Theoretical Study on Quantum Mechanical Aspects of Gravity based on Quantum Field Theory

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PhD Thesis

Theoretical Study on Quantum Mechanical Aspects of Gravity based on Quantum Field Theory

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Abstract

The unification of gravity theory and quantum mechanics (quantum gravity) is a major challenge in modern physics. Theoretically, quantum gravity is expected to reveal extreme phenomena, such as the early universe, and the quantum aspect of a black hole. Quantum gravity still shows no signs of completion because experimental evidence for the quantum effects of gravity has never been observed so far.

Recent proposals for low-energy tabletop experiments have the potential to be game-changers. The rapid development of quantum technology has raised hope that the study of gravitational phenomena induced by quantum matter may be within its scope. In particular, pioneered by Bose *et al.* and Matletto and Vedral, the possibility of detecting gravity-induced entanglement between two quantum masses is a popular topic

However, the studies of Bose *et al.* and Matletto and Vedral did not take into account the dynamical degree of freedom of quantum gravity (graviton) at all. It has been pointed out that if the graviton is not considered, the properties of the theory of relativity and quantum mechanics may become incompatible. In addition, there has been no discussion on how the dynamical degrees of freedom of graviton affect entanglement generation.

In this thesis, we focus on revealing the quantum mechanical aspects of gravity based on a quantum field theory. In particular, by comparison with the quantized electromagnetic field case, we clarified the relationship between the existence of graviton and entanglement generation in a low-energy regime. We found that two quantum phenomena, vacuum fluctuation and quantum superposition of bremsstrahlung owing to the transverse mode of the electromagnetic field (photon field), appear in the formula of a quantum correlation. Furthermore, we demonstrated that causality, a property of the theory of relativity, and complementarity, a property of quantum mechanics, are consistent because there are no quantum correlations between objects that interact via dynamical fields. Finally, we quantitatively clarified that the superposition of gravitational fields is correlated with the superposition state of the particles.

Our results suggest that the dynamical degrees of freedom of the gravitational field may play an essential role in the quantum effects of gravity in tabletop experiments in low-energy regimes. We anticipate a deeper understanding of the decoherence induced by dynamical fields to be important in finding a consistent theory of gravity and quantum mechanics.

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Publication List

The work presented in this thesis is based on the following four publications [1, 2, 3, 4].

[1] **Yuuki Sugiyama**, Akira Matsumura, and Kazuhiro Yamamoto,
"Quantumness of gravitational field: A perspective on monogamy relation",
arXiv:2401.03867.

[2] **Yuuki Sugiyama**, Akira Matsumura, and Kazuhiro Yamamoto,
"Quantum uncertainty of gravitational field and entanglement in superposed massive particles",
Phys. Rev. D **108**, 105019 (2023), arXiv:2308.03903.

[3] **Yuuki Sugiyama**, Akira Matsumura, and Kazuhiro Yamamoto,
"Consistency between causality and complementarity guaranteed by Robertson inequality in quantum field theory",
Phys. Rev. D **106**, 125002 (2022), arXiv:2206.02506.

[4] **Yuuki Sugiyama**, Akira Matsumura, and Kazuhiro Yamamoto,
"Effects of photon field on entanglement generation in charged particles",
Phys. Rev. D **106**, 045009 (2022), arXiv:2203.09011.

The following six publications [5, 6, 7, 8, 9, 10] are not directly related to this thesis.

[5] Tomoya Shichijo, Nobuyuki Matsumoto, Akira Matsumura, Daisuke Miki, **Yuuki Sugiyama**, and Kazuhiro Yamamoto,
"Quantum state of a suspended mirror coupled to cavity light -Wiener filter analysis of the pendulum and rotational modes-",
arXiv:2303.04511.

[6] **Yuuki Sugiyama**, Tomoya Shichijo, Nobuyuki Matsumoto, Akira Matsumura, Daisuke Miki, and Kazuhiro Yamamoto,
"Effective description of a suspended mirror coupled to cavity light: Limitations of Q enhancement due to normal-mode splitting by an optical spring",
Phys. Rev. A **107**, 033515 (2023), arXiv:2212.11056.

[7] Daisuke Miki, Nobuyuki Matsumoto, Akira Matsumura, Tomoya Shichijo, **Yuuki Sugiyama**, Kazuhiro Yamamoto, and Naoki Yamamoto,
"Generating quantum entanglement between macroscopic objects with continuous

measurement and feedback control”,
Phys. Rev. A **107**, 032410 (2023), arXiv:2210.13169.

[8] Masahiro Hotta, Yasusada Nambu, **Yuuki Sugiyama**, Kazuhiro Yamamoto, and Go Yusa,
“Expanding Edges of Quantum Hall Systems in a Cosmology Language -Hawking Radiation from de Sitter Horizon in Edge Modes-”,
Phys. Rev. D **105**, 105009 (2022), arXiv:2202.03731.

[9] Koki Yamashita, Yue Nan, **Yuuki Sugiyama**, and Kazuhiro Yamamoto,
“Large-scale structure with superhorizon isocurvature dark energy”,
Phys. Rev. D **105**, 083531 (2022), arXiv:2112.12552.

[10] **Yuuki Sugiyama**, Kazuhiro Yamamoto, and Tsutomu Kobayashi,
“Gravitational waves in Kasner spacetimes and Rindler wedges in Regge-Wheeler gauge: Unruh effect”,
Phys. Rev. D **103**, 083503 (2021), arXiv:2012.15004.

Notation

Here is summary for notation. For this thesis, the metric signature is chosen to be $(+,-,-,-)$. Greek letters take values $0, 1, 2, 3$ denoting the indices of the spacetime coordinates. We will be working in the natural units $c = \hbar = 1$ excepting in Chapter 2 while we recover c and $\hbar$ as necessary. The standard mathematical and physical notations are adopted.

A $\Rightarrow$ B : A is sufficient for B

c : speed of light

$\hbar$: reduced Planck's constant

G : Newton's gravitational constant

e : elementary charge

$\eta_{\mu\nu}$: Minkowski spacetime metric

$g_{\mu\nu}$: Curved spacetime metric, which satisfies with $g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}$

$\Gamma^{\lambda}_{\mu\nu}$: Christoffel connections, $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\big(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu}\big)$

$\boldsymbol{\nabla}_{\mu}$: Covariant derivative in the μ direction, which generally leads to $\boldsymbol{\nabla}_{\lambda}g_{\mu\nu} = 0$

${R^{\mu}}_{\alpha\nu\beta}$: Rieman tensor, ${R^{\mu}}_{\alpha\nu\beta} = \partial_{\nu}\Gamma^{\mu}_{\alpha\beta} - \partial_{\beta}\Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\lambda\nu}\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\mu}_{\lambda\beta}\Gamma^{\lambda}_{\alpha\nu}$

$R_{\alpha\beta} = {R^{\mu}}_{\alpha\mu\beta}$: Ricci tensor

$R = g^{\mu\nu}R_{\mu\nu}$: Ricci curvature

$\mathbb{1}_{n\times n}$: $n \times n$ identiy operator

Contents

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1 Introduction

The construction of a unified theory of gravity theory and quantum mechanics is one of the most important attempts in modern physics. This theory is called the quantum gravity theory. Gravity is described by general relativity as the geometry of spacetime, whereas quantum mechanics is a theory that explains the laws of physics in the microscopic world. Currently, four fundamental forces (electromagnetism, strong force, weak force, and gravity) are confirmed to exist in nature. In particular, all three forces, except gravity, have been successfully described by quantum field theories, consistent with quantum mechanics.

Although the quantum gravity theory is considered essential for understanding extreme situations, it has not yet been completed. For example, at the beginning of the universe, the curvature of the spacetime will diverges. Standard cosmology, which assumes the classical Einstein equation, suggests the existence of a singularity where the spacetime curvature diverges [11, 12]. Under physical assumptions based on general relativity, a theorem was proposed that singularity generally exists in an expanding universe [13]. Another example is the quantum aspects of black holes: particle creation near the horizon of a black hole leads to the evaporation of black holes with thermal radiation [14]. In particular, when the initial state of the black hole is considered to be pure state quantum matter, this process is interpreted as a time evolution from a pure state to a mixed state, where the final state is thermal. Thus this time evolution is inconsistent with the unitarity of quantum mechanics, which is known as information loss problem. Many ideas have been proposed to resolve this problem [15, 16, 17, 18, 19]; however, this has not been solved because not only is the theory describing the structure of spacetime and matter inside a black hole still unknown, but its interaction with matter outside the black hole is also unclear. Quantum gravity theory is expected to potentially provide an answer to information loss problem. Various candidate theories (e.g., the superstring theory [20]) for quantum gravity theory have been proposed so far. However, the validity of the proposed theories remains unclear.

One of the reasons for the difficulty in constructing a quantum gravity theory may be that the phenomena originating from quantum gravitational fields have not yet been experimentally discovered. General relativity predicted the existence of gravitational waves, which are dynamical degrees of freedom in spacetime. In 2015, the existence of gravitational waves was confirmed through direct observations using a gravitational wave interferometer [21]. However, the essence of quantum mechanics is that a particle is a wave and a wave is a particle. Therefore, if the general relativity and quantum mechanics are unified, quantum of gravitational field (graviton) should exist. However, the verification of the existence of graviton requires accelerators with energies much higher than those currently attainable [22, 23] and thus faces technical difficulties. In recent years, methods for the indirect detection of gravitons using gravitational wave interferometry have been proposed [24, 25, 26, 27, 28, 29]; however, these methods have not yet been realized.

It has been pointed out that gravity may not obey quantum mechanics. Diósi [30, 31] and Penrose [32, 33] have suggested that quantum state of a macroscopic object may decohere due to its own gravitational energy. This implies that gravity is intrinsically fluctuating, not quantum fluctuation. In this model, the Newtonian gravitational potential term is the origin of the non-unitary dynamics. A more detailed theoretical model is presented in Sec. 2.3.2.

Developments in quantum technology from meso to macroscopic mechanical systems have led to studies that may confirm the quantum mechanical nature of gravity in laboratory-scale, low-energy experiments [34, 35]. Feynman predicted that a particle in a superposition state of position would also make another particle be in a superposition state of position via gravitational interactions [36]. This is interpreted as the realization of a superposition state of the gravitational potentials induced by the particles. Bose *et al.,* [37] and Marletto and Vedral [38] have devised a model to verify the superposition state of gravitational potentials (see also the review article [39]). They considered the time evolution between two particles in a superposition state of positions interacting due to the gravitational potential. Then they found that two particles that are initially not in an entangled state can become entangled due to the gravitational potential. This quantum entangled state is interpreted as being induced by the superposition state of the gravitational potential. Inspired by their work, various studies have been conducted to test the quantum gravity effect on a laboratory scale. For instance, Refs. [40, 41, 42, 43, 44, 45, 46] considered the quantum effects of gravity in a table-top experiment. Moreover, the authors in Refs. [5, 6, 7, 47, 48, 49, 50] discussed the entanglement generation between macroscopic objects using an optomechanical system [51, 52, 53]. However, the proposals of Bose *et al.,* and Marletto and Vedral did not take into account the degrees of freedom of the graviton, and it was noted that the properties of relativity and quantum mechanics may be inconsistent without considering the graviton [54, 55, 56].

This doctoral thesis aims to clarify the quantum-mechanical aspects of gravity from the viewpoint of quantum field theory. In particular, in order to focus on revealing the relationship between the property of gravity as a quantum field and the meaning of the quantum superposition of gravitational potentials, we attempt to investigate the following three questions:

(i) How do the degrees of freedom of a dynamical field affect quantum entanglement generation? (Section 3)

(ii) How are the properties of relativity and quantum mechanics consistent when the field degrees of freedom are considered? (Section 4)

(iii) How is the superposition state of the gravitational field quantified? (Section 5)

First, to clarify question (i), we construct a model that can discuss quantum entanglement generation by considering the dynamical field degrees of freedom. Looking ahead to the extension to the gravitational field, we consider quantum electrodynamics that can be analyzed rigorously. In particular, we discuss the entanglement generation between two charged particles interacting with an electromagnetic field. We assume that the initial states of the two non-relativistic charged particles each in a superposition of two trajectories and the electromagnetic field are separable states and analyze their time evolution. The gauge fixing of the electromagnetic field is chosen to preserve the covariance with a view to extending to a gravitational field. Under the above assumptions, we derived a formula to quantify the entanglement and discuss the entanglement generation between charged particles. Then we investigate the behavior of negativity by assuming specific trajectories of the charged particles and performing calculations concretely. As a result, we demonstrate that entanglement generation between two charged particles is suppressed by decoherence due to vacuum fluctuations of the electromagnetic field. In addition, we found that the quantum superposition of bremsstrahlung from a superposed trajectory affects the signature of the quantum coherence between the two particles. However, the entanglement is not generated because the vacuum fluctuations of the electromagnetic field dominate over the signature of the entanglement. Similar features are expected in the entanglement generation between two masses in the framework of a quantized gravitational field. We expect that even in the quantized gravitational field, vacuum fluctuation and quantum superposition of bremsstrahlung, owing to the gravitational field, will appear.

Next, based on the constructed model above, we consider question (ii). In particular, we focus on how causality, a property of relativity, and complementarity, a property of quantum mechanics, are consistent. In this study, we discuss the consistency of the above two properties in both the electromagnetic and linearized gravitational field cases. We show that the causality can be understood in terms of the properties of the retarded Green's function. In addition, by introducing inequalities to quantitatively express complementarity, we investigate the condition that causality and complementarity are consistent. We then prove that the inequality expressing complementarity is guaranteed by the uncertainty relation of the dynamical fields. To further deepen our understanding of this condition, we focused on the relationship between the quantum correlation (entanglement) between two particles caused by electromagnetic and gravitational fields. We found that the uncertainty relation of the electromagnetic and gravitational fields led to vacuum fluctuations and that the two particles do not become entangled. Furthermore, we numerically demonstrate that the condition in which the two particles are quantum-uncorrelated guarantees consistency with complementarity.

Finally, we investigate question (iii). In particular, we consider the trade-off relationship between quantum entanglement in a composite system of two particles and a quantized gravitational field. Consequently, it is found that the strength of entanglement between the particle and the gravitational field changes due to the strength of entanglement between the two particles. This relationship is known as a monogamy relation. Using the monogamy relation, we analyze the behavior of the quantum superposition state of the gravitational field. Then it is quantitatively discussed that the larger the width of the superposition of the particles, the better the gravitational field is also superposed.

Our analysis will play an important role in the construction of a quantum field theory of gravity that respects the properties of quantized gravitational fields (decoherence, superposition of bremsstrahlung, consistency of causality and complementarity).

This thesis is organized as follows. In Chapter. 2, we present the theoretical background of this thesis. Chapters 3-5 constitute the main part of this thesis. In Chapter 3, we clarify how the dynamical field affects entanglement generation using a quantum field theoretic approach. In Chapter 4, we consider the consistency of the properties of the causality and complementarity and reveal its condition. In Chapter 5, we discuss the quantification of the quantum superposition of the gravitational field. Chapter 6 presents the conclusions of this thesis and remarks on the future prospects. In Appendix A, we present a formula for the $1/c$ expansion of the phase shift in the non-relativistic regime, c is the speed of light. In Appendix B, we provide detailed computations in Sec. 3.2 and 5.1. In Appendix C, we prove the statement in (4.43). In Appendix D, we numerically demonstrates the condition (4.48).

Finally, in Appendix E, the proofs of inequalities (5.5) and (5.18) are presented.

2 Theoretical Background

Here, we provide the theoretical background of this thesis. In particular, we briefly introduce general relativity, quantum mechanics, and theories that combine them. Finally, we consider the quantum field theory (quantum electrodynamics).

2.1 General relativity

General relativity is the simplest theory that describes the dynamics of gravity as the geometry of spacetime. This theory has been used in various verification experiments since it was proposed by Einstein. The general relativity assumes the following two principles: the equivalence principle, and the general principle of relativity. The equivalence principle guarantees that we can vanish the effect of gravity at any one point in spacetime. Specifically, this means that the we can consider the Minkowski metric in the vicinity of any one point in spacetime. The coordinate system around this point can be regarded as an inertial frame (the local inertial frame). The general principle of relativity states that physical laws are identical in any local inertial frame. This assertion indicates that the equation of motion is expressed in a covariant tensor form with respect to the general coordinate transformation.

According to this theory, spacetime behaves as a dynamical variable. Mathematically, the dynamical variable of the spacetime is introduced as

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{2.1}$$

Here the line element ds in curved spacetime characterizes the length from a spacetime point x^μ to $x^\mu + dx^\mu$, and $g_{\mu\nu}(x)$ is the metric tensor. In the following, we will see the equation of motion with respect to $g_{\mu\nu}(x)$. The metric tensor $g_{\mu\nu}$ is a covariant tensor and transforms by an arbitrary coordinate transformation $x^\mu \to x^{\mu\prime} = f^\mu(x)$ as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^{\mu\prime}}\frac{\partial x^\beta}{\partial x^{\nu\prime}}g_{\alpha\beta}. \tag{2.2}$$

Here we first consider the motion of a free falling particle in curved spacetime. In Minkowski spacetime, this particle has a straight line trajectory. However, in curved spacetime, "straight line" is generalized: geodesics. The equation of motion of particle along the geodesic trajectory is given by a geodesic equation. The geodesic equation is derived as follows: We consider a particle moving the spacetime point from P_1 to P_2. The "straight line" is given as the path that minimizes the distance between two points P_1 and P_2. In curved spacetime, the distance between two points P_1 and P_2 is characterized by

$$\int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \sqrt{g_{\mu\nu} dx^\mu dx^\nu}. \tag{2.3}$$

Note that the line element ds is invariant under the general coordinate transformation. A "straight line" in curved spacetime is determined by the condition that minimizes $-\int ds$:

$$-\delta \int_{P_1}^{P_2} ds = 0. \tag{2.4}$$

This condition leads to the following geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \tag{2.5}$$

Here $d\tau = ds/c$ is the proper time, which characterizes the time measured in the frame where the particle is at rest (rest frame). This equation describes the straight line in the curved spacetime and determines the trajectory of a free falling particle. Note that, in the flat spacetime case ($g_{\mu\nu} = \eta_{\mu\nu}$), the Christoffel connections becomes $\Gamma^\mu_{\alpha\beta} = 0$, and this leads to $d^2 x^\mu / d\tau^2 = 0$. The solution of this equation is $x^\mu = a^\mu \tau + b^\mu$ with arbitrary coefficients a^μ and b^μ, that is, this result shows the straight line in flat spacetime.

Next we derive the equation describing the dynamics of the metric $g_{\mu\nu}$. We start the following Einstein–Hilbert action

$$S = \frac{c^4}{16\pi G} \int d^4 x \sqrt{-g} R + S_{\mathrm{m}}, \tag{2.6}$$

where we define $g = \det(g_{\mu\nu})$, and S_{m} is the action of a matter, which relates to energy-momentum tensor $T_{\mu\nu}$ as follows:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{2.7}$$

Taking the functional derivative of the Einstein-Hilbert action S with respect to $g_{\mu\nu}$, we obtain the following equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.8}$$

where we defined the Einstein tensor $G_{\mu\nu} := R_{\mu\nu} - R g_{\mu\nu}/2$. To derive Eq. (2.8), we used the following formulas

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}, \quad \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{2.9}$$

for the metric tensor, and

$$\begin{aligned} \delta R_{\mu\nu} &= \boldsymbol{\nabla}_\lambda \delta\Gamma^\lambda_{\mu\nu} - \boldsymbol{\nabla}_\nu \delta\Gamma^\lambda_{\mu\lambda} \\ &= \frac{1}{2}\big(\boldsymbol{\nabla}^\lambda \boldsymbol{\nabla}_\nu \delta g_{\nu\lambda} + \boldsymbol{\nabla}^\lambda \boldsymbol{\nabla}_\mu \delta g_{\lambda\nu} - \boldsymbol{\nabla}_\nu \boldsymbol{\nabla}_\mu \delta(g^{\lambda\rho} g_{\lambda\rho}) - \boldsymbol{\nabla}^\lambda \boldsymbol{\lambda}_\lambda \delta g_{\mu\nu}\big), \end{aligned} \tag{2.10}$$

$$\delta R = \boldsymbol{\nabla}^\mu \boldsymbol{\nabla}^\nu \delta g_{\mu\nu} - \boldsymbol{\nabla}^\lambda \boldsymbol{\nabla}_\lambda (g^{\mu\nu} \delta g_{\mu\nu}) \tag{2.11}$$

for the Ricci tensor and Ricci scalar, respectively. The left-hand-side of the above equation (2.8) characterizes a geometric quantity of the spacetime and determines the dynamics of the metric tensor $g_{\mu\nu}$, whereas the right-hand-side shows the energy and momentum of the matter. Thus, the Einstein equation (2.8) represents the equivalence between the spacetime curvature and the energy of matter. Note that Eq (2.8) automatically satisfies the conservation of the energy-momentum tensor. Using the properties of the Rieman tensor $R_{\mu\alpha\nu\beta} = -R_{\alpha\mu\nu\beta} = -R_{\mu\alpha\beta\nu}$ and Bianchi identity

$$\boldsymbol{\nabla}_\lambda R^\mu{}_{\alpha\nu\beta} + \boldsymbol{\nabla}_\nu R^\mu{}_{\alpha\beta\lambda} + \boldsymbol{\nabla}_\beta R^\mu{}_{\alpha\lambda\nu} = 0, \tag{2.12}$$

the divergence of the Einstein tensor

$$\boldsymbol{\nabla}^\nu G_{\mu\nu} = 0 \tag{2.13}$$

is always satisfied. This result leads to the conservation of the energy-momentum tensor.

$$\boldsymbol{\nabla}^\nu T_{\mu\nu} = 0. \tag{2.14}$$

Finally, deeply understanding the dynamics of gravity, we consider the linear perturbation of the metric from the Minkowski spacetime as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} \approx \eta_{\mu\nu} - h_{\mu\nu}, \tag{2.15}$$

where, in the equation of the right-hand side equation, we neglected terms of a higher order than $\mathcal{O}(h^2)$. This approximation makes the Einstein tensor $G_{\mu\nu}$ to be

$$G_{\mu\nu} \approx \frac{1}{2}\Big[-\Box h_{\mu\nu} + \partial_\nu \partial^\alpha h_{\mu\alpha} + \partial_\mu \partial^\alpha h_{\nu\alpha} - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\big(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h\big)\Big], \tag{2.16}$$

where $R_{\mu\nu}$ and R are approximated as

$$R_{\mu\nu} \approx \frac{1}{2}\Big(\partial_\alpha \partial_\mu h^\alpha_\nu - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - \Box h_{\mu\nu}\Big), \tag{2.17}$$

$$R \approx \Big(\partial^\mu \partial^\nu h_{\mu\nu} - \Box h\Big) \tag{2.18}$$

with $\Box := \eta^{\rho\sigma}\partial_\rho \partial_\sigma$ and $h := \eta^{\rho\sigma} h_{\rho\sigma}$. Then the Einstein equation (2.8) becomes

$$-\Box h_{\mu\nu} + \partial_\nu \partial^\alpha h_{\mu\alpha} + \partial_\mu \partial^\alpha h_{\nu\alpha} - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\big(\partial^\alpha \partial^\beta h_{\alpha\beta} - \Box h\big) = \frac{16\pi G}{c^4} T^{(1)}_{\mu\nu}, \tag{2.19}$$

where $T^{(1)}_{\mu\nu}$ is of the first order of the energy-momentum tensor in the perturbations for consistency with the weak field approximation. Furthermore, we can simplify the above equation by introducing the new field $\psi_{\mu\nu}$ defined as

$$\psi_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h. \tag{2.20}$$

This field makes it easier the above linearized Einstein equation as

$$-\Box \psi_{\mu\nu} + \partial_\nu \partial^\alpha \psi_{\mu\alpha} + \partial_\mu \partial^\alpha \psi_{\nu\alpha} - \eta_{\mu\nu}\partial^\alpha \partial^\beta \psi_{\alpha\beta} = \frac{16\pi G}{c^4} T^{(1)}_{\mu\nu}. \tag{2.21}$$

By choosing the Lorentz gauge condition $\partial^\mu \psi_{\mu\nu} = 0$, we obtain the following wave equation

$$\Box \psi_{\mu\nu} = -\frac{16\pi G}{c^4} T^{(1)}_{\mu\nu}. \tag{2.22}$$

We can solve the above equation with respect to $\psi_{\mu\nu}$ by Green's function method, and then the solution is

$$\psi_{\mu\nu} = \frac{4G}{c^4} \int d^3y \frac{T^{(1)}_{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{y}|/c, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{2.23}$$

Note that by rewriting $\psi_{\mu\nu}$ in terms of $h_{\mu\nu}$, $h_{\mu\nu}$ becomes

$$\begin{aligned} h_{\mu\nu} &= \psi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\psi \\ &= \frac{4G}{c^4} \int d^3y \frac{\mathcal{T}_{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{y}|/c, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \end{aligned} \tag{2.24}$$

with $\mathcal{T}_{\mu\nu} := T^{(1)}_{\mu\nu} - \eta_{\mu\nu}T^{(1)}/2$ and $T^{(1)} := \eta^{\mu\nu}T^{(1)}_{\mu\nu}$. This equation shows that the energy and momentum of the matter is the source of the gravity. The dynamical degrees of freedom of gravity are gravitational waves, which were first directly observed in 2015 by using the Michelson type interferometer.

2.2 Quantum mechanics: Perspective on the quantum information theory

Quantum mechanics is the theory that describes the laws of physics in the microscopic world. In addition to general relativity, quantum mechanics, which has been experimentally verified with high accuracy, is one of the two pillars of modern physics. In quantum mechanics, light and matter, etc. are interpreted as having both particle and wave properties. For instance, Young's interference experiment and other experiments showed that light classically behaves as an electromagnetic wave, i.e., a wave. However, with the discovery of the photoelectric effect [57] and Compton effect [58], it was also experimentally determined that light has properties as a particle. Furthermore, electrons are known to have properties as a particle owing to the oil drop experiment [59]. In contrast, electron diffraction experiments [60, 61, 62, 63, 64, 65] have revealed that electrons behave as waves, as shown schematically in Fig. 1. These experimental results demonstrate the duality between particles and waves.

In relation to particle-wave duality, the uncertainty principle is another important concept in quantum mechanics. The uncertainty principle implies that there is a common limit to the product of minimum error between some pairs, such as the position and momentum of a particle. This implies that these pairs of variables are not perfectly determined simultaneously. These particle-wave duality and the uncertainty principle were more generally

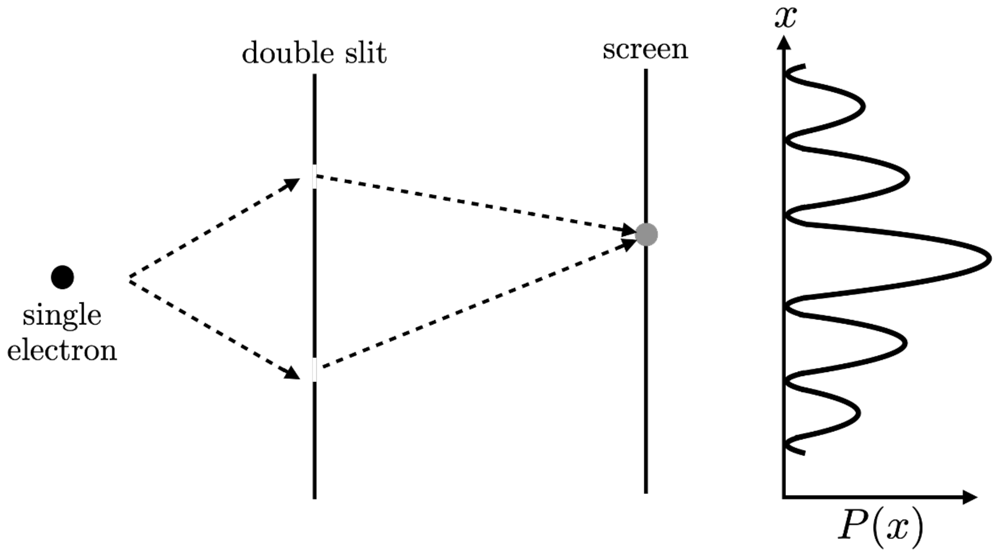


FIG 1: A schematic picture of a double-slit experiment using electrons. The electron beam gun repeatedly sends a single electron to a screen. The electrons passing through both slits accumulate on the screen and produce an interference pattern as if they were interfering with itself obeying a probability $P(x)$.

summarized by the concept of complementarity. In this thesis, we define complementarity as "the property of a particle that changes particle's state when its position is measured." In Chapter 4, we will present a study on the consistency of complementarity in this sense and relativistic causality.

The state of a quantum mechanical system is represented by a state vector $|\psi\rangle$ belonging to the complex Hilbert space $\mathcal{H}$. Hilbert space is a linear space in which the norm of the state vector is defined and complete (the vector is continuous). Because of the linearity in Hilbert space, we can consider the superposition state of several state vectors, as for instance

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2.25}$$

where the coefficients α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$ due to the condition of the norm $\langle\psi|\psi\rangle = 1$. This property is the *quantum superposition state*, and the quantum mechanics allows for the simultaneous existence of different states.

The time evolution of a quantum mechanical system is described by a unitary transformation on Hilbert space. The state vector $|\psi(t)\rangle$ evolves in time according to the Schrödinger equation as follows:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{2.26}$$

where $\hat{H}$ is the operator that describes a quantum mechanical system. By solving this equation, the following result is obtained

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle, \tag{2.27}$$

where $\hat{U}(t) := e^{-i\hat{H}t/\hbar}$ is the unitary operator, which satisfies $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \mathbb{1}$, and it describes the time evolution of this system. As Eq. (2.27), the quantum state represented by a single state vector is pure state. In contrast, a mixed state is a superposition of pure states with a certain probability. It is convenient to use a density matrix to describe the mixed state, including the pure state. The density matrix ρ is defined as

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \tag{2.28}$$

where p_i represents the probability that the system is in pure state $|\psi_i\rangle$ and satisfies $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. The density matrix ρ is regarded as a collection of pure states $|\psi_i\rangle$ with the probability p_i. Note that if all $p_i = 0$ except one value, then the density matrix becomes $\rho = |\psi_i\rangle\langle\psi_i|$, that is, the system is in a pure state. Otherwise, it is in a mixed state, that is, there exists nonzero values of p_i. The pure state generally changes to a mixed state because of interactions with the surrounding environment and other factors. This phenomenon is known as quantum decoherence.

One of the important key concepts in quantum mechanics is quantum entanglement. For simplicity, we consider two quantum systems A and B each possessing two quantum states $|0\rangle$ and $|1\rangle$. This system is often called a 2-qubit system. For instance, if the state of the system is represented as $|\Psi\rangle_\mathrm{s} = |0\rangle_\mathrm{A}|0\rangle_\mathrm{B}$, this state is separable. On the other hand, the entangled state $|\Psi\rangle_\mathrm{e}$ is introduced as follows:

$$|\Psi\rangle_\mathrm{e} = \frac{1}{\sqrt{2}}\Big(|0\rangle_\mathrm{A}|0\rangle_\mathrm{B} + |1\rangle_\mathrm{A}|1\rangle_\mathrm{B}\Big). \tag{2.29}$$

This state shows that if a measurement is performed in system A and reveals that the state is $|0\rangle_\mathrm{A}$ ($|1\rangle_\mathrm{A}$), the state in system B can also be determined to be $|0\rangle_\mathrm{B}$ ($|1\rangle_\mathrm{B}$), independent of the distance. The feature of the entanglement is that it can exhibit similar correlations not only on a particular basis but also based on the quantum superposition states of that basis. We introduce orthogonal basis with $|0\rangle_i$ and $|1\rangle_i$ ($i = \mathrm{A}, \mathrm{B}$) rotated by θ as follows:

$$\begin{cases} |\theta\rangle_i = \cos\theta|0\rangle_i + \sin\theta|1\rangle_i \\ |\theta_\perp\rangle_i = -\sin\theta|0\rangle_i + \cos\theta|1\rangle_i \end{cases} \tag{2.30}$$

with $i = \mathrm{A}, \mathrm{B}$. Then the state $|\Psi\rangle_\mathrm{e}$ becomes

$$|\Psi\rangle_\mathrm{e} = \frac{1}{\sqrt{2}}\Big(|0\rangle_\mathrm{A}|0\rangle_\mathrm{B} + |1\rangle_\mathrm{A}|1\rangle_\mathrm{B}\Big) = \frac{1}{\sqrt{2}}\Big(|\theta\rangle_\mathrm{A}|\theta\rangle_\mathrm{B} + |\theta_\perp\rangle_\mathrm{A}|\theta_\perp\rangle_\mathrm{B}\Big). \tag{2.31}$$

This result shows that the entanglement is independent of the basis of the two subsystems and that this correlation is a quantum correlation.

The properties of an operation LOCC [66, 67] is important for understanding the entanglement. The LOCC protocol is as follows. We consider two local systems, A and B. We assume that these two systems can only exchange classical mechanical information (e.g., telephone) with each other. Then, A and B can perform physical operations such as quantum measurements and unitary transformations, respectively, only on their local systems. This operation is called local operations and classical communication (LOCC), and has the property of not generating entangled states.

In the following, to judge whether it is an entangled state or not quantitatively, we introduce two tools to quantify the entanglement: von Neumann entropy and negativity.

2.2.1 The von Neumann entropy

Here, we introduce the von Neumann entropy or entanglement entropy. We consider a pure state $|\Psi\rangle_\mathrm{AB}$ consisting of quantum states A and B, and the definition of the von Neumann entropy S is defined by

$$S = -\mathrm{Tr}_\mathrm{A}\big[\rho_\mathrm{A}\ln\rho_\mathrm{A}\big] = -\mathrm{Tr}_\mathrm{B}\big[\rho_\mathrm{B}\ln\rho_\mathrm{B}\big], \tag{2.32}$$

where $\rho_\mathrm{A} := \mathrm{Tr}_\mathrm{B}\big[|\Psi\rangle_\mathrm{AB}\langle\Psi|\big]$ ($\rho_\mathrm{B} := \mathrm{Tr}_\mathrm{A}\big[|\Psi\rangle_\mathrm{AB}\langle\Psi|\big]$) is reduced density matrix of the state B (A), respectively. The von Neumann entropy S measures the strength of the quantum correlation between subsystem A and its complement system B.

Note that the von Neumann entropy is valid only for pure states. This is because it may take a non-zero value for a separable state: for a mixed state $|\Psi\rangle_{\mathrm{AB}}\langle\Psi| = \mathbb{1}_{4\times 4}/4 = \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}$ with $\rho_{\mathrm{A}} = \rho_{\mathrm{B}} = \mathbb{1}_{2\times 2}/2$, the von Neumann entropy becomes $S = \ln 2$, whereas, for the entangled state (2.29), the von Neumann entropy is also the same result. In the following, we consider the negativity. This is a measure of the entanglement of the two systems and is also applicable to mixed states.

2.2.2 Negativity

Here, we present the negativity $\mathscr{N}$ [68]. We consider a density matrix ρ of a bipartite system AB. The negativity is defined as follows:

$$\mathscr{N} = \sum_{\lambda_i<0} |\lambda_i|, \tag{2.33}$$

where λ_i are the negative eigenvalues of the partial transposition $\rho^{\mathrm{T_A}}$ with the elements $\langle a|\langle b|\rho^{\mathrm{T_A}}|a'\rangle|b'\rangle = \langle a'|\langle b|\rho|a\rangle|b'\rangle$ in a basis $\{|a\rangle|b\rangle\}_{a,b}$ of the system AB. If the negativity does not disappear, then the system is entangled, which follows by the positive partial transpose criterion [69, 70]. Additionally, nonzero negativity is a necessary and sufficient condition for the entanglement of a two-qubit or a qubit-qutrit system [69]. Particularly, there is only one negative eigenvalue λ_{min} of the partial transposed density matrix of a two-qubit system [71, 72]. We can rewrite the negativity as

$$\mathscr{N} = \max\big[-\lambda_{\mathrm{min}}, 0\big]. \tag{2.34}$$

2.3 Effect of the gravity on quantum mechanics

Here we demonstrate how classical or quantized gravity affects quantum mechanics. In particular, we review the COW experiment, the Schrödinger-Newton equation, and the BMV experimental proposal.

2.3.1 COW experiment

The COW experiment [73] shows that the classical gravitational field originating from the Earth induces the phase shift of a quantum particle. We consider a monochromatic neutron beam with wavelength λ at room temperature injected into a neutron interferometer shown

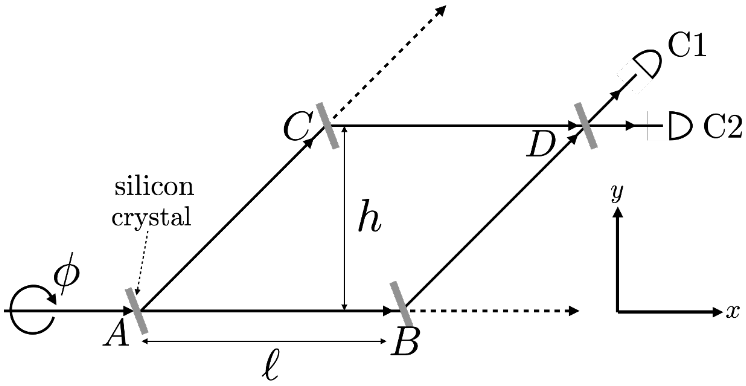


FIG 2: A schematic experimental setup of the COW experiment [73]. The dashed line shows the transmitted wave at the crystal surface.

in Fig. 2 along a horizontal line with momentum $\hbar k_0$. The neutron beam incident on a silicon crystal is split into two wave packets, a transmitted wave and a diffracted wave, by Bragg reflection at the crystal surface at point A and follows partial beam paths ABD and ACD. Then the interferometer is rotated by an angle ϕ around the direction of the incident beam. Finally, the wave packets of neutron recombine and interfere at point D, where the point has higher gravitational potential above the Earth than the entry point A, and the potential is given by $m_{\rm n} gh \sin\phi$ with the neutron mass $m_{\rm n}$ and the gravitational acceleration g. The total energy of the neutron passing through paths ABD and ACD is conserved as,

$$E = \frac{\hbar^2 k_0^2}{2m_{\rm n}} = \frac{\hbar^2 k^2}{2m_{\rm n}} + m_{\rm n} gh \sin\phi. \tag{2.35}$$

Because part of the kinetic energy of the neutron is converted to potential energy, the kinetic energy, that is, the wavenumber of the neutron is reduced on the path ACD . Thus the difference of the path ACD and ABD β becomes [39, 74, 75]

$$\beta := k\ell - k_0\ell \approx -\frac{\lambda m_{\rm n}^2 g}{2\pi\hbar^2} A \sin\phi = -q_{\rm cow} \sin\phi, \tag{2.36}$$

where we defined the area of the parallelogram $A := \ell h$, and we used the approximation $k \approx k_0 = 2\pi/\lambda$ due to the small contribution of the gravity. The setup shown in Fig. 2 corresponds to the usual Mach–Zehnder interferometer, and the intensity difference ΔI is proportional to $\cos\beta$ (for instance, see [76]). The oscillation frequency of ΔI is related to the parameters of $q_{\rm cow}$ and rotation angle ϕ. Fig. 3 shows the difference of the intensity (or counting rate) at counters C1 and C2 as a function of angle ϕ. This result demonstrates

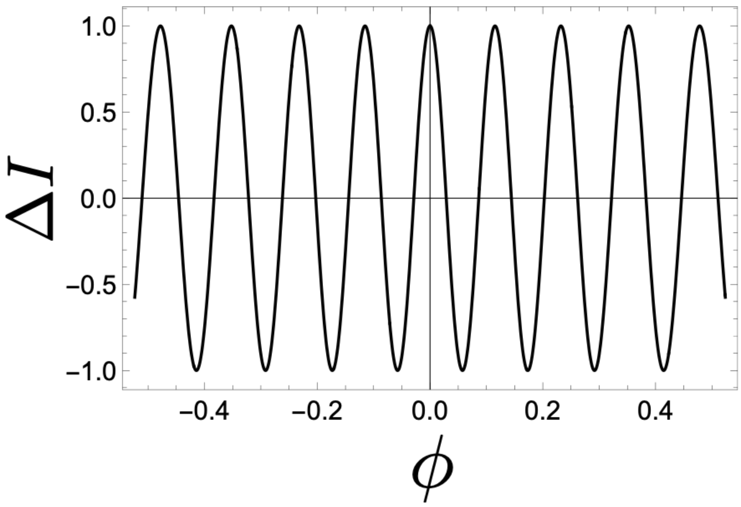


FIG 3: Demonstration of the result of the difference intensity $\Delta I = \cos\beta$ in COW experiment [73] as a function of rotation angle $-\pi/6 \leq \phi \leq \pi/6$. The following parameters were used: $\lambda \simeq 1.4 \times 10^{-10}$ m, $m_{\rm n} \simeq 1.7 \times 10^{-27}$ kg, $g \simeq 9.8$ m/s^2, $\hbar \simeq 1.1 \times 10^{-34}$ J $\cdot$ s, $A = \ell h = 2d(d + a\cos\theta)\tan\theta \simeq 1.0 \times 10^{-3}$ m^2 with $a = 2.0 \times 10^{-3}$ m, $d = 3.5 \times 10^{-2}$ m, $\theta = 22.1\pi/180$ rad.

that the classical gravitational potential changes the phase of the neutron wave function.

2.3.2 Schrödinger-Newton equation

One of the simplest models of classical gravity coupled to quantum matter is known as semi-classical gravity [77, 78]. We consider a single quantum particle coupled with a classical gravity. This system is described by the following semi-classical Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle, \tag{2.37}$$

where the right-hand-side of the above equations is modified by an expectation value of operator valued energy-momentum tensor $\hat{T}_{\mu\nu}$ compared with Eq. (2.8). This equation shows that the quantum particle, which is taken an expectation value, creates the gravity. In the non-relativistic limit, the state $|\psi\rangle$ is evolved under the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{x}) = \Big(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + m\phi(t,\boldsymbol{x})\Big)\psi(t,\boldsymbol{x}), \tag{2.38}$$

where $\phi(t, \boldsymbol{x})$ is the Newtonian potential, which is determined by the Poison equation

$$\boldsymbol{\nabla}^2 \phi(t, \boldsymbol{x}) = \frac{4\pi G m}{c^2} \langle \psi | \hat{T}_{00} | \psi \rangle = 4\pi G m |\psi(t, \boldsymbol{x})|^2 \tag{2.39}$$

with $\hat{T}_{00} = mc^2 |\boldsymbol{x}\rangle\langle \boldsymbol{x}|$ and $\langle \boldsymbol{x} | \psi \rangle = \psi(t, \boldsymbol{x})$. By solving the above equation, the Newtonian potential $\phi(t, \boldsymbol{x})$ becomes

$$\phi(t, \boldsymbol{x}) = -Gm \int d^3 y \frac{|\psi(t, \boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{2.40}$$

Thus, the above Schrödinger equation (2.38) is rewritten as the following Schrödinger-Newton equation [79]

$$i\hbar \frac{\partial}{\partial t} \psi(t, \boldsymbol{x}) = \left[-\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 - Gm^2 \int d^3 y \frac{|\psi(t, \boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|} \right] \psi(t, \boldsymbol{x}). \tag{2.41}$$

In this formalism, the gravitational potential is treated classically and the influence of gravity on quantum particles is explained by assuming that the mass probability density of the particle appears as a source. The Schrödinger-Newton equation was studied to discuss the collapse of the wave function due to measurement [30, 31, 32, 33]. Unlike ordinary linear quantum mechanics, the feature of the Schrödinger-Newton equation is its nonlinearity with respect to the wave function $\psi(t, \boldsymbol{x})$, which breaks the unitarity and causes the decoherence via self gravitational field. This modifications of linear quantum mechanics and classical gravity invite criticism [80]. However, the validity of this approach has not yet been theoretically or experimentally ruled out. Recently, it has been discussed in relationship with the suggestion of whether gravity might need quantization or not [81].

Both of the above two examples are discussed in terms of classical (not operator valued) gravitational fields. Thus, if we prepare multiple quantum particles, the entanglement between them would not be created by the LOCC. We then present an example in which two quantum particles interact with a gravitational field as a position operator and generate entanglement.

2.3.3 BMV experimental proposal

Here we briefly review the BMV experimental proposal [37, 38]. This proposal involves the experimental setting of two matter-wave interferometers that test the quantum aspects of

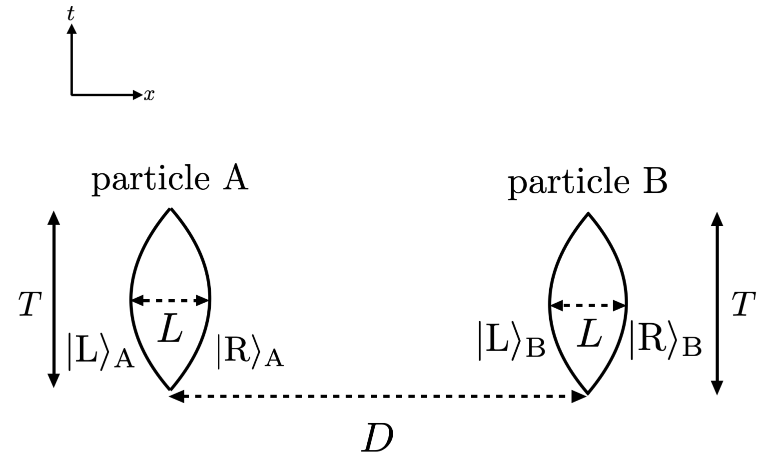


FIG 4: Configuration of trajectories of two charged particles. The length scale of each superposition is L, the coordinate time during which each particle is superposed is T, and the particles are initially separated by the distance D.

gravity via entanglement generation. We consider two massive particles A and B each in a superposition of two trajectories (see Fig. 4). These particles are coupled with each other using the Newtonian potential. The total Hamiltonian is

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{V}_{\mathrm{AB}}, \quad \hat{V}_{\mathrm{AB}} = -\frac{Gm_{\mathrm{A}}m_{\mathrm{B}}}{|\hat{\mathbf{x}}_{\mathrm{A}} - \hat{\mathbf{x}}_{\mathrm{B}}|}, \tag{2.42}$$

where $\hat{H}_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}}$ are the Hamiltonians of the charged particles A and B, $\hat{V}_{\mathrm{AB}}$ is the interaction Hamiltonian between them with the masses m_{A} and m_{B}, and $\hat{\mathbf{x}}_{\mathrm{A}}$ and $\hat{\mathbf{x}}_{\mathrm{B}}$ denote each position operator of the two massive particles. Note that, in the following, if we replace the coupling constant $Gm_{\mathrm{A}}m_{\mathrm{B}}$ with $e_{\mathrm{A}}e_{\mathrm{B}}$ where e_{A} and e_{B} are the electric charges of the two charged particles A and B, the Coulomb potential version of the BMV proposal can be discussed. We stress that the Newtonian potential $\hat{V}_{\mathrm{AB}}$ is an operator of the position operators $\hat{\mathbf{x}}_{\mathrm{A}}$ and $\hat{\mathbf{x}}_{\mathrm{B}}$. In the following computation, we do not need the explicit forms of $\hat{H}_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}}$. As we will mention after Eq. (2.45), they are implicitly given by specifying the trajectories of each particle. Each of the two charged particles at $t = 0$ is in the spatially superposed state

$$|\Psi(0)\rangle = \frac{1}{2}|\mathrm{C}\rangle_{\mathrm{A}}(|\uparrow\rangle_{\mathrm{A}} + |\downarrow\rangle_{\mathrm{A}})|\mathrm{C}\rangle_{\mathrm{B}}(|\uparrow\rangle_{\mathrm{B}} + |\downarrow\rangle_{\mathrm{B}}), \tag{2.43}$$

where $|\uparrow\rangle_j$ ($|\downarrow\rangle_j$) are the spin degrees of freedom of the massive particle j with $j = \mathrm{A}, \mathrm{B}$, and $|\mathrm{C}\rangle_{\mathrm{A}}$ and $|\mathrm{C}\rangle_{\mathrm{B}}$ denote the localized particle wave function of A and B, respectively. For $t < 0$, the massive particles A and B are localized around each trajectory, whose states are described by $|\mathrm{C}\rangle_{\mathrm{A}}$ and $|\mathrm{C}\rangle_{\mathrm{B}}$, respectively. Then the Newtonian potential for $t<0$ is not in a quantum superposition and behaves classically. In this case the states of A and B are uncorrelated with the Newtonian potential. Now, we assume that each particle is manipulated through an inhomogeneous magnetic field ($|\mathrm{C}\rangle_j\,|\uparrow\rangle_j \to |\psi_{\mathrm{L}}\rangle_j\,|\uparrow\rangle_j\,, |\mathrm{C}\rangle_j\,|\downarrow\rangle_j \to |\psi_{\mathrm{R}}\rangle_j\,|\downarrow\rangle_j$) to create spatially superposed states with $|\psi_{\mathrm{L}}\rangle_j\,|\uparrow\rangle_j$, and $|\psi_{\mathrm{R}}\rangle_j\,|\downarrow\rangle_j$, which is understood as the Stern-Gerlach effect. In the following, $|\mathrm{C}\rangle_j\,|\uparrow\rangle_j$ and $|\mathrm{C}\rangle_j\,|\downarrow\rangle_j$ are represented by $|\mathrm{L}\rangle_j$ and $|\mathrm{R}\rangle_j$ with $j=\mathrm{A},\mathrm{B}$ for simplicity. The initial state is rewritten as

$$|\Psi(0)\rangle=\frac{1}{2}(|\mathrm{L}\rangle_{\mathrm{A}}+|\mathrm{R}\rangle_{\mathrm{A}})(|\mathrm{L}\rangle_{\mathrm{B}}+|\mathrm{R}\rangle_{\mathrm{B}}). \tag{2.44}$$

We note that $|\mathrm{R}\rangle_{\mathrm{A}}$ ($|\mathrm{R}\rangle_{\mathrm{B}}$) and $|\mathrm{L}\rangle_{\mathrm{A}}$ ($|\mathrm{L}\rangle_{\mathrm{B}}$) are the states of wave packets localized around classical trajectories $\boldsymbol{x}=\boldsymbol{X}_{\mathrm{AR}}(t=0)$ ($\boldsymbol{x}=\boldsymbol{X}_{\mathrm{BR}}(t=0)$) and $\boldsymbol{x}=\boldsymbol{X}_{\mathrm{AL}}(t=0)$ ($\boldsymbol{x}=\boldsymbol{X}_{\mathrm{BL}}(t=0)$), respectively. After each particle has passed through an inhomogeneous magnetic field, the states $|\mathrm{L}\rangle_j$ and $|\mathrm{R}\rangle_j$ are regarded as the localized states of the particle $j=\mathrm{A},\mathrm{B}$ around the left trajectory and the right trajectory shown in Fig.4, respectively. We assume the following approximation,

$$\hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{A}}(t)|\mathrm{P}\rangle_{\mathrm{A}}\approx\mathbf{X}_{\mathrm{AP}}(t)|\mathrm{P}\rangle_{\mathrm{A}},\quad \hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{B}}(t)|\mathrm{Q}\rangle_{\mathrm{B}}\approx\mathbf{X}_{\mathrm{BQ}}(t)|\mathrm{Q}\rangle_{\mathrm{B}}, \tag{2.45}$$

where $\hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{A}}(t)=e^{it(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}\hat{\mathbf{x}}_{\mathrm{A}}e^{-it(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}$ and $\hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{B}}(t)=e^{it(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}\hat{\mathbf{x}}_{\mathrm{B}}e^{-it(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}$ are the position operators in the interaction picture. These assumptions are valid [82] when the de Broglie wavelength λ_{dB} of the massive particle is much smaller than the width Δx of its wave packet ($\lambda_{\mathrm{dB}}\ll\Delta x$). The trajectories of the particles, $\boldsymbol{X}_{\mathrm{AP}}(t)$ and $\boldsymbol{X}_{\mathrm{BQ}}(t)$, are determined by the Hamiltonians $\hat{H}_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}}$. In our computation, we specify the trajectories by hand.

The evolved state $|\Psi(T)\rangle$ is

$$\begin{aligned}|\Psi(T)\rangle&=e^{-i\hat{H}T}|\Psi(0)\rangle\\&=e^{-iT(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}\mathrm{T}\exp\left[i\int_0^T dt\frac{Gm_{\mathrm{A}}m_{\mathrm{B}}}{|\hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{A}}(t)-\hat{\mathbf{x}}^{\mathrm{I}}_{\mathrm{B}}(t)|}\right]|\Psi(0)\rangle\\&\approx\frac{1}{2}e^{-iT(\hat{H}_{\mathrm{A}}+\hat{H}_{\mathrm{B}})}\sum_{\mathrm{P,Q=R,\ L}}e^{i\Phi_{\mathrm{PQ}}}|\mathrm{P}\rangle_{\mathrm{A}}\otimes|\mathrm{Q}\rangle_{\mathrm{B}},\end{aligned} \tag{2.46}$$

where T is the time-ordered product, and the approximation (2.45) was used in the third line. The phase shift

$$\Phi_{\rm PQ} = \int_0^T dt \frac{G m_{\rm A} m_{\rm B}}{|\mathbf{X}_{\rm AP}(t) - \mathbf{X}_{\rm BQ}(t)|} \tag{2.47}$$

is induced by the Newtonian potential between particles A and B. The density matrix of those particles is

$$\rho_{\rm c} = |\Psi(T)\rangle\langle\Psi(T)| = \frac{1}{4} \sum_{\rm P,Q=R,L} \sum_{\rm P',Q'=R,L} e^{i\Phi_{\rm PQ} - i\Phi_{\rm P'Q'}} \, |{\rm P_f}\rangle_{\rm A}\langle {\rm P'_f}| \otimes |{\rm Q_f}\rangle_{\rm B}\langle {\rm Q'_f}|, \tag{2.48}$$

where $|{\rm P_f}\rangle_{\rm A} = e^{-i\hat{H}_{\rm A}T}|{\rm P}\rangle_{\rm A}$ and $|{\rm Q_f}\rangle_{\rm B} = e^{-i\hat{H}_{\rm B}T}|{\rm Q}\rangle_{\rm B}$ are the states of the massive particles A and B moving along trajectories P and Q, respectively.

We then adopt the negativity $\mathscr{N}$ [68] to determine whether the state of two massive particles is entangled or not. The minimum eigenvalue of the partial transpose of the density matrix (2.48) is

$$\lambda_{\rm min} = -\frac{1}{2}\left| \sin\left[\frac{\Phi_{\rm c}}{2}\right]\right|, \tag{2.49}$$

where $\Phi_{\rm c}$ is given as

$$\begin{aligned}\Phi_{\rm c} = G m_{\rm A} m_{\rm B} \int_0^T dt \Big(& \frac{1}{|\mathbf{X}_{\rm AR}(t) - \mathbf{X}_{\rm BR}(t)|} - \frac{1}{|\mathbf{X}_{\rm AR}(t) - \mathbf{X}_{\rm BL}(t)|} \\ & - \frac{1}{|\mathbf{X}_{\rm AL}(t) - \mathbf{X}_{\rm BR}(t)|} + \frac{1}{|\mathbf{X}_{\rm AL}(t) - \mathbf{X}_{\rm BL}(t)|} \Big).\end{aligned} \tag{2.50}$$

To evaluate $\Phi_{\rm c}$ and the negativity (2.34), we consider the trajectories

$$\boldsymbol{X}_{\rm AP}(t) = \Big[\epsilon_{\rm P} X(t), 0, 0\Big]^{\rm T}, \quad \boldsymbol{X}_{\rm BQ} = \Big[\epsilon_{\rm Q} X(t) + D, 0, 0\Big]^{\rm T}, \quad X(t) = 8L\Big(1 - \frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2 \tag{2.51}$$

where $\epsilon_{\rm R} = -\epsilon_{\rm L} = 1$, L is the length scale of each superposition, T is the time scale during which each particle is superposed and D is the initial distance between those particles (see Fig. 4). The function $X(t)$ is chosen so that each particle has no velocity at $t = 0$ and $t = T$ to avoid the UV divergence in our computation in the following sections. This point is discussed in more detail around Eq.(3.27). There can be other possible choice for superposition and trajectories. For example, the authors in Ref. [83] considered two particles in superposition states of multiple trajectories, and discussed the entanglement generation due to the Newtonian potential. The result indicated that multiple trajectories cases are more resilient to decoherence than the two trajectories case. In the present paper, for simplicity, we consider the entanglement generation between two charged particles.

When the trajectories of the particles are specified by Eq. (2.51), the quantity $\Phi_{\rm c}$ is given by

$$\Phi_{\rm c} = G m_{\rm A} m_{\rm B} \int_0^T dt \Big[\frac{2}{D} - \Big(\frac{1}{D - 2X(t)} + \frac{1}{D + 2X(t)} \Big) \Big]. \tag{2.52}$$

Now, we recover the light velocity c and the reduced Planck constant $\hbar$. We focus on the two regimes $cT \gg D \sim L$ and $cT \gg D \gg L$, in which the charged particles move with non-relativistic velocities ($cT \gg L$). In the regime $cT \gg D \sim L$, the above formula of $\Phi_{\rm c}$ and the minimum eigenvalue (2.49) are computed numerically. In the regime $cT \gg D \gg L$, the quantity $\Phi_{\rm c}$ (2.52) and the minimum eigenvalue (2.49) are approximated as

$$\Phi_{\rm c} \approx -\frac{256 G m_{\rm A} m_{\rm B}}{315 \pi \hbar c} \Big(\frac{L}{cT} \Big)^2 \Big(\frac{cT}{D} \Big)^3, \quad \lambda_{\rm min} \approx -\frac{64 G m_{\rm A} m_{\rm B}}{315 \pi \hbar c} \Big(\frac{L}{cT} \Big)^2 \Big(\frac{cT}{D} \Big)^3, \tag{2.53}$$

where $\mathcal{O}(L^3/D^3)$ was ignored, and the Taylor expansion $\sin \Phi_{\rm c}/2 \approx \Phi_{\rm c}/2$ was used.

Fig. 5 (a) and (b) show the negativity in the regime $cT \gg D \sim L$ and $cT \gg D \gg L$. These results show that the negativity decreases as the ratio D/cT increases. Because the negativity is always positive, the two massive particles A and B interacting with the Newtonian potential are entangled in the regimes $cT \gg D \sim L$ and $cT \gg D \gg L$.

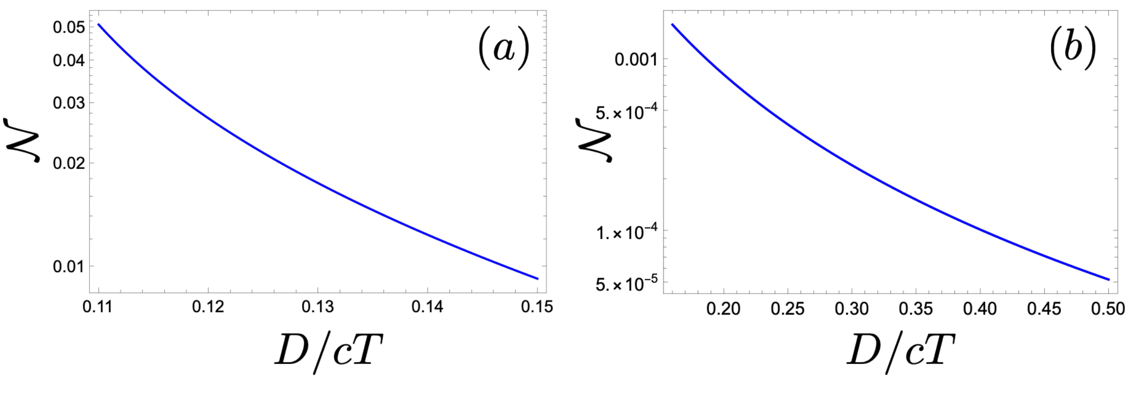


FIG 5: Negativity $\mathscr{N}$ induced by the Newtonian potential between the massive particles. We adopted $L/cT = 0.1$ and $G m_{\rm A} m_{\rm B}/\hbar c = 0.1$.

The entanglement generation here is understood to be caused by the Newtonian potential (2.42) treated as an operator of the positions of two massive particles A and B, which allows the quantum superposition of Newtonian potentials associated with the superposition of the massive particles. In the context of quantum information theory, LOCC explained in Sec. 2.2 cannot create the entanglement between two systems. Translating to the BMV setting, this means that it is impossible to create entanglement through classical interaction. It immediately follows that if the Newtonian interaction entangles two massive particles, then the interaction is quantum and is not described by LOCC.

The main theme of this thesis is how the quantum field of gravity relates to its low-energy regime. In the next subsection, we introduce the quantum field theory and consider the treatment of field theoretic approaches before proceeding to the main text of this thesis.

2.4 Quantum field theory

Here, we briefly introduce the quantization of quantum electrodynamics as an example of quantum field theory. In particular, we focus on the quantization of electrodynamics in Minkowski space-time using the Becchi-Rouet-Stora-Tyutin (BRST) formalism to preserve the covariance of the theory with a view toward its extension to quantum gravity theory.

2.4.1 BRST formalism

The Lagrangian density of the QED in BRST formalism is written as follows:

$$\mathscr{L} = \mathscr{L}_{\text{QED}} + \mathscr{L}_{\text{GF+FP}}, \quad \mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{2.54}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the $U(1)$ gauge field A_μ, ψ is the Dirac field with mass m, $\bar{\psi} = \psi^\dagger\gamma^0$, γ^μ is the gamma matrix satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, which includes the electromagnetic interaction term with the coupling constant e, and $\mathscr{L}_{\text{GF+FP}}$ is the gauge fixing and Faddeev-Popov ghost term. The Lagrangian density $\mathscr{L}_{\text{QED}}$ is invariant under the following transformation

$$\psi \to e^{-ie\theta(x)}\psi \simeq (1 - ie\theta(x))\psi \equiv \psi + \delta\psi, \quad A_\mu \to A_\mu + \partial_\mu\theta(x) \equiv A_\mu + \delta A_\mu, \tag{2.55}$$

where $\theta(x)$ is a real function. To give the gauge fixing and Faddeev-Popov ghost term $\mathscr{L}_{\text{GF+FP}}$, we define $\theta(x) \equiv \lambda C(x)$, where λ and $C(x)$ are the global and local Grassmann numbers. Field $C(x)$ is a scalar field, but it satisfies the anti-commutation relations $\{C(x), C(y)\} = 0$, which is the Faddeev-Popov ghost field. We rewrite $\delta\psi$ and δA_μ as follows

$$\delta\psi(x) = \lambda(-ieC(x)\psi(x)) \equiv \lambda\delta_{\rm B}\psi(x), \quad \delta A_\mu = \lambda(\partial_\mu C(x)) \equiv \lambda\delta_{\rm B}A_\mu, \quad \delta_{\rm B}C(x) = 0, \tag{2.56}$$

where the operator $\delta_{\rm B}$ is defined so that the nilpotency $\delta_{\rm B}^2 = 0$ satisfies. We also introduce the anti-ghost field $\bar{C}(x)$ and the Nakanishi-Lautrup field $B(x)$. They satisfy

$$\delta_{\rm B}\bar{C}(x) = iB(x), \quad \delta_{\rm B}B(x) = 0, \tag{2.57}$$

where α is an arbitrary parameter. The transformation of Eqs. (2.56) and (2.57) is referred to as the BRST transformation. We can choose the gauge fixing and Faddeev-Popov ghost terms as follows:

$$\mathscr{L}_{\rm GF+FP} = -i\delta_{\rm B}(\bar{C}F), \quad F = \partial^\mu A_\mu + \frac{1}{2}\alpha {\rm B}. \tag{2.58}$$

Consequently, the full Lagrangian density in BRST formalism is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu\psi - m)\psi + \frac{1}{2}\alpha B^2 - \partial^\mu B A_\mu - i\partial^\mu\bar{C}\partial_\mu C. \tag{2.59}$$

The equations of motion for fields $A_\mu, B, C, \bar{C}$ are given by the Euler-Lagrange equations,

$$0 = \partial^\nu F_{\nu\mu} - J_\mu - \partial_\mu B, \tag{2.60}$$

$$0 = \partial^\mu A_\mu + \alpha B, \tag{2.61}$$

$$0 = \Box C = \Box\bar{C}, \tag{2.62}$$

where $J_\mu = e\bar{\psi}\gamma_\mu\psi$. The fields $C(x)$ and $\bar{C}(x)$ follow the free evolution and do not interact with the other fields. Substituting (2.61) into (2.59), we arrive at the following Lagrangian density,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu\psi - m)\psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - i\partial^\mu\bar{C}\partial_\mu C, \tag{2.63}$$

and the BRST transformations are summarized as

$$\delta_B A_\mu = \partial_\mu C, \quad \delta_B \psi = -ieC\psi, \delta_B C = 0, \delta_B \bar{C} = \frac{i}{\alpha}(\partial_\mu A^\mu). \tag{2.64}$$

Because of the BRST transformation, the Lagrangian density has a global symmetry (BRST symmetry)

$$\lambda \delta_B \mathscr{L} = 0. \tag{2.65}$$

Associated with this global symmetry, there is a conserved current referred to as the BRST current $J^\mu_{\rm B}$ defined by

$$J^\mu_{\rm B} = \sum_{\rm I} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi_{\rm I})} \delta_{\rm B} \Phi_{\rm I} = -F^{\mu\nu}\partial_\nu C - \frac{1}{\alpha}\partial_\nu A^\nu \partial^\mu C + J^\mu C, \tag{2.66}$$

where $\Phi_{\rm I} = \{A_\mu, \psi, C, \bar{C}\}$. The BRST charge $Q_{\rm B}$ is given by

$$Q_{\rm B} \equiv \int d^3x J^0_{\rm B}(x) = \int d^3x \Big[(\partial_i C)F^{i0} + J^0 C - \frac{1}{\alpha}(\partial_\mu A^\mu)\dot{C}\Big]. \tag{2.67}$$

We perform the canonical quantization procedure in the Feynman gauge ($\alpha = 1$). The canonical conjugate momenta are defined as

$$\pi^\mu_A \equiv \frac{\partial \mathscr{L}}{\partial \dot{A}_\mu} = -F^{0\mu} - (\partial_\nu A^\nu)\eta^{0\mu} \ , \pi_\psi \equiv \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 \ , \pi_c \equiv \frac{\partial \mathscr{L}}{\partial \dot{C}} = i\dot{\bar{C}} \ , \pi_{\bar{c}} \equiv \frac{\partial \mathscr{L}}{\partial \dot{\bar{C}}} = i\dot{C}, \tag{2.68}$$

where " $\cdot$ " denotes the derivative with respect to time $x^0 = t$. The commutation relations are assigned as follows

$$\begin{aligned} \{\hat{\psi}(x), \hat{\pi}_\psi(y)\}|_{x^0=y^0} &= i\delta^3(\mathbf{x}-\mathbf{y}), \\ \{\hat{C}(x), \hat{\pi}_c(y)\}|_{x^0=y^0} &= i\delta^3(\mathbf{x}-\mathbf{y}), \\ \{\hat{\bar{C}}(x), \hat{\pi}_{\bar{c}}(y)\}|_{x^0=y^0} &= i\delta^3(\mathbf{x}-\mathbf{y}), \\ [\hat{A}_\mu(x), \hat{\pi}^\nu_A(y)]|_{x^0=y^0} &= i\delta^\nu_\mu \delta^3(\mathbf{x}-\mathbf{y}). \end{aligned}$$

The quantized BRST charge is given by

$$\hat{Q}_{\rm B} = \int d^3x[(\partial_i\hat{C})\hat{F}^{i0} + \hat{J}^0\hat{C} - (\partial_\mu\hat{A}^\mu)\dot{\hat{C}}] = \int d^3x[-(\partial_i\hat{\pi}^i)\hat{C} + \hat{J}^0\hat{C} + i\hat{\pi}^0\hat{\pi}_{\bar{c}}]. \tag{2.69}$$

It is well known that when we quantize a gauge theory while maintaining the Lorentz covariance, a state space $\mathcal{V}$ with an indefinite metric is required. For the standard probabilistic interpretation of quantum mechanics , a physical state $|\Psi_{\rm phys}\rangle$ has no negative norm. This state with a non-negative norm is identified by imposing the following condition (the BRST condition)

$$\hat{Q}_{\rm B}|\Psi_{\rm phys}\rangle = 0, \tag{2.70}$$

where the physical state $|\Psi_{\rm phys}\rangle$ satisfies $\langle\Psi_{\rm phys}|\Psi_{\rm phys}\rangle \geq 0$.

2.4.2 BRST charge in the interaction picture and in the Schrödinger picture

We derive a useful form of the BRST charge for our computation. Using Eq. (2.69), we obtain the BRST charge in the interaction picture,

$$\hat{Q}^{\rm I}_{\rm B}(t) = e^{i\hat{H}_0t}\hat{Q}_{\rm B}e^{-i\hat{H}_0t} = \int d^3x[-(\partial_i\hat{\pi}^{i\rm I})\hat{C} + \hat{J}^0_{\rm I}\hat{C}^{\rm I} + i\hat{\pi}^{0\rm I}\hat{\pi}^{\rm I}_{\bar{c}}], \tag{2.71}$$

where $\hat{\phi}^{\rm I} = e^{i\hat{H}_0t}\hat{\phi}e^{-i\hat{H}_0t}, \hat{\phi} = \{\hat{A}_\mu, \hat{\pi}^\mu, \hat{C}, \hat{\bar{C}}, \hat{\pi}_c, \hat{\pi}_{\bar{c}}, \hat{J}^0\}$, and they satisfy the Heisenberg equation

$$i\dot{\hat{\phi}}^{\rm I} = [\hat{\phi}^{\rm I}, \hat{H}_0]. \tag{2.72}$$

The gauge field $\hat{A}^{\rm I}_\mu(x)$ and the ghost field $\hat{C}^{\rm I}(x)$ satisfy the Klein-Gordon equation. The solutions are

$$\hat{A}^{\rm I}_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}}(\hat{a}_\mu(\boldsymbol{k})e^{ik\cdot x} + \text{h.c.}), \tag{2.73}$$

$$\hat{C}^{\rm I}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k^0}}(\hat{c}(\boldsymbol{k})e^{ik\cdot x} + \text{h.c.}), \tag{2.74}$$

where $k^0 = |\boldsymbol{k}|$, $\hat{a}_\mu(\boldsymbol{k})$ and $\hat{c}(\boldsymbol{k})$ are the annihilation operators of the gauge field $\hat{A}^{\rm I}_\mu(x)$, and the ghost field $\hat{C}^{\rm I}(x)$, respectively. The annihilation operators $\hat{a}_\mu(\boldsymbol{k})$, $\hat{c}(\boldsymbol{k})$, and the creation operators satisfy

$$\left[\hat{a}_\mu(\boldsymbol{k}), \hat{a}^\dagger_\nu(\boldsymbol{k}')\right] = \eta_{\mu\nu}\delta(\boldsymbol{k}-\boldsymbol{k}'), \quad \left\{\hat{c}(\boldsymbol{k}), \hat{c}^\dagger(\boldsymbol{k}')\right\} = \delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{2.75}$$

Substituting (2.73) and (2.74) into Eq. (2.69), we obtain the BRST charge in the interaction picture

$$\hat{Q}^{\rm I}_{\rm B}(t) = \int \frac{d^3k}{\sqrt{(2\pi)^3}}\left[\left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\hat{\tilde{J}}^0_{\rm I}(t,\boldsymbol{k})}{\sqrt{2k^0}}e^{ik^0t}\right)c^\dagger(\boldsymbol{k}) + \text{h.c.}\right], \tag{2.76}$$

where $\hat{\tilde{J}}^0_{\rm I}(t,\boldsymbol{k})$ is the Fourier transformation of $\hat{J}^0_{\rm I}(t,\boldsymbol{x})$

$$\hat{J}^0_{\rm I}(t,\boldsymbol{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3}}\hat{\tilde{J}}^0_{\rm I}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2.77}$$

Using the BRST charge in the interaction picture and ((2.69), the BRST charge in the Schrödinger picture is obtained as

$$\hat{Q}_{\rm B} = e^{-i\hat{H}_0t}\hat{Q}^{\rm I}_{\rm B}(t)e^{i\hat{H}_0t} = \int \frac{d^3k}{\sqrt{(2\pi)^3}}\left[\left(k^\mu \hat{a}_\mu(\boldsymbol{k} + \frac{\hat{\tilde{J}}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)c^\dagger(\boldsymbol{k}) + \text{h.c.}\right], \tag{2.78}$$

where we used

$$e^{-i\hat{H}_0t}\hat{a}_\mu(\boldsymbol{k})e^{i\hat{H}_0t} = \hat{a}_\mu(\boldsymbol{k})e^{ik^0t}, \quad e^{-i\hat{H}_0t}\hat{c}^\dagger(\boldsymbol{k})e^{i\hat{H}_0t} = \hat{c}^\dagger(\boldsymbol{k})e^{-ik^0t}, \quad e^{-i\hat{H}_0t}\hat{\tilde{J}}^0_{\rm I}(t,\boldsymbol{k})e^{i\hat{H}_0t} = \hat{\tilde{J}}^0(\boldsymbol{k}), \tag{2.79}$$

Here, $\hat{\tilde{J}}^0$ is the Fourier transform of the matter current in the Schrödinger picture.

2.4.3 BRST condition for our models with charged particles

We use the explicit form of the BRST charge in the Schrödinger picture (2.78) to derive the BRST condition for our models. Assuming a physical state $|\Psi_{\rm phys}\rangle = |\Psi'_{\rm phys}\rangle \otimes |0\rangle_c$, where $|0\rangle_c$ is the ground state of the ghost field, and using (2.78), we can reduce the BRST condition (2.70) as

$$\left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\hat{\tilde{J}}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)|\Psi'_{\rm phys}\rangle = 0. \tag{2.80}$$

When $|\Psi'_{\rm phys}\rangle$ is the initial state given in (3.2), (2.80) gives the equation,

$$\begin{aligned} 0 &= \left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\hat{\tilde{J}}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)|\Psi'_{\rm phys}\rangle \\ &= \left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\hat{\tilde{J}}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)\frac{1}{\sqrt{2}}\big(|{\rm R}\rangle + |{\rm L}\rangle\big) \otimes |\alpha\rangle_{\rm EM} \\ &\approx \frac{1}{\sqrt{2}}\big(|{\rm R}\rangle + |{\rm L}\rangle\big) \otimes \left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\tilde{J}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)|\alpha\rangle_{\rm EM}, \end{aligned} \tag{2.81}$$

where the approximation (3.4) was used in the second line, and note that $\tilde{J}^0_{\rm R}(\boldsymbol{k}) = \tilde{J}^0_{\rm L}(\boldsymbol{k}) = \tilde{J}^0(\boldsymbol{k})$ at the initial time. Hence the initial coherent state of the electromagnetic field must satisfy

$$\left(k^\mu \hat{a}_\mu(\boldsymbol{k}) + \frac{\tilde{J}^0(\boldsymbol{k})}{\sqrt{2k^0}}\right)|\alpha\rangle_{\rm EM} = 0. \tag{2.82}$$

Because the displacement operator $\hat{D}(\alpha)$ given in (3.3) has the following relation

$$\hat{D}^\dagger(\alpha)\hat{a}_\mu(\boldsymbol{k})\hat{D}(\alpha) = \hat{a}_\mu(\boldsymbol{k}) + \alpha_\mu(\boldsymbol{k}), \tag{2.83}$$

we obtain the constraint for the complex function $\alpha^\mu(\boldsymbol{k})$ as

$$k^\mu \alpha_\mu(\boldsymbol{k}) = -\frac{\tilde{J}^0(\boldsymbol{k})}{\sqrt{2k^0}}. \tag{2.84}$$

This is the BRST condition for the model of a single charged particle. The BRST condition for the model of two charged particles is obtained using the same procedure.

3 Effect of photon field on entanglement generation

In this section, based on QED, we evaluate the entanglement generation between two charged particles [4]. We first introduce the model of a single charged particle interacting with an electromagnetic field, and then extend it to the model of two charged particles, which corresponds to the BMV proposal setting, explained in Sec. 2.3.3. The results in the next two subsections are based on the first principle analysis of the QED. Our analysis may be useful to understand how the entanglement generation based on the operator valued Newtonian potential Eq. (2.42) is related to the quantum field theory of gravitational field. We will see that the contribution from the Coulomb potential is reproduced in the behavior of the entanglement and is consistent with the result of the non-relativistic limit. This implies that the operator valued Coulomb potential is originated from the quantum field theory of the electromagnetic field. As we will see below, this entanglement generation is driven by the fact that an electromagnetic field can be in a superposition state associated with the superposition states of currents of the charged particles, which shows the quantum nature of the electromagnetic field.

3.1 Dynamics of charged particles coupled with an electromagnetic field

We consider the dynamics of charged particles coupled with an electromagnetic field, where the charged particles are each in a superposition of trajectories. After a brief review of the model of a single charged particle, we extend it to the model of two charged particles. For the covariant quantization of the electromagnetic field, we use the BRST formalism [84] in the Feynman gauge. The details of the BRST formalism are presented in Sec. 2.4.

3.1.1 Model of a single charged particle

We consider a single charged particle and an electromagnetic field coupled to it. The total Hamiltonian in the Schrödinger picture is

$$\hat{H} = \hat{H}_{\rm p} + \hat{H}_{\rm EM} + \hat{V}, \quad \hat{V} = \int d^3x \hat{J}_\mu(\mathbf{x})\hat{A}^\mu(\mathbf{x}), \tag{3.1}$$

where $\hat{H}_{\rm p}$ is the Hamiltonian of the charged particle, $\hat{H}_{\rm EM}$ is the free Hamiltonian of the electromagnetic field, and $\hat{V}$ is their interaction Hamiltonian. $\hat{J}_\mu$ is the current operator of the charged particle, and $\hat{A}^\mu$ is the electromagnetic field operator (the U(1) gauge field).

We assume that the charged particle is superposed in two different trajectories R and L. The charged particle is initially in the superposed state of $|{\rm R}\rangle$ and $|{\rm L}\rangle$, where $|{\rm R}\rangle(|{\rm L}\rangle)$ is the state that the particle will go through a trajectory R (L).

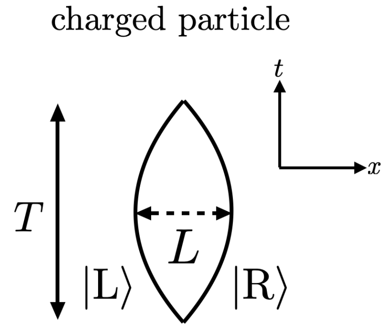


FIG 6: Configuration of a single charged particle trajectory.

The electromagnetic field is assumed to be initially in a coherent state. Then the total initial state at the time $t = 0$ is

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|{\rm R}\rangle + |{\rm L}\rangle\big) \otimes |\alpha\rangle_{\rm EM}, \tag{3.2}$$

where $|\alpha\rangle_{\rm EM} = \hat{D}(\alpha)|0\rangle_{\rm EM}$ is the coherent state of the electromagnetic field. Here, $|0\rangle_{\rm EM}$ is the vacuum state satisfying $\hat{a}_\mu(\boldsymbol{k})|0\rangle_{\rm EM} = 0$, and $\hat{D}(\alpha)$ is the unitary operator referred to as a displacement operator defined as

$$\hat{D}(\alpha) = \exp\left[\int d^3k(\alpha^\mu(\boldsymbol{k})\hat{a}^\dagger_\mu(\boldsymbol{k}) - h.c.)\right], \tag{3.3}$$

where the complex function $\alpha^\mu(\boldsymbol{k})$ characterizes the amplitude and phase of the initial electromagnetic field. The form of the complex function $\alpha^\mu(\boldsymbol{k})$ is restricted by the auxiliary condition in the BRST formalism. Because we will find that the entanglement between two charged particles does not depend on $\alpha^\mu(\boldsymbol{k})$ in Sec. III A, the details on $\alpha^\mu(\boldsymbol{k})$ are omitted here. The details are presented in Appendix A. The coherent state $|\alpha\rangle_{\rm EM}$ is interpreted as a state in which there is a mode of the electromagnetic field following Gauss's law due to the presence of charged particles.

We assume that the current operator $\hat{J}^\mu_{\rm I}(x) = e^{i\hat{H}_0 t}\hat{J}^\mu(0,\boldsymbol{x})e^{-i\hat{H}_0 t}$ in the interaction picture defined with $\hat{H}_0 = \hat{H}_{\rm p} + \hat{H}_{\rm EM}$ is approximated by a classical current as

$$\hat{J}^\mu_{\rm I}(x)|{\rm P}\rangle \approx J^\mu_{\rm P}(x)|{\rm P}\rangle, \quad J^\mu_{\rm P}(x) = e\int d\tau \frac{dX^\mu_{\rm P}}{d\tau}\delta^{(4)}(x - X_{\rm P}(\tau)), \tag{3.4}$$

where ${\rm P} = {\rm R}, {\rm L}$, e is an electric charge, and $X^\mu_{\rm P}(\tau)$ represents each trajectory of the charged particle. This approximation is valid when the following two assumptions are satisfied [82]:

1. The de Brogile wavelength is smaller than the width of the particle wavepacket.
2. The Compton wavelength $\lambda_{\rm C}$ of the charged particle is much shorter than the wavelength of the photon field $\lambda_{\rm EM}$ emitted from the charged particle ($\lambda_{\rm C} \ll \lambda_{\rm EM}$).

The first assumption justifies the localized classical trajectories $J^\mu_{j{\rm R}}(x)$ and $J^\mu_{j{\rm L}}(x)$. The second one neglects the process of a pair creation and annihilation.

The evolution of the initial state $|\Psi(0)\rangle$ is

$$\begin{aligned}|\Psi(T)\rangle &= e^{-i\hat{H}T}|\Psi(0)\rangle \\ &= e^{-i\hat{H}_0 T}\,{\rm T}\exp\Big[-i\int_0^T dt\hat{V}_{\rm I}(t)\Big]|\Psi(0)\rangle \\ &= e^{-i\hat{H}_0 T}\,{\rm T}\exp\Big[-i\int_0^T dt\int d^3x \hat{J}^\mu_{\rm I}(x)\hat{A}^{\rm I}_\mu(x)\Big]\frac{1}{\sqrt{2}}\sum_{\rm P=R,L}|{\rm P}\rangle\otimes|\alpha\rangle_{\rm EM} \\ &\approx \frac{e^{-i\hat{H}_0 T}}{\sqrt{2}}\sum_{\rm P=R,L}|{\rm P}\rangle\otimes\hat{U}_{\rm P}|\alpha\rangle_{\rm EM},\end{aligned} \tag{3.5}$$

where the approximation in (3.4) was used in the fourth line, $\hat{V}_{\rm I}(t) = e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}_0 t}$ and $\hat{A}^{\rm I}_\mu(x) = e^{i\hat{H}_0 t}\hat{A}(0,\boldsymbol{x})e^{-i\hat{H}_0 t}$. "T" in the second and third lines denotes the time ordered product. The operator $\hat{U}_{\rm P}$ is given by

$$\begin{aligned}\hat{U}_{\rm P} &= {\rm T}\exp\Big[-i\int_0^T dt\int d^3x J^\mu_{\rm P}(x)\hat{A}^{\rm I}_\mu(x)\Big] \\ &= \exp\Big[-i\int d^4x J^\mu_{\rm P}(x)\hat{A}^{\rm I}_\mu(x) - \frac{i}{2}\int d^4x\int d^4y J^\mu_{\rm P}(x)J^\nu_{\rm P}(y)G^{\rm r}_{\mu\nu}(x,y)\Big],\end{aligned} \tag{3.6}$$

where in the second line we used the Magnus expansion [85]

$$\mathrm{T}\exp\Big[-i\int_0^T dt\hat{V}_{\mathrm{I}}(t)\Big]=\exp\Big[\sum_{k=1}^{\infty}\Omega_k(T,0)\Big] \tag{3.7}$$

with

$$\Omega_1(T,0)=-i\int_0^T dt\hat{V}_{\mathrm{I}}(t),\quad \Omega_2(T,0)=\frac{(-i)^2}{2}\int_0^T dt_1\int_0^{t_1}dt_2[\hat{V}_{\mathrm{I}}(t_1),\hat{V}_{\mathrm{I}}(t_2)], \tag{3.8}$$

and $\Omega_{k\geq 3}(T,0)$ given by higher commutators, for example, $[[\hat{V}_{\mathrm{I}}(t_1),\hat{V}_{\mathrm{I}}(t_2)],\hat{V}_{\mathrm{I}}(t_3)]$. We note that the commutator $[\hat{V}_{\mathrm{I}}(t_1),\hat{V}_{\mathrm{I}}(t_2)]$ is proportional to the identity operator and commutes with $\hat{V}_{\mathrm{I}}(t)$ for any given time t. Hence, the terms $\Omega_{k\geq 3}(T,0)$ involving higher commutators vanish in Eq. (3.6). $G^{\mathrm{r}}_{\mu\nu}(x,y)$ in Eq. (3.6) is the retarded Green's function given by

$$G^{\mathrm{r}}_{\mu\nu}(x,y)=-i[\hat{A}^{\mathrm{I}}_{\mu}(x),\hat{A}^{\mathrm{I}}_{\nu}(y)]\theta(x^0-y^0). \tag{3.9}$$

We obtain the reduced density matrix of the charged particle as

$$\rho_{\mathrm{P}}=\mathrm{Tr}_{\mathrm{EM}}[|\Psi(T)\rangle\langle\Psi(T)|]=\frac{1}{2}\sum_{\mathrm{P},\mathrm{P}'=\mathrm{R},\mathrm{L}}{}_{\mathrm{EM}}\langle\alpha|\hat{U}^{\dagger}_{\mathrm{P}'}\hat{U}_{\mathrm{P}}|\alpha\rangle_{\mathrm{EM}}\,|\mathrm{P}_{\mathrm{f}}\rangle\langle\mathrm{P}'_{\mathrm{f}}|=\frac{1}{2}\sum_{\mathrm{P},\mathrm{P}'=\mathrm{R},\mathrm{L}}e^{-\Gamma_{\mathrm{P}'\mathrm{P}}+i\Phi_{\mathrm{P}'\mathrm{P}}}|\mathrm{P}_{\mathrm{f}}\rangle\langle\mathrm{P}'_{\mathrm{f}}|, \tag{3.10}$$

where $|\mathrm{P}_{\mathrm{f}}\rangle=e^{-i\hat{H}_{\mathrm{P}}T}|\mathrm{P}\rangle$ is the state of the charged particle, which moved along the trajectory $\mathrm{P}(=\mathrm{R},\mathrm{L})$. $\Gamma_{\mathrm{P}'\mathrm{P}}$ and $\Phi_{\mathrm{P}'\mathrm{P}}$ are

$$\Gamma_{\mathrm{P}'\mathrm{P}}=\frac{1}{4}\int d^4x\int d^4y\left(J^{\mu}_{\mathrm{P}'}(x)-J^{\mu}_{\mathrm{P}}(x)\right)\left(J^{\nu}_{\mathrm{P}'}(y)-J^{\nu}_{\mathrm{P}}(y)\right)\left\langle\left\{\hat{A}^{\mathrm{I}}_{\mu}(x),\hat{A}^{\mathrm{I}}_{\nu}(y)\right\}\right\rangle, \tag{3.11}$$

$$\begin{aligned}\Phi_{\mathrm{P}'\mathrm{P}}=&\int d^4x\left(J^{\mu}_{\mathrm{P}'}(x)-J^{\mu}_{\mathrm{P}}(x)\right)A_{\mu}(x)\\&-\frac{1}{2}\int d^4x\int d^4y\left(J^{\mu}_{\mathrm{P}'}(x)-J^{\mu}_{\mathrm{P}}(x)\right)\left(J^{\nu}_{\mathrm{P}'}(y)+J^{\nu}_{\mathrm{P}}(y)\right)G^{\mathrm{r}}_{\mu\nu}(x,y),\end{aligned} \tag{3.12}$$

where $\langle\{\hat{A}^{\mathrm{I}}_{\mu}(x),\hat{A}^{\mathrm{I}}_{\nu}(y)\}\rangle$ is the two-point function of the vacuum given by

$$\left\langle\left\{\hat{A}^{\mathrm{I}}_{\mu}(x),\hat{A}^{\mathrm{I}}_{\nu}(y)\right\}\right\rangle=\frac{\eta_{\mu\nu}}{4\pi^2}\left(\frac{1}{-\left(x^0-y^0-i\epsilon\right)^2+|\boldsymbol{x}-\boldsymbol{y}|^2}+\frac{1}{-\left(x^0-y^0+i\epsilon\right)^2+|\boldsymbol{x}-\boldsymbol{y}|^2}\right) \tag{3.13}$$

with the UV cutoff parameter ϵ, and the field $A_\mu(x)$ is

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}}(\alpha_\mu(\boldsymbol{k})e^{ik_\nu x^\nu} + c.c.). \tag{3.14}$$

The computation of the inner product ${}_{\rm EM}\langle\alpha|\hat{U}^\dagger_{\rm P'}\hat{U}_{\rm P}|\alpha\rangle_{\rm EM}$ in (3.10) and the derivation of Eqs. (3.11) and (3.12) are presented as follows: The inner product is rewritten as

$$\begin{aligned}{}_{\rm EM}\langle\alpha|\hat{U}^\dagger_{\rm P'}\hat{U}_{\rm P}|\alpha\rangle_{\rm EM} &= {}_{\rm EM}\langle 0|\hat{D}^\dagger(\alpha)\hat{U}^\dagger_{\rm P'}\hat{D}(\alpha)\hat{D}^\dagger(\alpha)\hat{U}_{\rm P}\hat{D}(\alpha)|0\rangle_{\rm EM} \\ &= {}_{\rm EM}\langle 0|(\hat{D}^\dagger(\alpha)\hat{U}_{\rm P'}\hat{D}(\alpha))^\dagger(\hat{D}^\dagger(\alpha)\hat{U}_{\rm P}\hat{D}(\alpha))|0\rangle_{\rm EM}, \end{aligned} \tag{3.15}$$

where we used $|\alpha\rangle = \hat{D}(\alpha)|0\rangle_{\rm EM}$, and the identity operator $\hat{I} = \hat{D}(\alpha)\hat{D}^\dagger(\alpha)$ was inserted between the unitary operators $\hat{U}^\dagger_{P'}$ and $\hat{U}_{\rm P}$ in the first equality. Because the displacement operator $\hat{D}(\alpha)$ satisfies Eq. (2.83), we obtain

$$\hat{D}^\dagger(\alpha)\hat{A}^{\rm I}_\mu(x)\hat{D}(\alpha) = \hat{A}^{\rm I}_\mu(x) + A_\mu(x), \tag{3.16}$$

where $A_\mu(x)$ is defined in Eq. (3.14). Subsequently, we obtain

$$\begin{aligned}&\hat{D}^\dagger(\alpha)\hat{U}_{\rm P}(x)\hat{D}(\alpha) \\ &= \exp\left[-\frac{i}{2}\int d^4x\int d^4y J^\mu_{\rm P}(x)J^\nu_{\rm P}(y)G^{\rm r}_{\mu\nu}(x,y)\right]\hat{D}^\dagger(\alpha)\exp\left[-i\int d^4x J^\mu_{\rm P}(x)\hat{A}^{\rm I}_\mu(x)\right]\hat{D}(\alpha) \\ &= \exp\left[-\frac{i}{2}\int d^4x\int d^4y J^\mu_{\rm P}(x)J^\nu_{\rm P}(y)G^{\rm r}_{\mu\nu}(x,y)\right]\exp\left[-i\int d^4x J^\mu_{\rm P}(x)\hat{D}^\dagger(\alpha)\hat{A}^{\rm I}_\mu(x)\hat{D}(\alpha)\right] \\ &= \exp\left[-\frac{i}{2}\int d^4x\int d^4y J^\mu_{\rm P}(x)J^\nu_{\rm P}(y)G^{\rm r}_{\mu\nu}(x,y) - i\int d^4x J^\mu_{\rm P}(x)A_\mu(x)\right] \\ &\times\exp\left[-i\int d^4x J^\mu_{\rm P}(x)\hat{A}^{\rm I}_\mu(x)\right], \end{aligned} \tag{3.17}$$

where the formula of the unitary operator $\hat{U}_{\rm P}$ (3.6) was substituted and $G^{\rm r}_{\mu\nu}(x,y)$ denotes the retarded Green's function given in Eq. (3.9). In the third equality we used Eq. (3.16).

We further obtain

$$
\begin{aligned}
&(\hat{D}^\dagger(\alpha)\hat{U}_{\mathrm{P}'}\hat{D}(\alpha))^\dagger(\hat{D}^\dagger(\alpha)\hat{U}_{\mathrm{P}}\hat{D}(\alpha)) \\
&= \exp\left[\frac{i}{2}\int d^4x\int d^4y\big(J^\mu_{\mathrm{P}'}(x)J^\nu_{\mathrm{P}'}(y) - J^\mu_{\mathrm{P}}(x)J^\nu_{\mathrm{P}}(y)\big)G^{\mathrm{r}}_{\mu\nu}(x,y) + i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)A_\mu(x)\right] \\
&\quad\times\exp\left[i\int d^4x J^\mu_{\mathrm{P}'}(x)\hat{A}^{\mathrm{I}}_\mu(x)\right]\exp\left[-i\int d^4x J^\mu_{\mathrm{P}}(x)\hat{A}^{\mathrm{I}}_\mu(x)\right] \\
&= \exp\left[\frac{i}{2}\int d^4x\int d^4y\big(J^\mu_{\mathrm{P}'}(x)J^\nu_{\mathrm{P}'}(y) - J^\mu_{\mathrm{P}}(x)J^\nu_{\mathrm{P}}(y)\big)G^{\mathrm{r}}_{\mu\nu}(x,y) + i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)A_\mu(x)\right] \\
&\quad\times\exp\left[i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\hat{A}^{\mathrm{I}}_\mu(x) + \frac{1}{2}\int d^4x d^4y J^\mu_{\mathrm{P}'}(x)J^\nu_{\mathrm{P}}(y)[\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)]\right] \\
&= \exp\left[\frac{i}{2}\int d^4x\int d^4y\big(J^\mu_{\mathrm{P}'}(x)J^\nu_{\mathrm{P}'}(y) - J^\mu_{\mathrm{P}}(x)J^\nu_{\mathrm{P}}(y)\big)G^{\mathrm{r}}_{\mu\nu}(x,y) + i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)A_\mu(x)\right] \\
&\quad\times\exp\left[i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\hat{A}^{\mathrm{I}}_\mu(x) + \frac{i}{2}\int d^4x d^4y\big(J^\mu_{\mathrm{P}'}(x)J^\nu_{\mathrm{P}}(y) - J^\nu_{\mathrm{P}'}(y)J^\mu_{\mathrm{P}}(x)\big)G^{\mathrm{r}}_{\mu\nu}(x,y)\right] \\
&= \exp\left[i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)A_\mu(x) + \frac{i}{2}\int d^4x\int d^4y\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\big(J^\nu_{\mathrm{P}'}(y) + J^\nu_{\mathrm{P}}(y)\big)G^{\mathrm{r}}_{\mu\nu}(x,y)\right] \\
&\quad\times\exp\left[i\int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\hat{A}^{\mathrm{I}}_\mu(x)\right] \\
&= \exp\left[i\Phi_{\mathrm{P}'\mathrm{P}} + i\hat{\Theta}_{\mathrm{P}\mathrm{P}'}\right], \qquad (3.18)
\end{aligned}
$$

where the Baker–Campbell–Hausdorff formula $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+[\hat{A},\hat{B}]/2+\cdots}$ was used in the second equality, and the relation $[\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)] = iG^{\mathrm{r}}_{\mu\nu}(x,y) - iG^{\mathrm{r}}_{\nu\mu}(y,x)$ was substituted in the third equality. "$\cdots$" in the Baker–Campbell–Hausdorff formula indicates the terms involving the higher commutators of $\hat{A}$ and $\hat{B}$. In our case, the commutator $[\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)]$ is proportional to the identity operator, so the higher commutators vanish. In the last equality, we defined $\hat{\Theta}_{\mathrm{P}\mathrm{P}'}$ and $\Phi_{\mathrm{P}'\mathrm{P}}$ as

$$
\hat{\Theta}_{\mathrm{P}\mathrm{P}'} = \int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\hat{A}^{\mathrm{I}}_\mu(x), \qquad (3.19)
$$

$$
\begin{aligned}
\Phi_{\mathrm{P}'\mathrm{P}} &= \int d^4x\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)A_\mu(x) \\
&\quad+\frac{1}{2}\int d^4x\int d^4y\big(J^\mu_{\mathrm{P}'}(x) - J^\mu_{\mathrm{P}}(x)\big)\big(J^\nu_{\mathrm{P}'}(y) + J^\nu_{\mathrm{P}}(y)\big)G^{\mathrm{r}}_{\mu\nu}(x,y). \qquad (3.20)
\end{aligned}
$$

Using the cumulant expansion for a given density matrix ρ,

$$\langle e^{i\lambda\hat{A}}\rangle_\rho = \mathrm{Tr}[\rho e^{i\lambda\hat{A}}] = \exp\left[i\lambda\langle\hat{A}\rangle_\rho - \frac{1}{2}\lambda^2\langle\big(\hat{A}-\langle\hat{A}\rangle_\rho\big)^2\rangle_\rho + \cdots\right], \tag{3.21}$$

where λ is a c-number parameter, $\hat{A}$ is an operator, and "$\cdots$" is the term with the third or higher cumulant, we can compute the inner product (3.15) as

$$\begin{aligned}
&{}_{\mathrm{EM}}\langle 0|\hat{D}^\dagger(\alpha)\hat{U}^\dagger_{\mathrm{P}'}\hat{U}_{\mathrm{P}}\hat{D}(\alpha)|0\rangle_{\mathrm{EM}} \\
&= e^{i\Phi_{\mathrm{P}'\mathrm{P}}}\ {}_{\mathrm{EM}}\langle 0|e^{i\hat{\Theta}_{\mathrm{PP}'}}|0\rangle_{\mathrm{EM}} \\
&= e^{i\Phi_{\mathrm{P}'\mathrm{P}}}\exp\left[i\langle\hat{\Theta}_{\mathrm{PP}'}\rangle - \frac{1}{2}\langle(\hat{\Theta}_{\mathrm{PP}'}-\langle\hat{\Theta}_{\mathrm{PP}'}\rangle)^2\rangle + \cdots\right] \\
&= e^{i\Phi_{\mathrm{P}'\mathrm{P}}}\exp\left[-\frac{1}{2}\int d^4x d^4y \big(J^\mu_{\mathrm{P}'}(x)-J^\mu_{\mathrm{P}}(x)\big)\big(J^\nu_{\mathrm{P}'}(y)-J^\nu_{\mathrm{P}}(y)\big)\ {}_{\mathrm{EM}}\langle 0|\hat{A}^{\mathrm{I}}_\mu(x)\hat{A}^{\mathrm{I}}_\nu(y)|0\rangle_{\mathrm{EM}}\right] \\
&= e^{-\Gamma_{\mathrm{P}'\mathrm{P}}+i\Phi_{\mathrm{P}'\mathrm{P}}}.
\end{aligned} \tag{3.22}$$

We used Eq. (3.18) and the cumulant expansion with $\rho = |0\rangle_{\mathrm{EM}}\langle 0|$, $\lambda = 1$ and $\hat{A} = \hat{\Theta}_{\mathrm{PP}'}$ in the first and second lines, respectively. $\langle\cdot\rangle$ denotes the vacuum expectation value. In the third equality, we substituted Eq. (3.19), and the term "$\cdots$" with the n-th cumulant for $n \geq 3$ vanishes because the free vacuum state $|0\rangle_{\mathrm{EM}}$ is Gaussian. In the last equality, we defined $\Gamma_{\mathrm{P}'\mathrm{P}}$ as

$$\begin{aligned}
\Gamma_{\mathrm{P}'\mathrm{P}} &= \frac{1}{2}\int d^4x d^4y \big(J^\mu_{\mathrm{P}'}(x)-J^\mu_{\mathrm{P}}(x)\big)\big(J^\nu_{\mathrm{P}'}(y)-J^\nu_{\mathrm{P}}(y)\big)\ {}_{\mathrm{EM}}\langle 0|\hat{A}^{\mathrm{I}}_\mu(x)\hat{A}^{\mathrm{I}}_\nu(y)|0\rangle_{\mathrm{EM}} \\
&= \frac{1}{4}\int d^4x d^4y \big(J^\mu_{\mathrm{P}'}(x)-J^\mu_{\mathrm{P}}(x)\big)\big(J^\nu_{\mathrm{P}'}(y)-J^\nu_{\mathrm{P}}(y)\big)\langle\big\{\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)\big\}\rangle.
\end{aligned}$$

Replacing the currents J^μ_{P} and $J^\mu_{\mathrm{P}'}$ with J^μ_{PQ} and $J^\mu_{\mathrm{P}'\mathrm{Q}'}$ in the above procedure, we can also derive (3.38) . It is obvious that $\Gamma_{\mathrm{RR}} = \Gamma_{\mathrm{LL}} = \Phi_{\mathrm{RR}} = \Phi_{\mathrm{LL}} = 0$. However, Γ_{RL} and Φ_{RL} are given as

$$\begin{aligned}
\Gamma_{\mathrm{RL}} &= \frac{1}{4}\int d^4x \int d^4y (J^\mu_{\mathrm{R}}(x)-J^\mu_{\mathrm{L}}(x))(J^\nu_{\mathrm{R}}(y)-J^\nu_{\mathrm{L}}(y))\langle\big\{\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)\big\}\rangle \\
&= \frac{e^2}{4}\oint_{\mathrm{C}} dx^\mu \oint_{\mathrm{C}} dy^\mu \langle\big\{\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)\big\}\rangle
\end{aligned} \tag{3.23}$$

and

$$\Phi_{\mathrm{RL}} = \int d^4x \left(J^\mu_{\mathrm{R}}(x) - J^\mu_{\mathrm{L}}(x)\right) A_\mu(x) - \frac{1}{2}\int d^4x \int d^4y (J^\mu_{\mathrm{R}}(x) - J^\mu_{\mathrm{L}}(x))(J^\nu_{\mathrm{R}}(y) + J^\nu_{\mathrm{L}}(y))G^{\mathrm{r}}_{\mu\nu}(x,y)$$
$$= e\oint_{\mathrm{C}} dx_\mu A^\mu(x) - \frac{e}{2}\oint_{\mathrm{C}} dx_\mu (A^\mu_{\mathrm{R}}(x) + A^\mu_{\mathrm{L}}(x)), \tag{3.24}$$

where $\oint_{\mathrm{C}} dx_\mu = \int_{\mathrm{R}} dx_\mu - \int_{\mathrm{L}} dx_\mu$ is the integral along the closed trajectory composed of trajectories R and L. Here, $A^\mu_{\mathrm{P}}(x)$ is the retarded potential given by

$$A^\mu_{\mathrm{P}}(x) = \int d^4y G^r_{\mu\nu}(x,y) J^\nu_{\mathrm{P}}(y). \tag{3.25}$$

According to (3.23), Γ_{RL} is always positive, and the interference terms of ρ_{P} (off-diagonal components) decay for a large Γ_{RL}. The quantity Γ_{RL} is referred to as the decoherence functional. The quantity $\Phi_{\mathrm{RL}} = -\Phi_{\mathrm{LR}}$ gives the phase shift in the interference pattern of the charged particle.

Assuming the following trajectories of the charged particle

$$X^\mu_{\mathrm{P}}(t) = \left[t, \epsilon_{\mathrm{P}} X(t), 0, 0\right]^{\mathrm{T}}, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \quad X(t) = 8L\left(1 - \frac{t}{T}\right)^2\left(\frac{t}{T}\right)^2, \tag{3.26}$$

where L and T are the length and time scales of the trajectories (also see Fig. 6), we obtain the decoherence functional as

$$\Gamma_{\mathrm{RL}} \approx \frac{32e^2}{3\pi^2}\left(\frac{L}{T}\right)^2, \tag{3.27}$$

when the charged particle has a non-relativistic velocity $L/T \ll 1$. We mention here the reason to choose $X(t)$ in Eq. (3.26). According to the function $X(t)$, the particle at $t = 0$ and $t = T$ has zero velocity and is smoothly superposed and recombined. The smoothness of the trajectory avoids a divergence in the calculations of decoherence, which guarantees our results in a form independent of an UV cutoff. The authors in [86] discussed the relation between the smoothness of particle trajectories and the UV divergence in decoherence effect. They reported that the decoherence functional computed assuming smooth trajectories is free from the UV cutoff and of the order of $O(e^2v^2)$, where v is the characteristic velocity of particle. This is consistent with our result written by the characteristic velocity L/T. The physical meaning of Γ_{RL} is interpreted in the following two ways. First, we consider that decoherence occurs through photon emission. The number of emitted photons is estimated as

$$\frac{WT}{\nu} = WT^2 \sim e^2 \Big(\frac{L}{T^2}\Big)^2 T^2 = e^2 \Big(\frac{L}{T}\Big)^2, \tag{3.28}$$

where $\nu = 1/T$ is the energy of a single photon in the unit $\hbar = 1$, and $W \sim e^2(L/T^2)^2$ is the Larmor formula of the power of radiation emitted from a non-relativistic charged particle. This formula shows the number of emitted photons during the time T. When this number exceeds one, i.e., $WT/\nu \geq 1$, the decoherence becomes significant. The decoherence due to bremsstrahlung was also discussed in [82]. This result is qualitatively interpreted as follows: if the superposed time T is sufficiently long, the acceleration of the charged particles becomes small, and therefore the number of photons emitted by the accelerated charged particle decreases. This result in the small decoherence. On the other hand, when the superposition scale L is large, i.e., if the acceleration of the charged particle is significant, the number of emitted photons increases. Thus, the decoherence works well. Second, we can deduce that the decoherence is due to the vacuum fluctuations of the transverse mode of the electromagnetic field (photon field) [87, 88]. The fluctuating photon field leads to dephasing effects,

$$\langle e^{i\phi_i} \rangle = e^{-\langle \phi_i^2 \rangle/2} \sim e^{-(e\Delta ELT)^2/2}, \tag{3.29}$$

where ϕ_i with $i = \mathrm{R}, \mathrm{L}$ is the phase shift due to the fluctuating photon field,

$$\hat{\phi}_i = \int d^4x \Delta J^\mu_i(x) \hat{A}^{\mathrm{I}}_\mu(x), \tag{3.30}$$

and $\langle \phi_{\mathrm{A}}^2 \rangle = \langle \phi_{\mathrm{B}}^2 \rangle \sim (e\Delta ELT)^2$ is its variance. ΔE is the vacuum fluctuation of the electric component of the photon field, which is estimated as $\Delta E \sim 1/T^2$ in [89]. The variance of the phase shift is

$$(e\Delta ELT)^2 \sim \Big(e\frac{1}{T^2}LT\Big)^2 = e^2 \Big(\frac{L}{T}\Big)^2. \tag{3.31}$$

This result is equivalent to Eq. (3.28), and the decoherence becomes significant when $(e\Delta ELT)^2 \geq 1$ is satisfied.

3.1.2 Model of two charged particles

Here, we extend the previous model to the model of two charged particles (for example, see Fig. 4). The total Hamiltonian in the Schrödinger picture is composed of the local Hamiltonians of each charged particle $\hat{H}_{\mathrm{A}}$ and $\hat{H}_{\mathrm{B}}$, the free Hamiltonian of the electromagnetic field $\hat{H}_{\mathrm{EM}}$ and the interaction term $\hat{V}$ as

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{EM}} + \hat{V}, \quad \hat{V} = \int d^3x \Big(\hat{J}^{\mu}_{\mathrm{A}}(\boldsymbol{x}) + \hat{J}^{\mu}_{\mathrm{B}}(\boldsymbol{x}) \Big) \hat{A}^{\mu}(\boldsymbol{x}), \tag{3.32}$$

where $\hat{J}^{\mu}_{\mathrm{A}}$ and $\hat{J}^{\mu}_{\mathrm{B}}$ are the current operators of each particle, which are coupled with the electromagnetic field operator $\hat{A}^{\mu}$. We consider the following initial condition at $t = 0$,

$$|\Psi(0)\rangle = \frac{1}{2} \sum_{\mathrm{P,\ Q=R,\ L}} |\mathrm{P}\rangle_{\mathrm{A}} |\mathrm{Q}\rangle_{\mathrm{B}} |\alpha\rangle_{\mathrm{EM}}, \tag{3.33}$$

where each particle is in superposition $|\mathrm{R}\rangle_{\mathrm{A}} + |\mathrm{L}\rangle_{\mathrm{A}}$ and $|\mathrm{R}\rangle_{\mathrm{B}} + |\mathrm{L}\rangle_{\mathrm{B}}$, and the electromagnetic field is in a coherent state $|\alpha\rangle_{\mathrm{EM}}$. We assume that the current operators $\hat{J}^{\mu}_{i\mathrm{I}}(x) = e^{i\hat{H}_0 t} \hat{J}^{\mu}_{i}(0, \boldsymbol{x}) e^{-i\hat{H}_0 t}$ in the interaction picture with respect to $\hat{H}_0 = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{EM}}$ are approximated by the following classical currents as

$$\hat{J}^{\mu}_{\mathrm{AI}}(x)\, |\mathrm{P}\rangle_{\mathrm{A}} \approx J^{\mu}_{\mathrm{AP}}(x)\, |\mathrm{P}\rangle_{\mathrm{A}}\,, \quad \hat{J}^{\mu}_{\mathrm{BI}}(x)\, |\mathrm{Q}\rangle_{\mathrm{B}} \approx J^{\mu}_{\mathrm{BQ}}(x)\, |\mathrm{Q}\rangle_{\mathrm{B}}\,, \tag{3.34}$$

$$J^{\mu}_{\mathrm{AP}}(x) = e \int d\tau \frac{dX^{\mu}_{\mathrm{AP}}}{d\tau} \delta^{(4)} \left(x - X_{\mathrm{AP}}(\tau)\right), \quad J^{\mu}_{\mathrm{BQ}}(x) = e \int d\tau \frac{dX^{\mu}_{\mathrm{BQ}}}{d\tau} \delta^{(4)} \big(x - X_{\mathrm{BQ}}(\tau)\big)\,, \tag{3.35}$$

where $X^{\mu}_{\mathrm{AP}}(\tau)$ and $X^{\mu}_{\mathrm{BQ}}(\tau)$ with $\mathrm{P, Q = R, L}$ represent the trajectories of each particle. The initial state evolves as follows:

$$\begin{aligned} |\Psi(T)\rangle &= \exp\big[-i\hat{H}T \big] |\Psi(0)\rangle \\ &= e^{-i\hat{H}_0 T} \mathrm{T} \exp\big[-i \int_0^T dt \hat{V}_{\mathrm{I}}(t) \big] |\Psi(0)\rangle \\ &\approx e^{-i\hat{H}_0 T} \frac{1}{2} \sum_{\mathrm{P,Q=R,L}} |\mathrm{P}\rangle_{\mathrm{A}} |\mathrm{Q}\rangle_{\mathrm{B}} \hat{U}_{\mathrm{PQ}} |\alpha\rangle_{\mathrm{EM}}, \end{aligned} \tag{3.36}$$

where we used the approximations (3.34) in the third line. The unitary operator $\hat{U}_{\rm PQ}$ is given by

$$\begin{aligned}\hat{U}_{\rm PQ} &= {\rm T}\exp\left[-i\int_0^T dt\int d^3x\left(J^\mu_{\rm 1P}+J^\mu_{\rm 2Q}\right)\hat{A}^{\rm I}_\mu(x)\right] \\ &= \exp\left[-i\int d^4x J^\mu_{\rm PQ}(x)\hat{A}^{\rm I}_\mu(x)-\frac{i}{2}\int d^4x\int d^4y J^\mu_{\rm PQ}(x)J^\nu_{\rm PQ}(y)G^{\rm r}_{\mu\nu}(x,y)\right], \qquad (3.37)\end{aligned}$$

where the Magnus expansion was used, and $J^\mu_{\rm PQ}=J^\mu_{\rm AP}+J^\mu_{\rm BQ}$. Tracing out the degrees of freedom of the electromagnetic field to focus on the quantum state of the charged particles, we obtain the reduced density matrix of particles A and B,

$$\begin{aligned}\rho_{AB} &= {\rm Tr}_{\rm EM}[|\Psi(T)\rangle\langle\Psi(T)|] \\ &= \frac{1}{4}\sum_{\rm P,Q=R,L}\sum_{\rm P',Q'=R,L}{}_{\rm EM}\langle\alpha|\hat{U}^\dagger_{\rm P'Q'}\hat{U}_{\rm PQ}|\alpha\rangle_{\rm EM}\,|{\rm P_f}\rangle_{\rm A}\langle{\rm P'_f}|\otimes|{\rm Q_f}\rangle_{\rm B}\langle{\rm Q'_f}| \\ &= \frac{1}{4}\sum_{\rm P,Q=R,L}\sum_{\rm P',Q'=R,L} e^{-\Gamma_{\rm P'Q'PQ}+i\Phi_{\rm P'Q'PQ}}\,|{\rm P_f}\rangle_{\rm A}\langle{\rm P'_f}|\otimes|{\rm Q_f}\rangle_{\rm B}\,\langle{\rm Q'_f}|, \qquad (3.38)\end{aligned}$$

where $|{\rm P_f}\rangle_{\rm A}=e^{-i\hat{H}_{\rm A}T}|{\rm P}\rangle_{\rm A}$ and $|{\rm Q_f}\rangle_{\rm B}=e^{-i\hat{H}_{\rm B}T}|{\rm Q}\rangle_{\rm B}$ are the states of the charged particles A and B, which moved along the trajectories P and Q, respectively. The quantities $\Gamma_{\rm P'Q'PQ}$ and $\Phi_{\rm P'Q'PQ}$ are

$$\Gamma_{\rm P'Q'PQ}=\frac{1}{4}\int d^4x\int d^4y(J^\mu_{\rm P'Q'}(x)-J^\mu_{\rm PQ}(x))(J^\nu_{\rm P'Q'}(y)-J^\nu_{\rm PQ}(y))\langle\{\hat{A}^{\rm I}_\mu(x),\hat{A}^{\rm I}_\nu(y)\}\rangle, \qquad (3.39)$$

$$\begin{aligned}\Phi_{\rm P'Q'PQ} &= \int d^4x(J^\mu_{\rm P'Q'}(x)-J^\mu_{\rm PQ}(x))A_\mu(x) \\ &\quad -\frac{1}{2}\int d^4x\int d^4y(J^\mu_{\rm P'Q'}(x)-J^\mu_{\rm PQ}(x))(J^\nu_{\rm P'Q'}(y)+J^\nu_{\rm PQ}(y))G^{\rm r}_{\mu\nu}(x,y), \qquad (3.40)\end{aligned}$$

where $\langle\{\hat{A}^{\rm I}_\mu(x),\hat{A}^{\rm I}_\nu(y)\}\rangle$ and $G^{\rm r}_{\mu\nu}(x,y)$ are the two-point function (3.13) and the retarded Green's function (3.9). $A_\mu(x)$ is the coherent electromagnetic field (3.14). The above formulas (3.39) and (3.40) are given by replacing the currents $J^\mu_{\rm P}$ and $J^\mu_{\rm P'}$ in Eqs. (3.11) and (3.12) with $J^\mu_{\rm PQ}$ and $J^\mu_{\rm P'Q'}$, respectively. In the next section, we derive the entanglement negativity of the two charged particles. We also demonstrate the entanglement behavior for a couple of typical configurations of the particle's trajectories.

3.2 Entanglement behavior of two charged particles

3.2.1 Formula of the negativity of two charged particles

We evaluate the entanglement negativity with the formula (2.34). As explained in Sec. 2.2.2, the negativity is obtained by composing the partial transposition of the density matrix. To see the difference of the eigenvalue by the partial transposition, we first compute the eigenvalues of the density matrix ρ_{AB}, respectively. They are obtained directly as follows:

$$\Lambda_1^{\pm} = \frac{1}{4}\left[1 - e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cosh[\Gamma_{\mathrm{c}}] \pm \left\{\left(e^{-\Gamma_{\mathrm{A}}} - e^{-\Gamma_{\mathrm{B}}}\right)^2 + 4e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\sin^2\left[\frac{\Theta}{2}\right] + e^{-2\Gamma_{\mathrm{A}}-2\Gamma_{\mathrm{B}}}\sinh^2[\Gamma_{\mathrm{c}}]\right\}^{\frac{1}{2}}\right], \tag{3.41}$$

$$\Lambda_2^{\pm} = \frac{1}{4}\left[1 + e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cosh[\Gamma_{\mathrm{c}}] \pm \left\{\left(e^{-\Gamma_{\mathrm{A}}} - e^{-\Gamma_{\mathrm{B}}}\right)^2 + 4e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cos^2\left[\frac{\Theta}{2}\right] + e^{-2\Gamma_{\mathrm{A}}-2\Gamma_{\mathrm{B}}}\sinh^2[\Gamma_{\mathrm{c}}]\right\}^{\frac{1}{2}}\right]. \tag{3.42}$$

In particular, the quantity Θ is

$$\Theta = -\frac{i}{2}\int d^4x \int d^4y \Delta J^{\mu}_{\mathrm{A}}(x)\Delta J^{\nu}_{\mathrm{B}}(y)[\hat{A}^{\mathrm{I}}_{\mu}(x), \hat{A}^{\mathrm{I}}_{\nu}(y)] = -\frac{i}{2}[\hat{\phi}_{\mathrm{A}}, \hat{\phi}_{\mathrm{B}}] \tag{3.43}$$

with the commutation relation of the operators $\hat{\phi}_{\mathrm{A}}$ and $\hat{\phi}_{\mathrm{B}}$,

$$\begin{aligned}
[\hat{\phi}_{\mathrm{A}}, \hat{\phi}_{\mathrm{B}}] &= \int d^4x d^4y (J^{\mu}_{\mathrm{AR}}(x) - J^{\mu}_{\mathrm{AL}}(x))(J^{\nu}_{\mathrm{BR}} - J^{\nu}_{\mathrm{BL}}(y))[\hat{A}^{\mathrm{I}}_{\mu}(x), \hat{A}^{\mathrm{I}}_{\nu}(y)] \\
&= \int d^4x d^4y (J^{\mu}_{\mathrm{AR}}(x) - J^{\mu}_{\mathrm{AL}}(x))(J^{\nu}_{\mathrm{BR}} - J^{\nu}_{\mathrm{BL}}(y))[\hat{A}^{\mathrm{I}}_{\mu}(x), \hat{A}^{\mathrm{I}}_{\nu}(y)]\theta(x^0 - y^0) \\
&+ \int d^4x d^4y (J^{\mu}_{\mathrm{AR}}(x) - J^{\mu}_{\mathrm{AL}}(x))(J^{\nu}_{\mathrm{BR}} - J^{\nu}_{\mathrm{BL}}(y))[\hat{A}^{\mathrm{I}}_{\mu}(x), \hat{A}^{\mathrm{I}}_{\nu}(y)]\theta(y^0 - x^0) \\
&= i\int d^4x \Delta J^{\mu}_{\mathrm{A}} \Delta A_{\mathrm{B}\mu} - i\int d^4x \Delta J^{\mu}_{\mathrm{B}} \Delta A_{\mathrm{A}\mu}, \\
&= ie\oint_{\mathrm{C_A}} dx_{\mu} \Delta A^{\mu}_{\mathrm{B}}(x) - ie\oint_{\mathrm{C_B}} dx_{\mu} \Delta A^{\mu}_{\mathrm{A}}(x),
\end{aligned} \tag{3.44}$$

where we inserted the step functions $\theta(x^0 - y^0) + \theta(y^0 - x^0)$ in the second line, and we changed variables as $x^{\mu} \leftrightarrow y^{\mu}$ and indices as $\mu \leftrightarrow \nu$ of the second term in the third line.

$\Delta J_i^\mu = J_{i\mathrm{R}}^\mu - J_{i\mathrm{L}}^\mu$ and $J_{i\mathrm{P}}^\mu$ is the current of the particle $i\,(=\mathrm{A},\mathrm{B})$ on the trajectory $\mathrm{P}\,(=\mathrm{R},\mathrm{L})$. The quantity $\Delta A_i^\mu = A_{i\mathrm{R}}^\mu - A_{i\mathrm{L}}^\mu$ is the difference between the retarded potentials defined by

$$A_{i\mathrm{P}}^\mu(x) = \int d^4y G_{\mu\nu}^r(x,y) J_{i\mathrm{P}}^\nu(y). \tag{3.45}$$

The line integral along the closed trajectory $\oint_{\mathrm{C}_i} dx_\mu$ is defined by $\oint_{\mathrm{C}_i} dx_\mu = \int_{i\mathrm{R}} dx_\mu - \int_{i\mathrm{L}} dx_\mu$, where $i\mathrm{P}$ denotes the trajectory P of the particle i. Next, the eigenvalues of the partial transposition $\rho_{AB}^{\mathrm{T_A}}$ with the components $\langle \mathrm{P}'|\langle \mathrm{Q}'|\rho_{AB}^{\mathrm{T_A}}|\mathrm{P}\rangle|\mathrm{Q}\rangle = \langle \mathrm{P}|\langle \mathrm{Q}'|\rho_{AB}|\mathrm{P}'\rangle|\mathrm{Q}\rangle$ are

$$\lambda_\pm = \frac{1}{4}\left[1 - e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\cosh[\Gamma_\mathrm{c}] \pm \left\{\left(e^{-\Gamma_\mathrm{A}} - e^{-\Gamma_\mathrm{B}}\right)^2 + 4e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\sin^2(\Phi/2) + e^{-2\Gamma_\mathrm{A}-2\Gamma_\mathrm{B}}\sinh^2[\Gamma_\mathrm{c}]\right\}^{\frac{1}{2}}\right], \tag{3.46}$$

$$\lambda'_\pm = \frac{1}{4}\left[1 + e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\cosh[\Gamma_\mathrm{c}] \pm \left\{\left(e^{-\Gamma_\mathrm{A}} - e^{-\Gamma_\mathrm{B}}\right)^2 + 4e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\cos^2(\Phi/2) + e^{-2\Gamma_\mathrm{A}-2\Gamma_\mathrm{B}}\sinh^2[\Gamma_\mathrm{c}]\right\}^{\frac{1}{2}}\right]. \tag{3.47}$$

Thus, by doing the partial transposition, Θ is changed to Φ defined in Eq. (3.51) below. We note that λ_- is the minimum eigenvalue λ_{min}, and hence the negativity of the two charged particles is

$$\begin{aligned} \mathscr{N} &= \max[-\lambda_{\mathrm{min}}, 0], \\ \lambda_{\mathrm{min}} &= \frac{1}{4}\left[1 - e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\cosh[\Gamma_\mathrm{c}] - \left\{\left(e^{-\Gamma_\mathrm{A}} - e^{-\Gamma_\mathrm{B}}\right)^2 + 4e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\sin^2(\Phi/2) + e^{-2\Gamma_\mathrm{A}-2\Gamma_\mathrm{B}}\sinh^2[\Gamma_\mathrm{c}]\right\}^{\frac{1}{2}}\right]. \end{aligned} \tag{3.48}$$

Because the density matrix ρ_{AB} of the charged particles is regarded as that of a two-qubit system, the negativity completely determines whether the particles are entangled or not.

The quantities Γ_i ($i = \mathrm{A}, \mathrm{B}$), Γ_c and Φ are given as

$$\begin{aligned}\Gamma_i &= \frac{1}{4}\int d^4x \int d^4y \Delta J_i^\mu(x) \Delta J_i^\nu(y) \langle \{\hat{A}_\mu^{\mathrm{I}}(x), \hat{A}_\nu^{\mathrm{I}}(y)\}\rangle \\ &= \frac{e^2}{4}\oint_{\mathrm{C}_i} dx^\mu \oint_{\mathrm{C}_i} dy^\nu \langle \{\hat{A}_\mu^{\mathrm{I}}(x), \hat{A}_\nu^{\mathrm{I}}(y)\}\rangle, \end{aligned} \tag{3.49}$$

$$\begin{aligned}\Gamma_{\mathrm{c}} &= \frac{1}{2}\int d^4x \int d^4y \Delta J_{\mathrm{A}}^\mu(x) \Delta J_{\mathrm{B}}^\nu(y) \langle \{\hat{A}_\mu^{\mathrm{I}}(x), \hat{A}_\nu^{\mathrm{I}}(y)\}\rangle \\ &= \frac{e^2}{2}\oint_{\mathrm{C_A}} dx^\mu \oint_{\mathrm{C_B}} dy^\nu \, \langle \{\hat{A}_\mu^{\mathrm{I}}(x), \hat{A}_\nu^{\mathrm{I}}(y)\}\rangle, \end{aligned} \tag{3.50}$$

$$\begin{aligned}\Phi &= \frac{1}{2}\int d^4x \int d^4y \Big\{ \Delta J_{\mathrm{A}}^\mu(x) \Delta J_{\mathrm{B}}^\nu(y) + \Delta J_{\mathrm{B}}^\mu(x) \Delta J_{\mathrm{A}}^\nu(y) \Big\} G_{\mu\nu}^r(x,y) \\ &= \frac{e}{2}\Big(\oint_{\mathrm{C_A}} dx_\mu \Delta A_{\mathrm{B}}^\mu(x) + \oint_{\mathrm{C_B}} dx_\mu \Delta A_{\mathrm{A}}^\mu(x) \Big). \end{aligned} \tag{3.51}$$

The quantities Γ_{A} and Γ_{B} depend on the trajectories of each particle and have the similar form to Γ_{RL} (3.23). These are the decoherence functionals appearing in the interference terms of each charged particle. In Appendix B.1, Γ_{A} and Γ_{B} are computed explicitly. Γ_{c} is characterized by the correlation function between the photon field coupled to particle A and the photon field coupled to particle B. Φ is computed from the phase shifts by the retarded potentials of the electromagnetic field $A_{i\mathrm{P}}^\mu$, which is analogous to the Aharanov-Bohm effect. Γ_{c} and Φ depend on the relative configuration of the trajectories of particles A and B. In Appendices B.2 and B.3, we explicitly evaluate Γ_{c} and Φ assuming two specific configurations of particles, which we refer to as the linear configuration (Figs. 7 and 9) and the parallel configuration (Figs. 11 and 13) in this paper. The quantities Γ_i, Γ_{c} and Φ are independent of the complex function $\alpha_\mu(\boldsymbol{k})$ of the initial coherent state of the electromagnetic field, and hence the negativity $\mathscr{N}$ also does not depend on $\alpha_\mu(\boldsymbol{k})$. Hence, as mentioned around Eq. (3.3), the entanglement between the particles does not depend on $\alpha_\mu(\boldsymbol{k})$. Using the Stokes's theorem to rewrite the line integrals in Eqs. (3.49), (3.50), and (3.51) by the surface integrals, we can express the quantities Γ_i, Γ_c and Φ in terms of the field strengths as

$$\Gamma_i = \frac{e^2}{16}\int_{\mathrm{S}_i} d\sigma^{\mu\nu}\int_{\mathrm{S}_i} d\sigma'^{\,\alpha\beta}\langle\{\hat{F}^{\mathrm{I}}_{\mu\nu}(x),\hat{F}^{\mathrm{I}}_{\alpha\beta}(x')\}\rangle, \tag{3.52}$$

$$\Gamma_\mathrm{c} = \frac{e^2}{16}\int_{\mathrm{S}_1} d\sigma^{\mu\nu}\int_{\mathrm{S}_2} d\sigma'^{\alpha\beta}\,\langle\{\hat{F}^{\mathrm{I}}_{\mu\nu}(x),\hat{F}^{\mathrm{I}}_{\alpha\beta}(x')\}\rangle, \tag{3.53}$$

$$\Phi = \frac{e}{4}\Big(\int_{\mathrm{S_A}} d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\mathrm{B}}(x) + \int_{\mathrm{S_B}} d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\mathrm{A}}(x)\Big), \tag{3.54}$$

where S_i is the surface surrounded by the closed trajectory C_i, $\hat{F}^{\mathrm{I}}_{\mu\nu} = \partial_\mu \hat{A}^{\mathrm{I}}_\nu - \partial_\nu \hat{A}^{\mathrm{I}}_\mu$, and $\Delta F^{\mu\nu}_i = F^{\mu\nu}_{i\mathrm{R}} - F^{\mu\nu}_{i\mathrm{L}}$ with the retarded field strengths $F^{\mu\nu}_{i\mathrm{P}} = \partial^\mu A^\nu_{i\mathrm{P}} - \partial^\nu A^\mu_{i\mathrm{P}}$.

In the following subsections, computing the quantities Γ_i, Γ_c and Φ, we present the minimum eigenvalue (3.48) and entanglement negativity $\mathscr{N}$ of the charged particles. Hereafter, we restore the reduced Planck constant $\hbar$ and the light velocity c to determine the non-relativistic limit of our analysis.

3.2.2 Linear configuration

We consider the linear configurations shown in Fig. 7 and Fig. 9. The parameters T, L, and D represent the time of maintaining a superposition state of each particle, the length of separation between the superposed trajectories of each particle, and the initial distance between the charged particles A and B, respectively.

3.2.2.1 $cT \gg D \sim L$ or $cT \gg D \gg L$ regimes

To evaluate the minimum eigenvalue λ_{min}, which gives the negativity of the two charged particles, we compute the quantities Γ_i, Γ_c and Φ by specifying the trajectories of the particles. We consider the following trajectories

$$X^\mu_{\mathrm{AP}} = [t, \epsilon_\mathrm{P} X(t), 0, 0]^{\mathrm{T}}, \quad X^\mu_{\mathrm{BQ}}(t) = \Big[t, \epsilon_\mathrm{Q} X(t) + D, 0, 0\Big]^{\mathrm{T}}, \quad \epsilon_\mathrm{R} = -\epsilon_\mathrm{L} = 1, \tag{3.55}$$

$$X(t) = 8L\Big(1-\frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2, \tag{3.56}$$

where X^μ_{AP} and X^μ_{BQ} with $\mathrm{P}, \mathrm{Q} = \mathrm{R}, \mathrm{L}$ describe the trajectories of particles A and B, respectively. Fig. 7 schematically shows the configuration of the particles. In the regimes $cT \gg D \sim L$ and $cT \gg D \gg L$, the quantities Γ_i, Γ_c, and Φ are evaluated. As we show in

Appendix B.1, assuming the above trajectories, we can compute Γ_i for $cT \gg L$ as

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2. \tag{3.57}$$

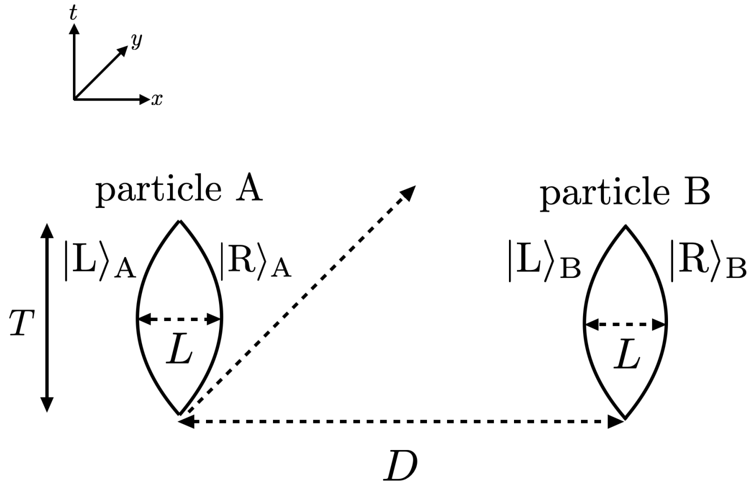


FIG 7: Linear configuration in the regimes $cT \gg D \sim L$ and $cT \gg D \gg L$. The left panel shows the entire view of the linear configuration.

In the regime $cT \gg D \sim L$, the quantity $\Gamma_{\rm c}$ is analytically obtained as

$$\Gamma_{\rm c} \approx \frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2, \tag{3.58}$$

and the quantity Φ is numerically computed from the formula

$$\Phi \approx -\frac{e^2}{4\pi\hbar}\int_0^T dt\left[\frac{2}{D}\left(1-\frac{v^2}{c^2}\right)-\left(1+\frac{v^2}{c^2}\right)\left(\frac{1}{D-2X(t)}+\frac{1}{D+2X(t)}\right)\right], \tag{3.59}$$

where $v = dX/dt$. Substituting Eqs. (3.57), (3.58), and (3.59) into Eq. (3.48), we evaluate the minimum eigenvalue $\lambda_{\rm min}$ and the negativity $\mathscr{N}$. The behavior is shown by the red curve in Fig. 8 (a). The derivation of Eqs. (3.58) and (3.59) is presented in Appendix B.2.1. In the regime $cT \gg D \gg L$, the quantities $\Gamma_{\rm c}$ and Φ are estimated as

$$\Gamma_{\rm c} \approx \frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\left(1+4\left(\frac{D}{cT}\right)^2\ln\left[\frac{D}{cT}\right]\right), \quad \Phi \approx \frac{64e^2}{315\pi\hbar c}\left(\frac{L}{cT}\right)^2\left\{\left(\frac{cT}{D}\right)^3+6\frac{cT}{D}\right\}, \tag{3.60}$$

and we obtain the following eigenvalue (3.48)

$$\begin{aligned}\lambda_{\rm min} &\approx \frac{1}{4}\Big[\Gamma_{\rm A}+\Gamma_{\rm B}-\sqrt{(\Gamma_{\rm A}-\Gamma_{\rm B})^2+\Phi^2+\Gamma_{\rm c}^2}\Big]\\ &\approx \frac{16e^2}{3\pi^2\hbar c}\Big(\frac{L}{cT}\Big)^2\\ &\quad -\frac{1}{4}\sqrt{\bigg[\frac{64e^2}{315\pi\hbar c}\Big(\frac{L}{cT}\Big)^2\Big\{\Big(\frac{cT}{D}\Big)^3+6\frac{cT}{D}\Big\}\bigg]^2+\bigg[\frac{64e^2}{3\pi^2\hbar c}\Big(\frac{L}{cT}\Big)^2\Big(1+4\Big(\frac{D}{cT}\Big)^2\ln\Big[\frac{D}{cT}\Big]\Big)\bigg]^2},\end{aligned} \tag{3.61}$$

where in the first line we assumed that Γ_i, Γ_c, and Φ are small, and Eqs. (3.57) and (3.60) were substituted in the second line. Eq. (3.60) is derived in Appendix B.2.1. The term $\Gamma_{\rm A}+\Gamma_{\rm B}$ in the first line of Eq. (3.61) (or the first positive term in the second line) makes $\lambda_{\rm min}$ positive and reduces the negativity. In contrast, the second term given by Φ and $\Gamma_{\rm c}$ (or the second term in the second line) decreases $\lambda_{\rm min}$, where Φ is much larger than $\Gamma_{\rm c}$ because of $\Gamma_{\rm c}/\Phi \approx (D/cT)^3 \ll 1$. The quantity Φ reflects the contribution of the Coulomb potential (proportional to D^{-3} term) and its relativistic correction (proportional to D^{-1} term).

The panels (a) and (b) in Fig. 8 show the negativity in the regimes $cT \gg D \sim L$ and $cT \gg D \gg L$, respectively. The blue curve in each panel presents the behavior of the negativity in Fig. 5, which is given in the non-relativistic limit and has no contributions from the dynamical degrees of freedom of the electromagnetic field. The red curve shows the behavior of the negativity computed from our analysis. In the panel (a) under the regime $cT \gg D \sim L$, the red curve is similar to the blue curve. This means that the Coulomb potential is dominant to determine the negativity in this regime, and the relativistic corrections are small. However, in the panel (b) under the regime $cT \gg D \gg L$, there is the parameter region without the negativity. This is because the decoherence effects $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are more dominant than the term Φ mainly determined by the Coulomb potential. In this regime, the computation of the negativity in the non-relativistic limit is not valid.

3.2.2.2 $D \gg cT \gg L$ regime

Subsequently, we present the formula of the minimum eigenvalue $\lambda_{\rm min}$ in the regime $D \gg$

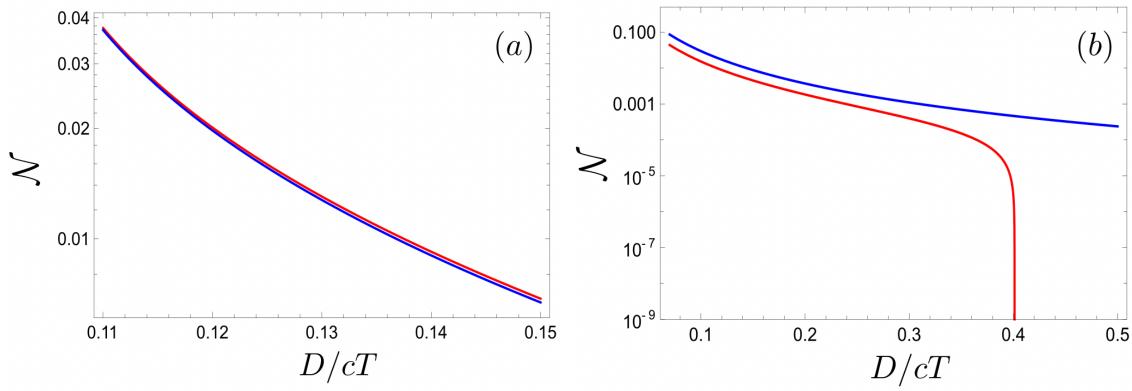


FIG 8: Negativity $\mathscr{N}$ for the linear configuration. (a) is the case $cT \gg D \sim L$ while (b) is the case $cT \gg D \gg L$. We adopted $L/cT = 0.1$.

$cT \gg L$. We assume the trajectories of the charged particles A and B given by

$$X^{\mu}_{\mathrm{AP}}(t) = \Big[t, \epsilon_{\mathrm{P}} X(t), 0, 0\Big]^T, \quad X^{\mu}_{\mathrm{BQ}}(t) = \Big[t, \epsilon_{\mathrm{Q}} X(t-D) + D, 0, 0\Big]^T, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \tag{3.62}$$

$$X(t) = 8L\Big(1 - \frac{t}{T}\Big)^2 \Big(\frac{t}{T}\Big)^2, \tag{3.63}$$

where X^{μ}_{BQ} is defined in $D/c \leq t \leq T + D/c$. The whole configuration of the trajectories is shown in Fig. 9, in which the superposition of particle B is formed after particle A is superposed. The trajectories of the particles are arranged to be causally connected.

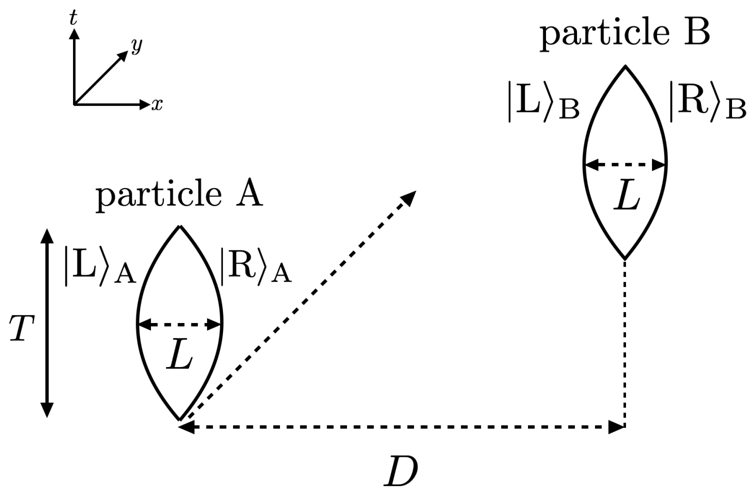


FIG 9: Linear configuration in the $D \gg cT \gg L$ regime.

We obtain the following formulas for the regime $D \gg cT \gg L$,

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2, \quad \Gamma_{\rm c} \approx -\frac{32e^2}{225\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\left(\frac{cT}{D}\right)^4, \quad \Phi \approx \frac{16e^2}{315\pi\hbar c}\left(\frac{L}{cT}\right)^2\left(\frac{cT}{D}\right)^3, \tag{3.64}$$

where $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are the same as those given in (3.57) because they depend only on each particle motion, and the explicit derivation of $\Gamma_{\rm c}$ and Φ is presented in Appendix B.2.2. We can then compute the eigenvalue (3.48) as

$$\begin{aligned} \lambda_{\rm min} &\approx \frac{1}{4}\left[\Gamma_{\rm A} + \Gamma_{\rm B} - \sqrt{(\Gamma_{\rm A} - \Gamma_{\rm B})^2 + \Phi^2 + \Gamma_{\rm c}^2}\right] \\ &\approx \frac{16e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2 - \frac{16e^2}{315\pi\hbar c}\left(\frac{L}{cT}\right)^2\left(\frac{cT}{D}\right)^3, \end{aligned} \tag{3.65}$$

where in the first equality, the minimum eigenvalue was approximated by assuming that $\Gamma_i (i = {\rm A}, {\rm B})$, $\Gamma_{\rm c}$, and Φ are small. In the second equality, we substituted (3.64) and neglected $\Gamma_{\rm c}$ because of $\Gamma_{\rm c}/\Phi \approx cT/D \ll 1$ for the regime $D \gg cT \gg L$. The positive term in the right hand side of Eq. (3.65), which is given by the decoherence functional Γ_i, comes from the vacuum fluctuations of the photon field. The negative term in Eq. (3.65) is given by the quantity Φ depending on the phase shifts due to the retarded field (see the formula of Φ (3.51) and the discussion around (3.45)).

Fig. 10 shows the minimum eigenvalue (3.65) for a fixed $L/cT = 0.1$ as a function of D/cT in the regime $D \gg cT \gg L$. The minimum eigenvalue is always positive, and hence the charged particles A and B are not entangled. This result shows that the decoherence due to the vacuum fluctuation of the photon field suppresses the entanglement generation due to the retarded field. In Sec. 3.3, we will discuss that the retarded field corresponds to the longitudinal mode, that is, the non-dynamical part of the electromagnetic field.

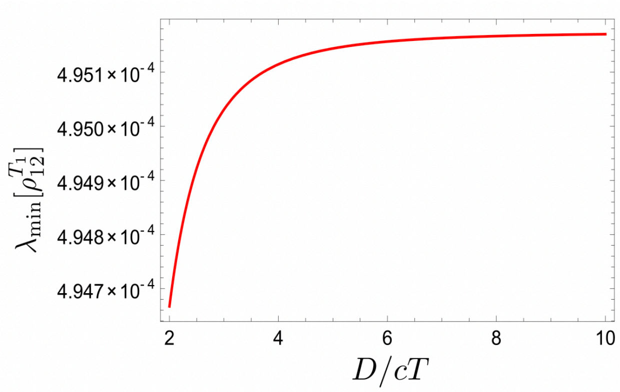


FIG 10: Minimum eigenvalue $\lambda_{\rm min}[\rho_{\rm AB}^{\rm T_A}]$ for the linear configuration in the regime $D \gg cT \gg L$. We adopted $L/cT = 0.1$.

3.2.3 Parallel configuration

Here, we consider the parallel configurations shown in Fig. 11 and Fig. 13. The parameters T, L, and D play the same role as those in the linear configuration, which are the typical scales appearing in the trajectories of the particles.

3.2.2.1 $cT \gg D \gg L$ or $cT \gg D \gg L$ regimes

We first consider the trajectories of the two particles A and B as

$$X^\mu_{\rm AP}(t) = \Big[t, \epsilon_{\rm P} X(t), 0, 0\Big]^{\rm T}, \quad X^\mu_{\rm BQ}(t) = \Big[t, \epsilon_{\rm Q} X(t), D, 0\Big]^{\rm T}, \quad \epsilon_{\rm R} = -\epsilon_{\rm L} = 1, \tag{3.66}$$

$$X(t) = 8L\Big(1-\frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2. \tag{3.67}$$

The schematic configuration is shown in Fig. 11. We examine the quantities $\Gamma_i (i = {\rm A, B})$, $\Gamma_{\rm c}$, and Φ for the regimes $cT \gg L \gg D$ and $cT \gg D \gg L$ to estimate the minimum eigenvalue $\lambda_{\rm min}$. Even in this configuration, the decoherence functionals $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ for $cT \gg L$ are identical to those in Eq. (3.57), that is,

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\Big(\frac{L}{cT}\Big)^2. \tag{3.68}$$

This is because the decoherence functionals are given by the local motions of each charged particle. In the following, we evaluate $\Gamma_{\rm c}$ and Φ for each of the regimes $cT \gg L \gg D$ and

$cT \gg D \gg L$.

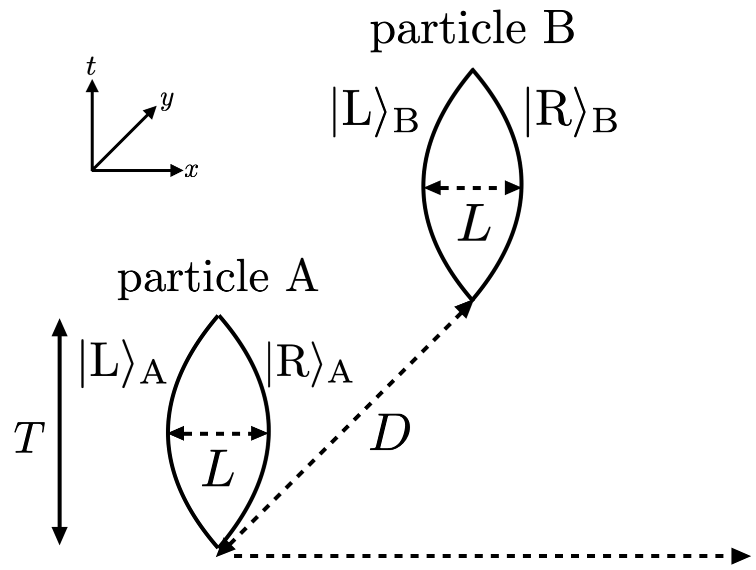


FIG 11: Parallel configuration in $cT \gg D \gg L$ regime.

In the regime $cT \gg L \gg D$, the quantities Γ_c and Φ are

$$\Gamma_\mathrm{c} \approx \frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2, \quad \Phi \approx \frac{e^2}{4\pi\hbar c}\frac{cT}{D}\left\{1 - \frac{64}{105}\left(\frac{L}{cT}\right)^2\right\}, \tag{3.69}$$

which are derived in Appendix B.3.1. The minimum eigenvalue (3.48) for the regime $cT \gg L \gg D$ is given as

$$\begin{aligned}\lambda_\mathrm{min}[\rho_\mathrm{AB}^{\mathrm{T_A}}] &\approx \frac{1}{4}\left[\Gamma_\mathrm{A} + \Gamma_\mathrm{B} - \sqrt{(\Gamma_\mathrm{A} - \Gamma_\mathrm{B})^2 + 4\sin^2\left[\frac{\Phi}{2}\right] + \Gamma_\mathrm{c}^2}\right] \\ &\approx \frac{16e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2 - \frac{1}{4}\sqrt{\left\{2\sin\left[\frac{e^2}{4\pi\hbar c}\frac{cT}{D}\left(1 - \frac{64}{105}\left(\frac{L}{cT}\right)^2\right)\right]\right\}^2 + \left\{\frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\right\}^2}.\end{aligned} \tag{3.70}$$

In the above equation, the first term coming from $\Gamma_\mathrm{A}+\Gamma_\mathrm{B}$ increases the minimum eigenvalue, whereas the second term given by Φ and Γ_c decreases it. It should be noted that the quantity Φ can be $\Phi \gg 1$ because of $cT/D(1 - L^2/(cT)^2) \approx cT/D \gg 1$ for the regime $cT \gg L \gg D$.

In the regime $cT \gg D \gg L$, the quantities $\Gamma_{\rm c}$ and Φ are

$$\Gamma_{\rm c} \approx \frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\left\{1+4\left(\frac{D}{cT}\right)^2\ln\left[\frac{D}{cT}\right]\right\}, \quad \Phi \approx \frac{32e^2}{315\pi\hbar c}\left(\frac{L}{cT}\right)^2\left\{\left(\frac{cT}{D}\right)^3 - 6\frac{cT}{D}\right\}. \tag{3.71}$$

These formulas are derived in Appendix B.3. The minimum eigenvalue (3.48) for the regime $cT \gg D \gg L$ is approximated as

$$\begin{aligned}\lambda_{\rm min}[\rho_{\rm AB}^{\rm T_A}] &\approx \frac{1}{4}\left[\Gamma_{\rm A}+\Gamma_{\rm B}-\sqrt{(\Gamma_{\rm A}-\Gamma_{\rm B})^2+\Phi^2+\Gamma_{\rm c}^2}\right] \\ &\approx \frac{16e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2 \\ &\quad -\frac{1}{4}\sqrt{\left[\frac{32e^2}{315\pi\hbar c}\left(\frac{L}{cT}\right)^2\left\{\left(\frac{cT}{D}\right)^3-\frac{6cT}{D}\right\}\right]^2+\left[\frac{64e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\left\{1+4\left(\frac{D}{cT}\right)^2\ln\left[\frac{D}{cT}\right]\right\}\right]^2},\end{aligned} \tag{3.72}$$

This minimum eigenvalue has the very similar feature to that obtained in the case of the linear configuration. The first positive contribution in (3.72) comes from the decoherence functional Γ_i quantifying the decoherence due to the vacuum fluctuations of the photon field. The second negative contribution in (3.72) is computed from $\Gamma_{\rm c}$ and Φ, which is mostly from Φ because of $\Gamma_{\rm c}/\Phi \approx (D/cT)^3 \ll 1$. The quantities $\Gamma_{\rm c}$ and Φ stem from the vacuum correlation of the photon field and the phase shifts due to the retarded field, respectively.

The panels in Fig. 12 (a) and (b) present the behavior of the negativity in the regimes $cT \gg L \gg D$ and $cT \gg D \gg L$, respectively. The blue curve shows the negativity in the non-relativistic limit, which corresponds to the electromagnetic version of the BMV experiment. The red curve is given by our analysis. The behavior of the negativity in Fig. 12 (a) means that our analysis is consistent with the non-relativistic result. However, in Fig. 12 (b), due to the decoherence, the parameter region without the negativity appears, and hence the computation in the non-relativistic limit becomes invalid in $cT \gg D \gg L$.

3.2.2.1 $D \gg cT \gg L$ regime

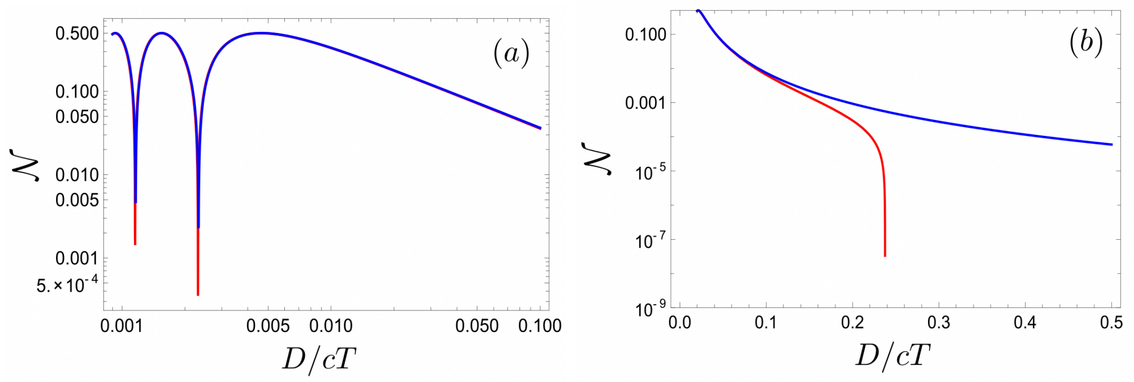


FIG 12: Negativity $\mathcal{N}$ for the parallel configuration. (a) is the case $cT \gg L \gg D$, whereas (b) is the case $cT \gg D \gg L$. We adopted $L/cT = 0.1$.

We consider the trajectories of two charged particles A and B as

$$X^{\mu}_{\mathrm{AP}}(t) = \Big[t, \epsilon_{\mathrm{P}} X(t), 0, 0\Big]^{\mathrm{T}}, \quad X^{\mu}_{\mathrm{BP}}(t) = \Big[t, \epsilon_{\mathrm{P}} X(t - D/c), D, 0\Big]^{\mathrm{T}}, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \tag{3.73}$$

$$X(t) = 8L\Big(1 - \frac{t}{T}\Big)^2 \Big(\frac{t}{T}\Big)^2, \tag{3.74}$$

where X^{μ}_{AP} and X^{μ}_{BQ} with $\mathrm{P}, \mathrm{Q} = \mathrm{R}, \mathrm{L}$ describe the trajectory of each particle. Here, X^{μ}_{BQ} is defined in $D/c \leq t \leq T + D/c$. The spacetime configuration of the particles is presented in Fig. 13. We examine the minimum eigenvalue in the regime $D \gg cT \gg L$.

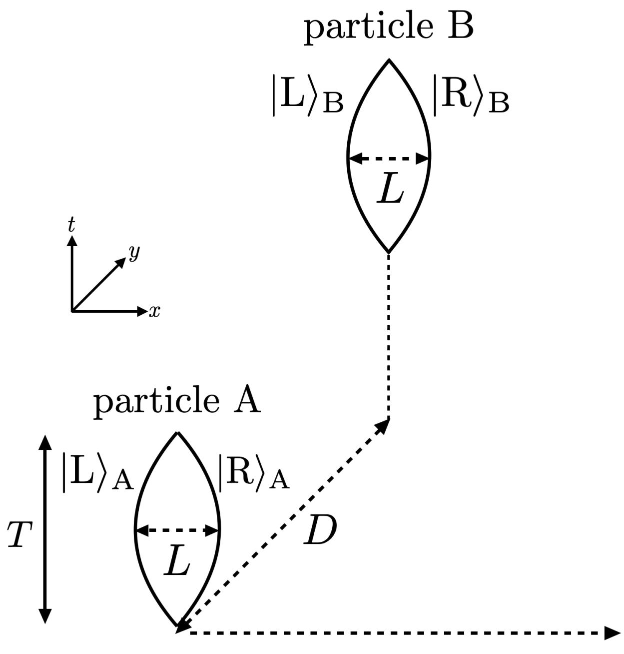


FIG 13: Parallel configuration in $D \gg cT \gg L$ regime.

We have the following formulas of $\Gamma_{\rm A}$, $\Gamma_{\rm B}$, $\Gamma_{\rm c}$ and Φ for the regime $D \gg cT \gg L$,

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2, \quad \Phi \approx -\frac{64e^2}{105\pi\hbar c}\left(\frac{L}{cT}\right)^2\frac{cT}{D}, \quad \Gamma_{\rm c} \approx -\frac{32e^2}{225\pi^2\hbar c}\left(\frac{L}{cT}\right)^2\left(\frac{cT}{D}\right)^4, \tag{3.75}$$

where $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are not at all different from those given in (3.57) or (3.68), and the quantities $\Gamma_{\rm c}$ and Φ are derived in Appendix B.3.2. Then, we can compute the minimum eigenvalue (3.48) as

$$\begin{aligned}\lambda_{\rm min}[\rho_{\rm AB}^{\rm T_A}] &\approx \frac{1}{4}\left[\Gamma_{\rm A} + \Gamma_{\rm B} - \sqrt{(\Gamma_{\rm A} - \Gamma_{\rm B})^2 + \Phi^2 + \Gamma_{\rm c}^2}\right] \\ &\approx \frac{16e^2}{3\pi^2\hbar c}\left(\frac{L}{cT}\right)^2 - \frac{16e^2}{105\pi\hbar c}\left(\frac{L}{cT}\right)^2\frac{cT}{D},\end{aligned} \tag{3.76}$$

where the first term coming from the decoherence functional Γ_i increases the minimum eigenvalue, and the second term given by Φ decreases it. In the second equality, we neglected $\Gamma_{\rm c}$ because of $\Gamma_{\rm c}/\Phi \approx (cT/D)^3 \ll 1$. Fig. 14 shows the minimum eigenvalue (3.48) as a function of D/cT in the regime $D \gg cT \gg L$, which is always positive. Similar to the result in the case of the linear configuration (see Fig. 14), the negativity remains zero, and the entanglement between the charged particles A and B does not appear in the regime $D \gg cT \gg L$. We come to the same conclusion that the decoherence due to the vacuum fluctuations of the photon field prevents the entanglement generation due to the retarded field.

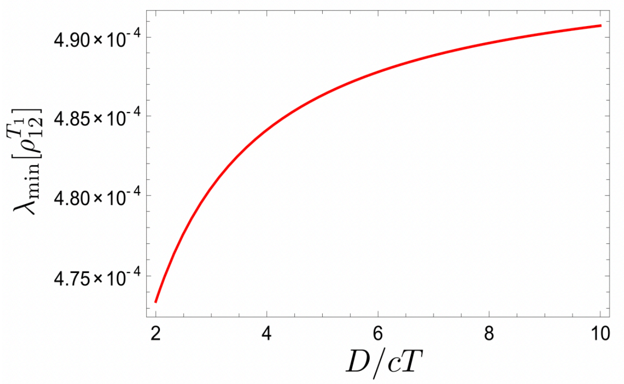


FIG 14: Minimum eigenvalue $\lambda_{\rm min}[\rho_{\rm AB}^{\rm T_A}]$ for the parallel configuration in the regime $D \gg cT \gg L$. We adopted $L/cT = 0.1$.

It is important to note that the parameter dependence appearing in the formulas of the minimum eigenvalue (3.65) and (3.76) is different. The second terms of (3.65) and (3.76) are proportional to $-cTL^2/D^3$ and $-L^2/D(cT)$, respectively. The latter is regarded as a consequence of the quantum superposition of bremsstrahlung, as we will discuss in the next section.

3.3 Discussion

Before the main discussion in this section, we first mention a basic property of the field strength of a charged particle. Generally, the field strength of a charged particle is decomposed into two terms $F^{\mu\nu} = F_{\rm v}^{\mu\nu} + F_{\rm a}^{\mu\nu}$, which are given as

$$F_{\rm v}^{\mu\nu}(x) = -\frac{e}{4\pi}\frac{(x^\mu - X^\mu(t_{\rm r}))v^\nu(t_{\rm r}) - (\mu \leftrightarrow \nu)}{\gamma^2[(x - X(t_{\rm r}))\cdot v(t_{\rm r})]^3}, \tag{3.77}$$

$$F_{\rm a}^{\mu\nu}(x) = \frac{e}{4\pi[(x - X(t_{\rm r}))\cdot v(t_{\rm r})]^2}\left[(x^\mu - X^\mu(t_{\rm r}))\Big(a^\nu(t_{\rm r}) - \frac{(x - X(t_{\rm r}))\cdot a(t_{\rm r})}{(x - X(t_{\rm r}))\cdot v(t_{\rm r})}v^\nu(t_{\rm r})\Big) - (\mu \leftrightarrow \nu)\right], \tag{3.78}$$

where X^μ is the spacetime position of the particle, $v^\mu = dX^\mu/dt$ is the velocity , $a^\mu = dv^\mu/dt$ is the acceleration, and $\gamma = 1/\sqrt{-v^\mu v_\mu}$ is the Lorentz factor. The retarded time $t_{\rm r}$ is given by $-(t - t_{\rm r}) + |\boldsymbol{x} - \boldsymbol{X}(t_{\rm r})| = 0$. The above equations are obtained as follows: The current of a charged particle is given as a four-vector current in a covariant form with

$$J^\mu(x) = e\int d\tau \frac{dX^\mu}{d\tau}\delta^{(4)}(x - X(\tau)), \tag{3.79}$$

where $X^\mu(\tau)$ is the trajectory of the charged particle parameterized by a proper time τ. Using this current and the retarded Green's function,

$$G^r_{\mu\nu}(x,y) = -\frac{\eta_{\mu\nu}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\delta\big(|\boldsymbol{x} - \boldsymbol{y}| - (x^0 - y^0)\big), \tag{3.80}$$

we obtain the retarded potential as

$$A^\mu(x) = \int d^4y G^{{\rm r}\mu}{}_\nu(x,y)J^\nu(y) = \frac{e}{4\pi}\frac{u^\mu(\tau_{\rm r})}{(x - X(\tau_{\rm r}))\cdot u(\tau_{\rm r})}, \tag{3.81}$$

where $u^\mu = dX^\mu/d\tau$ is the four-velocity of the charge, and τ_r is determined by the light-cone condition

$$-(t - X^0(\tau_\mathrm{r})) + |\boldsymbol{x} - \boldsymbol{X}(\tau_\mathrm{r})| = 0. \tag{3.82}$$

From the definition of the field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we obtain

$$F^{\mu\nu} = F^{\mu\nu}_\mathrm{v} + F^{\mu\nu}_\mathrm{a}, \tag{3.83}$$

$$F^{\mu\nu}_\mathrm{v} = -\frac{e}{4\pi}\frac{(x^\mu - X^\mu(\tau_\mathrm{r}))u^\nu(\tau_\mathrm{r}) - (x^\nu - X^\nu(\tau_\mathrm{r}))u^\mu(\tau_\mathrm{r})}{[(x - X(\tau_\mathrm{r}))\cdot u(\tau_\mathrm{r})]^3}, \tag{3.84}$$

$$\begin{aligned} F^{\mu\nu}_\mathrm{a} = {} & \frac{e}{4\pi[(x - X(\tau_\mathrm{r}))\cdot u(\tau_\mathrm{r})]^2}\Big((x^\mu - X^\mu(\tau_\mathrm{r}))\big(\dot{u}^\nu(\tau_\mathrm{r}) - \frac{(x - X(\tau_\mathrm{r}))\cdot \dot{u}(\tau_\mathrm{r})}{(x - X(\tau_\mathrm{r}))\cdot u(\tau_\mathrm{r})}u^\nu(\tau_\mathrm{r})\big) \\ & - (x^\nu - X^\nu(\tau_\mathrm{r}))\big(\dot{u}^\mu(\tau_\mathrm{r}) - \frac{(x - X(\tau_\mathrm{r}))\cdot \dot{u}(\tau_\mathrm{r})}{(x - X(\tau_\mathrm{r}))\cdot u(\tau_\mathrm{r})}u^\mu(\tau_\mathrm{r})\big)\Big), \end{aligned} \tag{3.85}$$

where $\dot{u}^\mu = du^\mu/d\tau$ is the four-acceleration. We use the coordinate time t instead of the proper time τ to rewrite the above field strengths. The four-vector and four-acceleration as a function of t are

$$u^\mu = \frac{dX^\mu}{d\tau} = \gamma\frac{dX^\mu}{dt} = \gamma v^\mu, \quad \dot{u}^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{d\gamma}{dt}v^\mu + \gamma^2 a^\mu, \tag{3.86}$$

where v^μ and a^μ are the velocity and acceleration measured in the coordinate time t, and γ is the Lorentz factor. These are defined by

$$v^\mu = \frac{dX^\mu}{dt} = \left[1, \frac{d\boldsymbol{X}}{dt}\right]^\mathrm{T}, \quad a^\mu = \frac{dv^\mu}{dt} = \left[0, \frac{d^2\boldsymbol{X}}{dt^2}\right]^\mathrm{T}, \quad \gamma = \frac{1}{\sqrt{-v^2}} = \frac{1}{\sqrt{1 - \boldsymbol{v}^2}}. \tag{3.87}$$

We then determine the following retarded potential and its field strength as

$$A^\mu(x) = \frac{e}{4\pi}\frac{v^\mu(t_\mathrm{r})}{(x - X(t_\mathrm{r}))\cdot v(t_\mathrm{r})}, \tag{3.88}$$

$$F^{\mu\nu}_\mathrm{v} = -\frac{e}{4\pi}\frac{(x^\mu - X^\mu(t_\mathrm{r}))v^\nu(t_\mathrm{r}) - (x^\nu - X^\nu(t_\mathrm{r}))v^\mu(t_\mathrm{r})}{\gamma^2[(x - X(t_\mathrm{r}))\cdot v(t_\mathrm{r})]^3}, \tag{3.89}$$

$$\begin{aligned} F^{\mu\nu}_\mathrm{a} = {} & \frac{e}{4\pi[(x - X(t_\mathrm{r}))\cdot v(t_\mathrm{r})]^2}\Big[(x^\mu - X^\mu(t_\mathrm{r}))\Big(a^\nu(t_\mathrm{r}) - \frac{(x - X(t_\mathrm{r}))\cdot a(t_\mathrm{r})}{(x - X(t_\mathrm{r}))\cdot v(t_\mathrm{r})}v^\nu(t_\mathrm{r})\Big) \\ & - (x^\nu - X^\nu(t_\mathrm{r}))\Big(a^\mu(t_\mathrm{r}) - \frac{(x - X(t_\mathrm{r}))\cdot a(t_\mathrm{r})}{(x - X(t_\mathrm{r}))\cdot v(t_\mathrm{r})}v^\mu(t_\mathrm{r})\Big)\Big], \end{aligned} \tag{3.90}$$

where the retarded time $t_{\rm r}$ is given by

$$-(t-t_{\rm r})+|\boldsymbol{x}-\boldsymbol{X}(t_{\rm r})|=0. \tag{3.91}$$

The field strength $F_{\rm v}^{\mu\nu}$ independent of acceleration has the longitudinal mode of the retarded field. In fact, the inner product of the unit vector $\boldsymbol{n}=(\boldsymbol{x}-\boldsymbol{X}(t_{\rm r}))/|\boldsymbol{x}-\boldsymbol{X}(t_{\rm r})|$ in the propagation direction and the electric field $\boldsymbol{E}_{\rm v}$ with $E^i_{\rm v}=F^{0i}_{\rm v}$ does not vanish, $\boldsymbol{n}\cdot\boldsymbol{E}_{\rm v}\neq 0$. The field strength $F_{\rm a}^{\mu\nu}$ proportional to the acceleration only has the transverse modes of the retarded field. This is because the propagation direction vector $\boldsymbol{n}$, the electric field $\boldsymbol{E}_{\rm a}$ with $E^i_{\rm a}=F^{0i}_{\rm a}$ and the magnetic field $\boldsymbol{B}_{\rm a}$ with $B^i_{\rm a}=\varepsilon^{0i}{}_{jk}F^{jk}_{\rm a}/2$ ($\varepsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor) satisfy

$$\boldsymbol{n}\cdot\boldsymbol{E}_{\rm a}=F^{0i}_{\rm a}n_i=\frac{F^{0\mu}_{\rm a}(x_\mu-X_\mu(t_{\rm r}))}{|\boldsymbol{x}-\boldsymbol{X}(t_{\rm r})|}=0, \tag{3.92}$$

$$\boldsymbol{n}\cdot\boldsymbol{B}_{\rm a}=\frac{1}{2}\varepsilon^{0i}{}_{jk}F^{jk}_{\rm a}n_i=\frac{\varepsilon^{0}{}_{\mu\nu\rho}F^{\nu\rho}_{\rm a}(x^\mu-X^\mu(t_{\rm r}))}{2|\boldsymbol{x}-\boldsymbol{X}(t_{\rm r})|}=0, \tag{3.93}$$

where the last equality of the first equation holds by the light cone condition $-(t-t_{\rm r})+|\boldsymbol{x}-\boldsymbol{X}(t_{\rm r})|=0$.

With the above knowledge, we next discuss the origin of the second terms in (3.65) and (3.76) computed from the quantity Φ. We derived those terms by assuming the regime $D\gg cT\gg L$ for each case of the linear and parallel configurations. The regime $D\gg cT$ is regarded as the wave zone in which the distance between two charged particles D is much larger than the wavelength of the photon field $\lambda_p=cT$ emitted from each charged particle. Hence it is important to understand how the radiative field affects the quantity Φ. Let us revisit the formula (3.54) of Φ expressed in terms of the field strengths,

$$\Phi=\frac{e}{4}\Big(\int_{\rm S_A}d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\rm B}(x)+\int_{\rm S_B}d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\rm A}(x)\Big), \tag{3.94}$$

where S_i is the surface surrounded by the spacetime trajectories of the particle $i(={\rm A},{\rm B})$, and $\Delta F^{\mu\nu}_i=F^{\mu\nu}_{i\rm R}-F^{\mu\nu}_{i\rm L}$. Here, $F^{\mu\nu}_{i\rm P}=\partial^\mu A^\nu_{i\rm P}-\partial^\nu A^\mu_{i\rm P}$ are the retarded field strengths of the charged particle i moving along the trajectory ${\rm P}(={\rm R},{\rm L})$. As mentioned in the above paragraph, the field strengths of the particle i moving the trajectory P, $F^{\mu\nu}_{i\rm P}$, are separated into two parts $F^{\mu\nu}_{i\rm P}=F^{\mu\nu}_{i\rm P,v}+F^{\mu\nu}_{i\rm P,a}$, and then the quantity Φ is also given as $\Phi=\Phi_{\rm v}+\Phi_{\rm a}$

with

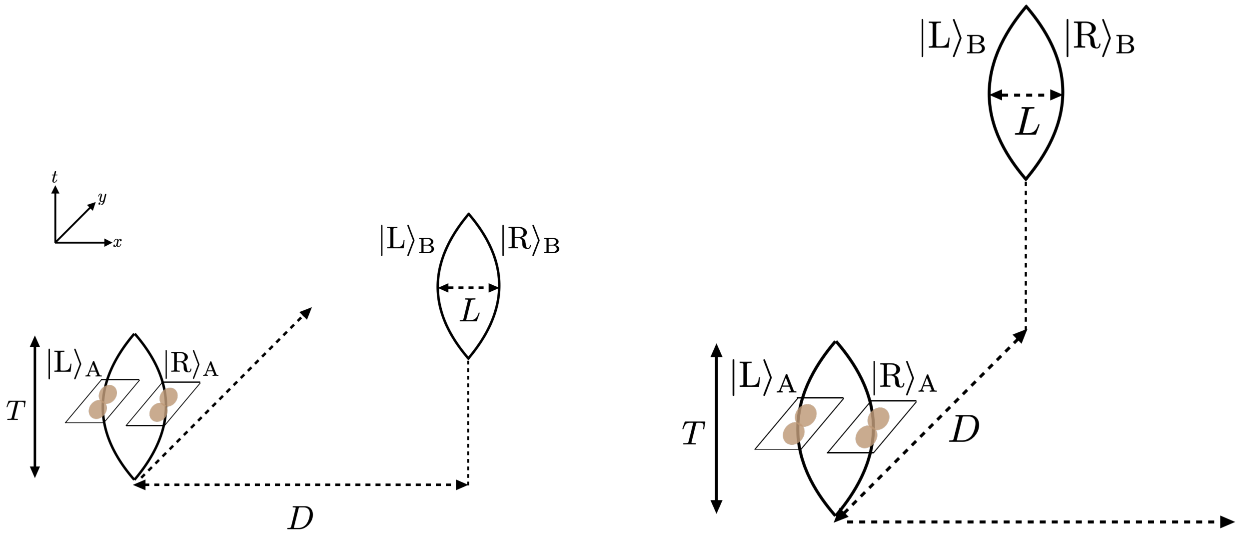
$$\Phi_{\rm v} = \frac{e}{4}\Big(\int_{\rm S_A} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm B,v}(x) + \int_{\rm S_B} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm A,v}(x)\Big), \tag{3.95}$$

$$\Phi_{\rm a} = \frac{e}{4}\Big(\int_{\rm S_A} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm B,a}(x) + \int_{\rm S_B} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm A,a}(x)\Big), \tag{3.96}$$

where $\Delta F^{\mu\nu}_{i,\rm v} = F^{\mu\nu}_{i\rm R,v} - F^{\mu\nu}_{i\rm L,v}$ and $\Delta F^{\mu\nu}_{i,\rm a} = F^{\mu\nu}_{i\rm R,a} - F^{\mu\nu}_{i\rm L,a}$. The term $\Phi_{\rm v}$ depends on the longitudinal mode (non-dynamical part) of the retarded electromagnetic field, and $\Phi_{\rm a}$ comes from the transverse modes (dynamical parts) of the retarded electromagnetic field of the accelerated charged particles. In the linear and parallel configurations, $\Phi_{\rm v}$ for the regime $D \gg cT \gg L$ has the same formula (see (B.19) and (B.35)), whereas $\Phi_{\rm a}$ for the regime $D \gg cT \gg L$ depends on each configuration: $\Phi_{\rm a}$ vanishes in the linear configuration, but it does not in the parallel configuration. To observe this, we focus on the fact that $\Phi_{\rm a}$ in the configurations shown in Fig. 15 is given as

$$\Phi_{\rm a} = \frac{e}{4}\int_{\rm S_B} dt dx \Delta F^{01}_{\rm A,a} = \frac{e}{4}\int_{\rm S_B} dt dx (E^x_{{\rm AR},a} - E^x_{\rm AL,a}), \tag{3.97}$$

where $E^x_{\rm AP,a} = F^{01}_{\rm AP,a}$ is the x-component of the electric field induced by the accelerated motion of the charged particle A on the trajectory P $(= \rm R, L)$. Here, the first term in the formula of $\Phi_{\rm a}$ in (3.96) vanished by assuming that the retarded field sourced by particle B is causally disconnected with particle A. Following the Larmor radiation formula, the electromagnetic wave emitted from the charged particle A cannot propagate in the direction of the particle acceleration [90]. The shaded region in Fig. 15 shows the angular distribution of the photon field of the charged particle A on each trajectory. In the linear configuration, because each particle moves along the x-axis, the electromagnetic wave from particle A does not propagate to particle B. This leads to $E^x_{\rm AR,a} = E^x_{\rm AL,a} = 0$ and hence $\Phi_{\rm a} = 0$. In the parallel configuration, because the electromagnetic wave from particle A can reach particle B, the electric fields $E^x_{\rm AR,a}$ and $E^x_{\rm AL,a}$ generated by the superposed particle A give a nontrivial $\Phi_{\rm a}$. Hence, the origin of $\Phi_{\rm a}$ is regarded as the quantum superposition of bremsstrahlung from the charged particle B in a superposition state. As observed in the previous section, the quantity $\Phi(= \Phi_{\rm v} + \Phi_{\rm a})$ decreases the minimum eigenvalue $\lambda_{\rm min}$. This suggests that the effect of the quantum superposition of bremsstrahlung appears in the formula of the entanglement. As observed in the previous section, the decoherence due to the vacuum fluctuation of the



(a) Linear configuration

(b) Parallel configuration

FIG 15: Angular distribution of the photon field induced by each trajectory of the accelerating charged particle A for linear configuration (a) and parallel configuration (b) on the $x - y$ plane at a constant time.

photon field suppresses the entanglement generation in the charged particles.

4 Quantum uncertainty of field in superposed particles

In this section, we reveal how the properties of relativity and quantum mechanics are consistent based on the discussion of a gedanken experiment based on Refs. [2, 3]. In particular, we focus on the consistency between causality and complementarity.

The gedanken experiment involving the quantum superposition of a massive object, as discussed in Refs. [54, 55, 56, 89, 91, 92], has garnered considerable attention. In the gedanken experiment, the quantum superposition of the gravitational potential induced by the object leads to inconsistency between causality and complementarity. This inconsistency is resolved by considering the quantized dynamical degrees of freedom of electromagnetic/gravitational field [54, 55, 56, 92]. A deep understanding of the gedanken experiment may allow one to clarify the manner by which the quantum nature of the gravitational potential correlates with the quantization of the gravitational field.

4.1 Brief Review of the gedanken experiment

We consider two quantum systems (Alice's particle and Bob's particle) separated by a distance D interacting through the electromagnetic/gravitational potential (Fig. 16). In Alice's system, her particle is prepared in a quantum superposition of two locations and starts to recombine during time $T_{\rm A}$. At $t = T_{\rm A}$, Alice performs an interference experiment and assesses whether it will be successful (whether the interference pattern of her particle will be observed). If the superposition state of Alice's particle is preserved, then the interference experiment will be successful; however, if the superposition state is not preserved, then the experiment will not be successful. In Bob's system, Bob chooses whether he releases his particle or not at $t = 0$. When he releases his particle, it is affected by the electromagnetic/gravitational potential due to Alice's particle and is thus displaced. Because Alice's particle is in the superposition of the two paths, the magnitude of the potential perceived by Bob's particle changes depending on the path traversed by her particle. Thus, Bob can use his particle to measure which-path Alice's particle took.

Let us assume that Alice's interference experiment during the time $T_{\rm A}$ and Bob's choice and measurement during the time $T_{\rm B}$ are performed in a spacelike separated region satisfying $D > T_{\rm A}$ and $D > T_{\rm B}$ (Fig. 16). If Bob releases his particle and can measure the position of Alice's particle, then, by complementarity, the superposition state of Alice's particle collapses and the particle decoheres. Thus, the interference experiment is not successful. By

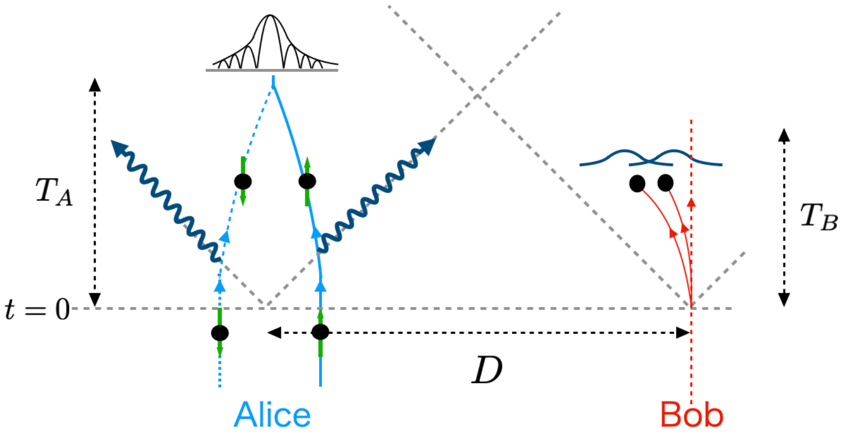


FIG 16: Setup for the gedanken experiment. D represents the distance between Alice's system and Bob's system. T_{A} is a time scale for recombining particle A, and T_{B} is a time scale that particle B in Bob's system will be superposed when he released it. Here, we assume $D > T_{\mathrm{A}}$ and $D > T_{\mathrm{B}}$, in which Alice and Bob perform their actions in spacelike separated regions.

contrast, when Bob decides not to release his particle and does not measure the path undertaken by Alice's particle, then her particle will preserve the superposition state and her interference experiment will be successful. This indicates that causality is violated because Bob's choice is known by Alice when her particle is in a region where his actions have no influence causally. However, if the causality holds, then Alice's interference experiment is successful (she observes the interference pattern of her particle). In this case, without decohering Alice's particle, Bob can use his released particle to obtain the which-path information of her particle. This results in a violation in complementarity. The inconsistency between causality and complementarity can be resolved by considering the vacuum fluctuations of the electromagnetic/gravitational field and the emissions of photons/gravitons, which was demonstrated as an order estimation in Refs. [54, 55, 92]. Bob's measurement to acquire the which-path information of Alice's particle is limited by the vacuum fluctuations of a quantized electromagnetic/gravitational field, and Alice's interference experiment fails because of the decoherence induced by the entangling radiation of photons/gravitons. Here, the entangling radiation refers to the radiation emitted from and entangled with Alice's particle. This suggests that a quantized electromagnetic/gravitational field is sufficient to avoid the inconsistency between causality and complementarity.

Here we reanalyze the consistency between causality and complementarity by assuming a situation similar to that in Fig. 16. This is an extension of the study explained in Sec. 3, which investigated the effect of vacuum fluctuations of a photon field on the electromagnetic version of the BMV proposal. We consider the electromagnetic and gravitational versions of a similar gedanken experiment based on QED and the quantum theory of linearized gravity, respectively. Finally, we also discuss the relationship among the causality, complementarity, and the entanglement between two particles via negativity (3.48).

4.2 Extension to quantum theory of linearized gravity version

Here we consider two charged/massive particles, A and B, which are non-relativistic and obey the framework of quantum mechanics; the electromagnetic/gravitational field coupled to the particles is assumed to be a quantum field.

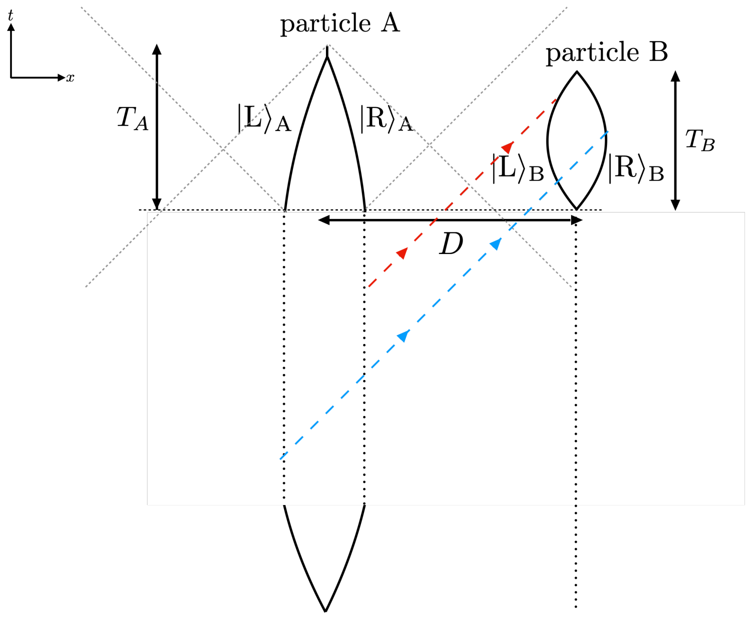


FIG 17: Configuration of our model. We specify regimes $D > T_A$ and $D > T_B$, in which the retarded Green's function propagating from particle B to A vanishes. Particle A traverse via the right or left path $|R\rangle_A(|L\rangle_A)$ and induces an electromagnetic/gravitational field along each path (as shown by the dashed red or blue line). The retarded field caused by particle A affects particle B traversing via the left ($|L\rangle_B$) or right ($|R\rangle_B$) path.

We first review the result of the system of two charged particle coupled with an electromagnetic field discussed in Sec. 3. The initial state of the particles defined as (3.33) is assumed to be each in spatially localized superposition (Fig. 17), which might be realized via the Stern-Gerlach effect, as explained in Sec. 2.3.3. Additionally, we assume that no initial entanglement occurs between the particles and the electromagnetic/gravitational field. [1] Furthermore, the wave packets of these particles are sufficiently far apart to form local paths within each device.

Then, we can define the current of particle A (B) as $J^\mu_{\mathrm{AP}}(x)$ ($J^\mu_{\mathrm{BQ}}(x)$) localized around their paths of P (Q)($= R, L$). Under the assumptions above, as shown in Sec. 3.1.2, the decoherence and the entanglement between the two particles for the electromagnetic version can be described by the following quantities (4.1)−(4.3)

$$\begin{aligned}
\Gamma^{\mathrm{EM}}_i &= \frac{1}{4}\int d^4x \int d^4y \Delta J^\mu_i(x)\Delta J^\nu_i(y)\langle\{\hat{A}^{\mathrm{I}}_\mu(x), \hat{A}^{\mathrm{I}}_\nu(y)\}\rangle \\
&= \frac{1}{4}\langle\{\hat{\phi}^{\mathrm{EM}}_i, \hat{\phi}^{\mathrm{EM}}_i\}\rangle = \frac{1}{2}\langle 0|(\hat{\phi}^{\mathrm{EM}}_i)^2|0\rangle, \qquad (4.1)\\
\Gamma^{\mathrm{EM}}_{\mathrm{c}} &= \frac{1}{2}\int d^4x \int d^4y \Delta J^\mu_{\mathrm{A}}(x)\Delta J^\nu_{\mathrm{B}}(y)\langle\{\hat{A}^{\mathrm{I}}_\mu(x), \hat{A}^{\mathrm{I}}_\nu(y)\}\rangle = \frac{1}{2}\langle\{\hat{\phi}^{\mathrm{EM}}_{\mathrm{A}}, \hat{\phi}^{\mathrm{EM}}_{\mathrm{B}}\}\rangle, \qquad (4.2)\\
\Phi^{\mathrm{EM}} &= \frac{1}{2}\int d^4x d^4y \Big\{\Delta J^\mu_{\mathrm{A}}(x)\Delta J^\nu_{\mathrm{B}}(y) + \Delta J^\mu_{\mathrm{B}}(x)\Delta J^\nu_{\mathrm{A}}(y)\Big\} G^{\mathrm{r}}_{\mu\nu}(x,y) \\
&= \frac{1}{2}(\Phi^{\mathrm{EM}}_{\mathrm{AB}} + \Phi^{\mathrm{EM}}_{\mathrm{BA}}), \qquad (4.3)
\end{aligned}$$

where we defined $\Delta J^\mu_i = J^\mu_{i\mathrm{R}} - J^\mu_{i\mathrm{L}}$ with $i = \mathrm{A}, \mathrm{B}$. The operators $\hat{\phi}^{\mathrm{EM}}_i$ introduced in Eq. (3.30) describes the phase shifts due to the quantum fluctuations of the electromagnetic field, respectively. Here, $\langle\big\{\hat{A}^{\mathrm{I}}_\mu(x), \hat{A}^{\mathrm{I}}_\mu(y)\big\}\rangle$ and $G^{\mathrm{r}}_{\mu\nu}(x,y)$ are the two-point function of the vacuum state $|0\rangle$ and the retarded Green's function with respect to the quantized electromagnetic field in the interaction picture. The quantities $\Phi^{\mathrm{EM}}_{\mathrm{AB}}$ and $\Phi^{\mathrm{EM}}_{\mathrm{BA}}$ are defined as

$$\begin{aligned}
\Phi^{\mathrm{EM}}_{\mathrm{AB}} &= \int d^4x d^4y \Delta J^\mu_{\mathrm{A}}(x)\Delta J^\nu_{\mathrm{B}}(y) G^{\mathrm{r}}_{\mu\nu}(x,y), \\
\Phi^{\mathrm{EM}}_{\mathrm{BA}} &= \int d^4x d^4y \Delta J^\mu_{\mathrm{B}}(x)\Delta J^\nu_{\mathrm{A}}(y) G^{\mathrm{r}}_{\mu\nu}(x,y). \qquad (4.4)
\end{aligned}$$

These results are equivalent to Eqs (3.49), (3.50), and (3.51), and we subscript "EM" to

[1] In the case of gravity, Alice's particle may be entangled with her apparatus because of the conservation of energy-momentum. For a rigorous description, we must consider the effects induced in a laboratory, as discussed in [93]. However, the authors of [54] argued that laboratory effects need not be considered because the state of the laboratory does not produce a significant decoherence. The authors of [94] discussed the effect of the gravitational potential arising from the apparatus in a laboratory on the relative phases that cause entanglement between two particles. If the laboratory apparatus is sufficiently heavy, then it does not shift significantly. Thus the apparatus will not be superposed state and not contribute to the relative phases. Moreover, if the gravitational field created by the apparatus is homogeneous, then the phase shift due to its field will also not affect the relative phase.

emphasize that this is the case for electromagnetic fields. Using these quantities, the formula of the minimum eigenvalue $\lambda_{\min}^{\mathrm{EM}}$

$$\lambda_{\min}^{\mathrm{EM}} = \frac{1}{4}\Big[1 - e^{-\Gamma_{\mathrm{A}}^{\mathrm{EM}} - \Gamma_{\mathrm{B}}^{\mathrm{EM}}} \cosh[\Gamma_{\mathrm{c}}^{\mathrm{EM}}] - \Big\{ \big(e^{-\Gamma_{\mathrm{A}}^{\mathrm{EM}}} - e^{-\Gamma_{\mathrm{B}}^{\mathrm{EM}}}\big)^2 + 4e^{-\Gamma_{\mathrm{A}}^{\mathrm{EM}} - \Gamma_{\mathrm{B}}^{\mathrm{EM}}} \sin^2\left[\frac{\Phi^{\mathrm{EM}}}{2}\right] + e^{-2\Gamma_{\mathrm{A}}^{\mathrm{EM}} - 2\Gamma_{\mathrm{B}}^{\mathrm{EM}}} \sinh^2[\Gamma_{\mathrm{c}}^{\mathrm{EM}}] \Big\}^{\frac{1}{2}} \Big]. \tag{4.5}$$

Later, we extend the above formula to the gravitational version to judge whether two massive particles are entangled or not by computing the negativity.

Then we consider the system of two massive particles, which interacts with a quantized gravitational field. To achieve this, we focus on the similarity between the electromagnetic and gravitational fields, and naively extend the results presented the above Eqs. (4.1), (4.2), and (4.3) to a gravitational field. In this extension, we introduce several important assumptions. We consider a linearized regime of gravity by expanding the metric of spacetime around the Minkowski spacetime metric $\eta_{\mu\nu}$. The full spacetime metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the metric perturbation satisfying $|h_{\mu\nu}| \ll 1$.

This metric perturbation is justified as follows: Let us now consider the gravitational interaction of two particles with the same masses m. Based on the linearized gravity theory [95], the energy-momentum tensor of a particle $\mathcal{T}_{\mu\nu}$ induces the fluctuation part of the metric

$$h_{\mu\nu} \sim G \int d^3y \frac{\mathcal{T}_{\mu\nu}(t_{\mathrm{r}}, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{4.6}$$

Here $t_{\mathrm{r}} = t - |\boldsymbol{x} - \boldsymbol{y}|$ is the retarded time, which represents the delay with respect to propagation from the source point $\boldsymbol{y}$ to a spacetime point $\boldsymbol{x}$. The components of $h_{\mu\nu}$ are evaluated as

$$h_{00} \sim G\frac{m}{R}, \quad h_{0i} \sim h_{00}\Big(\frac{L}{T}\Big)\boldsymbol{e}_i, \quad h_{ij} \sim h_{00}\Big(\frac{L}{T}\Big)^2 \boldsymbol{e}_i\boldsymbol{e}_j, \tag{4.7}$$

where R characterizes the typical length scale of particle A and B satisfying $R \lesssim L$. $(L/T)\boldsymbol{e}_i$ denote the characteristic velocity of the system with the i direction of the unit vector $\boldsymbol{e}_i$. We regard the length scale R as the typical size of the particle. Considering the non-relativistic condition $L/T \ll 1$, the condition $|h_{\mu\nu}| \ll 1$ is valid when

$$1 \gg G\frac{m}{L} = \frac{g}{mL} = g\frac{\lambda_{\rm C}}{L} \tag{4.8}$$

is satisfied. Here we introduced $g = Gm^2$ of the coupling constant between the two particles with the gravitational constant G. $\lambda_{\rm C} = 1/m$ is the Compton wave length of the two particles. In the following analysis, we choose the parameter with which the above condition (4.8) is satisfied. Similar to the electromagnetic case, we can define the energy-momentum tensors of particle A (B) as $\mathcal{T}^{\mu\nu}_{\rm AP}(x)$ ($\mathcal{T}^{\mu\nu}_{\rm BQ}(x)$) localized around their paths of P (Q)$(= R, L)$, and the gravitational field version of the quantities $\Gamma^{\rm EM}_i$, $\Gamma^{\rm EM}_{\rm c}$, and $\Phi^{\rm EM}$ are presented as follows:

$$\begin{aligned}
\Gamma^{\rm GR}_i &= \frac{1}{4}\int d^4x \int d^4y \Delta\mathcal{T}^{\mu\nu}_i(x)\Delta\mathcal{T}^{\rho\sigma}_i(y)\langle\{\hat{h}^{\rm I}_{\mu\nu}(x), \hat{h}^{\rm I}_{\rho\sigma}(y)\}\rangle \\
&= \frac{1}{4}\langle\{\hat{\phi}^{\rm GR}_i, \hat{\phi}^{\rm GR}_i\}\rangle = \frac{1}{2}\langle 0|(\hat{\phi}^{\rm GR}_i)^2|0\rangle, && (4.9)\\
\Gamma^{\rm GR}_{\rm c} &= \frac{1}{2}\int d^4x \int d^4y \Delta\mathcal{T}^{\mu\nu}_{\rm A}(x)\Delta\mathcal{T}^{\rho\sigma}_{\rm B}(y)\langle\{\hat{h}^{\rm I}_{\mu\nu}(x), \hat{h}^{\rm I}_{\rho\sigma}(y)\}\rangle = \frac{1}{2}\langle\{\hat{\phi}^{\rm GR}_{\rm A}, \hat{\phi}^{\rm GR}_{\rm B}\}\rangle, && (4.10)\\
\Phi^{\rm GR} &= \frac{1}{2}\int d^4x d^4y \Big\{\Delta\mathcal{T}^{\mu\nu}_{\rm A}(x)\Delta\mathcal{T}^{\rho\sigma}_{\rm B}(y) + \Delta\mathcal{T}^{\mu\nu}_{\rm B}(x)\Delta\mathcal{T}^{\rho\sigma}_{\rm A}(y)\Big\}G^{\rm r}_{\mu\nu\rho\sigma}(x,y) \\
&= \frac{1}{2}(\Phi^{\rm GR}_{\rm AB} + \Phi^{\rm GR}_{\rm BA}), && (4.11)
\end{aligned}$$

where we defined $\Delta\mathcal{T}^{\mu\nu}_i = \mathcal{T}^{\mu\nu}_{i\rm R} - \mathcal{T}^{\mu\nu}_{i\rm L}$ with $i = {\rm A, B}$. The operators $\hat{\phi}^{\rm GR}_i$ also describes the phase shifts due to the quantum fluctuations of gravitational field. This is explicitly expressed as

$$\hat{\phi}^{\rm GR}_i = \int d^4x \Delta\mathcal{T}^{\mu\nu}_i(x)\hat{h}^{\rm I}_{\mu\nu}(x). \tag{4.12}$$

Here, $\langle\big\{\hat{h}^{\rm I}_{\mu\nu}(x), \hat{h}^{\rm I}_{\mu\nu}(y)\big\}\rangle$ and $G^{\rm r}_{\mu\nu\rho\sigma}(x,y)$ are the gravitational version of the two point function and retarded Green's function, respectively [96]. The gravitational version of the quantities $\Phi^{\rm GR}_{\rm AB}$ and $\Phi^{\rm GR}_{\rm BA}$ can be similarly expressed as

$$\begin{aligned}
\Phi^{\rm GR}_{\rm AB} &= \int d^4x d^4y \Delta\mathcal{T}^{\mu\nu}_{\rm A}(x)\Delta\mathcal{T}^{\rho\sigma}_{\rm B}(y)G^{\rm r}_{\mu\nu\rho\sigma}(x,y), \\
\Phi^{\rm GR}_{\rm BA} &= \int d^4x d^4y \Delta\mathcal{T}^{\mu\nu}_{\rm B}(x)\Delta\mathcal{T}^{\rho\sigma}_{\rm A}(y)G^{\rm r}_{\mu\nu\rho\sigma}(x,y). && (4.13)
\end{aligned}$$

The gravitational version of the minimum eigenvalue $\lambda^{\rm GR}_{\rm min}$ can be obtained by replacing $\Gamma^{\rm EM}_i$, $\Gamma^{\rm EM}_{\rm c}$, and $\Phi^{\rm EM}$ in the result above (4.5) with $\Gamma^{\rm GR}_i$, $\Gamma^{\rm GR}_{\rm c}$, and $\Phi^{\rm GR}$, respectively, as shown:

$$\lambda^{\rm GR}_{\rm min} = \frac{1}{4}\Big[1 - e^{-\Gamma^{\rm GR}_{\rm A}-\Gamma^{\rm GR}_{\rm B}} \cosh[\Gamma^{\rm GR}_{\rm c}] - \Big\{\big(e^{-\Gamma^{\rm GR}_{\rm A}} - e^{-\Gamma^{\rm GR}_{\rm B}}\big)^2 + 4e^{-\Gamma^{\rm GR}_{\rm A}-\Gamma^{\rm GR}_{\rm B}} \sin^2\left[\frac{\Phi^{\rm GR}}{2}\right] + e^{-2\Gamma^{\rm GR}_{\rm A}-2\Gamma^{\rm GR}_{\rm B}} \sinh^2[\Gamma^{\rm GR}_{\rm c}]\Big\}^{\frac{1}{2}}\Big], \tag{4.14}$$

where $\Gamma^{\rm GR}_i$, $\Gamma^{\rm GR}_{\rm c}$, and $\Phi^{\rm GR}$ are defined as shown in Eqs. (4.9), (4.10), and (4.11). The results of Eqs. (4.5) and (4.14) are extended as $\lambda_{\rm min}$ presented as Eq. (3.48) herein.

In the following, we present the inequality representing complementarity, the uncertainty relation, and one of the entanglement measure: negativity. The inequality, the uncertainty relation, and the negativity for the electromagnetic case are evaluated from the quantum state of the charged particles determined using $\Gamma^{\rm EM}_i\,(i = {\rm A}, {\rm B})$, $\Gamma^{\rm EM}_{\rm c}$, $\Phi^{\rm EM}_{\rm AB}$ and $\Phi^{\rm EM}_{\rm BA}$. By replacing these quantities with $\Gamma^{\rm GR}_i\,(i = {\rm A}, {\rm B})$, $\Gamma^{\rm GR}_{\rm c}$, $\Phi^{\rm GR}_{\rm AB}$ and $\Phi^{\rm GR}_{\rm BA}$, we obtain the formulas for gravitational case. Subsequently, we adopt simple notations $\Gamma_i\,(i = {\rm A}, {\rm B})$, $\Gamma_{\rm c}$, $\Phi_{\rm AB}$ and $\Phi_{\rm BA}$ to describe the quantities above for the electromagnetic and gravitational cases in a unified manner.

4.3 Complementarity inequality in QED

4.3.1 A brief proof of the complementarity inequality

Here, we present the QED results for the complementarity inequality. We first introduce the visibility $\mathcal{V}^{\rm EM}_{\rm A}$ of charged particle A and the distinguishability $\mathcal{D}^{\rm EM}_{\rm B}$ which quantifies the which-path information of particle A acquired through charged particle B. These two quantities are useful for expressing complementarity. Additionally, we discuss the relationship with the Robertson inequality in the last subsection. According to Refs. [97, 98], the visibility $\mathcal{V}^{\rm EM}_{\rm A}$ and the distinguishability $\mathcal{D}^{\rm EM}_{\rm B}$ satisfy the inequality,

$$(\mathcal{V}^{\rm EM}_{\rm A})^2 + (\mathcal{D}^{\rm EM}_{\rm B})^2 \leq 1. \tag{4.15}$$

This inequality expresses the complementarity: if the distinguishability is unity, $D^{\rm EM}_{\rm B} = 1$, the visibility $\mathcal{V}^{\rm EM}_{\rm A}$ vanishes, and if the visibility is unity, $\mathcal{V}^{\rm EM}_{\rm A} = 1$, the distinguishability $\mathcal{D}^{\rm EM}_{\rm B}$ vanishes. In the following, we give a simple proof of the above inequality by using the definitions of visibility and distinguishability:

Proof. We prove the inequality (4.15) between visibility and distinguishability. First, we derive the visibility for the state given in

$$|\Psi(T)\rangle = \frac{1}{2}\sum_{\mathrm{P,Q=R,L}} |\mathrm{P_f}\rangle_{\mathrm{A}}|\mathrm{Q_f}\rangle_{\mathrm{B}}\, e^{-i\hat{H}_0 T}\hat{U}_{\mathrm{PQ}}|\alpha\rangle_{\mathrm{EM}}$$
$$= \frac{1}{\sqrt{2}}|\mathrm{R_f}\rangle_{\mathrm{A}}|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,EM}} + \frac{1}{\sqrt{2}}|\mathrm{L_f}\rangle_{\mathrm{A}}|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,EM}}, \tag{4.16}$$

where we rewrite the state (3.36) for later convenience and defined

$$|\Omega_{\mathrm{P}}\rangle_{\mathrm{B,EM}} = \frac{1}{\sqrt{2}}\sum_{\mathrm{Q=R,L}} |\mathrm{Q_f}\rangle_{\mathrm{B}} e^{-i\hat{H}_0 T}\hat{U}_{\mathrm{PQ}}|\alpha\rangle_{\mathrm{EM}}. \tag{4.17}$$

The visibility of charged particle A is calculated with respect to the reduced density matrix $\rho_{\mathrm{A}}^{\mathrm{EM}}$ as

$$\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}} = 2|{}_{\mathrm{A}}\langle \mathrm{L_f}|\rho_{\mathrm{A}}^{\mathrm{EM}}|\mathrm{R_f}\rangle_{\mathrm{A}}|$$
$$= 2|\mathrm{Tr}_{\mathrm{B,EM}}[{}_{\mathrm{A}}\langle \mathrm{L_f}|\Psi(T)\rangle\langle\Psi(T)|\mathrm{R_f}\rangle_{\mathrm{A}}]|$$
$$= |{}_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}| \equiv |\alpha|. \tag{4.18}$$

We next evaluate the distinguishability of charged particle B. For a trace distance $\mathcal{D}(\rho,\sigma)$ with arbitrary density operators ρ and σ, we use the fact that the trace-preserving quantum operations are contractive [99]:

$$\mathcal{D}(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \leq \mathcal{D}(\rho,\sigma), \tag{4.19}$$

where $\mathcal{E}$ is a trace-preserving quantum operation. This inequality means that the operation $\mathcal{E}$ makes it difficult to distinguish between the two quantum states ρ and σ, i.e., the trace distance does not increase. Then, the distinguishability is bounded as

$$\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}} = \frac{1}{2}\mathrm{Tr_B}|\rho_{\mathrm{BR}}^{\mathrm{EM}} - \rho_{\mathrm{BL}}^{\mathrm{EM}}|$$
$$= \frac{1}{2}\mathrm{Tr_B}|\mathrm{Tr_{EM}}[|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}|] - \mathrm{Tr_{EM}}[|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|]|$$
$$\leq \frac{1}{2}\mathrm{Tr_B}||\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}||, \tag{4.20}$$

where the inequality (4.19) was used in the third line because the partial trace is a trace-preserving quantum operation. Here we defined $\rho_{\mathrm{BP}}^{\mathrm{EM}} = \mathrm{Tr}_{\mathrm{EM}}[|\Omega_{\mathrm{P}}\rangle_{\mathrm{B,EM}}\langle\Omega_{\mathrm{P}}|]$ with $\mathrm{P} = \mathrm{R}, \mathrm{L}$ and, the trace distance $\mathrm{Tr}|\hat{O}| = \sum_i |\lambda_i|$ is given by the eigenvalues λ_i of a Hermitian operator $\hat{O}$. The general property of the trace distance is presented in [99]. The density operator $\rho_{\mathrm{BP}}^{\mathrm{EM}}$ characterizes the state of particle B when particle A moves along the path P. The vector $|\Omega_{\mathrm{P}}\rangle_{\mathrm{B,EM}}$ describes the composite state of particle B and the electromagnetic field when particle A moves along the path P and is introduced by rewriting the state (4.17). If the distinguishability vanishes, $\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}} = 0$, and the two density operators $\rho_{\mathrm{BR}}^{\mathrm{EM}}$ and $\rho_{\mathrm{BL}}^{\mathrm{EM}}$ are identical. This means that Bob cannot know which trajectory particle A has taken from the state of particle B. However, if $\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}} = 1$, the density operators $\rho_{\mathrm{BR}}^{\mathrm{EM}}$ and $\rho_{\mathrm{BL}}^{\mathrm{EM}}$ are orthogonal to each other ($\rho_{\mathrm{BR}}^{\mathrm{EM}}\rho_{\mathrm{BL}}^{\mathrm{EM}} = 0$). Then, by measuring the state of particle B, Bob can guess which trajectory particle A has passed through. In this sense, the distinguishability $\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}}$ quantifies the amount of which-path information of particle A. The meaning of the distinguishability mentioned above was discussed in [97]. To obtain the eigenvalues of the operator $|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|$, we define the orthonormal basis $\{|u_{\mathrm{A}}\rangle, |u_{\mathrm{B}}\rangle\}$ using the Gram-Schmidt orthonormalization as:

$$|u_{\mathrm{A}}\rangle = |\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}, \quad |u_{\mathrm{B}}\rangle = \frac{|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}} - \alpha|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}}{\sqrt{1-|\alpha|^2}}, \tag{4.21}$$

where the overlap α is defined in (4.18). In this basis, the operator $|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|$ can be rewritten as

$$\begin{aligned}|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}| &= |u_{\mathrm{A}}\rangle\langle u_{\mathrm{A}}| - (\alpha|u_{\mathrm{A}}\rangle + \sqrt{1-|\alpha|^2}|u_{\mathrm{B}}\rangle)(\alpha^*\langle u_{\mathrm{A}}| + \sqrt{1-|\alpha|^2}\langle u_{\mathrm{B}}|) \\ &= \begin{pmatrix} 1-|\alpha|^2 & \alpha\sqrt{1-|\alpha|^2} \\ \alpha^*\sqrt{1-|\alpha|^2} & -(1-|\alpha|^2) \end{pmatrix},\end{aligned} \tag{4.22}$$

in the orthonormal basis $\{|u_{\mathrm{A}}\rangle, |u_{\mathrm{B}}\rangle\}$. Thus, the eigenvalues of this matrix $\lambda_{\mathrm{A,B}}$ are

$$\lambda_{\mathrm{A}} = \sqrt{1-|\alpha|^2}, \quad \lambda_{\mathrm{B}} = -\sqrt{1-|\alpha|^2}, \tag{4.23}$$

and the distinguishability $\mathcal{D}_{\mathrm{B}}$ is suppressed by the sum of these eigenvalues as follows:

$$\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}} \leq \frac{1}{2}\mathrm{Tr}_{\mathrm{B}}||\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|| = \frac{1}{2}(|\lambda_{\mathrm{A}}| + |\lambda_{\mathrm{B}}|) = \sqrt{1-|\alpha|^2}. \tag{4.24}$$

Substituting (4.18) into (4.24), we find the relationship

$$(\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}})^2 + (\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}})^2 \leq 1. \tag{4.25}$$

Therefore, the visibility of charged particle A and the distinguishability of charged particle B follow the inequality (4.15).

4.3.2 Concrete computation of the visibility and distinguishability

Here we computed the visibility ($\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}}$) and the distinguishability ($\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}}$). The visibility $\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}}$ describes the extent to which the coherence of charged particle A remains when Alice performs an interference experiment. The distinguishability $\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}}$ characterizes how particle B can distinguish the path of particle A from the state of particle B. The visibility $\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}}$ of charged particle A is expressed as

$$\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}} = 2|{}_{\mathrm{A}}\langle \mathrm{L_f}|\rho_{\mathrm{A}}^{\mathrm{EM}}|\mathrm{R_f}\rangle_{\mathrm{A}}|. \tag{4.26}$$

The quantum state of particle A $\rho_{\mathrm{A}}^{\mathrm{EM}}$ is obtained by tracing out the degrees of freedom of particle B and the electromagnetic field:

$$\begin{aligned}\rho_{\mathrm{A}}^{\mathrm{EM}} &= \mathrm{Tr}_{\mathrm{B,EM}}[|\Psi(T)\rangle\langle\Psi(T)|] \\ &= \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{2}e^{-\Gamma_{\mathrm{A}}^{\mathrm{EM}}+i\Phi_{\mathrm{A}}^{\mathrm{EM}}}\left(e^{-i\int d^4x\Delta J_{\mathrm{A}}^{\mu}(x)A_{\mathrm{BR}\mu}(x)} + e^{-i\int d^4x\Delta J_{\mathrm{A}}^{\mu}(x)A_{\mathrm{BL}\mu}(x)}\right) \\ * & 1 \end{pmatrix},\end{aligned} \tag{4.27}$$

where we used the basis $\{|\mathrm{R_f}\rangle_{\mathrm{A}}, |\mathrm{L_f}\rangle_{\mathrm{A}}\}$ to represent the density matrix, and $*$ is the complex conjugate of the (R,L) component. $A_{i\mathrm{P}}^{\mu}(x) = \int d^4y G_{\mu\nu}^{r}(x,y)J_{i\mathrm{P}}^{\nu}(y)$ is the retarded potential with $i = \mathrm{A},\mathrm{B}$ and $\mathrm{P} = \mathrm{R},\mathrm{L}$ introduced in (3.45). The quantity $\Phi_{\mathrm{A}}^{\mathrm{EM}}$ is expressed as

$$\Phi_{\mathrm{A}}^{\mathrm{EM}} = \int d^4x\Delta J_{\mathrm{A}}^{\mu}(x)A_{\mu}(x) - \frac{1}{2}\int d^4xd^4y\Delta J_{\mathrm{A}}^{\mu}(x)(J_{\mathrm{AR}}^{\nu}(y) + J_{\mathrm{AL}}^{\nu}(y))G_{\mu\nu}^{\mathrm{r}}(x,y). \tag{4.28}$$

The density operator $\rho_{\mathrm{A}}^{\mathrm{EM}}$ is directly obtained by tracing out the degree of freedom of the particle B in the density matrix $\rho_{\mathrm{AB}}^{\mathrm{EM}}$ (3.38). The quantity $\Gamma_{\mathrm{A}}^{\mathrm{EM}}$ defined in Eq. (4.1) characterizes the decoherence effect due to the radiation of the on-shell photon emitted by particle A [4, 56]. The result (4.27) with the retarded electromagnetic field A_{BQ}^{μ} of particle B implies that the effect of particle B can propagate to Alice's system. However, in the spacelike case $D > T_{\rm A}$ and $D > T_{\rm B}$ (see Fig. 17), the photon field induced by particle B does not reach particle A, i.e., $A^{\mu}_{\rm BQ}(x) = 0$, i.e., $[\hat{A}_{\mu}(x), \hat{A}_{\nu}(y)] = 0$ due to micro causality $(x^0 - y^0)^2 - (\boldsymbol{x} - \boldsymbol{y})^2 < 0$. Thus, the density operator (4.27) becomes

$$\rho^{\rm EM}_{\rm A} = \frac{1}{2}\begin{pmatrix} 1 & e^{-\Gamma^{\rm EM}_{\rm A} + i\Phi^{\rm EM}_{\rm A}} \\ e^{-\Gamma^{\rm EM}_{\rm A} - i\Phi^{\rm EM}_{\rm A}} & 1 \end{pmatrix}. \tag{4.29}$$

This result indicates that the process of charged particle B during the time $T_{\rm B}$ does not affect the interference experiment on charged particle A by causality. Note that, given the law of charge conservation, we also have to consider the contribution from charged particle B before the time $T_{\rm B}$. Even by considering this, we can see that the density operator $\rho_{\rm A}$ does not depend on influences from spacelike separated regions. In the derivation of the above equations, for simplicity, we only discussed the contribution from particle B during the time $T_{\rm B}$. From the definition of the visibility (4.26) and (4.27), we obtain the visibility as follows:

$$\mathcal{V}^{\rm EM}_{\rm A} = e^{-\Gamma^{\rm EM}_{\rm A}} \left| \cos\left(\frac{\Phi^{\rm EM}_{\rm AB}}{2}\right) \right|, \tag{4.30}$$

which leads to $\mathcal{V}^{\rm EM}_{\rm A} = e^{-\Gamma^{\rm EM}_{\rm A}}$ in the region where $D > T_{\rm A}$ and $D > T_{\rm B}$.

Next, we compute the distinguishability $\mathcal{D}^{\rm EM}_{\rm B}$. The definition of the distinguishability $\mathcal{D}^{\rm EM}_{\rm B}$ is expressed as

$$\mathcal{D}^{\rm EM}_{\rm B} = \frac{1}{2}{\rm Tr}_{\rm B}|\rho^{\rm EM}_{\rm BR} - \rho^{\rm EM}_{\rm BL}|. \tag{4.31}$$

The eigenvalues of the density matrix $\rho^{\rm EM}_{\rm BR} - \rho^{\rm EM}_{\rm BL}$ are

$$\begin{aligned} \lambda_{\pm} &= \pm\frac{1}{2}\left| e^{-\Gamma^{\rm EM}_{\rm B} + i\Phi^{\rm EM}_{\rm B} - i\int d^4x \Delta J^{\mu}_{\rm B} A_{{\rm R}\mu}} - e^{-\Gamma^{\rm EM}_{\rm B} + i\Phi^{\rm EM}_{\rm B} - i\int d^4x \Delta J^{\mu}_{\rm B} A_{{\rm L}\mu}} \right| \\ &= \pm e^{-\Gamma^{\rm EM}_{\rm B}} \left| \sin\left(\frac{1}{2}\int d^4x \Delta J^{\mu}_{\rm B}(x) \Delta A_{{\rm A}\mu}(x)\right) \right|. \end{aligned} \tag{4.32}$$

Thus, the distinguishability is expressed as

$$\mathcal{D}^{\rm EM}_{\rm B} = \frac{1}{2}(|\lambda_+| + |\lambda_-|) = e^{-\Gamma^{\rm EM}_{\rm B}} \left| \sin\left(\frac{\Phi^{\rm EM}_{\rm BA}}{2}\right) \right|. \tag{4.33}$$

As proven in (4.15), there is a trade-off relationship between the visibility $\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}}$ and the distinguishability $\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}}$, as indicated by the following inequality:

$$(\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}})^2 + (\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}})^2 \leq 1. \tag{4.34}$$

Therefore, the electromagnetic version of the complementarity inequality is written as

$$(\mathcal{V}_{\mathrm{A}}^{\mathrm{EM}})^2 + (\mathcal{D}_{\mathrm{B}}^{\mathrm{EM}})^2 = e^{-2\Gamma_{\mathrm{A}}^{\mathrm{EM}}} \cos^2\left(\frac{\Phi_{\mathrm{AB}}^{\mathrm{EM}}}{2}\right) + e^{-2\Gamma_{\mathrm{B}}^{\mathrm{EM}}} \sin^2\left(\frac{\Phi_{\mathrm{BA}}^{\mathrm{EM}}}{2}\right) \leq 1. \tag{4.35}$$

Note that by replacing the quantities Γ_i^{EM}, $\Gamma_{\mathrm{c}}^{\mathrm{EM}}$, and Φ^{EM} in the above result with Γ_i^{GR}, $\Gamma_{\mathrm{c}}^{\mathrm{GR}}$, and Φ^{GR}, respectively; the inequality (4.35) in the gravitational version is written as

$$(\mathcal{V}_{\mathrm{A}}^{\mathrm{GR}})^2 + (\mathcal{D}_{\mathrm{B}}^{\mathrm{GR}})^2 = e^{-2\Gamma_{\mathrm{A}}^{\mathrm{GR}}} \cos^2\left(\frac{\Phi_{\mathrm{AB}}^{\mathrm{GR}}}{2}\right) + e^{-2\Gamma_{\mathrm{B}}^{\mathrm{GR}}} \sin^2\left(\frac{\Phi_{\mathrm{BA}}^{\mathrm{GR}}}{2}\right) \leq 1. \tag{4.36}$$

By adopting the simple notations $\Gamma_i\,(i = \mathrm{A}, \mathrm{B})$, Γ_{c}, Φ_{AB} and Φ_{BA}, we also describe the inequalities Eqs. (4.35) and (4.36) for the electromagnetic and gravitational cases in a unified manner as follows

$$\mathcal{V}_{\mathrm{A}}^2 + \mathcal{D}_{\mathrm{B}}^2 = e^{-2\Gamma_{\mathrm{A}}} \cos^2\left(\frac{\Phi_{\mathrm{AB}}}{2}\right) + e^{-2\Gamma_{\mathrm{B}}} \sin^2\left(\frac{\Phi_{\mathrm{BA}}}{2}\right) \leq 1. \tag{4.37}$$

In particular, for the case $D > T_{\mathrm{A}}$ and $D > T_{\mathrm{B}}$, the retarded photon field of particle B vanishes ($A_{\mathrm{BP}}^{\mu} = 0$), which leads to $\Phi_{\mathrm{AB}} = 0$, and the above inequality becomes

$$\mathcal{V}_{\mathrm{A}}^2 + \mathcal{D}_{\mathrm{B}}^2 = e^{-2\Gamma_{\mathrm{A}}} + e^{-2\Gamma_{\mathrm{B}}} \sin^2\left(\frac{\Phi_{\mathrm{BA}}}{2}\right) \leq 1. \tag{4.38}$$

This inequality is consistent with the existence of the quantum radiation emitted from particle A ($\Gamma_{\mathrm{A}} > 0$) and the vacuum fluctuations of the photon field around particle B ($\Gamma_{\mathrm{B}} > 0$) when the causality holds. If we can remove the two effects ($\Gamma_{\mathrm{A}} = \Gamma_{\mathrm{B}} = 0$), this inequality would be violated as long as the retarded electromagnetic field of particle A does not vanish ($A_{\mathrm{AP}}^{\mu} \neq 0$ and then $\Phi_{\mathrm{BA}} \neq 0$). Hence, if the two effects vanish, then complementarity is violated, and the paradox would appear. In the following subsection, we will discuss that the inequality (4.38) is never violated by the (Schrödinger-)Robertson uncertainty relation associated with the electromagnetic field.

4.4 Relationship with uncertainty relation

Here we discuss the relationship with the complementarity inequality (4.38) and the following Schrödinger-Robertson uncertainty relation

$$(\Delta\phi_{\rm A})^2(\Delta\phi_{\rm B})^2 \geq \frac{1}{4}\Big(\langle\{\hat{\phi}_{\rm A},\hat{\phi}_{\rm B}\}\rangle\Big)^2 + \frac{1}{4}\Big|\langle[\hat{\phi}_{\rm A},\hat{\phi}_{\rm B}]\rangle\Big|^2, \tag{4.39}$$

where $\Delta\hat{\phi}_{\rm A}$ and $\Delta\hat{\phi}_{\rm B}$ are the variances of the operators $\hat{\phi}_{\rm A}$ [(3.30)] and $\hat{\phi}_{\rm B}$ [(4.12)], respectively. The commutation relation is equivalent to $[\hat{\phi}_{\rm A},\hat{\phi}_{\rm B}] = i(\Phi_{\rm AB} - \Phi_{\rm BA})$, which is also extended to the gravitational version of Eq. (3.44) by the same manner. Subsequently, the following Schrödinger-Robertson uncertainty relation can be obtained:

$$\Gamma_{\rm A}\Gamma_{\rm B} \geq \frac{\Gamma_{\rm c}^2}{4} + \frac{1}{16}(\Phi_{\rm AB} - \Phi_{\rm BA})^2, \tag{4.40}$$

where we used $(\Delta\phi_i)^2 = \langle 0|\hat{\phi}_i^2|0\rangle - (\langle 0|\hat{\phi}_i|0\rangle)^2 = 2\Gamma_i$ and $\langle\{\hat{\phi}_{\rm A},\hat{\phi}_{\rm B}\}\rangle = 2\Gamma_{\rm c}$, followed by Eqs. (4.1), (4.9) and (4.2), (4.10). The expectation value of the commutator, $\langle[\hat{\phi}_{\rm A},\hat{\phi}_{\rm B}]\rangle = i(\Phi_{\rm AB} - \Phi_{\rm BA})$, is the result of Eq. (3.44). The inequality above shows that the product of $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ has a lower bound expressed by $\Gamma_{\rm c}$, $\Phi_{\rm AB}$, and $\Phi_{\rm BA}$.

In particular, in the region $D > T_{\rm A}$ and $D > T_{\rm B}$ where there is no retarded propagation of photon field from Bob's system to Alice's system, we obtain the following Robertson inequality as

$$\Gamma_{\rm A}\Gamma_{\rm B} \geq \frac{\Gamma_{\rm c}^2}{4} + \frac{1}{16}\Phi_{\rm BA}^2 \tag{4.41}$$

$$\geq \frac{1}{16}\Phi_{\rm BA}^2. \tag{4.42}$$

This means that the quantities $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ do not vanish simultaneously if $\Phi_{\rm BA} \neq 0$. Additionally, we can show that the Robertson inequality (4.42) is a sufficient condition for the inequality (4.38):

$$\Gamma_{\rm A}\Gamma_{\rm B} \geq \frac{1}{16}\Phi_{\rm BA}^2 \quad \Longrightarrow \quad e^{-2\Gamma_{\rm A}} + e^{-2\Gamma_{\rm B}}\sin^2\left(\frac{\Phi_{\rm BA}}{2}\right) \leq 1. \tag{4.43}$$

The analytical and numerical proof of this statement is presented in Appendix C. This result implies that the Robertson inequality among $\Gamma_{\rm A}$, $\Gamma_{\rm B}$ and $\Phi_{\rm BA}$, which reflects the non-commutative property of the photon/gravitational fields, guarantees the complementarity described by the inequality between the visibility $\mathcal{V}_{\rm A}$ and the distinguishability $\mathcal{D}_{\rm B}$.

In the above analysis, we discovered that the uncertainty relation represented by Robertson's inequality for a quantized field guarantees the consistency between causality and complementarity. In the following subsection, we provide a detailed discussion regarding consistency by focusing on the entanglement between two charged/massive particles.

4.5 Role of entanglement on uncertainty relation of field and complementarity

In this subsection, we reveal how the uncertainty relation relates to complementarity, using the entanglement between two particles A and B introduced in (3.48). In our gedanken experiment, we consider the region where Bob's effect does not propagate to Alice's system. Using the retarded Green's function, which describes the causal influence of a source, this is quantified as $\Phi_{\rm AB}=0$ [2, 3, 4, 100, 101]. This result reflects a general property of the retarded Green's function, which holds for both electromagnetic and gravitational fields. Therefore, the complementarity inequality (4.38), Schrödinger-Robertson uncertainty relation (4.41), and $\lambda_{\rm min}$ (3.48) are expressed as follows, respectively:

$$e^{-2\Gamma_{\rm A}}+e^{-2\Gamma_{\rm B}}\sin^2\left(\frac{\Phi_{\rm BA}}{2}\right)\leq 1, \tag{4.44}$$

$$\Gamma_{\rm A}\Gamma_{\rm B}\geq\frac{\Gamma_{\rm c}^2}{4}+\frac{\Phi_{\rm BA}^2}{16}, \tag{4.45}$$

$$\begin{aligned}\lambda_{\rm min}=\frac{1}{4}\Big[1-e^{-\Gamma_{\rm A}-\Gamma_{\rm B}}\cosh[\Gamma_{\rm c}]-\Big\{\left(e^{-\Gamma_{\rm A}}-e^{-\Gamma_{\rm B}}\right)^2\\ +4e^{-\Gamma_{\rm A}-\Gamma_{\rm B}}\sin^2\left[\frac{\Phi_{\rm BA}}{4}\right]+e^{-2\Gamma_{\rm A}-2\Gamma_{\rm B}}\sinh^2[\Gamma_{\rm c}]\Big\}^{\frac{1}{2}}\Big].\end{aligned} \tag{4.46}$$

In Sec. 4.4, we discovered that the inequality presented in (4.45) is the sufficient condition for the complementarity inequality (Eq. (4.44)) in the electromagnetic case (strictly, we used the Robertson uncertainty relation $\Gamma_{\rm A}\Gamma_{\rm B}\geq\Phi_{\rm BA}^2/16$, which follows by (4.45)). To reveal the relationship between the inequality (4.45) and complementarity inequality (4.44), we consider the role of entanglement. We first focus on the relationship between the uncertainty relation of the electromagnetic/gravitational field (Eq. (4.45)) and $\lambda_{\rm min}$. Let us consider the limit of small coupling constants for the electromagnetic/gravitational cases. The quantities Γ_i, $\Gamma_{\rm c}$, and $\Phi_{\rm BA}$ depending on the coupling constants are small, and the approximate form of $\lambda_{\rm min}$ can be expressed as

$$\lambda_{\rm min} \approx \frac{1}{4}\left[\Gamma_{\rm A}+\Gamma_{\rm B}-\sqrt{\left(\Gamma_{\rm A}+\Gamma_{\rm B}\right)^2-4\left(\Gamma_{\rm A}\Gamma_{\rm B}-\frac{\Gamma_{\rm c}^2}{4}-\frac{\Phi_{\rm BA}^2}{16}\right)}\,\right], \tag{4.47}$$

which is valid for $\Gamma_i \ll 1$ $(i={\rm A},{\rm B})$, $|\Gamma_{\rm c}| \leq \Gamma_{\rm A}+\Gamma_{\rm B} \ll 1$, and $|\Phi_{\rm BA}| \ll 1$. The inside of the square root is always positive because of $(\Gamma_{\rm A}+\Gamma_{\rm B})^2 - 4(\Gamma_{\rm A}\Gamma_{\rm B} - \Gamma_{\rm c}^2/4 - \Phi_{\rm BA}^2/16) = (\Gamma_{\rm A}-\Gamma_{\rm B})^2 + \Gamma_{\rm c}^2 + \Phi_{\rm BA}^2/4 \geq 0$. On the other hand, the sign of $\Gamma_{\rm A}\Gamma_{\rm B} - \Gamma_{\rm c}^2/4 - \Phi_{\rm BA}^2/16$ inside of the square root in Eq. (4.47) determines the sign of $\lambda_{\rm min}$, i.e., appearance of entanglement between two particles. From the Schrödinger-Robertson uncertainty relation (4.45), $\Gamma_{\rm A}\Gamma_{\rm B} - \Gamma_{\rm c}^2/4 - \Phi_{\rm BA}^2/16$ must be non-negative. Therefore, the Schrödinger-Robertson uncertainty relation and the entanglement between the particles A and B appear to be correlated. This observation, which is obtained on the basis of the approximation, is extended to more general relationship among the complementarity inequality, Schrödinger-Robertson uncertainty relation, and the non-entanglement property between the two particles. Namely, we can demonstrate the following relationship of the sufficient conditions numerically (for a more detailed explanation, see Appendix D):

$$\Gamma_{\rm A}\Gamma_{\rm B} \geq \frac{\Gamma_{\rm c}^2}{4}+\frac{\Phi_{\rm BA}^2}{16} \quad \Longrightarrow \quad \lambda_{\rm min} \geq 0 \quad \Longrightarrow \quad e^{-2\Gamma_{\rm A}}+e^{-2\Gamma_{\rm B}}\sin^2\left(\frac{\Phi_{\rm BA}}{2}\right) \leq 1. \tag{4.48}$$

The relationship of the above sufficient conditions (4.48) are depicted in Fig. 18, which are obtained under the causality condition that the Bob's action is spacelike separated from Alice, i.e., $\Phi_{\rm AB}=0$. The relationship presented in (4.48) mean as follows: The Schrödinger-Robertson uncertainty relation implies the existence of the vacuum fluctuations of electromagnetic/gravitational field because $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ must be non-zero since $\Phi_{\rm BA}$ is non-zero. The origin of the non-zero values of $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ is the decoherence of the superposition of each particle, which is supposed to come from the entanglement between the particles and the electromagnetic/gravitational field. This decoherence causes no generation of the entanglement between particles A and B, i.e., $\lambda_{\rm min} \geq 0$. Because particles A and B are not entangled, Bob is not able to sufficiently get the which-path information of Alice's particle. Therefore, the complementarity inequality holds.

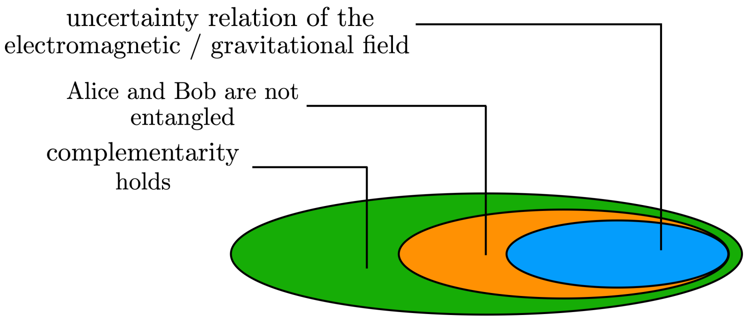


FIG 18: Inclusion relationship of the uncertainty relation (blue region), the condition of the non generation of entanglement (orange region), and the complementarity inequality (green region) are inclusive.

5 Quantification of quantumness of the gravitational field

In this section, we consider how the superposition state of the gravitational field is quantified. This study is based on the paper [1]. First, we introduce the setup of our system: two massive particles each in a superposed state coupled to a quantized gravitational field. Then we consider a trade-off relation (monogamy relation) between the negativity and the conditional von Neumann entropy in specific two configurations as shown in Figs. 9 and 7. Based on the above relation, we derive the condition to be entangled state between the particle and gravitational field. Finally, we use quantum discord to analyze quantum correlations between particles in superposition states and gravitational field and discuss its behavior.

5.1 Setup of two particles system coupled with gravitational field

We first consider the system of two massive quantum particles to be each in a superposed state, where they are interacting through quanitzed gravitational field. Note that the two particles are non-relativistic, and the gravitational field is treated as linearized gravity. This system can be regarded as the system explained in Chapter 4. Thus the quantum state of this system is described by the $\Gamma_i^{\rm GR}$, $\Gamma_{\rm c}^{\rm GR}$, $\Phi_{\rm AB}^{\rm GR}$, and $\Phi_{\rm BA}^{\rm GR}$ as introduced in Eqs. (4.9), (4.10), and (4.11). In our analysis, we apply unified notation of electromagnetic and gravitational case: $\Gamma_i^{\rm GR}$, $\Gamma_{\rm c}^{\rm GR}$, $\Phi_{\rm AB}^{\rm GR}$, and $\Phi_{\rm BA}^{\rm GR}$ are equivalent to Γ_i, $\Gamma_{\rm c}$, $\Phi_{\rm AB}$, and $\Phi_{\rm BA}$.

For later convenience, it is useful to evaluate the quantities Γ_i, $\Gamma_{\rm c}$, $\Phi_{\rm AB}$, and $\Phi_{\rm BA}$ by order estimation where we ignore the numerical factors because we analyze quantitatively. In the configurations shown in Figs. 9 and 7, the characteristic parameters of the system are given by the superposed time T, the width of the superposition L, and the distance of two particles D. Thus the quantities Γ_i, $\Gamma_{\rm c}$, $\Phi_{\rm AB}$, and $\Phi_{\rm BA}$ are described by these parameters. For instance, the quantities $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are estimated as the number of graviton emitted by the quadrupole radiation during time T per the energy of a single graviton

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx g^2 \left(\frac{L}{T}\right)^4 , \tag{5.1}$$

whereas the electromagnetic case of the quantities $\Gamma^{\rm EM}_{\rm A}$ and $\Gamma^{\rm EM}_{\rm B}$, which corresponds to the dipole radiation, are

$$\Gamma^{\rm EM}_{\rm A} = \Gamma^{\rm EM}_{\rm B} \approx e^2 \left(\frac{L}{T}\right)^2 \tag{5.2}$$

with the electric charge e. The quantities $\Gamma_{\rm A}$ and $\Gamma_{\rm B}$ are determined by the parameters of their system A and B. In contrast, the quantities $\Phi_{\rm AB}$, $\Phi_{\rm BA}$, and $\Gamma_{\rm c}$ characterize the correlation between two particles. Thus the distance of the systems A and B is important. In the regime $D \gg T \gg L$, the quantities $\Phi_{\rm AB}$, $\Phi_{\rm BA}$, and $\Gamma_{\rm c}$ will be given by

$$\Phi_{\rm AB} = 0, \quad \Phi_{\rm BA} \approx g^2 \left(\frac{L}{T}\right)^2 \left(\frac{T}{D}\right)^3 , \quad |\Gamma_{\rm c}| \approx g^2 \left(\frac{L}{T}\right)^4 \left(\frac{T}{D}\right)^4 , \tag{5.3}$$

where $\Phi_{\rm AB} = 0$ is understood as the vanishing of the retarded Green's function propagating from the particle B to A. The quantity $\Phi_{\rm BA}$ can regarded as the Newtonian potential induced by the massive particle. $\Gamma_{\rm c}$ is referred to the result of the order estimation (B.14) presented in Appendix B.2. In the regime $T \gg D \gg L$, the quantities $\Phi_{\rm AB}$, $\Phi_{\rm BA}$, and $\Gamma_{\rm c}$ will be of order

$$\Phi_{\rm AB} = \Phi_{\rm BA} \approx g^2 \left(\frac{L}{T}\right)^2 \left(\frac{T}{D}\right)^3 , \quad |\Gamma_{\rm c}| \approx g^2 \left(\frac{L}{T}\right)^4 . \tag{5.4}$$

Note that, in the regime $T \gg D \gg L$, $\Phi_{\rm AB}$ can be equivalent to $\Phi_{\rm BA}$ because of the symmetric configuration of the systems A and B (see Fig. 7). The quantity $\Gamma_{\rm c}$ is estimated by using Eq. (B.8) in Appendix B.2, where we ignored the term proportional to D/T because of $D/T \ll 1$.

5.2 Quantumness of gravitational due to monogamy relation

Here we discuss the entanglement between the particle A and the gravitational field from the viewpoint of entanglement monogamy. To judge whether the particle A and the gravitational field are entangled or not, we consider the entanglement of formation $E_{\rm f}(\rho_{\rm A,g})$. The entanglement of formation is one of the quantity judging two quantum states are in an entangled state or not. For example, if $E_{\rm f}(\rho_{\rm A,g}) > 0$, then the particle A and the gravitational field is entangled. However, if the entanglement of formation vanishes: $E_{\rm f}(\rho_{\rm A,g}) = 0$, the particle A and the gravitational field are not entangled. The entanglement of formation has lower bound due to the conditional von Neumann entropy $S(\rm A|B)$ [102] as

$$E_{\rm f}(\rho_{\rm A,g}) \geq S({\rm A}|{\rm B}), \tag{5.5}$$

where the proof of this inequality is presented in Appendix E. Thus the inequality (5.5) indicates that the particle A and the gravitational field are entangled when $S({\rm A}|{\rm B}) > 0$ is satisfied. The conditional von Neumann entropy $S(\rm A|B)$ is given by the analogy with the classical conditional entropy as

$$S({\rm A}|{\rm B}) := S(\rho_{\rm AB}) - S(\rho_{\rm B}). \tag{5.6}$$

The von Neumann entropy $S(\rho_{\rm X})$ measures how strong the correlation is between subsystem X and its complement system $\bar{X}$. In classical theory, the conditional entropy is always positive, but, in quantum theory, it can be negative [103]. The negativity of the conditional von Neumann entropy is roughly interpreted as entanglement. The von Neumann entropy $S(\rho_{\rm B})$ is computed as follows:

$$S(\rho_{\rm B}) = -\sum_{s=\pm} \Lambda_{\rm s}[\rho_{\rm B}] \log[\Lambda_{\rm s}[\rho_{\rm B}]], \quad \Lambda_{\pm}[\rho_{\rm B}] = \frac{1}{2}\left[1 \pm e^{-\Gamma_{\rm B}} \cos\left(\frac{\Phi_{\rm BA}}{2}\right)\right], \tag{5.7}$$

where the eigenvalues $\Lambda_{\pm}[\rho_{\rm B}]$ are obtained by calculating the eigenvalues of the density matrix $\rho_{\rm B}^{\rm EM} = {\rm Tr}_{\rm A}[\rho_{\rm AB}^{\rm EM}]$ in the electromagnetic field case and extending it to the gravitational field. The von Neumann entropy $S(\rho_{\rm AB})$ is also derived as

$$S(\rho_{\rm AB}) = -\sum_{s=\pm} \Big(\Lambda_1^{\rm s}[\rho_{\rm AB}] \log[\Lambda_1^{\rm s}[\rho_{\rm AB}]] + \Lambda_2^{\rm s}[\rho_{\rm AB}] \log[\Lambda_2^{\rm s}[\rho_{\rm AB}]]\Big) \tag{5.8}$$

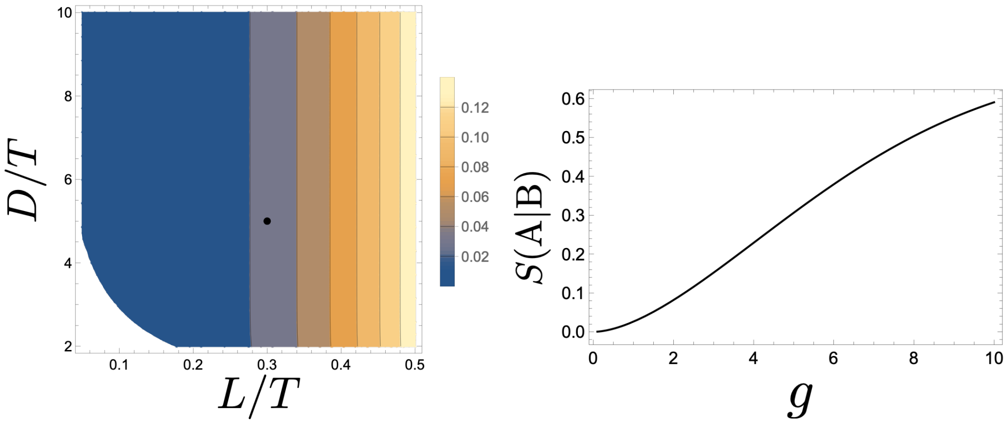


FIG 19: Left panel: Contour plots of the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ as functions of L/T and D/T, where we adopted the coupling constant $g = 1$. The black circle is a point when $D/T = 5$ with $L/T = 3/10$. Right panel: conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ as a function of coupling constant g. This graph corresponds to the black circle in the left panel.

with the eigenvalues of the density matrix ρ_{AB} obtained from Eqs. (3.41) and (3.42), which are extended to the gravitational version

$$\Lambda_1^{\pm}[\rho_{\mathrm{AB}}] = \frac{1}{4}\Big[1 - e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cosh[\Gamma_{\mathrm{c}}] \pm \Big\{\left(e^{-\Gamma_{\mathrm{A}}} - e^{-\Gamma_{\mathrm{B}}}\right)^2 + 4e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\sin^2\left[\frac{\Phi_{\mathrm{AB}} - \Phi_{\mathrm{BA}}}{4}\right] + e^{-2\Gamma_{\mathrm{A}}-2\Gamma_{\mathrm{B}}}\sinh^2[\Gamma_{\mathrm{c}}]\Big\}^{\frac{1}{2}}\Big], \tag{5.9}$$

$$\Lambda_2^{\pm}[\rho_{\mathrm{AB}}] = \frac{1}{4}\Big[1 + e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cosh[\Gamma_{\mathrm{c}}] \pm \Big\{\left(e^{-\Gamma_{\mathrm{A}}} - e^{-\Gamma_{\mathrm{B}}}\right)^2 + 4e^{-\Gamma_{\mathrm{A}}-\Gamma_{\mathrm{B}}}\cos^2\left[\frac{\Phi_{\mathrm{AB}} - \Phi_{\mathrm{BA}}}{4}\right] + e^{-2\Gamma_{\mathrm{A}}-2\Gamma_{\mathrm{B}}}\sinh^2[\Gamma_{\mathrm{c}}]\Big\}^{\frac{1}{2}}\Big]. \tag{5.10}$$

In the following analysis, we evaluate the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ in two regimes $D \gg T \gg L$ (Fig. 9) and $T \gg D \gg L$ (Fig. 7).

5.2.1 $D \gg T \gg L$ regime

The left panel of the figure 19 denotes the parameters dependence of the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$. To obtain a qualitative understanding of the behavior of the left panel in Fig. 19, we approximate the conditional von Neumann entropy as

$$S(\mathrm{A}|\mathrm{B}) \approx \frac{\Gamma_{\mathrm{B}}}{2}\left(1 - \log\left[\frac{\Gamma_{\mathrm{B}}}{2}\right]\right), \tag{5.11}$$

where we used $\Gamma_\mathrm{A} = \Gamma_\mathrm{B} \ll 1$ and $\Phi_\mathrm{BA} \ll 1$, and we assumed the condition $\Phi_\mathrm{BA}/4\Gamma_\mathrm{B} \ll 1$. The above equation (5.11) is independent of the quantity of Φ_BA, i.e., D/T and its amount depends only on $\Gamma_\mathrm{B} = g^2(L/T)^4$. This figure represents that $S(\mathrm{A}|\mathrm{B})$ is always positive and does not depend on the distance between two particles. The independence of the distance D can be understood by introducing an entanglement measure: negativity. The negativity $\mathcal{N}_\mathrm{AB}$ characterizes the entanglement between two particles [68, 71]. In particular, two particles A and B are regarded as the two-qubit in our system and then defined as follows:

$$\mathcal{N}_\mathrm{AB} = \max[-\lambda_\mathrm{min}, 0] \tag{5.12}$$

with the minimum eigenvalue λ_min

$$\begin{aligned}\lambda_\mathrm{min} = \frac{1}{4}\Bigg[&1 - e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\cosh[\Gamma_\mathrm{c}] - \Bigg\{\left(e^{-\Gamma_\mathrm{A}} - e^{-\Gamma_\mathrm{B}}\right)^2 \\ &+ 4e^{-\Gamma_\mathrm{A}-\Gamma_\mathrm{B}}\sin^2\left[\frac{\Phi_\mathrm{AB}+\Phi_\mathrm{BA}}{4}\right] + e^{-2\Gamma_\mathrm{A}-2\Gamma_\mathrm{B}}\sinh^2[\Gamma_\mathrm{c}]\Bigg\}^{\frac{1}{2}}\Bigg].\end{aligned} \tag{5.13}$$

If $\mathcal{N}_\mathrm{AB} = 0$ or $\lambda_\mathrm{min} \geq 0$ holds, two particles are not entangled. In Refs. [2, 4], we pointed out that negativity $\mathcal{N}_\mathrm{AB}$ vanishes in the regime $D \gg T \gg L$ because of the existence of the vacuum fluctuations Γ_A and Γ_B. Thus, no correlation between A and B is interpreted as disentanglement of them. Note that the white region in Fig. 19 may show that the approximation to derive the quantities Γ_A, Γ_B, Γ_c, and Φ_BA is bad.

The right panel of the figure 19 shows the behavior of $S(\mathrm{A}|\mathrm{B})$ versus the coupling constant g, respectively. In the limit of $g \to 0$ there is no interaction among the particle A, B, and the graviational field, so that the quantum state ρ_AB and its reduced density matrix ρ_B become pure state, i.e., $S(\rho_\mathrm{AB}) = S(\rho_\mathrm{B}) = 0$. In contrast, in the limit of $g \to \infty$, the decoherences Γ_A and Γ_B are dominant, and then the quantum states ρ_AB and ρ_B approaches the classical mixed state

$$\rho_\mathrm{AB} \to \frac{1}{4}\mathbb{1}_{4\times4}, \quad \rho_\mathrm{B} \to \frac{1}{2}\mathbb{1}_{2\times2} \tag{5.14}$$

with $n \times n$ identity matrix $\mathbb{1}_{n\times n}$. These limits lead to $S(\mathrm{A}|\mathrm{B}) \to \log 2$ for $g \to \infty$. Thus, in the region $D \gg T \gg L$, the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ is always positive. Therefore, $E_\mathrm{f}(\rho_\mathrm{A,g}) > 0$ is constantly fulfilled because of the inequality (5.5).

The condition $E_\mathrm{f}(\rho_\mathrm{A,g}) > 0$ can also be understood from the view point of the monogamy relation. $\lambda_{\min} \geq 0$ gives the concrete reason of the positivity of the conditional von Neumann entropy as

$$0 = \mathcal{N}_{\mathrm{AB}} = \lambda_{\min} \geq 0 \quad \Rightarrow \quad 0 = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) \geq -S(\mathrm{A}|\mathrm{B}), \tag{5.15}$$

where the inequality (E.7) was used in the right-hand-side. Note that, in general, $\mathcal{N}_{\mathrm{AB}} > 0$ is equivalent to $E_{\mathrm{f}}(\rho_{\mathrm{AB}}) > 0$ if the composite system AB is two-qubit system, which leads to $\mathcal{N}_{\mathrm{AB}} = 0 \Leftrightarrow E_{\mathrm{f}}(\rho_{\mathrm{AB}}) = 0$ due to the contraposition. Combine with the inequality (5.5) and the above relation (5.15), we obtain the following result

$$0 = \mathcal{N}_{\mathrm{AB}} = \lambda_{\min} \geq 0 \quad \Rightarrow \quad E_{\mathrm{f}}(\rho_{\mathrm{AB}}) = 0 \quad \Rightarrow \quad S(\mathrm{A}|\mathrm{B}) > 0 \quad \Rightarrow \quad E_{\mathrm{f}}(\rho_{\mathrm{A,g}}) > 0, \tag{5.16}$$

where we used $S(\mathrm{A}|\mathrm{B}) > 0$ is satisfied in the regime $D \gg T \gg L$. This condition means that the particle A and gravitational field is always entangled when two particles A and B are not entangled. Thus, we consider that the entanglement between particle A, which is in the superposition state, and the gravitational field can be regarded as a quantum nature of the gravitational field since it accompanies particle A and causes the superposition of the gravitational field.

5.2.2 $T \gg D \gg L$ regime

The parameter dependence of the conditional von Neumann entropy is depicted in Fig. 20. The upper panels of Fig. 20 represent the contour plots of the conditional von Neumann entropy versus L/T and D/T with the coupling constant $g = 1$ (left panel) and $g = 3$ (right panel). We also show the borderline representing by bold black line in the upper right panel of Fig. 20, where the negativity $\mathcal{N}_{\mathrm{AB}}$ vanishes. In the left panel, $S(\mathrm{A}|\mathrm{B}) > 0$ is satisfied in this parameter region. However, in the right panel, there are three regions: $S(\mathrm{A}|\mathrm{B}) < 0$ and $\mathcal{N}_{\mathrm{AB}} > 0$, $S(\mathrm{A}|\mathrm{B}) > 0$ and $\mathcal{N}_{\mathrm{AB}} > 0$, and $S(\mathrm{A}|\mathrm{B}) > 0$ and $\mathcal{N}_{\mathrm{AB}} = 0$. In the region $S(\mathrm{A}|\mathrm{B}) < 0$ and $\mathcal{N}_{\mathrm{AB}} > 0$, the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ is negative so we cannot judge the entanglement of formation $E_{\mathrm{f}}(\rho_{\mathrm{A,g}})$ is positive or not. In other words, we do not know whether the particle A and graviton are entangled pr not. However, the negativity $\mathcal{N}_{\mathrm{AB}}$ is positive, and then two particles A and B are entangled. The region $S(\mathrm{A}|\mathrm{B}) > 0$ and $\mathcal{N}_{\mathrm{AB}} > 0$ means that the particle A and graviton and two particles A and B are entangled state. In the region $S(\mathrm{A}|\mathrm{B}) > 0$ and $\mathcal{N}_{\mathrm{AB}} = 0$, we can understand two particles A and B are

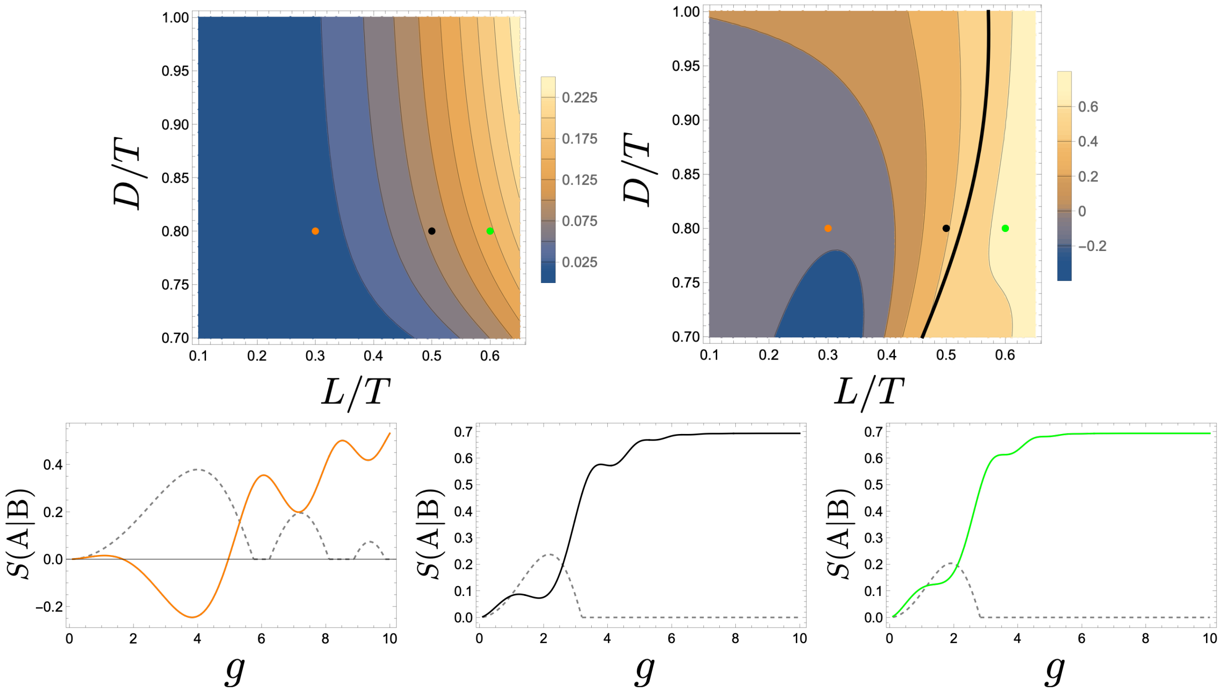


FIG 20: Upper panel: Contour plot of the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ as functions of L/T (horizontal axis) and D/T (vertical axis), where we adopted the coupling constant $g = 1$ (left panel) and $g = 3$ (right panel). The orange, black, and green circle are points when $L/T = 3/10$, $L/T = 5/10$, and $L/T = 6/10$ with $D/T = 8/10$. The bold black line is the boundary where the negativity $\mathcal{N}_{\mathrm{AB}}$ vanishes. Lower panel: Left, center, and right panels show the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ as a function of coupling constant g. The black dashed line depicts the negativity $\mathcal{N}_{\mathrm{AB}}$. This graph corresponds to the orange, black, and green circles in the upper panel.

not entangled, but the particle A and graviton are entangled state.

Three typical points, orange, black, and green dotted in the upper panels of Fig. 20 behave as the lower panels of Fig. 20 including the negativity $\mathcal{N}_{\mathrm{AB}}$ depicted by black dashed line. Each of them saturate, whereas the negativity vanish due to the decoherence when the coupling constant g becomes large. This behavior is also interpreted as one of the monogamy of the conditional von Neumann entropy and the negativity, respectively.

5.3 Behavior of Quantum discord

Here, we investigate the behavior of the quantum superposition of gravitational field using quantum discord [104, 105, 106]. The quantum discord is known to be a measure of all quantum correlations, including entanglement. The quantum discord of composite system AB is defined by the difference between the quantum mutual information $\mathcal{I}(\mathrm{A}, \mathrm{B})$ and the classical correlation $\mathcal{J}(\mathrm{A}, \mathrm{B})$

$$\mathcal{D}(\mathrm{A}, \mathrm{B}) = \mathcal{I}(\mathrm{A}, \mathrm{B}) - \mathcal{J}(\mathrm{A}, \mathrm{B}). \tag{5.17}$$

The non vanishing of the quantum discord is related to the quantum superposition principle [104]. In particular, we focus on the quantum discord between the particle A and the gravitational field $\mathcal{D}(\mathrm{A},\ \mathrm{g})$, which may be the evidence of the quantum superposition of the gravitational field, i.e., the quantumness of gravitational field. To simplify calculations, we represent $\mathcal{D}(\mathrm{A},\ \mathrm{g})$ by using the entanglement of formation $E_{\mathrm{f}}(\rho_{\mathrm{AB}})$ and the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ as

$$\mathcal{D}(\mathrm{A},\ \mathrm{g}) = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + S(\mathrm{A}|\mathrm{B}), \tag{5.18}$$

where the detail derivation is presented in Appendix E. The above equation (5.18) shows that the quantum correlation between the particle and the gravitational field is determined by the parameters of the systems A and B, which is one of the feature of the monogamy. Note that, in two-qubit system, there is a formula related to the entanglement of formation with respect to two-qubit state ρ_{AB} as

$$E_{\mathrm{f}}(\rho_{\mathrm{AB}}) = h\left(\frac{1+\sqrt{1-C^2(\rho_{\mathrm{AB}})}}{2}\right), \tag{5.19}$$

where we defined $h(x) := -x\log_2 x - (1-x)\log_2(1-x)$. $C(\rho_{\mathrm{AB}})$ is concurrence, which measures the degree of entanglement in the mixed state [66, 107, 108]. The concurrence for a mixed state of qubit system is introduced as

$$C(\rho_{\mathrm{AB}}) := \max\{0, \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\} \tag{5.20}$$

with $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. Here α_i $(i = 1, \ldots, 4)$ are the square root of eigenvalues of the non-Hermitian matrix $\rho_{\mathrm{AB}}(\sigma_y^{\mathrm{A}} \otimes \sigma_y^{\mathrm{B}})\rho_{\mathrm{AB}}^*(\sigma_y^{\mathrm{A}} \otimes \sigma_y^{\mathrm{B}})$. ρ_{AB}^* is the complex conjugate of ρ_{AB}, and σ_y^{A} (σ_y^{B}) is the Pauli matrix, which works for the local system A (B). In the following, we study the behavior of the quantum discord $\mathcal{D}(\mathrm{A},\ \mathrm{g})$ in two regions: $D \gg T \gg L$ and $T \gg D \gg L$.

5.3.1 $D \gg T \gg L$ regime

In this regime, two particles A and B are not entangled, i.e., $E_{\mathrm{f}}(\rho_{\mathrm{AB}}) = 0$. Thus, the quantum discord is exactly equivalent to conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B})$ due to Eq. (5.18).

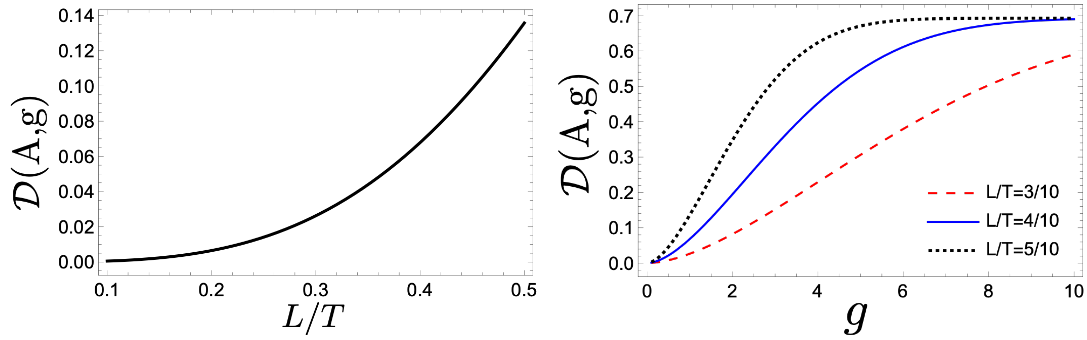


FIG 21: Quantum discord $\mathcal{D}(\mathrm{A}, \mathrm{g})$ as a function of L/T (left panel) with $g = 1$ and the coupling g (right panel). We adopted $D/T = 5$.

Fig. 21 depicts the behavior of the quantum discord $\mathcal{D}(\mathrm{A}, \mathrm{g})$ as a function of L/T (left panel) and coupling constant g, respectively. The result of the left panel shows that when the length scale of the superposition of the particle A L increases, the gravitational field also becomes well quantum superposition state. The right panel of Fig. 21 can be understood as follows: if the coupling constant g is increasing, the decoherence is dominant, so that the entanglement between two particles vanishes. However, the interaction between the particle A and the gravitational field becomes stronger, i.e., they are well correlated. This result is consistent with the (5.16).

5.3.2 $T \gg D \gg L$ regime

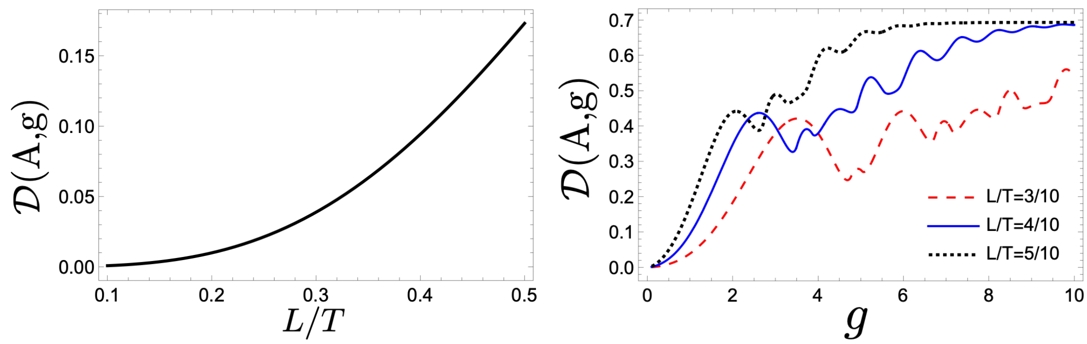


FIG 22: Quantum discord $\mathcal{D}(\mathrm{A}, \mathrm{g})$ as a function of L/T (left panel) with $g = 1$ and the coupling g (right panel). We adopted $D/T = 7/10$.

Fig. 22 also shows the behavior of the quantum discord $\mathcal{D}(A, g)$ as a function of L/T (left panel) and g (right panel). We come to the same conclusion in the region $D \gg T \gg L$ that increasing the superposition width of the particle A leads to the well superposition state of the gravitational field. Moreover, when the coupling constant g increases, the decoherence works dominant and destroy the entanglement between two particles. Note that, in this regime, the two particles A and B are slightly entangled, which reduces the correlation between the particle A and the gravitational field. From the view point of the monogamy, this disentanglement makes the entanglement between the particle A and the gravitational field strong.

6 Conclusion

The unification of quantum mechanics and gravity theory is an important problem in modern physics for explaining extreme situations, such as the quantum aspects of black holes and the early universe. Several theories have been proposed on the low- and high-energy scales. However, there is no experimental evidence of quantum gravity; thus, we cannot test these theories. In this thesis, we clarified the quantum mechanical aspects of gravity based on quantum field theory. In particular, we considered the role of the dynamical field for entanglement generation between two particles, the consistency of quantum mechanics and relativistic theory, and the quantitative understanding of the quantumness of the gravitational field.

In Chapter 3, we evaluated the effect of the dynamical electromagnetic field (photon field) on the entanglement generation between two charged particles each in a superposition state. The BMV experiment, explained in Chapter 2, is a proposal to detect entanglement generation due to Newtonian gravity, which originates from the nondynamical component of gravity. To understand entanglement generation in the context of quantum field theory, we evaluated the entanglement generation between two charged particles coupled to an electromagnetic field based on QED, motivated by the theoretical similarity between gravity and electromagnetism. We obtained a formula for the entanglement negativity between two charged particles, each in a superposition of two trajectories. This explicitly demonstrated the effect of a quantized electromagnetic field on the entanglement generation between two charged particles. Our analysis automatically includes contributions not only from the longitudinal mode (non-dynamical part) but also from the transverse mode (dynamical part) of the electromagnetic field. As expected, we demonstrated that the entanglement generation induced by the Coulomb potential was reproduced in the non-relativistic limit of our formula. We also demonstrate how relativistic corrections to Coulomb entanglement arise. In particular, vacuum fluctuations in the photon field cause quantum decoherence, which becomes significant when decoherence due to photon emission simultaneously becomes significant. When the two charged particles are separated by a long distance, the decoherence effect dominates, and entanglement generation is suppressed. However, when two particles are separated by the distance of a wave zone, the superposition of the electromagnetic wave from the other charged particle influences the quantum coherence signature. We also found that the quantum superposition of bremsstrahlung from a superposed trajectory affects the signature of the quantum coherence between the two particles; however, entanglement is not generated because the vacuum fluctuations of the photon field dominate over the signature of the entanglement.

Similar features are expected to appear in the entanglement generation between the two masses in the framework of the quantized gravitational field. The vacuum fluctuations of the gravitational field and quantum superposition of gravitational radiation are expected to be involved in the entanglement generation between the two masses. In Chapter 4, the framework discussed in this chapter was extended to the theory of gravity to clarify the dynamical effects of a quantized gravitational field.

Chapter 4 considered the gedanken experiment to reveal the relationship between complementarity and causality. Here, we revisited the resolution of the paradox proposed by Refs. [54, 55, 56, 89, 91] indicated that the quantum superposition of the gravitational potential may result in inconsistency between causality and complementarity. The authors of [54, 55, 56] argued that this inconsistency is resolved by vacuum fluctuations and the entangling radiation of the electromagnetic/gravitational field. We conclude that the inconsistency of causality and complementarity does not appear from the viewpoint of the Schrödinger-Robertson uncertainty relation and complementarity inequality. The analysis based on quantum field theory explicitly demonstrated the intuitively legitimate result that causality holds and that operations on Bob's system at a spacelike distance do not affect Alice's interference experiment at all by deriving Alice's reduced density operator. On the other hand, to find the validity of complementarity, we first derived visibility and distinguishability, which represent the degree of success of Alice's interference experiment and the degree of distinction of Alice's quantum state, respectively. We then argued that there is an inequality between these quantities, which is guaranteed by the Robertson inequality associated with the non-commutative property of the quantized electromagnetic field. This inequality describes the limit of complementarity in resolving this paradox.

Based on the QED/quantum theory of linearized gravity, we also analyzed the gedanken experiment in connection with the Schrödinger-Robertson uncertainty relation and complementarity inequality, focusing on the entanglement between two particles. We discovered that the Schrödinger-Robertson uncertainty relation in an electromagnetic/gravitational field prohibited the generation of entanglement between two particles when causality was fulfilled. Additionally, we numerically demonstrated that the condition in which the two particles are not entangled guarantees complementarity. The essence of the inconsistency between causality and complementarity is the assumption that the entanglement between Alice and Bob occurs in a region where Bob's information cannot causally propagate to Alice. Our results showed that the two particles cannot be entangled because of the existence of the quantized electromagnetic/gravitational field, which resolves the paradox and preserves the consistency between causality and complementarity.

Thus, the essence of the resolution for the paradox in this Gedanken experiment is the existence of vacuum fluctuation and entangling radiation, which cause decoherence. This decoherence is assumed to be induced by entanglement between the particle and the field. However, determining whether a particle and a field are entangled is a nontrivial task. It is important to discuss the condition that the particle and field are generally entangled. Furthermore, the structure of the entanglement between the particle and the field can be further investigated using various quantities of quantum information.

In Chapter 5, we considered the structure of the entanglement between the particle and the field. In this study, we analyzed the dynamics of a two-particle system interacting with a gravitational field and revealed the entanglement structure between the particle and gravitational field based on the quantum theory of linearized gravity. We derived the inequality in which the conditional von Neumann entropy between two particles gives a lower bound on the entanglement between the particle and the gravitational field. Furthermore, we found that the conditional von Neumann entropy has a tradeoff relationship with the negativity between the two particles. Thus, we showed that the particle and field are always entangled if the two particles are not entangled. In addition, we computed the quantum discord to quantitatively evaluate the quantum correlations between the particle and the gravitational field. Quantum discord characterizes the quantum superposition of the gravitational field. Consequently, as the width of the superposition state of the particle increases, the superposition of the gravitational field becomes significant.

In this doctoral thesis, we considered the roles of the nondynamical and dynamical degrees of freedom of fields in entanglement generation and decoherence. Thanks to the treatment of the quantum field theory, we revealed that the decoherence is induced by a dynamical quantum field. Therefore, the decoherence effect may be essential to the quantum theory of gravity. In the future, we intend to gain a deeper understanding of the decoherence induced by quantum fields, which will help elucidate the theory of quantum gravity. We consider it important to understand the vacuum fluctuation of the field coupled with the spin degrees of freedom of a particle and decoherence using a measuring instrument, which we neglected in this thesis.

We expect that our results will contribute to the construction of quantum gravity theory in which quantum mechanics and general relativity are consistent.

A 1/c expansion of Φ

We present the $1/c$ expansion of the quantity

$$\Phi = \frac{e}{2\hbar c}\Big(\oint_{\mathrm{C_A}} dx_\mu \Delta A^\mu \mathrm{B}(x) + \oint_{\mathrm{C_B}} dx_\mu \Delta A^\mu_{\mathrm{A}}(x)\Big), \tag{A.1}$$

where

$$\Delta A^\mu_i(x) = \sum_{\mathrm{P=R,L}} \epsilon_{\mathrm{P}} \frac{e}{4\pi}\Big[\frac{v^\mu_{i\mathrm{P}}(t_{i\mathrm{P}})}{(x - X_{i\mathrm{P}}(t_{i\mathrm{P}}))\cdot v_{i\mathrm{P}}(t_{i\mathrm{P}})}\Big], \tag{A.2}$$

and $v^\mu = [c, \boldsymbol{v}]^{\mathrm{T}}$, $\epsilon_{\mathrm{R}} = 1$, $\epsilon_{\mathrm{L}} = -1$ and $t_{i\mathrm{P}}$ satisfies the light cone condition $-c(t - t_{i\mathrm{P}}) + |\boldsymbol{x} - \boldsymbol{X}_{i\mathrm{P}}(t_{i\mathrm{P}})| = 0$. We restored the reduced Planck constant $\hbar$ and the light velocity c. Substituting (A.2) into (A.1), we obtain

$$\begin{aligned}\Phi &= \frac{e^2}{8\pi\hbar c}\Big(\oint_{\mathrm{C_A}} dx_\mu \sum_{\mathrm{Q=R,L}} \epsilon_{\mathrm{Q}}\Big[\frac{v^\mu_{\mathrm{BQ}}(t_{\mathrm{BQ}})}{(x - X_{\mathrm{BQ}}(t_{2\mathrm{Q}}))\cdot v_{\mathrm{BQ}}(t_{2\mathrm{Q}})}\Big] + (\mathrm{A} \leftrightarrow \mathrm{B})\Big)\\ &= \frac{e^2}{8\pi\hbar}\int dt \sum_{\mathrm{P,Q=R,L}} \epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}\Big[\frac{v_{\mathrm{AP}}(t)\cdot v_{\mathrm{BQ}}(t_{\mathrm{BQ}})}{c(X_{\mathrm{AP}}(t) - X(t_{\mathrm{BQ}}))\cdot v_{\mathrm{BQ}}(t_{\mathrm{BQ}})}\Big] + (\mathrm{A} \leftrightarrow \mathrm{B}),\end{aligned} \tag{A.3}$$

where we changed the integral as $\oint_{\mathrm{C}_i} dx^\mu = \sum_{\mathrm{P=R,L}} \epsilon_{\mathrm{P}} \int (dX^\mu_{i\mathrm{P}}/dt)dt = \sum_{\mathrm{P=R,L}} \epsilon_{\mathrm{P}} \int v^\mu_{i\mathrm{P}}(t)dt(i = \mathrm{A}, \mathrm{B})$ in the second line. The integrands have the form

$$\begin{aligned}\frac{v_{\mathrm{A}}(t)\cdot v_{\mathrm{B}}(t_{\mathrm{r}})}{c(X_{\mathrm{A}}(t) - X_{\mathrm{B}}(t_{\mathrm{r}}))\cdot v_{\mathrm{B}}(t_{\mathrm{r}})} &= \frac{c^2 - \boldsymbol{v}_{\mathrm{A}}(t)\cdot \boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}})}{c(-c(t - t_{\mathrm{r}}) + (\boldsymbol{X}_{\mathrm{A}}(t) - \boldsymbol{X}_{\mathrm{B}}(t_{\mathrm{r}}))\cdot \boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}}))}\\ &= \frac{-1}{|\boldsymbol{X}_{\mathrm{A}}(t) - \boldsymbol{X}_{\mathrm{B}}(t_{\mathrm{r}})| - (\boldsymbol{X}_{\mathrm{A}}(t) - \boldsymbol{X}_{\mathrm{B}}(t_{\mathrm{r}}))\cdot \boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}})/c}\Big(1 - \frac{\boldsymbol{v}_{\mathrm{A}}(t)\cdot \boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}})}{c^2}\Big),\end{aligned} \tag{A.4}$$

where the light cone condition $-c(t-t_{\rm r})+|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r})|=0$ was used in the second line. The $1/c$ expansion of the retarded time $t_{\rm r}$ is

$$\begin{aligned}
t_{\rm r} &= t-\frac{1}{c}|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r})| \\
&= t-\frac{1}{c}\sqrt{(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r}))^2} \\
&= t-\frac{1}{c}\sqrt{\left(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)+\frac{\boldsymbol{v}_{\rm B}(t)}{c}|\boldsymbol{X}_{\rm A}-\boldsymbol{X}_{\rm B}(t)|\right)^2}+O\Big(\frac{1}{c^3}\Big) \\
&= t-\frac{1}{c}\sqrt{(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t))^2+(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t))\cdot\frac{2\boldsymbol{v}_{\rm B}(t)}{c}|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)|}+O\Big(\frac{1}{c^3}\Big) \\
&= t-\frac{|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)|}{c}\left(1+\frac{\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)}{|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)|}\cdot\frac{\boldsymbol{v}_{\rm B}(t)}{c}\right)+O\Big(\frac{1}{c^3}\Big) \\
&= t-\frac{r(t)}{c}-\boldsymbol{r}(t)\cdot\frac{\boldsymbol{v}(t)}{c^2}+\mathcal{O}\Big(\frac{1}{c^3}\Big),
\end{aligned} \tag{A.5}$$

where $\boldsymbol{r}(t)=\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t)$ and $r(t)=|\boldsymbol{r}(t)|$. The denominator of the integrand (A.4) is

$$\begin{aligned}
&|\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r})|-(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r}))\cdot\frac{\boldsymbol{v}_{\rm B}(t_{\rm r})}{c} \\
&= \sqrt{(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r}))^2}-(\boldsymbol{X}_{\rm A}(t)-\boldsymbol{X}_{\rm B}(t_{\rm r}))\cdot\frac{\boldsymbol{v}_{\rm B}(t_{\rm r})}{c} \\
&= \sqrt{\left(\boldsymbol{r}+\boldsymbol{v}_{\rm B}\Big(\frac{r}{c}+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c^2}\Big)-\frac{r^2\boldsymbol{a}_{\rm B}}{2c^2}\right)^2}-\Big(\boldsymbol{r}+\boldsymbol{v}_{\rm B}\frac{r}{c}\Big)\cdot\frac{1}{c}\Big(\boldsymbol{v}_{\rm B}-\frac{r}{c}\boldsymbol{a}_{\rm B}\Big)+O\Big(\frac{1}{c^3}\Big) \\
&= \sqrt{r^2+2\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}\Big(\frac{r}{c}+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c^2}\Big)-2\boldsymbol{r}\cdot\frac{r^2\boldsymbol{a}_{\rm B}}{2c^2}+\frac{r^2v_{\rm B}^2}{c^2}}-\Big(\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c}+\frac{rv_{\rm B}^2}{c^2}-\frac{r}{c^2}\boldsymbol{r}\cdot\boldsymbol{a}_{\rm B}\Big)+O\Big(\frac{1}{c^3}\Big) \\
&= r\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{r^2}\Big(\frac{r}{c}+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c^2}\Big)-\boldsymbol{r}\cdot\frac{\boldsymbol{a}_{\rm B}}{2c^2}+\frac{v_{\rm B}^2}{2c^2}-\frac{(\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B})^2}{2r^2c^2}\right)-\Big(\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c}+\frac{rv_{\rm B}^2}{c^2}-\frac{r}{c^2}\boldsymbol{r}\cdot\boldsymbol{a}_{\rm B}\Big)+O\Big(\frac{1}{c^3}\Big) \\
&= r\left[1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{r^2}\Big(\frac{r}{c}+\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{c^2}\Big)-\boldsymbol{r}\cdot\frac{\boldsymbol{a}_{\rm B}}{2c^2}+\frac{v_{\rm B}^2}{2c^2}-\frac{(\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B})^2}{2r^2c^2}-\frac{\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B}}{rc}-\frac{v_{\rm B}^2}{c^2}+\frac{\boldsymbol{r}\cdot\boldsymbol{a}_{\rm B}}{c^2}\right] \\
&= r\left[1+\frac{(\boldsymbol{r}\cdot\boldsymbol{v}_{\rm B})^2}{2r^2c^2}-\frac{v_{\rm B}^2}{2c^2}+\frac{\boldsymbol{r}\cdot\boldsymbol{a}_{\rm B}}{2c^2}\right]+\mathcal{O}\Big(\frac{1}{c^3}\Big),
\end{aligned} \tag{A.6}$$

and the numerator of (A.4) is

$$1-\frac{\boldsymbol{v}_{\rm A}(t)\cdot\boldsymbol{v}_{\rm B}(t_{\rm r})}{c^2}=1-\frac{\boldsymbol{v}_{\rm A}\cdot\boldsymbol{v}_{\rm B}}{c^2}+\mathcal{O}\Big(\frac{1}{c^3}\Big), \tag{A.7}$$

where the light cone condition and the Taylor expansion were used and the argument t was omitted. Then, (A.4) reduces to

$$
\begin{aligned}
&\frac{v_1^{\mu}(t)v_{\mathrm{B}\mu}(t_{\mathrm{r}})}{c(X_{\mathrm{A}}(t)-X_{\mathrm{B}}(t_{\mathrm{r}}))\cdot v_{\mathrm{B}}(t_{\mathrm{r}})} \\
&=\frac{-1}{|\boldsymbol{X}_{\mathrm{A}}(t)-\boldsymbol{X}_{\mathrm{B}}(t_{\mathrm{r}})|-(\boldsymbol{X}_{\mathrm{A}}(t)-\boldsymbol{X}_{\mathrm{B}}(t_{\mathrm{r}}))\cdot\boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}})/c}\Big(1-\frac{\boldsymbol{v}_{\mathrm{A}}(t)\cdot\boldsymbol{v}_{\mathrm{B}}(t_{\mathrm{r}})}{c^2}\Big) \\
&=\frac{-1}{r\Big[1+\frac{(\boldsymbol{r}\cdot\boldsymbol{v}_{\mathrm{B}})^2}{2r^2c^2}-\frac{v_{\mathrm{B}}^2}{2c^2}+\frac{\boldsymbol{r}\cdot\boldsymbol{a}_{\mathrm{B}}}{2c^2}\Big]}\Big(1-\frac{\boldsymbol{v}_{\mathrm{A}}\cdot\boldsymbol{v}_{\mathrm{B}}}{c^2}\Big)+O\Big(\frac{1}{c^3}\Big) \\
&=-\frac{1}{r}\Big[1-\frac{(\boldsymbol{r}\cdot\boldsymbol{v}_{\mathrm{B}})^2}{2r^2c^2}+\frac{v_{\mathrm{B}}^2}{2c^2}-\frac{\boldsymbol{r}\cdot\boldsymbol{a}_{\mathrm{B}}}{2c^2}-\frac{\boldsymbol{v}_{\mathrm{A}}\cdot\boldsymbol{v}_{\mathrm{B}}}{c^2}\Big]+O\Big(\frac{1}{c^3}\Big) \\
&\approx-\frac{1}{|\boldsymbol{X}_{\mathrm{A}}-\boldsymbol{X}_{\mathrm{B}}|}\Big[1-\frac{\boldsymbol{v}_{\mathrm{A}}\cdot\boldsymbol{v}_{\mathrm{B}}}{c^2}+\frac{1}{2c^2}\Big\{v_{\mathrm{B}}^2-\Big(\frac{\boldsymbol{X}_{\mathrm{A}}-\boldsymbol{X}_{\mathrm{B}}}{|\boldsymbol{X}_{\mathrm{A}}-\boldsymbol{X}_{\mathrm{B}}|}\cdot\boldsymbol{v}_{\mathrm{B}}\Big)^2\Big\}-\frac{(\boldsymbol{X}_{\mathrm{A}}-\boldsymbol{X}_{\mathrm{B}})\cdot\boldsymbol{a}_{\mathrm{B}}}{2c^2}\Big].
\end{aligned}
\tag{A.8}
$$

We find that the $1/c$ expansion of Φ is

$$
\begin{aligned}
\Phi&=\frac{e^2}{8\pi\hbar}\int dt\sum_{\mathrm{P,Q=R,L}}\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}\Big[\frac{v_{\mathrm{AP}}(t)\cdot v_{\mathrm{BQ}}(t_{\mathrm{BQ}})}{c(X_{\mathrm{AP}}(t)-X(t_{\mathrm{BQ}}))\cdot v_{\mathrm{BQ}}(t_{\mathrm{BQ}})}\Big]+(\mathrm{A}\leftrightarrow\mathrm{B}) \\
&\approx-\frac{e^2}{8\pi\hbar}\int dt\sum_{\mathrm{P,Q=R,L}}\frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\Big[1-\frac{\boldsymbol{v}_{\mathrm{AP}}\cdot\boldsymbol{v}_{\mathrm{BQ}}}{c^2} \\
&\quad+\frac{1}{2c^2}\Big\{v_{\mathrm{BQ}}^2-\Big(\frac{\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\cdot\boldsymbol{v}_{\mathrm{BQ}}\Big)^2\Big\}-\frac{(\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}})\cdot\boldsymbol{a}_{\mathrm{BQ}}}{2c^2}\Big]+(\mathrm{A}\leftrightarrow\mathrm{B}).
\end{aligned}
\tag{A.9}
$$

For the non-relativistic limit $c\to\infty$, the quantity Φ is

$$
\begin{aligned}
\Phi&\to-\frac{e^2}{8\pi\hbar}\int dt\sum_{\mathrm{P,Q=R,L}}\frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}+(\mathrm{A}\leftrightarrow\mathrm{B}) \\
&=-\frac{e^2}{4\pi\hbar}\int dt\Big(\frac{1}{|\boldsymbol{X}_{\mathrm{AR}}-\boldsymbol{X}_{\mathrm{BR}}|}-\frac{1}{|\boldsymbol{X}_{\mathrm{AR}}-\boldsymbol{X}_{\mathrm{BL}}|}-\frac{1}{|\boldsymbol{X}_{\mathrm{AL}}-\boldsymbol{X}_{\mathrm{BR}}|}+\frac{1}{|\boldsymbol{X}_{\mathrm{AL}}-\boldsymbol{X}_{\mathrm{BL}}|}\Big).
\end{aligned}
\tag{A.10}
$$

This result is equivalent to the quantity (2.50) (in the unit $\hbar=1$) computed in the non-relativistic regime.

B Detail derivation of $\Gamma_{\mathbf{RL}}$, $\Gamma_{\mathbf{A}}$, $\Gamma_{\mathbf{B}}$, $\Gamma_{\mathbf{c}}$ and Φ

We present the detailed calculation of $\Gamma_{\rm RL}, \Gamma_{\rm A}, \Gamma_{\rm B}, \Gamma_{\rm c}$, and Φ. In this calculation, we assume that the charged particle has the non-relativistic velocity. We recover the constants c and $\hbar$ when we show the result of the calculation or use the formula of the $1/c$ expansion of Φ derived as (A.9).

B.1 Computations of $\Gamma_{\mathbf{RL}}$, $\Gamma_{\mathbf{A}}$ and $\Gamma_{\mathbf{B}}$

We first calculate the quantity $\Gamma_{\rm RL}$. We assume the following trajectories

$$X^{\mu}_{\rm P}(t) = \Big[t, \epsilon_{\rm P} X(t), 0, 0\Big]^{\rm T}, \quad \epsilon_{\rm R} = -\epsilon_{\rm L} = 1, \quad X(t) = 8L\Big(1-\frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2. \tag{B.1}$$

Using Eq. (3.23), we obtain

$$\begin{aligned}
\Gamma_{\rm RL} &= \frac{e^2}{4}\oint_{\rm C} dx^{\mu}\oint_{\rm C} dy^{\mu}\langle\{\hat{A}^{\rm I}_{\mu}(x), \hat{A}^{\rm I}_{\nu}(y)\}\rangle \\
&\approx \frac{e^2}{4}\oint_{\rm C} dx^{\mu}\oint_{\rm C} dy^{\mu}\langle\{\hat{A}^{\rm I}_{\mu}(x^0,\mathbf{0}), \hat{A}^{\rm I}_{\nu}(y^0,\mathbf{0})\}\rangle \\
&= \frac{e^2}{4}\oint_{\rm C} dx^{\mu}\oint_{\rm C} dy^{\mu}\frac{\eta_{\mu\nu}}{4\pi^2}\Big(\frac{1}{-(t-t'-i\epsilon)^2}+\frac{1}{-(t-t'+i\epsilon)^2}\Big) \\
&= \frac{e^2}{16\pi^2}\int_0^T dt\Big(\frac{dX^{\mu}_{\rm R}}{dt}-\frac{dX^{\mu}_{\rm L}}{dt}\Big)\int_0^T dt'\Big(\frac{dX_{{\rm R}\,\mu}}{dt'}-\frac{dX_{{\rm L}\,\mu}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2}+\frac{1}{-(t-t'+i\epsilon)^2}\Big) \\
&= \frac{e^2}{16\pi^2}\int_0^T dt\int_0^T dt'\Big(\frac{d\boldsymbol{X}_{\rm R}}{dt}-\frac{d\boldsymbol{X}_{\rm L}}{dt}\Big)\cdot\Big(\frac{d\boldsymbol{X}_{\rm R}}{dt'}-\frac{d\boldsymbol{X}_{\rm L}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2}+\frac{1}{-(t-t'+i\epsilon)^2}\Big) \\
&= \frac{32e^2}{3\pi^2}\frac{L^2}{T^2},
\end{aligned} \tag{B.2}$$

where we took the limit $\epsilon \to 0$ after the integration, and in the second line we used the dipole approximation [86, 109] which ignores the spatial dependence of the photon field. The dipole approximation is valid when the wave length of the photon field $\lambda_{\rm p} = T$ is considerably larger than the typical size ($\sim L$) of the region where the charge exists. This condition is always satisfied if we assume the non-relativistic velocity $L/T \ll 1$.

We next consider the quantity Γ_i (3.49) given in the model of two charged particles. Because of the time and spatial translation invariance of the vacuum state, Γ_i is independent of the choice of the origin. Assuming that each of the charged particles A and B follows the trajectories defined by (B.1) up to the choice of the origin of the time or spatial axis, we can evaluate Γ_{A} and Γ_{B} as

$$\Gamma_{\mathrm{A}} = \Gamma_{\mathrm{B}} = \Gamma_{\mathrm{RL}} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \tag{B.3}$$

where we recovered the constants c and $\hbar$.

B.2 Computations of Γ_{c} and Φ for the linear configuration

B.2.1 $T \gg D \sim L$ or $T \gg D \gg L$ regimes

Here, we focus on the regime $T \gg D \sim L$ or $T \gg D \gg L$ for the linear configuration. We assume the trajectories of two charged particles A and B as follows

$$X^{\mu}_{\mathrm{AP}} = [t, \epsilon_{\mathrm{P}} X(t), 0, 0]^{\mathrm{T}}, \quad X^{\mu}_{\mathrm{BQ}}(t) = \Big[t, \epsilon_{\mathrm{Q}} X(t) + D, 0, 0\Big]^{\mathrm{T}}, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \tag{B.4}$$

$$X(t) = 8L\Big(1 - \frac{t}{T}\Big)^2 \Big(\frac{t}{T}\Big)^2. \tag{B.5}$$

The parameters L and D should be $D > L \geq 2X(t)$ to avoid overlapping each trajectory of particles A and B. First, we focus on the regime $T \gg D \sim L$. The quantity Γ_{c} is computed by Eq. (3.50) as

$$\begin{aligned}
\Gamma_c &= \frac{e^2}{2} \oint_{\mathrm{C_A}} dx^{\mu} \oint_{\mathrm{C_B}} dy^{\nu} \, \langle \{ \hat{A}^{\mathrm{I}}_{\mu}(x), \hat{A}^{\mathrm{I}}_{\nu}(y) \} \rangle \\
&\approx \frac{e^2}{2} \oint_{\mathrm{C_A}} dx^{\mu} \oint_{\mathrm{C_B}} dy^{\nu} \, \langle \{ \hat{A}^{\mathrm{I}}_{\mu}(x^0, \mathbf{0}), \hat{A}^{\mathrm{I}}_{\nu}(y^0, \mathbf{0}) \} \rangle \\
&= \frac{e^2}{2} \oint_{\mathrm{C_A}} dx^{\mu} \oint_{\mathrm{C_B}} dy^{\nu} \, \frac{\eta_{\mu\nu}}{4\pi^2} \Big(\frac{1}{-(x^0 - y^0 - i\epsilon)^2} + \frac{1}{-(x^0 - y^0 + i\epsilon)^2} \Big) \\
&= \frac{e^2}{8\pi^2} \int_0^T dt \Big(\frac{dX^{\mu}_{1\mathrm{R}}}{dt} - \frac{dX^{\mu}_{1\mathrm{L}}}{dt} \Big) \int_0^T dt' \Big(\frac{dX_{2\mathrm{R}\,\mu}}{dt'} - \frac{dX_{2\mathrm{L}\,\mu}}{dt'} \Big) \Big(\frac{1}{-(t - t' - i\epsilon)^2} + \frac{1}{-(t - t' + i\epsilon)^2} \Big) \\
&= \frac{e^2}{8\pi^2} \int_0^T dt \int_0^T dt' \Big(\frac{d\boldsymbol{X}_{1\mathrm{R}}}{dt} - \frac{d\boldsymbol{X}_{1\mathrm{L}}}{dt} \Big) \cdot \Big(\frac{d\boldsymbol{X}_{2\mathrm{R}}}{dt'} - \frac{d\boldsymbol{X}_{2\mathrm{L}}}{dt'} \Big) \Big(\frac{1}{-(t - t' - i\epsilon)^2} + \frac{1}{-(t - t' + i\epsilon)^2} \Big) \\
&= \frac{64e^2}{3\pi^2} \frac{L^2}{T^2},
\end{aligned} \tag{B.6}$$

where the dipole approximation was used in the second line because of the condition $T \gg L$. The quantity Φ is evaluated using the result of (A.9) as

$$
\begin{aligned}
\Phi = & -\frac{e^2}{8\pi\hbar}\int dt \sum_{\mathrm{P,Q=R,L}} \frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\Big[1-\frac{\boldsymbol{v}_{\mathrm{AP}}\cdot\boldsymbol{v}_{\mathrm{BQ}}}{c^2} \\
& +\frac{1}{2c^2}\Big\{v_{\mathrm{BQ}}^2-\Big(\frac{\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\cdot\boldsymbol{v}_{\mathrm{BQ}}\Big)^2\Big\}-\frac{(\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}})\cdot\boldsymbol{a}_{\mathrm{BQ}}}{2c^2}\Big]+(\mathrm{A}\leftrightarrow\mathrm{B}) \\
= & -\frac{e^2}{8\pi\hbar}\int dt \sum_{\mathrm{P,Q=R,L}} \frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{|D-(\epsilon_{\mathrm{P}}-\epsilon_{\mathrm{Q}})X(t)|}\Big[1-\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}\frac{v^2(t)}{c^2}-\epsilon_{\mathrm{Q}}\frac{\{-D+(\epsilon_{\mathrm{P}}-\epsilon_{\mathrm{Q}})X(t)\}a(t)}{2c^2}\Big] \\
& +(\mathrm{A}\leftrightarrow\mathrm{B}) \\
= & -\frac{e^2}{4\pi\hbar}\int dt\Big[\frac{2}{D}\Big(1-\frac{v^2}{c^2}\Big)-\Big(1+\frac{v^2}{c^2}\Big)\Big(\frac{1}{|D-2X(t)|}+\frac{1}{|D+2X(t)|}\Big) \\
& +\frac{a(t)}{2c^2}\Big(\frac{D-2X(t)}{|D-2X(t)|}-\frac{D+2X(t)}{|D+2X(t)|}\Big)\Big] \\
= & -\frac{e^2}{4\pi\hbar}\int dt\Big[\frac{2}{D}\Big(1-\frac{v^2}{c^2}\Big)-\Big(1+\frac{v^2}{c^2}\Big)\Big(\frac{1}{D-2X(t)}+\frac{1}{D+2X(t)}\Big)\Big],
\end{aligned}
\tag{B.7}
$$

where we have recovered the natural units c and $\hbar$ to show the result of the $1/c$ expansion. Next, we consider the regime $T \gg D \gg L$. In this regime, we obtain the Γ_{c} and Φ using (3.50) and (B.7) as follows,

$$
\begin{aligned}
\Gamma_c = & \frac{e^2}{2}\oint_{\mathrm{C_A}} dx^\mu \oint_{\mathrm{C_B}} dy^\nu \,\langle\{\hat{A}^{\mathrm{I}}_\mu(x),\hat{A}^{\mathrm{I}}_\nu(y)\}\rangle \\
= & \frac{e^2}{2}\oint_{\mathrm{C_A}} dx^\mu \oint_{\mathrm{C_B}} dy^\nu \,\frac{\eta_{\mu\nu}}{4\pi^2}\Big(\frac{1}{-(x^0-y^0-i\epsilon)^2+|\boldsymbol{x}-\boldsymbol{y}|^2}+\frac{1}{-(x^0-y^0+i\epsilon)^2+|\boldsymbol{x}-\boldsymbol{y}|^2}\Big) \\
\approx & \frac{e^2}{2}\oint_{\mathrm{C_A}} dx^\mu \oint_{\mathrm{C_B}} dy^\nu \,\frac{\eta_{\mu\nu}}{4\pi^2}\Big(\frac{1}{-(x^0-y^0-i\epsilon)^2+D^2}+c.c.\Big) \\
= & \frac{e^2}{8\pi^2}\int_0^T dt\Big(\frac{dX^\mu_{1\mathrm{R}}}{dt}-\frac{dX^\mu_{1\mathrm{L}}}{dt}\Big)\int_0^T dt'\Big(\frac{dX_{2\mathrm{R}\,\mu}}{dt'}-\frac{dX_{2\mathrm{L}\,\mu}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2+D^2}+c.c.\Big) \\
= & \frac{e^2}{8\pi^2}\int_0^T dt\int_0^T dt'\Big(\frac{d\boldsymbol{X}_{1\mathrm{R}}}{dt}-\frac{d\boldsymbol{X}_{1\mathrm{L}}}{dt}\Big)\cdot\Big(\frac{d\boldsymbol{X}_{2\mathrm{R}}}{dt'}-\frac{d\boldsymbol{X}_{2\mathrm{L}}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2+D^2}+c.c.\Big) \\
\approx & \frac{64e^2}{3\pi^2}\frac{L^2}{T^2}\Big(1+\frac{4D^2}{T^2}\ln\Big[\frac{D}{T}\Big]\Big),
\end{aligned}
\tag{B.8}
$$

where the distance between the particles $|\boldsymbol{x}-\boldsymbol{y}|$ was approximated as D because of $D \gg L$ in the third line, and in the final line we took the limit $\epsilon \to 0$ and the leading order of

$T/D \ll 1$ after the integration, and

$$\begin{aligned}
\Phi &= -\frac{e^2}{4\pi\hbar}\int dt\Big[\frac{2}{D}\Big(1-\frac{v^2}{c^2}\Big)-\Big(1+\frac{v^2}{c^2}\Big)\Big(\frac{1}{D-2X(t)}+\frac{1}{D+2X(t)}\Big)\Big]\\
&\approx -\frac{e^2}{4\pi\hbar}\int dt\Big[\frac{2}{D}\Big(1-\frac{v^2}{c^2}\Big)-\frac{2}{D}\Big(1+\frac{v^2}{c^2}\Big)\Big(1+\frac{4X^2(t)}{D^2}\Big)\Big]\\
&\approx -\frac{e^2}{4\pi\hbar}\int dt\Big[-\frac{4}{D}\frac{v^2}{c^2}-\frac{8X^3(t)}{D^3}\Big]\\
&= \frac{64e^2}{315\pi\hbar c}\Big(\frac{L}{cT}\Big)^2\Big(\frac{6cT}{D}+\Big(\frac{cT}{D}\Big)^3\Big),
\end{aligned} \tag{B.9}$$

where we took the leading order of $4X^2(t)/D^2 \sim O(L^2/D^2) \ll 1$ in the second line, and neglected $\mathcal{O}(L^4/D^4)$ in the last line. Therefore, we obtain the result in the linear configuration in $cT \gg D \gg L$ regime as

$$\Gamma_{\mathrm{A}} = \Gamma_{\mathrm{B}} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Gamma_{\mathrm{c}} \approx \frac{64e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}\Big(1+\frac{4D^2}{(cT)^2}\ln\Big[\frac{D}{cT}\Big]\Big), \tag{B.10}$$

$$\Phi \approx \frac{64e^2}{315\pi\hbar c}\Big(\frac{L}{cT}\Big)^2\Big(\frac{6cT}{D}+\Big(\frac{cT}{D}\Big)^3\Big). \tag{B.11}$$

B.2.2 $D \gg T \gg L$ regime

Here, we focus on the regime $D \gg T \gg L$ and calculate the quantities Γ_{c} and Φ. We assume the following trajectories of the two charged particles A and B as

$$X^{\mu}_{\mathrm{AP}}(t) = \Big[t, \epsilon_{\mathrm{P}}X(t), 0, 0\Big]^T, \quad X^{\mu}_{\mathrm{BQ}}(t) = \Big[t, \epsilon_{\mathrm{Q}}X(t-D)+D, 0, 0\Big]^T, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \tag{B.12}$$

$$X(t) = 8L\Big(1-\frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2, \tag{B.13}$$

where $X^\mu_{\rm BQ}$ is defined in $D \le t \le T + D$. First, we calculate the quantity $\Gamma_{\rm c}$ in this regime by using (3.50) as

$$
\begin{aligned}
\Gamma_c &= \frac{e^2}{2}\oint_{\rm C_A} dx^\mu \oint_{\rm C_B} dy^\nu \, \langle\{\hat{A}^{\rm I}_\mu(x), \hat{A}^{\rm I}_\nu(y)\}\rangle \\
&= \frac{e^2}{2}\oint_{\rm C_A} dx^\mu \oint_{\rm C_B} dy^\nu \, \frac{\eta_{\mu\nu}}{4\pi^2}\Big(\frac{1}{-(x^0-y^0-i\epsilon)^2+|\boldsymbol{x}-\boldsymbol{y}|^2} + \frac{1}{-(x^0-y^0+i\epsilon)^2+|\boldsymbol{x}-\boldsymbol{y}|^2}\Big) \\
&\approx \frac{e^2}{2}\oint_{\rm C_A} dx^\mu \oint_{\rm C_B} dy^\nu \, \frac{\eta_{\mu\nu}}{4\pi^2}\Big(\frac{1}{-(x^0-y^0-i\epsilon)^2+D^2} + c.c.\Big) \\
&= \frac{e^2}{8\pi^2}\int_0^T dt \Big(\frac{dX^\mu_{\rm 1R}}{dt} - \frac{dX^\mu_{\rm 1L}}{dt}\Big)\int_D^{T+D} dt' \Big(\frac{dX_{{\rm 2R}\,\mu}}{dt'} - \frac{dX_{{\rm 2L}\,\mu}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2+D^2} + c.c.\Big) \\
&= \frac{e^2}{8\pi^2}\int_0^T dt \int_D^{T+D} dt' \Big(\frac{d\boldsymbol{X}_{\rm 1R}}{dt} - \frac{d\boldsymbol{X}_{\rm 1L}}{dt}\Big)\cdot\Big(\frac{d\boldsymbol{X}_{\rm 2R}}{dt'} - \frac{d\boldsymbol{X}_{\rm 2L}}{dt'}\Big)\Big(\frac{1}{-(t-t'-i\epsilon)^2+D^2} + c.c.\Big) \\
&\approx \frac{e^2}{8\pi^2}\frac{4}{D^2}\int_0^T dt \int_D^{T+D} dt' \frac{dX(t)}{dt}\cdot\frac{dX(t'-D)}{dt'}\Big\{1 + \frac{(t-t'-i\epsilon)^2}{D^2} + 1 + \frac{(t-t'+i\epsilon)^2}{D^2}\Big\} \\
&= \frac{e^2}{2\pi^2 D^4}\int_0^T dt \int_D^{T+D} dt' \frac{dX(t)}{dt}\cdot\frac{dX(t'-D)}{dt'}\Big\{(t-t'-i\epsilon)^2 + (t-t'+i\epsilon)^2\Big\} \\
&= -\frac{32e^2}{225\pi^2}\frac{L^2T^2}{D^4}.
\end{aligned}
\tag{B.14}
$$

where the distance between the particles $|\boldsymbol{x}-\boldsymbol{y}|$ was approximated as D because of $D \gg L$ in the third line. We used the geometric series expansion because of $|(t-t^{'} \pm i\epsilon)| < T \ll D$ in the third to last line, and in the final line, we took the limit $\epsilon \to 0$ after the integration. We next calculate the quantity Φ using Eq. (3.54) in this regime. The quantity Φ is

$$
\begin{aligned}
\Phi &= \frac{e}{4}\Big(\int_{\rm S_A} d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\rm B}(x) + \int_{\rm S_B} d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\rm A}(x)\Big) \\
&= \frac{e}{4}\int_{\rm S_B} d\sigma_{\mu\nu}\Delta F^{\mu\nu}_{\rm A}(x) \\
&= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)+D}^{X_{\rm BR}(t)+D} dx\, \Delta F^{01}_1(t,x,0,0) \\
&= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx\, \Delta F^{01}_1(t,x+D,0,0),
\end{aligned}
\tag{B.15}
$$

where the region $S_{\rm B} = \{D \le t \le T + D,\ X_{\rm BL}(t) + D \le x \le X_{\rm BR}(t) + D,\ y = 0,\ z = 0\}$, and the first term in the first line vanishes because, in this configuration, particle A does not experience the retarded field of particle B. We changed the variable $x \to x + D$ in the final line. The quantity Φ is decomposed into two terms $\Phi = \Phi_{\rm v} + \Phi_{\rm a}$, which are given as follows (see Eqs. (3.89) and (3.90))

$$
\begin{aligned}
\Phi_{\rm v} &= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx\, \Delta F^{01}_{\rm A,v}(t, x+D, 0, 0) \\
&= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\left[\frac{e}{4\pi}\frac{(t-t_{\rm AP})v_{\rm AP}(t_{\rm AP}) - (x+D-X_{\rm AP}(t_{\rm AP}))}{\gamma_{\rm AP}^2[t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\right],
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
\Phi_{\rm a} &= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx\, \Delta F^{01}_{\rm A,a}(t, x+D, 0, 0) \\
&= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\frac{e}{4\pi[t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm 1P}))v_{\rm AP}(t_{\rm AP})]^2} \\
&\quad\times\Bigg[(t-t_{\rm AP})\Bigg(a_{\rm AP}(t_{\rm AP}) + \frac{(x+D-X_{\rm AP}(t_{\rm AP}))a_{\rm AP}(t_{\rm AP})}{t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})}v_{\rm AP}(t_{\rm AP})\Bigg) \\
&\quad - \frac{(x+D-X_{\rm AP}(t_{\rm AP}))^2 a_{\rm AP}(t_{\rm AP})}{(t-t_{\rm AP})-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})}\Bigg)\Bigg], \\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\left[\frac{(t-t_{\rm AP})^2-(x+D-X_{\rm AP}(t_{\rm AP}))^2}{[t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\right]a_{\rm AP}(t_{\rm AP}),
\end{aligned}
\tag{B.17}
$$

where the retarded time $t_{\rm AP}$ is approximated by neglecting $\mathcal{O}(L^2/D^2)$ as

$$
t_{\rm AP} = t - |\boldsymbol{x} - \boldsymbol{X}_{\rm AP}(t_{\rm AP})| = t - \sqrt{(x+D-X_{\rm AP}(t_{\rm AP}))^2} \approx t - D, \tag{B.18}
$$

where $(x - X_{\rm AP}(t_{\rm AP})) \sim \mathcal{O}(L)$. For $D \gg cT \gg L$, we can approximate $\Phi_{\rm v}$ as

$$\begin{aligned}
\Phi_{\rm v} &= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{e}{4\pi}\frac{(t-t_{\rm AP})v_{\rm AP}(t_{\rm AP}) - (x+D-X_{\rm AP}(t_{\rm AP}))}{\gamma_{\rm AP}^2[t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\\
&\approx \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{v_{\rm AP}(t-D)}{D^2} + \frac{X_{\rm AP}(t-D)}{D^3} - \frac{x+D}{D^3}\Big]\\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt(X_{\rm BR}(t)-X_{\rm BL}(t))\Big[\frac{v_{\rm AR}(t-D)-v_{\rm AL}(t-D)}{D^2} + \frac{X_{\rm AR}(t-D)-X_{\rm AL}(t-D)}{D^3}\Big]\\
&= \frac{16e^2}{315\pi}\frac{L^2T}{D^3}. \qquad \text{(B.19)}
\end{aligned}$$

Moreover, in the second line of the above equation, we substituted the retarded condition (B.18) into Eq. (B.19) and approximated the denominator as

$$\begin{aligned}
&\gamma_{\rm AP}^2[t-t_{\rm AP}-(x+D-X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]\\
&\quad\approx \big(1-v_{\rm AP}^2(t-D)\big)^{-1}[D-(x+D-X_{\rm AP}(t-D))v_{\rm AP}(t-D)]\\
&\quad= D\big(1-v_{\rm AP}^2(t-D)\big)^{-1}[1-(1+(x-X_{\rm AP}(t-D))/D)v_{\rm AP}(t-D)]\\
&\quad\approx D, \qquad \text{(B.20)}
\end{aligned}$$

where $v_{\rm AP} \sim \mathcal{O}(L/T)$, $v_{\rm AP}^2 \sim \mathcal{O}(L^2/T^2)$, and $(x - X_{\rm AP})/D \sim \mathcal{O}(L/D)$ were neglected in the last line. However, the quantity $\Phi_{\rm a}$ is exactly equal to zero because of the retarded time condition (B.18). This result indicates that in the context of equation (3.97), the electric field $E_x^{\rm AR,a}$ ($E_x^{\rm AL,a}$) is equal to zero because the electromagnetic wave cannot propagate the direction of the acceleration of the charged particle A. Therefore, we summarize the result in the linear configuration in $D \gg cT \gg L$ regime as follows

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Gamma_{\rm c} \approx -\frac{32e^2}{225\pi^2\hbar c}\frac{L^2(cT)^2}{D^4}, \quad \Phi \approx \frac{16e^2}{315\pi\hbar c}\frac{L^2(cT)}{D^3}. \qquad \text{(B.21)}$$

B.3 Computation of Γ_c and Φ for parallel configuration

B.3.1 $T \gg L \gg D$ or $T \gg D \gg L$ regimes

Here, we focus on the regimes $T \gg L \gg D$ or $T \gg D \gg L$ and calculate the quantities Γ_c and Φ. We assume the following trajectories of the two charged particles A and B as

$$X^{\mu}_{\mathrm{AP}}(t) = \Big[t, \epsilon_{\mathrm{P}} X(t), 0, 0\Big]^{\mathrm{T}}, \quad X^{\mu}_{\mathrm{BQ}}(t) = \Big[t, \epsilon_{\mathrm{Q}} X(t), D, 0\Big]^{\mathrm{T}}, \quad \epsilon_{\mathrm{R}} = -\epsilon_{\mathrm{L}} = 1, \tag{B.22}$$

$$X(t) = 8L\Big(1-\frac{t}{T}\Big)^2\Big(\frac{t}{T}\Big)^2, \tag{B.23}$$

In these regimes, the approximate form of Γ_c is equal to (B.8). Neglecting $\mathcal{O}(D^2/T^2)$ in $T \gg L \gg D$, we obtain the quantity Γ_c as

$$\Gamma_c \approx \frac{64e^2}{3\pi^2}\frac{L^2}{T^2}, \tag{B.24}$$

The quantity Φ up to $\mathcal{O}(1/c^2)$ obtained from (A.9) is

$$\begin{aligned}
\Phi = &-\frac{e^2}{8\pi\hbar}\int dt \sum_{\mathrm{P,Q=R,L}} \frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\Big[1-\frac{\boldsymbol{v}_{\mathrm{AP}}\cdot\boldsymbol{v}_{\mathrm{BQ}}}{c^2} \\
&+\frac{1}{2c^2}\Big\{v^2_{\mathrm{BQ}}-\Big(\frac{\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}}{|\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}}|}\cdot\boldsymbol{v}_{\mathrm{BQ}}\Big)^2\Big\}-\frac{(\boldsymbol{X}_{\mathrm{AP}}-\boldsymbol{X}_{\mathrm{BQ}})\cdot\boldsymbol{a}_{\mathrm{BQ}}}{2c^2}\Big]+(\mathrm{A}\leftrightarrow\mathrm{B}) \\
=&-\frac{e^2}{8\pi\hbar}\int dt \sum_{\mathrm{P,Q=R,L}} \frac{\epsilon_{\mathrm{P}}\epsilon_{\mathrm{Q}}}{\sqrt{(X_{\mathrm{AP}}-X_{\mathrm{BQ}})^2+D^2}}\Big[1-\frac{v_{\mathrm{AP}}v_{\mathrm{BQ}}}{c^2} \\
&+\frac{1}{2c^2}\Big\{v^2_{\mathrm{BQ}}-\Big(\frac{X_{\mathrm{AP}}-X_{\mathrm{BQ}}}{\sqrt{(X_{\mathrm{AP}}-X_{\mathrm{BQ}})^2+D^2}}v_{\mathrm{BQ}}\Big)^2\Big\}-\frac{(X_{\mathrm{AP}}-X_{\mathrm{BQ}})a_{\mathrm{BQ}}}{2c^2}\Big]+(\mathrm{A}\leftrightarrow\mathrm{B}) \\
=&-\frac{e^2}{4\pi\hbar}\int dt\Big(\frac{2}{D}\Big[1-\frac{v^2}{2c^2}\Big]-\frac{2}{\sqrt{4X^2+D^2}}\Big[1+\Big(1+\frac{D^2}{2(4X^2+D^2)}\Big)\frac{v^2}{c^2}+\frac{Xa}{c^2}\Big]\Big).
\end{aligned} \tag{B.25}$$

For $cT \gg L \gg D$, the quantity Φ is approximated as

$$\begin{aligned}\Phi &= -\frac{e^2}{4\pi\hbar}\int dt\Big(\frac{2}{D}\Big[1-\frac{v^2}{2c^2}\Big] - \frac{2}{\sqrt{4X^2+D^2}}\Big[1+\Big(1+\frac{D^2}{2(4X^2+D^2)}\Big)\frac{v^2}{c^2}+\frac{Xa}{c^2}\Big]\Big) \\ &\approx -\frac{e^2}{4\pi\hbar}\int dt\frac{2}{D}\Big[1-\frac{v^2}{2c^2}\Big] \\ &= -\frac{e^2}{2\pi\hbar c}\frac{cT}{D}\Big(1-\frac{64L^2}{105(cT)^2}\Big),\end{aligned} \tag{B.26}$$

where we neglected $\mathcal{O}(D/L)$ in the second line. In the regime $cT \gg D \gg L$, we obtain

$$\begin{aligned}\Phi &= -\frac{e^2}{4\pi\hbar}\int dt\Big(\frac{2}{D}\Big[1-\frac{v^2}{2c^2}\Big] - \frac{2}{\sqrt{4X^2+D^2}}\Big[1+\Big(1+\frac{D^2}{2(4X^2+D^2)}\Big)\frac{v^2}{c^2}+\frac{Xa}{c^2}\Big]\Big) \\ &\approx -\frac{e^2}{4\pi\hbar}\int dt\Big[\frac{4X^2}{D^3}-\frac{4v^2+2Xa}{c^2D}\Big] \\ &= -\frac{32e^2}{315\pi\hbar c}\frac{cTL^2}{D^3}\Big(1-\frac{6D^2}{(cT)^2}\Big),\end{aligned} \tag{B.27}$$

where we used the Taylor expansion $(4X^2+D^2)^\alpha \approx D^{2\alpha}(1+4\alpha X^2/D^2)$ in the first line and neglected $\mathcal{O}(L^3/T^3)$ in the second line. Consequently, $\Gamma_{\rm A}, \Gamma_{\rm B}, \Gamma_{\rm c}$, and Φ in the parallel configuration are obtained as

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Gamma_{\rm c} \approx \frac{64e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Phi \approx -\frac{e^2}{2\pi\hbar c}\frac{cT}{D}\left(1-\frac{64L^2}{105(cT)^2}\right), \tag{B.28}$$

for $cT \gg L \gg D$, and

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Gamma_{\rm c} \approx \frac{64e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}\left(1+\frac{4D^2}{(cT)^2}\ln\left[\frac{D}{cT}\right]\right), \tag{B.29}$$

$$\Phi \approx -\frac{32e^2}{315\pi\hbar c}\frac{cTL^2}{D^3}\left(1-\frac{6D^2}{(cT)^2}\right), \tag{B.30}$$

for $cT \gg D \gg L$, respectively.

B.3.2 $D \gg T \gg L$ regime

Here, we consider the $D \gg T \gg L$ regime and calculate the quantities Γ_c and Φ. In this regime, the trajectories of the two charged particles A and B are assumed as follows

$$X^\mu_{\rm AP}(t) = \Big[t, \epsilon_{\rm P} X(t), 0, 0\Big]^{\rm T}, \quad X^\mu_{\rm BP}(t) = \Big[t, \epsilon_{\rm P} X(t-D), D, 0\Big]^{\rm T}, \quad \epsilon_{\rm R} = -\epsilon_{\rm L} = 1, \tag{B.31}$$

$$X(t) = 8L\Big(1 - \frac{t}{T}\Big)^2 \Big(\frac{t}{T}\Big)^2, \tag{B.32}$$

where $X^\mu_{\rm BQ}$ is defined in $D \le t \le T + D$. The quantity Γ_c is equal to the Eq. (B.14) because we can approximate the difference of the distance of the two charged particles $|\boldsymbol{x} - \boldsymbol{y}| \approx D$ and use the geometric series expansion because of $|(t - t' \pm i\epsilon)|/D < T/D \ll 1$ in this regime (detailed derivation, see the Eq. (B.14)). The quantity Φ is obtained as

$$\begin{aligned}
\Phi &= \frac{e}{4}\Big(\int_{\rm S_A} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm B}(x) + \int_{\rm S_B} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm A}(x)\Big) \\
&= \frac{e}{4}\int_{\rm S_B} d\sigma_{\mu\nu} \Delta F^{\mu\nu}_{\rm A}(x) \\
&= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx\, \Delta F^{01}_{\rm A}(t, x, D, 0),
\end{aligned} \tag{B.33}$$

where we note that the region $S_{\rm B} = \{D \le t \le T + D, X_{\rm BL}(t) \le x \le X_{\rm BR}, y = D, z = 0\}$; in this configuration of interest, the first term in the first line vanishes because the retarded field from particle B is causally disconnected with particle A. The retarded time $t_{\rm AP}$ is approximated as

$$t_{\rm AP} = t - |\boldsymbol{x} - \boldsymbol{X}_{\rm AP}(t_{\rm AP})| = t - \sqrt{(x - X_{\rm AP}(t_{\rm AP}))^2 + D^2} \approx t - D - \frac{(x - X_{\rm AP}(t - D))^2}{2D}, \tag{B.34}$$

where $(x - X_{\rm AP}(t_{\rm AP})) \sim \mathcal{O}(L)$ and $\mathcal{O}(L^2/D^2)$ was neglected. We therefore obtain the quantity $\Phi_{\rm v}$ and $\Phi_{\rm a}$ as

$$\begin{aligned}
\Phi_{\rm v} &= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{e}{4\pi}\frac{(t-t_{\rm AP})v_{\rm AP}(t_{\rm AP}) - (x - X_{\rm 1P}(t_{\rm AP}))}{\gamma_{\rm AP}^2[t - t_{\rm AP} - (x - X_{\rm 1P}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\Big] \\
&\approx \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{v_{\rm AP}(t-D)}{D^2} - \frac{x - X_{\rm AP}(t-D)}{D^3}\Big] \\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt (X_{\rm BR}(t) - X_{\rm BL}(t))\Big[\frac{v_{\rm AR}(t-D) - v_{\rm AL}(t-D)}{D^2} + \frac{X_{\rm AR}(t-D) - X_{\rm AL}(t-D)}{D^3}\Big] \\
&= \frac{16e^2}{315\pi}\frac{L^2 T}{D^3}, \qquad \text{(B.35)}
\end{aligned}$$

where in the second line of the above equation, the denominator was approximated in the same manner performed in (B.20), and

$$\begin{aligned}
\Phi_{\rm a} &= \frac{e}{2}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\frac{e}{4\pi[t - t_{\rm AP} - (x - X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^2} \\
&\quad \times \Big[(t - t_{\rm AP})\Big(a_{\rm AP}(t_{\rm AP}) + \frac{(x - X_{\rm AP}(t_{\rm AP}))a_{\rm AP}(t_{\rm AP})}{t - t_{\rm AP} - (x - X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})}v_{\rm AP}(t_{\rm AP})\Big) \\
&\quad - \frac{(x - X_{\rm AP}(t_{\rm AP}))^2 a_{\rm AP}(t_{\rm AP})}{(t - t_{\rm AP}) - (x - X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})}\Big)\Big] \\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{(t - t_{\rm AP})^2 - (x - X_{\rm AP}(t_{\rm AP}))^2}{[t - t_{\rm AP} - (x - X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\Big] a_{\rm AP}(t_{\rm AP}) \\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\Big[\frac{D^2}{[t - t_{\rm AP} - (x - X_{\rm AP}(t_{\rm AP}))v_{\rm AP}(t_{\rm AP})]^3}\Big] a_{\rm AP}(t_{\rm AP}) \\
&\approx \frac{e^2}{8\pi}\int_D^{T+D} dt \int_{X_{\rm BL}(t)}^{X_{\rm BR}(t)} dx \sum_{\rm P=R,L} \epsilon_{\rm P}\frac{a_{\rm AP}(t-D)}{D} \\
&= \frac{e^2}{8\pi}\int_D^{T+D} dt (X_{\rm BR}(t) - X_{\rm BL}(t))\Big[\frac{a_{\rm AR}(t-D) - a_{\rm AL}(t-D)}{D}\Big] \\
&= -\frac{64e^2}{105\pi}\frac{L^2}{DT}, \qquad \text{(B.36)}
\end{aligned}$$

where we substituted the retarded time condition (B.34) into the second line of the above equation and neglected the $\mathcal{O}(L^2/D^2)$ and $v \sim \mathcal{O}(L/T)$ in the third line of the denominator.

Consequently, the quantity Φ is

$$\Phi \approx -\frac{64e^2}{105\pi}\frac{L^2}{DT}\left(1-\frac{T^2}{12D^2}\right) \approx -\frac{64e^2}{105\pi}\frac{L^2}{DT}, \tag{B.37}$$

where we neglected the second term because of $D \gg T$ in the last equality. Thus, $\Gamma_{\rm A}, \Gamma_{\rm B}, \Gamma_{\rm c}$, and Φ in the parallel configuration in the regime $D \gg cT \gg L$ are

$$\Gamma_{\rm A} = \Gamma_{\rm B} \approx \frac{32e^2}{3\pi^2\hbar c}\frac{L^2}{(cT)^2}, \quad \Gamma_{\rm c} \approx -\frac{32e^2}{225\pi^2\hbar c}\frac{L^2(cT)^2}{D^4}, \quad \Phi \approx -\frac{64e^2}{105\pi\hbar c}\frac{L^2}{D(cT)}. \tag{B.38}$$

C Proof of the statement in (4.43)

We numerically prove the statement in (4.43). Using the Robertson inequality(4.42), $\Gamma_{\rm A}\Gamma_{\rm B} \geq \Phi_{\rm AB}^2/16$, we have

$$1 - e^{-2\Gamma_{\rm A}} - e^{-2\Gamma_{\rm B}} \sin^2\left(\frac{\Phi_{\rm BA}}{2}\right) \geq 1 - e^{-2\Gamma_{\rm A}} - e^{-\Phi_{\rm BA}^2/8\Gamma_{\rm A}} \sin^2\left(\frac{\Phi_{\rm BA}}{2}\right) = f(X, Y), \quad \text{(C.1)}$$

where we defined the function $f(X, Y)$ with $X = e^{-2\Gamma_{\rm A}}$ and $Y = e^{-\Phi_{\rm BA}^2/8\Gamma_{\rm A}}$ as follows:

$$f(X, Y) = 1 - X - Y \sin^2\left(\sqrt{\log X \log Y}\right). \quad \text{(C.2)}$$

As it is sufficient to consider that $\Gamma_{\rm A} > 0$ and $\Phi_{\rm BA} > 0$, we can assume that $0 < X < 1$ and $0 < Y < 1$.

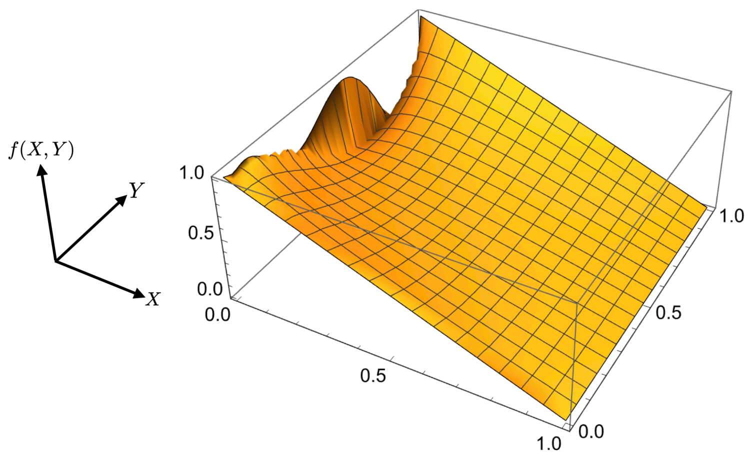


FIG 23: Behavior of the function $f(X, Y)$ where the region $0 < X < 1$ and $0 < Y < 1$.

Fig. 23 shows the behavior of the function $f(X, Y)$, which is positive in the regions $0 < X < 1$ and $0 < Y < 1$. Since the function $f(X, Y)$ is positive, the inequality $e^{-2\Gamma_{\rm A}} + e^{-2\Gamma_{\rm B}} \sin^2(\Phi_{\rm BA}/2) \leq 1$ in (4.38) is satisfied. Hence, the Robertson inequality (4.42) is the sufficient condition for the inequality (4.38), and the statement in (4.43) holds. In the following, we show that the function $f(X, Y)$ is always positive in an analytic manner.

Proof. Now let derive the partial derivatives to find the gradient for $f(X, Y)$, and the results are

$$\frac{\partial f(X,Y)}{\partial X} = -1 - \frac{Y \log Y \sin\left(\sqrt{\log X \log Y}\right)\cos\left(\sqrt{\log X \log Y}\right)}{X\sqrt{\log X \log Y}}, \tag{C.3}$$

$$\frac{\partial f(X,Y)}{\partial Y} = -\left(\frac{\log X \cos\left(\sqrt{\log X \log Y}\right)}{\sqrt{\log X \log Y}} + \sin\left(\sqrt{\log X \log Y}\right)\right)\sin\left(\sqrt{\log X \log Y}\right). \tag{C.4}$$

We are looking for the gradient is zero:

$$0 = \log X \cos\left(\sqrt{\log X \log Y}\right) + \sqrt{\log X \log Y}\sin\left(\sqrt{\log X \log Y}\right), \tag{C.5}$$

and

$$\begin{aligned} 0 &= -X\sqrt{\log X \log Y} - Y \log Y \sin\left(\sqrt{\log X \log Y}\right)\cos\left(\sqrt{\log X \log Y}\right) \\ &= -X\left(\log X \log Y\right) - Y \log Y \left(\left(\sqrt{\log X \log Y}\right)\sin\left(\sqrt{\log X \log Y}\right)\right)\cos\left(\sqrt{\log X \log Y}\right), \end{aligned} \tag{C.6}$$

where we multiplied by the factor $\sqrt{\log X \log Y}$ in the second line. Substituting (C.5) into (C.6), we obtain the following condition

$$0 = \left(\log X \log Y\right)\left(-X - Y\sin^2\left(\sqrt{\log X \log Y}\right) + Y\right). \tag{C.7}$$

Case 1: $\log X \log Y = 0$, i.e., $X = 1$ or $Y = 1$. When $X = 1$, by definition of the function $f(X,Y)$, we have

$$f(1,Y) = 0, \tag{C.8}$$

where we used $\log 1 = 0$ and $\sin 0 = 0$ for arbitrary value Y. Note that when $Y \to 0$, then $\sqrt{\log X \log Y}$ is non-trivial. However, due to $Y \to 0$, $f(1,Y)$ becomes 0. When $Y = 1$,

$$f(X,1) = 1 - Y > 0, \tag{C.9}$$

where we used $\log 1 = 0$ and $\sin 0 = 0$ for arbitrary values X. Note that when $X \to 0$, then $\sqrt{\log X \log Y}$ is also non-trivial. However, in this case, $f(X,Y)$ is

$$\lim_{X \to 0} f(X,Y)|_{Y=1} = 1 - \sin^2\left(\sqrt{\log X \log Y}\right) > 0. \tag{C.10}$$

Thus, in case 1, $f(X,Y)$ is always positive.
Case 2: $-X - Y\sin^2\left(\sqrt{\log X \log Y}\right) + Y = 0$. Then $f(X,Y)$ becomes

$$\begin{aligned} f(X,Y) &= 1 - X - Y\sin^2\left(\sqrt{\log X \log Y}\right) \\ &= 1 - Y > 0. \end{aligned} \tag{C.11}$$

Thus, in case 2, $f(X,Y)$ is also always positive. In either case, $f(X,Y) \geq 0$, so the result is proven.

D Demonstration of the relationship expressed in the relationship (4.48)

In this Appendix, the relationship expressed in the relationship (4.48) is demonstrated in a numerical manner. For convenience, we rewrite $\lambda_{\min}$ as

$$\begin{aligned}\lambda_{\min} &= \frac{1}{4}\Big[1 - e^{-\Gamma_A-\Gamma_B}\cosh[\Gamma_c] - \Big\{\big(e^{-\Gamma_A} - e^{-\Gamma_B}\big)^2 \\ &\quad + 4e^{-\Gamma_A-\Gamma_B}\sin^2\left[\frac{(\Phi_{AB}+\Phi_{BA})}{4}\right] + e^{-2\Gamma_A-2\Gamma_B}\sinh^2[\Gamma_c]\Big\}^{\frac{1}{2}}\Big] \\ &= C\Big(\sinh[\Gamma_A]\sinh[\Gamma_B] - \sinh^2\left[\frac{\Gamma_c}{2}\right] - \sin^2\left[\frac{\Phi_{BA}}{4}\right]\Big),\end{aligned} \tag{D.1}$$

where the coefficient C is expressed as

$$\begin{aligned}C = e^{-\Gamma_A-\Gamma_B}\Big[1 - e^{-\Gamma_A-\Gamma_B}\cosh[\Gamma_c] + \Big\{\big(e^{-\Gamma_A} - e^{-\Gamma_B}\big)^2 \\ + 4e^{-\Gamma_A-\Gamma_B}\sin^2\left[\frac{(\Phi_{AB}+\Phi_{BA})}{4}\right] + e^{-2\Gamma_A-2\Gamma_B}\sinh^2[\Gamma_c]\Big\}^{1/2}\Big]^{-1}.\end{aligned} \tag{D.2}$$

The coefficient C is always positive because $1 - e^{-\Gamma_A-\Gamma_B}\cosh[\Gamma_c] > 0$ since $\Gamma_A + \Gamma_B \geq |\Gamma_c|$. Therefore, the condition $\lambda_{\min} \geq 0$ is equivalent to

$$\sinh[\Gamma_A]\sinh[\Gamma_B] - \sinh^2\left[\frac{\Gamma_c}{2}\right] - \sin^2\left[\frac{\Phi_{BA}}{4}\right] \geq 0. \tag{D.3}$$

Hereinafter, we regard the inequality (D.3) as $\lambda_{\min} \geq 0$ and demonstrate the relationship shown in the relationship (4.48). The relationship (4.48) can be divided into two components ((D.4) and (D.5)) as follows:

$$\Gamma_A\Gamma_B \geq \frac{\Gamma_c^2}{4} + \frac{\Phi_{BA}^2}{16} \Longrightarrow \sinh[\Gamma_A]\sinh[\Gamma_B] - \sinh^2\left[\frac{\Gamma_c}{2}\right] - \sin^2\left[\frac{\Phi_{BA}}{4}\right] \geq 0, \tag{D.4}$$

$$\sinh[\Gamma_A]\sinh[\Gamma_B] - \sinh^2\left[\frac{\Gamma_c}{2}\right] - \sin^2\left[\frac{\Phi_{BA}}{4}\right] \geq 0 \Longrightarrow e^{-2\Gamma_A} + e^{-2\Gamma_B}\sin^2\left[\frac{\Phi_{BA}}{2}\right] \leq 1. \tag{D.5}$$

In the following two subsections, we examine whether the relationships above (D.4) and (D.5) are satisfied.

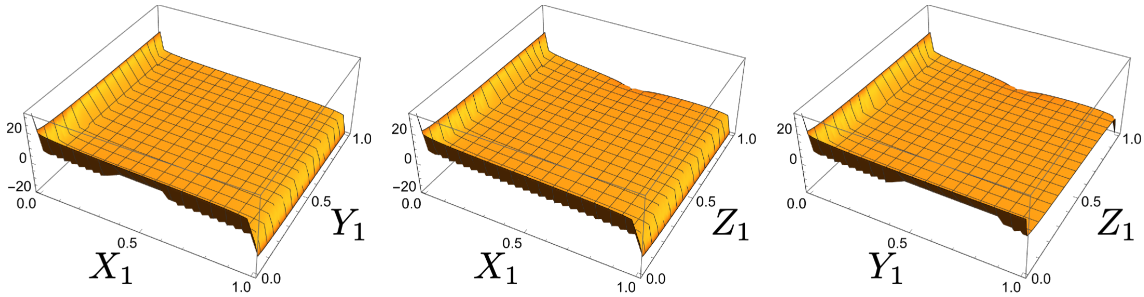


FIG 24: Behavior of the functions of $\log[F(X_1, Y_1, 1/2)]$ (left), $\log[F(X_1, 1/2, Z_1)]$ (center), and $\log[F(1/2, Y_1, Z_1)]$ (right).

D.1 Demonstration of the relationship expressed in (D.4)

First, we demonstrate the relationship expressed in (D.4). Substituting the left-hand side of the inequality expressed in (D.4) into the right-hand side, we obtain the following inequality:

$$\sinh\left[\Gamma_{\mathrm{A}}\right]\sinh\left[\Gamma_{\mathrm{B}}\right] - \sinh^2\left[\frac{\Gamma_{\mathrm{c}}}{2}\right] - \sin^2\left(\frac{\Phi_{\mathrm{BA}}}{4}\right)$$
$$\geq \sinh\left[\Gamma_{\mathrm{A}}\right]\sinh\left[\frac{\Gamma_{\mathrm{c}}^2}{4\Gamma_{\mathrm{A}}} + \frac{\Phi_{\mathrm{BA}}^2}{16\Gamma_{\mathrm{A}}}\right] - \sinh^2\left[\frac{\Gamma_{\mathrm{c}}}{2}\right] - \sin^2\left[\frac{\Phi_{\mathrm{BA}}}{4}\right]. \quad \text{(D.6)}$$

The goal of this subsection is to demonstrate that the right-hand side of the above inequality is always positive. Next, we define variables $X_1 := e^{-\Gamma_{\mathrm{A}}}, Y_1 := e^{-\Gamma_{\mathrm{c}}^2/4\Gamma_{\mathrm{A}}}$, and $Z_1 := e^{-\Phi_{\mathrm{BA}}^2/16\Gamma_{\mathrm{A}}}$. Note that the ranges of X_1, Y_1, Z_1 are limited to $0 < X_1 < 1$, $0 < Y_1 < 1$, and $0 < Z_1 < 1$, respectively. Therefore, we compute the minimum of the following function:

$$F(X_1, Y_1, Z_1) := \frac{1}{4}\left(\frac{1}{X_1} - X_1\right)\left(\frac{1}{Y_1 Z_1} - Y_1 Z_1\right) - \sinh^2\left[\sqrt{\log X_1 \log Y_1}\right] - \sin^2\left[\sqrt{\log X_1 \log Z_1}\right]. \quad \text{(D.7)}$$

The function $\log[F(X_1, Y_1, Z_1)]$ is depicted in Figure 24. The minimum value of the function $F(X_1, Y_1, Z_1)$ is zero at $X_1 = 1$ based on a numerical program written using Mathematica. These result shows that the minimum of the function $F(X_1, Y_1, Z_1)$ is larger than zero, i.e., $F(X_1, Y_1, Z_1) \geq 0$. Thus, the relationship shown in (D.4) is proven.

D.2 Demonstration of the relationship expressed in (D.5)

Next, we also demonstrate the relationship expressed in (D.5). The strategy used is the same as that used for (D.4), i.e., we demonstrated that the minimum of the right-most side of the inequality is greater than zero. The left-hand-side of (D.5) can be rewritten as

$$\sinh[\Gamma_{\rm B}] \geq \frac{\sinh^2[\Gamma_{\rm c}/2]}{\sinh[\Gamma_{\rm A}]} + \frac{\sin^2[\Phi_{\rm BA}/4]}{\sinh[\Gamma_{\rm A}]}, \tag{D.8}$$

where $\sinh[\Gamma_{\rm A}] > 0$ because of $\Gamma_{\rm A} > 0$. Solving the inequality above with respect to $e^{\Gamma_{\rm B}}$ yields

$$e^{\Gamma_{\rm B}} \geq C + \sqrt{1 + C^2}, \tag{D.9}$$

where $C := (\sinh^2[\Gamma_{\rm c}/2]\sinh[\Gamma_{\rm A}]+\sin^2[\Phi_{\rm BA}/4])/\sinh[\Gamma_{\rm A}]$. Substituting the inequality in (D.9) into the right-hand side of (D.5) leads to the following inequality

$$1 - e^{-2\Gamma_{\rm A}} - e^{-2\Gamma_{\rm B}} \sin^2\left[\frac{\Phi_{\rm BA}}{2}\right] \geq 1 - e^{-2\Gamma_{\rm A}} - \frac{\sin^2\left[\Phi_{\rm BA}/2\right]}{\left(C + \sqrt{1 + C^2}\right)^2} =: G(X_2, Y_2, Z_2), \tag{D.10}$$

where we defined the function $G(X_2, Y_2, Z_2)$ as

$$G(X_2, Y_2, Z_2) := 1 - X_2^2 - \frac{\sin^2\left[2\sin^{-1}[Z_2]\right]}{\left(\tilde{C} + \sqrt{1 + \tilde{C}^2}\right)^2}. \tag{D.11}$$

Here, $X_2 := e^{-\Gamma_{\rm A}}, Y_2 := e^{-\Gamma_{\rm c}/2}$, $Z_2 := \sin[\Phi_{\rm BA}/4]$ $(0 < X_2 < 1,\ 0 < Y_2 < 1,\ 0 < Z_2 < 1)$, and

$$\tilde{C} := \frac{\left(1/Y_2 - Y_2\right)^2}{2\left(1/X_2 - X_2\right)} + \frac{2Z_2^2}{\left(1/X_2 - X_2\right)}. \tag{D.12}$$

Therefore, we focus on the minimum of the function $G(X_2, Y_2, Z_2)$ and show that the minimum value is greater than zero. Fig. 25 shows the behavior of the function $G(X_2, Y_2, Z_2)$. The minimum value of the function $G(X_2, Y_2, Z_2)$ is zero in the limit $X_2 \to 1$ based on a numerical program written using Mathematica. Because the function $G(X_2, Y_2, Z_2)$ is always positive, the inequality $G(X_2, Y_2, Z_2) \geq 0$ is satisfied. Thus, the relationship expressed in (D.5) is proven. Furthermore, based on the relationship shown in (D.4) and (D.5), the relationship (4.48) is proven.

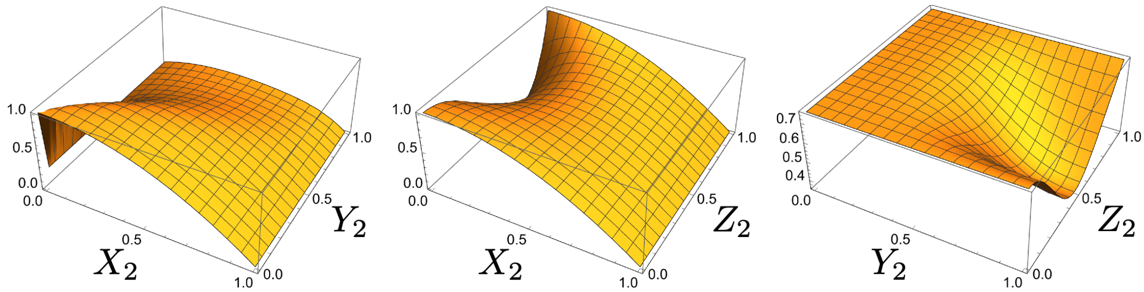


FIG 25: Behavior of the function of $G(X_2, Y_2, 1/2)$ (left), $G(X_2, 1/2, Z_2)$ (center), and $G(1/2, Y_2, Z_2)$ (right).

E Proofs of inequality (5.5) and Eq. (5.18)

The goal of this Appendix is to prove inequality (5.5) and Eq. (5.18). In order to achieve the goal, it is convenient to introduce the Koashi-Winter relation [110] of a pure tripartite system $|\Psi_{\rm ABE}\rangle$ as follows:

$$S(\rho_{\rm A}) = E_{\rm f}(\rho_{\rm AE}) + \mathcal{J}({\rm A}, {\rm B}), \tag{E.1}$$

where the entanglement of formation $E_{\rm f}(\rho_{\rm AE})$ is defined by

$$E_{\rm f}(\rho_{\rm AE}) := \min_{\{p_i, |\psi_{\rm AE}\rangle_i\}} \sum_i p_i S({\rm Tr}_{\rm E}[|\psi_{\rm AE}\rangle_i\langle\psi_{\rm AE}|]) \tag{E.2}$$

with states $|\psi_{\rm AE}\rangle_i$ due to Schmidt decomposition $|\Psi_{\rm ABE}\rangle = \sum_i \sqrt{p_i}|\psi_{\rm AE}\rangle_i \otimes |\psi_{\rm B}\rangle_i$ satisfying $\sum_i p_i = 1$ and $p_i \geq 0$. The minimization is taken over all ensembles $\{p_i, |\psi_{\rm AE}\rangle_i\}$ such that $\sum_i p_i |\psi_{\rm AE}\rangle_i\langle\psi_{\rm AE}| = \rho_{\rm AE}$. Roughly speaking, this entanglement of formation characterizes at least how many maximally entangled states $|\psi_{\rm AE}\rangle$ required to generate the state ${\rm Tr}_{\rm E}[|\psi_{\rm AE}\rangle\langle\psi_{\rm AE}|]$. $\mathcal{J}(A, B)$ in the second term of Eq. (E.1) is the classical correlation, which is seen as the amount of information about the subsystem A that can be obtained via performing a measurement on the other subsystem B, and is defined by

$$\mathcal{J}({\rm A}, {\rm B}) := S(\rho_{\rm A}) - \min_{\{\Pi_i\}} \sum_i p_i S(\rho_{{\rm A}|i}), \tag{E.3}$$

where $S(\rho_{\mathrm{A}|i})$ is the von Neumann entropy of the post-measurement state $\rho_{\mathrm{A}|i}$ with the probability p_i defined below

$$\rho_{\mathrm{A}|i} := \frac{1}{p_i}\mathrm{Tr}_{\mathrm{B}}\left[(\mathbb{1}_{\mathrm{A}} \otimes \Pi_i^{\mathrm{B}})\rho_{\mathrm{AB}}(\mathbb{1}_{\mathrm{A}} \otimes \Pi_i^{\mathrm{B}})\right], \quad p_i := \mathrm{Tr}_{\mathrm{AB}}\left[(\mathbb{1}_{\mathrm{A}} \otimes \Pi_i^{\mathrm{B}})\rho_{\mathrm{AB}}(\mathbb{1}_{\mathrm{A}} \otimes \Pi_i^{\mathrm{B}})\right]. \quad \text{(E.4)}$$

Π_i is the positive operator valued measure (POVM) acting on the subsystem B. The condition $\min_{\{\Pi_i\}}$ is introduced not to disturb the all state, i.e., we must choose the projective operator $\{\Pi_i\}$ so as to reduce the dependence on the projection measure. Note that, as a difference of classical theory, the measurement in subsystem B disturbs subsystem A. When we measure the state of subsystem B, the wave function collapses, and the state of subsystem B determines, that is, the projective measure makes a condition to the state of subsystem A.

The classical correlation $\mathcal{J}(\mathrm{A},\mathrm{B})$ is related to the quantum discord $\mathcal{D}(\mathrm{A},\mathrm{B})$[104]. The definition of the quantum discord is the difference between the quantum mutual information $\mathcal{I}(\mathrm{A},\mathrm{B}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}})$ and the classical correlation $\mathcal{J}(\mathrm{A},\mathrm{B})$:

$$\mathcal{D}(\mathrm{A},\mathrm{B}) = \mathcal{I}(\mathrm{A},\mathrm{B}) - \mathcal{J}(\mathrm{A},\mathrm{B}). \quad \text{(E.5)}$$

The quantum mutual information $\mathcal{I}(\mathrm{A},\mathrm{B})$ quantifies the total amount of correlations between the two subsystems A and B. We note that the quantum mutual information is always non-negative due to the subadditivity of von Neumann entropy. In classical theory, $\mathcal{D}(\mathrm{A},\mathrm{B}) = 0$ is always correct, but, in quantum theory, it can become $\mathcal{D}(A,B) > 0$.

By using Eq. (E.1) and (E.5), we can prove the inequality $E_{\mathrm{f}}(\rho_{\mathrm{AE}}) \geq S(\mathrm{A}|\mathrm{B})$ as follows:

$$\begin{aligned} E_{\mathrm{f}}(\rho_{\mathrm{AE}}) &= S(\rho_{\mathrm{A}}) - \mathcal{J}(A,B) \\ &= S(\rho_{\mathrm{A}}) - \mathcal{I}(\mathrm{A},\mathrm{B}) + \mathcal{I}(\mathrm{A},\mathrm{B}) - \mathcal{J}(\mathrm{A},\mathrm{B}) \\ &= S(\rho_{\mathrm{A}}) - \mathcal{I}(\mathrm{A},\mathrm{B}) + \mathcal{D}(\mathrm{A},\mathrm{B}) \\ &\geq S(\rho_{\mathrm{A}}) - \mathcal{I}(\mathrm{A},\mathrm{B}) \\ &= S(\rho_{\mathrm{AB}}) - S(\rho_{\mathrm{B}}) = S(\mathrm{A}|\mathrm{B}), \end{aligned} \quad \text{(E.6)}$$

where we inserted the quantum mutual information $\mathcal{I}(\mathrm{A},\mathrm{B})$ in the second line and $\mathcal{D}(\mathrm{A},\mathrm{B}) \geq 0$ was used in the fourth line. An other reorder version of the inequality (E.6) is also computed as

$$
\begin{aligned}
E_{\mathrm{f}}(\rho_{\mathrm{AB}}) &\geq S(\rho_{\mathrm{AE}}) - S(\rho_{\mathrm{E}}) \\
&= S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \\
&= -S(\mathrm{A}|\mathrm{B}),
\end{aligned} \tag{E.7}
$$

where, in the second line, we used the properties $S(\rho_{\mathrm{E}}) = S(\rho_{\mathrm{AB}})$ and $S(\rho_{\mathrm{AE}}) = S(\rho_{\mathrm{B}})$, which holds because the state $|\psi_{\mathrm{ABE}}\rangle$ is pure state. Note that these properties are always satisfied because of the invariance of the von Neumann entropy under the Unitary evolution when the initial state is pure state. In the last line, we inserted the definition of the conditional von Neumann entropy $S(\mathrm{A}|\mathrm{B}) := S(\rho_{\mathrm{AB}}) - S(\rho_{\mathrm{B}})$.

Furthermore, we can show the equation $\mathcal{D}(\mathrm{A,E}) = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + S(\mathrm{A}|\mathrm{B})$ as follows:

$$
\mathcal{D}(\mathrm{A,E}) = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + S(\rho_{\mathrm{E}}) - S(\rho_{\mathrm{AE}}) = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + S(\rho_{\mathrm{AB}}) - S(\rho_{\mathrm{B}}) = E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + S(\mathrm{A}|\mathrm{B}), \tag{E.8}
$$

where, in the first equality, we used an other reorder version of the Koashi-Winter relation (E.1) with respect to B and E

$$
\begin{aligned}
S(\rho_{\mathrm{A}}) &= E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + \mathcal{J}(\mathrm{A,E}) \\
&= E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + \mathcal{I}(\mathrm{A,E}) - \mathcal{I}(\mathrm{A,E}) + \mathcal{J}(\mathrm{A,E}) \\
&= E_{\mathrm{f}}(\rho_{\mathrm{AB}}) + \mathcal{I}(\mathrm{A,E}) - \mathcal{D}(\mathrm{A,E}).
\end{aligned} \tag{E.9}
$$

In inequality. (5.5) and Eq. (5.18), we regard the state E as the state of the gravitational field. Therefore, inequality (5.5) and Eq. (5.18) is proven.

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