \qquad \hat{c}^{(n+1)}(t) = \sum_{k=0}^{n} {}_{n}C_k J(t)^{(n-k)} c^{(k+1)}(t).$$

By (3.8) and (3.9), we get $\hat{c}^{(n)}(0) = J(0)c^{(n)}(0)$ and $\hat{c}^{(n+1)}(t) = {}_nC_{n-1}J'(0)c^{(n)}(0) + J(0)c^{(n+1)}(0)$. Since $J'(t) = J_x(t)x'(t) + J_y(t)y'(t)$ and $c'(0) = \mathbf{0}$, we have $J'(0) = \mathbf{0}$. Thus we get $\hat{c}^{(n+1)}(t) = J(0)c^{(n+1)}(0)$. Therefore we obtain

$$\det(\hat{c}^{(n)}(0), \hat{c}^{(n+1)}(0)) = \det J(0) \det(c^{(n)}(0), c^{(n+1)}(0)) \neq 0.$$

Next, we show that the condition (3.6) does not depend on the choice of a local diffeomorphism ψ. By Fact 3.3.6, we have

$$\tilde{c}^{(n+1)}(0) = \sum_{k=1}^{n+1} \sum_{\substack{i_1+\cdots+i_k=n+1 \\ i_1,\cdots,i_k \neq 0}} \left(\frac{(n+1)!}{k!} (x^{(k)}(0)) \prod_{l=1}^{k} \frac{\psi^{(i_l)}(0)}{i_l!}, \frac{(n+1)!}{k!} (y^{(k)}(0)) \prod_{l=1}^{k} \frac{\psi^{(i_l)}(0)}{i_l!} \right).$$

Thus we get

$$\tilde{c}^{(n)}(0) = (\psi'(0))^n c^{(n)}(0), \quad \tilde{c}^{(n+1)}(0) = Q c^{(n)}(0) + (\psi'(0))^{n+1} c^{(n+1)}(0), \tag{3.10}$$

where Q is some constant. Hence we obtain

$$\det(\tilde{c}^{(n)}(0), \tilde{c}^{(n+1)}(0)) = (\psi'(0))^{2n+1} \det(c^{(n)}(0), c^{(n+1)}(0)) \neq 0$$

and have the assertion. □

Proof of Theorem 3.3.2. By using Proposition 3.3.3, Fact 3.2.3 and Corollary 3.2.4, we have the assertion. □

As an application of Proposition 3.3.3, we can get the following result with respect to $(n, n+1)$-cusps.

Corollary 3.3.7. *Let $c : I \to \mathbb{R}^2$ be a front without inflection points. Assume that $t = 0$ is a singular point of c, that is, $(dc/dt)(0) = \mathbf{0}$. If $t = 0$ is a singular point of $Ev(c), \cdots, Ev^{n-1}(c)$, then γ is not $\mathcal{A}$-equivalent to $Cusp_{(n,n+1)}$ at $t = 0$.*

Proof. Suppose that c is $\mathcal{A}$-equivalent to $Cusp_{(n,n+1)}$ at $t = 0$. By Proposition 3.3.3, c satisfies the conditions (3.5) and (3.6). On the other hand, since $t = 0$ is a singular point of $Ev(c)$, $\cdots, Ev^{n-1}(c)$, we see that the following by Fact 3.2.3.

$$\frac{d^2 c}{dt^2}(0) = \cdots = \frac{d^n c}{dt^n}(0) = \mathbf{0}.$$

This is a contradiction. Therefore we have the assertion. □

3.3.2. *C^1 diffeomorphism forms of $(n, n+1)$ cusps.* The purpose of this step is to prove the following proposition.

Proposition 3.3.8. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c satisfies (3.5) and (3.6) if and only if γ is $\mathcal{A}$-equivalent to $C_{(n,n+1)}(s) = (s^n, s^{n+1} Y_0(s))$ at $s = 0$, where Y_0 is a C^∞ function and $Y_0(0) \neq 0$.*

To prove this proposition, we need the following result ([45, Proposition A.1]).

Fact 3.3.9. (*The division lemma*) *Let $f : U \to \mathbb{R}$ be a C^∞ function defined on a convex open neighborhood U of the origin of $\mathbb{R}^2$. If there exists a non-negative integer k such that*

$$f(u, 0) = \frac{\partial f}{\partial v}(u, 0) = \frac{\partial^2 f}{\partial v^2}(u, 0) = \cdots = \frac{\partial^k f}{\partial v^k}(u, 0) = 0,$$

then there exists a C^∞ function $g : U \to \mathbb{R}$ such that

$$f(u, v) = v^{k+1} g(u, v), \quad (u, v) \in U.$$

Proof of Proposition 3.3.8. If c is $\mathcal{A}$-equivalent to $C_{(n,n+1)}(s) = (s^n, s^{n+1} Y_0(s))$ at $s = 0$, then clearly $C_{(n,n+1)}$ satisfies (3.5) and (3.6). By Proposition 3.3.3, c also satisfies (3.5) and (3.6).

Assume that c satisfies (3.5) and (3.6). We put $c(t) = (x(t), y(t))$. By the division lemma, there exist C^∞ functions x_1 and y_1 such that $x(t) = t^n x_1(t)$, $y(t) = t^n y_1(t)$. Since $c^{(n)}(0) \neq 0$, we may assume that $x_1(0) \neq 0$. In the following, we assume that $x_1(t) > 0$ in an open interval around $t = 0$. Define a function $s : \mathbb{R} \to \mathbb{R}$ by $s(t) := t\sqrt[n]{x_1(t)}$, then s is a C^∞ local diffeomorphism on an open interval around $t = 0$ (If $x_1(t) < 0$, we define $s(t) := t\sqrt[n]{-x_1(t)}$). Thus c can be written as $\tilde{c}(s) = (s^n, s^n Y_1(s))$ using s, where $Y_1(s) := \dfrac{y_1(t(s))}{x_1(t(s))}$. Define a C^∞ function $\tilde{Y}(s) = Y_1(s) - Y_1(0)$. By using the division lemma, there exists a C^∞ function Y_0 such that $\tilde{Y}(s) = sY_0(s)$ and hence we get $Y_1(s) = Y_1(0) + sY_0(s)$. Since $\dfrac{dt}{ds} \neq 0$ at $s = 0$, $\det\,(c^{(n)}(0), c^{(n+1)}(0)) \neq 0$ and

$$Y_1'(s) = \frac{y_1'(t(s))x_1(t(s)) - x_1'(t(s))y_1(t(s))}{x_1(t(s))^2}\frac{dt(s)}{ds},$$

we have $Y_1'(0) = Y_0(0) \neq 0$. Therefore we obtain

$$\tilde{c}(s) = (s^n, s^n(Y_1(0) + sY_0(s)))$$

Define a C^∞ diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(u, v) = (u, v - uY_1(0)).$$

Then $f \circ \tilde{c}(s) = (s^n, s^{n+1}Y_0(s))$. This concludes the proof. □

3.4. **Proof of Theorem 3.3.1.** If $c = Cusp_{(n,n+1)}$, then it is clear that the conditions (3.5) and (3.6) hold. Hence, by Proposition 3.3.3, if c is C^1-equivalent to $Cusp_{(n,n+1)}$ at $t = 0$, then the conditions (3.5) and (3.6) hold.

Conversely, assume that c satisfies(3.5) and (3.6). By Proposition 3.3.8, c is $\mathcal{A}$-equivalent to $C_{(n,n+1)}(s) = (s^n, s^{n+1}Y_0(s))$ at $s = 0$, where Y_0 is a C^∞ function and $Y_0(0) \neq 0$. Define a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) := (x, z)$, where z is the uniquely-determined C^1 function of y satisfying $y = zY_0(z^{\frac{1}{n+1}})$. Then we see that

$$\frac{dy}{dz} = Y_0(z^{\frac{1}{n+1}}) + z\frac{dY_0}{ds}(z^{\frac{1}{n+1}})\frac{1}{n+1}z^{\frac{1}{n+1}-1} = Y_0(z^{\frac{1}{n+1}}) + \frac{1}{n+1}z^{\frac{1}{n+1}}\frac{dY_0}{ds}(z^{\frac{1}{n+1}}).$$

By $\dfrac{dy(0)}{dz} = Y_0(0) \neq 0$, y is a C^1 local diffeomorphism on an open interval of $z = 0$. Thus φ is a C^1 local diffeomorphism on an open neighborhood of the origin of $\mathbb{R}^2$. In particular, note that the only z corresponding to $y = s^{n+1}Y_0(s)$ is $z = s^{n+1}$. Therefore we obtain $\varphi(s^n, s^{n+1}Y_0(s)) = (s^n, s^{n+1})$ and have the assertion. □

3.5. **Examples of evolutes of** $(4, 5)$**-cusp.** Let $c(t) = (t^4, t^5)$ be the $(4, 5)$-cusp. Since $\nu(t) = \dfrac{1}{\sqrt{16 + 25t^2}}(-5t, 4)$ and $\mu(t) = \dfrac{-1}{\sqrt{16 + 25t^2}}(4, 5t)$, we have

$$l(t) = \frac{20}{16 + 25t^2} \quad \beta(t) = -t^3\sqrt{16 + 25t^2}. \tag{3.11}$$

By using (3.4) and (3.11), we get the representations of the evolute, the second evolute and the third evolute of γ as follows.

$$
\begin{aligned}
Ev(c)(t) &= \left(-3t^4 - \frac{25}{4}t^6, \frac{16}{5}t^3 + 6t^5\right), \\
Ev(Ev(c))(t) &= \left(-\frac{192}{25}t^2 - 39t^4 - \frac{175}{4}t^6, -\frac{32}{5}t^3 - 39t^5 - \frac{375}{8}t^7\right), \\
Ev(Ev(Ev(c)))(t) &= \left(\frac{192}{25}t^2 + 141t^4 + \frac{925}{2}t^6 + \frac{13125}{32}t^8, -\frac{1536}{125}t - \frac{752}{5}t^3 - 444t^5 - 375t^7\right).
\end{aligned}
$$

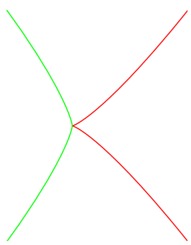

FIGURE 15.
The red curve is the $(4, 5)$ cusp and the green curve is $Ev(c)$.

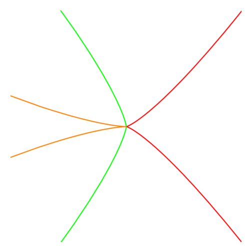

FIGURE 16.
The orange curve is $Ev(Ev(c))$.

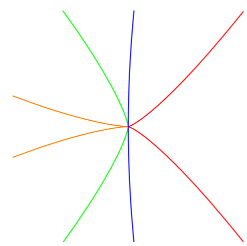

FIGURE 17.
The blue curve is $Ev(Ev(Ev(c)))$.

4. Criterion for $(4,5)$-cusps and the $(4,5;\pm 7)$-cuspidal curvature

4.1. **Introduction.** One of the most basic singularities that appear in plane curves are cusps. A plane curve c is (n,m)-cusp at $t=0$ if c is $\mathcal{A}$-equivalent to $Cusp_{(n,m)}(t) = (t^n, t^m)$ at $t=0$. Classifications with respect to cusps were known in the sense of complex-analytically equivalence by Bruce and Gaffney ([4, Theorem 3.8]). Namely,

Fact 4.1.1. *The following are representatives of the complex C^{ω}-equivalent simple germs $(\mathbb{C},0) \to (\mathbb{C}^2,\mathbf{0})$:*

(1) A_{2k} $: t \to (t^2, t^{2k+1})$,

(2) E_{6k} $: t \to (t^3, t^{3k+1} + t^{3k+p+2}); 0 \le p \le k-2, \quad t \to (t^3, t^{3k+1})$,

(3) E_{6k+2} $: t \to (t^3, t^{3k+2} + t^{3k+p+4}); 0 \le p \le k-2, \quad t \to (t^3, t^{3k+2})$,

(4) W_{12} $: t \to (t^4, t^5 + t^7), \quad t \to (t^4, t^5)$,

W_{18} $: t \to (t^4, t^7 + t^9), \quad t \to (t^4, t^7 + t^{13}), \quad t \to (t^4, t^7)$,

(5) $W^{\#}_{1,2q-1}$ $: t \to (t^4, t^6 + t^{2q+5}); q \ge 1$.

For the definition of *simple*, see [4, Definition 2.6]. In general, it is difficult to examine types of (n,m)-cusps in the sense of $\mathcal{A}$-equivalence (or analytically equivalence). However, simple criteria for $(2,3)$-cusps, $(2,5)$-cusps, $(2,7)$-cusps, $(3,4)$-cusps and $(3,5)$-cusps are known ([45, Theorem 1.3.2], [37, Theorem 1.23], [13, Theorem A.1], [45, Theorem 1.3.4], [27, Fact 2.1] and [15]). Namely,

Fact 4.1.2. *Let $c : I \subset \mathbb{R} \to \mathbb{R}^2$ be a C^∞ curve.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(2,3)}(t) = (t^2, t^3)$ at $t=0$ if and only if $c'(0) = \mathbf{0}$ and* $\det(c''(0), c'''(0)) \neq 0$.

(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(2,5)}(t) = (t^2, t^5)$ at $t=0$ if and only if $c'(0) = \mathbf{0}$,* $\det(c''(0), c'''(0)) = 0$ *and*

$$3\det(c''(0), c^{(5)}(0))c''(0) - 10\det(c''(0), c^{(4)}(0))c'''(0) \neq \mathbf{0}.$$

(3) *c is $\mathcal{A}$-equivalent to $Cusp_{(2,7)}(t) = (t^2, t^7)$ at $t=0$ if and only if there exist real numbers $k, l \in \mathbb{R}$ such that $c'(0) = \mathbf{0}$, $c''(0) \neq \mathbf{0}$, $c'''(0) = kc''(0)$, $c^{(5)}(0) - \dfrac{10}{3}kc^{(4)}(0) = \ell c''(0)$ and*

$$\det\left(c''(0), c^{(7)}(0) - 7kc^{(6)}(0) - \left(7\ell - \frac{70}{3}k^3\right)c^{(4)}(0)\right) \neq 0.$$

(4) *c is $\mathcal{A}$-equivalent to $Cusp_{(3,4)}(t) = (t^3, t^4)$ at $t=0$ if and only if $c'(0) = c''(0) = \mathbf{0}$ and* $\det(c''(0), c^{(4)}(0)) \neq 0$.

(5) *c is $\mathcal{A}$-equivalent to $Cusp_{(3,5)}(t) = (t^3, t^5)$ at $t=0$ if and only if $c'(0) = c''(0) = \mathbf{0}$,* $\det(c'''(0), c^{(4)}(0)) = 0$ *and* $\det(c'''(0), c^{(5)}(0)) \neq 0$.

(6) *If c satisfies the conditions $c'(0) = \mathbf{0}, c''(0) \neq \mathbf{0}, c'''(0) = \cdots = c^{(n-1)}(0) = \mathbf{0}$ and*

$$\det(c''(0), c^{(n)}(0)) \neq 0,$$

then c is $\mathcal{A}$-equivalent to $Cusp_{(2,n)}(t) = (t^2, t^n)$ at $t=0$, where n is an odd number greater than or equal to 9.

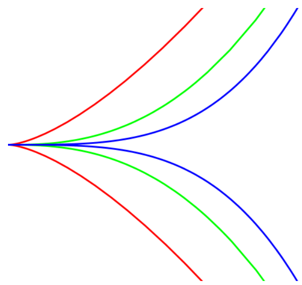

FIGURE 18.
The red curve is $Cusp_{2,3}(t)$, the green curve is $Cusp_{(2,5)}(t)$ and the blue curve is $Cusp_{(2,7)}(t)$.

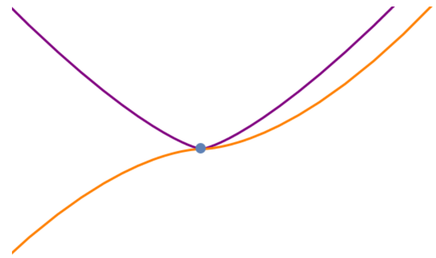

FIGURE 19.
The purple curve is $Cusp_{(3,4)}(t)$ and the orange curve is $Cusp_{(3,5)}(t)$.

On the other hand, we gave a criterion for $(n, n+1)$-cusps in the sense of C^1-equivalence (Theorem 3.1.3) that looks like a natural generalization of Fact 4.1.2 (1) and (4). However, in the sense of $\mathcal{A}$-equivalence, such a simple generalization does not hold for $(4,5)$-cusps. In fact, Theorem 4.1.3 below shows that the curves $(t^4, t^5 + t^7)$, $(t^4, t^5 - t^7)$ and (t^4, t^5) satisfy the conditions $c'(0) = c''(0) = c^{(3)}(0) = 0$ and $\det(c^{(4)}(0), c^{(5)}(0)) \neq 0$, but they are not $\mathcal{A}$-equivalent each other at $t = 0$. This implies that any criterion for $(4,5)$-cusps needs to include imformation on the term with order higher than 5, and therefore it is more complicated than the cusps which appeared in Fact 4.1.2.

In this chapter, we prove the following theorem.

Theorem 4.1.3. *([30]) Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Let A and κ_q be constants that will be defined in Definition 4.2.4.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5)}(t) = (t^4, t^5)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q = 0$.*
(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;+7)}(t) = (t^4, t^5 + t^7)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q > 0$.*
(3) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;-7)}(t) = (t^4, t^5 - t^7)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q < 0$.*
(4) *$Cusp_{(4,5)}$, $Cusp_{(4,5;+7)}$ and $Cusp_{(4,5;-7)}$ are not $\mathcal{A}$-equivalent to at $t = 0$. However, they are C^1-equivalent each other to at $t = 0$.*

To prove this theorem, we newly introduce an invariant κ_q of $(4,5)$-cusp other than the $(4,5)$-cuspidal curvature (Definition 4.2.4). For the $(n, n+1)$-*cuspidal curvature*, see ([28, Appendix A]). This new invariant κ_q is called the $(4, 5; \pm 7)$-*cuspidal curvature*. We give complete classification with respect to $(4,5)$-cusps by the sign of κ_q (Theorem 4.5.2).

4.2. $(4, 5; \pm 7)$**-cuspidal curvature.** We give a necessary and sufficient condition for a plane curve to be $\mathcal{A}$-*equivalent* to $Cusp_{(4,5)}$ at the origin. Namely, we prove the following.

Theorem 4.2.1. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c is $\mathcal{A}$-equivalent to $Cusp_{(4,5)}(t) = (t^4, t^5)$ at $t = 0$ if and only if $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and*

$$-77B^2 + 105AD + 60AC = 0, \tag{4.1}$$

where $A = \det(c^{(5)}(0), c^{(4)}(0))$, $B = \det(c^{(6)}(0), c^{(4)}(0))$, $C = \det(c^{(7)}(0), c^{(4)}(0))$ and $D = \det(c^{(6)}(0), c^{(5)}(0))$.

Before we prove this theorem, we check the following.

Proposition 4.2.2. *Under the conditions $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, the condition (4.1) does not depend on the choices of coordinates on the source nor on the target.*

Proof. First, we show that the condition (4.1) does not depend on the choice of a C^∞ local diffeomorphism $\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. We put $\Phi = (\Phi_1, \Phi_2)$ and $\gamma(t) = (x(t), y(t))$. Moreover, we set $\hat{c}(t) = \Phi \circ c(t) = \Phi(x(t), y(t))$. By (3.8), we have the followings.

$$\hat{c}^{(4)}(t) = \sum_{k=0}^{3} {}_3C_k J(t)^{(3-k)} c^{(k+1)}(t), \quad \hat{c}^{(5)}(t) = \sum_{k=0}^{4} {}_4C_k J(t)^{(4-k)} c^{(k+1)}(t),$$
$$\hat{c}^{(6)}(t) = \sum_{k=0}^{5} {}_5C_k J(t)^{(5-k)} c^{(k+1)}(t), \quad \hat{c}^{(7)}(t) = \sum_{k=0}^{6} {}_6C_k J(t)^{(6-k)} c^{(k+1)}(t).$$

Note that we see that $J'(0) = J''(0) = J'''(0) = \mathbf{0}$ by $c'(0) = c''(0) = c'''(0) = \mathbf{0}$. Thus we have the following.

$$\hat{c}^{(4)}(0) = J(0)\gamma^{(4)}(0), \quad \hat{c}^{(5)}(0) = J(0)c^{(5)}(0),$$
$$\hat{c}^{(6)}(0) = J(0)c^{(6)}(0), \quad \hat{c}^{(7)}(0) = J(0)c^{(7)}(0).$$

Setting $\hat{A} = \det(\hat{c}^{(5)}(0), \hat{c}^{(4)}(0))$, $\hat{B} = \det(\hat{c}^{(6)}(0), \hat{c}^{(4)}(0))$, $\hat{C} = \det(\hat{c}^{(7)}(0), \hat{c}^{(4)}(0))$ and $\hat{D} = \det(\hat{c}^{(6)}(0), \hat{c}^{(5)}(0))$, we have

$$-77\hat{B}^2 + 105\hat{A}\hat{D} + 60\hat{A}\hat{C} = (\det J(0))^2(-77B^2 + 105AD + 60AC) = 0. \tag{4.2}$$

Next, we show that the condition (4.1) does not depend on the choice of a C^∞ local diffeomorphism $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. We set $\tilde{c}(t) = \varsigma(\psi(t))$. By $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $c(t)$ can be written as

$$c(t) = (t^4 z(t), t^4 w(t)), \tag{4.3}$$

where $z(t)$ and $w(t)$ are C^∞ functions. Then we have the followings.

$$A = -2880z(0)w'(0) + 2880w(0)z'(0),$$
$$B = -8640z(0)w''(0) + 8640w(0)z''(0),$$
$$C = -20160z(0)w'''(0) + 20160w(0)z'''(0),$$
$$D = -43200z'(0)w''(0) + 43200w'(0)z''(0).$$

Thus we see that

$$
\begin{aligned}
-77B^2+105AD+60AC &= -5748019200(-z(0)w''(0)+w(0)z''(0))^2 \\
&- 13063680000(z(0)w'(0)-w(0)z'(0))(-z'(0)w''(0)+w'(0)z''(0)) \\
&+ 3483648000(z(0)w'(0)-w(0)z'(0))(z(0)w'''(0)-w(0)z'''(0)).
\end{aligned}
$$

Setting $\tilde{A}=\det(\tilde{c}^{(5)}(0),\tilde{c}^{(4)}(0))$, $\tilde{B}=\det(\tilde{c}^{(6)}(0),\tilde{c}^{(4)}(0))$, $\tilde{C}=\det(\tilde{c}^{(7)}(0),\tilde{c}^{(4)}(0))$ and $\tilde{D}=\det(\tilde{c}^{(6)}(0),\tilde{c}^{(5)}(0))$, we obtain the following by the same calculations.

$$
-77\tilde{B}^2+105\tilde{A}\tilde{D}+60\tilde{A}\tilde{C}=(\psi'(0))^{20}(-77B^2+105AD+60AC)=0. \tag{4.4}
$$

Therefore we have the assertion. □

Remark 4.2.3. *By (4.2) and (4.4), we see that the expression $-77B^2+105AD+60AC$ does not change sign by coordinate transformations on the source nor on the target.*

Definition 4.2.4. *([30]) Under the conditions $c'(0)=c''(0)=c'''(0)=\mathbf{0}$ and $c^{(4)}(0)\neq\mathbf{0}$, we define the $(4,5;\pm7)$-cuspidal curvature as follow:*

$$
\kappa_q=\frac{-77B^2+105AD+60AC}{\|c^{(4)}(0)\|^5}, \tag{4.5}
$$

where $A=\det(c^{(5)}(0),c^{(4)}(0))$, $B=\det(c^{(6)}(0),c^{(4)}(0))$, $C=\det(c^{(7)}(0),c^{(4)}(0))$ and $D=\det(c^{(6)}(0),c^{(5)}(0))$.

Proposition 4.2.5. *The number κ_q does not depend on the choices of parameter transformations on the source nor of congruent transformations on the target.*

Proof. By (3.10) and (4.4), we see that the number κ_q is an invariant with respect to parameter transformations. It is clear that the number κ_q is independent of congruent transformations by (4.2). □

The $(4,5;\pm7)$-cuspidal curvature may depend on a coordinate transformation on the target. However, this never changes the sign by coordinate transformations.

4.3. $(4,5)$**-cusps and Whitney's lemma.** The purpose of this step is to prove the following proposition.

Proposition 4.3.1. *$\alpha(s)=(s^4,s^5)+M(s)$ is $\mathcal{A}$-equivalent to $Cusp_{(4,5)}(s)=(s^4,s^5)$ at $s=0$, where $M(s)$ is a $\mathbb{R}^2$-valued C^∞ function satisfying $M(0)=M'(0)\cdots=M^{(7)}(0)=\mathbf{0}$ and $M^{(11)}(0)=\mathbf{0}$.*

To prove this proposition, we need the following result ([45, Theorem 3.1.7], [50]).

Fact 4.3.2. *(Whitney's lemma) Let $f:U\subset\mathbb{R}^2\to\mathbb{R}$ be a C^∞ function defined on an open neighborhood of the origin of $\mathbb{R}^2$. If f satisfies $f(u,v)=f(u,-v)$, then there exists a C^∞ function g defined on an open neighborhood of the origin of $\mathbb{R}^2$ such that $f(u,v)=g(u,v^2)$ holds near the origin.*

Lemma 4.3.3. *Let k be an integer greater than or equal to 1. Then for any C^∞ function $f:\mathbb{R}\to\mathbb{R}$, there exist C^∞ functions $g_l:I\subset\mathbb{R}\to\mathbb{R}$ $(l=1,2,\cdots,2^k)$ such that*

$$
f(t)=\sum_{l=1}^{2^k}t^{l-1}g_l(t^{2^k}), \tag{4.6}
$$

where I is an open interval around $t = 0$.

Proof. We use the mathematical induction and Whitney's lemma. In fact, we difine functions

$$u(t) := \frac{f(t) + f(-t)}{2}, \quad v(t) := \frac{f(t) - f(-t)}{2}.$$

Since $u(t)$ is an even function, there exists a C^∞ function g_1 such that $u(t) = g_1(t^2)$ by Whitney's lemma. Moreover, since $v(t)$ is an odd function, there exists a C^∞ function g_2 such that $u(t) = tg_2(t^2)$ by the division lemma and Whitney's lemma. Note that $f(t) = u(t) + v(t)$. Therefore we see that

$$f(t) = g_1(t^2) + tg_2(t^2)$$

holds on an open interval containing $t = 0$. By the same argument as above for g_1 and g_2, we have the assertion. □

Proof of Proposition 4.3.1. By Lemma 4.3.3, there exist $\mathbb{R}^2$-valued C^∞ functions $M_1(s)$, $M_2(s)$, $M_3(s)$ and $M_4(s)$ defined on an open interval containing $s = 0$ such that

$$M(s) = M_1(s^4) + sM_2(s^4) + s^2M_3(s^4) + s^3M_4(s^4). \tag{4.7}$$

Since $M(s)$ satisfies $M(0) = M'(0) = \cdots = M^{(7)}(0) = \mathbf{0}$ and $M^{(11)}(0) = \mathbf{0}$, there exist $\mathbb{R}^2$-valued C^∞ functions $\tilde{M}_1(s)$, $\tilde{M}_2(s)$, $\tilde{M}_3(s)$ and $\tilde{M}_4(s)$ defined on an open interval containing $s = 0$ such that

$$M_i(s) = s^2\tilde{M}_i(s), M_4(s) = s^3\tilde{M}_4(s), \quad (i = 1, 2, 3).$$

Substituting these results into (4.7), we get the following.

$$M(s) = s^8\tilde{M}_1(s^4) + s^9\tilde{M}_2(s^4) + s^{10}\tilde{M}_3(s^4) + s^{15}\tilde{M}_4(s^4).$$

Define a C^∞ map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi(X, Y) = X^2\tilde{M}_1(X) + XY\tilde{M}_2(X) + Y^2\tilde{M}_3(X) + Y^3\tilde{M}_4(X).$$

Then we see that $\psi(s^4, s^5) = M(s)$. Define a C^∞ diffeomorphism $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(X, Y) = (X, Y) + \psi(X, Y).$$

Therefore we get

$$\Psi(s^4, s^5) = (s^4, s^5) + M(s) = \alpha(s). \tag{4.8}$$

This concludes the proof. □

4.4. **Eliminations of the 7-th order and the 11-th order terms.** The essential problems of a criterion for (4, 5)-cusps are the treatments of the 7-th order and 11-th order terms. It is due to the fact that $7, 11 \notin \{4p + 5q \,;\, p, q \in \mathbb{Z}_{>0}\}$. In this step, we discuss the method of them.

Lemma 4.4.1. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c satisfies (3.5) and (3.6) if and only if c is $\mathcal{A}$-equivalent to $c_0(t) = (t^n, t^{n+1}) + h_{n+3}(t)$ at $t = 0$, where h_{n+3} is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_{n+3}(0) = h'_{n+3}(0) \cdots = h^{(n+2)}_{n+3}(0) = \mathbf{0}$.*

Proof. If c is $\mathcal{A}$-equivalent to $c_0(t) = (t^n, t^{n+1}) + h_{n+3}(t)$ at $t = 0$, then c satisfies (3.5) and (3.6) by Proposition 3.3.3.

We assume that c satisfies (3.5) and (3.6). By $c'(0) = c''(0) = \cdots = c^{(n-1)}(0) = \mathbf{0}$, c can be written as

$$c(t) = (a_n t^n + a_{n+1}t^{n+1} + a_{n+2}t^{n+2}, b_n t^n + b_{n+1}t^{n+1} + b_{n+2}t^{n+2}) + H_{n+3}(t),$$

where H_{n+3} is a $\mathbb{R}^2$-valued C^∞ function satisfying $H_{n+3}(0) = H'_{n+3}(0) \cdots = H^{(n+2)}_{n+3}(0) = \mathbf{0}$. Since $\det(c^{(n)}(0), c^{(n+1)}(0)) \neq 0$, there exists a linear coordinate transformation of $\mathbb{R}^2$ such that c is $\mathcal{A}$-equivalent to

$$c_L(t) = (t^n + \hat{a}_{n+2}t^{n+2}, t^{n+1} + \hat{b}_{n+2}t^{n+2}) + \hat{H}_{n+3}(t), \tag{4.9}$$

where $\hat{H}_{n+3}$ is a $\mathbb{R}^2$-valued C^∞ function satisfying $\hat{H}_{n+3}(0) = \hat{H}'_{n+3}(0) \cdots = \hat{H}^{(n+2)}_{n+3}(0) = \mathbf{0}$. Define a parameter transformation $\tau : \mathbb{R} \to \mathbb{R}$ by

$$\tau(\tilde{t}) = \tilde{t} + c_1\tilde{t}^2 + c_2\tilde{t}^3, \tag{4.10}$$

where $c_1 := -\dfrac{\hat{b}_{n+2}}{n+1}$, $c_2 := -\dfrac{1}{n}\left(\hat{a}_{n+2} + \dfrac{n(n-1)}{2(n+1)^2}\hat{b}^2_{n+2}\right)$. By (4.9) and (4.10), we obtain

$$c_L \circ \tau(\tilde{t}) = \left(\tilde{t}^n - \frac{n}{n+1}\hat{b}_{n+2}\tilde{t}^{n+1}, \tilde{t}^{n+1}\right) + \tilde{H}_{n+3}(\tilde{t}),$$

where $\tilde{H}_{n+3}$ is a $\mathbb{R}^2$-valued C^∞ function satisfying $\tilde{H}_{n+3}(0) = \tilde{H}'_{n+3}(0) \cdots = \tilde{H}^{(n+2)}_{n+3}(0) = \mathbf{0}$. Define a C^∞ diffeomorphism $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\tilde{f}(x, y) = \left(x + \frac{n}{n+1}\hat{b}_{n+2}y, y\right).$$

Composing $c_L \circ \tau$ and $\tilde{f}$, we have

$$\tilde{f} \circ c_L \circ \tau(\tilde{t}) = (\tilde{t}^n, \tilde{t}^{n+1}) + h_{n+3}(\tilde{t}), \tag{4.11}$$

where h_{n+3} is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_{n+3}(0) = h'_{n+3}(0) \cdots = h^{(n+2)}_{n+3}(0) = \mathbf{0}$. Therefore we have the assertion. $\square$

Corollary 4.4.2. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c satisfies (3.5) and (3.6) if and only if there exists a real number T such that c is $\mathcal{A}$-equivalent to $\tilde{c}(s) = (s^n, s^{n+1} + Ts^{n+3}) + h_{n+4}(s)$ at $s = 0$, where h_{n+4} is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_{n+4}(0) = h'_{n+4}(0) \cdots = h^{(n+3)}_{n+4}(0) = \mathbf{0}$.*

Proof. If c is $\mathcal{A}$-equivalent to $\tilde{c}(s) = (s^n, s^{n+1} + Ts^{n+3}) + h_{n+4}(s)$ at $s = 0$, then c also satisfies (3.5) and (3.6) by Proposition 3.3.3.

We assume that c satisfies (3.5) and (3.6). By Lemma 4.4.1, c is $\mathcal{A}$-equivalent to a $c_0(t) = (t^n, t^{n+1}) + h_{n+3}(t)$ at $t = 0$, where h_{n+3} is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_{n+3}(0) = h'_{n+3}(0) \cdots = h^{(n+2)}_{n+3}(0) = \mathbf{0}$. Thus we can write

$$h_{n+3}(t) = (\tilde{a}_{n+3}t^{n+3} + \cdots, \tilde{b}_{n+3}t^{n+3} + \cdots).$$

Define a parameter transformation $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(s) = s - \frac{\tilde{a}_{n+3}}{n} s^4.$$

Composing c_0 and φ, we obtain

$$c_0 \circ \varphi(s) = (s^n, s^{n+1} + \tilde{b}_{n+3} s^{n+3}) + \tilde{h}_{n+4}(s), \tag{4.12}$$

where $\tilde{h}_{n+4}$ is a $\mathbb{R}^2$-valued C^∞ function satisfying $\tilde{h}_{n+4}(0) = \tilde{h}'_{n+4}(0) \cdots = \tilde{h}^{(n+3)}_{n+4}(0) = \mathbf{0}$. Setting $T = \tilde{b}_{n+3}$ and $h_{n+4}(s) = \tilde{h}_{n+4}(s)$, we have the assertion. □

Now, we are ready to discuss the elimination of the 7-th order and 11-th order terms. The following result plays an essential role in the proof of a criterion for $(4, 5)$-cusps given by Lemma 4.4.3.

Lemma 4.4.3. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1) if and only if c is $\mathcal{A}$-equivalent to $c_1(s) = (s^4, s^5) + h_8(s)$ at $s = 0$, where h_8 is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_8(0) = h'_8(0) = \cdots = h^{(7)}_8(0) = \mathbf{0}$.*

Proof. If c is $\mathcal{A}$-equivalent to $c_1(s) = (s^4, s^5) + h_8(s)$ at $s = 0$, then c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1) by Proposition 3.3.3 and Proposition 4.2.2.

Conversely, assume that c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1). By Corollary 4.4.2, there exists a real number T such that c is $\mathcal{A}$-equivalent to $\tilde{c}(s) = (s^4, s^5 + Ts^7) + h_8(s)$ at $s = 0$. Calculating the expression $-77B^2 + 105AD + 60AC$ for c_1, we have

$$-77B^2 + 105AD + 60AC = 20901888000T. \tag{4.13}$$

Thus we get T = 0 and see that c is $\mathcal{A}$-equivalent to c_1 at $s = 0$. This completes the proof. □

Lemma 4.4.4. *Let $c : I \to \mathbb{R}^2$ be a C^∞ curve. Then c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1) if and only if c is $\mathcal{A}$-equivalent to $c_2(s) = (s^4, s^5) + K_8(s)$ at $s = 0$, where K_8 is a $\mathbb{R}^2$-valued C^∞ function satisfying $K_8(0) = K'_8(0) = \cdots = K^{(7)}_8(0) = \mathbf{0}$ and $K^{(11)}_8(0) = \mathbf{0}$.*

Proof. If c is $\mathcal{A}$-equivalent to $c_2(s) = (s^4, s^5) + K_8(s)$ at $s = 0$, then clearly c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1) by Lemma 4.4.3.

Conversely, assume that c satisfies $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and (4.1). By Lemma 4.4.3, c is $\mathcal{A}$-equivalent to $c_1(s) = (s^4, s^5) + h_8(s)$, where $h_8(t) = (\tilde{a}_8 t^8 + \cdots, \tilde{b}_8 t^8 + \cdots)$. Define a coordinate transformation $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(s) = s - \frac{\tilde{b}_{11}}{5} s^7 - \frac{\tilde{a}_{11}}{4} s^8.$$

Composing c_1 and φ, the coefficients of the 11-th order term are $(0, 0)$. Therefore we have the assertion. □

Remark 4.4.5. *If c satisfies the conditions $c'(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$, then we can always eliminate the 11-th order term from c. In fact, by Lemma 4.4.1, c is $\mathcal{A}$-equivalent to $c_0(t) = (t^4, t^5) + h_7(t)$ at $t = 0$, where $h_7(t) = (\tilde{a}_7 t^7 + \cdots, \tilde{b}_7 t^7 + \cdots)$. Define a coordinate transformation $\varphi : \mathbb{R} \to \mathbb{R}$ by*

$$\varphi(s)=s-\frac{\tilde{a}_7}{4}s^4-\frac{1}{40}(5\tilde{a}_7^2-16\tilde{a}_7\tilde{b}_8+8\tilde{b}_{11})s^7+\frac{1}{4}(2\tilde{a}_7\tilde{a}_8-\tilde{a}_{11})s^8.$$

Composing c_0 and φ, the coefficients of the 7-th order term are $(0,\tilde{b}_7)$ and the coefficients of the 11-th order term are $(0,0)$.

4.5. **Proof of Theorem 4.2.1.** If c satisfies $c'(0)=c''(0)=c'''(0)=\mathbf{0}$, $A\neq 0$ and (4.1), then c is $\mathcal{A}$-equivalent to $c_2(s)=(s^4,s^5)+K_8(s)$ at $s=0$ by Lemma 4.4.4. By Proposition 4.3.1, we see that c is also $\mathcal{A}$-equivalent to $Cusp_{(4,5)}$ at $s=0$.

Assume that c is $\mathcal{A}$-equivalent to $Cusp_{(4,5)}$ at $s=0$. Using Proposition 3.3.3 and Proposition 4.2.2, we get the conclusion. □

Theorem 4.5.1. *Let $c: I\to\mathbb{R}^2$ be a C^∞ curve.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;+7)}(t)=(t^4,t^5+t^7)$ if and only if $c'(0)=c''(0)=c'''(0)=\mathbf{0}$, $A\neq 0$ and $-77B^2+105AD+60AC>0$,*
(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;-7)}(t)=(t^4,t^5-t^7)$ if and only if $c'(0)=c''(0)=c'''(0)=\mathbf{0}$, $A\neq 0$ and $-77B^2+105AD+60AC<0$,*
(3) *$Cusp_{(4,5;+7)}$ and $Cusp_{(4,5;-7)}$ are not $\mathcal{A}$-equivalent at $t=0$,*

where $A=\det(c^{(5)}(0),c^{(4)}(0))$, $B=\det(c^{(6)}(0),c^{(4)}(0))$, $C=\det(c^{(7)}(0),c^{(4)}(0))$ and $D=\det(c^{(6)}(0),c^{(5)}(0))$.

Proof. By Corollary 4.4.2, c satisfies $c'(0)=c''(0)=c'''(0)=\mathbf{0}$ and $A\neq 0$ if and only if there exists a real number T such that c is $\mathcal{A}$-equivalent to $\tilde{c}(s)=(s^4,s^5+Ts^7)+h_8(s)$ at $s=0$, where h_8 is a $\mathbb{R}^2$-valued C^∞ function satisfying $h_8(0)=h_8'(0)\cdots=h_8^{(7)}(0)=\mathbf{0}$. We note that $T>0$ (resp. $T<0$) if and only if $-77B^2+105AD+60AC>0$ (resp. $-77B^2+105AD+60AC<0$) by (4.13).

(1) By the above discussion, c satisfies $c'(0)=c'(0)=c'''(0)=\mathbf{0}$, $A\neq 0$ and $-77B^2+105AD+60AC>0$ if and only if there exists a positive real number T such that c is $\mathcal{A}$-equivalent to $\tilde{c}(s)=(s^4,s^5+Ts^7)+h_8(s)$ at $s=0$. We set $p=\sqrt{T}$ and
$$L=\begin{pmatrix}p^4 & 0\\ 0 & p^5\end{pmatrix}.$$
Then we see that
$$L\circ\tilde{c}\left(\frac{s}{p}\right)=(s^4,s^5+s^7)=Cusp_{(4,5;+7)}(s).$$
If c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;+7)}$ at $t=0$, then c satisfies $c'(0)=c''(0)=c'''(0)=\mathbf{0}$, $A\neq 0$ and $-77B^2+105AD+60AC>0$ by Proposition 3.3.3 and Remark 4.2.3.
(2) By a similar argument as (1), we get the assertion.
(3) We immediately have the conclusion by Remark 4.2.3.

□

Summarizing Theorem 3.3.1, Theorem 4.2.1 and Theorem 4.5.1 by using κ_q, we obtain the following results.

Theorem 4.5.2. *Let $c: I\to\mathbb{R}^2$ be a C^∞ curve.*

(1) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5)}(t) = (t^4, t^5)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q = 0$.*
(2) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;+7)}(t) = (t^4, t^5 + t^7)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q > 0$.*
(3) *c is $\mathcal{A}$-equivalent to $Cusp_{(4,5;-7)}(t) = (t^4, t^5 - t^7)$ if and only if $c(0) = c''(0) = c'''(0) = \mathbf{0}$, $A \neq 0$ and $\kappa_q < 0$.*
(4) *$Cusp_{(4,5)}$, $Cusp_{(4,5;+7)}$ and $Cusp_{(4,5;-7)}$ are not $\mathcal{A}$-equivalent to at $t = 0$. However, they are C^1-equivalent each other to at $t = 0$.*

Remark 4.5.3. *$Cusp_{(4,5;+7)}$ and $Cusp_{(4,5;-7)}$ are complex-analytically equivalent at $t = 0$. In fact, we set a matrix*

$$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad (i = \sqrt{-1}).$$

Then we see that

$$\mathcal{L} \circ Cusp_{(4,5;-7)}(is) = (s^4, s^5 + s^7) = Cusp_{(4,5;+7)}(s).$$

4.6. **Examples.**

(1) $Cusp_{(4,5)}(t)$, $p_1(t) = (t^4 + t^7, t^5)$ and $p_2(t) = (t^4 - t^7, t^5)$ are $\mathcal{A}$-*equivalent* at $t = 0$ by Theorem 4.2.1.
(2) $Cusp_{(4,5;+7)}(t)$, $q_1(t) = (t^4+t^7, t^5+t^7)$, $q_2(t) = (t^4-t^7, t^5+t^7)$ and $q_3(t) = (t^4-t^6, t^5)$ are $\mathcal{A}$-*equivalent* at $t = 0$ by Theorem 4.5.1 (1)
(3) $Cusp_{(4,5;-7)}(t)$, $c_1(t) = (t^4+t^7, t^5-t^7)$, $c_2(t) = (t^4-t^7, t^5-t^7)$, $c_3(t) = (t^4, t^5+t^6)$, $c_4(t) = (t^4, t^5 - t^6)$, and $c_5(t) = (t^4 + t^6, t^5)$ are $\mathcal{A}$-*equivalent* at $t = 0$ by Theorem 4.5.1 (2)

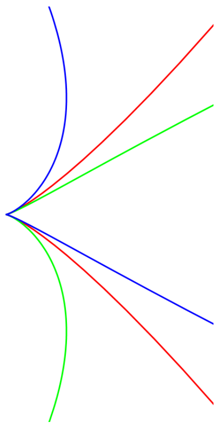

FIGURE 20.
The red curve is $Cusp_{(4,5)}(t)$, the green curve is $p_1(t)$ and the blue curve is $p_2(t)$.

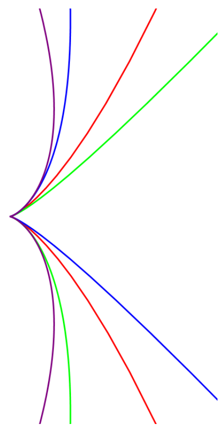

FIGURE 21.
The red curve is $Cusp_{(4,5;+7)}(t)$, the green curve is $q_1(t)$, the blue curve is $q_2(t)$ and the purple curve is $q_3(t)$.

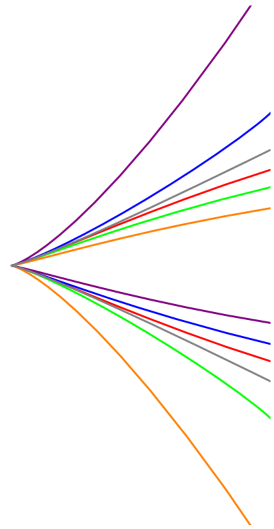

FIGURE 22.
The red curve is $Cusp_{(4,5;-7)}(t)$, the green curve is $c_1(t)$, the blue curve is $c_2(t)$, the purple curve is $c_3(t)$, the orange curve is $c_4(t)$ and the gray curve is $c_5(t)$.

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