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Introduction

In this paper, we investigate the asymptotic behavior of oscillatory integrals of the form

$$I_\varphi(t;F) = \int_{\mathbb{R}^n} e^{itF(x)}\varphi(x)dx. \tag{i-1}$$

Here t is a real parameter and F, φ are real-valued smooth (infinitely differentiable) functions defined on a small open neighborhood U of the origin in $\mathbb{R}^n$, which are called the *phase* and *amplitude*, respectively.

We assume that support of φ is contained in U for the convergence of the integral. The points where the gradient of F (usually denoted by (∇F)) vanishes are called *critical points*. If F has no critical point on the support of φ, $I_\varphi(t;F) = \mathcal{O}(t^{-N})$ holds for any $N \in \mathbb{N}$ (see Prpositions 2.1.2, 2.1.7). So we assume that F satisfies $F(0) = |(\nabla F)(0)| = 0$. When $n = 1$, the asymptotic behavior as t tends to infinity is completely understood for non-flat functions. In this case, F can be expressed as $F(\phi(x)) = x^k$ for some $k \in \mathbb{N}$ by change of variable ϕ and this k appears in the asymptotic behavior of $I_\varphi(t;F)$ ([31], see also Proposition 2.1.4 in this paper). However, the general dimensional case is much more complicated and there still remain many open problems. In analysis of this case, we need some assumptions to the phase functions. As a special case, if F is nondegenerate (i.e., the Hessian matrix of F is invertible), the asymptotic expansion of $I_\varphi(t;F)$ is obtained by using Morse's lemma (see Proposition 2.1.8).

On the other hand, when F is degenerate, the other tools and methods are required. If F is real analytic, by using Hironaka's resolution of singularities, the form of the asymptotic expansion of $I_\varphi(t;F)$:

$$I_\varphi(t;F) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha k}(\varphi) t^{\alpha} (\log t)^{k-1} \quad \text{as } t \to +\infty, \tag{i-2}$$

is obtained (Jeanquartier [14], Malgrange [25]). Here α runs through finite number of arithmetic progressions consisting of negative rational numbers. Since Hironaka's theorem does not give quantitative resolution of singularities, we cannot know precise properties of each terms in (i-2). A.N.Varchenko [34] constructs a method to compute the above arithmetic progressions from the geometrical information of the Newton polyhedron of F by using the theory of toric varieties. The main analysis in this paper is based on his work. After that, many generalization of his work have been obtained, for instance, smooth phase case [17], weighted amplitude case [5],[18] and so on. For readers, there is a good survey [21] by E. León-Cardenal.

The aim of this paper is to generalize the above result of Varchenko to the case where the phase function is replaced by $f(x)/g(x)$, i.e.,

$$I_\varphi(t;f,g) := \int_{\mathbb{R}^n\setminus g^{-1}(0)} e^{it\frac{f(x)}{g(x)}}\varphi(x)dx \tag{i-3}$$

where f,g are real analytic functions defined on U satisfying that $f(0)=|(\nabla f)(0)|=0$ and $g(0)=|(\nabla g)(0)|=0$, U and φ are the same as in (i-1). We call this integral the *oscillatory integral* attached to $(f/g,\varphi)$. In one-dimensional case, we can apply the same argument of analytic case and obtain some kinds of series expression which imply the singularities of the denominator g appear in the smoothness of $I_\varphi(t;f,g)$ at the origin (see Section 7.3). Note that if g does not vanish on U, (i-3) is reduced to the analytic phase case (i-1).

By using a simultaneous resolution of singularities to $f^{-1}(0)\cup g^{-1}(0)$, W.Veys and W.A.Zúñiga-Galindo [35] show that if the support of φ is contained in a sufficiently small open neighborhood of the origin, then $I_\varphi(t;f,g)$ has two types of asymptotic expansion, that is, the case when its parameter t tends to infinity and zero. More precisely, for any positive integer N,

$$I_\varphi(t;f,g) = \sum_{\alpha<N}\sum_{k=1}^{n} C_{\alpha,k}(\varphi)t^{-\alpha}(\log t)^{k-1} + \mathcal{O}(t^{-N}) \qquad \text{as } t\to+\infty, \tag{i-4}$$

$$I_\varphi(t;f,g) - \psi_N(t) = \sum_{\substack{\beta<N\\ \beta\notin\mathbb{N}}} D_{\beta,1}(\varphi)t^\beta + \sum_{\beta<N}\sum_{l=2}^{n+1} D_{\beta,l}(\varphi)t^\beta(\log t)^{l-1} \qquad \text{for } t\in\mathbb{R}\setminus\{0\} \tag{i-5}$$

hold, where $\psi_N(t)$ is a C^N function satisfying that $\psi_N(0)=\int_{\mathbb{R}^n}\varphi(x)dx$ and α,β run through finite number arithmetic progressions consisting of positive rational numbers. As is the case in the analytic case, we cannot know the properties of each terms in (i-4), (i-5) from the information of f and g. Let us focus the equation (i-5). From (i-5), we see that $I_\varphi(t; f, g)$ is smooth on $\mathbb{R}\setminus\{0\}$, however, at $t = 0$, $I_\varphi(t; f, g)$ has non-smooth part which correspond to the right hand side in (i-5). We call this part *singular part* of $I_\varphi(t; f, g)$ and denote by $S_\varphi(t; f, g)$ (see Definition 3.0.3). In the analytic phase case, it is easy to see that $I_\varphi(t; F)$ is smooth at $t = 0$ and this implies that the singular part of $I_\varphi(t; F)$ does not appear. From this fact, the influence of the denominator g also seems to appear in singular part of $I_\varphi(t; f, g)$. Therefore, in the analysis of asymptotic behavior of $I_\varphi(t; f, g)$, the leading terms of (i-4) and $S_\varphi(t; f, g)$ are very important. In order to investigate the properties of these leading terms, we define the following indices.

Definition 1. Let f, g be real analytic functions for which the oscillatory integral (i-3) admits the asymptotic expansions of the form (i-4), (i-5). Then, the *oscillatory index at infinity* $\xi_\infty(f, g)$ and the *oscillatory index at zero* $\xi_0(f, g)$ are defined as follows:

$$\xi_\infty(f, g) := \min\{\alpha : C_{\alpha,k}(\varphi) \neq 0 \text{ for some } \varphi, k\},$$
$$\xi_0(f, g) := \min\{\beta : D_{\beta,l}(\varphi) \neq 0 \text{ for some } \varphi, l\}$$

and the *multiplicity* of each index $\eta_\infty(f, g)$, $\eta_0(f, g)$ are defined by

$$\eta_\infty(f, g) := \max\{k : C_{\xi_\infty(f,g),k}(\varphi) \neq 0 \text{ for some } \varphi\},$$
$$\eta_0(f, g) := \max\{l : D_{\xi_0(f,g),l}(\varphi) \neq 0 \text{ for some } \varphi\}.$$

Our main purpose is

- to construct an algorithm to compute the arithmetic progressions where α, β in (i-4), (i-5) move from the information of f and g.
- To determine or precisely estimate the above oscillatory indices and their multiplicities by means of the information of f and g.

Another main object of our investigation in this paper is the following integrals

$$Z_{\mathbb{R}}(s; F, \varphi) := \int_{\mathbb{R}^n} |F(x)|^s \varphi(x) dx \quad (s \in \mathbb{C}), \tag{i-6}$$

for detail definition, see Chapter 7. This integral converges locally uniformly on the right-half plane and defines a holomorphic function there, which is called *local zeta function* attached to (F, φ). The central question of this function is its analytic continuation. In general, if F is an analytic function, $Z_{\mathbb{R}}(s; F, \varphi)$ can be meromorphically continued to the whole complex plane and has poles on the negative real axis (see [3], [10]). Our main interest is a relationship between the locations and orders of such poles and the properties of F, φ. It is known (see, for instance, [17], [18], or Section 7.1 in this paper) that the properties of poles of local zeta function $Z_{\mathbb{R}}(s; F, \varphi)$ is deeply connected to the asymptotic analysis of oscillatory integral (i-1). The work of Varchenko is essentially constructing a method to compute the locations and orders of poles of local zeta function from the information of Newton polyhedron of F. We attempt to generalize this method to rational functions. For rational case, we consider the following integrals

$$Z_{\mathbb{R}}(s; f, g, \varphi) = \int_{\mathbb{R}^n \setminus D} \left| \frac{f(x)}{g(x)} \right|^s \varphi(x) dx, \tag{i-7}$$

where f, g, φ are the same as in (i-3) and $D := f^{-1}(0) \cup g^{-1}(0)$. It is shown that this integral converges on some domain in $\mathbb{C}$ and defines a holomorphic function there, which is called local zeta function attached to $(f/g, \varphi)$. As is the case of analytic function, this function is meromorphically continued to $\mathbb{C}$ and has poles on real axis. The substantial analysis in this paper is to investigate and describe properties of poles of the local zeta function $Z_{\mathbb{R}}(s; f, g, \varphi)$ by means of Newton polyhedra of f and g by using simultaneous toric desingularization. In Chapter 6, we also investigate the case when f/g is meromorphic function (i.e., f, g are holomorphic functions) and the integral (i-7) is considered on $\mathbb{C}^n$.

Local zeta function itself is a mathematically interesting object and there have been many researches of this function. In connection with number theory, local zeta functions for p-adic field are enthusiastically investigated [6], [27], [7], [12], [23], [26]. Theory of p-adic case is established by J.Igusa [13] and often called Igusa zeta function. There are some generalization of local zeta functions for multi-functions. In [30], C.Sabbah introduces several variables version of local zeta function, which is called multivariate local zeta function. The author shows there exists meromorphic continuation to whole $\mathbb{C}^l$ and its poles are contained in the union of some hyperplanes. Later, in [24], F.Loeser defines multivariate local zeta function for local field of characteristic zero. In [22], E.L.Cardenal, W.Veys and W.A.Zúñiga-Galindo consider the local zeta functions for analytic mapping and generalize the work of Varchenko.

This paper is organized as follows. In Chapter 1, we explain many important words and their elementary properties, which are often used in this paper. In Chapter 2, in order to show our motivation to the investigation of this paper, we roughly explain the earlier results concerning with the analytic phase case in [14],[25],[31],[34] and rational phase case in [35]. In Chapter 3, we state our main results relating to the estimate and determination of the oscillation index and its multiplicity. In Chapter 4, we explain how to construct simultaneous resolution of singularities with respect to several functions by using the theory of toric varieties. In Chapter 5, we construct an appropriate fan which are the most suitable to investigate and describe the properties of poles of $Z_K(s; f, g, \varphi)$. From this fan, two important sets of integer vectors which are used to express the sets of candidate poles of $Z_K(s; f, g, \varphi)$ are obtained. We also define and investigate two important subfans which will play important roles to compute the coefficients of the leading terms of asymptotic expansion (i-4) and (i-5). In Chapter 6, we investigate the poles of local zeta function $Z_K(s; f, g, \varphi)$ by using of the results in previous two chapters. Here, we give the positions of all candidate poles and some sufficient conditions where the positions and orders of the (e-)leading poles (see Definition 6.0.5) are explicitly determined. To do this, we compute the explicit formulae of the coefficients of terms of Laurent expansions. In Chapter 7, after an exact relationship between oscillatory integrals and local zeta functions is recalled, we will show some theorems concerning with the Mellin transform and the Fourier transform which help us obtain the explicit formulae of the coefficients of the leading terms in (i-4) and (i-5). As a result, proofs of the theorems in Chapter 3 will be given. Furthermore, we consider more general case which contains a non-smooth phase case in one-dimension.

Notation and Symbols

- We denote by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively. We write $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ express the real part of s and imaginary part of s, respectively. We define $1/0 := \infty$ and $1/\infty := 0$.

- We use the multi-index as follows. For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$, define
$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n, \quad |x| = \sqrt{x_1^2 + \cdots + x_n^2},$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$
$$\langle \alpha\rangle = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad 0! = 1.$$

- For $A, B \subset \mathbb{R}$, $A \leq B (A < B)$ means $x \leq y (x < y)$ for any $x \in A$ and $y \in B$.

- For $A, B \subset \mathbb{R}^n$ and $c \in \mathbb{R}$, we set
$$A + B = \{a + b \in \mathbb{R}^n : a \in A \text{ and } b \in B\}, \quad c \cdot A = \{ca \in \mathbb{R}^n : a \in A\}.$$
Moreover, $\mathrm{Int}(A)$ expresses the interior of the set A.

- We express by $\mathbf{1}$ the vector $(1, ..., 1)$ or the set $\{(1, ..., 1)\}$.

- For a set A, $\mathcal{P}(A)$ is the set of all subsets of A.

- For a finite set A, $\#A$ means the cardinality of A.

- For a nonnegative real number r and a subset I in $\{1, ..., n\}$, the map $T_I^r : \mathbb{R}^n \to \mathbb{R}^n$ is defined by
$$(z_1, ..., z_n) = T_I^r(x_1, ..., x_n) \text{ with } z_j := \begin{cases} r & \text{for } j \in I, \\ x_j & \text{otherwise.} \end{cases}$$

We define $T_I := T_I^0$. For a set A in $\mathbb{R}^n$, the image of A by T_I is denoted by $T_I(A)$. When $A = \mathbb{R}^n$ or $\mathbb{Z}_+^n$, its image is expressed as

$$T_I(A) = \{x \in A : x_j = 0 \text{ for } j \in I\}.$$

- We use $\mathcal{O}$ as big O notation. That is: $f(x) = \mathcal{O}(g(x))$ $(x \to \infty)$ if there exist $M, N > 0$ such that

$$|f(x)| \leq M|g(x)| \qquad \text{for } N < x$$

and $f(x) = \mathcal{O}(g(x))$ $(x \to a)$ if there exist $M, \delta > 0$ such that

$$|f(x)| \leq M|g(x)| \qquad \text{for } |x - a| < \delta.$$

Contents

Chapter 1

Newton polyhedra and Newton data

In this chapter, we define Newton polyhedra of analytic functions and some values derived from the geometrical information of Newton polyhedra. First, let us recall important concepts about convex rational polyhedra. Refer to [36] for a general theory of convex polyhedra.

1.1 Newton Polyhedra

1.1.1 Polyhedra

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a hyperplane and a closed half-space in $\mathbb{R}^n$ defined by

$$H(a, l) := \{x \in \mathbb{R}^n : \langle a, x \rangle = l\},$$
$$H^+(a, l) := \{x \in \mathbb{R}^n : \langle a, x \rangle \geq l\},$$

respectively. It is clear that $H^+(a, l)$ is a convex set in $\mathbb{R}^n$ and $H(a, l)$ is the topological boundary of $H^+(a, l)$ unless $a = 0$.

Remark 1.1.1. It follows from the definition of $H^+(\cdot, \cdot)$ that for $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$,

$$H^+(a, l + d \cdot \langle a \rangle) = H^+(a, l) + d \cdot \mathbf{1} \text{ for } d \geq 0.$$

In the case of hyperplane $H(a, l)$, analogous equation can be obtained.

Definition 1.1.2. $P \subset \mathbb{R}^n$ is called *(convex rational) polyhedron* if P is expressed as an intersection of some closed half-space, that is,

$$P = \bigcap_{j=1}^{N} H^+(a^j, l_j)$$

for $(a^j, l_j) \in \mathbb{Z}^n \times \mathbb{Z}$ $(j = 1, ..., N)$.

Definition 1.1.3. A pair $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$ is *valid* for P if P is contained in $H^+(a, l)$. A set $\gamma \subset P$ is called *face* if $\gamma = H(a, l) \cap P$ for some valid pairs $(a, l) \in \mathbb{Z}^n \times \mathbb{Z}$.

Remark 1.1.4. (i) Since $\mathbb{R}^n = H^+(0, 0)$, $\mathbb{R}^n$ is a polyhedron and $(0, 0)$ is valid for any polyhedron. Thus, polyhedron P is a trivial face and the other faces are called *proper faces*.

(ii) The pair $(0, -1)$ is valid for any polyhedron and $H(0, -1) \cap P = \emptyset$. This implies that the empty set is also a face of the polyhedron P.

The *boundary* of a polyhedron P, denoted by ∂P, is the union of all proper faces of P. For a face F, ∂F is similarly defined.

From the definitions above, we can easily know that every proper face γ is contained in $\bigcap_{j=1}^{M} H(a^j, l_j)$ for some $\{(a^j, l_j) \in \mathbb{Z}^n \times \mathbb{Z}\}$ and $M \in \mathbb{N}$. We write

$$\mathcal{F}[P] = \{\text{the set of all nonempty faces of } P\}.$$

Definition 1.1.5. The *dimension* of a face F is the dimension of its affine hull and denoted $\dim(F)$. The faces of dimensions $0, 1$ and $\dim(P) - 1$ are called *vertices, edges* and *facets*, respectively.

Lemma 1.1.6 (Lemma 3.1 in [5])**.** *Let P_1, P_2 be n-dimensional polyhedra in $\mathbb{R}^n$. If $P_1 \subset P_2$, then $P_1 \cap \partial P_2$ is the union of proper faces of P_1.*

Every polyhedron treated in this paper satisfies a condition in the following lemma.

Lemma 1.1.7 (Lemma 2.2 in [18])**.** *Let $P \subset \mathbb{R}^n_+$ be a polyhedron. Then the following conditions are equivalent.*

(i) $P + \mathbb{R}^n_+ \subset P$.

(ii) There exists a finite set of pairs $\{(a^j, l_j)\}_{j=1}^N \subset \mathbb{Z}^n_+ \times \mathbb{Z}_+$ such that $P = \bigcap_{j=1}^{N} H^+(a^j, l_j)$.

1.1.2 Newton polyhedra

Let $K = \mathbb{R}$ or $\mathbb{C}$. Let us define the Newton polyhedron of a K-analytic function f and some important functions associated with the Newton polyhedron. In this paper, $\mathbb{R}$-*analytic* means "real analytic" and $\mathbb{C}$-*analytic* means "holomorphic".

Let f be a K-analytic function defined on a neighborhood of the origin in K^n, which has the Taylor series at the origin:

$$f(x) \sim \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha \qquad \text{with } c_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}. \tag{1.1.1}$$

We define the set S_f by

$$S_f := \{\alpha \in \mathbb{Z}_+^n : c_\alpha \neq 0 \text{ in } (1.1.1)\}.$$

Definition 1.1.8. The *Newton polyhedron* $\Gamma_+(f)$ of f is defined to be the convex hull of the set $\bigcup\{\alpha + \mathbb{R}_+^n : \alpha \in S_f\}$.

It is known [36] that the Newton polyhedron of f is a polyhedron. The union of compact faces of the Newton polyhedron $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$, while the topological boundary of $\Gamma_+(f)$ is denoted by $\partial\Gamma_+(f)$.

Definition 1.1.9. For any face γ of $\Gamma_+(f)$, the γ-*part of* f is a function $f_\gamma(x)$ defined by

$$f_\gamma(x) = \sum_{\alpha\in\gamma\cap\mathbb{Z}_+^n} c_\alpha x^\alpha. \tag{1.1.2}$$

Note that the series in (1.1.2) is always convergent when f is K-analytic.

Remark 1.1.10. (i) Let us consider the case when f is a smooth function. When $K = \mathbb{R}$, the Newton polyhedron of f can be similarly defined. However, when $K = \mathbb{C}$, the above definition is not available since its Taylor series may contain terms of the form $c_{\alpha,\beta} x^\alpha \overline{x}^\beta$. There exists an extended definition containing such a case, for instance, see [29].

(ii) When f is assumed to be smooth, the above definition of γ-part is not available since the series (1.1.2) may not converge for non-compact face γ. In [5], [18], the authors introduce definition of γ-part for non-analytic smooth functions which satisfy some conditions concerning with the limit.

1.2 The Newton data with respect to the pair (f, g)

In this section, we define the Newton distances and the Newton multiplicities with respect to the pair of two K-analytic functions (f, g), which play important roles in the investigation of this paper. Throughout this section, let f, g be K-analytic functions defined on a small neighborhood of the origin.

Definition 1.2.1. The *Newton distances* with respect to the pair (f, g) are defined by

$$\begin{aligned} d_\infty(f,g) &:= \min\{d \geq 0 : (\Gamma_+(g) + d \cdot \mathbf{1}) \subset \Gamma_+(f)\}, \\ d_0(f,g) &:= \min\{d \geq 0 : (\Gamma_+(f) + d \cdot \mathbf{1}) \subset \Gamma_+(g)\}. \end{aligned} \tag{1.2.1}$$

Remark 1.2.2. From the above definition, we can see the following.

- $d_\infty(f, g) = 0$ if and only if $\Gamma_+(g) \subset \Gamma_+(f)$.
- $d_0(f, g) = 0$ if and only if $\Gamma_+(f) \subset \Gamma_+(g)$.

The Newton distances have the following another expressions.

Lemma 1.2.3. *The Newton distances can be expressed as follows.*

(i) $d_\infty(f, g) = \max\{d \geq 0 : \partial\Gamma_+(f) \cap (\Gamma_+(g) + d \cdot \mathbf{1}) \neq \emptyset\}$.

(ii) $d_0(f, g) = \max\{d \geq 0 : \partial\Gamma_+(g) \cap (\Gamma_+(f) + d \cdot \mathbf{1}) \neq \emptyset\}$.

Proof. We only consider the case of $d_\infty(f, g)$. Let $A := \min\{d \geq 0 : (\Gamma_+(g) + d \cdot \mathbf{1}) \subset \Gamma_+(f)\}$ and $B := \max\{d \geq 0 : \partial\Gamma_+(f) \cap (\Gamma_+(g) + d \cdot \mathbf{1}) \neq \emptyset\}$. We will prove two side inequations.

($A \leq B$) Assume that $\partial\Gamma_+(f) \cap (\Gamma_+(g) + A \cdot \mathbf{1}) = \emptyset$. Then, from the definition of A, $(\Gamma_+(g) + A \cdot \mathbf{1}) \subsetneq \Gamma_+(f)$ holds. There exists positive constant $0 < \delta < A$ satisfying that $(\Gamma_+(g) + \delta \cdot \mathbf{1}) \subset \Gamma_+(f)$ and this is contradicted to the minimality of A. Hence $\partial\Gamma_+(f) \cap (\Gamma_+(g) + A \cdot \mathbf{1}) \neq \emptyset$ holds and this leads $A \leq B$.

($B \leq A$) Since for all $d > A$, both $(\Gamma_+(g) + d \cdot \mathbf{1}) \subsetneq \Gamma_+(f)$ and $\partial\Gamma_+(f) \cap (\Gamma_+(g) + d \cdot \mathbf{1}) = \emptyset$ hold, we have the following relation:

$$\{d \geq 0 : \partial\Gamma_+(f) \cap (\Gamma_+(g) + d \cdot \mathbf{1}) \neq \emptyset\} < \{d \geq 0 : (\Gamma_+(g) + d \cdot \mathbf{1}) \subsetneq \Gamma_+(f)\}.$$

Taking the infimum of right side, we have

$$\{d \geq 0 : \partial\Gamma_+(f) \cap (\Gamma_+(g) + d \cdot \mathbf{1}) \neq \emptyset\} \leq A.$$

Taking the maximum of left side, we have the desired inequation. $\square$

From the viewpoint of geometry of two Newton polyhedra, the above expressions are obvious. We define the two affine maps $\Phi_\infty, \Phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Phi_\infty(\alpha) := \alpha + d_\infty(f, g) \cdot \mathbf{1},$$
$$\Phi_0(\alpha) := \alpha + d_0(f, g) \cdot \mathbf{1}.$$

We define the subsets of $\partial\Gamma_+(f), \partial\Gamma_+(g)$ as

$$\begin{aligned} \Gamma_*(f) &:= \partial\Gamma_+(f) \cap \Phi_\infty(\Gamma_+(g)) \ (= \partial\Gamma_+(f) \cap (\Gamma_+(g) + d_\infty(f, g) \cdot \mathbf{1})), \\ \Gamma_*(g) &:= \partial\Gamma_+(g) \cap \Phi_0(\Gamma_+(f)) \ \ (= \partial\Gamma_+(g) \cap (\Gamma_+(f) + d_0(f, g) \cdot \mathbf{1})). \end{aligned} \tag{1.2.2}$$

From the expressions in Lemma 1.2.3, the sets in (1.2.2) are not empty unless one Newton polyhedron is completely contained in another one. We note that the above sets are not necessarily the union of proper faces of each Newton polyhedron.

Remark 1.2.4. From Lemma 1.1.6, we see that $\Gamma_*(f)$(resp. $\Gamma_*(g)$) is a set of proper faces of $\Phi_\infty(\Gamma_+(g))$ (resp. $\Phi_0(\Gamma_+(f))$).

Let us define the Newton multiplicities of $d_\infty(f, g), d_0(f, g)$ and the sets of important faces of $\Gamma_+(f)$ and $\Gamma_+(g)$, which will play important roles in the investigation of multiplicities of the oscillation index. Let $\mathcal{F}[f]$ (resp. $\mathcal{F}[g]$) be the set of faces of $\Gamma_+(f)$ (resp. $\Gamma_+(g)$). We define two maps

$$\gamma_f : \partial\Gamma_+(f) \to \mathcal{F}[f], \qquad \tau_g : \partial\Gamma_+(g) \to \mathcal{F}[g],$$

as follows: for $\alpha \in \partial\Gamma_+(f)$, let $\gamma_f(\alpha)$ be the face of $\Gamma_+(f)$ whose relative interior contains α. It is clear that such a face can be uniquely determined. For $\beta \in \partial\Gamma_+(g)$, $\tau_g(\beta)$ is determined in the same way. Then, by using these maps, define

$$\mathcal{F}_*[f] := \{\gamma_f(\alpha) \in \mathcal{F}[f] : \alpha \in \Gamma_*(f)\},$$
$$\mathcal{F}_*[g] := \{\tau_g(\beta) \in \mathcal{F}[g] : \beta \in \Gamma_*(g)\}.$$

When $\Gamma_*(f)$ (resp. $\Gamma_*(g)$) is empty, we define $\mathcal{F}_*[f] = \emptyset$ (resp. $\mathcal{F}_*[g] = \emptyset$).

Definition 1.2.5. The *Newton multiplicities* of $d_\infty(f, g)$ and $d_0(f, g)$ are defined by

$$m_\infty(f, g) := \max\{n - \dim(\gamma) : \gamma \in \mathcal{F}_*[f]\},$$
$$m_0(f, g) := \max\{n - \dim(\tau) : \tau \in \mathcal{F}_*[g]\}.$$

If $\mathcal{F}_*[f] = \emptyset$ (resp. $\mathcal{F}_*[g] = \emptyset$), we define $m_\infty(f, g) = 0$ (resp. $m_0(f, g) = 0$).

We call the pair $(d_\infty(f, g), m_\infty(f, g))$ and $(d_0(f, g), m_0(f, g))$ the *Newton data* with respect to the pair (f, g). Note that these values depend on the choice of coordinate.

Definition 1.2.6. The set of *principal faces at infinity* of $\Gamma_+(f)$ is defined as

$$\mathcal{F}_\infty[f] := \{\gamma \in \mathcal{F}_*[f] : n - \dim(\gamma) = m_\infty(f,g)\}.$$

Define

$$\mathcal{F}_\infty[g] := \{\Phi_\infty^{-1}(\gamma) \cap \Gamma_+(g) : \gamma \in \mathcal{F}_\infty[f]\}. \tag{1.2.3}$$

It is easy to see that every element of the above set is a face of $\Gamma_+(g)$, which is called *principal face at infinity* of $\Gamma_+(g)$.

Similarly, we define the sets of *principal faces at zero* of $\Gamma_+(f)$ and $\Gamma_+(g)$ as follows.

Definition 1.2.7.

$$\mathcal{F}_0[g] := \{\tau \in \mathcal{F}_*[g] : n - \dim(\tau) = m_0(f,g)\},$$
$$\mathcal{F}_0[f] := \{\Phi_0^{-1}(\tau) \cap \Gamma_+(f) : \tau \in \mathcal{F}_0[g]\}.$$

For $\gamma_\infty \in \mathcal{F}_\infty[f]$ and $\tau_0 \in \mathcal{F}_0[g]$, we define two maps Ψ_∞, Ψ_0 as follows:

$$\Psi_\infty : \mathcal{F}_\infty[f] \to \mathcal{F}_\infty[g] \text{ as } \Psi_\infty(\gamma_\infty) := \Phi_\infty^{-1}(\gamma_\infty) \cap \Gamma_+(g),$$
$$\Psi_0 : \mathcal{F}_0[g] \to \mathcal{F}_0[f] \quad \text{as } \Psi_0(\tau_0) := \Phi_0^{-1}(\tau_0) \cap \Gamma_+(f).$$

It is easy to see that these maps are bijective. We say that $\gamma_\infty \in \mathcal{F}_\infty[f]$ (resp. $\tau_0 \in \mathcal{F}_0[g]$) *is associated to* $\tau_\infty \in \mathcal{F}_\infty[g]$ (resp. $\gamma_0 \in \mathcal{F}_0[f]$) if $\Psi_\infty(\gamma_\infty) = \tau_\infty$ (resp. $\Psi_0(\tau_0) = \gamma_0$) hold. Roughly speaking, when τ_∞ has an intersection with the image of γ_∞ by the map Φ_∞, we say γ_∞ is associated to τ_∞.

Remark 1.2.8. Let us consider the case of $g(0) \neq 0$. Then $\Gamma_+(g) = \mathbb{R}_+^n$ and it follows from the definitions that $d_0(f,g) = m_0(f,g) = 0$. In this case, since $d_\infty(f,g)$ and $m_\infty(f,g)$ are independent of g, we simply denote them by d_f and m_f, respectively. It is easy to see the followings.

- The Newton distance d_f is determined by the point $q = (d_f, ..., d_f)$, which is the intersection of the diagonal line $\alpha_1 = \cdots = \alpha_n$ with $\partial\Gamma_+(f)$.
- The principal face of $\Gamma_+(f)$ is the smallest face γ_* of $\Gamma_+(f)$ containing the point q, which is uniquely determined.
- $m_f = n - \dim(\gamma_*)$.

In Section 2.2, we introduce the result of this analytic case, which is a seminal work of Varchenko.

1.2.1 Classification and examples

From the viewpoint of geometrical relationship between the two polyhedra, the situation will be classified into the following four cases. We will explain the characteristic of each case and give simple examples. In the examples below we consider the case of $n = 2$ and polynomials f and g which does not have zero except the origin.

(1) The case of $\Gamma_+(f) = \Gamma_+(g)$

At first, let us consider the case where two Newton polyhedra have the same shape. In this case, it is easy to see that

$$d_\infty(f, g) = d_0(f, g) = 0, \quad m_\infty(f, g) = m_0(f, g) = n,$$
$$\mathcal{F}_\infty[f] = \mathcal{F}_\infty[g] = \mathcal{F}_0[f] = \mathcal{F}_0[g] = \text{the set of vertices of } \Gamma_+(f)(= \Gamma_+(g)).$$

Example 1. Let $f(x) = (x_1^2 + x_2^2)^2$ and $g(x) = x_1^4 + x_2^4$. Then $\Gamma_+(f) = \Gamma_+(g)$ and

$$d_\infty(f, g) = d_0(f, g) = 0, \quad m_\infty(f, g) = m_0(f, g) = 2,$$
$$\mathcal{F}_\infty[f] = \mathcal{F}_\infty[g] = \mathcal{F}_0[f] = \mathcal{F}_0[g] = \{(4, 0), (0, 4)\}.$$

(2) The case of $\Gamma_+(f) \subset \Gamma_+(g)$

Next, let us consider the case where the Newton polyhedron of f is contained in that of g. In this case, from the definition of the Newton data, we can see

$$d_0(f, g) = m_0(f, g) = 0, \quad \mathcal{F}_0[f] = \mathcal{F}_0[g] = \emptyset,$$
$$d_\infty(f, g) > 0,\ m_\infty(f, g) \in \{1, ..., n\}, \quad \mathcal{F}_\infty[f] \neq \emptyset,\ \mathcal{F}_\infty[g] \neq \emptyset.$$

Example 2. Let $f(x) = x_1^6 + x_2^6$ and $g(x) = x_1^2 + x_2^2$. Then, we see that

$$d_\infty(f, g) = 2,\ m_\infty(f, g) = 1,$$
$$\mathcal{F}_\infty[f] = \{\alpha \in \mathbb{R}_+^2 : \alpha_1 + \alpha_2 = 6\},\ \mathcal{F}_\infty[g] = \{\alpha \in \mathbb{R}_+^2 : \alpha_1 + \alpha_2 = 2\}.$$

The important case: $f(0) = 0, g(0) \neq 0$ is contained in this case. Since $\Gamma_+(g) = \mathbb{R}_+^n$, Newton distance and multiplicity of (f, g) coincide with those appeared in the studies of classical case (see [34] or Section 2.2 in this paper).

(3) The case of $\Gamma_+(g) \subset \Gamma_+(f)$

The case where the roles of f and g are exchanged in the case (2) can be similarly dealt with. More precisely,

$$d_\infty(f, g) = m_\infty(f, g) = 0, \quad \mathcal{F}_\infty[f] = \mathcal{F}_\infty[g] = \emptyset,$$
$$d_0(f, g) > 0,\ m_0(f, g) \in \{1, ..., n\}, \quad \mathcal{F}_0[f] \neq \emptyset,\ \mathcal{F}_0[g] \neq \emptyset.$$

The case when $g(0) = 0, f(0) \neq 0$ is the simplest situation of rational case and contained in this case.

(4) The case of $\Gamma_+(f) \not\subset \Gamma_+(g)$ and $\Gamma_+(g) \not\subset \Gamma_+(f)$

Finally, we consider the most interesting case where the properties of both Newton polyhedra are appearing. In this case, we can see that

$$d_\infty(f,g) > 0, m_\infty(f,g) \in \{1,...,n\}, \quad d_0(f,g) > 0, m_0(f,g) \in \{1,...,n\}.$$

Example 3. Let $f(x) = x_1^2 + x_2^4$ and $g(x) = x_1^4 + x_2^2$. Then,

$$d_\infty(f,g) = d_0(f,g) = 2/3, m_\infty(f,g) = m_0(f,g) = 1,$$
$$\mathcal{F}_\infty[f] = \{(2,0)\}, \mathcal{F}_\infty[g] = \{\alpha \in \mathbb{R}_+^2 : \alpha_1 + 2\alpha_2 = 4\},$$
$$\mathcal{F}_\infty[g] = \{(0,2)\}, \mathcal{F}_\infty[f] = \{\alpha \in \mathbb{R}_+^2 : 2\alpha_1 + \alpha_2 = 4\}.$$

Example 4. We consider more complicated example. Let $f(x) = x_1^8 x_2^2 + x_1^4 x_2^4 + x_2^{10}$ and $g(x) = x_1^8 + x_1^2 x_2^2$. Then, the Newton polyhedra of f, g and figures describing the case when each polyhedron moves until it is contained in another one are as follows.

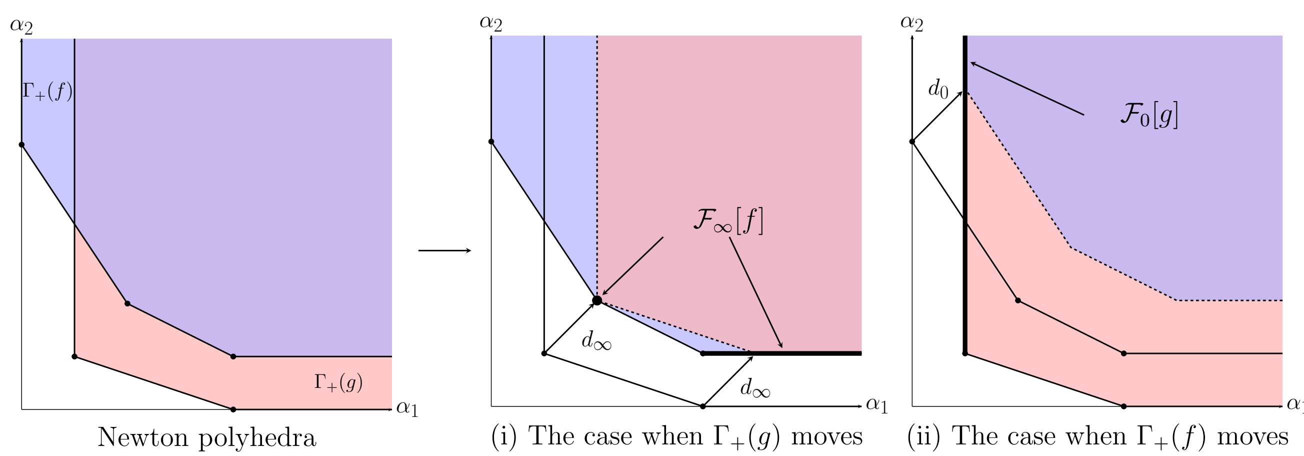

Figure 1 : the image of Newton data of the pair (f, g)

By a simple computation, from (i) in Figure 1, we have

$$d_\infty(f,g) = 2, m_\infty(f,g) = 2,$$
$$\mathcal{F}_\infty[f] = \{(4,4)\} \cup \{(\alpha_1, 2) \in \mathbb{R}_+^2 : \alpha_1 \geq 4\},$$
$$\mathcal{F}_\infty[g] = \{(2,2)\} \cup \{(\alpha_1, 0) \in \mathbb{R}_+^2 : \alpha_1 \geq 4\}.$$

Similarly, from (ii) in Figure 1, we have

$$
\begin{aligned}
&d_0(f, g) = 2, m_0(f, g) = 1,\\
&\mathcal{F}_0[g] = \{(2, \alpha_2) \in \mathbb{R}_+^2 : \alpha_2 \geq 2\}, \mathcal{F}_0[f] = \{(0, \alpha_2) \in \mathbb{R}_+^2 : \alpha_2 \geq 10\}.
\end{aligned}
$$

Chapter 2

Earlier results

In this chapter, we study some properties of the oscillatory integrals and the result of Varchenko. After that, we introduce earlier results of rational case.

2.1 Analysis of smooth phase case

Let f be a smooth function defined on an open neighborhood U of the origin in $\mathbb{R}^n$. At first, let us start one-dimensional case. In this case, almost all asymptotic behavior of $I_\varphi(t; f)$ as its parameter $t \to \infty$, including an asymptotic expansion, is already obtained. We define important class of function often appearing in this paper.

Definition 2.1.1. Let f be a function defined on $\mathbb{R}^n$.

- We say f is *rapidly decreasing* as $x \to \infty$ if $\lim_{x\to\infty} x^\alpha f(x) = 0$ for any $\alpha \in \mathbb{Z}_+^n$.
- If f is assumed to be smooth on $\mathbb{R}^n$ and for any $\beta \in \mathbb{Z}_+^n$, $\partial^\beta f(x)$ is rapidly decreasing as $x \to \infty$, then f is called a *rapidly decreasing function.*

We denote the set of all rapidly decreasing functions defined on $D \subset \mathbb{R}^n$ by $\mathcal{S}(D)$, which is called Schwartz space.

Proposition 2.1.2. *Let $n = 1$ and φ be a smooth function satisfying that Supp$(\varphi) \subset (a, b)$. If $f'(x) \neq 0$ for all $x \in [a, b]$, then for any $N \in \mathbb{N}$*

$$I_\varphi(t; f) = \int_a^b e^{itf(x)}\varphi(x)dx = \mathcal{O}(t^{-N}) \quad \text{as } t \to \infty.$$

Proof. See the proof of Proposition 1 in Chapter 8 in [31]. □

It follows from the convergence of the integrals

$$i^M \int_a^b (f(x))^M e^{itf(x)} \varphi(x) dx$$

for any $M \in \mathbb{Z}_+$ that $I_\varphi(t; f)$ is a smooth function on $\mathbb{R}$. Putting this fact and Proposition 2.1.2, we see that $I_\varphi(t; f)$ is a rapidly decreasing function if f' does not vanish.

As we see in Proposition 2.1.2, the asymptotic behavior of $I_\varphi(t; f)$ is essentially affected by the existence of the point where the first derivative of phase function f vanishes. Next proposition is called van der Corput lemma, which directly indicates an influence of flatness of the phase function.

Proposition 2.1.3 (van der Corput lemma)**.** *Let $n = 1$ and φ be a smooth function satisfying that Supp(φ) $\subset (a, b)$. If $|f^{(k)}(x)| \geq 1$ for some $k \in \mathbb{N}$, then*

$$|I_\varphi(t; f)| \leq C_k t^{-1/k} \quad (t \geq 1)$$

for positive constant C_k which is independent of t. If $k = 1$, f' is required to be monotonic.

Proof. See the proof of Proposition 2 in Chapter 8 in [31]. □

Moreover, explicit asymptotic expansion of $I_\varphi(t; f)$ is obtained.

Proposition 2.1.4. *Suppose that f satisfies $f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0$ and $f^{(k)}(x_0) \neq 0$ for $k \geq 2$. Then, we have the following:*

$$I_\varphi(t; f) \sim t^{-1/k} \sum_{j=0}^{\infty} a_j t^{-j/k} \quad \textit{as } t \to \infty. \tag{2.1.1}$$

Proof. See the proof of Proposition 3 in Chapter 8 in [31]. □

Remark 2.1.5. The first coefficient of the asymptotic expansion (2.1.1) is given by

$$a_0 = \left(\frac{2\pi k!}{-if^{(k)}(x_0)} \right)^{1/k} \varphi(x_0).$$

So if φ does not vanish at $x = x_0$, the leading term of the asymptotic expansion (2.1.1) is $a_0 t^{-1/k}$.

In the case of $n \geq 2$ is more complicated. To discuss this case, we define critical point as follows.

Definition 2.1.6. For a smooth function f, the point x_0 is called a *critical point* if

$$(\nabla f)(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \cdots, \frac{\partial f}{\partial x_n}(x_0) \right) = (0, ..., 0).$$

In the case of $n \geq 2$, similar result is obtained if the phase function does not have critical point.

Proposition 2.1.7. *Suppose that φ has a sufficiently small support and f has no critical point on Supp(φ). Then for any $N \in \mathbb{N}$*

$$I_\varphi(t; f) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx = \mathcal{O}(t^{-N}) \quad \text{as } t \to \infty.$$

Applying Proposition 2.1.2 (if necessary, choosing a coordinate system), one can prove this proposition. This proposition shows that the behavior of the oscillatory integral essentially depends on the properties of critical point of the phase function. Indeed, in the results appearing below, some conditions are assumed to critical point of f.

Proposition 2.1.8. *Suppose that f has a critical point at the origin and the support of φ is contained in sufficiently small neighborhood of the origin. If the Hessian matrix*

$$H_f(0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(0) \end{pmatrix}$$

is invertible (such critical point is called nondegenerate), then

$$I_\varphi(t; f) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{-j} \quad \text{as } t \to \infty, \tag{2.1.2}$$

where each a_j is a constant depending on f and φ.

Proof. See the proof of Proposition 6 in Chapter 8 in [31]. □

This proposition is proved by applying Morse's lemma and transforming f into the form $y_1^2 + \cdots + y_m^2 - (y_{m+1}^2 + \cdots + y_n^2)$ for some $0 \leq m \leq n$.

Remark 2.1.9. The first coefficient of the asymptotic expansion (2.1.2) is given by

$$a_0 = \frac{(2\pi i)^{n/2} \cdot \varphi(0)}{\sqrt{|\det H_f(0)|}}.$$

So if φ does not vanish at the origin, the leading term of the asymptotic expansion (2.1.2) is $a_0 t^{-n/2}$.

2.2 Analysis of degenerate case and result of Varchenko

In this section, we assume that

- f is a real analytic function defined on a sufficiently small open neighborhood U of the origin in $\mathbb{R}^n$ and satisfy that $f(0) = |(\nabla f)(0)| = 0$.
- f has no critical point apart from the origin on U.
- φ is a smooth function whose support is contained in U.

Theorem 2.2.1 ([14], [25]). *The oscillatory integral $I_\varphi(t; f)$ admits an asymptotic expansion of the form:*

$$I_\varphi(t; f) \sim \sum_{\alpha} \sum_{k=1}^{n} C_{\alpha,k}(\varphi) t^{-\alpha} (\log t)^{k-1} \qquad \text{as } t \to \infty, \tag{2.2.1}$$

where α runs through finitely many arithmetic progressions consisting of positive rational numbers and a map $\varphi \mapsto C_{\alpha,k}(\varphi)$ is distribution.

This result is obtained by an application of Hironaka's resolution of singularities for analytic functions.

Theorem 2.2.2 ([11], [3]). *There exists a proper real analytic mapping π from some n-dimensional real analytic manifold Y to $\mathbb{R}^n$ such that at each point of the set $\pi^{-1}(0)$, there exist local coordinates $y = (y_1, ..., y_n)$ satisfying the following properties:*

(1) There exist nonnegative integers l_j such that

$$f(\pi(y)) = \pm \prod_{j=1}^{n} y_j^{l_j}. \tag{2.2.2}$$

(2) The Jacobian of the mapping π has the form

$$J_\pi(y) = \pm \prod_{j=1}^{n} y_j^{m_j - 1}, \tag{2.2.3}$$

where m_j are positive integers.

(3) $\pi : Y \setminus \pi^{-1}(0) \to U \setminus \{0\}$ is a diffeomorphism.

Since Hironaka's resolution theorem is existence theorem, we cannot have explicit order of each term in (2.2.1), even the leading term (explicit values of (l_j, m_j) in Theorem 2.2.2 can not be obtained). To discuss the leading term of (2.2.1), we define *oscillation index* $\xi(f)$ and its *multiplicity* $\eta(f)$ of $I_\varphi(t; f)$ as follows.

$$\xi(f) := \min\{\alpha : C_{\alpha,k} \neq 0 \text{ for some } k, \varphi\},$$
$$\eta(f) := \max\{k : C_{\xi(f),k} \neq 0 \text{ for some } \varphi\}.$$

Newton polyhedron has only information of multi-index of the Taylor series of of f, further condition, concerning with the coefficients of the Taylor series, us needed. The following condition is very crucial in the theory of Varchenko.

Definition 2.2.3. Let $K = \mathbb{R}$ or $\mathbb{C}$. f is *nondegenerate* over K with respect to the Newton polyhedron $\Gamma_+(f)$ if for every compact face γ of $\Gamma_+(f)$, f_γ satisfies

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \cdots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, ..., 0) \text{ on the set } (K \setminus \{0\})^n.$$

This nondegeneracy condition is introduced by Kouchinirenko in [20].

Remark 2.2.4. The above nondegeneracy condition depends on K. For example, let us consider a function $f(x_1, x_2) = (x_1^2 + x_2^2)^2$. Then, $\Gamma_+(f)$ has only one compact face γ and $f_\gamma = f$. The gradient of f_γ is $(\nabla f_\gamma)(x) = (4x_1(x_1^2 + x_2^2), 4x_2(x_1^2 + x_2^2))$ and $(\nabla f_\gamma)(x) = (0, 0)$ is equivalent to $x_1 = ix_2$. This implies f is nondegenerate over $\mathbb{R}$ but degenerate over $\mathbb{C}$ with respect to $\Gamma_+(f)$.

Theorem 2.2.5 (Varchenko, [34])**.** *Suppose that f is nondegenerate over $\mathbb{R}$ with respect to $\Gamma_+(f)$. Let d_f, m_f and γ_* be as in Remark 1.2.8 then the followings hold:*

(i) the arithmetic progression $\{\alpha\}$ appearing in Theorem 2.2.1 is obtained from geometrical informations of $\Gamma_+(f)$.

(ii) There exists a positive constant $C(\varphi)$ satisfying that

$$|I_\varphi(t;f)| \leq C(\varphi)t^{-1/d_f}(\log t)^{m_f-1} \quad (t \geq 2).$$

In particular, $-\xi(f) \leq -1/d_f$.

(iii) Assume that φ satisfies $\varphi(0) \neq 0$. If at least one of the following conditions is satisfied:

(a) $d_f > 1$;

(b) f is nonnegative or nonpositive on U;

(c) $1/d_f$ is not an odd integer and f_{γ_*} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$,

then $\xi(f) = 1/d_f$ and $\eta(f) = m_f$.

These results are obtained by investigating the properties of the following integral

$$Z_{\mathbb{R}}(s;F,\varphi) = \int_{\mathbb{R}^n} |f(x)|^s \varphi(x)dx \quad (s \in \mathbb{C}),$$

where f, φ are the same of $I_\varphi(t;f)$. From the convergence of the integral, this integral defines a holomorphic function on the right half plane $\{\mathrm{Re}(s) > 0\}$, which is called local zeta function. In analysis of this function, resolution of singularities of f are used to know the property of its analytic continuation. In fact, $Z_{\mathbb{R}}(s;F,\varphi)$ is analytically continued to the whole complex plane as a meromorphic function and its poles appear in the order of each term in the asymptotic expansion (2.2.1). An exact relationship between $I_\varphi(t;f)$ and $Z_{\mathbb{R}}(s;F,\varphi)$ is written in [16], [35] or Chapter 7 in this paper. However, as we mentioned before, resolution of singularities cannot be given explicitly for general analytic function f. The work of Varchenko is essentially to give a method of constructing a quantitative resolution of singularities under the nondegeneracy condition in Definition 2.2.3.

Remark 2.2.6. There is an oscillatory integrals with complex phase. In this case, $I_\varphi(\tau;f)$ is defined as follows

$$I_\varphi(\tau;f) := \int_\Gamma e^{\tau f(z)}\varphi(z)dz \quad (\tau \in \mathbb{R}), \tag{2.2.4}$$

where functions f and φ are holomorphic functions defined on an open neighborhood U of a critical point of f and Γ is an n-dimensional chain lying on U. Then, when τ tends to infinity, it is shown in [25] that $I_\varphi(\tau;f)$ admits the following asymptotic expansion

$$I_\varphi(\tau;f) \sim \sum_\alpha \sum_{k=1}^{n} C_{\alpha,k}\tau^\alpha(\log\tau)^{k-1}, \tag{2.2.5}$$

where α runs through finitely many arithmetic expressions consisting of negative rational numbers. Furthermore, the components (α, k) in (2.2.5) are connected to the eigenvalues of (classical) monodromy operator of f at its critical point. Malgrange shows in [25] that for each α, $\exp(-2\pi\alpha)$ is an eigenvalue of monodromy of f and if $C_{\alpha,k} \neq 0$, the size of the Jordan block of $\exp(-2\pi\alpha)$ is not smaller than k.

2.3 Known result of rational case

We recall a part of the work of W.Veys and W.A.Zúñiga-Galindo in [35] for the case of rational functions. By using a simultaneous resolution of singularities, they determine the forms of two asymptotic expansions of $I_\varphi(t; f, g)$ as its parameter tends to zero and infinity. In this section, we assume that

- f, φ and U are same as in section 2.2.
- g is a real analytic function defined on U and satisfies that $g(0) = |(\nabla g)(0)| = 0$. Moreover, g has no critical point apart from the origin on U.

Theorem 2.3.1 ([35]). *Let m_λ be the order of a pole λ of $Z_{\mathbb{R}}(s; f, g, \varphi)$ as in* (i-7), *then we have the following:*

(i) $I_\varphi(t; f, g)$ has an asymptotic expansion as $t \to \infty$ of the form:

$$I_\varphi(t; f, g) \sim \sum_{\alpha} \sum_{k=1}^{m_\alpha} C_{\alpha,k} t^{-\alpha} (\log t)^{k-1}, \tag{2.3.1}$$

where $-\alpha$ runs through all negative poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$.

(ii) $I_\varphi(t; f, g)$ has an asymptotic expansion as $t \to 0$ of the form:

$$I_\varphi(t; f, g) - C \sim \sum_{\beta \notin \mathbb{Z}} \sum_{k=1}^{m_\beta} D_{\beta,k} t^{\beta} (\log t)^{k-1} + \sum_{\lambda \in \mathbb{N}} \sum_{k=1}^{m_\lambda+1} D_{\lambda,k} t^{\lambda} (\log t)^{k-1}, \tag{2.3.2}$$

where $C = \int_{\mathbb{R}^n} \varphi(x)dx$ is a constant and β runs through all positive poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$. If λ is not a pole of $Z_{\mathbb{R}}(s; f, g, \varphi)$, we put $m_\lambda = 0$.

Remark 2.3.2. In [35], the authors investigate the meromorphic case (i.e., f, g are holomorphic functions and the integral is considered on $\mathbb{C}$) and non-archimedean case (i.e., the integrals on p-adic local fields), and obtain similar results.

As is the result of analytic case, from the above theorem, we cannot know exact order of each term of asymptotic expansions. Our main results enable us to determine orders of leading terms in (2.3.1), (2.3.2) in an analogous way to the result of Varchenko.

Chapter 3

Main results

Let us state our main results relating to the estimate and determination of oscillatory indices $\xi_\infty(f,g)$ and $\xi_0(f,g)$. Our main results extends Theorem 2.2.5 to rational phase case. Let U be a small open neighborhood of the origin in $\mathbb{R}^n$. In this section, we assume that

- f and g are real analytic functions defined on U, which satisfy that $f(0) = |(\nabla f)(0)| = 0$ and $g(0) = |(\nabla g)(0)| = 0$. In other words, f and g have a critical point at the origin.
- f and g have no critical point apart from the origin on U.
- φ is a smooth function whose support is contained in U.

First, we give an estimate for $I_\varphi(t; f, g)$ when its parameter is sufficiently large.

Theorem 3.0.1. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron. If the support of φ is contained in a sufficiently small neighborhood of the origin, then*

$$|I_\varphi(t; f, g)| \leq C(\varphi)\, t^{-1/d_\infty(f,g)} (\log t)^{m_\infty(f,g)-1} \quad (t \geq 2).$$

In particular, we have $-\xi_\infty(f,g) \leq -1/d_\infty(f,g)$.

We shall give some conditions where the oscillation index $\xi_\infty(f,g)$ and its multiplicity $\eta_\infty(f,g)$ are determined by means of Newton data with respect to the pair (f,g).

Theorem 3.0.2. *Suppose that f, g, φ satisfy the conditions in Theorem 3.0.1 and φ satisfies $\varphi(0) \neq 0$. Moreover, at least one of the following three conditions is satisfied:*

(a) $d_\infty(f,g) > 1$*;*

(b) f *is nonnegative or nonpositive on* U*;*

(c) $1/d_\infty(f,g)$ *is not an odd integer and* f_{γ_∞} *does not vanish on* $U \cap (\mathbb{R} \setminus \{0\})^n$ *for some principal face at infinity* $\gamma_\infty \in \mathcal{F}_\infty[f]$*,*

then $\xi_\infty(f,g) = 1/d_\infty(f,g)$ *and* $\eta_\infty(f,g) = m_\infty(f,g)$*.*

Next, let us consider the case when t tends to zero. From the second part of the asymptotic expansion (2.3.2), we see that $I_\varphi(t; f, g)$ has a smooth part near the origin, which correspond to the terms of $\lambda \in \mathbb{N}$ and $k = 0$. In order to clarify an influence of the singularity of phase function, we decompose $I_\varphi(t; f, g)$ into two part, regular part and singular part. By Borel's theorem, there exists a C^∞ function ψ, small $\delta > 0$ and a positive constant C not depending on t satisfying that $|\psi(t) - \psi_N(t)| \leq Ct^{N+\delta}$ for any N, where ψ_N is as in (i-5).

Definition 3.0.3. We call $\psi(t)$ the *regular part* of $I_\varphi(t; f, g)$ and $S_\varphi(t; f, g) := I_\varphi(t; f, g) - \psi(t)$ the *singular part* of $I_\varphi(t; f, g)$.

Note that the regular part $\psi(t)$ cannot be uniquely determined since $\psi(t)+$ (flat function) has same Taylor series. From (2.3.2) and the above definition, it is easy to see that the singular part $S_\varphi(t; f, g)$ has the following asymptotic expansion

$$S_\varphi(t; f, g) \sim \sum_\beta \sum_{k=1}^{m_\beta} b_{\beta,k} t^\beta (\log t)^{k-1} \quad \text{as } t \to 0,$$

where β, m_β are the same as in Theorem 2.3.1. Our main results are concerned with the leading term of this asymptotic expansion. At first, we give an estimate for $S_\varphi(t; f, g)$ for small parameter.

Theorem 3.0.4. *Suppose that* f, g, φ *satisfy the conditions in Theorem 3.0.1, then*

$$|S_\varphi(t; f, g)| \leq D(\varphi)\, t^{1/d_0(f,g)} |\log t|^{m_0(f,g)-1} \quad (0 \leq t \leq 1/2).$$

In particular, we have $\xi_0(f,g) \geq 1/d_0(f,g)$*.*

From the above theorem, we can see the next corollary relating to the regularity of $I_\varphi(t; f, g)$ at zero.

Corollary 3.0.5. *Suppose that f, g, φ satisfy the conditions in Theorem 3.0.1 and let k be nonnegative integer with $k < 1/d_0(f,g) \leq k+1$. Then $I_\varphi(t; f, g)$ is C^k function at 0.*

Finally, we shall give some conditions where $\xi_0(f,g)$ and $\eta_0(f,g)$ are determined by means of Newton data with respect to the pair (f,g).

Theorem 3.0.6. *Suppose that f, g, φ satisfy the conditions in Theorem 3.0.1 and φ satisfies $\varphi(0) \neq 0$. Moreover,*

(i) if $1/d_0(f,g)$ is <u>not an integer</u> and at least one of the following three conditions is satisfied:

(a) $d_0(f,g) > 1$;

(b) g is nonnegative or nonpositive on U;

(c) g_{τ_0} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$ for some principal face at zero $\tau_0 \in \mathcal{F}_0[g]$,

then $\xi_0(f,g) = 1/d_0(f,g)$ and $\eta_0(f,g) = m_0(f,g)$.

(ii) If $1/d_0(f,g)$ is <u>an integer</u> and at least one of the following two conditions is satisfied:

(d) g is nonnegative or nonpositive on U;

(e) $1/d_0(f,g)$ is an even integer and g_{τ_0} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$ for some principal face at zero $\tau_0 \in \mathcal{F}_0[g]$,

then $\xi_0(f,g) = 1/d_0(f,g)$ and $\eta_0(f,g) = m_0(f,g) + 1$.

Remark 3.0.7. (i) From the proof of the above theorems, we can see that under the same assumptions, the sets of arithmetic progressions $\{-\alpha\}$ and $\{\beta\}$ in the asymptotic expansions (i-4), (i-5) belong to the following sets:

$$\{-\alpha\} \subset \bigcup_{a \in V_+} \mathcal{P}_{\mathbb{R}}(a) \cup (-\mathbb{N}), \tag{3.0.1}$$

$$\{\beta\} \subset \bigcup_{a \in V_-} \mathcal{P}_{\mathbb{R}}(a) \cup \mathbb{N}, \tag{3.0.2}$$

where $V_\pm \subset \mathbb{Z}_+^n$ are finite sets of vectors as in (5.1.5) and $\mathcal{P}_{\mathbb{R}}(a)$ is arithmetic progression depend on $a \in \mathbb{Z}_+^n$ defined in (6.0.2).

(ii) It is known [16] that $-1/d_\infty(f,g)$ is the maximum element of the first set in (3.0.1) and $1/d_0(f,g)$ is the minimum element of the first set in (3.0.2).

Remark 3.0.8. Let us consider the case when $d_\infty(f,g) = 0$. In this case, Theorem 3.0.1 implies for any $N \in \mathbb{N}$, the following inequation holds

$$|I_\varphi(t; f, g)| \leq C(\varphi) t^{-N} \quad (t \geq 2).$$

This means $I_\varphi(t; f, g)$ is rapidly decreasing as $t \to \infty$ and the singularity of f does not appear in this asymptotic expansion. Similarly, if $d_0(f,g) = 0$, for any $N \in \mathbb{N}$, we have

$$|S_\varphi(t; f, g)| \leq D(\varphi) t^{N} \quad (t \leq 1/2)$$

from Theorem 3.0.6. This means $I_\varphi(t; f, g)$ is smooth at the origin and the singularity of g does not appear.

Remark 3.0.9. The limits

$$\lim_{t \to \infty} t^{1/d_\infty(f,g)} (\log t)^{-m_\infty(f,g)+1}\, I_\varphi(t; f, g)$$

and

$$\lim_{t \to 0} t^{-1/d_0(f,g)} (\log t)^{-m_0(f,g)+1}\, I_\varphi(t; f, g) \quad (1/d_0(f,g) \notin \mathbb{Z}_{>0}),$$
$$\lim_{t \to 0} t^{-1/d_0(f,g)} (\log t)^{-m_0(f,g)}\, I_\varphi(t; f, g) \quad (1/d_0(f,g) \in \mathbb{Z}_{>0})$$

are explicitly computed in Chapter 7. The conditions in Theorems 3.0.2, 3.0.6 are sufficient conditions where such limits do not vanish.

Remark 3.0.10. It is needless to say that the behavior of $I_\varphi(t; f, g)$ is independent of the exchanges of the integral variables. Therefore, if there exists a coordinate in which f and g satisfy the assumptions in each theorems, then the respected assertion holds.

Chapter 4

Toric resolution

In analysis of Varchenko in [34], the theory of toric varieties plays an important role. In this section, we recall the fundamental terminologies and the method to construct a toric variety from a given fan.

4.1 Cones and fans

Definition 4.1.1. A *rational polyhedral cone* $\sigma \subset \mathbb{R}^n$ is a cone generated by finitely many elements of $\mathbb{Z}^n$, i.e., there exist $u_1, ..., u_k \in \mathbb{Z}^n$ such that

$$\sigma = \{\lambda_1 u_1 + \cdots + \lambda_k u_k \in \mathbb{R}^n : \lambda_1, ..., \lambda_k \geq 0\}. \tag{4.1.1}$$

Furthermore, if the vectors $\{u_1, ..., u_k\}$ in (4.1.1) are linearly independent and primitive integer vectors, i.e., the greatest common divisor of each component of u_j is equal to 1, the set $\{u_1, ..., u_k\}$ is called the *skeleton* of σ.

We say that σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. By regarding a cone as a polyhedron in $\mathbb{R}^n$, the definitions of *dimension, face, edge, facet* for the cone are given in the same way as in Definition 1.1.5. It is clear that the skeleton of σ generates σ itself and that the number of the elements of skeleton of k-dimensional cone is not less than k.

Definition 4.1.2. Σ is a *fan* if Σ is a finite collection of cones in $\mathbb{R}^n$ with the following properties:

(i) Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone;

(ii) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$;

(iii) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

For a fan Σ, the *support* of Σ is defined by

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma.$$

For $k = 1, \ldots, n$, we denote by $\Sigma^{(k)}$ the set of k-dimensional cones in Σ. For fans $\Sigma_1, ..., \Sigma_m$, we define $\Sigma := \{\sigma_1 \cap \cdots \cap \sigma_m : \sigma_j \in \Sigma_j\}$. Then Σ is also a fan.

Definition 4.1.3. Let Σ be a fan in $\mathbb{R}^n_+$. The fan $\hat{\Sigma}$ is called *simplicial subdivision* of Σ if $\hat{\Sigma}$ satisfies the following properties:

(i) The fans Σ and $\hat{\Sigma}$ have the same support;

(ii) Each cones of $\hat{\Sigma}$ lies in some cone of Σ;

(iii) The skeleton of any cone belonging to $\hat{\Sigma}$ can be completed to a base of the lattice dual to $\mathbb{Z}^n$.

Note that for k-dimensional cone $\sigma \in \hat{\Sigma}^{(k)}$, the number of the elements of its skeleton is equal to k.

Remark 4.1.4. It is known [19] that for an arbitrary fan Σ, we can find a simplicial subdivision of Σ by a unimodular triangulation. In fact, the simplicial subdivision of $\hat{\Sigma}$ of Σ is not uniquely determined. There can exist infinitely many simplicial subdivisions for one fan.

4.2 Fan associated with polyhedra

Let P be an n-dimensional polyhedron satisfying that $P + \mathbb{R}^n_+ \subset P \subset \mathbb{R}^n_+$. We explain a method to construct a fan from polyhedron P. We denote by $(\mathbb{R}^n)^\vee$ the dual space of $\mathbb{R}^n$ with respect to the standard inner product. For $a = (a_1, ..., a_n) \in (\mathbb{R}^n)^\vee$ with $a_j \geq 0$, we define

$$l(a) = \min\{\langle a, \alpha \rangle : \alpha \in P\},$$
$$\gamma(a) = \{\alpha \in P : \langle a, \alpha \rangle = l(a)\}(= H(a, l(a)) \cap P).$$

We define a relation $\sim$ in $(\mathbb{R}^n)^\vee$ by $a \sim a' \Leftrightarrow \gamma(a) = \gamma(a')$. Then, we immediately see that the relation "$\sim$" is an equivalence relation and for any face γ of P, there is an equivalence class $\gamma^\vee$ which is defined by

$$\begin{aligned}\gamma^\vee &:= \{a \in (\mathbb{R}^n)^\vee : \gamma(a) = \gamma \text{ and } a_j \geq 0 \text{ for } j = 1, ..., n\} \\ (&= \{a \in (\mathbb{R}^n)^\vee : \gamma = H(a, l(a)) \cap P \text{ and } a_j \geq 0 \text{ for } j = 1, ..., n\}).\end{aligned} \tag{4.2.1}$$

Here $P^\vee = \{0\}$. The closure of $\gamma^\vee$, denoted by $\overline{\gamma^\vee}$, is expressed as

$$\overline{\gamma^\vee} = \{a \in (\mathbb{R}^n)^\vee : \gamma \subset H(a, l(a)) \cap P \text{ and } a_j \geq 0 \text{ for } j = 1, ..., n\}. \tag{4.2.2}$$

Proposition 4.2.1. *Let γ be a k-dimensional face of P expressed as*

$$\gamma = \bigcap_{j=1}^{m} H(a^j, l(a^j)) \cap P$$

for $\{a^1, ..., a^m\} \subset \mathbb{Z}_+^n$. Then, $\overline{\gamma^\vee}$ is an $(n-k)$-dimensional strongly convex rational polyhedral cone in $(\mathbb{R}^n)^\vee$ and the set $\{a^1, ..., a^m\}$ is its skeleton.

Moreover, the collection of $\overline{\gamma^\vee}$ for all faces of P gives a fan Σ_P, which is called the fan associated with the polyhedron P.

Note that $|\Sigma_P| = \mathbb{R}_+^n$. Furthermore, let us consider n-dimensional polyhedra $P_1, ..., P_m \subset \mathbb{R}_+^n$ satisfying $P_j + \mathbb{R}_+^n \subset P_j \subset \mathbb{R}_+^n$. Let Σ_{P_j} be the fan associated with P_j. It is known that the collection $\sigma_1 \cap \cdots \cap \sigma_m$ for all $\sigma_j \in \Sigma_{P_j}$ gives a fan, which is called the *fan associated with the polyhedra* $P_1, ..., P_m$. We remark that any simplicial subdivision of this fan is also a simplicial subdivision of Σ_{P_j} for each j.

In order to make the relationship between a face of P and an n-dimensional cone in $\hat{\Sigma}$ more understandable, we introduce the following two maps.

Let $\hat{\Sigma}$ be a simplicial subdivision of Σ and $a^1(\sigma), ..., a^n(\sigma)$ be the skeleton of $\sigma \in \hat{\Sigma}^{(n)}$. Two maps

$$\gamma : \mathcal{P}(\{1, ..., n\}) \times \hat{\Sigma}^{(n)} \to \mathcal{F}[P], \qquad I : \mathcal{F}[P] \times \hat{\Sigma}^{(n)} \to \mathcal{P}(\{1, ..., n\}) \tag{4.2.3}$$

are defined as

$$\gamma(I, \sigma) := \bigcap_{j \in I} H(a^j(\sigma), l(a^j(\sigma))) \cap P,$$

$$I(\gamma, \sigma) := \{j : \gamma \subset H(a^j(\sigma), l(a^j(\sigma))). \tag{4.2.4}$$

If $I = \emptyset$, we define $\gamma(\emptyset, \sigma) := P$. Note that $I(P, \sigma) = \emptyset$.

Lemma 4.2.2. *For $\sigma \in \hat{\Sigma}^{(n)}$, $\gamma \in \mathcal{F}[P]$ and $I \in \mathcal{P}(\{1,...,n\})$, we have the following.*

(i) $\gamma \subset \gamma(I(\gamma,\sigma),\sigma)$ *and* $\dim(\gamma) \leq n - \#I(\gamma,\sigma)$.

(ii) $\gamma = \gamma(I,\sigma) \Rightarrow I \subset I(\gamma,\sigma) \Rightarrow \dim(\gamma) \leq n - \#I$.

Proof. (i) By the definitions of $\gamma(I,\sigma)$ and $I(\gamma,\sigma)$, we have

$$\gamma(I(\gamma,\sigma),\sigma) = \bigcap_{j \in I(\gamma,\sigma)} H(a^j(\sigma), l(a^j(\sigma))) \cap P \supset \gamma.$$

From the above relation, one can find that $\dim(\gamma) \leq \dim(\bigcap_{j \in I(\gamma,\sigma)} H(a^j(\sigma), l(a^j(\sigma)))) = n - \#I(\gamma,\sigma)$.

(ii) The first implication is shown as follows:

$$\begin{aligned}\gamma = \gamma(I,\sigma) \Rightarrow \gamma = \bigcap_{j \in I} H(a^j(\sigma), l(a^j(\sigma))) \cap P \\ \Rightarrow \gamma \subset H(a^j(\sigma), l(a^j(\sigma))) \text{ for all } j \in I \Rightarrow I \subset I(\gamma,\sigma).\end{aligned}$$

From the inequation in (i) and $\#I \leq \#I(\gamma,\sigma)$, the second implication in (ii) is obvious. □

Next, let us consider the case when $\dim(\gamma) = n - \#I(\gamma,\sigma)$. For a face γ of P, we define a set of cones as

$$\hat{\Sigma}^{(n)}(\gamma) := \{\sigma \in \hat{\Sigma}^{(n)} : \dim(\gamma) = n - \#I(\gamma,\sigma)\}. \tag{4.2.5}$$

Lemma 4.2.3. *For $\sigma \in \hat{\Sigma}^{(n)}$, $\gamma \in \mathcal{F}[P]$ and $I \in \mathcal{P}(\{1,...,n\})$, we have the following.*

(i) $\#I(\gamma,\sigma) = \dim(\gamma^\vee \cap \sigma)$.

(ii) $\hat{\Sigma}^{(n)}(\gamma) = \{\sigma \in \hat{\Sigma}^{(n)} : \dim(\gamma^\vee \cap \sigma) = \dim(\gamma^\vee)\} \neq \emptyset$.

(iii) If $\sigma \in \hat{\Sigma}^{(n)}(\gamma)$, *then* $\gamma = \gamma(I(\gamma,\sigma),\sigma)$.

Proof. (i) For any $j \in I(\gamma,\sigma)$, the face γ is contained in the hyperplane $H(a^j(\sigma), l(a^j(\sigma)))$. From the definition (4.2.2), we have $a^j(\sigma) \in \overline{\gamma^\vee}$ and this implies $a^j(\sigma) \in \overline{\gamma^\vee} \cap \sigma$. So there exists a bijection from $I(\gamma,\sigma)$ to the set of linearly independent vectors $\{a^j(\sigma) : j \in I(\gamma,\sigma)\} \subset \overline{\gamma^\vee} \cap \sigma$.

(ii) From the equation in (i) and $\dim(\gamma^\vee) = n - \dim(\gamma)$, we see that $\dim(\gamma^\vee) = \#I(\gamma,\sigma) = \dim(\gamma^\vee \cap \sigma)$. Since the support of $\hat{\Sigma}$ is $\mathbb{R}^n_+$, there exists σ satisfying that $\dim(\gamma^\vee \cap \sigma) = \dim(\gamma^\vee)$. So $\hat{\Sigma}^{(n)}(\gamma)$ is not empty.

(iii) We have the following inequation,

$$\dim(\gamma) \leq \dim(\gamma(I(\gamma,\sigma),\sigma)) \overset{(1)}{\leq} \dim\left(\bigcap_{j\in I(\gamma,\sigma)} H(a^j(\sigma), l(a^j(\sigma)))\right)$$
$$= n - \#I(\gamma,\sigma) \overset{(2)}{=} \dim(\gamma).$$

The inequality (1) is obtained from the assertion (i) in Lemma 4.2.2 and the equality (2) comes from the assumption. This implies $\dim(\gamma) = \dim(\gamma(I(\gamma,\sigma),\sigma))$. Since $\gamma \subset \gamma(I(\gamma,\sigma),\sigma)$ by Lemma 4.2.2-(i), the above dimensional equation yields $\gamma = \gamma(I(\gamma,\sigma),\sigma)$. □

4.3 Resolution of singularities associated with $\hat{\Sigma}$

Here, we explain the method to construct a toric resolution of singularities for K-analytic functions defined on $K = \mathbb{R}$ or $\mathbb{C}$. It is known that this method is available for the case when K is a local field with characteristic zero, for instance, p-adic field. If you want to know about this and its applications to the non-archimedean local zeta function, see [35].

Let Σ be a fan satisfying $|\Sigma| = \mathbb{R}^n_+$ and $\hat{\Sigma}$ be one of a simplical subdivision of Σ. For each $\sigma \in \hat{\Sigma}^{(n)}$, let $a^1(\sigma), ..., a^n(\sigma)$ be the skeleton of σ. We set the coordinates of $a^j(\sigma)$ as

$$a^j(\sigma) = (a^j_1(\sigma), ..., a^j_n(\sigma)).$$

We denote the copy of K^n by $K^n(\sigma)$, which is associated with a cone σ. We define the map $\pi_K(\sigma) : K^n(\sigma) \to K^n$ as follows: $\pi_K(\sigma)(y_1, ..., y_n) = (x_1, ..., x_n)$ with

$$x_j = \prod_{k=1}^n y_k^{a^k_j(\sigma)} = y_1^{a^1_j(\sigma)} \cdots y_n^{a^n_j(\sigma)}, \quad j = 1, ..., n.$$

Let $Y_{\hat{\Sigma}}$ be the union of $K^n(\sigma)$ for σ which are glued along the image of $\pi_K(\sigma)$. In detail, $Y_{\hat{\Sigma}}$ is a quotient space $\bigsqcup_{\sigma\in\hat{\Sigma}} K^n(\sigma)/\sim$ with the equivalence relation defined by $y = (y_1, ..., y_n) \sim y' = (y'_1, ..., y'_n) \Leftrightarrow \pi_K(\sigma)(y) = \pi_K(\tau)(y')$ for $y \in K^n(\sigma), y' \in K^n(\tau)$. It is known (see [9]) that

- $Y_{\hat{\Sigma}}$ is an n-dimensional algebraic manifold;
- The map $\pi_K : Y_{\hat{\Sigma}} \to K^n$ defined on each coordinate $K^n(\sigma)$ as $\pi_K(\sigma) : K^n(\sigma) \to K^n$ is proper;

- Each $K^n(\sigma)$ is densely embedded in $Y_{\hat{\Sigma}}$.

Remark 4.3.1. The following conditions are equivalent.

- $a^1(\sigma), ..., a^n(\sigma)$ can be completed to a base of the lattice dual to $\mathbb{Z}^n$.
- $\det(a^j_k(\sigma))_{1\le j,k\le n} = \pm 1$.
- The inverse map $\pi_K(\sigma)^{-1}$ is rational.

Definition 4.3.2. The manifold $Y_{\hat{\Sigma}}$ is called the *toric variety* associated with $\hat{\Sigma}$.

Later, we see that the pair $(Y_{\hat{\Sigma}}, \pi_K)$ satisfies the properties in Theorem 2.2.2. The pair $(Y_{\hat{\Sigma}}, \pi_K)$ is called the *K-resolution of singularities associated to* $\hat{\Sigma}$.

The following lemma is useful for the analysis in Chapter 6.

Lemma 4.3.3. *(i) The set of the points in $K^n(\sigma)$ in which $\pi_K(\sigma)$ is not an isomorphism is a union of coordinate hyperplanes.*

(ii) The Jacobian of the mapping $\pi_K(\sigma)$ is

$$J_{\pi_K(\sigma)}(y) = \pm \prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle - 1}.$$

Proof. Here, we prove only (ii). From the definition of the Jacobian and the determinant, we have

$$\begin{aligned}
|J_{\pi_K(\sigma)}(y)| &= \begin{vmatrix} a_1^1(\sigma) y_1^{a_1^1(\sigma)-1} y_2^{a_1^2(\sigma)} \cdots y_n^{a_1^n(\sigma)} & \cdots & a_1^n(\sigma) y_1^{a_1^1(\sigma)} y_2^{a_1^2(\sigma)} \cdots y_n^{a_1^n(\sigma)-1} \\ a_2^1(\sigma) y_1^{a_2^1(\sigma)-1} y_2^{a_2^2(\sigma)} \cdots y_n^{a_2^n(\sigma)} & \cdots & a_2^n(\sigma) y_1^{a_2^1(\sigma)} y_2^{a_2^2(\sigma)} \cdots y_n^{a_2^n(\sigma)-1} \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma) y_1^{a_n^1(\sigma)-1} y_2^{a_n^2(\sigma)} \cdots y_n^{a_n^n(\sigma)} & \cdots & a_n^n(\sigma) y_1^{a_n^1(\sigma)} y_2^{a_n^2(\sigma)} \cdots y_n^{a_n^n(\sigma)-1} \end{vmatrix} \\
&= \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle} \right) \cdot \begin{vmatrix} a_1^1(\sigma) y_1^{-1} & \cdots & a_1^n(\sigma) y_n^{-1} \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma) y_1^{-1} & \cdots & a_n^n(\sigma) y_n^{-1} \end{vmatrix} \\
&= \left(\prod_{j=1}^{n} y_j^{\langle a^j(\sigma)\rangle - 1} \right) \cdot \begin{vmatrix} a_1^1(\sigma) & \cdots & a_1^n(\sigma) \\ \vdots & \ddots & \vdots \\ a_n^1(\sigma) & \cdots & a_n^n(\sigma) \end{vmatrix}.
\end{aligned} \tag{4.3.1}$$

The determinant of the last matrix in (4.3.1) is equal to 1 or -1, because of the property of the simplicial subdivision (the property (iii) in Definition 4.1.3 and Remark 4.3.1). $\square$

Lemma 4.3.4. *If $\gamma = \gamma(I, \sigma)$ as in* (4.2.3) *and $\{a^1(\sigma), ..., a^n(\sigma)\}$ is a skeleton of σ, then f_γ satisfies the following equation on $\{x \in K^n : f_\gamma(x) = 0\}$:*

$$a_1^j(\sigma)x_1 \frac{\partial f_\gamma}{\partial x_1} + \cdots + a_n^j(\sigma)x_n \frac{\partial f_\gamma}{\partial x_n} = 0 \quad \textit{for } j \in I.$$

Proof. From the form of f_γ in (1.1.2), we see that f_γ is a quasi-homogeneous polynomial with a weight $\{a_1^j(\sigma), ..., a_n^j(\sigma)\}$ for each $j \in I$. Since $\langle a^j, \alpha \rangle = l_f(a^j(\sigma))$ for $\alpha \in \gamma$, we have

$$f_\gamma(t^{a_1^j(\sigma)}x_1, ..., t^{a_n^j(\sigma)}x_n) = t^{l_f(a^j(\sigma))} f_\gamma(x)$$

with a parameter t. Differentiating both side by t, we have

$$\sum_{k=1}^n a_k^j(\sigma) t^{a_k^j(\sigma)-1} x_k \frac{\partial f_\gamma}{\partial x_k}(t, x) = l_f(a^j(\sigma)) t^{l_f(a^j(\sigma))-1} f_\gamma(x). \tag{4.3.2}$$

Substituting $t = 1$, $f_\gamma(x) = 0$ to (4.3.2), we have the desired equation. □

The next lemma is concerned with the property of the map $\pi_K(\sigma)$ when the face $\gamma(I, \sigma)$ is compact.

Lemma 4.3.5. *If $\gamma = \gamma(I, \sigma)$, the following conditions are equivalent.*

(i) *γ is compact.*

(ii) $\pi_K(\sigma)(T_I(K^n)) = 0$.

Proof. See the proof of Proposition 8.6 in [18]. □

4.4 Resolution of singularities with respect to two functions

Let Σ_f, Σ_g be fans associated to the polyhedron $\Gamma_+(f)$, $\Gamma_+(g)$ and $\Sigma := \{\sigma_1 \cap \sigma_2 : \sigma_1 \in \Sigma_f, \sigma_2 \in \Sigma_g\}$. In this section, let us recall some lemmas on the simultaneous resolution of singularities for two functions introduced in [18]. Hereafter, we use the symbol $l_h(a)$ for $a \in \mathbb{R}_+^n$ and K-analytic function h defined near the origin in K^n defined as

$$l_h(a) := \min\{\langle a, \alpha \rangle : \alpha \in \Gamma_+(h)\}. \tag{4.4.1}$$

and for $I \subset \{1, ..., n\}$

$$T_I(K^n) := \{y \in K^n : y_j = 0 \text{ if } j \in I\},$$
$$T_I^1(K^n) := \{y \in K^n : y_j = 1 \text{ if } j \in I\}.$$

Lemma 4.4.1 (Proposition 7.4. in [18]). *Let $\hat{\Sigma}$ be a simplicial subdivision of the fan Σ and $\sigma \in \hat{\Sigma}^{(n)}$ be an n-dimensional cone whose skeleton is $a^1(\sigma), ..., a^n(\sigma) \in \mathbb{Z}_+^n$. Then, there exists K-analytic functions f_σ, g_σ defined on the set $\pi_K(\sigma)^{-1}(U)$ such that $f_\sigma(0) \cdot g_\sigma(0) \neq 0$ and*

$$\begin{aligned} f(\pi_K(\sigma)(y)) &= \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma))}\right) \cdot f_\sigma(y), \\ g(\pi_K(\sigma)(y)) &= \left(\prod_{j=1}^n y_j^{l_g(a^j(\sigma))}\right) \cdot g_\sigma(y) \end{aligned} \tag{4.4.2}$$

for $y \in \pi_K(\sigma)^{-1}(U)$.

Proof. Here, we treat only the case of f. Substituting $\pi_K(\sigma)$ into the Taylor series of f, we have

$$\begin{aligned} f(\pi_K(\sigma)(y)) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \left(\prod_{k=1}^n y_1^{a_k^1(\sigma)} \cdots y_n^{a_k^n(\sigma)}\right)^\alpha &= \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \prod_{j=1}^n y_j^{a_1^j(\sigma)\alpha_1 + a_2^j(\sigma)\alpha_2 + \cdots + a_n^j(\sigma)\alpha_n} \\ &= \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \prod_{j=1}^n y_j^{\langle a^j(\sigma), \alpha \rangle}. \end{aligned} \tag{4.4.3}$$

Since the multi-index α lies in $\Gamma_+(f)$ and from the definition of $l_f(a)$ in (4.4.1), all the terms in (4.4.3) can be divided by $y_j^{l_f(a^j(\sigma))}$ for $j = 1, ..., n$. Thus, one can find that

$$f(\pi_K(\sigma)(y)) = \prod_{j=1}^n y_j^{l_f(a^j(\sigma))} \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \prod_{j=1}^n y_j^{\langle a^j(\sigma), \alpha \rangle - l_f(a^j(\sigma))} = \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma))}\right) \cdot f_\sigma(y).$$

From the construction of $\hat{\Sigma}$, there exists $\alpha_0 \in \Gamma_+(f) \cap \mathbb{Z}_+^n$ satisfying that $\langle a^j(\sigma), \alpha_0 \rangle - l_f(a^j(\sigma)) = 0$ for any $j = 1, ..., n$ and this implies $f_\sigma(0) = c_{\alpha_0} \neq 0$. □

For $h = f, g$, we define a map $\gamma_h : \mathcal{P}(\{1, ..., n\}) \times \hat{\Sigma}^{(n)} \to \mathcal{F}[h]$ as follows:

$$\gamma_h(I, \sigma) := \bigcap_{j \in I} H(a^j(\sigma), l_h(a^j(\sigma))) \cap \Gamma_+(h).$$

Lemma 4.4.2. *Let f_σ and g_σ be as in Lemma 4.4.1. If $\gamma_1 = \gamma_f(I,\sigma)$, $\gamma_2 = \gamma_g(I,\sigma)$, then we have*

$$\begin{aligned} f_{\gamma_1}(\pi(\sigma)(T_I^1(y))) &= \left(\prod_{j\notin I} y_j^{l_f(a^j(\sigma))}\right) \cdot f_\sigma(T_I(y)), \\ g_{\gamma_2}(\pi(\sigma)(T_I^1(y))) &= \left(\prod_{j\notin I} y_j^{l_g(a^j(\sigma))}\right) \cdot g_\sigma(T_I(y)). \end{aligned} \tag{4.4.4}$$

Proof. By substituting $y = T_I^1(y)$ to (4.4.2) and the relation $f_\sigma(T_I^1(y)) = f_\sigma(T_I(y))$, we have (4.4.4). □

Note that $\pi_K(\sigma)(T_I^1(y)) \in (K \setminus \{0\})^n$ if $y \in T_I(K^n)$. From Lemma 4.4.1, we see that f, g can be expressed as normal crossing form near the origin of each $K^n(\sigma)$. Then, in order to complete the resolution, we have to consider the zero set of f_σ, g_σ at each coordinate axis far from the origin. For this, we need Newton nondegeneracy condition appeared in Chapter 2.

Proposition 4.4.3. *If f and g are nondegenerate over K with respect to their Newton polyheda and a set $I \subset \{1,\ldots,n\}$ satisfies $\pi_K(\sigma)(T_I(K^n)) = 0$, then the sets $\{y \in T_I(K^n) : f_\sigma(y) = 0\}$ and $\{y \in T_I(K^n) : g_\sigma(y) = 0\}$ are nonsingular, that is, the restriction of the gradient of functions f_σ and g_σ to $T_I(K^n)$ does not vanish at the points of the set $\{y \in T_I(K^n) : f_\sigma(y) = 0\}$ and $\{y \in T_I(K^n) : g_\sigma(y) = 0\}$, respectively.*

Proof. We only prove the case of f. Let $\gamma = \gamma_f(I,\sigma)$. Since we consider the coordinate axis far from the origin, we see that $\prod_{j\notin I} y_j^{l_f(a^j(\sigma))} \neq 0$. Thus, the relation (4.4.4) implies

$$\begin{aligned} f_\sigma(T_I(y)) = 0 &\Longleftrightarrow f_\gamma(\pi_K(\sigma)(T_I^1(y))) = 0, \\ \frac{\partial}{\partial y_j} f_\sigma(T_I(y)) = 0 &\Longleftrightarrow \frac{\partial}{\partial y_j} f_\gamma(\pi_K(\sigma)(T_I^1(y))) = 0 \quad \text{for } j \notin I. \end{aligned} \tag{4.4.5}$$

Therefore, it suffices to investigate the zero set of $f_\gamma(\pi_K(\sigma)(T_I^1(y)))$. We denote a coordinate of $\pi_K(\sigma)(T_I^1(y))$ by $(\tilde{y}_1, ..., \tilde{y}_n)$, i.e., $\tilde{y}_k = \prod_{j\notin I} y_j^{a_k^j(\sigma)}$. Then, by chain rule, each partial derivative of $f_\gamma(\tilde{y})$ with respect to y_j for $j \notin I$ is

$$\frac{\partial}{\partial y_j} f_\gamma(\tilde{y}) = \sum_{k=1}^n \frac{\partial}{\partial \tilde{y}_k} f_\gamma(\tilde{y}) \cdot \frac{\partial \tilde{y}_k}{\partial y_j} = \frac{1}{y_j} \sum_{k=1}^n a_k^j(\sigma) \tilde{y}_k \frac{\partial}{\partial \tilde{y}_k} f_\gamma(\tilde{y}) \quad \text{for } j \notin I. \tag{4.4.6}$$

On the other hand, from $f_\gamma(\tilde{y}) = 0$ and Lemma 4.3.4, we have

$$\sum_{k=1}^{n} a_k^j(\sigma)\tilde{y}_k \frac{\partial}{\partial \tilde{y}_k} f_\gamma(\tilde{y}) = 0 \quad \text{for } j \in I. \tag{4.4.7}$$

If the gradient of $f_\sigma(T_I(y))$ vanishes, from (4.4.5), (4.4.6) and (4.4.7) we have

$$\begin{pmatrix} a_1^1(\sigma) & \cdots & a_n^1(\sigma) \\ a_1^2(\sigma) & \cdots & a_n^2(\sigma) \\ \vdots & \ddots & \vdots \\ a_1^n(\sigma) & \cdots & a_n^n(\sigma) \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \frac{\partial}{\partial \tilde{y}_1} f_\gamma(\tilde{y}) \\ \tilde{y}_2 \frac{\partial}{\partial \tilde{y}_2} f_\gamma(\tilde{y}) \\ \vdots \\ \tilde{y}_n \frac{\partial}{\partial \tilde{y}_n} f_\gamma(\tilde{y}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.4.8}$$

Since $n \times n$ matrix $(a_k^j(\sigma))_{1 \le j,k \le n}$ is invertible, (4.4.8) implies all partial derivatives satisfy $\frac{\partial}{\partial \tilde{y}_j} f_\gamma(\tilde{y}) = 0$ for $\tilde{y} \neq 0$. This is contradicted to the nondegeneracy of f. □

Remark 4.4.4. Let $b = (b_1, ..., b_n)$, $c = (c_1, ..., c_n)$ be points on $T_I(K^n)$ satisfying $f_\sigma(b) = 0$, $g_\sigma(b) \neq 0$ and $f_\sigma(c) \neq 0$, $g_\sigma(c) = 0$. Then, from the implicit function theorem, there exist local diffeomorphisms ϕ_b, ϕ_c defined on an each neighborhood of b, c such that

(i) $y = \phi_b(u)$ with $b = \phi_b(b)$ and

$$\left(\frac{f}{g} \circ \pi_K(\sigma) \circ \phi_b\right)(u) = (u_i - b_i)\left(\prod_{j \in I} u_j^{l_f(a^j(\sigma))}\right),$$

where $y_j = u_j$ for $j \in I$ and $i \notin I$.

(ii) $y = \phi_c(v)$ with $c = \phi_c(c)$ and

$$\left(\frac{f}{g} \circ \pi_K(\sigma) \circ \phi_c\right)(v) = \frac{1}{(v_i - c_i)}\left(\prod_{j \in I} v_j^{l_f(a^j(\sigma))}\right),$$

where $y_j = v_j$ for $j \in I$ and $i \notin I$.

Mixing Lemma 4.4.2, Proposition 4.4.3 and Remark 4.4.4, we see that the pair $(Y_{\hat{\Sigma}}, \pi_K)$ satisfy the properties in Theorem 2.2.2 for both f and g. We call the pair $(Y_{\hat{\Sigma}}, \pi_K)$ *simultaneous resolution of singularities* with respect to f and g.

Lemma 4.4.5 (Lemma 7.5. in [18]). *Let $a = (a_1, ..., a_n) \in \mathbb{R}_+^n$ and $F := f \cdot g$. Let $\gamma_h(a) := H(a, l_h(a)) \cap \Gamma_+(h)$ for $h = F, f, g$. Then we have* *(i)* $F_{\gamma_F(a)}(x) = f_{\gamma_f(a)}(x) \cdot g_{\gamma_g(a)}(x)$.

(ii) $l_F(a) = l_g(a) + l_f(a)$.

Proof. Let $h = F, f, g$ have a Taylor series at the origin of the form

$$h(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_h(\alpha) x^\alpha.$$

For a positive t, we have

$$h(t^{a_1}x_1, ..., t^{a_n}x_n) = \sum_{\alpha \in \mathbb{Z}_+^n} c_h(\alpha) t^{\langle a, \alpha \rangle} x^\alpha = t^{l_h(a)} h_t(x),$$

where $h_t(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_h(\alpha) t^{\langle a,\alpha\rangle - l_h(a)} x^\alpha$. It is easy to see that $h_t(x)$ satisfies $h_0(x) = h_{\gamma_h(a)}(x)$. We have the following equation;

$$t^{l_F(a)} F_t(x) = t^{l_f(a)+l_g(a)} f_t(x) g_t(x) \Longleftrightarrow F_t(x) = t^{l_f(a)+l_g(a)-l_F(a)} f_t(x) g_t(x). \tag{4.4.9}$$

Considering the limit as $t \to 0$ in (4.4.9) and be careful to the fact that $F_0(x) \not\equiv 0$, the assertions (i),(ii) are shown. □

Lemma 4.4.6 (Lemma 7.8. in [18]). *If $f \cdot g$ is nondegenerate over K with respect to its Newton polyhedoron, then so is each f and g.*

Proof. Suppose that f is not nondegenerate. Then there exists a compact face γ of $\Gamma_+(f)$ and a point $x_* \in (K \setminus \{0\})^n$ such that $\frac{\partial f_\gamma}{\partial x_j}(x_*) = 0$ for $j = 1, ..., n$. Let $a \in \mathbb{R}_{>0}$ satisfy that $(a, l_f(a))$ is a valid pair defining γ. Moreover, from Lemma 4.3.4, $\sum_{j=1}^n a_j x_j \frac{\partial f_\gamma}{\partial x_j}(x) = f_\gamma(x)$, we have $f_\gamma(x_*) = 0$. Since $F_\gamma(x) = f_{\gamma_f(a)}(x) \cdot g_{\gamma_g(a)}(x)$ from Lemma 4.4.5-(i), we have

$$\frac{\partial F_{\gamma(a)}}{\partial x_j}(x_*) = \frac{\partial f_{\gamma(a)}}{\partial x_j}(x_*) \cdot g_{\gamma_g(a)}(x_*) + f_{\gamma(a)}(x_*) \cdot \frac{\partial g_{\gamma_g(a)}}{\partial x_j}(x_*) = 0$$

for $j = 1, ..., n$. This is contradicted to the nondegeneracy of F. □

Remark 4.4.7. In general, the converse of the assertion in Lemma 4.4.6 does not hold (for example, $f(x) = x_1(x_1 - x_2)$ and $g(x) = (x_1 - x_2)$). So the assumption of nondegeneracy of $f \cdot g$ is a little stronger to say f_σ and g_σ do not have other singularities. In fact, the nondegeneracy of $f \cdot g$ is necessary to avoid another difficulty (see Lemma 4.4.8).

From Remark 4.4.4, f_σ and g_σ can be expressed normal crossing form at the points on the coordinate axis far from the origin under nondegeneracy condition. However, if f_σ and g_σ vanish at the same point, we might have to consider further resolution. The following lemma states that such situation does not occur under the assumption of nondegeneracy of $f \cdot g$.

Lemma 4.4.8. *Suppose that $F := f \cdot g$ is nondegenerate over K with respect to its Newton polyhedron and for any $\sigma \in \hat{\Sigma}^{(n)}$, a set $I \subset \{1, ..., n\}$ satisfies $\pi_K(\sigma)(T_I(K^n)) = 0$, then $f_\sigma(y)$ and $g_\sigma(y)$ do not vanish simultaneously on $T_I(K^n)$.*

Proof. Assume that $f_\sigma(b) = g_\sigma(b) = 0$ for some $b \in T_I(K^n)$. From Lemma 4.4.2., we obtain

$$f_{\gamma_1}(B) = g_{\gamma_2}(B) = 0,$$

where $B = \pi_K(\sigma)(T_I^1(b)) \in (K \setminus \{0\})^n$ and γ_1, γ_2 are as in Lemma 4.4.2. By Lemma 4.4.5, there exists $\gamma \in \mathcal{F}[F]$ such that $F_\gamma(x) = f_{\gamma_1}(x) \cdot g_{\gamma_2}(x)$. Then, we have

$$\frac{\partial F_\gamma}{\partial x_j}(B) = \frac{\partial f_{\gamma_1}}{\partial x_j}(B) \cdot g_{\gamma_2}(B) + f_{\gamma_1}(B) \cdot \frac{\partial g_{\gamma_2}}{\partial x_j}(B) = 0$$

for $j = 1, ..., n$. This shows that F is not nondegenerate. $\square$

Chapter 5

Some Important Fans

In this chapter, as a preparation for the analysis of local zeta function, we construct some important fans which have appropriate properties reflecting the geometrical relationship between two Newton polyhedra. Furthermore, we define and investigate some fans which are concerned with the coefficients of the leading terms in the asymptotic expansions (i-4), (i-5). Indeed, we will have formulae of such coefficients by using cones belonging to the fans treated here.

5.1 Important sets for candidate poles

5.1.1 Construction of an appropriate fan

In this section, we construct a fan which is appropriate in our analysis of local zeta function in chapter 6. Let f, g be K-analytic functions defined near the origin in K^n.

We define the sets of vectors as

$$\mathcal{V}_{\pm}(f,g) := \{a \in \mathbb{R}_+^n : \pm(l_f(a) - l_g(a)) > 0\},$$

where $l_f(\cdot), l_g(\cdot)$ are as in (4.4.1).

Lemma 5.1.1. *The following conditions are equivalent.*

(i) $d_\infty(f,g) = 0 \iff \Gamma_+(g) \subset \Gamma_+(f) \iff \mathcal{V}_+(f,g) = \emptyset.$

(ii) $d_0(f,g) = 0 \iff \Gamma_+(f) \subset \Gamma_+(g) \iff \mathcal{V}_-(f,g) = \emptyset.$

Proof. We only prove the assertion (i). The first equivalence is obvious from Remark 1.2.2. The second equivalence is proved as follows.

($\Leftarrow$) $\mathcal{V}(f,g) = \emptyset$ implies $l_f(a) - l_g(a) \leq 0$ for all $a \in \mathbb{R}^n_+$. Then, we have a relation

$$H^+(a, l_g(a)) \subset H^+(a, l_f(a)) \qquad \text{for all } a \in \mathbb{R}^n_+. \tag{5.1.1}$$

Since Newton polyhedron is a polyhedron, (5.1.1) implies $\Gamma_+(g) \subset \Gamma_+(f)$.

($\Rightarrow$) Assume that $l_f(a') - l_g(a') > 0$ for some $a' \in \mathbb{R}^n_+$. By the same argument, we have a relation $H^+(a', l_f(a')) \subsetneq H^+(a', l_g(a'))$. Then, $H^+(a', l_g(a')) \cap \Gamma_+(g) =: \tau(a')$ is a face of $\Gamma_+(g)$ and

$$\Gamma_+(g) \setminus \Gamma_+(f) \supset \Gamma_+(g) \setminus H^+(a', l_f(a')) \supset \tau(a') \setminus H^+(a', l_f(a')) = \tau(a').$$

This implies $\Gamma_+(g) \not\subset \Gamma_+(f)$. $\square$

For f, g, we define

$$\Gamma_+(f,g) := \text{the convex hull of the set } \Gamma_+(f) \cup \Gamma_+(g).$$

Note that $\Gamma_+(f,g)$ is a polyhedron. Let $\Sigma_\natural$ be the fan associated with the polyhedron $\Gamma_+(f,g)$. We define a subset of $\Sigma_\natural$ as follows

$$\Sigma_D := \{\sigma \in \Sigma_\natural : l_f(a) = l_g(a) \text{ holds for any } a \in \sigma\}.$$

From this definition, it is easy to see that Σ_D is a subfan of $\Sigma_\natural$.

Lemma 5.1.2. *Every cone $\sigma \in \Sigma_\natural$ satisfies only one of the following conditions:*

(i) $\mathrm{Int}(\sigma) \subset \mathcal{V}_+(f,g)$*;*

(ii) $\mathrm{Int}(\sigma) \subset \mathcal{V}_-(f,g)$*;*

(iii) $\sigma \in \Sigma_D$*.*

Proof. Let σ be a cone in $\Sigma_\natural$ and $a^1(\sigma), \ldots, a^k(\sigma)$ be skeleton of σ. Let γ_σ be a face of $\Gamma_+(f,g)$ defined by

$$\gamma_\sigma = \left(\bigcap_{j=1}^{k} H(a^j(\sigma), l_*(a^j(\sigma))) \right) \cap \Gamma_+(f,g), \tag{5.1.2}$$

where $l_*(a) := \min\{\langle a, \alpha\rangle : \alpha \in \Gamma_+(f,g)\}$. For a polyhedron P, we denote the set of vertices of P by $V(P)$. Since $\Gamma_+(f,g)$ is the convex hull of $\Gamma_+(f) \cup \Gamma_+(g)$, we see that $V(\Gamma_+(f,g)) \subset V(\Gamma_+(f)) \cup V(\Gamma_+(g))$ and from this fact, we have the following classification of γ_σ:

(A) $\gamma_\sigma \cap V(\Gamma_+(f,g)) \subset V(\Gamma_+(g))$;

(B) $\gamma_\sigma \cap V(\Gamma_+(f,g)) \subset V(\Gamma_+(f))$;

(C) $\gamma_\sigma \cap V(\Gamma_+(g)) \cap V(\Gamma_+(f)) \neq \emptyset$.

Then, we will show that (A) $\Rightarrow$ (i), (B) $\Rightarrow$ (ii) and (C) $\Rightarrow$ (iii).

At first, we consider the condition (A). From the definitions of $l_f(\cdot)$ and $l_g(\cdot)$, it suffices to show that $\Gamma_+(f) \subsetneq H^+(a, l_g(a))$ for any $a \in \mathrm{Int}(\sigma)$. The condition (A) implies that the vertices of γ_σ are vertices of $\Gamma_+(g)$ only, then for any $a \in \mathrm{Int}(\sigma)$, we have

$$l_*(a) = l_g(a).$$

The construction from the face of polyhedron to its dual cone as in (4.2.1) implies that γ_σ is expressed as

$$\gamma_\sigma = H(a, l_g(a)) \cap \Gamma_+(f,g).$$

Thus, $(a, l_g(a))$ is valid for $\Gamma_+(f,g)$ and we have following relations:

$$H^+(a, l_g(a)) \supset \Gamma_+(f,g) \supset \Gamma_+(f). \tag{5.1.3}$$

If $H(a, l_g(a)) \cap \Gamma_+(f) \neq \emptyset$, this set is a face of $\Gamma_+(f)$ and from the relation (5.1.3), γ_σ contains this set. Then, γ_σ contains the vertex of $\Gamma_+(f)$ and this is contradiction.

(B) $\Rightarrow$ (ii) is similarly proved.

For the proof of (C) $\Rightarrow$ (iii), since the interior of σ can be shown by the same argument as above, it suffices to consider the boundary of σ. Since the boundary of σ is also a cone in $\Sigma_\natural$, we only have to consider the skeleton of σ. The condition (C) implies there exists $\alpha \in V(\Gamma_+(f))$ and $\beta \in V(\Gamma_+(g))$ satisfying that $\alpha, \beta \in \gamma_\sigma$. Then, from equation (5.1.2), we have

$$\alpha, \beta \in H(a^j(\sigma), l_*(a^j(\sigma))) \quad \text{for any } j$$

and this implies

$$l_*(a^j(\sigma)) = l_f(a^j(\sigma)) = l_g(a^j(\sigma)) \quad \text{for any } j.$$

$\square$

5.1.2 Important fans and sets of vectors for candidate poles

Let $F = f \cdot g$ and Σ_F be a fan associated with $\Gamma_+(F)$. Let Σ be a fan constructed from the fans Σ_F and $\Sigma_\natural$, i.e., $\Sigma := \{\sigma_1 \cap \sigma_2 : \sigma_1 \in \Sigma_F, \sigma_2 \in \Sigma_\natural\}$. Moreover, let $\hat{\Sigma}$ be a simplicial subdivision of Σ, which will be the most important fan in our analysis.

From Lemma 5.1.2, we can define the following three subfans of $\hat{\Sigma}$.

$$\hat{\Sigma}_\pm := \{\tau \in \hat{\Sigma} : \tau \text{ is a face of } \sigma \in \hat{\Sigma}^{(n)} \text{ satisfying } \mathrm{Int}(\sigma) \subset \mathcal{V}_\pm(f,g)\}.$$
$$\hat{\Sigma}_* := \{\tau \in \hat{\Sigma} : \tau \text{ is a face of } \sigma \in \hat{\Sigma}^{(n)} \text{ satisfying } \sigma \subset \Sigma_D\}.$$

It is easy to obtain the following decomposition

$$\hat{\Sigma} = \hat{\Sigma}_+ \cup \hat{\Sigma}_- \cup \hat{\Sigma}_*. \tag{5.1.4}$$

For an n-dimensional cone $\sigma \in \hat{\Sigma}$, we denote by $a^1(\sigma), ..., a^n(\sigma)$ the skeleton of σ. Let $V_\pm$ be the two sets of vectors in $\mathbb{R}^n_+$ defined by

$$V_\pm := \{a^j(\sigma) \in \hat{\Sigma}^{(1)}_\pm \cap \mathcal{V}_\pm(f,g) : \sigma \in \hat{\Sigma}^{(n)}, j = 1, ..., n\}. \tag{5.1.5}$$

From the property of fan associated with polyhedron, we see that $V_\pm \subset \mathbb{Z}^n_+$. These sets of vectors will be used to express the sets of candidate poles of local zeta functions.

5.2 Important fans for leading poles

In this section, we define and consider important fans for computation of the coefficients of the leading poles of $Z_K(s; f, g, \varphi)$.

5.2.1 Newton distances and distance between the two Newton polyhedra

We define the symbol $d(f, g; a)$ as

$$d(f, g; a) := (l_f(a) - l_g(a))/\langle a \rangle,$$

where $l_f(\cdot), l_g(\cdot)$ are as in (4.4.1) and $a \in \mathbb{R}^n_+$. Let us explain the geometrical meaning of $d(f, g; a)$. For $h = f$ or g, we denote by $p(a)$ the point of the intersection of the hyperplane

$H(a, l_h(a))$ with the diagonal line $\{(t, ..., t) \in \mathbb{R}^n_+ : t \geq 0\}$. Then $l_h(a)/\langle a \rangle$ is a value satisfying that $p(a) = (l_h(a)/\langle a \rangle, ..., l_h(a)/\langle a \rangle)$. Since $H(a, l_h(a)) \cap \Gamma_+(h)$ is always a face of $\Gamma_+(h)$, $l_h(a)/\langle a \rangle$ can be regarded as a distance from the origin to $\Gamma_+(h)$ in the direction of a. So $d(f, g; a)$ stands for the some kinds of distance from $\Gamma_+(g)$ to $\Gamma_+(f)$ in the direction of a.

Proposition 5.2.1. *Let $d_\infty(f, g)$ and $d_0(f, g)$ be as in* (1.2.1)*, then we have the following:*

(i) $d_\infty(f, g) = \max\{d(f, g; a) : a \in \mathbb{R}^n_+\}$.

(ii) $d_0(f, g) = \max\{d(g, f; a) : a \in \mathbb{R}^n_+\}$.

Proof. We only consider the case of $d_\infty(f, g)$. Let $\Gamma_+(f)$ and $\Gamma_+(g)$ be expressed as

$$\Gamma_+(f) = \bigcap_{a \in \mathbb{R}^n_+} H^+(a, l_f(a)), \quad \Gamma_+(g) = \bigcap_{a \in \mathbb{R}^n_+} H^+(a, l_g(a)).$$

By the definition of $d_\infty(f, g)$ and Lemma 5.1.1, we have the following equivalence,

$$\begin{aligned} &(\Gamma_+(g) + d_\infty(f, g) \cdot \mathbf{1}) \subset \Gamma_+(f) \\ \Longleftrightarrow\ & l_f(a) - l_g(a) - d_\infty(f, g) \cdot \langle a \rangle \leq 0 \quad \text{for all } a \in \mathbb{R}^n_+ \\ \Longleftrightarrow\ & d_\infty(f, g) \geq \frac{l_f(a) - l_g(a)}{\langle a \rangle} \quad \text{for all } a \in \mathbb{R}^n_+. \end{aligned} \tag{5.2.1}$$

The last inequation implies $d_\infty(f, g) \geq \max\{d(f, g; a) : a \in \mathbb{R}^n_+\}$.

Assume that $d_\infty(f, g) > \max\{d(f, g; a) : a \in \mathbb{R}^n_+\}$, then from (5.2.1), we have

$$\begin{aligned} &(\Gamma_+(g) + d_\infty(f, g) \cdot \mathbf{1}) \subsetneq \Gamma_+(f) \\ \Longleftrightarrow\ &(\Gamma_+(g) + d_\infty(f, g) \cdot \mathbf{1}) \cap \partial\Gamma_+(f) = \emptyset. \end{aligned}$$

This is contradicted to the expression of $d_\infty(f, g)$ in Lemma 1.2.3-(i). □

By using the $d(\cdot, \cdot; \cdot)$, we define important subfans of Σ. Let Σ_∞, Σ_0 be subsets of Σ defined by

$$\Sigma_\infty = \{\sigma \in \Sigma : d(f, g; a) = d_\infty(f, g) \text{ for all } a \in \sigma\}.$$
$$\Sigma_0 = \{\sigma \in \Sigma : d(g, f; a) = d_0(f, g) \text{ for all } a \in \sigma\}.$$

Note that these subsets Σ_∞, Σ_0 are not empty.

Remark 5.2.2. From the above definition, it is easy to see that Σ_∞ and Σ_0 are fans.

5.2.2 Properties of Σ_∞ and Σ_0

Lemma 5.2.3. *Let* $\{a^1(\sigma), ..., a^m(\sigma)\}$ *be a skeleton of* $\sigma \in \Sigma^{(k)} (k = 2, ..., n)$*. For all* $a \in \mathrm{Int}(\sigma)$*, the following inequation holds:*

$$\min_{1\leq j\leq m} d(f, g; a^j(\sigma)) \leq d(f, g; a) \leq \max_{1\leq j\leq m} d(f, g; a^j(\sigma)). \tag{5.2.2}$$

Proof. For any $a \in \mathrm{Int}(\sigma)$, there exist $k_i > 0$ $(i = 1, ..., m)$ such that $a = \sum_{i=1}^m k_i a^i(\sigma)$. From the construction of Σ, we can find the points $\xi \in \partial\Gamma_+(f)$ and $\eta \in \partial\Gamma_+(g)$ satisfying that $l_f(a^j(\sigma)) = \langle a^j(\sigma), \xi\rangle$ and $l_g(a^j(\sigma)) = \langle a^j(\sigma), \eta\rangle$ for all $j = 1, ..., m$. Then, from the definition of $d(f, g; a)$, we have

$$d(f, g; a) = \frac{\langle a, \xi - \eta\rangle}{\langle a\rangle} = \frac{\sum_{i=1}^m k_i\langle a^i(\sigma), \xi - \eta\rangle}{\sum_{i=1}^m \langle a^i(\sigma)\rangle}.$$

By a simple computation, we have

$$\begin{aligned}
&d(f, g; a) - d(f, g; a^j(\sigma)) \\
=&\frac{1}{C_j}\sum_{i=1, i\neq j}^{m} k_i \cdot \langle a^j(\sigma)\rangle \cdot \langle a^i(\sigma)\rangle \cdot \left(\frac{\langle a^i(\sigma), \xi - \eta\rangle}{\langle a^i(\sigma)\rangle} - \frac{\langle a^j(\sigma), \xi - \eta\rangle}{\langle a^j(\sigma)\rangle}\right) \\
=&\frac{1}{C_j}\sum_{i=1, i\neq j}^{m} k_i \cdot \langle a^j(\sigma)\rangle \cdot \langle a^i(\sigma)\rangle \cdot (d(f, g, a^i(\sigma)) - d(f, g, a^j(\sigma))),
\end{aligned} \tag{5.2.3}$$

where C_j is a positive constant denoted by $C_j = \langle a^j(\sigma)\rangle \cdot \sum_{i=1}^m k_i\langle a^i(\sigma)\rangle$. By checking the signature of the each term in (5.2.3), we can obtain the inequation (5.2.2). $\square$

Remark 5.2.4. From equation (5.2.3) in the above proof, equality in (5.2.2) holds if and only if $\min_{1\leq i\leq m} d(f, g; a^i(\sigma)) = \max_{1\leq i\leq m} d(f, g; a^i(\sigma)) = d(f, g; a^j(\sigma))$ for all j.

Lemma 5.2.3 and Remark 5.2.4 say that $d(f, g; a)$ has a property like maximum principle in theory of complex analysis. We obtain the following corollary from the above lemma and remark.

Corollary 5.2.5. *Under the same situation in Lemma 5.2.3, the following two conditions are equivalent.*

(i) $d(f, g, a) = \max_{1\leq j\leq m} d(f, g, a^j(\sigma)) = \min_{1\leq j\leq m} d(f, g, a^j(\sigma))$ *for some* $a \in Int(\sigma)$*.*

(ii) $d(f, g, a) = \max_{1\leq j\leq m} d(f, g, a^j(\sigma)) = \min_{1\leq j\leq m} d(f, g, a^j(\sigma))$ *for all* $a \in \sigma$*.*

From this corollary, we can check whether $\sigma \in \Sigma$ belongs to $\Sigma_\infty(\Sigma_0)$ or not by investigating only one vector $a \in \mathrm{Int}(\sigma)$.

In Chapter 6, we have to consider all cones containing vectors $a \in \mathbb{R}_+^n$ satisfying that $d(f,g;a) = d_\infty(f,g)$ or $d(g,f;a) = d_0(f,g)$. However, we find it sufficient to treat cones only belonging to Σ_∞ and Σ_0 from the next corollary.

Corollary 5.2.6. *The following equalities hold.*

$$|\Sigma_\infty| = \{a \in \mathbb{R}_+^n : d(f,g;a) = d_\infty(f,g)\}.$$
$$|\Sigma_0| = \{a \in \mathbb{R}_+^n : d(f,g;a) = d_0(f,g)\}.$$

Proof. We only consider the case of Σ_∞. Assume that there exist $a \in \mathbb{R}_+^n$ and $\sigma \notin \Sigma_\infty$ such that $a \in \mathrm{Int}(\sigma)$ and $d(f,g,a) = d_\infty(f,g)$. Then, from Corollary 5.2.5, we have $d(f,g,a) = d_\infty(f,g)$ for all $a \in \sigma$. This is contradiction. □

5.2.3 Properties of the principal faces

In order to investigate the properties of the leading poles of $Z_K(s;f,g,\varphi)$, we must understand more exact relationships between cones of the subfans of $\hat{\Sigma}$ and the faces of Newton polyhedra $\Gamma_+(f)$ and $\Gamma_+(g)$. In this subsection, after we focus cones of the fans Σ_∞, Σ_0 and the principal faces in $\mathcal{F}_\infty[f], \mathcal{F}_0[g]$, their relationships are investigated in detail.

For $\sigma \in \hat{\Sigma}^{(n)}$, we denote the skeleton of σ as $\{a^1(\sigma), ..., a^n(\sigma)\}$. Let

$$\begin{aligned} A_\infty(\sigma) &= \{j : d(f,g;a^j(\sigma)) = d_\infty(f,g)\} \subset \{1,...,n\}, \\ A_0(\sigma) &= \{j : d(g,f;a^j(\sigma)) = d_0(f,g)\} \subset \{1,...,n\} \end{aligned} \tag{5.2.4}$$

and for a cone $\sigma \in \hat{\Sigma}^{(n)}$ satisfying that $A_\infty(\sigma) \neq \emptyset$ or $A_0(\sigma) \neq \emptyset$, let $\gamma_\infty(\sigma), \gamma_0(\sigma)$ (resp. $\tau_\infty(\sigma), \tau_0(\sigma)$) be the faces of $\Gamma_+(f)$ (resp. $\Gamma_+(g)$) defined by

$$\gamma_\infty(\sigma) := \bigcap_{j \in A_\infty(\sigma)} H(a^j(\sigma), l_f(a^j(\sigma))) \cap \Gamma_+(f),$$
$$\gamma_0(\sigma) := \bigcap_{j \in A_0(\sigma)} H(a^j(\sigma), l_f(a^j(\sigma))) \cap \Gamma_+(f);$$

$$\tau_\infty(\sigma) := \bigcap_{j \in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma))) \cap \Gamma_+(g),$$
$$\tau_0(\sigma) := \bigcap_{j \in A_0(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma))) \cap \Gamma_+(g).$$

When $A_\infty(\sigma)$ or $A_0(\sigma)$ is empty, we define $\gamma_\infty(\sigma) = \tau_\infty(\sigma) := \emptyset$, $\gamma_0(\sigma) = \tau_0(\sigma) := \emptyset$, respectively.

Let $\hat{\Sigma}_\infty$ and $\hat{\Sigma}_0$ be simplicial subdivision of Σ_∞ and Σ_0 satisfying that $\hat{\Sigma}_\infty, \hat{\Sigma}_0 \subset \hat{\Sigma}$. Let $\tilde{\Sigma}_\infty^{(n)}, \tilde{\Sigma}_0^{(n)}$ be subsets of $\hat{\Sigma}^{(n)}$ defined by

$$\begin{aligned} \tilde{\Sigma}_\infty^{(n)} &:= \{\sigma \in \hat{\Sigma}_\infty^{(n)} : \#A_\infty(\sigma) = m_\infty(f,g)\}, \\ \tilde{\Sigma}_0^{(n)} &:= \{\sigma \in \hat{\Sigma}_0^{(n)} : \#A_0(\sigma) = m_0(f,g)\}. \end{aligned} \tag{5.2.5}$$

Lemma 5.2.7. *Suppose that γ_∞ is a principal face at infinity of $\Gamma_+(f)$ and τ_0 is a principal face at zero of $\Gamma_+(g)$. Then we have the followings.*

(i) $I(\gamma_\infty, \sigma) \subset A_\infty(\sigma)$, $I(\tau_0, \sigma) \subset A_0(\sigma)$ *for any* $\sigma \in \hat{\Sigma}^{(n)}$.

(ii) $\#A_\infty(\sigma) \leq m_\infty(f,g)$, $\#A_0(\sigma) \leq m_0(f,g)$ *for any* $\sigma \in \hat{\Sigma}^{(n)}$.

(iii) $\hat{\Sigma}^{(n)}(\gamma_\infty) \subset \tilde{\Sigma}_\infty^{(n)}$, $\hat{\Sigma}^{(n)}(\tau_0) \subset \tilde{\Sigma}_0^{(n)}$.

(iv) $\tilde{\Sigma}_\infty^{(n)}, \tilde{\Sigma}_0^{(n)} \neq \emptyset$.

Here, $I(\cdot,\cdot)$ and $\hat{\Sigma}^{(n)}(\cdot)$ are as in (4.2.4), (4.2.5).

Proof. Since we can prove in analogous way, we only consider the case of γ_∞.

(i) Suppose that $j \in I(\gamma_\infty, \sigma)$, i.e. $\gamma_\infty \subset H(a^j(\sigma), l_f(a^j(\sigma)))$. Let τ_∞ be a principal face at infinity of $\Gamma_+(g)$ associated to γ_∞, i.e., $\Psi_\infty(\gamma_\infty) = \tau_\infty$. From the definition of Ψ_∞, we have $\Phi_\infty(\tau_\infty) \subset \Psi_\infty^{-1}(\tau_\infty) = \gamma_\infty$ and

$$\begin{aligned} \gamma_\infty \subset H(a^j(\sigma), l_f(a^j(\sigma))) &\Longrightarrow \Phi_\infty(\tau_\infty) \subset H(a^j(\sigma), l_f(a^j(\sigma))) \\ &\Longleftrightarrow \tau_\infty \subset \Phi_\infty^{-1}(H(a^j(\sigma), l_f(a^j(\sigma)))). \end{aligned}$$

Because of $\Gamma_+(g) \subset \Phi_\infty^{-1}(H(a^j(\sigma), l_f(a^j(\sigma)))) = H(a^j(\sigma), l_f(a^j(\sigma)) - d_\infty(f,g)\cdot\mathbf{1})$ and the fact that τ_∞ is a nonempty proper face of $\Gamma_+(g)$, the definition of $l_g(\cdot)$ implies that $l_g(a^j(\sigma)) = l_f(a^j(\sigma)) - d_\infty(f,g)\cdot\langle a^j(\sigma)\rangle$. This shows $j \in A_\infty(\sigma)$.

(ii) The case when $A_\infty(\sigma) = \emptyset$ is obvious, so we assume that $A_\infty(\sigma) \neq \emptyset$. Since $j \in A_\infty(\sigma) \Leftrightarrow l_g(a^j(\sigma)) = l_f(a^j(\sigma)) - d_\infty(f,g)\cdot\langle a^j(\sigma)\rangle$ and $\Phi_\infty(\Gamma_+(g)) \subset \Gamma_+(f)$, we have

$$\Phi_\infty(\tau_\infty(\sigma)) \subset \Phi_\infty\left(\bigcap_{j\in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma)))\right) \cap \Phi_\infty(\Gamma_+(g))$$

$$\subset \bigcap_{j\in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma)) + d_\infty(f,g)\cdot\langle a^j(\sigma)\rangle) \cap \Gamma_+(f)$$
$$= \bigcap_{j\in A_\infty(\sigma)} H(a^j(\sigma), l_f(a^j(\sigma))) \cap \Gamma_+(f) = \gamma_\infty(\sigma). \tag{5.2.6}$$

Since $\Phi_\infty(\tau_\infty(\sigma))$ is a nonempty set in $\Gamma_*(f)$ and contained in the face $\gamma_\infty(\sigma)$ of $\Gamma_+(f)$, the definition of $\mathcal{F}_*[f]$ implies that there exists a face $\tilde{\gamma} \in \mathcal{F}_*[f]$ such that $\Phi_\infty(\tau_\infty(\sigma)) \subset \tilde{\gamma} \subset \gamma_\infty(\sigma)$. From the definition of $m_\infty(f,g)$, we have

$$\dim(\gamma_\infty(\sigma)) \geq \dim(\tilde{\gamma}) \geq n - m_\infty(f,g). \tag{5.2.7}$$

On the other hand, we have

$$\dim(\gamma_\infty(\sigma)) \leq \dim\left(\bigcap_{j\in A_\infty(\sigma)} H(a^j(\sigma), l_f(a^j(\sigma)))\right) = n - \#A_\infty(\sigma). \tag{5.2.8}$$

Putting (5.2.7), (5.2.8) together, we have $\#A_\infty(\sigma) \leq m_\infty(f,g)$.

Since the proofs of the assertions (iii),(iv) are same as those of Lemma 11.5-(iii),(iv) in [18], we omit them here. $\square$

Remark 5.2.8. From Lemma 5.2.7-(ii),(iv), we see that

$$\max\{\#A_\infty(\sigma) : \sigma \in \hat{\Sigma}^{(n)}\} = m_\infty(f,g),$$
$$\max\{\#A_0(\sigma) : \sigma \in \hat{\Sigma}^{(n)}\} = m_0(f,g).$$

Proposition 5.2.9. *(i) If $\sigma \in \tilde{\Sigma}_\infty^{(n)}$, then $\gamma_\infty(\sigma)$ (resp. $\tau_\infty(\sigma)$) is a principal face at infinity of $\Gamma_+(f)$ (resp. $\Gamma_+(g)$). Moreover, $\gamma_\infty(\sigma)$ is associated to $\tau_\infty(\sigma)$ (i.e., $\tau_\infty(\sigma) = \Psi_\infty(\gamma_\infty(\sigma))$).*

(ii) If $\sigma \in \tilde{\Sigma}_0^{(n)}$, then $\tau_0(\sigma)$ (resp. $\gamma_0(\sigma)$) is a positive principal face at zero of $\Gamma_+(g)$ (resp. $\Gamma_+(f)$). Moreover, $\tau_0(\sigma)$ is associated to $\gamma_0(\sigma)$ (i.e., $\gamma_0(\sigma) = \Psi_0(\tau_0(\sigma))$).

Proof. As we can prove in analogous way, we only consider the assertion (i). Suppose that $\sigma \in \tilde{\Sigma}_\infty^{(n)}$, i.e., $\#A_\infty(\sigma) = m_\infty(f,g)$.

<u>($\gamma_\infty(\sigma)$ is a principal face at infinity of $\Gamma_+(f)$)</u>

From (5.2.7), (5.2.8), we have $\dim(\gamma_\infty(\sigma)) = n - m_\infty(f,g)$ and, moreover, $\gamma_\infty(\sigma) = \tilde{\gamma} \in \mathcal{F}_*[f]$. It follows from these equations that $\gamma_\infty(\sigma)$ is a principal face at infinity of $\Gamma_+(f)$.

($\gamma_\infty(\sigma)$ is associated to $\tau_\infty(\sigma)$)

The following equation holds:

$$\Phi_\infty^{-1}(\gamma_\infty(\sigma)) = \bigcap_{j \in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma))) \cap \Phi_\infty^{-1}(\Gamma_+(f)).$$

According to the property $\Gamma_+(g) + d_\infty(f,g) \cdot \mathbf{1} \subset \Gamma_+(f)$, we have

$$\begin{aligned}\Phi_\infty^{-1}(\gamma_\infty(\sigma)) \cap \Gamma_+(g) &= \bigcap_{j \in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma))) \cap (\Gamma_+(f) - d_\infty(f,g) \cdot \mathbf{1}) \cap \Gamma_+(g) \\ &= \bigcap_{j \in A_\infty(\sigma)} H(a^j(\sigma), l_g(a^j(\sigma))) \cap \Gamma_+(g) = \tau_\infty(\sigma).\end{aligned}$$

From the above equation and the definition (1.2.3), we see that $\tau_\infty(\sigma)$ is a negative principal face of $\Gamma_+(g)$ associated to $\gamma_\infty(\sigma)$. $\square$

From Proposition 5.2.9, the map from $\tilde{\Sigma}_\infty^{(n)}$ to $\mathcal{F}_\infty[f]$ (resp. to $\mathcal{F}_\infty[g]$) is naturally defined (i.e., $\sigma \mapsto \gamma_\infty(\sigma)$ (resp. $\sigma \mapsto \tau_\infty(\sigma)$)). Similarly, the map from $\tilde{\Sigma}_0^{(n)}$ to $\mathcal{F}_0[f]$ (resp. to $\mathcal{F}_0[g]$) is defined by $\sigma \mapsto \gamma_0(\sigma)$ (resp. $\sigma \mapsto \tau_0(\sigma)$).

Lemma 5.2.10. *The above four maps are surjective.*

Proof. See the proof of Lemma 11.8 in [18]. $\square$

Chapter 6

Analysis of local zeta functions

The purpose of this chapter is to investigate the following integrals

$$Z_K(s; f, g, \varphi) = \int_{K^n \setminus D_K} \left| \frac{f(x)}{g(x)} \right|_K^s \varphi(x) |dx|_K \qquad (s \in \mathbb{C}), \tag{6.0.1}$$

where $K = \mathbb{R}$ or $\mathbb{C}$ and

- f, g are K-analytic functions defined on an open neighborhood U of the origin and $D_K = f^{-1}(0) \cup g^{-1}(0)$. Here, we recall that $\mathbb{R}$-analytic means "real analytic" and $\mathbb{C}$-analytic means "holomorphic".
- φ is a smooth function whose support is contained in U.
- $|\cdot|_K$ means $|\cdot|_{\mathbb{R}} = |\cdot|$ or $|\cdot|_{\mathbb{C}} = |\cdot|^2$, where $|\cdot|$ is a usual absolute values in $\mathbb{R}$ or $\mathbb{C}$.
- $|dx|_K$ means $|dx|_{\mathbb{R}} = dx_1 \wedge \cdots \wedge dx_n$ for $(x_1, ..., x_n) \in \mathbb{R}^n$ and $|dx|_{\mathbb{C}} = dx_1 \wedge \cdots \wedge dx_n \wedge d\overline{x}_1 \wedge \cdots \wedge d\overline{x}_n$ for $(x_1, ..., x_n) \in \mathbb{C}^n$.

Unlike the analytic case, the convergence of the integral in (6.0.1) is not followed from the compactness of support of φ. In fact, it is shown [1] that the above integral converges on some slit domain in $\mathbb{C}$.

Theorem 6.0.1 ([1]). *There exist positive constants α, β with $0 < \alpha, \beta \leq \infty$ such that the integral (6.0.1) converges locally uniformly on the region $\{-\alpha < \mathrm{Re}(s) < \beta\}$ and defines a holomorphic function there.*

This theorem is shown by using a simultaneous resolution of singularities. The holomorphic function defined on the region given in Theorem 6.0.1 is called *archimedean local zeta function* attached to $(f/g, \varphi)$. Furthermore, W.Veys and W.A.Zúñiga-Galindo [35] show that the above local zeta function has a meromorphic continuation to the whole complex plane and its poles are contained in finitely many arithmetic progressions consisting of rational numbers. We denote this meromorphic continuation by the same symbol $Z_K(s; f, g, \varphi)$.

As we mentioned in Introduction, poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$ are deeply connected to the asymptotics of oscillatory integrals attached to $(f/g, \varphi)$ in (i-3). Therefore, to describe the properties of these poles is very important.

6.0.1 Candidate poles

Let us state the results relating to the positions and the orders of candidate poles of $Z_K(s; f, g, \varphi)$. For this, we define the arithmetic progression derived from a vector defined by

$$\mathcal{P}_K(a) := \left\{ -\frac{\langle a \rangle + \delta_K \nu}{l_f(a) - l_g(a)} : \nu \in \mathbb{Z}_+ \right\} \subset \mathbb{Q}, \tag{6.0.2}$$

where $a \in \mathbb{Z}_+^n$, $l_f(\cdot), l_g(\cdot)$ are as in (4.4.1) and

$$\delta_K = \begin{cases} 1 & (K = \mathbb{R}) \\ 1/2 & (K = \mathbb{C}). \end{cases}$$

Theorem 6.0.2. *Suppose that $f \cdot g$ is nondegenerate over K with respect to its Newton polyhedron, then we have the followings.*

(i) The poles of the function $Z_K(s; f, g, \varphi)$ are contained in the set

$$\bigcup_{a \in V_+} \mathcal{P}_K(a) \cup \bigcup_{a \in V_-} \mathcal{P}_K(a) \cup (\delta_K \mathbb{Z} \setminus \{0, \pm 1/2\}), \tag{6.0.3}$$

where $V_\pm$ are as in (5.1.5). Note that $V_+ = \emptyset$ if and only if $d_\infty(f, g) = 0$ and $V_- = \emptyset$ if and only if $d_0(f, g) = 0$.

(ii) (a) If $d_\infty(f, g) > 0$, then the largest element of the first set in (6.0.3) is $-1/d_\infty(f, g)$.

(b) If $d_0(f, g) > 0$, then the smallest element of the second set in (6.0.3) is $1/d_0(f, g)$.

(iii) *(a) If* $d_\infty(f,g) > 0$*, the order of pole of* $Z_K(s; f, g, \varphi)$ *at* $s = -1/d_\infty(f,g)$ *is* at most

$$\begin{cases} m_\infty(f,g) & \text{if } 1/d_\infty(f,g) \text{ is not an integer,} \\ \min\{m_\infty(f,g)+1, n\} & \text{otherwise.} \end{cases}$$

(b) If $d_0(f,g) > 0$*, the order of pole of* $Z_K(s; f, g, \varphi)$ *at* $s = 1/d_0(f,g)$ *is* at most

$$\begin{cases} m_0(f,g) & \text{if } 1/d_0(f,g) \text{ is not an integer,} \\ \min\{m_0(f,g)+1, n\} & \text{otherwise.} \end{cases}$$

Here, the poles on $(\delta_K\mathbb{Z} \setminus \{0, \pm 1/2\})$ will be called *trivial poles*.

Remark 6.0.3. The set (6.0.3) is a set of candidate poles of $Z_K(s; f, g, \varphi)$ and there might be many poles which do not appear in actuality. As a known result, Denef and Sargos [8] show that in analytic case, if a is an additional vector obtained by process of simplicial subdivision, poles belonging to $P_K(a)$ do not appear. We believe that the same assertion holds in meromorphic case.

Remark 6.0.4. (a) As in the proof of Theorem 6.0.2, we see that the non-trivial poles, that is, lying on $\bigcup_{a\in V_+} \mathcal{P}_K(a) \cup \bigcup_{a\in V_-} \mathcal{P}_K(a)$ can be computed by using the theory of toric varieties based on the Newton polyhedra of f and g. This means that the list of non-trivial poles can be determined by the geometry of the Newton polyhedra of f and g. On the other hand, the existence of trivial poles cannot be determined from the information of $\Gamma_+(f)$ and $\Gamma_+(g)$ only.

(b) When $d_\infty(f,g) = 0$, the set V_+ is empty and this implies that $Z_K(s; f, g, \varphi)$ has no negative non-trivial poles. The same can be said for the positive non-trivial poles when $d_0(f,g) = 0$.

6.0.2 The leading poles

Let us consider the important poles of $Z_K(s; f, g, \varphi)$ which have crucial roles in both the properties of $Z_K(s; f, g, \varphi)$ and relationship between $Z_K(s; f, g, \varphi)$ and other mathematical fields.

Definition 6.0.5. (i) The largest negative pole in (6.0.3) is called the *negative leading pole*. Moreover, the largest negative non-trivial pole is called the *negative e-leading pole*.

(ii) The smallest positive pole in (6.0.3) is called the *positive leading pole*. Moreover, the smallest positive non-trivial pole is called the *positive e-leading pole*.

The word "e-leading pole" means "essential leading pole", which plays an essentially important role in the analysis of oscillatory integrals.

Note that $Z_K(s; f, g, \varphi)$ does not always have negative(positive) (e-)leading poles. If $Z_K(s; f, g, \varphi)$ has a negative leading pole at $s = -\alpha^*$ and a positive leading pole at $s = \beta^*$, $Z_K(s; f, g, \varphi)$ can be regarded as a holomorphic function on the region $\{s \in \mathbb{C} : -\alpha^* < \mathrm{Re}(s) < \beta^*\}$. In Chapter 7, we see that properties of the negative and positive e-leading poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$ are reflected to the orders of leading terms of asymptotic expansions (i-4), (i-5). In addition, in the recent studies of Bernstein-Sato polynomials in [32], [1], it is an important issue to express these leading poles of $Z_K(s; f, g, \varphi)$ by using appropriate informations of f and g. For these reasons, we attempt to describe conditions where each (e-)leading pole can be determined explicitly by means of Newton data defined in Chapter 1.

Theorem 6.0.6. *Suppose that $f \cdot g$ is nondegenerate over K with respect to its Newton polyhedron and φ satisfies that $\varphi(0) > 0$ and φ is nonnegative on its support.*

(i) Suppose that $d_\infty(f, g) > 0$. If at least one of the following two conditions is satisfied:

(a) $d_\infty(f, g) > 1$;

(b) $K = \mathbb{R}$ and f is nonnegative or nonpositive on U,

then the negative leading pole of $Z_K(s; f, g, \varphi)$ exists at $s = -1/d_\infty(f, g)$ and its order is equal to $m_\infty(f, g)$. Furthermore, if at least one of the three conditions (a), (b) and

(c) there exists $\gamma_\infty \in \mathcal{F}_\infty[f]$ such that f_{γ_∞} does not vanish on $U \cap (K \setminus \{0\})^n$

is satisfied, then the negative e-leading pole of $Z_K(s; f, g, \varphi)$ exists at $s = -1/d_\infty(f, g)$ and its order is equal to $m_\infty(f, g)$.

(ii) Suppose that $d_0(f, g) > 0$. If at least one of the following two conditions is satisfied:

(d) $d_0(f,g) > 1$;

(e) $K = \mathbb{R}$ *and* g *is nonnegative or nonpositive on* U,

then the positive leading pole of $Z_K(s; f, g, \varphi)$ *exists at* $s = 1/d_0(f,g)$ *and its order is equal to* $m_0(f,g)$. *Furthermore, if at least one of the three conditions* (d), (e) *and*

(f) there exists $\tau_0 \in \mathcal{F}_0[g]$ *such that* g_{τ_0} *does not vanish on* $U \cap (K \setminus \{0\})^n$

is satisfied, then the positive e-leading pole of $Z_K(s; f, g, \varphi)$ *exists at* $s = 1/d_0(f,g)$ *and its order is equal to* $m_0(f,g)$.

Remark 6.0.7. (i) We consider the case when $d_\infty(f,g) = 0$. From Remark 6.0.4-(b), negative poles of $Z_K(s; f, g, \varphi)$ are only trivial poles and from the proofs of Proposition 6.3.1 and Theorems 6.3.3, 6.3.6, we see that $Z_K(s; f, g, \varphi)$ has no negative trivial pole under the assumptions (i)-(b),(c). According to these facts, we can interpret that locating the negative leading pole at $s = -\infty$ indicates $Z_K(s; f, g, \varphi)$ is holomorphically extended to the left half plane. Similarly, $Z_K(s; f, g, \varphi)$ is holomorphically extended to the right half plane when its positive leading pole is at $s = +\infty$.

(ii) In Theorems 6.3.3, 6.3.4, 6.3.6, 6.3.7, we give explicit formulae for the coefficients of terms in Laurent expansion at e-leading poles under the same assumption in Theorem 6.0.6. These explicit formulae show that the above coefficients essentially depend on the principal face-parts $(f_{\gamma_\infty}, g_{\tau_\infty})$ and $(f_{\gamma_0}, g_{\tau_0})$. The conditions in Theorem 6.0.6 are sufficient conditions for the non-vanishing of these coefficients.

Example 5. Let us consider adapting above theorem to the functions f, g in subsection 1.2.1. It is easy to see that all the functions satisfy the assumptions in Theorem 6.0.6. Then, we have the followings:

(i) When $f(x) = (x_1^2 + x_2^2)^2$ and $g(x) = x_1^4 + x_2^4$, both Newton distances are equal to 0 and assumptions (i)-(b),(c) and (ii)-(e),(f) are satisfied. Then, we see that $Z_{\mathbb{R}}(s; f, g, \varphi)$ has leading poles at $s = \pm\infty$, which implies $Z_{\mathbb{R}}(s; f, g, \varphi)$ can be regarded as an entire function.

(ii) When $f(x) = x_1^6 + x_2^6$ and $g(x) = x_1^2 + x_2^2$, $d_\infty(f,g) = 2$ and $d_0(f,g) = 0$. Here, the assumptions (i)-(a) and (ii)-(e),(f) are satisfied, then $Z_{\mathbb{R}}(s; f, g, \varphi)$ has a pole at

$s = -1/2$ of order 1 as a negative leading pole and is holomorphically extended to the right half plane.

(iii) When $f(x) = x_1^2 + x_2^4$ and $g(x) = x_1^4 + x_2^2$, $Z_{\mathbb{R}}(s; f, g, \varphi)$ has a negative leasing pole at $s = -3/2$ of order 1 and a positive leading pole at $s = 3/2$ of order 1. Here, the assumptions (i)-(b),(c) and (ii)-(e),(f) are satisfied.

6.1 Poles of elementary functions

For the analysis of local zeta functions, we investigate poles of elementary integrals of the form

$$L(s) = \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l_j s + m_j - 1} \right) \psi(y, s) dy,$$

where $l_j \in \mathbb{Z}$, $m_j \in \mathbb{N}$ and $\psi(\cdot, s)$ is a C^∞ function of y in $\mathbb{R}^n$ for any $s \in \mathbb{C}$ and $\psi(y, \cdot)$ is an entire function on $\mathbb{C}$ for any $y \in \mathbb{R}^n$.

6.1.1 Positions and orders of poles of $L(s)$

Let $B, B_\pm$ be subsets of $\{1, ..., n\}$ defined by

$$B_\pm := \{j : \pm l_j > 0\}, \quad B := B_+ \cup B_-.$$

Remark 6.1.1. It is easy to see that $L(s)$ converges when $s \in \mathbb{C}$ satisfies $\mathrm{Re}(l_j s + m_j - 1) > -1$ for all j. Hence, $L(s)$ defines a holomorphic function on the region

$$\max_{j \in B_+} \{-m_j / l_j\} < \mathrm{Re}(s) < \min_{j \in B_-} \{-m_j / l_j\}.$$

Proposition 6.1.2. *$L(s)$ can be analytically continued to the whole complex plane as a meromorphic function and its poles belong to the set*

$$\left\{ -\frac{m_j + \nu_j}{l_j} : \nu_j \in \mathbb{Z}_+, j \in B \right\}. \tag{6.1.1}$$

Moreover, suppose that p belongs to the above set and let

$$A(p) = \left\{ j \in B : -\frac{m_j + \nu_j}{l_j} = p \text{ for some } \nu_j \in \mathbb{Z}_+ \right\}.$$

Then, if $L(s)$ has a pole at $s = p$, the order of the pole of $L(s)$ at p is at most $\# A(p)$.

Proof. For any $\nu = (\nu_1, ..., \nu_n) \in \mathbb{Z}_+^n$, by repeating the integration by parts , we have

$$L(s) = \left(\prod_{j=1}^{n} \prod_{\nu=0}^{\nu_j} \frac{1}{(l_j s + m_j + \nu)} \right) \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^{n} y_j^{l_j s + m_j + \nu_j} \right) \frac{\partial^{\langle \nu \rangle} \psi(y, s)}{\partial y_1^{\nu_1} \cdots \partial y_n^{\nu_n}} dy.$$

This shows $L(s)$ is analytically continued to the wider region $\mathrm{Re}(l_j s + m_j + \nu_j) > -1$ and the poles of $L(s)$ are contained in the set (6.1.1). Since $\nu \in \mathbb{Z}_+^n$ is arbitrary, we see that $L(s)$ can be analytically continued to the whole complex plane. Moreover, it is easy to see that $(l_j s + m_j + \nu_j)$ satisfying $p = -(m_j + \nu_j)/l_j$ appear at most $\# A(p)$ times. □

6.1.2 First coefficients of $L(s)$

Let us compute the coefficients of the terms of Laurent expansions of $L(s)$ at the important poles. When $B_\pm \neq \emptyset$, we define $p_+ := \max\{-m_j/l_j : j \in B_+\}$, $p_- := \min\{-m_j/l_j : j \in B_-\}$ and $A_\pm := A(p_\pm)$, respectively. Note that $p_\pm$ are the negative(positive) leading poles of $L(s)$ if the coefficients do not vanish.

Proposition 6.1.3. *The coefficients of $(s - p_\pm)^{-\# A_\pm}$ in the Laurent expansions of $L(s)$ are*

$$\begin{cases} \displaystyle\prod_{j=1}^{n} \frac{1}{l_j} \psi(0, p_\pm) \quad \text{if } A_\pm = \{1, ..., n\}, \\ \displaystyle\prod_{j \in A_\pm} \frac{1}{l_j} \int_{\mathbb{R}_+^{n - \# A_\pm}} \left(\prod_{j \notin A_\pm} y_j^{l_j p_\pm + m_j - 1} \right) \psi(T_{A_\pm}(y), p_\pm) \prod_{j \notin A_\pm} dy_j \quad \text{(otherwise)}. \end{cases}$$

Proof. By an integration by parts with respect to each y_j for $j \in A_\pm$ and the computation of $\lim_{s \to p_\pm} (s - p_\pm)^{\# A_\pm} L(s)$, we obtain this proposition. □

6.1.3 Trivial poles of $L(s)$

In the analysis in Section 6.2, we must consider the case when the coefficients of some poles have particular properties. This property will be understood through the following functions:

$$L_{1,\pm}(s) = \int_{\mathbb{R}_+^n} \left(y_n^s \prod_{j \in D_+} y_j^{l_j s + m_j - 1} \right) \psi(y_1, ..., y_{n-1}, \pm y_n) dy,$$

$$L_{2,\pm}(s) = \int_{\mathbb{R}_+^n} \left(y_n^{-s} \prod_{j \in D_-} y_j^{l_j s + m_j - 1} \right) \psi(y_1, ..., y_{n-1}, \pm y_n) dy,$$

where $D_\pm$ are subsets of $B \setminus \{n\}$.

Lemma 6.1.4. *For $\lambda \in \mathbb{N}$, let $A_{\pm\lambda}$ be subsets in $D_\pm$ defined by $A_\lambda = \{j : l_j\lambda + m_j - 1 \in \mathbb{N}\}$ and $A_{-\lambda} = \{j : -l_j\lambda + m_j - 1 \in -\mathbb{N}\}$. Then, we have the following:*

(i) The functions $L_{1,\pm}(s)$ have poles at $s = -\lambda$ of order not higher than $\#A_{-\lambda} + 1$. Moreover, the following holds : Let $a_{-\lambda}^{\pm}$ be the coefficients of $(s+\lambda)^{-\#A_{-\lambda}-1}$ in the Laurent expansions of $L_{1,\pm}(s)$ at $s = -\lambda$, respectively. Then the equation $a_{-\lambda}^{+} = (-1)^{\lambda-1}a_{-\lambda}^{-}$ holds.

(ii) The functions $L_{2,\pm}(s)$ have poles at $s = \lambda$ of order not higher than $\#A_\lambda + 1$. Moreover, the following holds : Let $a_\lambda^{\pm}$ be the coefficients of $(s-\lambda)^{-\#A_\lambda-1}$ in the Laurent expansions of $L_{2,\pm}(s)$ at $s = \lambda$, respectively. Then the equation $a_\lambda^{+} = -(-1)^{\lambda-1}a_\lambda^{-}$ holds.

Proof. See the proof of Lemma 9.4 in [18]. □

6.2 Proof of Theorem 6.0.2

6.2.1 The case of $K = \mathbb{R}$

At first, we consider the case of $K = \mathbb{R}$. It is easy to see that the following relation holds.

$$Z_{\mathbb{R}}(s; f, g, \varphi) = Z_+(s; f, g, \varphi) + Z_-(s; f, g, \varphi), \tag{6.2.1}$$

where

$$Z_\pm(s; f, g, \varphi) = \int_{\mathbb{R}^n \setminus g^{-1}(0)} \left(\frac{f(x)}{g(x)}\right)_\pm^s \varphi(x)dx \tag{6.2.2}$$

and

$$(f(x)/g(x))_+ = \max\{f(x)/g(x), 0\}, \qquad (f(x)/g(x))_- = \max\{-f(x)/g(x), 0\}.$$

By applying the orthant decomposition to the functions $Z_\pm(s; f, g, \varphi)$, we have

$$Z_\pm(s; f, g, \varphi) = \sum_{\theta \in \{-1,1\}^n} \zeta_\pm(s; \varphi_\theta, f_\theta, g_\theta) \tag{6.2.3}$$

with

$$\zeta_\pm(s; f, g, \varphi) := \int_{\mathbb{R}_+^n} \left(\frac{f(x)}{g(x)}\right)_\pm^s \varphi(x)dx \tag{6.2.4}$$

and

$$h_\theta(x) := h(\theta_1 x_1, ..., \theta_n x_n) \tag{6.2.5}$$

for $h = \varphi, f, g$, $\theta = (\theta_1, ..., \theta_n) \in \{-1, 1\}^n$.

By applying the real toric resolution of singularities $x = \pi_{\mathbb{R}}(y)$, $\zeta_\pm(s; f, g, \varphi)$ can be expressed as

$$\begin{aligned}\zeta_\pm(s; f, g, \varphi) &= \int_{\mathbb{R}_+^n} \left(\frac{f(x)}{g(x)}\right)_\pm^s \varphi(x) dx \\ &= \int_{Y_{\hat{\Sigma}} \cap \pi_{\mathbb{R}}^{-1}(\mathbb{R}_+^n)} \left(\frac{(f \circ \pi_{\mathbb{R}})(y)}{(g \circ \pi_{\mathbb{R}})(y)}\right)_\pm^s (\varphi \circ \pi_{\mathbb{R}})(y) |J_{\pi_{\mathbb{R}}}(y)| dy,\end{aligned}$$

where dy is a volume element in $Y_{\hat{\Sigma}}$. It is easy to see that there exists a set of C_0^∞ functions $\{\chi_\sigma : Y_{\hat{\Sigma}} \to \mathbb{R}_+ : \sigma \in \hat{\Sigma}^{(n)}\}$ satisfying the following properties:

- For each $\sigma \in \hat{\Sigma}^{(n)}$, the support of the function χ_σ is contained in $\mathbb{R}^n(\sigma)$ and χ_σ identically equals one in some neighborhood of the origin.
- $\sum_{\sigma \in \hat{\Sigma}^{(n)}} \chi_\sigma \equiv 1$ on the support of $\varphi \circ \pi_{\mathbb{R}}$.

A set of these functions is called *partition of unity on* $Y_{\hat{\Sigma}}$. Then, we have

$$\zeta_\pm(s; f, g, \varphi) = \sum_{\sigma \in \hat{\Sigma}^{(n)}} \zeta_\pm^{(\sigma)}(s; \varphi, f, g) \tag{6.2.6}$$

with

$$\begin{aligned}\zeta_\pm^{(\sigma)}(s; \varphi, f, g) &:= \int_{\mathbb{R}_+^n} \left(\frac{(f \circ \pi_{\mathbb{R}}(\sigma))(y)}{(g \circ \pi_{\mathbb{R}}(\sigma))(y)}\right)_\pm^s (\varphi \circ \pi_{\mathbb{R}}(\sigma))(y) \chi_\sigma(y) |J_{\pi_{\mathbb{R}(\sigma)}}(y)| dy \\ &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma)) - l_g(a^j(\sigma))} \cdot \frac{f_\sigma(y)}{g_\sigma(y)}\right)_\pm^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle - 1}\right| \hat{\chi}_\sigma(y) dy,\end{aligned} \tag{6.2.7}$$

where $\hat{\chi}_\sigma(y) = (\varphi \circ \pi_{\mathbb{R}}(\sigma))(y) \chi_\sigma(y)$.

Consider the functions $\zeta_\pm^{(\sigma)}(s; \varphi, f, g)$ for each $\sigma \in \hat{\Sigma}^{(n)}$. We easily see the existence of finite sets of C_0^∞ functions $\{\xi_k : \mathbb{R}^n \to \mathbb{R}_+\}$, $\{\eta_l : \mathbb{R}^n \to \mathbb{R}_+\}$ and $\{\kappa_m : \mathbb{R}^n \to \mathbb{R}_+\}$ satisfying the following conditions.

- The supports of ξ_k, η_l and κ_m are sufficiently small and $\sum_k \xi_k + \sum_l \eta_l + \sum_m \kappa_m \equiv 1$ on the support of $\hat{\chi}_\sigma$.

- For each k, the function f_σ/g_σ is always positive or negative on the support of ξ_k.
- For each l, the support of η_l contains just one point in the set $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : f_\sigma(y) = 0\}$ and the union of the support of η_l for all l contains the set $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : f_\sigma(y) = 0\}$.
- For each m, the support of κ_m contains just one point in the set $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : g_\sigma(y) = 0\}$ and the union of the support of κ_m for all m contains the set $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : g_\sigma(y) = 0\}$.

Remark 6.2.1. From Lemma 4.4.8, f_σ and g_σ do not vanish simultaneously on the $T_I(\mathbb{R}^n)$ which satisfies $\pi(\sigma)(T_I(\mathbb{R}^n)) = 0$. Hence, by shrinking their supports if necessary, we can select the supports of η_l and κ_m satisfying that $\mathrm{Supp}(\eta_l) \cap \mathrm{Supp}(\kappa_m) = \emptyset$ hold for all l and m.

By using the functions $\{\xi_k\}$, $\{\eta_l\}$ and $\{\kappa_m\}$, we have the following decomposition:

$$\zeta_\pm^{(\sigma)}(s;\varphi,f,g) = \sum_k I_{\pm,\sigma}^{(k)}(s) + \sum_l J_{\pm,\sigma}^{(l)}(s) + \sum_m K_{\pm,\sigma}^{(m)}(s) \tag{6.2.8}$$

with

$$\begin{aligned}
I_{\pm,\sigma}^{(k)}(s) &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma))-l_g(a^j(\sigma))} \cdot \frac{f_\sigma(y)}{g_\sigma(y)}\right)_\pm^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle-1}\right| \hat{\xi}_k(y)dy,\\
J_{\pm,\sigma}^{(l)}(s) &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma))-l_g(a^j(\sigma))} \cdot \frac{f_\sigma(y)}{g_\sigma(y)}\right)_\pm^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle-1}\right| \hat{\eta}_l(y)dy, \qquad (6.2.9)\\
K_{\pm,\sigma}^{(m)}(s) &= \int_{\mathbb{R}_+^n} \left(\prod_{j=1}^n y_j^{l_f(a^j(\sigma))-l_g(a^j(\sigma))} \cdot \frac{f_\sigma(y)}{g_\sigma(y)}\right)_\pm^s \left|\prod_{j=1}^n y_j^{\langle a^j(\sigma)\rangle-1}\right| \hat{\kappa}_m(y)dy,
\end{aligned}$$

where $\hat{\xi}_k(y) = \hat{\chi}_\sigma(y)\xi_k(y)$, $\hat{\eta}_l(y) = \hat{\chi}_\sigma(y)\eta_l(y)$ and $\hat{\kappa}_m(y) = \hat{\chi}_\sigma(y)\kappa_m(y)$. If the set $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : f_\sigma(y) = 0\} \cap \mathbb{R}_+^n$ (resp. $\{y \in \mathrm{Supp}(\hat{\chi}_\sigma) : g_\sigma(y) = 0\} \cap \mathbb{R}_+^n$) is empty, then the functions $J_{\pm,\sigma}^{(l)}(s)$ (resp. $K_{\pm,\sigma}^{(m)}(s)$) do not appear in the decomposition (6.2.8).

6.2.2 Poles of $I_{\pm,\sigma}^{(k)}(s)$, $J_{\pm,\sigma}^{(l)}(s)$ and $K_{\pm,\sigma}^{(m)}(s)$

Let us consider the positions of poles of $I_{\pm,\sigma}^{(k)}(s)$, $J_{\pm,\sigma}^{(l)}(s)$ and $K_{\pm,\sigma}^{(m)}(s)$ in (6.2.9). First, we consider properties of poles of the functions $I_{\pm,\sigma}^{(k)}(s)$. Since every y_j is nonnegative in the integrand, we have

$$I^{(k)}_{\pm,\sigma}(s) = \int_{\mathbb{R}^n_+} \prod_{j=1}^{n} y_j^{(l_f(a^j(\sigma))-l_g(a^j(\sigma)))s+\langle a^j(\sigma)\rangle-1} \left(\frac{f_\sigma(y)}{g_\sigma(y)}\right)^s_{\pm} \hat{\xi}_k(y)dy.$$

Let

$$A(\sigma) := \{j : l_f(a^j(\sigma)) - l_g(a^j(\sigma)) \neq 0\} \subset \{1, ..., n\}. \tag{6.2.10}$$

By applying Proposition 6.1.2 to (5.2.3), we can see that each $I^{(k)}_{\pm,\sigma}(s)$ can be analytically continued to the whole complex plane as meromorphic functions and their poles are contained in the set

$$\bigcup_{j\in A(\sigma)} \mathcal{P}_{\mathbb{R}}(a^j(\sigma)). \tag{6.2.11}$$

Second, we consider the case of the functions $J^{(l)}_{\pm,\sigma}(s)$. By applying Lemma 4.4.1 and Remark 4.4.4, $J^{(l)}_{\pm,\sigma}(s)$ can be expressed as follows:

$$\begin{aligned} J^{(l)}_{\pm,\sigma}(s) = \int_{\mathbb{R}^n_+} &\left((u_i - b_i) \prod_{j\in A_l(\sigma)} u_j^{l_f(a^j(\sigma))-l_g(a^j(\sigma))}\right)^s_{\pm} \\ &\times \left|\prod_{j\in A_l(\sigma)} u_j^{\langle a^j(\sigma)\rangle-1}\right| \hat{\eta}_l(u_1, ..., u_i - b_i, ..., u_n)du, \end{aligned}$$

where $A_l(\sigma) \subset \{1, ..., n\}$, $i \notin A_l(\sigma)$, $b_i > 0$ and $\hat{\eta}_l \in C_0^\infty(\mathbb{R}^n)$ has a support containing the origin. Consider further changing the integral variables, we have

$$J^{(l)}_{\pm,\sigma}(s) = \int_{\mathbb{R}^n_+} \left(u_i^s \prod_{j\in A_l(\sigma)} u_j^{(l_f(a^j(\sigma))-l_g(a^j(\sigma)))s+\langle a^j(\sigma)\rangle-1}\right) \hat{\eta}_l(u_1, ..., \pm u_i, ..., u_n)du. \tag{6.2.12}$$

By applying Proposition 6.1.2 to (6.2.12), we can see that each $J^{(l)}_{\pm,\sigma}(s)$ can be analytically continued to the whole complex plane as meromorphic functions and their poles are contained in the set

$$\bigcup_{j\in \tilde{A}_l(\sigma)} \mathcal{P}_{\mathbb{R}}(a^j(\sigma)) \cup (-\mathbb{N}), \tag{6.2.13}$$

where $\hat{A}_l(\sigma) := \{j \in A_l(\sigma) : l_f(a^j(\sigma)) - l_g(a^j(\sigma)) \neq 0\}$.

Finally, we consider the case of the functions $K^{(m)}_{\pm,\sigma}(s)$. In a similar fashion to the case of $J^{(l)}_{\pm,\sigma}$, we have

$$K^{(m)}_{\pm,\sigma}(s) = \int_{\mathbb{R}^n_+} \left(v_i^{-s} \prod_{j \in A_m(\sigma)} v_j^{(l_f(a^j(\sigma)) - l_g(a^j(\sigma)))s + \langle a^j(\sigma) \rangle - 1} \right) \hat{\kappa}_m(v_1, ..., \pm v_i, ..., v_n) dv, \quad (6.2.14)$$

where $A_m(\sigma) \subset \{1, ..., n\}$, $i \notin A_m(\sigma)$, $b_i > 0$ and $\hat{\kappa}_m \in C_0^\infty(\mathbb{R}^n)$ has a support containing the origin. By applying Proposition 6.1.2 to (6.2.14) again, we can see that each $K^{(m)}_{\pm,\sigma}(s)$ can be analytically continued to the whole complex plane as meromorphic functions and their poles are contained in the set

$$\bigcup_{j \in \tilde{A}_m(\sigma)} \mathcal{P}_{\mathbb{R}}(a^j(\sigma)) \cup \mathbb{N}, \quad (6.2.15)$$

where $\hat{A}_m(\sigma) := \{j \in A_m(\sigma) : l_f(a^j(\sigma)) - l_g(a^j(\sigma)) \neq 0\}$.

Considering the relation (6.2.1), (6.2.3) and above argument, the poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$ are contained in the union of the sets (6.2.11), (6.2.13) and (6.2.15). This implies the assertion-(i) in Theorem 6.0.2 for the case $K = \mathbb{R}$.

6.2.3 The case of $K = \mathbb{C}$

In the case of $K = \mathbb{C}$, $Z_{\mathbb{C}}(s; f, g, \varphi)$ can be written as

$$Z_{\mathbb{C}}(s; f, g, \varphi) = \int_{\mathbb{C}^n \setminus D_{\mathbb{C}}} \left| \frac{f(x)}{g(x)} \right|^{2s} \varphi(x) dx \wedge d\overline{x}.$$

Applying the complex toric resolution of singularities $x = \pi_{\mathbb{C}}(w)$ and using the partition of unity $\{\tilde{\chi}_\sigma : \tilde{Y}_{\hat{\Sigma}} \to \mathbb{R}_+ : \sigma \in \hat{\Sigma}^{(n)}\}$ on $\tilde{Y}_{\hat{\Sigma}}$, we have

$$Z_{\mathbb{C}}(s; f, g, \varphi) = \sum_{\sigma \in \hat{\Sigma}^{(n)}} \tilde{Z}_\sigma(s; f, g) \quad (6.2.16)$$

with

$$\tilde{Z}_\sigma(s; f, g) = \int_{\mathbb{C}^n} \left| \prod_{j=1}^n w_j^{l_f(a^j(\sigma)) - l_g(a^j(\sigma))} \cdot \frac{f_\sigma(w)}{g_\sigma(w)} \right|^{2s} \check{\chi}_\sigma(w) \left| \prod_{j=1}^n w_j^{\langle a^j(\sigma) \rangle - 1} \right|^2 dw \wedge d\overline{w}, \quad (6.2.17)$$

where $\check{\chi}_\sigma = (\varphi \circ \pi_{\mathbb{C}})(w) \tilde{\chi}_\sigma(w)$. Furthermore, for each σ, we find finite sets of C_0^∞ functions $\{\xi_k : \mathbb{C}^n \to \mathbb{R}_+\}$, $\{\eta_l : \mathbb{C}^n \to \mathbb{R}_+\}$ and $\{\kappa_m : \mathbb{C}^n \to \mathbb{R}_+\}$ as in section 5.2 replaced $\hat{\chi}_\sigma$ with

$\check{\chi}_\sigma$. Then, considering the polar coordinate $w_j = r_j e^{i\theta_j}$ with $r_j \geq 0$ and $\theta_j \in [0, 2\pi]$ and Remark 4.4.4, $\tilde{Z}_\sigma(s; f, g)$ is expressed as

$$\tilde{Z}_\sigma(s; f, g) = \sum_k \tilde{I}_\sigma^{(k)}(s) + \sum_l \tilde{J}_\sigma^{(l)}(s) + \sum_m \tilde{K}_\sigma^{(m)}(s),$$

where

$$\tilde{I}_\sigma^{(k)}(s) = \int_{\mathbb{R}_+^n} \prod_{j=1}^n r_j^{2(l_f(a^j(\sigma)) - l_g(a^j(\sigma)))s + 2\langle a^j(\sigma)\rangle - 1} \cdot \mathcal{H}_k(r, s) dr, \tag{6.2.18}$$

$$\tilde{J}_\sigma^{(l)}(s) = \int_{\mathbb{R}_+^n} r_i^{2s+1} \prod_{j \in A_l(\sigma)} r_j^{2(l_f(a^j(\sigma)) - l_g(a^j(\sigma)))s + 2\langle a^j(\sigma)\rangle - 1} \cdot \mathcal{H}_l(r) dr, \tag{6.2.19}$$

$$\tilde{K}_\sigma^{(m)}(s) = \int_{\mathbb{R}_+^n} r_i^{-2s+1} \prod_{j \in A_m(\sigma)} r_j^{2(l_f(a^j(\sigma)) - l_g(a^j(\sigma)))s + 2\langle a^j(\sigma)\rangle - 1} \cdot \mathcal{H}_m(r) dr \tag{6.2.20}$$

with

$$\begin{aligned}
\mathcal{H}_k(r, s) &= \int_{[0,2\pi]^n} \left| \frac{f_\sigma(re^{i\theta})}{g_\sigma(re^{i\theta})} \right|^{2s} \cdot \check{\chi}_\sigma(re^{i\theta}) \cdot \xi_k(re^{i\theta}) d\theta, \\
\mathcal{H}_l(r) &= \int_{[0,2\pi]^n} \check{\chi}_\sigma(re^{i\theta}) \cdot \tilde{\eta}_l(re^{i\theta}) d\theta, \\
\mathcal{H}_m(r) &= \int_{[0,2\pi]^n} \check{\chi}_\sigma(re^{i\theta}) \cdot \tilde{\kappa}_m(re^{i\theta}) d\theta.
\end{aligned}$$

Here, $re^{i\theta} = (r_1 e^{i\theta_1}, ..., r_n e^{i\theta_n})$ and $\tilde{\eta}_l(\cdot), \tilde{\kappa}_m(\cdot)$ are C_0^∞ functions whose supports contain the origin. From the above expressions, it is easy to see that $\mathcal{H}_k(\cdot, s)$ are C^∞ functions for any $s \in \mathbb{C}$, $\mathcal{H}_k(r, \cdot)$ are holomorphic functions for any $r \in \mathbb{R}_+$ and $\mathcal{H}_l(\cdot), \mathcal{H}_m(\cdot)$ are C^∞ functions. Therefore, we can apply same argument in the proof of real case to the integrals (6.2.18), (6.2.19) and (6.2.20). Consequently, in the case of $K = \mathbb{C}$, we see that the poles of $Z_{\mathbb{C}}(s; f, g, \varphi)$ are contained in the set (6.0.3).

Order of poles

Next, let us consider the orders of poles of $Z_K(s; f, g, \varphi)$. At first, we give the proof of the assertion-(ii) in Theorem 6.0.2.

Lemma 6.2.2.

$$\max\left\{ -\frac{\langle a\rangle}{l_f(a) - l_g(a)} : a \in V_+ \right\} = -\frac{1}{d_\infty(f, g)}. \tag{6.2.21}$$

$$\min\left\{ -\frac{\langle a\rangle}{l_f(a) - l_g(a)} : a \in V_- \right\} = \frac{1}{d_0(f, g)}. \tag{6.2.22}$$

Proof. We only consider the case of $d_\infty(f,g)$. By Proposition 5.2.1, for any $a \in \mathbb{R}_+^n$, we have the following equivalence:

$$d_\infty(f,g) \geq d(f,g,a) = \frac{l_f(a) - l_g(a)}{\langle a \rangle} \Longleftrightarrow -\frac{1}{d_\infty(f,g)} \geq -\frac{\langle a \rangle}{l_f(a) - l_g(a)}.$$

Moreover, from the construction of $\hat{\Sigma}$, we see the existence of $a \in V_+$ satisfying the equality in the last inequality. Therefore, we have the equation in the lemma. $\square$

Obviously, each element of the left sets in (6.2.21), (6.2.22) is equal to the first element of the arithmetic progression $\mathcal{P}_K(a)$. This implies the assertions (ii) in Theorem 6.0.2.

Let us consider the orders of poles of $Z_K(s; f, g, \varphi)$ at $s = -1/d_\infty(f,g)$ and $s = 1/d_0(f,g)$. From the equations (5.2.1), (5.2.2), it suffices to analyze the poles of $I_{\pm,\sigma}^{(k)}(s)$, $J_{\pm,\sigma}^{(l)}(s)$ and $K_{\pm,\sigma}^{(m)}(s)$. Applying Proposition 6.1.2 to the integrals (5.2.3), (5.2.5) and (5.2.6), we see that orders of poles at $s = -1/d_\infty(f,g)$, $1/d_0(f,g)$ of $I_{\pm,\sigma}^{(k)}(s)$ are at most $\#A_\infty(\sigma)$, $\#A_0(\sigma)$ and those of $J_{\pm,\sigma}^{(l)}(s)$ and $K_{\pm,\sigma}^{(m)}(s)$ are at most

$$\begin{cases} \min\{\#A_\infty(\sigma), n-1\} & -1/d_\infty(f,g) \notin \mathbb{Z}, \\ \min\{\#A_\infty(\sigma)+1, n\} & -1/d_\infty(f,g) \in \mathbb{Z}; \end{cases}$$

$$\begin{cases} \min\{\#A_0(\sigma), n-1\} & 1/d_0(f,g) \notin \mathbb{Z}, \\ \min\{\#A_0(\sigma)+1, n\} & 1/d_0(f,g) \in \mathbb{Z}. \end{cases}$$

Here, $A_\infty(\sigma)$ and $A_0(\sigma)$ are as in (5.2.4).

To show the assertions (iii) and (iv) in Theorem 6.0.2, it suffices to show the estimates $\#A_\infty(\sigma) \leq m_\infty(f,g)$ and $\#A_0(\sigma) \leq m_0(f,g)$. We can obtain these estimates from Lemma 5.2.7 and Proposition 6.1.2.

6.3 First coefficients of $Z_K(s; f, g, \varphi)$

In this section, we compute the coefficients of Laurent series of $Z_K(s; f, g, \varphi)$ at the (candidate) e-leading poles and give the conditions where the position and order of each e-leading pole are determined by means of Newton data.

6.3.1 The case of $K = \mathbb{R}$

At first, we compute the coefficients of $(s + 1/d_\infty(f, g))^{-m_\infty(f,g)}$ and $(s - 1/d_0(f, g))^{-m_0(f,g)}$ in the Laurent series of $Z_\pm(s; f, g, \varphi)$ for the case of $K = \mathbb{R}$. Respectively, we define

$$\begin{aligned} \mathcal{C}_\pm &= \lim_{s \to -1/d_\infty(f,g)} (s + 1/d_\infty(f, g))^{m_\infty(f,g)} \zeta_\pm(s; f, g, \varphi), \\ \mathcal{D}_\pm &= \lim_{s \to 1/d_0(f,g)} (s - 1/d_0(f, g))^{m_0(f,g)} \zeta_\pm(s; f, g, \varphi), \end{aligned} \tag{6.3.1}$$

where $\zeta_\pm(s; f, g, \varphi)$ is as in (6.2.4).

Here, we recall the definitions of important fans $\tilde{\Sigma}_\infty^{(n)}, \tilde{\Sigma}_0^{(n)}$ defined in Chapter 5.

$$\begin{aligned} \tilde{\Sigma}_\infty^{(n)} &:= \{\sigma \in \hat{\Sigma}_\infty^{(n)} : \#A_\infty(\sigma) = m_\infty(f, g)\}, \\ \tilde{\Sigma}_0^{(n)} &:= \{\sigma \in \hat{\Sigma}_0^{(n)} : \#A_0(\sigma) = m_0(f, g)\}. \end{aligned}$$

In this subsection, we use the following notation and symbols to decrease the complexity in the expression of the integrals.

- $\prod_{j \notin A} y_j^{a_j} dy_j$ means $\prod_{j \notin A} y_j^{a_j} \cdot \prod_{j \notin A} dy_j$ with $a_j \geq 0$.
- $L(A) := \prod_{j \in A} (l_f(a^j(\sigma)) - l_g(a^j(\sigma)))^{-1}$.
- $T_A(y) := \{(y_1, ..., y_n) \in \mathbb{R}^n : y_j = 0 \text{ if } j \in A\}$.
- $A_\infty^\sigma := A_\infty(\sigma)$, $A_0^\sigma := A_0(\sigma)$.

Here, A is a subset of $\{1, ..., n\}$.

First, let us consider the coefficients of $(s + 1/d_\infty(f, g))^{-m_\infty(f,g)}$.

Proposition 6.3.1. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and at least one of the following conditions is satisfied.*

(a) $d_\infty(f, g) > 1$;

(b) $f_\sigma(T_{A_\infty^\sigma}(y))$ does not vanish on $\mathbb{R}_+^n \cap \pi(\sigma)^{-1}(U)$ for any $\sigma \in \tilde{\Sigma}_\infty^{(n)}$.

Then, we have explicit formulae for coefficients $\mathcal{C}_\pm = \sum_{\sigma \in \tilde{\Sigma}_\infty^{(n)}} \mathcal{C}_\pm(\sigma) =: G_\pm(f, g, \varphi)$, where $G_\pm(f, g, \varphi)$ are as in (6.3.8), (6.3.10), (6.3.11), (6.3.12) *in the proof of this proposition.*

Proof. Let us compute the limit $\mathcal{C}_\pm$ explicitly. Let

$$M_j(\sigma) := -(l_f(a^j(\sigma)) - l_g(a^j(\sigma)))/d_\infty(f,g) + \langle a^j(\sigma)\rangle.$$

Respectively, we define

$$\mathcal{C}_\pm(\sigma) := \lim_{s\to -1/d_\infty(f,g)} (s+1/d_\infty(f,g))^{m_\infty(f,g)} \zeta_\pm^{(\sigma)}(s;\varphi,f,g),$$

where $\zeta_\pm^{(\sigma)}(s;\varphi,f,g)$ is as in (6.2.6). If $\sigma \notin \tilde{\Sigma}_\infty^{(n)}$, then $\mathcal{C}_\pm(\sigma) = 0$ since the orders of poles of $\zeta_\pm^{(\sigma)}(s;\varphi,f,g)$ at $s = -1/d_\infty(f,g)$ is less than $m_\infty(f,g)$. Thus, it suffices to consider the case when $\sigma \in \tilde{\Sigma}_\infty^{(n)}$. We divide the computation into the following two cases: $m_\infty(f,g) < n$ and $m_\infty(f,g) = n$.

(The case: $m_\infty(f,g) = N < n$)

First, we consider the case when condition (a) is satisfied. Considering the equation (5.2.2) and applying Proposition 6.1.3 to (5.2.3), (5.2.5) and (5.2.6), we have

$$\mathcal{C}_\pm(\sigma) = \sum_k I_\pm^{(k)}(\sigma) + \sum_l J_\pm^{(l)}(\sigma) + \sum_m K_\pm^{(m)}(\sigma) \tag{6.3.2}$$

with

$$I_\pm^{(k)}(\sigma) = L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{f_\sigma(T_{A_\infty^\sigma}(y))}{g_\sigma(T_{A_\infty^\sigma}(y))}\right)_\pm^{-1/d_\infty(f,g)} \hat{\xi}_k(T_{A_\infty^\sigma}(y)) \prod_{j\notin A_\infty^\sigma} y_j^{M_j(\sigma)-1} dy_j \tag{6.3.3}$$

and

$$J_\pm^{(l)}(\sigma) = L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \frac{\hat{\eta}_l(T_{A_\infty^\sigma}(u_1,\dots,\pm u_i,\dots,u_n))}{u_i^{1/d_\infty(f,g)}} \prod_{j\in A_l(\sigma)\setminus A_\infty^\sigma} u_j^{M_j(\sigma)-1} du_j, \tag{6.3.4}$$

$$K_\pm^{(m)}(\sigma) = L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \frac{\hat{\kappa}_m(T_{A_\infty^\sigma}(v_1,\dots,\pm v_i,\dots,v_n))}{v_i^{-1/d_\infty(f,g)}} \prod_{j\in A_m(\sigma)\setminus A_\infty^\sigma} v_j^{M_j(\sigma)-1} dv_j, \tag{6.3.5}$$

where $\hat{\xi}_k, \hat{\eta}_l, \hat{\kappa}_m, A_l(\sigma), A_m(\sigma), u_i, v_i$ are as in (5.2.3), (5.2.5), (5.2.6). The summations in (6.3.2) are taken for all k,l,m satisfying that $T_{A_\infty^\sigma}(\mathbb{R}^n)\cap \mathrm{Supp}(\xi_k) \neq \emptyset$ and $A_\infty^\sigma \subset A_l(\sigma), A_m(\sigma)$, respectively. Since $d_\infty(f,g) > 1$, the integrals in (6.3.3), (6.3.4) are convergent as improper integrals.

We remark that the values of $I_\pm^{(k)}(\sigma)$, $J_\pm^{(l)}(\sigma)$ and $K_\pm^{(m)}(\sigma)$ depend on the cut-off functions χ_σ, ξ_k, η_l, κ_m. In (6.3.3), (6.3.4) and (6.3.5), we deform the cut-off functions ξ_k, η_l and κ_m

as the volume of the support of η_l and κ_m tend to zero for all l,m. Then the limits of the values of $J_{l,\pm}(\sigma)$ and $K_{m,\pm}(\sigma)$ are zero and we have

$$\mathcal{C}_{\pm}(\sigma) = \sum_k I_{\pm}^{(k)}(\sigma). \tag{6.3.6}$$

Furthermore, considering the support of $\sum_k \xi_k(y)$, we have

$$\mathcal{C}_{\pm}(\sigma) = L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{f_\sigma(T_{A_\infty^\sigma}(y))}{g_\sigma(T_{A_\infty^\sigma}(y))}\right)_{\pm}^{-1/d_\infty(f,g)} \hat{\chi}_\sigma(T_{A_\infty^\sigma}(y)) \prod_{j\notin A_\infty^\sigma} y_j^{M_j(\sigma)-1} dy_j. \tag{6.3.7}$$

Furthermore, let us compute the limits $\mathcal{C}_\pm$ explicitly. If the cut-off function χ_σ is deformed as the volume of the support of χ_σ tends to zero, then $\mathcal{C}_\pm(\sigma)$ tends to zero. Since each $\mathbb{R}_+^n(\sigma)$ is densely embedded in $Y_{\hat{\Sigma}}$ and $\mathcal{C}_\pm = \sum_{\sigma\in\tilde{\Sigma}_\infty^{(n)}} \mathcal{C}_\pm(\sigma)$, for any fixed cone $\sigma \in \tilde{\Sigma}_\infty^{(n)}$, we have $\mathcal{C}_\pm = G_\pm(f,g,\varphi)$ with

$$G_\pm(f,g,\varphi) = L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{f_\sigma(T_{A_\infty^\sigma}(y))}{g_\sigma(T_{A_\infty^\sigma}(y))}\right)_{\pm}^{-1/d_\infty(f,g)} (\varphi\circ\pi(\sigma))(T_{A_\infty^\sigma}(y)) \prod_{j\notin A_\infty^\sigma} y_j^{M_j(\sigma)-1} dy_j. \tag{6.3.8}$$

We remark that the above integral does not depend on the cut-off functions. Let us give the other formulae of $G_\pm(f,g,\varphi)$, which are more directly expressed by f,g,φ with principal faces at infinity $\gamma_\infty := \gamma_\infty(\sigma) \in \mathcal{F}_\infty[f]$ and $\tau_\infty := \tau_\infty(\sigma) \in \mathcal{F}_\infty[g]$ which is associated to each other. From Lemma 4.4.2, we obtain

$$\begin{aligned} (f_{\gamma_\infty}\circ\pi_{\mathbb{R}}(\sigma))(T^1_{A_\infty^\sigma}(y)) &= \left(\prod_{j\notin A_\infty^\sigma} y_j^{l_f(a^j(\sigma))}\right)\cdot f_\sigma(T_{A_\infty^\sigma}(y)),\\ (g_{\tau_\infty}\circ\pi_{\mathbb{R}}(\sigma))(T^1_{A_\infty^\sigma}(y)) &= \left(\prod_{j\notin A_\infty^\sigma} y_j^{l_g(a^j(\sigma))}\right)\cdot g_\sigma(T_{A_\infty^\sigma}(y)). \end{aligned} \tag{6.3.9}$$

By using the above equations, (6.3.8) can be rewritten as

$$\begin{aligned} &G_\pm(f,g,\varphi)\\ &= L(A_\infty^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{(f_{\gamma_\infty}\circ\pi_{\mathbb{R}}(\sigma))(T^1_{A_\infty^\sigma}(y))}{(g_{\tau_\infty}\circ\pi_{\mathbb{R}}(\sigma))(T^1_{A_\infty^\sigma}(y))}\right)_{\pm}^{-1/d_\infty(f,g)} (\varphi\circ\pi_{\mathbb{R}}(\sigma))(T_{A_\infty^\sigma}(y)) \prod_{j\notin A_\infty^\sigma} y_j^{\langle a^j(\sigma)\rangle-1} dy_j. \end{aligned} \tag{6.3.10}$$

Secondly, we consider the case when the condition (b) is satisfied. In this case, $J^{(l)}_{\pm,\sigma}(s)$ do not appear in the decompositions (5.2.2) and the integral in (6.3.3) is convergent since

$f_\sigma(T_{A_\infty^\sigma}(y))$ does not vanish. Thus, we can obtain the equations (6.3.7) for $\sigma \in \tilde{\Sigma}_\infty^{(n)}$ by the same argument as in the case of condition (a). However, in this case, we must be careful that $\mathcal{C}_\pm(\sigma)$ do not always vanish even in the case when $\sigma \notin \tilde{\Sigma}_\infty^{(n)}$. If $-1/d_\infty(f,g)$ is an integer and $\sigma \notin \tilde{\Sigma}_\infty^{(n)}$ satisfies $\#A_\infty(\sigma) = m_\infty(f,g) - 1$, the assertion-(iii) in Theorem 6.0.2 implies that $J_{\pm,\sigma}^{(l)}(s)$ can have a pole at $s = -1/d_\infty(f,g)$ of order $m_\infty(f,g)$. Indeed, for such σ, the orders of poles of $I_{\pm,\sigma}^{(k)}(s)$ at $s = -1/d_\infty(f,g)$ are at most $m_\infty(f,g) - 1$, so the value of $\mathcal{C}_\pm(\sigma)$ derives from the integral $J_{\pm,\sigma}^{(l)}(s)$ only. Hence, coefficients $\mathcal{C}_\pm(\sigma)$ can be computed in a similar argument as in the proof of Proposition 6.1.3. From these computations, it is easy to see that these coefficients tend to zero if the volume of the support of χ_σ tends to zero. Therefore, the limits $\mathcal{C}_\pm$ can be similarly computed as in (6.3.8) and (6.3.10).

(The case: $m_\infty(f,g) = n$)

In this case, by the same argument, we have

$$G_\pm(f,g,\varphi) = L \cdot \left(\frac{f_\sigma(0)}{g_\sigma(0)}\right)_\pm^{-1/d_\infty(f,g)} \varphi(0), \tag{6.3.11}$$

where $\sigma \in \tilde{\Sigma}_\infty^{(n)}$ and $L = L(\{1,...,n\})$. From the equation (6.3.9), we obtain another expression corresponding to (6.3.10):

$$G_\pm(f,g,\varphi) = L \cdot \left(\frac{f_{\gamma_\infty}(\mathbf{1})}{g_{\tau_\infty}(\mathbf{1})}\right)_\pm^{-1/d_\infty(f,g)} \varphi(0). \tag{6.3.12}$$

□

For the coefficients of $(s - 1/d_0(f,g))^{-m_0(f,g)}$, we obtain similar result.

Proposition 6.3.2. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and at least one of the following conditions is satisfied.*

(a) $d_0(f,g) > 1$;

(b) $g_\sigma(T_{A_0^\sigma}(y))$ does not vanish on $\mathbb{R}_+^n \cap \pi(\sigma)^{-1}(U)$ for any $\sigma \in \tilde{\Sigma}_0^{(n)}$.

Then, we have explicit formulae for coefficients $\mathcal{D}_\pm = \sum_{\sigma\in\tilde{\Sigma}_0^{(n)}} \mathcal{D}_\pm(\sigma) =: H_\pm(f,g,\varphi)$, where $H_\pm(f,g,\varphi)$ are as in (6.3.13), (6.3.15), (6.3.14), (6.3.16) *in the proof of this proposition.*

Proof. Let

$$m_j(\sigma) := (l_f(a^j(\sigma)) - l_g(\sigma))/d_0(f,g) + \langle a^j(\sigma)\rangle.$$

Respectively, we define

$$\mathcal{D}_{\pm}(\sigma) := \lim_{s \to 1/d_0(f,g)} (s - 1/d_0(f,g))^{m_0(f,g)} \zeta_{\pm}^{(\sigma)}(s; \varphi, f, g).$$

By the same argument in the proof of Proposition 6.3.1, we obtain $H_{\pm}(f, g, \varphi)$ as follows: if $m_0(f, g) = N < n$,

$$H_{\pm}(f, g, \varphi) = L(A_0^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{f_\sigma(T_{A_0^\sigma}(y))}{g_\sigma(T_{A_0^\sigma}(y))} \right)_{\pm}^{1/d_0(f,g)} (\varphi \circ \pi(\sigma))(T_{A_0^\sigma}(y)) \prod_{j \notin A_0^\sigma} y_j^{m_j(\sigma)-1} dy_j \tag{6.3.13}$$

and if $m_0(f, g) = n$,

$$H_{\pm}(f, g, \varphi) = L \cdot \left(\frac{f_\sigma(0)}{g_\sigma(0)} \right)_{\pm}^{1/d_0(f,g)} \varphi(0). \tag{6.3.14}$$

where $\sigma \in \tilde{\Sigma}_0^{(n)}$ and $L = L(\{1, ..., n\})$.

We obtain the other formulae of $H_{\pm}(f, g, \varphi)$ in a similar fashion to the proof of Proposition 6.3.1 with principal faces at zero $\tau_0 := \tau_0(\sigma) \in \mathcal{F}_0[g]$ and $\gamma_0 := \gamma_0(\sigma) \in \mathcal{F}_0[f]$, which is associated to each other. If $m_0(f, g) = N < n$, then

$$\begin{aligned} & H_{\pm}(f, g, \varphi) \\ & = L(A_0^\sigma) \int_{\mathbb{R}_+^{n-N}} \left(\frac{(f_{\gamma_0} \circ \pi_{\mathbb{R}}(\sigma))(T^1_{A_0^\sigma}(y))}{(g_{\tau_0} \circ \pi_{\mathbb{R}}(\sigma))(T^1_{A_0^\sigma}(y))} \right)_{\pm}^{1/d_0(f,g)} (\varphi \circ \pi_{\mathbb{R}}(\sigma))(T_{A_0^\sigma}(y)) \prod_{j \notin A_0^\sigma} y_j^{\langle a^j(\sigma) \rangle - 1} dy_j \end{aligned} \tag{6.3.15}$$

and if $m_0(f, g) = n$, then

$$H_{\pm}(f, g, \varphi) = L \cdot \left(\frac{f_{\gamma_0}(\mathbf{1})}{g_{\tau_0}(\mathbf{1})} \right)_{\pm}^{1/d_0(f,g)} \varphi(0). \tag{6.3.16}$$

□

Finally, let us compute the coefficients of $(s+1/d_\infty(f,g))^{-m_\infty(f,g)}$ and $(s-1/d_0(f,g))^{-m_0(f,g)}$ in the Laurent series of $Z_{\pm}(s; f, g, \varphi)$ and $Z_{\mathbb{R}}(s; f, g, \varphi)$. Respectively, we define

$$\begin{aligned} C_{\pm} &= \lim_{s \to -1/d_\infty(f,g)} (s + 1/d_\infty(f,g))^{m_\infty(f,g)} Z_{\pm}(s; f, g, \varphi), \\ C &= \lim_{s \to -1/d_\infty(f,g)} (s + 1/d_\infty(f,g))^{m_\infty(f,g)} Z_{\mathbb{R}}(s; f, g, \varphi); \end{aligned}$$

$$\begin{aligned} D_{\pm} &= \lim_{s \to 1/d_0(f,g)} (s - 1/d_0(f,g))^{m_0(f,g)} Z_{\pm}(s; f, g, \varphi), \\ D &= \lim_{s \to 1/d_0(f,g)} (s - 1/d_0(f,g))^{m_0(f,g)} Z_{\mathbb{R}}(s; f, g, \varphi). \end{aligned}$$

Theorem 6.3.3. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and at least one of the following three conditions is satisfied:*

(a) $d_\infty(f,g) > 1$;

(b) f *is nonnegative or nonpositive on* U;

(c) f_{γ_∞} *does not vanish on* $U \cap (\mathbb{R} \setminus \{0\})^n$, *where* γ_∞ *is a principal face at infinity of* $\Gamma_+(f)$.

Then, we obtain explicit formulae for coefficients in the following:

$$C_\pm = \sum_{\theta \in \{-1,1\}^n} G_\pm(f_\theta, g_\theta, \varphi_\theta) \quad \text{and} \quad C = C_+ + C_-, \tag{6.3.17}$$

where f_θ, g_θ *and* φ_θ *are as in* (6.2.5) *and* $G_\pm(f,g,\varphi)$ *are as in Propositions 6.3.1.*

Furthermore, if φ *satisfies that* $\varphi(x) \geq 0$ *and* $\varphi(0) > 0$ *on its support, then* $C_\pm$ *are nonnegative and* $C = C_+ + C_-$ *is positive.*

Proof. From the equation (6.2.3), (6.2.4), we must show that the conditions (a),(b),(c) imply the conditions in Proposition 6.3.1 to obtain the formulae (6.3.17).

Since the case (a) is obvious, we only consider the cases (b) and (c). It suffices to show that the conditions (b) and (c) imply that $f_\sigma(T_{A^\sigma_-}(y))$ does not vanish on $\mathbb{R}^n \cap \pi_{\mathbb{R}}(\sigma)^{-1}(U)$ for any $\sigma \in \tilde{\Sigma}_\infty^{(n)}$.

Assume that for some $\sigma \in \tilde{\Sigma}_\infty^{(n)}$ there exists a point $b \in T_{A^\sigma_\infty(\mathbb{R}^n)} \cap \pi_{\mathbb{R}}(\sigma)^{-1}(U)$ such that $f_\sigma(b) = 0$. By Lemma 4.4.6, f is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and Proposition 4.4.3 implies that there exists points $b_1, b_2 \in T_{A^\sigma_\infty(\mathbb{R}^n)} \cap \pi_{\mathbb{R}}(\sigma)^{-1}(U)$ near b such that $f_\sigma(b_1) > 0$ and $f_\sigma(b_2) < 0$. From the equations (4.4.2), (6.3.9) and $g_\sigma(y)$ does not vanish near b, it is easy to see that the conditions (b) and (c) induce the contradiction to the existence of the above points b_1, b_2.

In order to see $C = C_+ + C_- \neq 0$ from the formula (6.3.10), it suffices to show that $(g_{\tau_\infty} \circ \pi_{\mathbb{R}}(\sigma))(T_{A^\sigma_\infty}(y))$ does not identically equal to zero near the origin. This follows from the equation (6.3.9) and $g_\sigma(0) \neq 0$. □

Theorem 6.3.4. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron and at least one of the following three conditions is satisfied:*

(a) $d_0(f,g) > 1$;

(b) g is nonnegative or nonpositive;

(c) g_{τ_0} does not vanish on $U \cap (\mathbb{R} \setminus \{0\})^n$, where τ_0 is a principal face at zero of $\Gamma_+(g)$.

Then, we obtain explicit formulae for coefficients in the following:

$$D_\pm = \sum_{\theta \in \{-1,1\}^n} H_\pm(f_\theta, g_\theta, \varphi_\theta) \quad \text{and} \quad D = D_+ + D_- \tag{6.3.18}$$

where f_θ, g_θ and φ_θ are as in (6.2.5) and $H_\pm(f, g, \varphi)$ are as in Proposition 6.3.2.

Furthermore, assume that φ satisfies that $\varphi(x) \geq 0$ and $\varphi(0) > 0$ on its support. If $m_0(f, g)$ is odd, then $D_\pm$ are nonpositive and $D = D_+ + D_-$ is negative. If $m_0(f, g)$ is even, then $D_\pm$ are nonnegative and $D = D_+ + D_-$ is positive

Proof. In a similar fashion to the proof of Theorem 6.3.3, we obtain the formulae (6.3.18) under the one of above three conditions. The signature of $D_\pm$ and D are seen by checking the signature of $L(A_0^\sigma)$. □

Theorems 6.3.3, 6.3.4 imply that $Z_{\mathbb{R}}(s; f, g, \varphi)$ has at $s = -1/d_\infty(f, g)$ (resp. $1/d_0(f, g)$) poles of orders $m_\infty(f, g)$ (resp. $m_0(f, g)$).

Poles on integers

Let us consider the properties of poles of $Z_\pm(s; f, g, \varphi)$ at integers. For $\lambda \in \mathbb{N}$, define

$$\begin{aligned} A_{\pm\lambda}(\sigma) &:= \{j \in A(\sigma) : -(l_f(a^j(\sigma)) - l_g(a^j(\sigma))\lambda + \langle a^j(\sigma) \rangle - 1 \in \pm\mathbb{N}\}, \\ \rho_{\pm\lambda} &:= \min\{\max\{\# A_{\pm\lambda}(\sigma) : \sigma \in \hat{\Sigma}^{(n)}\}, n - 1\}, \end{aligned}$$

where $A(\sigma)$ is as in (5.2.4).

Proposition 6.3.5. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{R}$ with respect to its Newton polyhedron. If the support of φ is contained in a sufficiently small neighborhood of the origin, then we have the following:*

(i) The orders of poles of $Z_\pm(s; f, g, \varphi)$ at $s = -\lambda \in -\mathbb{N}$ are at most $\rho_{-\lambda} + 1$. In particular, if $-\lambda > -1/d_\infty(f, g)$, then these orders are at most 1. Moreover, let $a_{-\lambda}^\pm$ be the coefficients of $(s + \lambda)^{-\rho_{-\lambda} - 1}$ in the Laurent series of $Z_\pm(s; f, g, \varphi)$ at $s = -\lambda$, respectively. Then we have $a_{-\lambda}^+ = (-1)^{\lambda - 1} a_{-\lambda}^-$ for $-\lambda \in -\mathbb{N}$.

(ii) *The orders of poles of $Z_{\pm}(s; f, g, \varphi)$ at $s = \lambda \in \mathbb{N}$ are at most $\rho_\lambda + 1$. In particular, if $\lambda < 1/d_0(f, g)$, then these orders are at most 1. Moreover, let $a_\lambda^{\pm}$ be the coefficients of $(s-\lambda)^{-\rho_\lambda - 1}$ in the Laurent series of $Z_{\pm}(s; f, g, \varphi)$ at $s = \lambda$, respectively. Then we have $a_\lambda^+ = -(-1)^{\lambda-1} a_\lambda^-$ for $\lambda \in \mathbb{N}$.*

Proof. Compared the forms of the integrals in (5.2.5), (5.2.6) and $L_{1,\pm}(s), L_{2,\pm}(s)$ in (5.1.4), we obtain above assertions by applying Lemma 6.1.4 to $J^{(l)}_{\pm,\sigma}(s)$ and $K^{(m)}_{\pm,\sigma}(s)$. □

6.3.2 The case of $K = \mathbb{C}$

Next, let us consider the case of $K = \mathbb{C}$. In this case, from the proof in subsection 6.2.3, there appear only one type decomposition as in (6.2.16), while two types decomposition "$\zeta_{\pm}^{(\sigma)}$" appear in the real case. Thus, the argument becomes a little simpler. In consequence, we obtain similar results to the real case.

We define

$$\begin{aligned} \tilde{C} &= \lim_{s \to -1/d_\infty(f,g)} (s + 1/d_\infty(f, g))^{m_\infty(f,g)} Z_{\mathbb{C}}(s; f, g, \varphi), \\ \tilde{D} &= \lim_{s \to 1/d_0(f,g)} (s - 1/d_0(f, g))^{m_0(f,g)} Z_{\mathbb{C}}(s; f, g, \varphi). \end{aligned} \tag{6.3.19}$$

Theorem 6.3.6. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{C}$ with respect to its Newton polyhedron and at least one of the following conditions is satisfied;*

(a) *$d_\infty(f, g) > 1$;*

(b) *$f_\sigma(T_{A_\infty^\sigma}(y))$ does not vanish on $\mathbb{C}^n \cap \pi_{\mathbb{C}}(\sigma)^{-1}$ for any $\sigma \in \tilde{\Sigma}_\infty^{(n)}$.*

Then, we have explicit formulae for coefficient $\tilde{C}$ as in (6.3.23), (6.3.24) *and* (6.3.25) *in the proof of this theorem. Furthermore, if the conditions (i) - (a) or (c) are satisfied, and φ satisfies the conditions in Theorem 6.0.6, then $\tilde{C}$ is positive.*

Proof. We define

$$\tilde{C}(\sigma) = \lim_{s \to -1/d_\infty(f,g)} (s + 1/d_\infty(f, g)) \tilde{Z}_\sigma(s; f, g), \tag{6.3.20}$$

where $\tilde{Z}_\sigma(s; f, g)$ is as in (6.2.17). In this proof, we use same notations appearing in the proof of Proposition 6.3.1, for instance, $M_j(\sigma), L(A_\infty^\sigma), T_{A_\infty^\sigma}(\cdot), T^1_{A_\infty^\sigma}(\cdot), A_l(\sigma), A_m(\sigma), L$. At first, we assume that the condition (a) is satisfied.

(The case : $m_\infty(f,g)=N<n$)

In a similar fashion to the proof of Proposition 6.3.1, we apply Proposition 6.1.3 to (6.2.18), (6.2.19) and (6.2.20). Then, we have

$$\tilde{C}(\sigma)=\sum_k \tilde{I}^{(k)}(\sigma)+\sum_l \tilde{J}^{(l)}(\sigma)+\sum_m \tilde{K}^{(m)}(\sigma),$$

where

$$\tilde{I}^{(k)}(\sigma)=\pi^N L(A_\infty^\sigma)\int_{\mathbb{R}_+^{n-N}}\int_{[0,2\pi]^{n-N}}\left|\frac{f_\sigma(T_{A_\infty^\sigma}(re^{i\theta}))}{g_\sigma(T_{A_\infty^\sigma}(re^{i\theta}))}\right|^{-2/d_\infty(f,g)}\check{\xi}_k(T_{A_\infty^\sigma}(re^{i\theta}))\prod_{j\notin A_\infty^\sigma} r_j^{2M_j(\sigma)-1}dr_j d\theta_j \tag{6.3.21}$$

and

$$\tilde{J}^{(l)}(\sigma)=\pi^N L(A_\infty^\sigma)\int_{\mathbb{R}_+^{n-N}}\int_{[0,2\pi]^{n-N}}\check{\eta}_l(T_{A_\infty^\sigma}(re^{i\theta}))r_i^{-2/d_\infty(f,g)}\prod_{j\in A_l(\sigma)\setminus A_\infty^\sigma} r_j^{2M_j(\sigma)-1}dr_j d\theta_j, \tag{6.3.22}$$

$$\tilde{K}^{(m)}(\sigma)=\frac{L(A_\infty^\sigma)}{2^N}\int_{\mathbb{R}_+^{n-N}}\left(\int_{[0,2\pi]^n}\check{\kappa}_m(T_{A_\infty^\sigma}(r),\theta)d\theta\right)r_i^{2/d_\infty(f,g)}\prod_{j\in A_m(\sigma)\setminus A_\infty^\sigma} r_j^{2M_j(\sigma)-1}dr_j,$$

where $\check{\xi}_k(\cdot)=\check{\chi}_\sigma(\cdot)\xi_k(\cdot)$, $\check{\eta}_l(\cdot)=\check{\chi}_\sigma(\cdot)\tilde{\eta}_l(\cdot)$, $\check{\kappa}_m(\cdot)=\check{\chi}_\sigma(\cdot)\tilde{\kappa}_m(\cdot)$. Note that the integrals (6.3.21), (6.3.22) are convergent as improper integrals because of $d_\infty(f,g)>2$. By deforming the support of ξ_k,η_l and κ_m as in the proof of Proposition 6.3.1, we have

$$\tilde{C}(\sigma)=\pi^N L(A_\infty^\sigma)\int_{\mathbb{R}_+^{n-N}}\int_{[0,2\pi]^{n-N}}\left|\frac{f_\sigma(T_{A_\infty^\sigma}(re^i\theta))}{g_\sigma(T_{A_\infty^\sigma}(re^{i\theta}))}\right|^{-2/d_\infty(f,g)}\check{\chi}_\sigma(T_{A_\infty^\sigma}(re^{i\theta}))\prod_{j\notin A_\infty^\sigma} r_j^{2M_j(\sigma)-1}dr_j d\theta_j.$$

Then, for any fixed cone $\sigma\in\tilde{\Sigma}_\infty^{(n)}$, by deforming the support of $\tilde{\chi}_\sigma$ in a similar fashion to the functions $\{\chi_\sigma\}$ and considering the polar coordinate exchange again, we have explicit formula for $\tilde{C}$ as

$$\tilde{C}=\frac{\pi^N L(A_\infty^\sigma)}{(2i)^{n-N}}\int_{\mathbb{C}^{n-N}}\left|\frac{f_\sigma(T_{A_\infty^\sigma}(w))}{g_\sigma(T_{A_\infty^\sigma}(w))}\right|^{-2/d_\infty(f,g)}(\varphi\circ\pi_{\mathbb{C}}(\sigma))(T_{A_\infty^\sigma}(w))\prod_{j\notin A_\infty^\sigma}|w_j|^{2M_j(\sigma)-2}dw_j\wedge d\overline{w}_j. \tag{6.3.23}$$

Furthermore, by using equations in (6.3.9), (6.3.23) can be rewritten as

$$\tilde{C} = \frac{\pi^N L(A^\sigma_\infty)}{(2i)^{n-N}} \int_{\mathbb{C}^{n-N}} \left| \frac{(f_{\gamma_\infty} \circ \pi_{\mathbb{C}}(\sigma))(T^1_{A^\sigma_\infty}(w))}{(g_{\tau_\infty} \circ \pi_{\mathbb{C}}(\sigma))(T^1_{A^\sigma_\infty}(w))} \right|^{-2/d_\infty(f,g)} (\varphi \circ \pi_{\mathbb{C}}(\sigma))(T_{A^\sigma_\infty}(w)) \times \prod_{j \notin A^\sigma_\infty} |w_j|^{2\langle a^j(\sigma)\rangle - 2} dw_j \wedge d\overline{w}_j. \tag{6.3.24}$$

(The case : $m_\infty(f,g) = n$)

We have similar formulae to (6.3.11), (6.3.12) in the case of $K = \mathbb{R}$.

$$\tilde{C} = \pi^n L \cdot \varphi(0) \cdot \left| \frac{f_\sigma(0)}{g_\sigma(0)} \right|^{-2/d_\infty(f,g)} = \pi^n L \cdot \varphi(0) \cdot \left| \frac{f_{\gamma_\infty}(\mathbf{1})}{g_{\tau_\infty}(\mathbf{1})} \right|^{-2/d_\infty(f,g)}. \tag{6.3.25}$$

These computations are applicable if the condition (b) is satisfied by the same reason discussed in the proof of Proposition 6.3.1. □

Similarly, the formulae of the coefficients of $(s - 1/d_0(f,g))^{-m_0(f,g)}$ are obtained.

Theorem 6.3.7. *Suppose that $f \cdot g$ is nondegenerate over $\mathbb{C}$ with respect to its Newton polyhedron and at least one of the following conditions is satisfied;*

(a) $d_0(f,g) > 1$;

(b) $f_\sigma(T_{A^\sigma_0}(y))$ *does not vanish on* $\mathbb{C}^n \cap \pi_{\mathbb{C}}(\sigma)^{-1}$ *for any* $\sigma \in \tilde{\Sigma}^{(n)}_0$.

Then, we have explicit formulae for coefficient $\tilde{D}$ as in (6.3.26), (6.3.27) *and* (6.3.28) *in the proof of this theorem. Furthermore, if the conditions (ii) - (a) or (c) are satisfied, and φ satisfies the conditions in Theorem 6.0.6, then $\tilde{D}$ is positive (resp. negative) when $m_0(f,g)$ is an even (resp. odd) integer.*

Proof. The notations $m_j(\sigma), L(A^\sigma_0), T_{A^\sigma_0}(\cdot), T^1_{A^\sigma_0}(\cdot), A_l(\sigma), A_m(\sigma), L$ are same as in the proof of Proposition 6.3.2. By the same argument in the proof of Theorem 6.3.6, we obtain explicit formulae for $\tilde{D}$ as follows:

if $m_0(f,g) = N < n$, then

$$\tilde{D} = \frac{\pi^N L(A^\sigma_0)}{(2i)^{n-N}} \int_{\mathbb{C}^{n-N}} \left| \frac{f_\sigma(T_{A^\sigma_0}(w))}{g_\sigma(T_{A^\sigma_0}(w))} \right|^{2/d_0(f,g)} (\varphi \circ \pi_{\mathbb{C}}(\sigma))(T_{A^\sigma_0}(w)) \prod_{j \notin A^\sigma_0} |w_j|^{2m_j(\sigma)-2} dw_j \wedge d\overline{w}_j. \tag{6.3.26}$$

Furthermore, we obtain the following other formulae of $\tilde{D}$ with principal faces at zero $\tau_0 := \tau_0(\sigma) \in \mathcal{F}_0[g]$ and $\gamma_0 := \gamma_0(\sigma) \in \mathcal{F}_0[f]$, which is associated to each other:

$$\begin{aligned}\tilde{D} = \frac{\pi^N L(A_0^\sigma)}{(2i)^{n-N}} \int_{\mathbb{C}^{n-N}} \left| \frac{(f_{\gamma_0} \circ \pi_{\mathbb{C}}(\sigma))(T^1_{A_0^\sigma}(w))}{(g_{\tau_0} \circ \pi_{\mathbb{C}}(\sigma))(T^1_{A_0^\sigma}(w))} \right|^{2/d_0(f,g)} & (\varphi \circ \pi_{\mathbb{C}}(\sigma))(T_{A_0^\sigma}(w)) \\ & \times \prod_{j \notin A_0^\sigma} |w_j|^{2\langle a^j(\sigma)\rangle - 2} dw_j \wedge d\overline{w}_j.\end{aligned} \tag{6.3.27}$$

If $m_0(f, g) = n$, then

$$\tilde{D} = \pi^n L \cdot \varphi(0) \cdot \left| \frac{f_\sigma(0)}{g_\sigma(0)} \right|^{2/d_0(f,g)} = \pi^n L \cdot \varphi(0) \cdot \left| \frac{f_{\gamma_0}(\mathbf{1})}{g_{\tau_0}(\mathbf{1})} \right|^{2/d_0(f,g)}. \tag{6.3.28}$$

$\square$

Chapter 7

Integral transforms and asymptotics of oscillatory integrals

In this chapter, we will show some theorems and lemmas on two important integral transforms. Indeed, these lemmas connect the poles of local zeta function $Z_{\mathbb{R}}(s; f, g, \varphi)$ with the asymptotic behavior of $I_{\varphi}(t; f, g)$ via Gelfand-Relay function $\mathcal{K}(u)$. After that, we will give the proofs of main theorems of this paper.

7.1 Mellin transform and Fourier transform

First, we introduce two integral transforms, Mellin transform and Fourier transform, which play important roles in the analysis of oscillatory integrals.

Definition 7.1.1. Let f be continuous and locally integrable function on $\mathbb{R}_+$. Then the *Mellin transform* of f is defined by

$$(\mathcal{M}f)(s) := \int_0^\infty x^{s-1} f(x) dx \quad (s \in \mathbb{C}). \tag{7.1.1}$$

Remark 7.1.2. Let f be as in Definition 7.1.1 and satisfy the following conditions

(i) $f(x) = \mathcal{O}(x^{-(a+\varepsilon)})$ as $x \to +0$;

(ii) $f(x) = \mathcal{O}(x^{-(b-\varepsilon)})$ as $x \to +\infty$,

where a, b are real constants with $a < b$ and $\varepsilon > 0$ is sufficiently small. Then, the integral in (7.1.1) absolutely converges and defines holomorphic function on the region $\{a < \mathrm{Re}(s) < b\}$.

The inverse transform of the Mellin transform is defined by the following contour integral

$$(\mathcal{M}^{-1}F)(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)ds,$$

where $a < c < b$. For the convergence of the above integral, $F(s)$ is required to be analytic in the strip $\{a < \mathrm{Re}(s) < b\}$ and tend to zero uniformly as $\mathrm{Im}(s) \to \pm\infty$ for $a < \mathrm{Re}(s) < b$.

The Fourier transform is defined as the integral below.

Definition 7.1.3. Let f be an absolutely integrable function. Then the *Fourier transform* of f is defined by

$$(\mathcal{F}f)(t) := \int_{-\infty}^{\infty} e^{itx}f(x)dx.$$

It is known that if f is a rapidly decreasing function, then so is $(\mathcal{F}f)$.

7.1.1 Relationship between $I_\varphi(t; f, g)$ and $Z_\mathbb{R}(s; f, g, \varphi)$

Now, let us consider the transformation of $I_\varphi(t; f, g)$ in (i-3) and $Z_\pm(s; f, g, \varphi)$ in (6.2.2).

Define the *Gelfand-Leray function* $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{K}(f, g, \varphi; u)(:= \mathcal{K}(u)) = \int_{W_u} \varphi(x)\omega,$$

where $W_u = \{x \in \mathbb{R}^n \setminus g^{-1}(0) : f(x)/g(x) = u\}$ for $u \in \mathbb{R}$ and ω is the surface element on W_u which is determined by $d(f/g) \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$.

Considering the change of variable $f(x)/g(x) = u$ and Fubini's theorem to $I_\varphi(t; f, g)$, we have

$$\begin{aligned} I_\varphi(t; f, g) = \int_{\mathbb{R}^n\setminus g^{-1}(0)} e^{it\frac{f(x)}{g(x)}}\varphi(x)dx &= \int_{W_u+W_{-u}} e^{it\frac{f(x)}{g(x)}}\varphi(x)dx \\ &= \int_{-\infty}^{\infty} e^{itu}\mathcal{K}(u)du. \end{aligned}$$

This shows that the oscillatory integral $I_\varphi(t; f, g)$ is a Fourier transform of $\mathcal{K}(u)$. On the other hand, consider the same change of variable and Fubini's theorem to $Z_\pm(s; f, g, \varphi)$, we have

$$Z_\pm(s;f,g,\varphi) = \int_{\mathbb{R}^n\setminus D_\mathbb{R}} \left(\frac{f(x)}{g(x)}\right)_\pm^s \varphi(x)dx = \int_0^\infty \int_{W_{\pm u}} (\pm u)^s \varphi(x) du \wedge \omega$$
$$= \int_0^\infty (\pm u)^s \mathcal{K}(\pm u) du.$$

Thus, we have

$$Z_\mathbb{R}(s;f,g,\varphi) = Z_+(s;f,g,\varphi) + Z_-(s;f,g,\varphi) = \int_{-\infty}^\infty u^s \mathcal{K}(u) du.$$

The last integral indicates that local zeta function $Z_\mathbb{R}(s;f,g,\varphi)$ is a Mellin transform of $\mathcal{K}(u)$. Since $I_\varphi(t;f,g)$ is a Fourier transform of $\mathcal{K}(u)$, our main object of investigation is $\mathcal{K}(u)$. In particular, we will focus on the influence of the critical points of f and g. To be more specific, since $u = f(x)/g(x)$, the properties of such singularities in $\mathcal{K}(u)$ are appearing when $|u|$ tends to zero and infinity, which correspond to the case when $f(x) \to 0$ and $g(x) \to 0$, respectively. In the next subsection, we investigate the properties of $\mathcal{K}(u)$ as an inverse Mellin transform of $Z_\mathbb{R}(s;f,g,\varphi)$.

7.1.2 Asymptotic expansion of the Gelfand-Relay function

Let us consider the asymptotic expansion of the Gelfand-Relay function $\mathcal{K}$ as its parameter $|u| \to 0$ and ∞.

Theorem 7.1.4. *Let $0 < p_1 < p_2 < \cdots$ be positive real numbers and k_j be nonnegative integers. Suppose that $Z_\pm(s;f,g,\varphi)$ has poles at $s = -p_j$ of order k_j for every j and $a_{j,k}^\pm$ are the coefficient of the term $(s+p_j)^{-k}$ of Laurent series of $Z_\pm(s;f,g,\varphi)$ at $s = -p_j$, respectively. Then, we have the asymptotic expansion of the form:*

$$\mathcal{K}(u) \sim \sum_{j=1}^\infty \sum_{k=1}^{k_j} A_{j,k} |u|^{p_j - 1} (\log|u|)^{k-1} \tag{7.1.2}$$

as $|u| \to 0$ and

$$A_{j,k} = \frac{(-1)^{k-1}}{(k-1)!} \cdot (a_{j,k}^+ + a_{j,k}^-).$$

Proof. According to Theorem 6.0.1, there exist positive constants a, b such that $Z_+(s;f,g,\varphi)$ is a holomorphic function on the region $\{-a < \mathrm{Re}(s) < b\}$. Furthermore, by an integration by parts to the integrals in (6.2.7), we see that $Z_+(s; f, g, \varphi)$ is uniformly dominated by $\mathrm{Im}(s)^{-1}$ for any s where $Z_+(s; f, g, \varphi)$ is holomorphic. Hence, we can apply the inverse Mellin transform to $Z_+(s; f, g, \varphi)$ as

$$\mathcal{K}(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-s-1} Z_+(s; f, g, \varphi) ds \tag{7.1.3}$$

for $-a < c < b$.

By applying the Cauchy's integral formula to (7.1.3), we move the integral contour to the left side. Then, for $\lambda \in \mathbb{R}$ with $\lambda < c$ and $\lambda \neq -p_j$, we have

$$\begin{aligned} \mathcal{K}(u) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-s-1} Z_+(s; f, g, \varphi) ds \\ &= \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} u^{-s-1} Z_+(s; f, g, \varphi) ds + \sum_{\lambda < -p_j < c} \frac{1}{2\pi i} \int_{\partial B_j} u^{-s-1} Z_+(s; f, g, \varphi) ds, \end{aligned} \tag{7.1.4}$$

where B_j is the sufficiently small circle with center $s = -p_j$. It is easy to see that the first integral in the right-hand side of (7.1.4) is estimated as

$$\left| \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} u^{-s-1} Z_+(s; f, g, \varphi) ds \right| \leq R \cdot u^{-\lambda-1}$$

with positive constant R. On the other hand, by the residue formula, the second term in (7.1.4) is

$$\sum_{\lambda < -p_j < c} \frac{1}{2\pi i} \int_{\partial B_j} u^{-s-1} Z_+(s; f, g, \varphi) ds = \sum_{\lambda < -p_j < c} u^{p_j - 1} \sum_{k=1}^{k_j} \frac{(-1)^{k-1}}{(k-1)!} a_{j,k}^{+} (\log u)^{k-1}.$$

Since λ is arbitrary, the following asymptotic expansion is obtained:

$$\mathcal{K}(u) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} \frac{(-1)^{k-1}}{(k-1)!} a_{j,k}^{+} u^{p_j - 1} (\log u)^{k-1} \tag{7.1.5}$$

as $u \to +0$. For $Z_-(s; f, g, \varphi)$, the same argument gives us the following asymptotic expansion:

$$\mathcal{K}(-u) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} \frac{(-1)^{k-1}}{(k-1)!} a_{j,k}^{-} (-u)^{p_j - 1} (\log(-u))^{k-1} \tag{7.1.6}$$

as $-u \to +0$. Owing to (7.1.5) and (7.1.6), the asymptotic expansion (7.1.2) holds.

□

Similarly, we have the asymptotic expansion of $\mathcal{K}(u)$ as $|u| \to \infty$.

Theorem 7.1.5. *Let $0 < q_1 < q_2 < \cdots$ be positive real numbers and k_j be nonnegative integers. Suppose that $Z_\pm(s; f, g, \varphi)$ has poles at $s = q_j$ of order k_j for every j and $b^{\pm}_{j,k}$ are the coefficient of the term $(s - q_j)^{-k}$ of Laurent series of $Z_\pm(s; f, g, \varphi)$ at $s = q_j$, respectively. Then, we have the asymptotic expansion of the form:*

$$\mathcal{K}(u) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} B_{j,k} |u|^{-q_j-1} (\log |u|)^{k-1} \tag{7.1.7}$$

as $|u| \to \infty$ and

$$B_{j,k} = \frac{(-1)^{k-1}}{(k-1)!} \cdot (b^{+}_{j,k} + b^{-}_{j,k}). \tag{7.1.8}$$

Proof. By deforming the integral contour in (7.1.3) to the right side, we have the above asymptotic expansion. □

7.1.3 Some Fourier transforms

For analysis in next section, we prepare some important lemmas concerning with the Fourier transform of some functions. To prove main theorems, we consider the Fourier transform of each term appearing in the asymptotic expansion (7.1.2) and (7.1.7). However, to find clear transformation of each term is difficult. So we consider the asymptotic formulae of each Fourier transform.

Throughout this subsection, we use the following notation: Let $f(t), g(t)$ be functions defined on an interval $I \subset \mathbb{R}$.

- $f(t) \equiv g(t) \bmod \mathcal{S}(I)$ means that $f(t) - g(t) \in \mathcal{S}(I)$.
- $f(t) \equiv g(t) \bmod C^\infty(I)$ means that $f(t) - g(t) \in C^\infty(I)$.
- When $n \le a < n + 1$ for $n \in \mathbb{Z}$, we set $\lfloor a \rfloor = n$.
- $\Gamma(\cdot)$ means a gamma function.

The following lemmas are useful for the estimate below.

Lemma 7.1.6. *For a real number λ, the following inequation holds:*

$$\int_t^\infty e^{-x}x^{\lambda-1}dx \le Ce^{-t/2} \quad \text{for } t \ge 1, \tag{7.1.9}$$

where C is a positive constant independent of t.

Proof. When $\lambda \le 1$, we have a direct estimate as

$$\int_t^\infty e^{-x}x^{\lambda-1}dx \le \int_t^\infty e^{-x} = e^{-t}$$

for $x \ge t \ge 1$. If $\lambda > 1$, an integration by parts implies

$$\int_t^\infty e^{-x}x^{\lambda-1}dx = e^{-t}t^{\lambda-1} + (\lambda-1)\int_t^\infty e^{-x}x^{\lambda-2}dx.$$

By repeating this process, we see that this case can be reduced to the case of $\lambda \le 1$, which implies inequation (7.1.9) holds for all λ. □

Lemma 7.1.7. *If $0 < \alpha < 1$, then*

$$\int_0^\infty e^{\pm ix^{\frac{1}{\alpha}}}dx = \alpha\int_0^\infty e^{\pm iy}y^{\alpha-1}dy = e^{\pm\frac{\alpha}{2}\pi i}\Gamma(\alpha+1).$$

Proof. The first equality is obtained by the change of variable $x^{\frac{1}{\alpha}} = y$. When $\alpha = 1/2$, the first integral is called the Fresnel integral. Thus, the first integral is generalization of the Fresnel integral and it can be computed in a similar way by using an elementary method of complex analysis. Details are written in [15], [28], [33]. □

Fourier transform of $x^{\lambda-1}$

Let $\lambda \in \mathbb{R}$, $\rho \in \mathbb{N}$ and $\chi_1 : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying that

$$\chi_1(x) = \begin{cases} 1 & \text{if } 0 \le x \le L, \\ 0 & \text{if } M \le x \end{cases}$$

for positive constants $L < M$. We define integrals $F^{(1)}_{\lambda,\rho}(t)$ as

$$F^{(1)}_{\lambda,\rho}(\pm t) := \int_0^\infty e^{\pm itx}|x|^{\lambda-1}(\log|x|)^{\rho-1}\chi_1(|x|)dx. \tag{7.1.10}$$

Note that the integral $F^{(1)}_{\lambda,\rho}(t) + F^{(1)}_{\lambda,\rho}(-t)$ is the Fourier transform of the function $x \mapsto |x|^{\lambda-1}(\log|x|)^{\rho-1}\chi_1(|x|)$ and each integral in (7.1.10) absolutely converges. It is easy to see that the relation

$$F^{(1)}_{\lambda,\rho}(\pm t) = \frac{\partial^{\rho-1}}{\partial\lambda^{\rho-1}} F^{(1)}_{\lambda,1}(\pm t) \tag{7.1.11}$$

holds. From this relationship, it suffices to investigate the asymptotic behavior of the case $\rho = 1$.

Lemma 7.1.8. *If $\lambda > 0$, then $F^{(1)}_{\lambda,1}(t) \equiv A_\lambda t^{-\lambda} \bmod \mathcal{S}(\mathbb{R})$, where $A_\lambda = e^{\frac{\lambda}{2}\pi i}\Gamma(\lambda)$.*

Proof. By change of integral variable $s = -itx$, we have a contour integral on the imaginary line as follows.

$$\begin{aligned} F^{(1)}_{\lambda,1}(t) = \int_0^\infty e^{itx}x^{\lambda-1}\hat{\chi}_1(x)dx &= \frac{1}{(-it)^\lambda}\int_0^{-it\infty} e^{-s}s^{\lambda-1}\hat{\chi}_1\left(\frac{s}{-it}\right)ds \\ &= e^{\frac{\lambda}{2}\pi i}t^{-\lambda}\int_0^{-it\infty} e^{-s}s^{\lambda-1}\hat{\chi}_1\left(\frac{is}{t}\right)ds. \end{aligned} \tag{7.1.12}$$

Here, we extend the real function χ_1 to the complex function $\hat{\chi}_1$ by defining $\hat{\chi}_1(s) := \chi_1(|s|)$ for $s \in \mathbb{C}$. Note that the support of $\hat{\chi}_1$ is contained in a disk $D(M) = \{s \in \mathbb{C} : |s| < M\}$. Let G be a domain in $\mathbb{C}$ whose boundary is anticlockwise oriented and consists of the following three curves;

$$\begin{aligned} &G_1 : s = x \quad (0 \le x \le M't), \\ &G_2 : s = iy \quad (-M't \le y \le 0), \\ &G_3 : s = M'te^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le 0\right), \end{aligned}$$

where M' is a positive constant with $M < M'$. Applying Green's theorem to the contour integral along the boundary of G, we have

$$\int_{G_1+G_2+G_3}\left\{e^{-s}s^{\lambda-1}\hat{\chi}_1\left(\frac{is}{t}\right)ds + 0\cdot d\overline{s}\right\} = -\iint_G \frac{\partial}{\partial\overline{s}}\left(e^{-s}s^{\lambda-1}\hat{\chi}_1\left(\frac{is}{t}\right)\right)dsd\overline{s}. \tag{7.1.13}$$

Since e^{-s} and $s^{\lambda-1}$ are holomorphic on G, the right integral in (7.1.13) becomes

$$-\iint_G e^{-s}s^{\lambda-1}\frac{\partial}{\partial\overline{s}}\hat{\chi}_1\left(\frac{is}{t}\right)dsd\overline{s}. \tag{7.1.14}$$

On the other hand, since the curve G_3 is outside of the support of $\hat{\chi}_1$, we see that the left integral in (7.1.13) is equal to

$$-\int_0^\infty e^{-x}x^{\lambda-1}\hat{\chi}_1\left(\frac{ix}{t}\right)dx + \int_0^{-it\infty} e^{-y}y^{\lambda-1}\hat{\chi}_1\left(\frac{iy}{t}\right)dy. \tag{7.1.15}$$

From (7.1.12), (7.1.13), (7.1.14) and (7.1.15), we have

$$F_{\lambda,1}^{(1)}(t) = \mathcal{I}(t) - \mathcal{J}(t)$$

with

$$\begin{aligned}
\mathcal{I}(t) &= e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \int_0^\infty e^{-x} x^{\lambda-1} \hat{\chi}_1\left(\frac{ix}{t}\right) dx, \\
\mathcal{J}(t) &= e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \iint_G e^{-s} s^{\lambda-1} \frac{\partial}{\partial \bar{s}} \hat{\chi}_1\left(\frac{is}{t}\right) ds d\bar{s}.
\end{aligned}$$

$\underline{(\mathcal{I}(t))}$

From the definition of $\hat{\chi}_1(s)$, we have

$$\begin{aligned}
\mathcal{I}(t) &= e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \int_0^\infty e^{-x} x^{\lambda-1} \chi_1\left(\frac{x}{t}\right) dx \\
&= e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \int_0^\infty e^{-x} x^{\lambda-1} dx - e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \int_0^\infty e^{-x} x^{\lambda-1} \left(1 - \chi_1\left(\frac{x}{t}\right)\right) dx \\
&= e^{\frac{\lambda}{2}\pi i} \Gamma(\lambda) t^{-\lambda} - e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \int_{Lt}^\infty e^{-x} x^{\lambda-1} \left(1 - \chi_1\left(\frac{x}{t}\right)\right) dx.
\end{aligned} \tag{7.1.16}$$

The last integral in (7.1.16) is estimated as

$$\left| \int_{Lt}^\infty e^{-x} x^{\lambda-1} \left(1 - \chi_1\left(\frac{x}{t}\right)\right) dx \right| \leq \int_{Lt}^\infty e^{-x} x^{\lambda-1} dx \leq C e^{-(Lt)/2} \tag{7.1.17}$$

by using Lemma 7.1.6. From (7.1.16) and (7.1.17), we see that $\mathcal{I}(t) - A_\lambda t^{-\lambda}$ is rapidly decreasing function.

$\underline{(\mathcal{J}(t))}$

Let $D(r)$ be a dick in $\mathbb{C}$ centered at the origin with radius r, that is

$$D(r) := \{s \in \mathbb{C} : |s| < r\},$$

where r is a positive real number. Note that $\frac{\partial}{\partial \bar{s}} \hat{\chi}_1 \equiv 0$ on $D(L)$, since $\hat{\chi}_1(s) \equiv 1$ on $D(L)$.

Consider the change of integral variable $s = t(x+iy)$ to $\mathcal{J}(t)$, we have

$$\begin{aligned}
\mathcal{J}(t) &= e^{\frac{\lambda}{2}\pi i} t^{-\lambda} \iint_G e^{-s} s^{\lambda-1} \frac{\partial}{\partial \bar{s}} \hat{\chi}_1\left(\frac{is}{t}\right) ds d\bar{s} \\
&= -2i e^{\frac{\lambda}{2}\pi i} \int_0^\infty \int_0^\infty e^{-tx-ity} (x+iy)^{\lambda-1} \overline{\partial} \hat{\chi}_1(ix-y) dx dy,
\end{aligned}$$

where $\overline{\partial}$ is a differential operator defined by $\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ and we use the relation $dsd\overline{s} = (2i)dxdy$. Define the function $\mathcal{H}(y)$ as

$$\mathcal{H}(y) = \int_0^\infty e^{-tx}(x+iy)^{\lambda-1}\overline{\partial}\hat{\chi}_1(ix-y)dx,$$

then the function $\mathcal{J}(t)$ can be regarded as a Fourier transform of $\mathcal{H}(y)$, that is,

$$\mathcal{J}(t) = -2ie^{i\frac{\lambda\pi}{2}}\int_0^\infty e^{-ity}\mathcal{H}(y)dy.$$

Since the support of $\overline{\partial}\hat{\chi}_1(ix-y)$ is contained in $D(M)\setminus D(L)$, $\mathcal{H}(y)$ is a C_0^∞ function and this implies $\mathcal{J}(t)$ is a rapidly decreasing function.

Consequently, the assertion in lemma is shown.

□

As a corollary of the above lemma, from the relation (7.1.11), we have the following asymptotic formulae.

Corollary 7.1.9.

$$F^{(1)}_{\lambda,\rho}(\pm t) = (\pm 1)^\lambda A_\lambda t^{-\lambda}(\log t)^{\rho-1} + \mathcal{O}(t^{-\lambda}(\log t)^{\rho-2})$$

as $t\to\infty$. Here, A_λ is as in Lemma 7.1.8.

Fourier transform of $x^{-\lambda-1}$

Let $\lambda\in\mathbb{R}_{>0}$, $\rho\in\mathbb{N}$ and $\chi_2:\mathbb{R}\to\mathbb{R}$ be a C^∞ function satisfying that

$$\chi_2(x) = \begin{cases} 1 & \text{if } M \le x, \\ 0 & \text{if } x \le L \end{cases} \tag{7.1.18}$$

for positive constant $L < M$. We define the integrals $F^{(2)}_{\lambda,\rho}(t)$ as

$$F^{(2)}_{\lambda,\rho}(\pm t) := \int_0^\infty e^{\pm itx}|x|^{-\lambda-1}(\log|x|)^{\rho-1}\chi_2(|x|)dx. \tag{7.1.19}$$

Note that the integral $F^{(2)}_{\lambda,\rho}(t) + F^{(2)}_{\lambda,\rho}(-t)$ is the Fourier transform of the function $x \mapsto |x|^{-\lambda-1}(\log|x|)^{\rho-1}\chi_2(|x|)$ and each integral in (7.1.19) absolutely converges. It is easy to see that the relation

$$F^{(2)}_{\lambda,\rho}(\pm t) = (-1)^{\rho-1}\frac{\partial^{\rho-1}}{\partial\lambda^{\rho-1}}F^{(2)}_{\lambda,1}(\pm t) \tag{7.1.20}$$

holds. Thus, it suffices to investigate the property of $F^{(2)}_{\lambda,1}(t)$ as well. Unlike Lemma 7.1.8, its asymptotic behavior is a little different whether λ is an integer or not.

At first, we state the case when λ is not an integer.

Lemma 7.1.10. *If $\lambda > -1$ is not an integer, then $F^{(2)}_{\lambda,1} \equiv B_\lambda t^\lambda \bmod C^\infty(\mathbb{R})$, where $B_\lambda = e^{\frac{-\lambda}{2}\pi i}\Gamma(-\lambda)$. Here, the branch of t^λ for $t < 0$ is chosen as $t^\lambda = e^{\pi\lambda i}|t|^\lambda$.*

Proof. $\underline{(-1 < \lambda < 0)}$ First, we consider the case where $\lambda \in (-1, 0)$. By using $\chi_2(x)$ in (7.1.18), $F^{(2)}_{\lambda,1}(t)$ is divided into two parts as follows,

$$F^{(2)}_{\lambda,1}(t) = \int_0^\infty e^{itx}x^{-\lambda-1}dx - \int_0^\infty e^{itx}x^{-\lambda-1}(1-\chi_2(x))dx. \tag{7.1.21}$$

Since the support of $(1-\chi_2(x))$ is compact, the second integral in (7.1.21) is a C^∞ function of t. When $t > 0$, by the change of integral variable, the first integral in (7.1.21) can be written as

$$\int_0^\infty e^{itx}x^{-\lambda-1}dx = t^\lambda \int_0^\infty e^{iy}y^{-\lambda-1}dy = t^\lambda e^{\frac{-\lambda}{2}\pi i}\Gamma(-\lambda) \tag{7.1.22}$$

by using Lemma 7.1.7. When $t < 0$, by choosing the branch of t^λ as in this lemma, we have

$$\int_0^\infty e^{itx}x^{-\lambda-1}dx = |t|^\lambda \int_0^\infty e^{-iy}y^{-\lambda-1}dy = |t|^\lambda e^{\frac{\lambda}{2}\pi i}\Gamma(-\lambda) = t^\lambda e^{\frac{-\lambda}{2}\pi i}\Gamma(-\lambda).$$

Therefore, the equation (7.1.22) holds for $\lambda \in (-1, 0)$.

$\underline{(0 < \lambda)}$

Next, let us consider the case where $\lambda \in \mathbb{R}_+ \setminus \mathbb{Z}$. Since $\chi_2(x) = 0$ for $x \leq L$, an integration by parts implies

$$\int_0^\infty e^{itx}x^{-\lambda-1}\chi_2(x)dx = \frac{it}{\lambda}\int_0^\infty e^{itx}x^{-\lambda}\chi_2(x)dx + \frac{1}{\lambda}\int_0^\infty e^{itx}x^{-\lambda}\chi_2'(x)dx. \tag{7.1.23}$$

From the definition of χ_2 in (7.1.18), the support of χ_2' is contained in $[L, M]$ and this leads to that the second integral in (7.1.23) is a C^∞ function. Since there uniquely exist $m \in \mathbb{N}$ and $q \in (-1, 0)$ such that $\lambda = q + m$, by repeating the above process m times, we have

$$F^{(2)}_{\lambda,\rho}(t) \equiv \frac{(it)^m}{\lambda(\lambda-1)\cdots(q+1)}\int_0^\infty e^{itx}x^{-q-1}\chi_2(x)dx \mod C^\infty(\mathbb{R}). \tag{7.1.24}$$

Since $q \in (-1, 0)$, we can apply (7.1.22) to (7.1.24) and we obtain $F^{(2)}_{\lambda,1}(t) \equiv \tilde{A}_\lambda t^\lambda \bmod C^\infty(\mathbb{R})$, where

$$\tilde{A}_\lambda = \frac{i^m}{\lambda(\lambda-1)\cdots(q+1)}A_q = \frac{(-i)^m}{(-\lambda)(-\lambda+1)\cdots(-q-1)}e^{-\frac{q}{2}\pi i}\Gamma(-q).$$

Using the property of gamma function, $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$ for $\alpha\in\mathbb{R}\setminus\mathbb{Z}$, we see that $\tilde{A}_\lambda=A_\lambda$ and the equation in the lemma holds for $\lambda\in\mathbb{R}_+\setminus\mathbb{Z}$. $\square$

On the other hand, when λ is an integer, a logarithmic function appears in the singularity of $F^{(2)}_{\lambda,1}(t)$.

Lemma 7.1.11. *If λ is a nonnegative integer, then $F^{(2)}_{\lambda,1}\equiv C_\lambda t^\lambda\log t \mod C^\infty(\mathbb{R}_+)$, where $C_\lambda=-i^\lambda/\lambda!$.*

Proof. It is sufficient to show the lemma in the case of $\lambda=0$. Indeed, we can easily deal with the general case in a similar argument to that in Lemma 7.1.10.

We divide the integral $F^{(2)}_{0,1}$ as follows.

$$\begin{aligned}
F^{(2)}_{0,1}(t)&=\int_L^{1/t}e^{itx}x^{-1}\chi_2(x)dx+\int_{1/t}^\infty e^{itx}x^{-1}\chi_2(x)dx\\
&=\int_L^{1/t}x^{-1}\chi_2(x)dx+\int_L^{1/t}(e^{itx}-1)x^{-1}\chi_2(x)dx+\int_{1/t}^\infty e^{itx}x^{-1}\chi_2(x)dx\\
&=: G(t)+H(t)+K(t)\quad\text{for } t\in\mathbb{R}_+.
\end{aligned}$$

For the integral $G(t)$, by an integration by parts, we have

$$\begin{aligned}
G(t)&=-\log t\cdot\chi_2(1/t)-\int_L^{1/t}\log x\cdot\chi_2'(x)dx\\
&\equiv-\log t\cdot\chi_2(1/t)\mod C^\infty(\mathbb{R}_+)\\
&\equiv-\log t\qquad\qquad\quad\ \mod C^\infty(\mathbb{R}_+).
\end{aligned}$$

The last equivalence is obtained by dividing $\chi_2(1/t)$ as $\chi_2(1/t)=1+(\chi_2(1/t)-1)$.

Next, let us show that $H(t)$ is a C^∞ function on $\mathbb{R}$. We define two C^∞ functions g,h as the following convergence series:

$$g(x,t)=\sum_{n=1}^\infty\frac{i^n}{n!}t^nx^{n-1},\quad h(x,t)=\sum_{n=1}^\infty\frac{i^n}{nn!}t^nx^n\quad(|xt|<1).$$

Note that $g(x,t)=(e^{itx}-1)x^{-1}$ and $\frac{\partial}{\partial x}h(x,t)=g(x,t)$ for $|xt|<1$. Then, by an integration by parts, we have

$$H(t)=\int_L^\infty g(x,t)\chi_2(x)dx=h(1/t,t)-\int_L^M h(x,t)\chi_2'(x)dx\quad\text{for } t\in\mathbb{R}_+.\tag{7.1.25}$$

It is easy to see that $H(1/t,t)$ is a constant and that the last integral in (7.1.25) is a C^∞ function of t on $\mathbb{R}_+$.

Finally, let us consider the integral $K(t)$. By changing the integral variable $y = tx$, we have

$$\begin{aligned} K(t) &= \int_{1/t}^{\infty} e^{itx} x^{-1} \chi_2(x) dx = \int_1^\infty e^{iy} y^{-1} \chi_2(y/t) dy \\ &= \int_1^\infty e^{iy} y^{-1} dy - \int_1^\infty e^{iy} y^{-1} (1 - \chi_2(y/t)) dy \quad \text{for } t \in \mathbb{R}_+. \end{aligned} \tag{7.1.26}$$

The first integral in (7.1.26) is a constant defined by a convergent improper integral. It is easy to see that the second integral in (7.1.26) is a C^∞ function of t on $\mathbb{R}_+$.

Putting together the above results, we can see that $F_{0,1}^{(2)}(t) + \log t$ is a C^∞ function on $\mathbb{R}_+$. □

From the relation (7.1.20), we have the following asymptotic formulae.

Corollary 7.1.12. *(i) If $\lambda > -1$ is not an integer, then*

$$F_{\lambda,\rho}^{(2)}(\pm t) = (\pm 1)^\lambda \cdot (-1)^{\rho-1} B_\lambda t^\lambda (\log t)^{\rho-1} + \mathcal{O}(t^\lambda (\log t)^{\rho-2}) \quad \text{as } t \to 0.$$

(ii) If λ is a nonnegative integer, then

$$F_{\lambda,\rho}^{(2)}(\pm t) = (\pm 1)^\lambda \cdot (-1)^{\rho-1} C_\lambda t^\lambda (\log t)^{\rho} + \mathcal{O}(t^\lambda (\log t)^{\rho-1}) \quad \text{as } t \to 0.$$

Here, B_λ, C_λ are as in Lemmas 7.1.10, 7.1.11.

Finally, we prepare a lemma for the analysis in next section which are useful to estimate the remainder terms.

Lemma 7.1.13. *Suppose that $h(x) = \mathcal{O}(x^{-\lambda-1}(\log x)^\rho)$ as $x \to \infty$, then the following holds:*

$$\int_0^\infty e^{itx} h(x) \chi_2(x) dx = \sum_{j=0}^{L} c_j t^j + \mathcal{O}(t^\lambda (\log t)^\rho) \quad \text{as } t \to 0, \tag{7.1.27}$$

where L is an integer satisfying that $L \le \lambda < L+1$.

Proof. We set $E_L(X) := e^X - \sum_{j=1}^{L} X^j/j! = X^{L+1} \sum_{j=L+1}^{\infty} X^{j-L-1}/j! = X^{L+1}\phi(X)$ with C^∞ function ϕ on $\mathbb{R}$. Let us consider the following integral

$$\int_M^\infty E_L(itx) h(x) \chi_2(x) dx = \int_M^{1/t} E_L(itx) h(x) \chi_2(x) dx + \int_{1/t}^\infty E_L(itx) h(x) \chi_2(x) dx. \tag{7.1.28}$$

Then, by a change of variable $y = tx$, the first integral of right-hand side in (7.1.28) is estimated as

$$\begin{aligned}\left|\int_M^{1/t} (itx)^{L+1}\phi(itx)h(x)\chi_2(x)dx\right| &\leq Ct^{L+1}\int_M^{1/t} x^{L-\lambda}(\log x)^\rho dx \\ &= Ct^{L+1}\int_{Mt}^{1}\left(\frac{y}{t}\right)^{L-\lambda}(\log y - \log t)^\rho \frac{1}{t}dy \\ &= Ct^\lambda \sum_{k=0}^{\rho}\binom{\rho}{k}(-\log t)^k \int_{Mt}^{1} y^{L-\lambda}(\log y)^{\rho-k-1}dy,\end{aligned} \tag{7.1.29}$$

where C is a positive constant and $\binom{\rho}{k}$ means a binomial coefficient. Each integral in the last term in (7.1.29) is estimated as

$$\int_{Mt}^{1} y^{L-\lambda}(\log y)^{\rho-k-1}dy \leq \int_0^1 y^{L-\lambda}(\log y)^{\rho-k-1}dy$$

and converges because of $L - \lambda > -1$ and it is easy to see that all the terms in (7.1.29) is dominated by $t^\lambda(\log t)^\rho$. The second integral is also estimated as

$$\begin{aligned}\left|\int_{1/t}^{\infty} E_L(itx)\phi(itx)h(x)\chi_2(x)dx\right| &\leq C'\int_{1/t}^{\infty} x^{-\lambda-1}(\log x)^\rho dx \\ &= C't^\lambda \sum_{k=0}^{\rho}\binom{\rho}{k}(-\log t)^k \int_1^\infty y^{-\lambda-1}(\log y)^{\rho-k-1}dy\end{aligned} \tag{7.1.30}$$

and the last integral is convergent because of $-\lambda - 1 < -1$. Thus, all the terms in (7.1.30) is dominated by $t^\lambda(\log t)^\rho$. For these arguments, we have the following equation:

$$\left|\int_M^\infty E_L(itx)h(x)\chi_2(x)dx\right| \leq C''t^\lambda(\log t)^\rho \quad \text{as } t \to 0$$

and this implies the equation (7.1.27). $\square$

7.2 Proof of main theorems

In this section, we apply the inverse Mellin transform to $Z_{\mathbb{R}}(s; f, g, \varphi)$ of real case and lemmas concerning the Fourier transform to $\mathcal{K}(u)$. Mixing the results obtained in Chapter 6, we will give the proof of main theorems in Chapter 3.

7.2.1 Asymptotic of $I_\varphi(t; f, g)$ as $t \to \infty$

At first, let us consider the case of the parameter t is sufficiently large. In this case, the negative poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$ appear in the asymptotic expansions.

By using Theorem 7.1.4 to $Z_{\mathbb{R}}(s; f, g, \varphi)$, we have asymptotic expansion of $\mathcal{K}(u)$ at $|u| \to 0$ as in (7.1.8). In particular, the coefficient A of $|u|^{-\alpha}(\log|u|)^{k-1}$ in this expansion is expressed as

$$A = \frac{(-1)^{k-1}}{(k-1)!}[A_+ + A_-],$$

where $A_\pm$ are the coefficients of Laurent expansion of $Z_\pm(s; f, g, \varphi)$ at $s = -\alpha$, that is,

$$Z_\pm(s; f, g, \varphi) = \frac{A_\pm}{(s+\alpha)^k} + \mathcal{O}\left(\frac{1}{(s+\alpha)^{k-1}}\right).$$

Applying Corollary 7.1.9 to each term in the asymptotic expansion of $\mathcal{K}(u)$, we have the asymptotic expansion of $I_\varphi(t; f, g)$ as $t \to \infty$ as in (i-4). It is obvious that the component $-\alpha$ runs through the all negative poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$ which are quantitatively computed in Theorem (6.0.2). Furthermore, the term in the asymptotic expansion of $I_\varphi(t; f, g)$ corresponding to the term $|u|^{-\alpha}(\log|u|)^{k-1}$ is $\tilde{A}t^{-\alpha}(\log t)^{k-1}$ and its coefficient is given by

$$\tilde{A} = \frac{\Gamma(\alpha)}{(k-1)!}[e^{i\pi\alpha/2}A_+ + e^{-i\pi\alpha/2}A_-], \tag{7.2.1}$$

where Γ is the Gamma function.

Remark 7.2.1. If α is not an odd integer, then

$$\mathrm{Re}(\tilde{A}) = \frac{2\Gamma(\alpha)\cos(\pi\alpha/2)}{(k-1)!}[A_+ + A_-].$$

In order to decide the vanishing of the coefficient, the above equation is helpful.

Next, let us consider the coefficient of leading term in the above expansion. From the relationship between $I_\varphi(t; f, g)$ and $Z_\pm(s; f, g, \varphi)$ and the equation (7.2.1), we give explicit formulae for the coefficient of the leading term of the asymptotic expansion (i-4) of $I_\varphi(t; f, g)$ at infinity.

Theorem 7.2.2. *If f, g and φ satisfy the conditions in Theorem 3.0.2, then we have*

$$\begin{aligned}&\lim_{t\to\infty} t^{1/d_\infty(f,g)}(\log t)^{-m_\infty(f,g)+1} \cdot I_\varphi(t; f, g)\\ &\quad = \frac{\Gamma(1/d_\infty(f,g))}{(m_\infty(f,g)-1)!}\left[e^{i\pi/(2d_\infty(f,g))}C_+ + e^{-i\pi/(2d_\infty(f,g))}C_-\right],\end{aligned}$$

where $C_\pm$ are as in (6.3.17).

7.2.2 Proof of Theorems 3.0.1 and 3.0.2

Applying the argument in Subsection 7.2.1 to the results relating to $Z_{\mathbb{R}}(s; f, g, \varphi)$ in Chapter 6, we obtain the Theorems 3.0.1 and 3.0.2.

Proof of Theorem 3.0.1. This theorem follows from Theorem 6.0.2. If $d_\infty(f, g) < 1$, there may appear terms derived from the negative trivial poles of $Z_{\mathbb{R}}(s; f, g, \varphi)$, whose decay rates are larger than $t^{-1/d_\infty(f,g)}(\log t)^{m_\infty(f,g)-1}$. However, Proposition 6.3.5 and the relationship (7.2.1) induce the cancellation of the coefficients of such terms and the assertion in Theorem 3.0.1 holds. □

Proof of Theorem 3.0.2. This theorem follows from Theorem 7.2.2 by considering the assertions in Theorem 6.3.3. If the condition (b) is satisfied, either $Z_+(s; f, g, \varphi)$ or $Z_-(s; f, g, \varphi)$ is equivalently zero and cancellation of the coefficient does not occur. Note that the necessity of the condition in (c), "$1/d_\infty(f, g)$ is not an odd integer", follows from Remark 7.2.1. □

7.2.3 Asymptotic expansion of $I_\varphi(t; f, g)$ as $t \to 0$

Next, we consider the case of the parameter t is sufficiently small. In this case, by applying Theorem 7.1.5 to $Z_\pm(s; f, g, \varphi)$ we have asymptotic expansion of $\mathcal{K}(u)$ at $|u| \to \infty$ as in (7.1.7). Then, we have asymptotic expansion of $I_\varphi(t; f, g)$ as $t \to 0$ as in (i-5) by applying Corollary 7.1.12 and Lemma 7.1.13 to each term of (7.1.7). Here, we can see more explicit relationship between the coefficients of the asymptotic expansion of $I_\varphi(t; f, g)$ ans the coefficients of Laurent expansions of $Z_\pm(s; f, g, \varphi)$.

Let $Z_\pm(s; f, g, \varphi)$ have the following Laurent expansion at $s = \beta$:

$$Z_\pm(s; f, g, \varphi) = \frac{B_\pm}{(s-\beta)^l} + \mathcal{O}\left(\frac{1}{(s-\beta)^{l-1}}\right),$$

then the corresponding parts of the asymptotic expansions $K(u)$ at $|u| \to \infty$ is

$$\frac{(-1)^{l-1}}{(l-1)!} B_\pm |u|^{-\beta-1}(\log |u|)^{l-1} + \mathcal{O}(|u|^{-\beta-1}(\log |u|)^{l-2}).$$

Be careful to whether β is an integer or not, we have the following two asymptotic formulae by applying Corollary 7.1.12 and Lemma 7.1.13 to these terms

$$\tilde{B}(\beta)t^\beta(\log t)^{l-1} + \sum_{j=0}^{\lfloor \beta \rfloor} \tilde{c}_j(\beta)t^j + \mathcal{O}(t^\beta(\log t)^{l-2}) \quad (\beta \notin \mathbb{Z}),$$

$$\tilde{B}(\beta)t^\beta(\log t)^l + \sum_{j=0}^{\beta-1}\tilde{c}_j(\beta)t^j + \mathcal{O}(t^\beta(\log t)^{l-1}) \qquad (\beta \in \mathbb{Z})$$

with

$$\tilde{B}(\beta) = \begin{cases} \dfrac{\Gamma(-\beta)}{(l-1)!}[e^{-\beta\pi i/2}B_+ + e^{\beta\pi i/2}B_-] & (\beta \notin \mathbb{Z}), \\ \dfrac{-1}{(l-1)!}\dfrac{i^\beta}{\beta!}[B_+ + (-1)^\beta B_-] & (\beta \in \mathbb{Z}). \end{cases} \tag{7.2.2}$$

Remark 7.2.3. If β is not an integer, then

$$\mathrm{Re}(\tilde{B}(\beta)) = \frac{\Gamma(-\beta)\cos(\beta\pi/2)}{(l-1)!}[B_+ + B_-].$$

In a similar fashion to the previous section, we obtain the explicit formula of the coefficient of leading term in the asymptotic expansion (i-5).

Theorem 7.2.4. *If f, g and φ satisfy the conditions in Theorem 3.0.6, then we have the followings:*

(i) If $1/d_0(f,g)$ is not an integer, then

$$\begin{aligned}\lim_{t\to 0} t^{-1/d_0(f,g)}&(\log t)^{-m_0(f,g)+1}\cdot I_\varphi(t; f, g) \\ &= \frac{\Gamma(-1/d_0(f,g))}{(m_0(f,g)-1)!}[e^{-i\pi/(2d_0(f,g))}D_+ + e^{i\pi/(2d_0(f,g))}D_-].\end{aligned}$$

(ii) If $1/d_0(f,g)$ is an integer, then

$$\begin{aligned}\lim_{t\to 0} t^{-1/d_0(f,g)}&(\log t)^{-m_0(f,g)}\cdot I_\varphi(t; f, g) \\ &= \frac{-1}{(m_0(f,g)-1)!}\frac{e^{\pi i/(2d_0(f,g))}}{(1/d_0(f,g))!}[D_+ + (-1)^{1/d_0(f,g)}D_-].\end{aligned} \tag{7.2.3}$$

Here, $D_\pm$ are as in (6.3.18).

7.2.4 Proof of Theorems 3.0.4 and 3.0.6

Proof of Theorem 3.0.4. This theorem follows from Theorem 6.0.2. If $d_0(f,g) < 1$, we see that Proposition 6.3.5 and the relationship (7.2.2) induce the cancellation of the coefficients of terms derived from positive trivial poles . □

Proof of Theorem 3.0.6. The assertions follow from the formulae in Theorem 7.2.4 by considering the assertions in Theorems 6.3.4. The condition in (ii)-(b), "$1/d_0(f,g)$ is an even integer" is necessary since the coefficient in (7.2.3) cancels when $D_+ = D_-$. □

7.3 Some general case in one-dimension

Here, we consider the case of $n = 1$. In this case, we can obtain the same results in Chapter 3 by using of a Taylor series of φ. Furthermore, through the argument below, we can extend Theorem 3.0.6 to more general phase functions containing even non-smooth functions at the origin. Precise explanation and arguments are written in [15].

Let us consider the following integral

$$I_\alpha(t) = \int_0^\infty e^{itx^{-\alpha}} \varphi(x) dx,$$

where α is a positive real number and φ is a C_0^∞ function on $\mathbb{R}$. In this section, we use the following notations.

- When $n - 1 < a \leq n$ for $n \in \mathbb{Z}$, we set $\lceil a \rceil = n$.
- For a smooth function f and $k \in \mathbb{Z}_+$, $f^{(k)}$ means the k-th derivative of f.

Theorem 7.3.1 ([15]). *(i) If α is a rational number, then for any positive integer N,*

$$I_\alpha(t) = \sum_{j=1}^N A_j t^{j/\alpha} + \sum_{j=1}^N B_j t^{j/\alpha} \log t + \psi_N(t) \quad \textit{for } t \in \mathbb{R}_+,$$

where $\psi_N(t)$ is a $C^{\lceil (N+1)/\alpha \rceil - 1}$ function on $\mathbb{R}_+$ and

$$\begin{aligned} A_j &= \frac{e^{(-j/2\alpha)\pi i}}{\alpha} \frac{\varphi^{(j-1)}(0)}{(j-1)!} \Gamma(-j/\alpha) \textit{ if } j/\alpha \notin \mathbb{N}, \quad A_j = 0 \textit{ if } j/\alpha \in \mathbb{N}, \\ B_j &= \frac{-1}{\alpha} \frac{\varphi^{(j-1)}(0)}{(j-1)!} \frac{i^{j/\alpha}}{(j/\alpha)!} \textit{ if } j/\alpha \in \mathbb{N}, \quad B_j = 0 \textit{ if } j/\alpha \notin \mathbb{N} \end{aligned} \tag{7.3.1}$$

for $j \in \mathbb{N}$, where Γ means the Gamma function.

(ii) If α is not a rational number, then for any positive number N,

$$I_\alpha(t) = \sum_{j=1}^N A_j t^{j/\alpha} + \phi_N(t) \quad \textit{for } t \in \mathbb{R}_+, \tag{7.3.2}$$

where $\phi_N(t)$ is a $C^{\lceil (N+1)/\alpha \rceil - 1}$ function on $\mathbb{R}$ and A_j are as in (7.3.1).

Remark 7.3.2. (1) If the branch of $t^{j/\alpha}$ for $t < 0$ is chosen as $t^{j/\alpha} = e^{ij\pi/\alpha}|t|^{j/\alpha}$, then the equation in (7.3.2) holds for all $t \in \mathbb{R}$.

(2) We can obtain a similar result to (7.3.1) in the case where $t \leq 0$. If $\alpha > 0$ is a rational number, then for any positive integer N,

$$I_\alpha(t) = \sum_{j=1}^{N} A_j t^{j/\alpha} + \sum_{j=1}^{N} B_j t^{j/\alpha} \log |t| + \psi_N(t) \quad \text{for } t \in \mathbb{R}_-,$$

where B_j are as in (7.3.1) and $\psi_N(t)$ is a $C^{\lceil (N+1)/\alpha \rceil - 1}$ function on $\mathbb{R}_-$.

After changing of integral variable $x^{-\alpha} = y$, by substituting the Taylor series of φ at the origin, we can apply Lemmas 7.1.10 and 7.1.11 and prove the above theorem. This method is not available for the case of $n \geq 2$. In the case of $n = 1$, it is easy to see that for any non-flat smooth functions $f(x), g(x)$, $f(x)/g(x)$ can be expressed as a monomial x^m with $m \in \mathbb{Z}$ by an implicit function theorem. Thus, the case when m is a negative integer is contained in the above theorem with $\alpha \in \mathbb{Z}_+$ (The case of $m \in \mathbb{Z}_+$ corresponds to Proposition 2.1.4). Furthermore, as a corollary, if $f(x)$ is a non-smooth function which can be expressed by the following particular form, we have similar result to Theorem 7.3.1.

Corollary 7.3.3. *Let $\alpha > 0$ be non-integer. Let f be a function which can be expressed as $f(x) = x^\alpha g(x)$ where g is a smooth function on $\mathbb{R}$ satisfying that $g(0) = g'(0) = \cdots = g^{(k-1)}(0) = 0$ and $g^{(k)}(0) \neq 0$. Then, for any positive integer N, the integral*

$$J_\alpha(t) = \int_0^\infty e^{it/f(x)} \varphi(x) dx$$

can be written as

$$J_\alpha(t) = \sum_{j=1}^{N} \tilde{A}_j t^{j/(\alpha+k)} + \sum_{j=1}^{N} \tilde{B}_j t^{j/(\alpha+k)} \log t + \psi_N(t) \quad \textit{for } t \in \mathbb{R}_+,$$

where ψ_N is a $C^{\lceil (N+1)/(\alpha+k) \rceil - 1}$ function on $\mathbb{R}_+$.

Note that the coefficients $\tilde{A}_j, \tilde{B}_j$ are slightly different from A_j, B_j in Theorem 7.3.1. These coefficients depend on α, k and the derivatives of g at the origin.

Refferences

[1] J. Àlvarez Montaner, M. González Villa, E. León-Cardenal and L. Núñez-Betancourt: Bernstein-Sato polynomial and related invariants for meromorphic functions, preprint, arXiv:2112.08492v1.

[2] V. I. Arnord, S. M. Gussein-Zade and A. N. Varčenko: *Singularités des applications différentiables, Vol.II*, Éditions Mir , Moscou, 1986.

[3] M. F. Atiyah: Resolution of singularities and division of distributions, Comm. Pure Appl. Math. **23** (1970), 145–150.

[4] M. Bocardo-Gaspar and W. A. Zúñiga-Galindo: Local zeta functions for rational functions and Newton polyhedra, Springer Proc. Math. Stat., **222** (2018), 363–385.

[5] K. Cho, J. Kamimoto and T. Nose: Asymptotic analysis of oscillatory integrals via the Newton polyhedra of the phase and the amplitude, J. Math. Soc. Japan, **65** (2013), 521–562.

[6] J. Denef: Report on Igusa's local zeta functions, Astérisque (1991), no. 201-203, Exp. No. 741, 359–386.

[7] J. Denef and K. Hoornaert: Newton Polyhedra and Igusa's Local Zeta Function, J. Number Theory **89** (2001), 31–64.

[8] J. Denef and P. Sargos: Polyèdra de Newton et distribution f_+^s. I, J. Anal. Math. **53** (1989), 201–218.

[9] W. Fulton: *Introduction to toric varieties*, Princeton University Press, Princeton, NJ, 1993.

[10] I. M. Gel'fand and G. E. Šilov: *Generalized Functions I*, 邦訳:共立出版 (1963-1964).

[11] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Ann. of Math. **79** (1964), 109–326.

[12] K. Hoornaert : Newton Polyhedra and the Poles of Igusa's Local Zeta Function, Bull.

Belg. Math. Soc. **9** (2002), 589–606.

[13] J. Igusa: *An Introduction to the Theory of Local Zeta Functions*, AMS/IP Stud. Adv. Math., **14** American Mathematical Society, Providence, RIInternational Press, Cambridge, MA, 2000, xii+232 pp.

[14] P. Jeanquartier: Développement asymptotique de la distribution de Dirac attaché à une fonction analytique, C. R. Acad. Sci. Paris Sér. **A-B 201** (1970), A1159–A1161.

[15] J. Kamimoto and H. Mizuno: Asymptotic expansion of oscillatory integrals with singular phases, Kyushu J. Math. **77** (2023), 319–329.

[16] J. Kamimoto and H. Mizuno: Newton polyhedra and Archimedean zeta functions for meromorphic functions, in preparation.

[17] J. Kamimoto and T. Nose: Toric resolution of singularities in a certain class of C^∞ functions and asymptotic analysis of oscillatory integrals, J. Math. Soc. Univ. Tokyo, **23** (2016), 425–485.

[18] J. Kamimoto and T. Nose: Newton polyhedra and weighted oscillatory integrals with smooth phases, Trans. Amer. Math. Soc., **368** (2016), 5301–5361.

[19] A. Kaneko: Newton diagrams, Singular points and Oscillatory integrals, Lecture note at Sophia University, 11 (1981) [in Japanese].

[20] A. G. Kouchinirenko: Polyédres de Newton et nombres de Milnor, Invent. Math., **32** (1976), 1–31.

[21] E. León-Cardenal: Archimedean zeta functions and oscillatory integrals, Contemp. Math., **778**, Amer. Math. Soc., 2022, 3–24.

[22] E. León-Cardenal, W. Veys and W. A. Zùñiga-Galindo: Poles of Archimedean zeta functions for analytic mappings, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 1–21.

[23] D. Lichtin and D. Meuser: Poles of a Local Zeta Function and Newton Polygons, Inventiones math. **73** (1983), 445–465.

[24] F. Loeser: Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 3, 435–471. MR 1011989.

[25] B. Malgrange: Intégrales asymptotiques et monodromie, Ann. Sci. École Norm. Sup. (4) **7** (1974), 405–430.

[26] D. Meuser: On the Poles of a Local Zeta Functions for Curves, Invent. math. **73** (1983), 445–665.

[27] D. Meuser: A survey of Igusa's local zeta function, Amer. J. Math. **138** (2016), no. 1, 149–179.

[28] T. Nagano and N. Miyazaki: Generalized Fresnel integrals as oscillatory integrals with positive real power phase functions and applications to asymptotic expansions, preprint, arxiv:2005.12754.

[29] M. Oka: Non-degenerate Mixed Functions, Kodai Math J. **32** (2010), 1–62.

[30] C. Sabbah: Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques, Compositio Math. **64** (1987), no. 2, 213–241.

[31] E. M. Stein: *Harmonic Analysis*, Princeton University Press, Princeton, New Jersey, 1993

[32] K. Takeuchi: On a Bernstein-Sato polynomial of a meromorphic function, Nagoya Math. J., **251** (2023), 715–733.

[33] T. Takeuchi: Asymptotic analysis of oscillatory integrals by using elementary methods, Master's thesis, Kyushu University, 2021 [in Japanese].

[34] A. N. Varchenko: Newton polyhedra and estimation of oscillating integrals, Funct. Anal Appl. **10-3** (1976), 175–196.

[35] W. Veys and W. A. Zúñiga-Galindo: Zeta functions and oscillatory integrals for meromorphic functions, Adv. Math. **311** (2017), 295–337.

[36] G. M. Ziegler: *Lectures on Polytopes*, Graduate Texts in Mathematics, **152**. Springer-Verlag, New York, 1995.