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Abstract

Borsuk-Ulam's theorem is a useful theorem of algebraic topology. It states that for any continuous mapping f from the n-sphere S^n to $\mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Recently Kawasaki (2023) introduced Borsuk-Ulam's theorem to nonlinear optimization. It applied Borsuk-Ulam's theorem to an n-tuple of parametric optimization problems with parameter $u \in S^n$, and presented an antipodal theorem for them. Further, it showed that given n convex sets in $\mathbb{R}^n$, it is possible to divide each width in half with a hyperplane. In this paper, we take another approach, which weakens the assumptions, simplifies the proofs, and includes a result beyond the scope of Kawasaki (2023). Further, we add a new insight to the ham-sandwich theorem.

Key Words and Phrases: Borsuk-Ulam's theorem, parametric optimization, ham-sandwich theorem.

1. Introduction

Borsuk (1933) proved an important theorem of algebraic topology. It states that for any continuous mapping f from the n-sphere S^n to the Euclidean space $\mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. As for its applications, the ham-sandwich theorem (Section 4), the necklace problem, and coloring of Kneser graph by Lovász (1978) are well-known, see e.g. Matoušek (2008). Kawasaki (2023) applied Borsuk-Ulam's theorem to an n-tuple of parametric optimization problems with parameter $u \in S^n$ by using a technique of the ham-sandwich theorem, and presented Theorems 1.1 and 1.2 below.

For any $\boldsymbol{u} = (u_1, \ldots, u_{n+1}) \in S^n$, we write $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\boldsymbol{u} = (u, u_{n+1})$. Then u is an element of n-disk D^n. We assign to $\boldsymbol{u} \in S^n$ a hyperplane $H_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle = u_{n+1}\}$ and two closed half-spaces:

$$H^+_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle \geq u_{n+1}\},\ H^-_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle \leq u_{n+1}\},$$

where $\langle u, x \rangle$ denotes the inner product $u_1x_1 + \cdots + u_nx_n$. It is clear that $H^+_{-\boldsymbol{u}} = H^-_{\boldsymbol{u}}$. When $\boldsymbol{u} = (u, 0)$, we denote them by

$$H^+_{u,0} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle \geq 0\},\ H^-_{u,0} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle \leq 0\}.$$

Let $A_i \subset \mathbb{R}^n$ $(i = 1, \ldots, n)$ be compact convex sets with non-empty interiors. We assumed the following strict convexity of A_i in Kawasaki (2023).

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(SC) $A_i \cap H = \{x\}$ for any boundary point x of A_i and for any supporting hyperplane H of A_i at x.

The n-tuple of optimal-value functions $\varphi = (\varphi_1, \ldots, \varphi_n) : S^n \to \mathbb{R}^n$ is defined by

$$\varphi_i(\boldsymbol{u}) := \begin{cases} \max\limits_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) - \min\limits_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) & (A_i \cap H_{\boldsymbol{u}}^+ \neq \emptyset), \\ \delta^*(u \mid A_i) - u_{n+1} & (A_i \cap H_{\boldsymbol{u}}^+ = \emptyset), \end{cases} \tag{1}$$

where $\delta^*(u \mid X) := \max\limits_{x \in X} \langle u, x \rangle$. We remark that the essential case of $\varphi_i(\boldsymbol{u})$ is the former. The latter is a device for φ_i to be continuous on the whole S^n. A function $f_i(x, \boldsymbol{u})$ is said to be antipodal w.r.t. $\boldsymbol{u}$ if $f_i(x, -\boldsymbol{u}) = -f_i(x, \boldsymbol{u})$ for any $(x, \boldsymbol{u}) \in \mathbb{R}^n \times S^n$. By applying Borsuk-Ulam's theorem to φ, we obtained the following antipodal theorem:

Theorem 1.1. *(Kawasaki (2023)) Assume that $f_i(x, \boldsymbol{u})$ is antipodal w.r.t. $\boldsymbol{u}$ and A_i satisfies (SC) for any $i = 1, \ldots, n$. Then there exists $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty, and*

$$\max_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) = \max_{x \in A_i \cap H_{\boldsymbol{u}}^-} f_i(x, \boldsymbol{u}) - \min_{x \in A_i \cap H_{\boldsymbol{u}}^-} f_i(x, \boldsymbol{u}) \tag{2}$$

for any $i = 1, \ldots, n$.

Further, by taking $f_i(x, \boldsymbol{u}) = \langle x, u \rangle$, we obtain the following theorem. It states that there exists a hyperplane that divides each A_i's width in half.

Theorem 1.2. *(Kawasaki (2023)) There exists $\boldsymbol{u} \in S^n$ such that both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty and*

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) \tag{3}$$

for any $i = 1, \ldots, n$, where $\delta_*(u \mid X) := \min\limits_{x \in X} \langle u, x \rangle$.

In order for $\varphi : S^n \to \mathbb{R}^n$ to be continuous, we required the continuity of the set-valued mapping $\boldsymbol{u} \mapsto A_i \cap H_{\boldsymbol{u}}^+$. So we needed assumption (SC) in Kawasaki (2023). In Section 2, we take another approach that does not go through the set-valued mapping. Then, assumption (SC) is unnecessary, and proofs simplify. In Section 3, we clarify the relationship between Theorem 1.2 and the corresponding Theorem 2.2. One will see the latter includes a result beyond the scope of the former. In Section 4, we apply our technique to the ham-sandwich theorem. One will see it brings a new insight to the ham-sandwich theorem.

2. Parametric Optimal-Value Functions

In the following, we assume that for any $i = 1, \ldots, n$

- $A_i \subset \mathbb{R}^n$ is a non-empty compact set. It is possible to be non-convex and empty interior.
- $f_i : \mathbb{R}^n \times D^n \to \mathbb{R}$ is continuous. It is antipodal w.r.t. $u \in D^n$ for any $x \in \mathbb{R}^n$.
- γ_i is a non-zero real constant.

We define a new optimal-value function $\kappa_i : S^n \to \mathbb{R}$ by

$$\kappa_i(\boldsymbol{u}) := \max_{x \in A_i} f_i(x,u) + \min_{x \in A_i} f_i(x,u) - 2\gamma_i u_{n+1}. \tag{4}$$

Then it is easily seen that κ_i is continuous. Applying Borsuk-Ulam's theorem to $\kappa = (\kappa_1, \dots, \kappa_n)$, we obtain the following antipodal theorem:

THEOREM 2.1. *(a) For any $\gamma_i \neq 0$ $(i = 1, \dots, n)$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that $u \neq \mathbf{0}$ and*

$$\max_{x \in A_i} f_i(x,u) + \min_{x \in A_i} f_i(x,u) = 2\gamma_i u_{n+1} \quad (i = 1, \dots, n). \tag{5}$$

(b) There exists non-zero $u \in D^n$ such that $\max_{x \in A_i} f_i(x,u) + \min_{x \in A_i} f_i(x,u)$ does not depend on $i = 1, \dots, n$.
(c) If there is no non-zero $v \in D^n$ such that

$$\max_{x \in A_i} f_i(x,v) + \min_{x \in A_i} f_i(x,v) = 0 \quad (i = 1, \dots, n), \tag{6}$$

then, for any $\gamma_i \neq 0$ $(i = 1, \dots, n)$, there exists non-zero $u \in D^n$ such that the ratio of $\max_{x \in A_i} f_i(x,u) + \min_{x \in A_i} f_i(x,u)$ $(i = 1, \dots, n)$ equals to the ratio of γ_i $(i = 1, \dots, n)$.

PROOF. By Borsuk-Ulam's theorem, there exists $\boldsymbol{u} \in S^n$ such that $\kappa_i(\boldsymbol{u}) = \kappa_i(-\boldsymbol{u})$ for any i. Namely,

$$\begin{aligned} & \max_{x \in A_i} f_i(x,u) + \min_{x \in A_i} f_i(x,u) - 2\gamma_i u_{n+1} \\ = \; & \max_{x \in A_i} f_i(x,-u) + \min_{x \in A_i} f_i(x,-u) + 2\gamma_i u_{n+1} \\ = \; & -\min_{x \in A_i} f_i(x,u) - \max_{x \in A_i} f_i(x,u) + 2\gamma_i u_{n+1}, \end{aligned}$$

which implies (5). Since $f_i(x,u)$ is antipodal w.r.t. u, it holds that $f_i(x,\mathbf{0}) = 0$ for any x. Suppose that $u = \mathbf{0}$, then it follows from (5) that

$$2\gamma_i u_{n+1} = \max_{x \in A_i} f_i(x,\mathbf{0}) + \min_{x \in A_i} f_i(x,\mathbf{0}) = 0,$$

which contradicts $(u, u_{n+1}) \neq \mathbf{0}$. Therefore, $u \neq \mathbf{0}$. (b) follows from (a) by taking $\gamma_i = 1$. (c) It follows from (a) and the assumption on (6) that $u_{n+1} \neq 0$. Hence, we obtain the desired ratio. This completes the proof.

In particular when $f_i(x,u) = \langle x, u \rangle$, Theorem 2.1 turns into an equal division theorem:

THEOREM 2.2. *(a) For any $\gamma_i \neq 0$ $(i = 1, \dots, n)$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that $u \neq \mathbf{0}$ and*

$$\delta^*(u \mid A_i) + \delta_*(u \mid A_i) = 2\gamma_i u_{n+1} \quad (i = 1, \dots, n). \tag{7}$$

(b) There exists non-zero $u \in D^n$ such that $\delta^*(u \mid A_i) + \delta_*(u \mid A_i)$ does not depend on $i = 1, \dots, n$.
(c) If there is no non-zero $v \in D^n$ such that

$$\delta^*(v \mid A_i) + \delta_*(v \mid A_i) = 0 \quad (i = 1, \dots, n), \tag{8}$$

then for any $\gamma_i \neq 0$ $(i = 1, \dots, n)$, there exists non-zero $u \in D^n$ such that the ratio of $\delta^*(u \mid A_i) + \delta_*(u \mid A_i)$ $(i = 1, \dots, n)$ equals to the ratio of γ_i $(i = 1, \dots, n)$.

Since the hyperplane

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \langle x, u \rangle = \frac{\delta^*(u \mid A_i) + \delta_*(u \mid A_i)}{2} \right\}$$

passes a kind of center of A_i, Theorem 2.2 (c) means that one can arrange $A_1, \ldots, A_n$ in order along the direction u, see Figure 2.

EXAMPLE 2.3. Figure 1 Left and Right represent Theorem 2.2 (b) and (8) in $\mathbb{R}^2$, respectively.

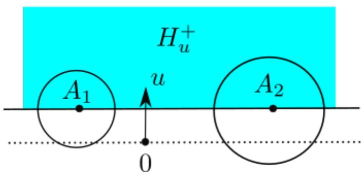

Figure 1: Left: $\delta^*(u \mid A_i) + \delta_*(u \mid A_i)$ $(i = 1, 2)$ are a non-zero constant. Right: $\delta^*(u \mid A_i) + \delta_*(u \mid A_i) = 0$ $(i = 1, 2)$.

Figure 2 represents Theorem 2.2 (c) in $\mathbb{R}^2$. For example, ratio $\gamma_1 : \gamma_2 = 1 : -2$ is substantialized by $u \in D^2$ in Left. Ratio $\gamma_1 : \gamma_2 = 1 : 2$ is substantialized by $u \in D^2$ in Right.

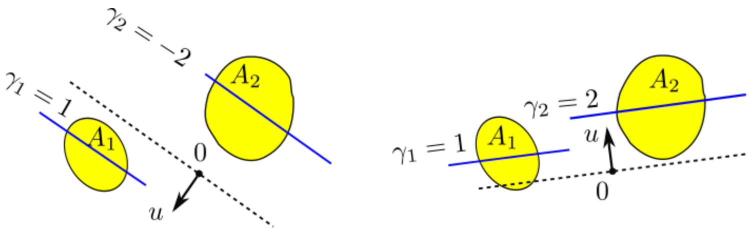

Figure 2: Left: $\gamma_1 : \gamma_2 = 1 : -2$. Right: $\gamma_1 : \gamma_2 = 1 : 2$.

3. Relationship between Theorems 1.2 and 2.2

In this section, we clarify the relationship between Theorems 1.2 and 2.2. For this aim, we take $f_i(x, \boldsymbol{u}) = f_i(x, u) = \langle u, x \rangle$.

LEMMA 3.1. *(a) If both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty, then*

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid A_i), \tag{9}$$

$$\delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) = \delta_*(u \mid A_i). \tag{10}$$

(b) If A_i is convex and both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty, then

$$\delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) = u_{n+1}. \tag{11}$$

Proof. (a)

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \max\{\langle u, x\rangle \mid x \in A_i,\ \langle u, x\rangle \geq u_{n+1}\} = \max_{x \in A_i} \langle u, x\rangle \ = \ \delta^*(u \mid A_i).$$

$$\delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) = \min\{\langle u, x\rangle \mid x \in A_i,\ \langle u, x\rangle \leq u_{n+1}\} = \min_{x \in A_i} \langle u, x\rangle \ = \ \delta_*(u \mid A_i).$$

(b) Take arbitrary $y \in A_i \cap H_{\boldsymbol{u}}^+$ and $z \in A_i \cap H_{\boldsymbol{u}}^-$, then $\langle u, z\rangle \leq u_{n+1} \leq \langle u, y\rangle$. Hence, there exists $0 \leq t \leq 1$ such that $\langle u, ty + (1-t)z\rangle = u_{n+1}$. Since A_i is convex, $x^* := ty + (1-t)z$ belongs to A_i and $\langle u, x^*\rangle = u_{n+1}$. Therefore,

$$\delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \min\{\langle u, x\rangle \mid x \in A_i,\ \langle u, x\rangle \geq u_{n+1}\} = \langle u, x^*\rangle = u_{n+1}. \tag{12}$$

$$\delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) = \max\{\langle u, x\rangle \mid x \in A_i,\ \langle u, x\rangle \leq u_{n+1}\} = \langle u, x^*\rangle = u_{n+1}. \tag{13}$$

This completes the proof.

Proposition 3.2. *Assume that A_i is convex and both $A_i \cap H_{\boldsymbol{u}}^+$ and $A_i \cap H_{\boldsymbol{u}}^-$ are non-empty, then Theorem 2.2 (7) with $\gamma_i = 1$ is equivalent to Theorem 1.2 (3).*

Proof. (7) with $\gamma_i = 1$ is written as

$$\delta^*(u \mid A_i) - u_{n+1} = u_{n+1} - \delta_*(u \mid A_i). \tag{14}$$

It follows from Lemma 3.1 that

$$\text{LHS of (14)} = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^+) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^+) = \text{LHS of (3)},$$

$$\text{RHS of (14)} = \delta^*(u \mid A_i \cap H_{\boldsymbol{u}}^-) - \delta_*(u \mid A_i \cap H_{\boldsymbol{u}}^-) = \text{RHS of (3)}.$$

Therefore, (7) is equivalent to (3).

Proposition 3.2 indicates that Theorem 2.2 (a) where $\gamma_i = 1$ is almost the same as Theorem 1.2. On the other hand, Theorem 2.2 (c) is beyond the scope of Theorem 1.2.

4. Variation

In this section, we apply the technique of Section 2 to the ham-sandwich theorem. Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be compact sets with positive Lebesgue measure μ. Then, the ham-sandwich theorem states that there exists $\boldsymbol{u} \in S^n$ such that

$$\mu(A_i \cap H_{\boldsymbol{u}}^+) = \mu(A_i \cap H_{\boldsymbol{u}}^-) \quad (i = 1, \ldots, n).$$

Now, let γ_i be non-zero constant, and define

$$\lambda_i(\boldsymbol{u}) := \mu\left(A_i \cap H_{\boldsymbol{u}}^+\right) - \mu\left(A_i \cap H_{\boldsymbol{u}}^-\right) - 2\gamma_i u_{n+1}. \tag{15}$$

Then, $\lambda = (\lambda_1, \ldots, \lambda_n) : S^n \to \mathbb{R}^n$ is continuous. Applying Borsuk-Ulam's theorem to λ, we obtain the following theorem.

THEOREM 4.1. *(a) For any $\gamma_i \neq 0$ $(i = 1, \ldots, n)$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that*

$$\mu\left(A_i \cap H_{\boldsymbol{u}}^+\right) - \mu\left(A_i \cap H_{\boldsymbol{u}}^-\right) = 2\gamma_i u_{n+1} \quad (i = 1, \ldots, n). \tag{16}$$

(b) There exists $\boldsymbol{u} \in S^n$ such that $\mu(A_i \cap H_{\boldsymbol{u}}^+) - \mu(A_i \cap H_{\boldsymbol{u}}^-)$ does not depend on $i = 1, \ldots, n$.
(c) If there is no non-zero $v \in D^n$ such that

$$\mu\left(A_i \cap H_{v,0}^+\right) = \mu\left(A_i \cap H_{v,0}^-\right) \quad (i = 1, \ldots, n), \tag{17}$$

then, for any $\gamma_i \neq 0$ $(i = 1, \ldots, n)$, there exists $\boldsymbol{u} \in S^n$ such that the ratio of $\mu(A_i \cap H_{\boldsymbol{u}}^+) - \mu(A_i \cap H_{\boldsymbol{u}}^-)$ $(i = 1, \ldots, n)$ equals to the ratio of γ_i $(i = 1, \ldots, n)$.

PROOF. (a) By Borsuk-Ulam's theorem, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that $\lambda_i(\boldsymbol{u}) = \lambda_i(-u)$ for any i. Namely,

$$\begin{aligned} & \mu\left(A_i \cap H_{\boldsymbol{u}}^+\right) - \mu\left(A_i \cap H_{\boldsymbol{u}}^-\right) - 2\gamma_i u_{n+1} \\ = \; & \mu\left(A_i \cap H_{-\boldsymbol{u}}^+\right) - \mu\left(A_i \cap H_{-\boldsymbol{u}}^-\right) + 2\gamma_i u_{n+1} \\ = \; & \mu\left(A_i \cap H_{\boldsymbol{u}}^-\right) - \mu\left(A_i \cap H_{\boldsymbol{u}}^+\right) + 2\gamma_i u_{n+1}, \end{aligned}$$

which implies (16). (b) follows from (a) by taking $\gamma_i = 1$. (c) Suppose that $u_{n+1} = 0$ in (16), then $\boldsymbol{u} = (u, 0)$ and u is non-zero. Hence (16) contradicts the assumption on (17). Thus $u_{n+1} \neq 0$, and we obtain the desired ratio. This completes the proof.

EXAMPLE 4.2. In Figure 3, since there is no line passing the origin that divides both A_1 and A_2 in half, Theorem 4.1 (c) is applicable. For example, ratio $\gamma_1 : \gamma_2 = 1 : 2$ is

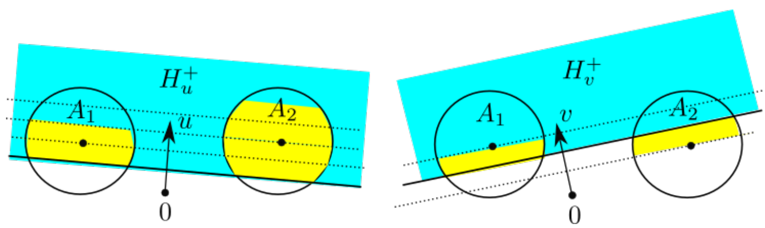

Figure 3: Theorem 4.1 (c). Left: $\gamma_1 : \gamma_2 = 1 : 2$. Right: $\gamma_1 : \gamma_2 = 1 : -1$.

substantialized by $u \in D^2$ in Figure 3 Left. Ratio $\gamma_1 : \gamma_2 = 1 : -1$ is substantialized by $v \in D^2$ in Figure 3 Right.

If we take

$$\lambda_i'(\boldsymbol{u}) := \mu\left(A_i \cap H_{u,0}^+\right) - \mu\left(A_i \cap H_{u,0}^-\right) - 2\gamma_i u_{n+1} \tag{18}$$

instead of (15), we obtain a theorem similar to Theorem 4.1. The difference is the dividing hyperplane $H_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u.x \rangle = u_{n+1}\}$. It does not necessarily pass the origin in Theorem 4.1, but passes the origin in Theorem 4.3. Since the proof is almost same with Theorem 4.1, we omit it.

THEOREM 4.3. *(a) For any* $\gamma_i \neq 0$ $(i = 1, \ldots, n)$*, there exists* $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ *such that*

$$\mu\left(A_i \cap H_{u,0}^+\right) - \mu\left(A_i \cap H_{u,0}^-\right) = 2\gamma_i u_{n+1} \quad (i = 1, \ldots, n). \tag{19}$$

(b) There exists non-zero $u \in D^n$ *such that* $\mu\left(A_i \cap H_{u,0}^+\right) - \mu\left(A_i \cap H_{u,0}^-\right)$ *does not depend on* $i = 1, \ldots, n$.
(c) If there is no non-zero $v \in D^n$ *such that*

$$\mu\left(A_i \cap H_{v,0}^+\right) = \mu\left(A_i \cap H_{v,0}^-\right) \quad (i = 1, \ldots, n), \tag{20}$$

then, for any $\gamma_i \neq 0$ $(i = 1, \ldots, n)$*, there exists* $u \in D^n$ *such that the ratio of* $\mu\left(A_i \cap H_{u,0}^+\right) - \mu\left(A_i \cap H_{u,0}^-\right)$ $(i = 1, \ldots, n)$ *equals to the ratio of* γ_i $(i = 1, \ldots, n)$.

EXAMPLE 4.4. In Figure 4, since there is no line passing the origin that divides both A_1 and A_2 in half, Theorem 4.3 (c) is applicable. For example, ratio $\gamma_1 : \gamma_2 = 1 : 2$ is substantialized by $u \in D^2$ in Figure 4 Left. Ratio $\gamma_1 : \gamma_2 = 1 : -1$ is substantialized by $v \in D^2$ in Figure 4 Right.

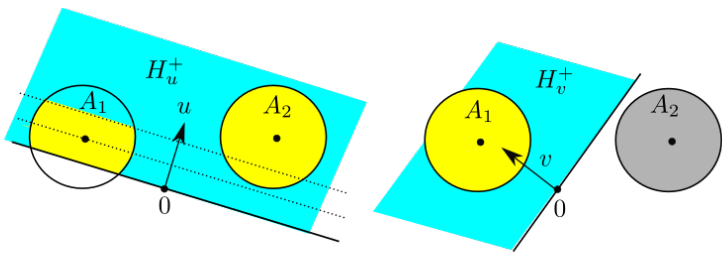

Figure 4: Theorem 4.3 (c). Left: $\gamma_1 : \gamma_2 = 1 : 2$. Right: $\gamma_1 : \gamma_2 = 1 : -1$.

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