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Double Flag Varieties and Representations of Quivers
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Double Flag Varieties and Representations of Quivers

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Abstract

For a complex reductive group G and a symmetric subgroup K, the direct product $G/P \times K/Q$ of partial flag varieties G/P and K/Q is called a double-flag variety for a symmetric pair (G, K). In this study, we classify P and Q using a finite number of K-orbits on a double flag variety $G/P \times K/Q$ for a symmetric pair (G, K) when $G = \mathrm{GL}_{m+n}$ and $K = \mathrm{GL}_m \times \mathrm{GL}_n$, and a description of K-orbits when the number of K-orbits of $G/P \times K/Q$ is finite. We address the quiver representation problem by providing a correspondence between K-orbits and isomorphism classes of appropriate quiver representations.

Keywords: Double flag variety, Symmetric pairs, Reductive group, Representation of quiver, Krull–Schmidt theorem, Tits quadratic form.

1. Introduction

For a complex reductive group G and a symmetric subgroup $K \subset G$, the direct product $G/P \times K/Q$ of partial flag varieties G/P and K/Q is referred to as a double-flag variety for a symmetric pair (G, K); the diagonal action of K on $G/P \times K/Q$ is a crucial object applied to branching rules of representations, e.g., [7]. Two specific challenges are as follows:

1. What are the quadruples (G, K, P, Q) such that there are only finite K-orbits on $G/P \times K/Q$?
2. Can we describe the K-orbits on $G/P \times K/Q$ when there are only finite K-orbits?

We address this challenge by attributing it to the quiver representation isoclass classification problem.

The development of G-actions on flag varieties is discussed. Let B be the Borel subgroup of G and W_G be the Weyl group of G. The Weyl group W_G describes the G-orbit of $G/B \times G/B$ using the Bruhat decomposition

$$G\backslash(G/B \times G/B) \cong B\backslash G/B = \coprod_{w \in W_G} BwB,$$

the G-orbit of $G/B \times G/B$ is described by the Weyl group W_G.

Let P_1 and P_2 be parabolic subgroups of G. We can generalize the Bruer decomposition by considering the G-action on $G/P_1 \times G/P_2$. The natural projection $BwB \mapsto P_2 w P_1$ decomposes $P_2\backslash G/P_1$. $G/P_1 \times G/P_2$ corresponds to a quiver of Dynkin-type A_m

$$\cdot \longrightarrow \cdot \longrightarrow \cdots \longrightarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longleftarrow \cdots \longleftarrow \cdot \longleftarrow \cdot\,,$$

with a finite number of G-orbits.

Consider the condition for k-pairs $(P_1, P_2, \ldots, P_k)$ of parabolic subgroups of G such that the number of G-orbits on multiple flag varieties $G/P_1 \times G/P_2 \times \cdots \times G/P_k$ is finite.

Because $G\backslash(G/P_1 \times G/P_2 \times G/B) \cong B\backslash(G/P_1 \times G/P_2)$ regarding $k = 3$ and $P_3 = B$, the following conditions on the pair (P_1, P_2) of parabolic subgroups of G are equivalent:

- The number of G-orbits on $G/P_1 \times G/P_2 \times G/B$ is finite.
- The number of B-orbits on $G/P_1 \times G/P_2$ is finite.

Particularly, this condition is equivalent to $G/P_1 \times G/P_2$ being spherical; Littelmann provided the complete classification in [8].

Magyar, Weyman, and Zelevinsky present the finiteness criteria for the number of G-orbits for the general linear group $G = \mathrm{GL}$ and the symplectic group $G = \mathrm{Sp}$ under general conditions in [9] and [10]. According to [10], the theory can be extended to the case of the special orthogonal group $G = \mathrm{SO}$. Extending it is feasible. However, the classification result is unwritten. Although not a complete classification, Matsuki provided an aspect of the classification in [11] regarding $G = \mathrm{SO}$. Magyar, Weyman, and Zelevinsky demonstrated in [9] and [10] that both are infinite types under the condition $k \geqq 4$. Thus, the challenge of classifying multiple flag varieties is also a problem of classifying triple flag varieties.

Subsequently, He–Nishiyama–Ochiai–Oshima [5] extended the setting to double-flag varieties $G/P \times K/Q$ for symmetric pairs, and provided a classification of P and Q such that the number of K-orbits is finite in exceptional situations where P or Q is a Borel subgroup. However, numerous problems remain unsolved, including the complete classification of P and Q, such that the number of K-orbits is finite when P and Q are arbitrary, and the described orbit decomposition as a parameterization of the orbit set.

Consider a double-flag variety for a symmetric pair with a finite number of K-orbits (e.g., [4]) to derive an analog of Steinberg's theory. Nishiyama and Ochiai, in [12], established the relationship between double-flag varieties for symmetric pairs and free representations of multiplicity and spherical varieties.

Next, we will provide additional information on the quiver representation varieties. Consider the representation of a quiver. If we determine the dimension of the linear space of each vertex, the entire representation is an algebraic variety. This algebraic variety is referred to as a quiver variety.

Example 1.1. *Consider a quiver of Dynkin-type* A_k *with arrows all pointing in a single direction, and let*

$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{k-1} \longrightarrow V_k$$

be a representation of that quiver. $\mathrm{Hom}_{\Bbbk}(V_1, V_2) \times \cdots \times \mathrm{Hom}_{\Bbbk}(V_{k-1}, V_k)$ *is a quiver variety and a natural action of* $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{k-1})$*.*

The closure of any $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{k-1})$-orbit of the algebraic variety $\mathrm{Hom}_{\Bbbk}(V_1, V_2) \times \cdots \times \mathrm{Hom}_{\Bbbk}(V_{k-1}, V_k)$ in Example 1.1 is normal, Cohen-Macaulay with rational singularities. This has been demonstrated by Abeasis, Del Fra, and Kraft in [1].

Upon adding the condition that the arrows in the quiver representation in Example 1.1 are injections, this quiver variety becomes a flag variety. Considering this case, Zelevinsky demonstrated in [13] that the $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{k-1})$-orbits correspond to the open subset of the Schubert variety.

Bobiński and Zwara demonstrated in [3] that the orbit closure of a quiver variety is also normal, Cohen–Macaulay with rational singularities when the condition on the orientation of the arrows in the representation of the quiver in Example 1.1 is removed. The key to the proof is to embed a Dynkin-type A Auslander–Reiten quiver of path algebra into a larger Dynkin-type A Auslander–Reiten quiver of path algebra.

The crucial point of uniquely mapping an orbit to an isomorphism class of quiver representations is the key to the proof of the classification of multiple flag varieties presented by Magyar, Weyman, and Zelevinsky in [9] and [10]. Kac's [6] method of uniquely providing an indecomposable representation of a quiver when the dimension of the quiver representation is fixed also aids in the classification of multiple flag varieties in [9].

In this study, we classified P and Q, such that the number of K-orbits is finite for $G = \mathrm{GL}_{m+n}$ and $K = \mathrm{GL}_m \times \mathrm{GL}_n$, and described the K-orbits in these cases. Furthermore, whereas earlier research investigated the problem using the Bruhat and K-orbit decomposition of flag varieties, this study. Investigates the problem using methods from the representation theory of quivers.

The joint-flag variety proposed in this study is a dense open subvariety of the triple-flag variety with an action of G. However, we observe the following anomalies. The joint-flag variety has a finite number of G-orbits, while the triple-flag variety has an infinite G-orbits. Concepts based on quiver representation theory are also used in designing this joint-flag variety. This discovery was possible through research encompassing flag varieties and quiver representations. Because we consider complex reductive algebraic groups in this study, the coefficient field $\Bbbk$ is complex.

Consider the integer partition $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ for a particular natural number n. Let $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ triplets and V be an n-dimensional linear space.

Section 2 presents the joint-flag variety $\mathrm{Jl}_{\mathbf{d}}(V)$ as a subvariety of the triple-flag variety $\mathrm{Fl}_{\mathbf{d}}(V)$ of $\mathrm{GL}(V)$. We consider the $\mathrm{GL}(V)$-action on $\mathrm{Jl}_{\mathbf{d}}(V)$. This section provides the necessary and sufficient condition for $\mathrm{Jl}_{\mathbf{d}}(V)$ to obtain a finite number of $\mathrm{GL}(V)$-orbits. Furthermore, we describe the $\mathrm{GL}(V)$-orbits when the number of $\mathrm{GL}(V)$-orbits is finite.

Section 3 provides a correspondence between the $\mathrm{GL}(V_1)$-orbits of the joint-flag variety $\mathrm{Jl}_{\mathbf{d}}(V_1)$ and the $\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits of the double flag variety $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ for a symmetric pair $(\mathrm{GL}(V_1), \mathrm{GL}(V_2) \times \mathrm{GL}(V_3))$. Based on this correspondence and the results of Section 2, we obtain the following two results:

1. A finiteness condition taha is both necessary and sufficient for the number of $\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits on $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$.
2. A defines K-orbits in $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ as a combination of indecomposable representations of the quiver when the number of $\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits on $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ is finite.

2. Joint Flag Varieties

Let $\mathbf{a} = (a_1, \ldots, a_p)$ be an integer partition containing zero parts. We define $|\mathbf{a}| := a_1 + \cdots + a_p$, $||\mathbf{a}||^2 := {a_1}^2 + \cdots + {a_p}^2$, $\mathbf{a}' := (a_1, \ldots, a_{p-1})$, $\ell(\mathbf{a}) := p$ referred to as the length of $\mathbf{a}$, and $(a^p) := \underbrace{(a, \ldots, a)}_{p \text{ parts}}$.

Let V be a $|\mathbf{a}|$-dimensional linear space. We indicate the flag variety $\mathrm{Fl}_{\mathbf{a}}(V)$, including all flags $(0 = A_0 \subset A_1 \subset \cdots \subset A_{p-1} \subset A_p = V)$ such that

$$\dim A_i - \dim A_{i-1} = a_i \quad (i = 1, \ldots, p).$$

Considering any triple of positive integers (p, q, r), let $Q_{r,p,q}$ be the following quiver:

$$\begin{tikzcd}
 & & & & y_{p-1} \arrow[dl, "\beta_{p-1}"'] & \cdots \arrow[l, "\beta_{p-2}"'] & y_1 \arrow[l, "\beta_1"'] \\
x_1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{r-2}"] & x_{r-1} \arrow[r, "\alpha_{r-1}"] & w & & & \\
 & & & & z_{q-1} \arrow[ul, "\gamma_{q-1}"] & \cdots \arrow[l, "\gamma_{q-2}"] & z_1 \arrow[l, "\gamma_1"]
\end{tikzcd}$$

and $\mathrm{rep}_{\Bbbk} Q_{r,p,q}$ be a category of finite-dimensional representations of $Q_{r,p,q}$.

Definition 2.1. Define the **joint flag category** $\mathscr{J}_{r,p,q}$ as follows: $\mathscr{J}_{r,p,q}$ is a full subcategory of $\mathrm{rep}_{\Bbbk} Q_{r,p,q}$, with objects in the following forms of $M \in \mathrm{rep}_{\Bbbk} Q_{r,p,q}$:

$$\begin{tikzcd}
 & & & & M(y_{p-1}) \arrow[dl, "M(\beta_{p-1})"'] & \cdots \arrow[l, "M(\beta_{p-2})"'] & M(y_1) \arrow[l, "M(\beta_1)"'] \\
M(x_1) \arrow[r, "M(\alpha_1)"] & \cdots \arrow[r, "M(\alpha_{r-2})"] & M(x_{r-1}) \arrow[r, "M(\alpha_{r-1})"] & M(w) & & & \\
 & & & & M(z_{q-1}) \arrow[ul, "M(\gamma_{q-1})"] & \cdots \arrow[l, "M(\gamma_{q-2})"] & M(z_1) \arrow[l, "M(\gamma_1)"]
\end{tikzcd}$$

where all arrows are injections, $\mathrm{Im}\, \beta_{p-1} \cap \mathrm{Im}\, \gamma_{q-1} = 0$, and $\mathrm{Im}\, \beta_{p-1} \bigoplus \mathrm{Im}\, \gamma_{q-1} = M(w)$.

Considering any positive integer p, q, r, let ${\Lambda_{r,p,q}}^J$ denote the additive semi-group of all triples of partitions $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ such that $(\ell(\mathbf{a}), \ell(\mathbf{b}), \ell(\mathbf{c})) = (r, p, q)$, and $|\mathbf{a}| = |\mathbf{b}'| + |\mathbf{c}'|$. When there is no risk of ambiguity, we eliminate the subscripts and define Λ^J, $\mathscr{J}$.

Also, category $\mathrm{rep}_{\Bbbk} Q_{r,p,q}$ is a Krull–Schmidt category ([2], Chapter I, Theorem 4.10), and $\mathscr{J}_{r,p,q}$ is an additive subcategory of $\mathrm{rep}_{\Bbbk} Q_{r,p,q}$ and is closed for the direct sum factor. Thus, $\mathscr{J}_{r,p,q}$ is a Krull–Schmidt category.

Definition 2.2. For any $\mathbf{d} \in \Lambda^J$, we define $\mathrm{Jl}_{\mathbf{d}}(V)$ as the subvariety of $\mathrm{Fl}_{\mathbf{d}}(V)$ constituting the whole of $(A, B, C) \in \mathrm{Fl}_{\mathbf{d}}(V)$ such that there exists some $M \in \mathscr{J}$ and $M((Q_{r,p,q})_0, (Q_{r,p,q})_1) =$

$$\begin{tikzcd}
& & & & B_{p-1} \arrow[dl, hook, "\alpha"'] & \cdots \arrow[l, hook] & B_1 \arrow[l] \\
A_1 \arrow[r, hook] & \cdots \arrow[r] & A_{r-1} \arrow[r, hook] & V & & & \\
& & & & C_{q-1} \arrow[ul, hook, "\beta"] & \cdots \arrow[l, hook] & C_1 \arrow[l]
\end{tikzcd} .$$

The variety $\mathrm{Jl}_{\mathbf{d}}(V)$ is called the **joint-flag variety**.

Definition 2.3. The object $M \in \mathscr{J}$ corresponding to the element $(A, B, C) \in \mathrm{Jl}_{\mathbf{d}}(V)$ is called **d-type**. We also denote M by (V, A, B, C).

Then the following are equivalent for $(A, B, C), (A', B', C') \in \mathrm{Jl}_{\mathbf{d}}(V)$:

1. $\mathrm{GL}(V)(A, B, C) = \mathrm{GL}(V)(A', B', C')$.
2. $(V, A, B, C) \cong (V, A', B', C')$ on $\mathscr{J}$.

Therefore, we obtain the following:

1. $\#\{\mathrm{GL}(V)(A, B, C) \mid (A, B, C) \in \mathrm{Jl}_{\mathbf{d}}(V)\} = \#\{$isomorphism classes of $\mathbf{d}$-type$\}$.
2. An exact description of the orbit decomposition of $\mathrm{Jl}_{\mathbf{d}}(V)$ can be provided by identifying all the isomorphism classes of $\mathbf{d}$-type.

We can now translate the problem of counting and describing the $\mathrm{GL}(V)$-orbits of $\mathrm{Jl}_{\mathbf{d}}(V)$ into the challenge of determining the isomorphism classes of appropriate quiver representations $Q_{r,p,q}$.

Definition 2.4. The **Tits quadratic form** is defined as the form

$$Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \dim \mathrm{GL}(V) - \dim \mathrm{Fl}_{\mathbf{a}}(V) - \dim \mathrm{Fl}_{\mathbf{b}}(V) - \dim \mathrm{Fl}_{\mathbf{c}}(V),$$

where $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$. A simple calculation demonstrates that

$$Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (||\mathbf{a}||^2 + ||\mathbf{b}||^2 + ||\mathbf{c}||^2 - (\dim V)^2)/2.$$

We obtain the following proposition immediately from ([9], Proposition 3.1).

Proposition 2.5. *Let $\mathbf{d} \in \Lambda^J$ is the dimension vector of an indecomposable object of $\mathscr{F}$ with $Q(\mathbf{d}) \geq 1$. Then $Q(\mathbf{d}) = 1$, and an indecomposable object of dimension vector $\mathbf{d}$ has a unique isomorphism class $I_{\mathbf{d}}$.*

Lemma 2.6. *The following equation holds:*

$$\dim \mathrm{Fl}_{\mathbf{d}}(V) = \dim \mathrm{Jl}_{\mathbf{d}}(V).$$

Proof. The following demonstrates the proposition. The triple-flag variety $\mathrm{Fl}_{\mathbf{d}}(V)$ is connected, and $\mathrm{Jl}_{\mathbf{d}}(V)$ is an open set of $\mathrm{Fl}_{\mathbf{d}}(V)$. □

Definition 2.7. A dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ is called a **finite type $\mathscr{J}$** if the number of $\mathrm{GL}_{|\mathbf{a}|}$-orbits in $\mathrm{Jl}_{\mathbf{d}}(V)$ is finite.

Definition 2.8. A dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ is called a **finite type $\mathscr{F}$** if the number of $\mathrm{GL}_{|\mathbf{a}|}$-orbits in $\mathrm{Fl}_{\mathbf{d}}(V)$ is finite.

Definition 2.9. A quiver of representation F is said to be **Schur indecomposable** if $\dim_{\Bbbk} \mathrm{Hom}\,(F, F) = 1$.

Proposition 2.10. *If a dimension vector $\mathbf{d}$ is of finite type $\mathscr{J}$ and satisfies $Q(\mathbf{d}) = 1$, then there exists a Schur indecomposable $I_{\mathbf{d}}$ with the dimension vector $\mathbf{d}$.*

Proof. Because $\mathbf{d}$ is of finite type, $\mathrm{Jl}_{\mathbf{d}}(V)$ has a Zariski open orbit Ω. Therefore, consider the isomorphism class $I_{\mathbf{d}}$ of $\mathrm{Jl}_{\mathbf{d}}(V)$ corresponding to Ω. Then consider any quiver representative F corresponding to $I_{\mathbf{d}}$. Subsequently, we obtain the following equation:

$$\begin{aligned}\dim_{\Bbbk} \mathrm{Hom}\,(I_{\mathbf{d}}, I_{\mathbf{d}}) &= \dim \mathrm{Stab}_{\mathrm{GL}(V)}(F) \\ &= \dim \mathrm{GL}(V) - \dim \mathrm{Jl}_{\mathbf{d}}(V) = Q(\mathbf{d}) = 1.\end{aligned}$$

Thus, $I_{\mathbf{d}}$ becomes Schur indecomposable. □

Definition 2.11. A non-zero dimension vector $\bar{\mathbf{d}} \in \Lambda^J$ is called a summand of $\mathbf{d} \in \Lambda^J$ of type $\mathscr{J}$ if $\mathbf{d} - \bar{\mathbf{d}} \in \Lambda^J$.

Definition 2.12. Considering a partition $\mathbf{a}$, we denote the partition obtained from $\mathbf{a}$ by eliminating all zero parts and reordering the non-zero parts in a weakly decreasing order by $\mathbf{a}^+$.

Considering objects F and F' of $\mathrm{Jl}_{\mathbf{d}}(V)$, if F and F' are isomorphic as representations of the quiver, then F and F' have identical $\mathrm{GL}(V)$-orbits, resulting in the following proposition.

Proposition 2.13. *A dimension vector* $\mathbf{d}$ *is of finite type* $\mathscr{J}$ *if and only if any summand* $\bar{\mathbf{d}}$ *of* $\mathbf{d}$ *of type* $\mathscr{J}$ *satisfies* $Q(\bar{\mathbf{d}}) \geq 1$.

Proof. First, consider $\mathbf{d}$ is of finite type $\mathscr{J}$. Then, any summand $\bar{\mathbf{d}}$ of $\mathbf{d}$ of type $\mathscr{J}$ is of finite type $\mathscr{J}$. Therefore, $Q(\bar{\mathbf{d}}) \geq 1$ is satisfied. Conversely, any summand $\bar{\mathbf{d}}$ of $\mathbf{d}$ of type $\mathscr{J}$ satisfies $Q(\bar{\mathbf{d}}) \geq 1$. Any object F of $\mathscr{J}$ has a unique decomposition into indecomposable objects. Considering this case, based on the hypotheses and Proposition 2.5, there are a finite number of isomorphism classes of indecomposable summands of F. Therefore, $\mathbf{d}$ is of finite type $\mathscr{J}$. $\square$

Considering a dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$, we use the representation

$$\mathbf{d}^+ := (\mathbf{a}^+, (\mathbf{b}'^+, |\mathbf{c}'|), (\mathbf{c}'^+, |\mathbf{b}'|)).$$

Theorem 2.14. *A triple* $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ *satisfies one of the following: Either* $\mathbf{d}$ *is of finite type* $\mathscr{J}$ *or* $\mathbf{d}$ *has a summand* $\bar{\mathbf{d}}$ *of type* $\mathscr{J}$ *where* $\bar{\mathbf{d}}^+$ *is among the following:* $((2^3), (1^3, 3), (1^3, 3))$, $((3^3), (2^2, 5), (1^5, 4))$, $((3^3), (1^5, 4), (2^2, 5))$, $((1^4), (1^2, 2), (1^2, 2))$, $((2^4), (3, 5), (1^5, 3))$, $((2^4), (1^5, 3), (3, 5))$, $((1^5), (2, 3), (1^3, 2))$, $((1^5), (1^3, 2), (2, 3))$, $((1^7), (3, 4), (2^2, 3))$, $((1^7), (2^2, 3), (3, 4))$.

Proof. First, we will demonstrate that $0 \neq \mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ satisfies approximately one of the following three conditions by dividing the cases for $\mathbf{d}$.

(I) $Q(\mathbf{d}) \geq 1$.

(II) $\mathbf{d}$ has at least one summand $\bar{\mathbf{d}}$ of type $\mathscr{J}$ where $\bar{\mathbf{d}}^+$ is among

$$((2^3), (1^3, 3), (1^3, 3)), ((3^3), (2^2, 5), (1^5, 4)), ((1^4), (1^2, 2), (1^2, 2)),$$
$$((2^4), (3, 5), (1^5, 3)), ((1^5), (2, 3), (1^3, 2)), \textit{and}\ ((1^7), (3, 4), (2^2, 3)).$$

(III) $\mathbf{d}$ is of finite type $\mathscr{F}$.

Here, $\mathbf{d} \in \Lambda^J$ satisfying (III) satisfies (I); however, the separation of (III) and (I) is for simplifying the proof. The condition that $\mathbf{d}$ satisfies (III) results from the finiteness determination condition for G-orbits of the triple-flag variety by [9], and $\mathbf{d}$ is of finite type $\mathscr{J}$.

In the following, we present a diagram depicting the division into cases. Subsequently, we investigate the relationship between each case and conditions (I), (II), and (III). Finally, we prove the proof using the condition by case division.

Let r, p, and q denote the number of parts of $\mathbf{a}^+$, $\mathbf{b}^+$, and $\mathbf{c}^+$, respectively. We also presume, without loss of generality, that $p \leq q$.

Note that tips (I), (II), and (III) of arrows $\Rightarrow$ and $\Leftarrow$ in Figures 1, 2, 3, 4, and 5 indicate that they satisfy (I), (II), and (III) under the conditions of case separation (however, as proved subsequently, when the arrow in the figure is labeled (II), there is one that satisfies condition (I); however, when the arrow is labeled (I), condition (II) is not satisfied).

$$
\begin{tikzcd}
(\mathrm{III}) & r=1 \arrow[l, Rightarrow] & HYPO \arrow[l] \arrow[dl] \arrow[dd] & & \\
(1)\ (\mathrm{I}) & r=2 \arrow[l, Rightarrow] & & & \\
(2)\ (\mathrm{I}) & \min(\mathbf{a}^+)=1 \arrow[l, Rightarrow] & r=3 \arrow[l] \arrow[d] & & \\
(\mathrm{III}) & p=1 \arrow[l, Rightarrow] & \min(\mathbf{a}^+)\neq 1 \arrow[l] \arrow[dl] \arrow[dd] \arrow[r] & p\geq 4 \arrow[r, Rightarrow] & (\mathrm{II})\ (1') \\
(3)\ (\mathrm{I}) & p=2 \arrow[l, Rightarrow] & & & \\
(4)\ (\mathrm{I}) & \min(\mathbf{a}^+)=2 \arrow[l, Rightarrow] & p=3 \arrow[l] \arrow[d] & & \\
(5)\ (\mathrm{I}) & \min(\mathbf{b}'^+)=1 \arrow[l, Rightarrow] & \min(\mathbf{a}^+)\neq 2 \arrow[l] \arrow[d] & & \\
(6)\ (\mathrm{I}) & 3\leq q\leq 5 \arrow[l, Rightarrow] & \min(\mathbf{b}'^+)\neq 1 \arrow[l] \arrow[r] & q\geq 6 \arrow[r, Rightarrow] & (\mathrm{II})\ (2')
\end{tikzcd}
$$

Figure 1: Illustration of the division cases for $r = 1, 2, 3$.

$$
\begin{tikzcd}
 & & HYPO \arrow[d] & & \\
\text{(III)} & p=1 \arrow[l, Rightarrow] & r=4 \arrow[l] \arrow[r] \arrow[d] & p\geq 3 \arrow[r, Rightarrow] & \text{(II) (3}'\text{)} \\
\text{(7) (I)} & \min(\mathbf{a}^+)=1 \arrow[l, Rightarrow] & p=2 \arrow[l] \arrow[d] & & \\
\text{(III)} & |\mathbf{b}'|=1 \arrow[l, Rightarrow] & \min(\mathbf{a}^+)\neq 1 \arrow[l] \arrow[dl] \arrow[dd] & & \\
\text{(8) (I)} & |\mathbf{b}'|=2 \arrow[l, Rightarrow] & & & \\
\text{(III)} & q=2 \arrow[l, Rightarrow] & |\mathbf{b}'|\geq 3 \arrow[l] \arrow[r] \arrow[dl] \arrow[ddl] & q\geq 6 \arrow[r, Rightarrow] & \text{(II) (4}'\text{)} \\
\text{(III)} & q=3 \arrow[l, Rightarrow] & & & \\
\text{(9) (I)} & 4\leq q\leq 5 \arrow[l, Rightarrow] & & &
\end{tikzcd}
$$

Figure 2: Illustration of the division cases for $r = 4$.

$$
\begin{tikzcd}
 & & HYPO \arrow[d] & & \\
\text{(III)} & p=1 \arrow[l, Rightarrow] & r=5 \arrow[l] \arrow[r] \arrow[d] & p\geq 3 \arrow[r, Rightarrow] & \text{(II) (5}'\text{)} \\
\text{(III)} & q=2 \arrow[l, Rightarrow] & p=2 \arrow[l] \arrow[dl] \arrow[dd] & & \\
\text{(III)} & q=3 \arrow[l, Rightarrow] & & & \\
\text{(III)} & |\mathbf{b}'|=1 \arrow[l, Rightarrow] & q\geq 4 \arrow[l] \arrow[r] & |\mathbf{b}'|\neq 1 \arrow[r, Rightarrow] & \text{(II) (6}'\text{)}
\end{tikzcd}
$$

Figure 3: Illustration of the division cases for $r = 5$.

$$\begin{tikzcd}
 & & HYPO \arrow[d] & & \\
\text{(III)} & p=1 \arrow[l, Rightarrow] & r=6 \arrow[l] \arrow[r] \arrow[d] & p\geq 3 \arrow[r, Rightarrow] & \text{(II) } (7') \\
\text{(III)} & |\mathbf{b}'|=1 \arrow[l, Rightarrow] & p=2 \arrow[l] \arrow[d] & & \\
\text{(III)} & q=2 \arrow[l, Rightarrow] & |\mathbf{b}'|\neq 1 \arrow[l] \arrow[r] \arrow[dl] & q\geq 4 \arrow[r, Rightarrow] & \text{(II) } (8') \\
(10)\text{ (I)} & q=3 \arrow[l, Rightarrow] & & &
\end{tikzcd}$$

Figure 4: Illustration of the division cases for $r=6$.

$$\begin{tikzcd}
 & & HYPO \arrow[d] & & \\
\text{(III)} & p=1 \arrow[l, Rightarrow] & r\geq 7 \arrow[l] \arrow[r] \arrow[d] & p\geq 3 \arrow[r, Rightarrow] & \text{(II) } (9') \\
\text{(III)} & |\mathbf{b}'|=1 \arrow[l, Rightarrow] & p=2 \arrow[l] \arrow[d] & & \\
\text{(III)} & q=2 \arrow[l, Rightarrow] & |\mathbf{b}'|\neq 1 \arrow[l] \arrow[r] \arrow[d] & q\geq 4 \arrow[r, Rightarrow] & \text{(II) } (10') \\
\text{(III)} & \min(\mathbf{b}'^{+})=2 \arrow[l, Rightarrow] & q=3 \arrow[l] \arrow[d] & & \\
\text{(III)} & \min(\mathbf{c}'^{+})=1 \arrow[l, Rightarrow] & \min(\mathbf{b}'^{+})\neq 2 \arrow[l] \arrow[r] & \min(\mathbf{c}'^{+})\neq 1 \arrow[r, Rightarrow] & \text{(II) } (11')
\end{tikzcd}$$

Figure 5: Illustration of the division cases for $r\geq 7$.

We initially demonstrate that $\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})$ satisfies (I) in cases (1) to (10). Considering $(A,B,C)\in \mathrm{Jl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$, let the dimensions of B_{p-1} and C_{q-1} be $x:=\dim B_{p-1}$ and $y:=\dim C_{q-1}$.

Then

$$\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})=(\mathbf{a},(\mathbf{b}':=(b_1,\ldots,b_{p-1}),y),(\mathbf{c}':=(c_1,\ldots,c_{q-1}),x)).$$

Therefore, we obtain

$$\begin{aligned}
Q(\mathbf{d})&=(||\mathbf{a}||^2+||\mathbf{b}||^2+||\mathbf{c}||^2-(x+y)^2)/2\\
&=(||\mathbf{a}||^2+||\mathbf{b}'||^2+||\mathbf{c}'||^2-2xy)/2.
\end{aligned}$$

Consider $\mathbf{a}^{\mathbb{R}}$, $\mathbf{b'}^{\mathbb{R}}$, and $\mathbf{c'}^{\mathbb{R}}$ as partitions of $x+y$, x, and y in the range of non-negative real numbers, respectively. Here, we presume that the conditions (1) to (10) (for the number of non-zero parts and the minimum value of non-zero parts) are satisfied for each case. Then there exists a $\mathbf{d}^{\mathbb{R}} := (\mathbf{a}^{\mathbb{R}}, \mathbf{b'}^{\mathbb{R}}, \mathbf{c'}^{\mathbb{R}})$ such that $Q(\mathbf{d}^{\mathbb{R}}) := (||\mathbf{a}^{\mathbb{R}}||^2 + ||\mathbf{b'}^{\mathbb{R}}||^2 + ||\mathbf{c'}^{\mathbb{R}}||^2 - 2xy)/2$ is minimized. Furthermore, except for reordering their respective parts, $\mathbf{a}^{\mathbb{R}}$, $\mathbf{b'}^{\mathbb{R}}$, and $\mathbf{c'}^{\mathbb{R}}$ are currently unique. Because $Q(\mathbf{d}) \geq 1$ if $Q(\mathbf{d}^{\mathbb{R}}) > 0$, we demonstrate that $Q(\mathbf{d}^{\mathbb{R}}) > 0$ for $\mathbf{d}$ in cases (1) to (10). The table of $\mathbf{d}^{\mathbb{R}^+}$ and $Q(\mathbf{d}^{\mathbb{R}})$ for cases (1) to (10) is presented bellow.

(1) $\mathbf{d}^{\mathbb{R}^+} = ((((x+y)/2)^2), (1^x, y), (1^y, x)),$
$(x+y \geq 2,\ y \geq x),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/2 + x + y$$
$$= ((x-y+1)^2)/2 + (4y-1)/2.$$

(2) $\mathbf{d}^{\mathbb{R}^+} = ((1, ((x+y-1)/2)^2), (1^x, y), (1^y, x)),$
$(x+y \geq 3,\ y \geq x),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y-1)^2)/2 + x + y + 1 = ((x-y)^2)/2 + 3/2.$$

(3) $\mathbf{d}^{\mathbb{R}^+} = ((((x+y)/3)^3), (x, y), (1^y, x)),$
$(x+y \geq 6,\ x \geq 1,\ y \geq 1),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/3 + x^2 + y = ((2x-y)^2)/3 + y.$$

(4) $\mathbf{d}^{\mathbb{R}^+} = ((2, ((x+y-2)/2)^2), ((x/2)^2, y), (1^y, x)),$
$(x+y \geq 6,\ x \geq 2,\ y \geq 2),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y-2)^2)/2 + x^2/2 + y + 4$$
$$= ((2x-y-2)^2)/4 + (y^2 - 8y + 20)/4.$$

(5) $\mathbf{d}^{\mathbb{R}^+} = ((((x+y)/3)^3), (1, x-1, y), (1^y, x)),$
$(x+y \geq 9,\ x \geq 2,\ y \geq 2),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/3 + (x-1)^2 + y + 1$$
$$= ((4x-2y-3)^2)/12 + 5/4.$$

(6) $(q=3)$ $\mathbf{d}^{\mathbb{R}^+} = ((((x+y)/3)^3), ((x/2)^2, y), ((y/2)^2, x)),$
$(x+y \geq 9,\ x \geq 4,\ y \geq 2),$
$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/3 + x^2/2 + y^2/2$$
$$= ((5x-4y)^2)/30 + (3y^2)/10.$$

$(q=4)$ $\mathbf{d}^{\mathbb{R}^+}\!= ((((x+y)/3)^3), ((x/2)^2, y), ((y/3)^3, x)),$

$(x+y \geq 9,\ x \geq 4,\ y \geq 3),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/3 + x^2/2 + y^2/3$$
$$= ((5x-4y)^2)/30 + (2y^2)/15.$$

$(q=5)$ $\mathbf{d}^{\mathbb{R}^+}\!= ((((x+y)/3)^3), ((x/2)^2, y), ((y/4)^4, x)),$

$(x+y \geq 9,\ x \geq 4,\ y \geq 4),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/3 + x^2/2 + y^2/4$$
$$= ((5x-4y)^2)/30 + (y^2)/20.$$

(7) $\mathbf{d}^{\mathbb{R}^+}\!= ((1, ((x+y-1)/3)^3), (x, y), (1^y, x)),$

$(x+y \geq 4,\ x \geq 1,\ y \geq 1),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y-1)^2)/3 + x^2 + y + 1$$
$$= ((4x-2y-1)^2)/12 + 5/4.$$

(8) $\mathbf{d}^{\mathbb{R}^+}\!= ((((2+y)/4)^4), (2, y), (1^y, 2)),$

$(y \geq 6),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -4y + ((2+y)^2)/4 + 2^2 + y = (y^2 - 8y + 20)/4.$$

(9) $(q=4)$ $\mathbf{d}^{\mathbb{R}^+}\!= ((((x+y)/4)^4), (x, y), ((y/3)^3, x)),$

$(x+y \geq 8,\ x \geq 3,\ y \geq 3),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/4 + x^2 + y^2/3$$
$$= ((5x-3y)^2)/20 + (2y^2)/15.$$

$(q=5)$ $\mathbf{d}^{\mathbb{R}^+}\!= ((((x+y)/4)^4), (x, y), ((y/4)^4, x)),$

$(x+y \geq 8,\ x \geq 3,\ y \geq 4),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/4 + x^2 + y^2/4$$
$$= ((5x-3y)^2)/20 + (y^2)/20.$$

(10) $\mathbf{d}^{\mathbb{R}^+}\!= ((((x+y)/6)^6), (x, y), ((y/2)^2, x)),$

$(x+y \geq 6,\ x \geq 2,\ y \geq 2),$

$$2Q(\mathbf{d}^{\mathbb{R}^+}) = -2xy + ((x+y)^2)/6 + x^2 + y^2/2$$
$$= ((7x-5y)^2)/42 + (y^2)/14.$$

The justification for the expression $Q(\mathbf{d}^{\mathbb{R}}) > 0$ can be determined considering that solving for $Q(\mathbf{d}^{\mathbb{R}}) \leq 0$ yields $(x, y) = (0, 0)$ or no solution.

Subsequently, considering $\mathbf{d}$ in the case $(1')$ to $(11')$, we demonstrate that there is a summand $\bar{\mathbf{d}}$ of $\mathbf{d}$ of type $\mathscr{J}$ such that $\bar{\mathbf{d}}^+$ is among the following:

$$((2^3),(1^3,3),(1^3,3)),((3^3),(2^2,5),(1^5,4)),((1^4),(1^2,2),(1^2,2)),$$
$$((2^4),(3,5),(1^5,3)),((1^5),(2,3),(1^3,2)),\text{and }((1^7),(3,4),(2^2,3)).$$

Regarding $(1')$, we obtain $q \geq p \geq 4$, $r = 3$, $\min(\mathbf{a}^+) \neq 1$; therefore, $b_{\ell(\mathbf{b})}, c_{\ell(\mathbf{c})} \geq 3$, and there exists $a_i, a_j, a_k \geq 2$ for some $1 \leq i < j < k \leq \ell(\mathbf{a})$, there exists $b_{i'}, b_{j'}, b_{k'} \geq 1$ for some $1 \leq i' < j' < k' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''}, c_{k''} \geq 1$ for some $1 \leq i'' < j'' < k'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{2}, \underline{0}, \overset{j}{2}, \underline{0}, \overset{k}{2}, \underline{0}), (\underline{0}, \overset{i'}{1}, \underline{0}, \overset{j'}{1}, \underline{0}, \overset{k'}{1}, \underline{0}, 3), (\underline{0}, \overset{i''}{1}, \underline{0}, \overset{j''}{1}, \underline{0}, \overset{k''}{1}, \underline{0}, 3))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^+ = ((2^3),(1^3,3),(1^3,3))$.

Regarding $(2')$, we obtain $p = 3$, $q \geq 6$, $r = 3$, $\min(\mathbf{a}^+) \neq 1, 2$, $min(\mathbf{b}'^+) \neq 1$; therefore, $b_{\ell(\mathbf{b})} \geq 5$, $c_{\ell(\mathbf{c})} \geq 4$, and there exists $a_i, a_j, a_k \geq 3$ for some $1 \leq i < j < k \leq \ell(\mathbf{a})$, there exists $b_{i'}, b_{j'} \geq 2$ for some $1 \leq i' < j' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''}, c_{k''}, c_{l''}, c_{m''} \geq 1$ for some $1 \leq i'' < j'' < k'' < l'' < m'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{3}, \underline{0}, \overset{j}{3}, \underline{0}, \overset{k}{3}, \underline{0}), (\underline{0}, \overset{i'}{2}, \underline{0}, \overset{j'}{2}, \underline{0}, 5), (\underline{0}, \overset{i''}{1}, \underline{0}, \overset{j''}{1}, \underline{0}, \overset{k''}{1}, \underline{0}, \overset{l''}{1}, \underline{0}, \overset{m''}{1}, \underline{0}, 4))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^+ = ((3^3),(2^2,5),(1^5,4))$.

Regarding $(3')$, $(5')$, $(7')$, $(9')$, we obtain $q \geq p \geq 3$, $r \geq 4$; therefore, $b_{\ell(\mathbf{b})}, c_{\ell(\mathbf{c})} \geq 2$, and there exists $a_i, a_j, a_k, a_l \geq 1$ for some $1 \leq i < j < k < l \leq \ell(\mathbf{a})$, there exists $b_{i'}, b_{j'} \geq 1$ for some $1 \leq i' < j' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''} \geq 1$ for some $1 \leq i'' < j'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{1}, \underline{0}, \overset{j}{1}, \underline{0}, \overset{k}{1}, \underline{0}, \overset{l}{1}, \underline{0}), (\underline{0}, \overset{i'}{1}, \underline{0}, \overset{j'}{1}, \underline{0}, 2), (\underline{0}, \overset{i''}{1}, \underline{0}, \overset{j''}{1}, \underline{0}, 2))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^+ = ((1^4),(1^2,2),(1^2,2))$.

Regarding $(4')$, we obtain $p = 2$, $q \geq 6$, $r = 4$, $\min(\mathbf{a}^+) \neq 1$, $|\mathbf{b}'| \geq 3$; therefore, $b_{\ell(\mathbf{b})} \geq 5$, $c_{\ell(\mathbf{c})} = |\mathbf{b}'| \geq 3$, and there exists $a_i, a_j, a_k, a_l \geq 2$ for some $1 \leq i < j < k < l \leq \ell(\mathbf{a})$, there exists $b_{i'} \geq 3$ for some $1 \leq i' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''}, c_{k''}, c_{l''}, c_{m''} \geq 1$ for some $1 \leq i'' < j'' < k'' < l'' < m'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{2}, \underline{0}, \overset{j}{2}, \underline{0}, \overset{k}{2}, \underline{0}, \overset{l}{2}, \underline{0}), (\underline{0}, \overset{i'}{3}, \underline{0}, 5), (\underline{0}, \overset{i''}{1}, \underline{0}, \overset{j''}{1}, \underline{0}, \overset{k''}{1}, \underline{0}, \overset{l''}{1}, \underline{0}, \overset{m''}{1}, \underline{0}, 3))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^{+} = ((2^4), (3, 5), (1^5, 3))$.

Regarding (6′), (8′), (10′), we obtain $p = 2$, $q \geq 4$, $r \geq 5$, $|\mathbf{b}'| \neq 1$; therefore, $b_{\ell(\mathbf{b})} \geq 3$, $c_{\ell(\mathbf{c})} \geq 2$, and there exists $a_i, a_j, a_k, a_l, a_m \geq 1$ for some $1 \leq i < j < k < l < m \leq \ell(\mathbf{a})$, there exists $b_{i'} \geq 2$ for some $1 \leq i' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''}, c_{k''} \geq 1$ for some $1 \leq i'' < j'' < k'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{1}, \underline{0}, \overset{j}{1}, \underline{0}, \overset{k}{1}, \underline{0}, \overset{l}{1}, \underline{0}, \overset{m}{1}, \underline{0}), (\underline{0}, \overset{i'}{2}, \underline{0}, 3), (\underline{0}, \overset{i''}{1}, \underline{0}, \overset{j''}{1}, \underline{0}, \overset{k''}{1}, \underline{0}, 2))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^{+} = ((1^5), (2, 3), (1^3, 2))$.

Regarding (11′), we obtain $p = 2$, $q = 3$, $r \geq 7$, $\min(\mathbf{b}'^{+}) \neq 1, 2$, $\min(\mathbf{c}'^{+}) \neq 1$; therefore, $b_{\ell(\mathbf{b})} \geq 4$, $c_{\ell(\mathbf{c})} \geq 3$, and there exists $a_i, a_j, a_k, a_l, a_m, a_n, a_o \geq 1$ for some $1 \leq i < j < k < l < m < n < o \leq \ell(\mathbf{a})$, there exists $b_{i'} \geq 3$ for some $1 \leq i' < \ell(\mathbf{b})$, there exists $c_{i''}, c_{j''} \geq 2$ for some $1 \leq i'' < j'' < \ell(\mathbf{c})$. Therefore, $\mathbf{d}$ has a summand

$$\bar{\mathbf{d}} = ((\underline{0}, \overset{i}{1}, \underline{0}, \overset{j}{1}, \underline{0}, \overset{k}{1}, \underline{0}, \overset{l}{1}, \underline{0}, \overset{m}{1}, \underline{0}, \overset{n}{1}, \underline{0}, \overset{o}{1}, \underline{0}), (\underline{0}, \overset{i'}{3}, \underline{0}, 4), (\underline{0}, \overset{i''}{2}, \underline{0}, \overset{j''}{2}, \underline{0}, 3))$$

of type $\mathscr{J}$, where $\bar{\mathbf{d}}^{+} = ((1^7), (3, 4), (2^2, 3))$.

Now, let $\mathbf{d} \in \Lambda^J$ be such that $\mathbf{d}^{+}$ is among the following:

$$((2^3), (1^3, 3), (1^3, 3)), ((3^3), (2^2, 5), (1^5, 4)), ((1^4), (1^2, 2), (1^2, 2)),$$
$$((2^4), (3, 5), (1^5, 3)), ((1^5), (2, 3), (1^3, 2)), \text{and } ((1^7), (3, 4), (2^2, 3)).$$

Then $Q(\mathbf{d}) = 0$; therefore, $\mathbf{d} \in \Lambda^J$ in case (II) is of infinite type $\mathscr{J}$ based on Proposition 2.13.

Next, we demonstrate that any $\mathbf{d} \in \Lambda^J$ in cases (1) to (10) does not satisfy II. Then $Q(\bar{\mathbf{d}}) \geq 1$ for any summand $\bar{\mathbf{d}}$ of $\mathbf{d}$ on $\mathscr{J}$ from the division into cases. Therefore, $\mathbf{d}$ in Cases (1) to (10) is of finite type $\mathscr{J}$ based on Proposition 2.13.

Considering Case (1), we obtain $r = 2$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (2), we obtain $r = 3$, and $\min(\mathbf{a}^{+}) = 1$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (3), we obtain $r = 3$, and $p = 2$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (4), we obtain $r = 3$, $p = 3$ and $\min(\mathbf{a}^{+}) = 2$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (5), we obtain $r = 3$, $p = 3$ and $\min(\mathbf{b}'^{+}) = 1$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (6), we obtain $r = 3$, $p = 3$ and $3 \leq q \leq 5$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (7), we obtain $r = 4$, $p = 2$ and $\min(\mathbf{a}^+) = 1$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (8), we obtain $r = 4$, $p = 2$ and $|\mathbf{b}'| = 2$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (9), we obtain $r = 4$, $p = 2$ and $4 \leq q \leq 5$. Therefore, $\mathbf{d}$ does not satisfy (II).

Considering Case (10), we obtain $r = 6$, $p = 2$ and $q = 3$. Therefore, $\mathbf{d}$ does not satisfy (II).

□

Lemma 2.15. *Fix two natural numbers m and n such that $m \leq n$. Let $\mathbf{a}$ be an integer partition of n such that $\ell(\mathbf{a}^+) = m$. Considering this case, the following are equivalent.*

(1) $\mathbf{a}$ *is an integer partition of n such that $||\mathbf{a}||^2$ is the minimum.*
(2) *The difference between any two parts of $\mathbf{a}^+$ is less than or equal to* 1.

Proof. First, presume that $\mathbf{a}$ is an integer partition of n such that $||\mathbf{a}||^2$ is the minimum. Presume that there exist non-zero parts a_1, a_2 of $\mathbf{a}$ such that $a_1 - a_2 \geq 2$. Then

$$\begin{aligned}(a_1 - 1)^2 + (a_2 + 1)^2 &= {a_1}^2 + {a_2}^2 - 2a_1 + 2a_2 + 2 \\ &= {a_1}^2 + {a_2}^2 - 2(a_1 - a_2 - 1) < {a_1}^2 + {a_2}^2.\end{aligned}$$

Let $\mathbf{b}$ be the integer partition of n obtained by replacing a_1 with $a_1 - 1$ and a_2 with $a_2 + 1$ among the parts of $\mathbf{a}$. Then $||\mathbf{b}||^2 < ||\mathbf{a}||^2$, which contradicts the assumption.

Conversely, if the difference between any two parts of $\mathbf{a}$ is less than or equal to one, then the integer partition $\mathbf{a}$ is unique except for reordering. Additionally, an integer partition of n that minimizes the sum of the squares of each part exists. Therefore, based on the proof that (2) is true if (1) is indicated earlier, we may conclude that $||\mathbf{a}||^2$ is the minimum. □

Theorem 2.16. *A dimension vector $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ is of finite type $\mathscr{J}$ and satisfies $Q(\mathbf{d}) = 1$ if and only if the $\mathbf{d}^+$ in the following list up to the* *order of* $\mathbf{b}$ *and* $\mathbf{c}$.

$$\begin{aligned}
&((1),(1),(1,0)),\quad ((1^6),(2,4),(2^2,2)),\\
&((1^{2x+1}),(x,x+1),(x,1,x))\ x\geq 2,\\
&((1^{2x}),(x,x),(x-1,1,x))\ x\geq 2,\\
&((1^{x}),(1,x-1),(1^{x-1},1))\ x\geq 2,\\
&((x,x-1,1),(1^{x},x),(1^{x},x))\ x\geq 2,\\
&((x^2,1),(1^{x},x+1),(1^{x+1},x))\ x\geq 2,\\
&((2^3),(1^2,4),(1^4,2)),\quad ((2^3),(2,1,3),(1^3,3)),\quad ((3,2^2),(2,1,4),(1^4,3)),\\
&((3^2,2),(2,1,5),(1^5,3)),\quad ((3^2,2),(2^2,4),(1^4,4)),\quad ((4,3,2),(2^2,5),(1^5,4)),\\
&((4^2,2),(2^2,6),(1^6,4)),\\
&((x^3),(x-1,1,2x),(1^{2x},x))\ x\geq 3,\\
&((x^2,x-1),(x-1,1,2x-1),(1^{2x-1},x))\ x\geq 4,\\
&((x,(x-1)^2),(x-1,1,2x-2),(1^{2x-2},x))\ x\geq 4,\\
&(((x-1)^3),(x-1,1,2x-3),(1^{2x-3},x))\ x\geq 4,\\
&((3^3),(2^2,5),(2,1^3,4)),\\
&(((x-1)^3,1),(x,2x-2),(1^{2x-2},x))\ x\geq 2,\\
&((x,(x-1)^2,1),(x,2x-1),(1^{2x-1},x))\ x\geq 2,\\
&((x^2,x-1,1),(x,2x),(1^{2x},x))\ x\geq 2,\\
&((x^3,1),(x,2x+1),(1^{2x+1},x))\ x\geq 2,\\
&((2^4),(2,6),(1^6,2)),\quad ((2^4),(3,5),(2,1^3,3)),\quad ((2,1^5),(3,4),(2^2,3)).
\end{aligned}$$

Proof. First, presume that $\mathbf{d}\in\Lambda^J$ is of finite type $\mathscr{J}$ and satisfies $Q(\mathbf{d})=1$. If $\mathbf{d}$ is of finite type $\mathscr{F}$ and satisfies $Q(\mathbf{d})=1$, then $\mathbf{d}^+$ is among the following based on [9]:

$$\begin{aligned}
&((1),(1),(1,0)),\quad ((1^6),(2,4),(2^2,2)),\\
&((1^{2x+1}),(x,x+1),(x,1,x))\ x\geq 2,\\
&((1^{2x}),(x,x),(x-1,1,x))\ x\geq 2,\\
&((1^{x}),(1,x-1),(1^{x-1},1))\ x\geq 2.
\end{aligned}$$

Therefore, identifying all the $\mathbf{d}$ in the cases (1) to (10) of Theorem 2.14 that satisfy $Q(\mathbf{d})=1$ is sufficient. Then, considering $\mathbf{d}^{\mathbb{R}^+}$ in each case from (1) to (10), we identify the pair (x,y) of natural numbers that satisfies $0<Q(\mathbf{d}^{\mathbb{R}^+})\leq 1$. Considering (x,y), is there $\mathbf{d}\in\Lambda^J$ that satisfies the

conditions of each case classification? If yes, use Lemma 2.15 to identify the $\mathbf{d} \in \Lambda^J$ for which $Q(\mathbf{d})$ is the minimum. This $\mathbf{d}$ is unique except for the rearrangement of parts, and we are only required to verify whether it is $Q(\mathbf{d}) = 1$. The table of (x, y) satisfying $0 < Q(\mathbf{d}^{\mathbb{R}^+}) \leq 1$ in each case from (1) to (10) is given as follows:

(1) $(x, y) = (1, 1),$
(2) $(x, y) = (1, 1),\ (1, 2),\ (x, x - z)\ 1 \geq z \geq -1,\ x \geq 2,$
(3) $(x, y) = (1, 1),\ (1, 2),$
(4) $(x, y) = (2, 2),\ (2, 3),\ (2, 4),\ (3, 3),\ (3, 4),\ (3, 5),\ (4, 4),\ (4, 5),\ (4, 6),$
(5) $(x, y) = (1, 1),\ (1, 2),\ (x, 2x - z)\ 3 \geq z \geq 0,\ x \geq 2,$
(6) $(q = 3)\ (x, y) = (1, 1),\ (1, 2),\ (2, 1),\ (2, 2),$
$(q = 4)\ (x, y) = (1, 1),\ (1, 2),\ (2, 1),\ (2, 2),\ (2, 3),\ (3, 3),$
$(q = 5)\ (x, y) = (1, 1),\ (1, 2),\ (2, 1),\ (2, 2),\ (2, 3),\ (2, 4),\ (3, 2),\ (3, 3),$
$(3, 4),\ (4, 4),\ (4, 5),\ (5, 6),$
(7) $(x, y) = (1, 1),\ (1, 2),\ (1, 3),\ (x, 2x - z)\ 2 \geq z \geq -1,\ x \geq 2,$
(8) $(x, y) = (2, y)\ 6 \geq y \geq 2,$
(9) $(q = 4)\ (x, y) = (1, 1),\ (1, 2),\ (1, 3),\ (2, 2),\ (2, 3),$
$(q = 5)\ (x, y) = (1, 1),\ (1, 2),\ (1, 3),\ (2, 2),\ (2, 3),\ (2, 4),\ (3, 4),\ (3, 5),$
$(4, 6),$
(10) $(x, y) = (1, 1),\ (1, 2),\ (2, 1),\ (2, 2),\ (2, 3),\ (2, 4),\ (3, 3),\ (3, 4),\ (4, 5).$

Subsequently, we provide a table of (x, y) such that among the (x, y) satisfying $0 < Q(\mathbf{d}^{\mathbb{R}^+}) \leq 1$ in each of the cases (1) to (10) stated earlier, there exists $\mathbf{d} \in \Lambda^J$ satisfying the conditions of each case separation for (x, y).

(1) $(x, y) = (1, 1),$
(2) $(x, y) = (1, 2),\ (x, x - z)\ 0 \geq z \geq -1,\ x \geq 2,$
(4) $(x, y) = (2, 4),\ (3, 3),\ (3, 4),\ (3, 5),\ (4, 4),\ (4, 5),\ (4, 6),$
(5) $(x, y) = (x, 2x)\ x \geq 3,$
$(x, y) = (x, 2x - z)\ 3 \geq z \geq 1,\ x \geq 4,$
(6) $(q = 5)\ (x, y) = (4, 5),\ (5, 6),$
(7) $(x, y) = (1, 3),\ (x, 2x - z)\ 2 \geq z \geq -1,\ x \geq 2,$
(8) $(x, y) = (2, 6),$

(9) $(q=5)$ $(x,y)=(3,5),\ (4,6)$,
(10) $(x,y)=(2,4),\ (3,3),\ (3,4),\ (4,5)$.

Finally, the table below presents the $\mathbf{d}^+$ of $\mathbf{d}\in\Lambda^J$ for which $Q(\mathbf{d})$ is minimized for each (x,y) in the table above.

(1) $(x,y)=(1,1)$ $\mathbf{d}^+=((1^2),(1,1),(1,1))$,
(2) $(x,y)=(1,2)$ $\mathbf{d}^+=((1^3),(1,2),(1^2,1))$,
$(x,y)=(x,x)$ $\mathbf{d}^+=((x,x-1,1),(1^x,x),(1^x,x))$ $x\geq 2$,
$(x,y)=(x,x+1)$ $\mathbf{d}^+=((x^2,1),(1^x,x+1),(1^{x+1},x))$ $x\geq 2$,
(4) $(x,y)=(2,4)$ $\mathbf{d}^+=((2^3),(1^2,4),(1^4,2))$,
$(x,y)=(3,3)$ $\mathbf{d}^+=((2^3),(2,1,3),(1^3,3))$,
$(x,y)=(3,4)$ $\mathbf{d}^+=((3,2^2),(2,1,4),(1^4,3))$,
$(x,y)=(3,5)$ $\mathbf{d}^+=((3^2,2),(2,1,5),(1^5,3))$,
$(x,y)=(4,4)$ $\mathbf{d}^+=((3^2,2),(2^2,4),(1^4,4))$,
$(x,y)=(4,5)$ $\mathbf{d}^+=((4,3,2),(2^2,5),(1^5,4))$,
$(x,y)=(4,6)$ $\mathbf{d}^+=((4^2,2),(2^2,6),(1^6,4))$,
(5) $(x,y)=(x,2x)$ $\mathbf{d}^+=((x^3),(x-1,1,2x),(1^{2x},x))$ $x\geq 3$,
$(x,y)=(x,2x-1)$ $\mathbf{d}^+=((x^2,x-1),(x-1,1,2x-1),(1^{2x-1},x))$
$x\geq 4$,
$(x,y)=(x,2x-2)$ $\mathbf{d}^+=((x,(x-1)^2),(x-1,1,2x-2),(1^{2x-2},x))$
$x\geq 4$,
$(x,y)=(x,2x-3)$ $\mathbf{d}^+=(((x-1)^3),(x-1,1,2x-3),(1^{2x-3},x))$
$x\geq 4$,
(6) $(q=5)$ $(x,y)=(4,5)$ $\mathbf{d}^+=((3^3),(2^2,5),(2,1^3,4))$,
$(x,y)=(5,6)$ $\mathbf{d}^+=((4^2,3),(3,2,6),(2^2,1^2,5))$,
(7) $(x,y)=(1,3)$ $\mathbf{d}^+=((1^4),(1,3),(1^3,1))$,
$(x,y)=(x,2x-2)$ $\mathbf{d}^+=(((x-1)^3,1),(x,2x-2),(1^{2x-2},x))$ $x\geq 2$,
$(x,y)=(x,2x-1)$ $\mathbf{d}^+=((x,(x-1)^2,1),(x,2x-1),(1^{2x-1},x))$ $x\geq 2$,
$(x,y)=(x,2x)$ $\mathbf{d}^+=((x^2,x-1,1),(x,2x),(1^{2x},x))$ $x\geq 2$,
$(x,y)=(x,2x+1)$ $\mathbf{d}^+=((x^3,1),(x,2x+1),(1^{2x+1},x))$ $x\geq 2$,
(8) $(x,y)=(2,6)$ $\mathbf{d}^+=((2^4),(2,6),(1^6,2))$,
(9) $(q=5)$ $(x,y)=(3,5)$ $\mathbf{d}^+=((2^4),(3,5),(2,1^3,3))$,

$$
\begin{aligned}
&\quad\; (x,y)=(4,6)\;\; \mathbf{d}^+=((3^2,2^2),(4,6),(2^2,1^2,4)),\\
(10)\;\; &(x,y)=(2,4)\;\; \mathbf{d}^+=((1^6),(2,4),(2^2,2)),\\
&(x,y)=(3,3)\;\; \mathbf{d}^+=((1^6),(3,3),(2,1,3)),\\
&(x,y)=(3,4)\;\; \mathbf{d}^+=((2,1^5),(3,4),(2^2,3)),\\
&(x,y)=(4,5)\;\; \mathbf{d}^+=((2^3,1^3),(4,5),(3,2,4)).
\end{aligned}
$$

Here $\mathbf{d}^+$ in the cases (1) $(x,y)=(1,1)$, (2) $(x,y)=(1,2)$, (7) $(x,y)=(1,3)$, and (10) $(x,y)=(2,4),(3,3)$ is of finite type $\mathscr{F}$. The cause for this overlap is that the method used in [9] to provide the list differs from the method used herein. A simple calculation demonstrates that $\mathbf{d}^+$ satisfies $Q(\mathbf{d}^+)=1$ except for (6) $(x,y)=(5,6)$, (9) $(x,y)=(4,6)$, and (10) $(x,y)=(4,5)$; therefore, the left-to-right proof is now complete. The converse can be determined from Theorem 2.14. $\square$

Theorem 2.17. *Let $(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda^J$ be of finite type $\mathscr{J}$. Next, there exists a natural bijection between $\mathrm{GL}_{|\mathbf{a}|}$-orbits in $\mathrm{Jl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$ and families $M=(m_{\mathbf{d}})$ of nonnegative integers satisfying*

$$\sum_{\mathbf{d}}(m_{\mathbf{d}})\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})$$

and indexed by $\mathbf{d}\in\Lambda^J$, where $\mathbf{d}$ is of finite type $\mathscr{J}$ and $Q(\mathbf{d})=1$.

Proof. It is evident from Propositions 2.5, 2.10, and 2.13. $\square$

3. Double Flag Varieties

Now introduce the following notation: For $\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda^J$,

$$\mathrm{Dl}_{\mathbf{d}}(V_2\bigoplus V_3):=\mathrm{Fl}_{\mathbf{a}}(V_1)\times\mathrm{Fl}_{\mathbf{b}'}(V_2)\times\mathrm{Fl}_{\mathbf{c}'}(V_3),$$

where $\dim V_2=|\mathbf{b}'|$, $\dim V_3=|\mathbf{c}'|$, and $V_1=V_2\bigoplus V_3$. The variety $\mathrm{Dl}_{\mathbf{d}}(V_2\bigoplus V_3)$ is called the **double-flag variety**.

Definition 3.1. A dimension vector $\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda^J$ is referred to as a **finite type** $\mathscr{D}$ if the number of $\mathrm{GL}(V_2)\times\mathrm{GL}(V_3)$-orbits in $\mathrm{Dl}_{\mathbf{d}}(V_2\bigoplus V_3)$ is finite.

Definition 3.2. Define the **double-flag category** $\mathscr{D}_{r,p,q}$ as follows: $\mathscr{D}_{r,p,q}$ is a full sub-category of $\mathscr{J}_{r,p,q}$, the objects of which are presented in the following forms of $(V_2 \bigoplus V_3, A', B', C') \in \mathscr{J}_{r,p,q}$:

$$\begin{tikzcd}[column sep=small]
 & & & & V_2 \arrow[dl, "\begin{pmatrix}1\\0\end{pmatrix}"'] & B_{p-1} \arrow[l] & \cdots \arrow[l] & B_1 \arrow[l] \\
A_1 \arrow[r] & \cdots \arrow[r] & A_{r-1} \arrow[r] & V_1 = V_2 \bigoplus V_3 & & & & \\
 & & & & V_3 \arrow[ul, "\begin{pmatrix}0\\1\end{pmatrix}"] & C_{q-1} \arrow[l] & \cdots \arrow[l] & C_1 \arrow[l]
\end{tikzcd} ,$$

where (A, B, C) is a triple of flags in (V_1, V_2, V_3) that fall within $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ for some $\mathbf{d} := (\mathbf{a}, (\mathbf{b}, \dim V_3), (\mathbf{c}, \dim V_2)) \in \Lambda^J$.

When there is no risk of ambiguity, we eliminate the subscripts and define $\mathscr{D}$.

Also, category $\mathscr{J}_{r,p,q}$ is a Krull–Schmidt category, and $\mathscr{D}_{r,p,q}$ is an additive subcategory of $\mathscr{J}_{r,p,q}$ and is closed for the direct sum factor. Thus, $\mathscr{D}_{r,p,q}$ is a Krull–Schmidt category.

Then, the following are equivalent for $(A, B, C), (A', B', C') \in \mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$:

(1) $(\mathrm{GL}(V_2) \times \mathrm{GL}(V_3))(A, B, C) = (\mathrm{GL}(V_2) \times \mathrm{GL}(V_3))(A', B', C')$

(2) $(V_2 \bigoplus V_3, A, B, C) \cong (V_2 \bigoplus V_3, A', B', C')$ *on* $\mathscr{D}$

Therefore, we obtain the following:

1. $\#\{\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)(A, B, C) \mid (A, B, C) \in \mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)\}$
   $= \#\{\text{isomorphism classes of } F_{\mathbf{d}} \in \mathscr{D}\}$.
2. An exact description of the orbit decomposition of $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ can be provided by identifying all the isomorphism classes of $F_{\mathbf{d}}$.

We can translate the challenge of the number and description of $\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits of $\mathrm{Dl}_{\mathbf{d}}(V_2 \bigoplus V_3)$ into the problem of identifying the isomorphism classes of the representations of the quiver.

Proposition 3.3. *The following categories a* $\mathscr{J}_{r,p,q}$ *and* $\mathscr{D}_{r,p,q}$ *are equivalent categories.*

Proof. Define functor $F : \mathscr{J}_{r,p,q} \longrightarrow \mathscr{D}_{r,p,q}$ as follows, for the following objects

$$\begin{tikzcd}[column sep=small]
 & & & & M(y_{p-1}) \arrow[dl, "M(\beta_{p-1})"'] & \cdots \arrow[l, "M(\beta_{p-2})"'] & M(y_1) \arrow[l, "M(\beta_1)"'] \\
M(x_1) \arrow[r, "M(\alpha_1)"] & \cdots \arrow[r, "M(\alpha_{r-2})"] & M(x_{r-1}) \arrow[r, "M(\alpha_{r-1})"] & M(w) & & & \\
 & & & & M(z_{q-1}) \arrow[ul, "M(\gamma_{q-1})"] & \cdots \arrow[l, "M(\gamma_{q-2})"] & M(z_1) \arrow[l, "M(\gamma_1)"]
\end{tikzcd}$$

of $\mathscr{J}_{r,p,q}$. The functor F transfers objects and morphisms to the branches B and C only, without changing the branch of A, as follows:

$$\begin{tikzcd}[column sep=small]
B : M(w) \arrow[d, "f"] & M(y_{p-1}) \arrow[l, "M(\beta_{p-1})"'] \arrow[d, "f_{p-1}"] & M(y_{p-2}) \arrow[l, "M(\beta_{p-2})"'] \arrow[d, "f_{p-2}"] & \cdot \arrow[l, "M(\beta_{p-3})"'] & M(y_1) \arrow[l, "M(\beta_1)"'] \arrow[d, "f_1"] \\
B' : M'(w) & M'(y_{p-1}) \arrow[l, "M'(\beta_{p-1})"'] & M'(y_{p-2}) \arrow[l, "M'(\beta_{p-2})"'] \arrow[d, "F"] & \cdot \arrow[l, "M'(\beta_{p-3})"'] & M'(y_1) \arrow[l, "M'(\beta_1)"'] \\
\operatorname{Im} M(\beta_{p-1}) \bigoplus \operatorname{Im} M(\gamma_{q-1}) \arrow[d, "f"] & \operatorname{Im} M(\beta_{p-1}) \arrow[l, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "f|_{M(\beta_{p-1})}"] & M(y_{p-2}) \arrow[l, "M(\beta_{p-1})M(\beta_{p-2})"'] \arrow[d, "f_{p-2}"] & \cdot \arrow[l, "M(\beta_{p-3})"'] & M(y_1) \arrow[l, "M(\beta_1)"'] \arrow[d, "f_1"] \\
\operatorname{Im} M'(\beta_{p-1}) \bigoplus \operatorname{Im} M'(\gamma_{q-1}) & \operatorname{Im} M'(\beta_{p-1}) \arrow[l, "\begin{pmatrix}1\\0\end{pmatrix}"] & M'(y_{p-2}) \arrow[l, "M'(\beta_{p-1})M'(\beta_{p-2})"'] & \cdot \arrow[l, "M'(\beta_{p-3})"'] & M'(y_1) \arrow[l, "M'(\beta_1)"']
\end{tikzcd}$$

Considering the same procedure as the B branch, we can also obtain a function F for the C branch. Then, M and $F(M)$ are isomorphic representations of a quiver, and F is an equivalence of categories. $\square$

Therefore, from Proposition 3.3, we immediately obtain the following Theorem 3.4 and 3.5 by appropriating the result of section 2.

Theorem 3.4. *A dimension vector* $\mathbf{d} \in \Lambda^J$ *is of finite type* $\mathscr{D}$ *if and only if* $\mathbf{d}$ *is of finite type* $\mathscr{J}$.

Theorem 3.5. *Let* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda^J$ *be of finite type* $\mathscr{D}$*. Then there exists a natural bijection between* $\mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$*-orbits in* $\mathrm{Dl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V_2 \bigoplus V_3)$ *and families* $M = (m_{\mathbf{d}})$ *of nonnegative integers satisfying*

$$\sum_{\mathbf{d}} (m_{\mathbf{d}})\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$$

and indexed by $\mathbf{d} \in \Lambda^J$*, where* $\mathbf{d}$ *is of finite type* $\mathscr{D}$ *and* $Q(\mathbf{d}) = 1$*.*

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