Local theta lift for p-adic unitary dual pairs U(2)×U(1) and U(2)×U(3)

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出版情報：Kyoto Journal of Mathematics. 59 (4), 2019-12. Duke University Press

LOCAL THETA LIFT FOR p-ADIC UNITARY DUAL PAIRS $\mathrm{U}(2) \times \mathrm{U}(1)$ AND $\mathrm{U}(2) \times \mathrm{U}(3)$

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ABSTRACT. In this paper we describe the local theta lift for p-adic unitary dual pairs $\mathrm{U}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$. We also describe the local theta lift for a pair of p-adic quaternionic unitary groups of rank one.

1. INTRODUCTION

The purpose of this paper is to describe the local theta lift for p-adic unitary dual pairs $\mathrm{U}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$ in terms of endoscopy. This is a complement to a result of Gelbart-Rogawski-Soudry [2]. Also Gan-Ichino [1] computed the local theta lift for unitary groups in almost equal rank case in terms of Vogan L-packets. The result in this paper provides another proof of their result in the cases of $\mathrm{U}(2) \times \mathrm{U}(1)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$. Our proof is based on some results in [2] and the endoscopic description of the anisotropic unitary group in two variables in Konno-Konno [3]. As an application, we obtain the description of the local theta lift for a pair of p-adic quaternionic unitary groups of rank one in terms of endoscopy.

To explain the problem in this paper, we recall some results in [2]. Let F be a non-archimedean local field of characteristic 0, and E a quadratic extension of F with associated quadratic character $\omega_{E/F}$. We denote the split (resp. anisotropic) 2-dimensional hermitian space over E by V_{sp} (resp. V_{an}) and a three-dimensional skew-hermitian space over E by W. We set $V = V_{sp}, V_{an}$. Rogawski gave the endoscopic descriptions of the irreducible admissible representations for $\mathrm{U}(V_{sp})$ and for $\mathrm{U}(W)$ in [10]. Using it, Gelbart-Rogawski-Soudry showed in [2] that the local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W)$ sends an L-packet to an L-packet. They also described the local pairing of an endoscopic representation π for $\mathrm{U}(W)$ by the behavior of the local theta lift of π to $\mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$. Namely, an explicit parametrization of the members of an endoscopic L-packet of $\mathrm{U}(W)$ is given in terms of the local theta lift to $\mathrm{U}(V)$.

We note that in [2] the local theta lift is investigated only for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W)$, because of the lack of the endoscopic description of the set $\mathrm{Irr}\,\mathrm{U}(V_{an})$ of equivalence classes of irreducible admissible representations of $\mathrm{U}(V_{an})$ in those days. But now, we have it at our disposal thanks to [3]. So it is natural to consider the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$ in terms of the endoscopic description.

We denote the local Langlands group of F by $L_F = W_F \times \mathrm{SU}_2(\mathbb{R})$, where W_F stands for the Weil group of F. For $G = \mathrm{U}(V)$ or $\mathrm{U}(W)$, $\Phi(G)$ means the set of equivalence classes of L-parameters $\phi : L_F \to {}^L G$, where ${}^L G$ is the L-group of G (see §2.2). The L-packets Π_ϕ was given in [10], when G is $\mathrm{U}(V_{sp})$ or $\mathrm{U}(W)$.

2010 *Mathematics Subject Classification.* 11F27, 11F70.
Key words and phrases. Local theta lift.

For $G = \mathrm{U}(V_{an})$, Π_ϕ was given in [2, §1] (see also §2.2) by Jacquet-Langlands correspondence. Let ϕ_E be the restriction of an L-parameter ϕ of $G = \mathrm{U}(V)$ or $\mathrm{U}(W)$ to L_E. Note that ϕ is uniquely determined by ϕ_E. Let ψ be a non-trivial character of F. Fix characters μ, η of $E^\times$ such that $\mu|_{F^\times} = \omega_{E/F}, \eta|_{F^\times} = \mathbb{1}$. Then we have the Weil representation $\omega_{\psi,V^\mu,W^\eta}$ and the local theta lift $\theta_{\psi,V^\mu,W^\eta}$ for $\mathrm{U}(V) \times \mathrm{U}(W)$ (see §2.1).

Briefly state the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(W)$ given by Rogawski [10] (See §4 for details). Let (H_0, s_0, ξ_0) be the unique non-trivial elliptic endoscopic datum for $\mathrm{U}(W)$ up to equivalence, where $H_0 = \mathrm{U}(V_{sp}) \times \mathrm{U}(1)$. For an L-parameter $\phi \in \Phi(\mathrm{U}(W))$, we define the finite set $\hat{\Pi}_\phi = \{\rho \in \Phi(H_0) \,|\, \xi_0 \circ \rho = \phi\}$. Rogawski defined the local pairing $\langle \rho, \pi \rangle = \pm$ for $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$ by using the trace formula in [10]. Then these $\langle \rho, \pi \rangle$'s determine each element of Π_ϕ.

Now we can explain our results. Let μ_1 and μ_2 be characters of $E^\times$ such that $\mu_1 \neq \mu_2$ and $\mu_i|_{F^\times} = \omega_{E/F}$. Then there exists a unique L-parameter φ_{μ_1,μ_2} of $\mathrm{U}(V_{an})$ such that $(\varphi_{\mu_1,\mu_2})_E = \mu_1 \oplus \mu_2$. We denote the L-packet for $\mathrm{U}(V_{an})$ associated to it by $\Pi_{\mu_1,\mu_2}(V_{an})$. Then the L-packet $\Pi_{\mu_1,\mu_2}(V_{an})$ has two elements $\tau(\mu_1,\mu_2)^+_{an}, \tau(\mu_1,\mu_2)^-_{an}$, where the signs are specified in §5.1. We want to compute the local theta lift $\theta_{\psi,V^\mu_{an},W^\eta}$ for $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$.

First we consider the case of $\mu = \mu_1$, that is, when the splitting character μ equals to one of components of E-restricted L-parameter.

Theorem 1.1 (Theorem 6.2, Corollary 6.5)**.** *We have*

$$\theta_{\psi,V^{\mu_1}_{an},W^\eta}(\tau(\mu_1,\mu_2)^\varepsilon_{an}) = \begin{cases} \pi(\eta, \eta\mu_1\mu_2^{-1})^- & \text{if } \varepsilon = +, \\ 0 & \text{if } \varepsilon = -. \end{cases}$$

Here the non-supercuspidal representation $\pi(\eta, \eta\mu_1\mu_2^{-1})^-$ *of* $\mathrm{U}(W)$ *will be explained in Theorem 6.2.*

Next we consider the case of $\mu \neq \mu_1, \mu_2$. Then by results in [2] we have

$$\pi^\pm := \theta_{\psi,V^\mu_{an},W^\eta}(\tau(\mu_1,\mu_2)^\pm_{an}) \neq 0$$

and the L-parameter ϕ of $\pi^\pm$ satisfies $\phi_E = \mu\eta\mu_1^{-1} \oplus \mu\eta\mu_2^{-1} \oplus \eta$. In this case, we have $\hat{\Pi}_\phi = \{\rho_0, \rho_1, \rho_2\}$. See §6 for the exact form of ρ_i's. Then the local pairings of $\pi^\pm$ are given as follows:

Theorem 1.2 (Corollary 6.7)**.** *We have for* $\varepsilon \in \{\pm\}$

$$\langle \rho_0, \pi^\varepsilon \rangle = -,$$
$$\langle \rho_1, \pi^\varepsilon \rangle = -\varepsilon,$$
$$\langle \rho_2, \pi^\varepsilon \rangle = \varepsilon.$$

These are our main results given in this paper. Moreover, by using Theorem 1.1, we can compute the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(1)$ (Theorem 6.4). Furthermore, by proving a relation between the Weil representation of $\mathrm{U}(V_{an}) \times \mathrm{U}(1)$ and that of a pair of quaternionic unitary groups of rank one (Proposition 7.6), we can compute the local theta lift for a pair of quaternionic unitary groups of rank one (Theorem 7.9).

The paper is organized as follows. In §2, we recall the Weil representation for unitary groups in general setting and the construction of the L-packets for $\mathrm{U}(V)$ and $\mathrm{U}(W)$. In §3, we compute the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(1)$. In §4, we recall some results in [2]. We also deduce the computation of the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W)$ from these and state the problem in this paper. In §5, we explain the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(V_{an})$ and prepare a global result to prove our main theorems. In §6, we prove our main theorems. In §7, we compute the local theta lift for a pair of quaternionic unitary groups of rank one.

Notation 1.3. Let E be a quadratic extension of F. The map σ stands for the non-trivial Galois automorphism of E over F. We will often write $\sigma(x) = x^\sigma$. Let $\mathrm{Tr}_{E/F}$ and $\mathrm{N}_{E/F}$ be the trace map and norm map of E over F, respectively. We denote the modulus of F by $|\cdot|_F$. Fix a non-trivial character ψ of F. Then $\lambda(E/F, \psi)$ is the Langlands λ-factor [7]. For each character χ of $F^\times$, $\Pi(E^\times, \chi)$ stands for the set of characters of $E^\times$ whose restriction to $F^\times$ are χ. For each $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we define a character η_u of $\mathrm{U}(1)$ by

$$\eta_u(z/\sigma(z)) = \eta(z), \quad z \in E^\times.$$

For a p-adic group G, we denote the set of equivalence classes of irreducible admissible representations of G by $\mathrm{Irr}\,G$. Fix a non-zero element ξ of E with $\mathrm{Tr}_{E/F}(\xi) = 0$ and $d_0 \in F^\times \smallsetminus \mathrm{N}_{E/F}(E^\times)$. Also fix two characters $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$ and $\mu \in \Pi(E^\times, \omega_{E/F})$ to construct Weil representations for unitary groups.

2. Preliminaries

2.1. **Weil representation for unitary groups.** In this subsection we recall the Weil representation for unitary groups in general setting.

Let V be a hermitian space and W a skew-hermitian space over E. The spaces V and W may be taken as follows:

$$V = (E^{\oplus m}, A), \qquad W = (E^{\oplus n}, B),$$
$$(v_1, v_2) = {}^*v_1 A v_2, \qquad \langle w_1, w_2 \rangle = w_1 B^* w_2,$$

where $A = {}^*A := {}^tA^\sigma \in \mathrm{GL}_m(E)$ and $B = -{}^*B \in \mathrm{GL}_n(E)$. Note that V is a space of column vectors, whereas W is of row vectors. Then the unitary groups of V and W are given by

$$\mathrm{U}(V) = \{h \in \mathrm{GL}_m(E) \mid {}^*hAh = A\},$$
$$\mathrm{U}(W) = \{g \in \mathrm{GL}_n(E) \mid gB^*g = B\}.$$

Here $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) acts on V (resp. W) on the left (resp. right).

Let

$$\mathbb{W} = V \otimes_E W$$

be the F-vector space equipped with the symplectic form

$$\langle\!\langle\ ,\ \rangle\!\rangle = \mathrm{Tr}_{E/F}\left((\ ,\ ) \otimes \langle\ ,\ \rangle^\sigma\right).$$

We identify $\mathbb{W}$ with $\mathrm{M}_{m,n}(E)$ by the isomorphism

$$V \otimes_E W \ni v \otimes w \mapsto v \cdot w \in \mathrm{M}_{m,n}(E). \tag{2.1}$$

Then the symplectic form on $\mathbb{W}$ is given by

$$\langle\!\langle X, X' \rangle\!\rangle = \mathrm{Tr}_{E/F}({}^*XAX'^*B), \quad X, X' \in \mathbb{W}.$$

The symplectic group of $\mathbb{W}$ is given by

$$\mathrm{Sp}(\mathbb{W}) = \{g \in \mathrm{GL}_F(\mathbb{W}) \mid \langle\!\langle wg, w'g \rangle\!\rangle = \langle\!\langle w, w' \rangle\!\rangle \text{ for all } w, w' \in \mathbb{W}\}.$$

Here $\mathrm{Sp}(\mathbb{W})$ acts on $\mathbb{W}$ on the right. The Heisenberg group $H(\mathbb{W})$ of $\mathbb{W}$ is given as follows:

$$H(\mathbb{W}) = \mathbb{W} \times F,$$
$$(w,t)\cdot(w',t') = \left(w + w', t + t' + \frac{\langle\langle w, w'\rangle\rangle}{2}\right)$$

for $w, w' \in \mathbb{W}$ and $t, t' \in F$.

Fix an irreducible unitary representation $(\tau_{\psi,V,W}, \mathcal{S}_{V,W})$ of $H(\mathbb{W})$ with central character ψ. For any $g \in \mathrm{Sp}(\mathbb{W})$, define the representation $(\tau_{\psi,V,W}\cdot g, \mathcal{S}_{V,W})$ of $H(\mathbb{W})$ by $\tau_{\psi,V,W}(w,t)g := \tau_{\psi,V,W}(wg,t)$ for $w \in \mathbb{W}, t \in F$. This is also an irreducible unitary representation of $H(\mathbb{W})$ with central character ψ. By Stone-von-Neumann's theorem, the unitary representation $\tau_{\psi,V,W}\cdot g$ is isomorphic to $\tau_{\psi,V,W}$. Thus we have the metaplectic group:

$$\mathrm{Mp}_{\psi}(\mathbb{W}) = \left\{ (g, M_g) \;\middle|\; \begin{array}{l} g \in \mathrm{Sp}(\mathbb{W}), \\ M_g \colon \tau_{\psi,V,W}\cdot g \cong \tau_{\psi,V,W} \text{ isomorphism} \end{array} \right\}.$$

The metaplectic group $\mathrm{Mp}_{\psi}(\mathbb{W})$ has the Weil representation $(\omega_{\psi,V,W}, \mathcal{S}_{V,W})$ defined by

$$\omega_{\psi,V,W}(g, M_g) := M_g.$$

Now let χ_V and χ_W be characters of $E^\times$ such that $\chi_V|_{F^\times} = \omega_{E/F}^{\dim_E V}$ and $\chi_W|_{F^\times} = \omega_{E/F}^{\dim_E W}$, respectively. Then Kudla [5] defined the splitting associated to χ_V and χ_W:

$$\iota_W^{\chi_W} \times \iota_V^{\chi_V} : \mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Mp}_{\psi}(\mathbb{W}).$$

Thus the Weil representation $(\omega_{\psi,V^{\chi_W},W^{\chi_V}}, \mathcal{S}_{V,W})$ of $\mathrm{U}(V)\times\mathrm{U}(W)$ associated to χ_V and χ_W is defined by

$$\omega_{\psi,V^{\chi_W},W^{\chi_V}} := \omega_{\psi,V,W} \circ \iota_W^{\chi_W} \times \iota_V^{\chi_V}.$$

To explain the mixed model of the Weil representation $\omega_{\psi,V^{\chi_W},W^{\chi_V}}$, we prepare a few notation. Without loss of generality, we may assume that

$$V = (E^{\oplus m}, \begin{pmatrix} & & 1_{m_0} \\ & A_1 & \\ 1_{m_0} & & \end{pmatrix}).$$

We put $V_0 = (E^{\oplus 2m_0}, \begin{pmatrix} & 1_{m_0} \\ 1_{m_0} & \end{pmatrix})$ and $V_1 = (E^{\oplus m_1}, A_1)$, where $m_1 = m - 2m_0$. We denote by X the subspace of V consisting of the elements of the form

$${}^t(x_1, \ldots, x_{m_0}, 0, \ldots, 0)$$

and X' the subspace of V consisting of the elements of the form

$${}^t(0, \ldots, 0, x_{m-m_0+1}, \ldots, x_m).$$

Thus we have

$$V = X \oplus V_1 \oplus X'. \tag{2.2}$$

Let $P_{X'} = M_{X'}N_{X'}$ be the maximal parabolic subgroup of $\mathrm{U}(V)$ stabilizing X', where $M_{X'}$ is the Levi component of $P_{X'}$ stabilizing X and $N_{X'}$ is the unipotent radical of $P_{X'}$. We have

$$M_{X'} = \left\{ m(a,h) = \begin{pmatrix} a & & \\ & h & \\ & & {}^*a^{-1} \end{pmatrix} \middle| \begin{array}{c} a \in \mathrm{GL}_{m_0}(E), \\ h \in \mathrm{U}(V_1) \end{array} \right\},$$

$$N_{X'} = \left\{ n(b,c) = \begin{pmatrix} 1_{m_0} & & \\ b & 1_{m_1} & \\ c - \frac{1}{2}{}^*bA_1b & -{}^*bA_1 & 1_{m_0} \end{pmatrix} \middle| \begin{array}{c} b \in \mathrm{M}_{m_1,m_0}(E), \\ c = -{}^*c \in \mathrm{M}_{m_0}(E) \end{array} \right\}.$$

We denote the space of Schwartz-Bruhat functions on $\mathbb{X} = X \otimes_E W$ by $\mathcal{S}(\mathbb{X})$. Here, as in (2.1), we identify $X \otimes_E W$ with $\mathrm{M}_{m_0,n}(E)$. Also fix an irreducible unitary representation $(\tau_{\psi,V_1,W}, \mathcal{S}_{V_1,W})$ of $H(\mathbb{W}_1)$, where $\mathbb{W}_1 = V_1 \otimes_E W = \mathrm{M}_{m_1,n}(E)$. Then the Weil representation $\omega_{\psi,V^{\chi_W},W^{\chi_V}}$ can be defined on $\mathcal{S}(\mathbb{X}) \otimes \mathcal{S}_{V_1,W}$ and has the following explicit formulas:

Theorem 2.1. *Let χ_{V_0}, χ_{V_1} be characters of $E^\times$ such that $\chi_V = \chi_{V_0}\chi_{V_1}$ and $\chi_{V_i}|_{F^\times} = \omega_{E/F}^{\dim_E V_i}$. For $\phi \otimes \phi' \in \mathcal{S}(\mathbb{X}) \otimes \mathcal{S}_{V_1,W}$, we have*

$$\begin{aligned}
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(m(a,h))\,\phi(x) \otimes \phi' \\
&\qquad\qquad = \chi_W(\det a)|\det a|_E^{-n/2}\phi(a^{-1}x) \cdot \omega_{\psi,V_1^{\chi_W},W^{\chi_{V_1}}}(h)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(n(b,c))\,\phi(x) \otimes \phi' \\
&\qquad\qquad = \psi\left(-\frac{1}{2}\mathrm{Tr}_{E/F}({}^*xcx^*B)\right)\phi(x) \cdot \tau_{\psi,V_1,W}(-bx,0)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}\,(g)\,\phi(x) \otimes \phi' \\
&\qquad\qquad = (\chi_{V_0})_u(\det g)\phi(xg) \cdot \omega_{\psi,V_1^{\chi_W},W^{\chi_{V_1}}}(g)\phi'.
\end{aligned}$$

Here $x \in \mathbb{X} = \mathrm{M}_{m_0,n}(E)$, $m(a,h) \in M_{X'}$, $n(b,c) \in N_{X'}$ and $g \in \mathrm{U}(W)$.

Similarly, we may assume that

$$W = (E^{\oplus n}, \begin{pmatrix} & & 1_{n_0} \\ & B_1 & \\ -1_{n_0} & & \end{pmatrix}).$$

We put $W_0 = (E^{\oplus 2n_0}, \begin{pmatrix} & 1_{n_0} \\ -1_{n_0} & \end{pmatrix})$ and $W_1 = (E^{\oplus n_1}, B_1)$, where $n_1 = n - 2n_0$. As in (2.2) , we have $W = Y \oplus W_1 \oplus Y'$, where $Y = \{(y_1, \dots, y_{n_0}, 0, \dots, 0) \mid y_i \in E\}$ and $Y' = \{(0, \dots, 0, y_{n-n_0+1}, \dots, y_n) \mid y_i \in E\}$.

Let $P_{Y'} = M_{Y'}N_{Y'}$ be the maximal parabolic subgroup of $\mathrm{U}(W)$ stabilizing Y', where

$$M_{Y'} = \left\{ m(a,g) = \begin{pmatrix} a & & \\ & g & \\ & & {}^*a^{-1} \end{pmatrix} \middle| \begin{array}{c} a \in \mathrm{GL}_{n_0}(E), \\ g \in \mathrm{U}(W_1) \end{array} \right\},$$

$$N_{Y'} = \left\{ n(b,c) = \begin{pmatrix} 1_{n_0} & b & c + \frac{1}{2}bB_1{}^*b \\ & 1_{n_1} & B_1{}^*b \\ & & 1_{n_0} \end{pmatrix} \middle| \begin{array}{c} b \in \mathrm{M}_{n_0,n_1}(E), \\ c = {}^*c \in \mathrm{M}_{n_0}(E) \end{array} \right\}.$$

We set $\mathbb{Y} = V \otimes_E Y = \mathrm{M}_{m,n_0}(E)$. Here, as in (2.1), we identified $V \otimes_E Y$ with $\mathrm{M}_{m,n_0}(E)$. Also fix an irreducible unitary representations $(\tau_{\psi,V,W_1}, \mathcal{S}_{V,W_1})$ of $H(\mathbb{W}_2)$, where $\mathbb{W}_2 = V \otimes_E W_1 = \mathrm{M}_{m,n_1}(E)$.

Theorem 2.2. *Let χ_{W_0}, χ_{W_1} be characters of $E^\times$ such that $\chi_W = \chi_{W_0}\chi_{W_1}$ and $\chi_{W_i}|_{F^\times} = \omega_{E/F}^{\dim_E W_i}$. For $\phi \otimes \phi' \in \mathcal{S}(\mathbb{Y}) \otimes \mathcal{S}_{V,W_1}$, we have the following explicit formulas:*

$$\begin{aligned}
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(h)\,\phi(y)\otimes\phi' \\
&\qquad\qquad = (\chi_{W_0})_u(\det h)\phi(h^{-1}y)\cdot\omega_{\psi,V^{\chi_{W_1}},W_1^{\chi_V}}(h)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(m(a,g))\,\phi(y)\otimes\phi' \\
&\qquad\qquad = \chi_V(\det a)|\det a|_E^{m/2}\phi(ya)\cdot\omega_{\psi,V^{\chi_{W_1}},W_1^{\chi_V}}(g)\phi', \\
&\omega_{\psi,V^{\chi_W},W^{\chi_V}}(n(b,c))\,\phi(y)\otimes\phi' \\
&\qquad\qquad = \psi\left(\frac{1}{2}\mathrm{Tr}_{E/F}({}^*yAyc)\right)\phi(y)\cdot\tau_{\psi,V,W_1}(yb,0)\phi'.
\end{aligned}$$

Here $y \in \mathbb{Y} = \mathrm{M}_{m,n_0}(E)$, $m(a,g) \in M_{Y'}$, $n(b,c) \in N_{Y'}$ and $h \in \mathrm{U}(V)$.

For an irreducible admissible representation π of $\mathrm{U}(V)$ or $\mathrm{U}(W)$, we denote the local theta lift of π with respect to $\omega_{\psi,V^{\chi_W},W^{\chi_V}}$ by $\theta_{\psi,V^{\chi_W},W^{\chi_V}}(\pi)$.

2.2. ***L*-packet.** In this subsection we recall the construction of L-packets for the unitary groups of our concern.

First we explain the unitary groups which we treat in this paper. For $a \in F^\times$, we define a one-dimensional skew-hermitian space:

$$W_a = (E, a\xi),$$

where $\xi \in E^\times$ is the fixed element as in Notation 1.3. Also define a three-dimensional skew-hermitian space:

$$W = (E^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \xi & \\ -1 & & \end{pmatrix}).$$

Let V_{sp} (resp. V_{an}) be the two-dimensional split (resp. anisotropic) hermitian space over E given by

$$\begin{aligned}
V_{sp} &= (E^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}), \\
V_{an} &= (E^{\oplus 2}, \begin{pmatrix} -d_0 & \\ & 1 \end{pmatrix}).
\end{aligned}$$

Note that the unitary group $\mathrm{U}(V_{an})$ is not quasi-split by our choice of d_0 (see notation). Thus we have four unitary groups $\mathrm{U}(W_a), \mathrm{U}(W), \mathrm{U}(V_{sp})$ and $\mathrm{U}(V_{an})$. We also denote the unitary similitude group of $V = V_{sp}, V_{an}$ by $\mathrm{GU}(V)$.

Next we recall the L-groups for these groups.

Lemma 2.3. *For $G = \mathrm{U}(W_a), \mathrm{U}(W), \mathrm{U}(V_{sp})$ or $\mathrm{U}(V_{an})$, the L-group ${}^LG$ is given as follows:*

$$\begin{aligned}
{}^LG &= \mathrm{GL}_n(\mathbb{C}) \rtimes W_F, \\
g \rtimes w \cdot g' \rtimes w' &= \begin{cases} gg' \rtimes ww' & w \in W_E, \\ gI_n{}^tg'^{-1}I_n^{-1} \rtimes ww' & w \notin W_E. \end{cases}
\end{aligned}$$

Here n is the dimension of the space associated to G and

$$I_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

We call a continuous homomorphism $\phi : L_F \to {}^L G$ over W_F an *L-parameter* if the projection of $\phi(W_F)$ to $\mathrm{GL}_n(\mathbb{C})$ consists of semi-simple elements of $\mathrm{GL}_n(\mathbb{C})$ and $\phi|_{\mathrm{SU}_2(\mathbb{R})} \to \mathrm{GL}_n(\mathbb{C})$ is analytic. Also, two L-parameters ϕ, ϕ' are called *equivalent* if there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $\phi' = \mathrm{Ad}(g) \circ \phi$. The set of equivalence classes of L-parameters of G is denoted by $\Phi(G)$. For any L-parameter ϕ of G, the restriction of ϕ to L_E is denoted by $\phi_E = \phi|_{L_E}$. Since it is well-known that ϕ is uniquely determined by ϕ_E, we also call ϕ_E an L-parameter of G.

Lemma 2.4. (*i*) *By local class field theory, we have the bijection:*

$$\Phi(\mathrm{U}(W_a)) \ni \phi \mapsto \phi_E \in \Pi(E^\times, \mathbb{1}_{F^\times}).$$

(*ii*) *By the bijection* $\Pi(E^\times, \mathbb{1}_{F^\times}) \ni \eta \mapsto \eta_u \in \mathrm{Irr}\,\mathrm{U}(W_a)$, *we have the bijection:*

$$\Phi(\mathrm{U}(W_a)) \to \mathrm{Irr}\,\mathrm{U}(W_a).$$

Thus we can identify $\Phi(\mathrm{U}(W_a))$ *with* $\mathrm{Irr}\,\mathrm{U}(W_a)$.

Finally we recall the construction of L-packets for $\mathrm{U}(V_{sp}), \mathrm{U}(V_{an}), \mathrm{U}(W)$.

Theorem 2.5 ([10])**.** *For* $G = \mathrm{U}(V_{sp}), \mathrm{U}(W)$, *we have a surjective map from* $\mathrm{Irr}\,G$ *onto* $\Phi(G)$ *with finite fibre.*

Then each L-packet for $\mathrm{U}(V_{sp})$ or $\mathrm{U}(W)$ is defined as a fibre Π of its map, and the image of Π under its map is the L-parameter associated to Π.

The construction of L-packets for $\mathrm{U}(V_{an})$ requires the Jaquet-Langlands correspondence. For $\tau \in \mathrm{Irr}\,\mathrm{U}(V_{an})$, there exists $\tau_s \in \mathrm{Irr}\,\mathrm{GU}(V_{an})$ such that $\tau \hookrightarrow \tau_s|_{\mathrm{U}(V_{an})}$. If τ_s^{JL} is the Jaquet-Langlands correspondence to $\mathrm{GU}(V_{sp})$, then there exists $\tau^{JL} \in \mathrm{Irr}\,\mathrm{U}(V_{sp})$ such that $\tau^{JL} \hookrightarrow \tau_s^{JL}|_{\mathrm{U}(V_{sp})}$. Define the L-parameter of τ to be that of τ^{JL}. This is well-defined. In this way, we obtain a map from $\mathrm{Irr}\,\mathrm{U}(V_{an})$ to $\Phi(\mathrm{U}(V_{an}))$. Then we can prove that it is a map with finite fibre. Thus, in the same way as above, we can define L-packets for $\mathrm{U}(V_{an})$.

3. Endoscopy and local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$

In this section we recall the classification of L-packets for $\mathrm{U}(V_{sp})$ and compute the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$.

3.1. **Endoscopy.** We denote by B the Borel subgroup of $\mathrm{U}(V_{sp})$ consisting of lower triangular matrices. Let

$$U = \left\{ u(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \middle| \, x \in E, \mathrm{Tr}_{E/F}(x) = 0 \right\}$$

be the unipotent radical of B and

$$T = \left\{ m(\alpha) = \begin{pmatrix} \alpha & \\ & {}^*\alpha^{-1} \end{pmatrix} \middle| \, \alpha \in E^\times \right\}$$

a maximal torus of $\mathrm{U}(V_{sp})$. Thus we have $B = TU$.

For each $b \in F^\times$, we define a non-trivial character $\psi^b_{U,\xi}$ of U by

$$\psi^b_{U,\xi} : U \ni u(x) \mapsto \psi(-bx\xi^{-1}) \in \mathbb{C}^1,$$

where ψ is the fixed character as in Notation 1.3. An irreducible admissible representation π of $\mathrm{U}(V_{sp})$ is called ψ^b-generic, or $\psi^b_{U,\xi}$-generic, if it has a non-zero homomorphism into $\mathrm{Ind}_U^{\mathrm{U}(V_{sp})} \psi^b_{U,\xi}$.

Theorem 3.1 ([10]). *An L-packet Π_φ of $\mathrm{U}(V_{sp})$ has two elements if and only if there exist characters $\mu_1, \mu_2 \in \Pi(E^\times, \omega_{E/F})$ such that $\varphi_E = \mu_1 \oplus \mu_2$. Otherwise the cardinality of Π_φ is one.*

When $\varphi_E = \mu_1 \oplus \mu_2$, we denote the L-packet Π_φ by $\Pi_{\mu_1,\mu_2} = \Pi_{\mu_1,\mu_2}(V_{sp})$ and call it an endoscopic L-packet. Also, its elements are called endoscopic representations.

An endoscopic L-packet Π_{μ_1,μ_2} has the unique ψ-generic (resp. ψ^{d_0}-generic) element, where d_0 is the fixed element as in Notation 1.3. We write it as $\tau(\mu_1,\mu_2)^+_{sp}$ (resp. $\tau(\mu_1,\mu_2)^-_{sp}$). Note that if $\mu_1 = \mu_2$, then $\tau(\mu_1,\mu_1)^+_{sp}$ is the unique ψ-generic irreducible subrepresentation of $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})} \mu_1$.

To compute the local theta lift for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$, we need a global result. Let $\tilde{E}$ be a quadratic extension of a number field $\tilde{F}$. Also its associated quadratic character of $\mathbb{A}^\times_{\tilde{F}}/\tilde{F}^\times$ is denoted by $\omega_{\tilde{E}/\tilde{F}}$, where $\mathbb{A}^\times_{\tilde{F}}$ is the idele group of $\tilde{F}$. For each character χ of $\mathbb{A}^\times_{\tilde{F}}/\tilde{F}^\times$, $\Pi(\mathbb{A}^\times_{\tilde{E}}, \chi)$ stands for the set of characters of $\mathbb{A}^\times_{\tilde{E}}/\tilde{E}^\times$ whose restriction to $\mathbb{A}^\times_{\tilde{F}}$ are χ. We have the following:

Theorem 3.2 ([10]). *Let $\tau = \otimes \tau_v$ be an irreducible cuspidal representation of $\mathrm{U}(V_{sp})(\mathbb{A}_{\tilde{F}})$. If there exist characters $\tilde{\mu}_1, \tilde{\mu}_2 \in \Pi(\mathbb{A}^\times_{\tilde{E}}, \omega_{\tilde{E}/\tilde{F}})$ such that $\tau_v \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}$ for almost all places v of $\tilde{F}$, then $\tau_v \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}$ for all places v.*

3.2. **Local theta lift.** In this subsection, we compute the local theta lift $\theta_{\psi,V^\mu_{sp},W^\eta_a}$ for $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$, where μ, η are the fixed characters as in Notation 1.3.

From Theorem 2.1, the Weil representation $\omega_{\psi,V^\mu_{sp},W^\eta_a}$ of $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$ on the space $\mathcal{S}(E)$ of Schwartz-Bruhat functions on E has the following explicit formulas:

$$\begin{aligned}
\omega_{\psi,V^\mu_{sp},W^\eta_a}(m(\alpha))\, f(t) &= \mu(\alpha)|\alpha|_E^{-1/2} f(\alpha^{-1}t),\\
\omega_{\psi,V^\mu_{sp},W^\eta_a}(u(x))\, f(t) &= \psi(a\xi x\, \mathrm{N}_{E/F}(t)) f(t),\\
\omega_{\psi,V^\mu_{sp},W^\eta_a}(g)\, f(t) &= \eta_u(g) f(tg).
\end{aligned}$$

Here $f \in \mathcal{S}(E)$, $t \in E$, $m(\alpha) \in T$, $u(x) \in U$ and $g \in \mathrm{U}(W_a)$.

For brevity, we write $\omega_\psi = \omega_{\psi,V^\mu_{sp},W^\eta_a}$. First we shall compute (twisted) Jacquet module of ω_ψ. For each $b \in F^\times$, we set

$$E_b = \left\{ t \in E^\times \,\middle|\, \mathrm{N}_{E/F}(t) = b\, \mathrm{N}_{E/F}(\xi^{-1}) \right\}.$$

One notes that if $b \notin \mathrm{N}_{E/F}(E^\times)$, then E_b is empty.

Lemma 3.3. *(i) The unnormalized Jacquet module $(\omega_\psi)_U$ of the Weil representation ω_ψ is isomorphic to $\mu|\cdot|_E^{-1/2} \boxtimes \eta_u$ as $T \times \mathrm{U}(W_a)$-module, where*

$$\mu|\cdot|_E^{-1/2} \boxtimes \eta_u : T \times \mathrm{U}(W_a) \ni m(\alpha) \times g \mapsto \mu(\alpha)|\alpha|_E^{-1/2}\eta_u(g) \in \mathbb{C}^\times.$$

(ii) The twisted Jacquet module $(\omega_\psi)_{U,\psi^b_{U,\xi}}$ of the Weil representation ω_ψ is isomorphic to $\eta_u \otimes \mathcal{S}(E_{a^{-1}b})$ as $\mathrm{U}(W_a)$-module, where the action of $\mathrm{U}(W_a)$ on $\mathcal{S}(E_{a^{-1}b})$ is given by right translation.

Proof. (i) If we define

$$\omega_\psi(U) = \{\omega_\psi(u(x))f - f \mid x \in F, f \in \mathcal{S}(E)\},$$

then $(\omega_\psi)_U = \omega_\psi/\omega_\psi(U)$. By the above explicit formulas, we have $\omega_\psi(U) = \{f \in \mathcal{S}(E) \mid f(0) = 0\}$. Thus the map $f \mapsto f(0)$ induces a $T \times \mathrm{U}(W_a)$-isomorphism $(\omega_\psi)_U \cong \mu|\cdot|_E^{-1/2} \boxtimes \eta_u$.

(ii) Similarly, we have $(\omega_\psi)_{U,\psi^b_{U,\xi}} = \omega_\psi/\omega_\psi(U, \psi^b_{U,\xi})$, where

$$\omega_\psi(U, \psi^b_{U,\xi}) = \left\{\omega_\psi(u(x))f - \psi^b_{U,\xi}(x)f \,\middle|\, x \in F, f \in \mathcal{S}(E)\right\}.$$

Moreover, we have $\omega_\psi(U, \psi^b_{U,\xi}) = \left\{f \in \mathcal{S}(E) \,\middle|\, f|_{E_{a^{-1}b}} = 0\right\}$. Thus $f \mapsto f|_{E_{a^{-1}b}}$ induces a $\mathrm{U}(W_a)$-isomorphism $(\omega_\psi)_{U,\psi^b_{U,\xi}} \cong \eta_u \otimes \mathcal{S}(E_{a^{-1}b})$. □

Corollary 3.4. *Let η' be an element of $\Pi(E^\times, \mathbb{1}_{F^\times})$.*
(i) The local theta lift $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u)$ is ψ^a-generic and not ψ^{ad_0}-generic.
(ii) If $\eta \neq \eta'$, then the local theta lift $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u)$ is an irreducible supercuspidal representation of $\mathrm{U}(V_{sp})$.
(iii) If $\eta = \eta'$, then the local theta lift $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta_u)$ is the irreducible ψ^a-generic subrepresentation $\tau(\mu,\mu)^{\omega_{E/F}(a)}_{sp}$ of $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\mu$.

Proof. We have a non-zero $\mathrm{U}(W_a)$-homomorphism $\omega_\psi \to \eta'_u$ by the explicit formulas. Thus we obtain $\tau = \theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u) \neq 0$. By the definition of the local theta lift, we have a surjective $\mathrm{U}(V_{sp}) \times \mathrm{U}(W_a)$-homomorphism

$$\omega_\psi \to \tau \boxtimes \eta'_u.$$

Taking its twisted Jacquet module with respect to $\psi^b_{U,\xi}$, we have a surjective $\mathrm{U}(W_a)$-homomorphism

$$\eta_u \otimes \mathcal{S}(E_{a^{-1}b}) \to \tau_{U,\psi^b_{U,\xi}} \boxtimes \eta'_u$$

by Lemma 3.3 (ii). Thus if $b = ad_0$, then $\tau_{U,\psi^b_{U,\xi}}$ is zero. Namely, τ is not ψ^{ad_0}-generic.

We assume that $\eta' \neq \eta$. Taking the Jacquet module of $\omega_\psi \to \tau \boxtimes \eta'_u$ with respect to U, we have $\tau_U = 0$ by Lemma 3.3 (i). Thus τ is supercuspidal. Here, it is well-known that any supercuspidal representation of $\mathrm{U}(V_{sp})$ is ψ^a-generic or ψ^{ad_0}-generic. Therefore τ is ψ^a-generic.

Next we assume that $\eta' = \eta$. By Frobenius reciprocity, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{U}(V_{sp})\times\mathrm{U}(W_a)}(\omega_\psi, \mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\mu \boxtimes \eta_u)\\
\cong\ &\mathrm{Hom}_{T\times\mathrm{U}(W_a)}((\omega_\psi)_U, \mu|\cdot|_E^{-1/2} \boxtimes \eta_u)\\
\cong\ &\mathbb{C}.
\end{aligned}$$

Thus τ is an irreducible subrepresentation of $\mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\mu$, that is, $\tau = \tau(\mu,\mu)^{\pm}_{sp}$. Since τ is not ψ^{ad_0}-generic, we have $\tau = \tau(\mu,\mu)^{\omega_{E/F}(a)}_{sp}$. Thus we obtain Corollary 3.4. □

Theorem 3.5. *For each $\eta'_u \in \mathrm{Irr}\,\mathrm{U}(W_a)$, the local theta lift $\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u)$ is the unique ψ^a-generic element of L-packet $\Pi_{\mu,\mu\eta\eta'^{-1}}$ of $\mathrm{U}(V_{sp})$. Namely,*

$$\theta_{\psi,V^\mu_{sp},W^\eta_a}(\eta'_u) = \tau(\mu, \mu\eta\eta'^{-1})^{\omega_{E/F}(a)}_{sp}.$$

Proof. Set $\tau = \theta_{\psi, V_{sp}^{\mu}, W_a^{\eta}}(\eta'_u)$. It is enough to show that τ is contained in $\Pi_{\mu, \mu\eta\eta'^{-1}}$. We have already proved it for $\eta = \eta'$. Thus we consider the case for $\eta \neq \eta'$. Our proof uses Theorem 3.2. Choose a number field $\tilde{F}$ and a quadratic field extension $\tilde{E}$ of $\tilde{F}$ such that $\tilde{E}_{v_0}/\tilde{F}_{v_0} = E/F$ for a finite place v_0 of $\tilde{F}$. Fix:

- $\tilde{a} \in \tilde{F}^\times$ such that $\tilde{a} \equiv a \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\xi} \in \tilde{E}^\times$ such that $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and $\tilde{\xi} \equiv \xi \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\psi}$ is a non-trivial character of $\mathbb{A}_{\tilde{F}}/\tilde{F}$ such that $\tilde{\psi}_{v_0} = \psi$;
- $\tilde{\mu} \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{v_0} = \mu$;
- $\tilde{\eta} \neq \tilde{\eta}' \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \mathbb{1})$ such that $\tilde{\eta}_{v_0} = \eta, \tilde{\eta}'_{v_0} = \eta'$.

Now we define two (skew) hermitian spaces over $\tilde{E}$:

$$\tilde{V}_{sp} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}) \text{ and } W_{\tilde{a}} = (\tilde{E}, \tilde{a}\tilde{\xi}).$$

Then the global theta lift $\tilde{\tau} = \theta_{\tilde{\psi}, \tilde{V}_{sp}^{\tilde{\mu}}, W_{\tilde{a}}^{\tilde{\eta}}}(\tilde{\eta}'_u)$ is non-zero and cuspidal, since the constant term of $\tilde{\tau}$ is zero and its $\tilde{\psi}$-Whittaker model is non-zero by the explicit formulas of the Weil representation of $\omega_{\tilde{\psi}, \tilde{V}_{sp}^{\tilde{\mu}}, W_{\tilde{a}}^{\tilde{\eta}}}$. Thus $\tilde{\tau}$ is an irreducible cuspidal representation of $\mathrm{U}(\tilde{V}_{sp})$. If we show the claim that the local theta lift $\tilde{\tau}_v$ of $\tilde{\eta}'_{u,v}$ is contained in $\Pi_{\tilde{\mu}_v, \tilde{\mu}_v\tilde{\eta}_v\tilde{\eta}_v'^{-1}}$ for almost all places v of $\tilde{F}$, then we obtain Theorem 3.5 by Theorem 3.2.

First suppose that a field extension $\tilde{E}_v/\tilde{F}_v$ and $\tilde{\eta}_v$, $\tilde{\eta}'_v$ are unramified. Then the claim immediately follows from $\tilde{\eta}_v = \tilde{\eta}'_v$.

Next suppose that $\tilde{E}_v/\tilde{F}_v$ is split. We omit the symbol v from the notation. Then we have $\tilde{E} \cong \tilde{F} \oplus \tilde{F}, \mathrm{U}(\tilde{V}_{sp}) \cong \mathrm{GL}_2(\tilde{F})$ and $\mathrm{U}(W_{\tilde{a}}) \cong \mathrm{GL}_1(\tilde{F})$. Also, its Weil representation $(\omega_{\tilde{\psi}}, \mathcal{S}(\tilde{F} \oplus \tilde{F}))$ is given as follows:

$$\omega_{\tilde{\psi}}(h) f(t) = \tilde{\mu}(\det h) |\det h|_{\tilde{F}}^{-1/2} f(h^{-1}t),$$
$$\omega_{\tilde{\psi}}(g) f(t) = \tilde{\eta}(g)|g|_{\tilde{F}} f(tg).$$

Here $f(t) \in \mathcal{S}(\tilde{F} \oplus \tilde{F})$, $h \in \mathrm{GL}_2(\tilde{F})$ and $g \in \mathrm{GL}_1(\tilde{F})$.

We want to show that $\tilde{\tau} = \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$. Let χ be a quasi-character of $\tilde{F}^\times$. For an element $f \in \mathcal{S}(\tilde{F} \oplus \tilde{F})$, we define a function Φ_f on $\mathrm{GL}_2(\tilde{F}) \times \mathrm{GL}_1(\tilde{F})$ by

$$\Phi_f(h, g) = \frac{1}{L(0, \chi)} \int_{\tilde{F}^\times} \omega_{\tilde{\psi}}(h, g) f \begin{pmatrix} t \\ 0 \end{pmatrix} \chi(t) dt^\times.$$

By the above explicit formulas, the map $f \mapsto \Phi_f$ induces a non-zero $\mathrm{GL}_2(\tilde{F}) \times \mathrm{GL}_1(\tilde{F})$-homomorphism

$$\omega_{\tilde{\psi}} \to \mathrm{Ind}(\tilde{\mu}\chi |\cdot|_{\tilde{F}}^{-1} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}\chi^{-1} |\cdot|_{\tilde{F}}.$$

If $\chi = \tilde{\eta}\tilde{\eta}'^{-1}|\cdot|_{\tilde{F}}$, then we have a non-zero homomorphism

$$\omega_{\tilde{\psi}} \to \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu}) \boxtimes \tilde{\eta}'.$$

Since $\tilde{\mu}, \tilde{\eta}$ and $\tilde{\eta}'$ are unitary, $\mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$ is irreducible. Thus we obtain $\tilde{\tau} = \mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu})$. Since $\mathrm{Ind}(\tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1} \boxtimes \tilde{\mu}) \in \Pi_{\tilde{\mu}, \tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1}}$, we have $\tilde{\tau} \in \Pi_{\tilde{\mu}, \tilde{\mu}\tilde{\eta}\tilde{\eta}'^{-1}}$. □

4. Endoscopy and local theta lift for $\mathrm{U}(2) \times \mathrm{U}(3)$

In this section we recall some results from [2], mainly about the relation between endoscopy and local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W)$. After that, we will state the problem in this paper.

First we define the subset of irreducible representations occurring in Weil representation $\omega_{\psi,V^\mu,W^\eta}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$. For $V = V_{sp}$ or V_{an}, we set

$$\mathcal{R}_{\psi,\mu,\eta}(V) = \{\tau \in \mathrm{Irr}\,\mathrm{U}(V) \,|\, \theta_{\psi,V^\mu,W^\eta}(\tau) \neq 0\}.$$

Similarly for W, we set

$$\mathcal{R}_{\psi,\mu,\eta}(W) = \{\pi \in \mathrm{Irr}\,\mathrm{U}(W) \,|\, \theta_{\psi,V^\mu,W^\eta}(\pi) \neq 0, V = V_{sp} \text{ or } V_{an}\}.$$

Let $\theta_{\psi,\mu,\eta}$ be the map defined by the local theta lifts $\theta_{\psi,V^\mu,W^\eta}$ for $V = V_{sp}, V_{an}$:

$$\theta_{\psi,\mu,\eta} \colon \mathcal{R}_{\psi,\mu,\eta}(V_{sp}) \sqcup \mathcal{R}_{\psi,\mu,\eta}(V_{an}) \to \mathcal{R}_{\psi,\mu,\eta}(W).$$

Theorem 4.1. *(i) The map $\theta_{\psi,\mu,\eta}$ is bijective.*
(ii) For $\tau \in \mathcal{R}_{\psi,\mu,\eta}(V)$ with L-parameter φ, the L-parameter of $\theta_{\psi,\mu,\eta}(\tau)$ is $\mu\eta\check{\varphi}_E \oplus \eta$, where $\check{\varphi}_E$ is the contragredient representation of φ_E.

Proof. (i) The injectivity follows from the Dichotomy Theorem [2, Theorem 1.2]. The surjectivity follows from the definition of the map.

(ii) This was proved in [2, §4]. □

The set $\mathcal{R}_{\psi,\mu,\eta}(V)$ is described in terms of the local theta lift for $\mathrm{U}(V) \times \mathrm{U}(W_1)$, where $W_1 = (E, \xi)$ (see §2.2).

Theorem 4.2 ([2, Lemma 4.2])**.** *For $\tau \in \mathrm{Irr}\,\mathrm{U}(V)$, the local theta lift $\theta_{\psi,\mu,\eta}(\tau)$ is non-zero if and only if $\theta_{\psi^{d_0},V^\mu,W_1^\eta}(\tau) = 0$, where d_0 is the fixed element as in Notation 1.3. Namely,*

$$\mathcal{R}_{\psi,\mu,\eta}(V) = \{\tau \in \mathrm{Irr}\,U(V) \mid \theta_{\psi^{d_0},V^\mu,W_1^\eta}(\tau) = 0\}.$$

To describe the set $\mathcal{R}_{\psi,\mu,\eta}(W)$, we prepare a few notation. An L-packet Π_ϕ of $\mathrm{U}(W)$ is called (χ-)endoscopic if it consists of infinite-dimensional representations and ϕ_E contains a one-dimensional representation χ of W_E. An element of such Π_ϕ is called (χ-)endoscopic.

Theorem 4.3 ([2, §4])**.** *The set $\mathcal{R}_{\psi,\mu,\eta}(W)$ consists of the η-endoscopic representations and $\eta_u \circ \det$.*

The local theta lift $\theta_{\psi,\mu,\eta}$ has the following property:

Theorem 4.4 ([2])**.** *For $\tau \in \mathcal{R}_{\psi,\mu,\eta}(V)$, $\theta_{\psi,\mu,\eta}(\tau)$ is generic if and only if $V = V_{sp}$, τ is ψ-generic and $\tau \neq \mathrm{Ind}_B^{\mathrm{U}(V_{sp})}\,\mu|\cdot|_E^{-1}$.*

Remark 4.5. The proofs of the above theorems in [2] do not require the endoscopic description of L-packets of $\mathrm{U}(V_{an})$. But Theorem 5.1 below on the cardinality of an L-packet requires (see [2, p.430]).

Next we explain the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(W)$ given by Rogawski ([10] and [8]). The unique non-trivial elliptic endoscopic datum (H_0, s_0, ξ_0) for $\mathrm{U}(W)$ up to equivalence is given as follows:

$$H_0 = \mathrm{U}(V_{sp}) \times \mathrm{U}(1),\ s_0 = \begin{pmatrix} -1_2 & \\ & 1 \end{pmatrix},$$

$$\xi_0 : {}^L H_0 \ni (h, h') \rtimes w \longmapsto \begin{cases} \begin{pmatrix} h\mu_0(w) & \\ & h' \end{pmatrix} \rtimes w & \text{if } w \in W_E, \\ \begin{pmatrix} & -h \\ h' & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma. \end{cases}$$

Here we have fixed $w_\sigma \in W_F \smallsetminus W_E$ and a character μ_0 of $E^\times$ such that $\mu_0|_{F^\times} = \omega_{E/F}$. For an L-parameter $\phi \in \Phi(\mathrm{U}(W))$, define $\hat{\Pi}_\phi = \{\rho \in \Phi(H_0) \,|\, \xi_0 \circ \rho = \phi\}$. Rogawski defined the local pairing $\langle \rho, \pi \rangle = \pm$ for $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$ by using the trace formula in [10]. Let $\mathcal{S}_\phi$ be the component group of the group S_ϕ, where S_ϕ is the quotient group of $\mathrm{Cent}_{\mathrm{GL}_3(\mathbb{C})}(\phi(L_F))$ by $\{\pm 1\}$. Then its local pairing $\langle \rho, \pi \rangle$ was rewritten in terms of $\mathcal{S}_\phi$ in [8]. It is given as follows. If $s \in \mathcal{S}_\phi$ is not the neutral element, there exist $\rho \in \hat{\Pi}_\phi$ and $w \in \mathrm{GL}_3(\mathbb{C})$ such that $\mathrm{Ad}(w) \circ \xi_0 \circ \rho = \phi$ and $\mathrm{Ad}(w)(s_0)$ equals s in $\mathcal{S}_\phi$. Then we set $\langle s, \cdot \rangle := \langle \rho, \cdot \rangle$. If s is the neutral element, then we set $\langle s, \cdot \rangle := +$. The following is the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(W)$.

Theorem 4.6 ([8])**.** *We have the bijection* $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle \in \mathrm{Irr}\, \mathcal{S}_\phi$.

Next we explain the relation between the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(W)$ and the local theta lift $\theta_{\psi,\mu,\eta}$. Let π be an element of an L-packet Π_ϕ, and ρ an element of $\hat{\Pi}_\phi$. In [2], Gelbart-Rogawski-Soudry defined another local pairing $\epsilon_\rho(\pi)$ by using the local theta lift $\theta_{\psi,\mu,\eta}$ as follows. $\rho \in \hat{\Pi}_\phi$ is an L-parameter of $H_0 = \mathrm{U}(V_{sp}) \times \mathrm{U}(1)$. Then we denote its $\mathrm{U}(1)$-part by ρ_η. The character of $E^\times/F^\times$ corresponding to ρ_η is denoted by η_ρ. Thus Π_ϕ is an η_ρ-endoscopic L-packet and π is an η_ρ-endoscopic. Then by Theorem 4.1 and Theorem 4.3, there exists a unique $V_{\pi,\rho} = V_{sp}$ or V_{an} such that $\theta_{\psi, V^\mu_{\pi,\rho}, W^{\eta_\rho}}(\pi) \neq 0$. Here $V_{\pi,\rho}$ is independent of the choices of characters ψ, μ by the following lemma:

Lemma 4.7. (*i*) *For $g \in \mathrm{GU}(V)$ with the similitude norm d, we have*

$$\theta_{\psi^d, V^\mu, W^\eta}(\pi) = \theta_{\psi, V^\mu, W^\eta}(\pi) \circ \mathrm{Ad}(g).$$

(*ii*) *For any character $\mu' \in \Pi(E^\times, \omega_{E/F})$, we have $\theta_{\psi, V^{\mu'}, W^\eta}(\pi) = (\mu'/\mu)_u \circ \det_{\mathrm{U}(V)} \otimes \theta_{\psi, V^\mu, W^\eta}(\pi)$.*

Thus we can define the local pairing $\epsilon_\rho(\pi) := +$ if $V_{\pi,\rho} = V_{sp}$ and $-$ otherwise. Namely, we have the following definition:

Definition 4.8. For $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$,

$$\epsilon_\rho(\pi) := \begin{cases} + & \text{if } \theta_{\psi, V^\mu_{sp}, W^{\eta_\rho}}(\pi) \neq 0, \\ - & \text{if } \theta_{\psi, V^\mu_{an}, W^{\eta_\rho}}(\pi) \neq 0. \end{cases}$$

The following theorem gives the relation between the endoscopic description of $\mathrm{Irr}\,\mathrm{U}(W)$ and the local theta lift $\theta_{\psi,\mu,\eta}$.

Theorem 4.9 ([2, Theorem 3.1])**.** *For $\rho \in \hat{\Pi}_\phi$ and $\pi \in \Pi_\phi$, we have $\langle \rho, \pi \rangle = \epsilon_\rho(\pi)$.*

The following also holds:

Proposition 4.10 ([2, Proposition 3.3]). *Let π be an element of an endoscopic L-packet Π_ϕ. Then π is generic if and only if $\epsilon_\rho(\pi) = +$ for any $\rho \in \hat{\Pi}_\phi$.*

Now we can state the problem of our concern. Consider the theta lift $\pi = \theta_{\psi,\mu,\eta}(\tau)$ of $\tau \in \operatorname{Irr} \mathrm{U}(V)$ with L-parameter φ. The L-parameter ϕ of π, when π is non-zero, is given by

$$\phi_E = \mu\eta\check{\varphi}_E \oplus \eta.$$

For $\rho \in \hat{\Pi}_\phi$, we would like to determine the local pairing $\epsilon_\rho(\pi)$ completely. In the cases of next proposition, the problem can be settled by the above results.

Proposition 4.11. *Let ρ_0 be the element of $\hat{\Pi}_\phi$ such that $\rho_{0,E} = \mu_0^{-1}\eta\mu\check{\varphi}_E \times \eta$.*
(i) If $V = V_{sp}$ and τ is ψ-generic, the local pairing $\epsilon_\rho(\pi) = +$ for any $\rho \in \hat{\Pi}_\phi$.
(ii) If φ_E is an irreducible representation of L_E, then we have $\hat{\Pi}_\phi = \{\rho_0\}$. Also, $\epsilon_{\rho_0}(\pi) = +$ if $V = V_{sp}$ and $-$ otherwise.
(iii) If $V = V_{sp}$ and τ is one dimensional, we have $\hat{\Pi}_\phi = \{\rho_0\}$ and $\epsilon_{\rho_0}(\pi) = +$.
(iv) If $V = V_{sp}$ and $\tau = \tau(\mu,\mu_1)^-_{sp}$ ($\mu_1 \in \Pi(E^\times, \omega_{E/F})$), then $\pi = 0$.
(v) If $V = V_{sp}$ and $\tau = \tau(\mu_1,\mu_2)^-_{sp}$ ($\mu \neq \mu_1, \mu_2 \in \Pi(E^\times, \omega_{E/F})$), then the cardinality of $\hat{\Pi}_\phi$ is two or three and the local pairing $\epsilon_{\rho_0}(\pi) = +$ and $\epsilon_\rho(\pi) = -$ for any $\rho \neq \rho_0, \in \hat{\Pi}_\phi$.

Proof. (i) If $\tau = \operatorname{Ind}_B^{\mathrm{U}(V_{sp})} \mu|\cdot|_E^{-1}$, then $\pi = \eta_u \circ \det$. The assertion is trivial. Otherwise π is generic by Theorem 4.4. The assertion follows from Proposition 4.10.

(ii), (iii) By Theorem 4.1, the first is trivial. The second follows from $V_{\pi,\rho_0} = V$.

(iv) By Theorem 3.5, we have

$$\begin{aligned}
&\theta_{\psi^{d_0}, V_{sp}^{\mu_1}, W_1^1}(\tau(\mu_1,\mu)^-_{sp}) \\
=&\theta_{\psi, V_{sp}^{\mu_1}, W_1^1}(\tau(\mu_1,\mu)^+_{sp}) \\
=&(\mu_1\mu^{-1})_u \\
\neq& 0.
\end{aligned}$$

Thus it follows from Theorem 4.2 that $\pi = 0$.

(v) π is not generic by Theorem 4.4. Moreover it is trivial that $\epsilon_{\rho_0}(\pi) = +$.

Assume that $\mu_1 \neq \mu_2$. Then we have $\hat{\Pi}_\phi = \{\rho_0, \rho_1, \rho_2\}$, where

$$\begin{aligned}
\rho_{0,E} &= \mu_0^{-1}\mu\eta(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta, \\
\rho_{1,E} &= \mu_0^{-1}(\eta \oplus \mu\eta\mu_2^{-1}) \times \mu\eta\mu_1^{-1}, \\
\rho_{2,E} &= \mu_0^{-1}(\mu\eta\mu_1^{-1} \oplus \eta) \times \mu\eta\mu_2^{-1}.
\end{aligned}$$

Since $\mathcal{S}_\phi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, each non-trivial element of $\mathcal{S}_\phi$ corresponds one-one to each ρ_i. Since $\langle\cdot,\pi\rangle$ is a character of $\mathcal{S}_\phi$ by Theorem 4.6, we have $\prod_s \langle s,\pi\rangle = +$, where s runs the non-trivial elements of $\mathcal{S}_\phi$. Thus we have $\langle\rho_0,\pi\rangle\langle\rho_1,\pi\rangle\langle\rho_2,\pi\rangle = +$, that is, $\epsilon_{\rho_0}(\pi)\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. Since $\epsilon_{\rho_0}(\pi) = +$, we obtain $\epsilon_{\rho_1}(\pi)\epsilon_{\rho_2}(\pi) = +$. If $\epsilon_{\rho_1}(\pi) = +$ and $\epsilon_{\rho_2}(\pi) = +$, then π is generic by Proposition 4.10. This is a contradiction. Therefore we have $\epsilon_{\rho_1}(\pi) = \epsilon_{\rho_2}(\pi) = -$.

We can also prove the case of $\mu_1 = \mu_2$ by the similar argument. □

Hence the remaining is the cases of $V = V_{an}$ and reducible φ_E. Note that these are the endoscopic cases. So we shall consider the endoscopic description of L-packets of $\mathrm{U}(V_{an})$ in the next section.

5. Endoscopy for $\mathrm{U}(V_{an})$

In this section, we consider the local endoscopy for $\mathrm{U}(V_{an})$ and a global result for a global anisotropic unitary group in two variables. These were studied by Konno-Konno [3] following the line of the article [6], which studied the endoscopy for SL(2).

5.1. **Local endoscopy.**

In this subsection, we recall the description of the local endoscopic L-packets of $\mathrm{U}(V_{an})$ from [3].

Theorem 5.1 ([3]). *An L-packet Π_φ of $\mathrm{U}(V_{an})$ has two elements if and only if there exist characters $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ such that $\varphi_E = \mu_1 \oplus \mu_2$. Otherwise the cardinality of Π_φ is one.*

When $\varphi_E = \mu_1 \oplus \mu_2$, we denote the L-packet Π_φ by $\Pi_{\mu_1,\mu_2} = \Pi_{\mu_1,\mu_2}(V_{an})$ and call it an endoscopic L-packet. Also, its elements are called endoscopic representations.

We set

$$T = \left\{ t(z,z') = \begin{pmatrix} zz' & \\ & z\sigma(z') \end{pmatrix} \in \mathrm{U}(V_{an}) \,\middle|\, \begin{matrix} z, z' \in E^\times, \\ \mathrm{N}_{E/F}(zz') = 1 \end{matrix} \right\}. \tag{5.1}$$

The endoscopic description of each endoscopic L-packet of $\mathrm{U}(V_{an})$ is given as follows:

Theorem 5.2 ([3]). *For any pair (μ_1, μ_2) of distinct characters $\mu_i \in \Pi(E^\times, \omega_{E/F})$, there exist inequivalent irreducible admissible representations $\tau(\mu_1,\mu_2)^\pm_{an}$ of $\mathrm{U}(V_{an})$ consisting the endoscopic L-packet Π_{μ_1,μ_2}. And they satisfy the following character identity:*

$$\begin{aligned} &\mathrm{Tr}\,\tau(\mu_1,\mu_2)^+_{an}(t(z,z')) - \mathrm{Tr}\,\tau(\mu_1,\mu_2)^-_{an}(t(z,z')) \\ &= \lambda\,(E/F,\psi)\,\omega_{E/F}\left(\frac{z'-\sigma(z')}{\xi}\right)\omega\left(z\,\mathrm{N}_{E/F}(z')\right)\frac{\omega'(\sigma(z')) - \omega'(z')}{|z'-\sigma(z')|_E^{1/2}}|z'|_E^{1/2} \end{aligned}$$

for any $t(z,z') \in T$ and $\omega, \omega' \in \mathrm{Irr}\,E^\times$ such that $\mu_1 = \omega\omega', \mu_2 = \omega(\omega' \circ \sigma)$.

It is easy to check the following:

Corollary 5.3. *(i) For any $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$ and $\varepsilon \in \{\pm\}$, we have $\tau(\mu_1,\mu_2)^\varepsilon_{an} = \tau(\mu_2,\mu_1)^{-\varepsilon}_{an}$.*
(ii) For each $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have $\eta_u \circ \det \otimes \tau(\mu_1,\mu_2)^\varepsilon_{an} = \tau(\eta\mu_1,\eta\mu_2)^\varepsilon_{an}$.
(iii) The representation $\tau(\mu_1^{-1},\mu_2^{-1})^{\omega_{E/F}(-1)\varepsilon}_{an}$ is the contragredient representation of $\tau(\mu_1,\mu_2)^\varepsilon_{an}$.
(iv) For $g \in \mathrm{GU}(V_{an})$ with the similitude norm a, we have $\tau(\mu_1,\mu_2)^\epsilon_{an} \circ \mathrm{Ad}(g) = \tau(\mu_1,\mu_2)^{\omega_{E/F}(a)\epsilon}_{an}$.

5.2. **Global endoscopy.** In this subsection, we prepare a global result to show our problem. Choose a number field $\tilde{F}$ and a quadratic field extension $\tilde{E}$ of $\tilde{F}$ such that

- $\tilde{F}_{v_0} = F$ and $\tilde{E}_{v_0} = E$ for a finite place v_0 of $\tilde{F}$;
- $\tilde{F}_{v_1} = \mathbb{R}$ and $\tilde{E}_{v_1} = \mathbb{C}$ for an infinite place v_1 of $\tilde{F}$.

Fix:

- $\tilde{\xi} \in \tilde{E}^\times$ such that $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\tilde{\xi}) = 0$ and $\tilde{\xi} \equiv \xi \mod \mathrm{N}_{E/F}(E^\times)$;
- $\tilde{\psi}$ is a non-trivial character of $\mathbb{A}_{\tilde{F}}/\tilde{F}$ such that $\tilde{\psi}_{v_0} = \psi$;
- $\tilde{d} \in \tilde{F}$ such that $\tilde{d} \notin \mathrm{N}_{\tilde{E}_v/\tilde{F}_v}(\tilde{E}_v^\times)$ at $v = v_0, v_1$ and $\tilde{d} \in \mathrm{N}_{\tilde{E}_v/\tilde{F}_v}(\tilde{E}_v^\times)$ at any place $v \neq v_0, v_1$.

We define two-dimensional hermitian spaces $\tilde{V}$ and $\tilde{V}_{sp}$ over $\tilde{E}$ as follows:

$$\tilde{V} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} -\tilde{d} & \\ & 1 \end{pmatrix}),$$
$$\tilde{V}_{sp} = (\tilde{E}^{\oplus 2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}).$$

By the above condition, we have $\tilde{V}_{v_0} \cong V_{an}$ and the signature of $\tilde{V}_{v_1}$ is $(2, 0)$. Moreover, at any place $v \neq v_0, v_1$, the group $\mathrm{U}(\tilde{V}_v)$ is a quasi-split unitary group or general linear group. Finally, for any $\mu_1 \neq \mu_2 \in \Pi(E^\times, \omega_{E/F})$, we choose the following global characters:

- $\tilde{\mu}_1 \neq \tilde{\mu}_2 \in \Pi(\mathbb{A}_{\tilde{E}}^\times, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{i,v_0} = \mu_i$ $(i = 1, 2)$ and $\tilde{\mu}_{1,v_1} \neq \tilde{\mu}_{2,v_1}$.

Now we explain the endoscopic descriptions of the global L-packets of $\mathrm{U}(\tilde{V}_{sp})$ and of $\mathrm{U}(\tilde{V})$ associated to $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

First we consider $\mathrm{U}(\tilde{V}_{sp})$. For any place v, we have the local L-packet $\Pi_{\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v}}(\tilde{V}_{sp})$ for $\mathrm{U}(\tilde{V}_{sp,v})$ associated to the L-parameter $\varphi_{E_v} = \tilde{\mu}_{1,v} \oplus \tilde{\mu}_{2,v}$. We omit v from the notation. If $\tilde{E}$ is a field, the local L-packet $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$ consists of two elements and have only one $\tilde{\psi}$-generic element. Here $\tilde{\psi}$-genericity is defined by using $\tilde{\xi}$ as in §3.1. Then we denote the $\tilde{\psi}$-generic element by $\tau(\tilde{\mu}_1, \tilde{\mu}_2)_{sp}^+$. The other element of $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$ is denoted by $\tau(\tilde{\mu}_1, \tilde{\mu}_2)_{sp}^-$. If $\tilde{E}$ is not a field, the local L-packet $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$ consists of one $\tilde{\psi}$-generic element. It is denoted by $\tau(\tilde{\mu}_1, \tilde{\mu}_2)_{sp}^+$.

We define a global L-packet $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$ of $\mathrm{U}(\tilde{V}_{sp})$ by

$$\left\{ \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^{\epsilon_v} \,\middle|\, \epsilon_v = + \text{ for almost all } v \right\}.$$

Theorem 5.4 ([3]). *(i) Let* $\tilde{\tau} = \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})_{sp}^{\epsilon_v}$ *be an element of* $\Pi_{\tilde{\mu}_1, \tilde{\mu}_2}(\tilde{V}_{sp})$. *Then* $\tilde{\tau}$ *is a cuspidal representation of* $\mathrm{U}(\tilde{V}_{sp})$ *if and only if* $\Pi_v \epsilon_v = +$.
(ii) Let v *be a place such that* $\tilde{E}_v$ *is a field. For* $\delta \in \tilde{E}^\times$ *such that* $\mathrm{Tr}_{\tilde{E}/\tilde{F}}(\delta) = 0$, *we set*

$$T_{\delta,v} = \left\{ t_\delta(z, z') = z \begin{pmatrix} x & -\tilde{\xi}^{-1} y \\ -\delta^2 \tilde{\xi} y & x \end{pmatrix} \in \mathrm{U}(\tilde{V}_{sp,v}) \,\middle|\, \begin{matrix} z, z' = x + y\delta \in \tilde{E}_v^\times \\ \mathrm{N}_{\tilde{E}_v/\tilde{F}_v}(zz') = 1 \end{matrix} \right\}.$$

Then we have the following character identity:

$$\begin{aligned}&\operatorname{Tr}\tau(\tilde\mu_{1,v},\tilde\mu_{2,v})^+_{sp}(t_\delta(z,z')) - \operatorname{Tr}\tau(\tilde\mu_{1,v},\tilde\mu_{2,v})^-_{sp}(t_\delta(z,z'))\\ &= \lambda\left(\tilde E_v/\tilde F_v,\tilde\psi_v\right)\omega_{\tilde E_v/\tilde F_v}\left(\frac{z'-\sigma(z')}{\sigma(\delta)-\delta}\right)\omega\left(z\,\mathrm{N}_{\tilde E_v/\tilde F_v}(z')\right)\frac{\omega'(\sigma(z'))+\omega'(z')}{|z'-\sigma(z')|^{1/2}_{\tilde E_v}}|z'|^{1/2}_{\tilde E_v}.\end{aligned}$$

Here $\omega,\omega'\in\operatorname{Irr}\tilde E_v^\times$ *such that* $\tilde\mu_{1,v}=\omega\omega'$, $\tilde\mu_{2,v}=\omega(\omega'\circ\sigma)$.

Next we consider $\mathrm{U}(\tilde V)$. For any place v, we have the local L-packet $\Pi_{\tilde\mu_{1,v},\tilde\mu_{2,v}}(\tilde V)$ for $\mathrm{U}(\tilde V_v)$ associated to the L-parameter $\tilde\mu_{1,v}\oplus\tilde\mu_{2,v}$. These local L-packets are described as follows.

If $v\neq v_0,v_1$, then $\tilde V_v\cong\tilde V_{sp,v}$. Thus we have $\mathrm{U}(\tilde V_v)\cong\mathrm{U}(\tilde V_{sp,v})$. We omit v from the notation. If $\tilde E$ is not a field, the local L-packet $\Pi_{\tilde\mu_1,\tilde\mu_2}(\tilde V)$ consists of one $\tilde\psi$-generic element. It is denoted by $\tau(\tilde\mu_1,\tilde\mu_2)^+$. If $\tilde E$ is a field, we define representations $\tau(\tilde\mu_1,\tilde\mu_2)^\pm$ of $\mathrm{U}(\tilde V)$ by the composition of $\tau(\tilde\mu_1,\tilde\mu_2)^\pm_{sp}$ and an isomorphism $\mathrm{U}(\tilde V)\cong\mathrm{U}(\tilde V_{sp})$ induced by $\tilde V\cong\tilde V_{sp}$. Then $\tau(\tilde\mu_1,\tilde\mu_2)^+$ is $\tilde\psi$-generic and $\Pi_{\tilde\mu_1,\tilde\mu_2}(\tilde V)=\{\tau(\tilde\mu_1,\tilde\mu_2)^\pm\}$. The following also holds:

Lemma 5.5. *Define a maximal torus T of $\mathrm{U}(\tilde V)$ as in (5.1). Then we have the following character identity:*

$$\begin{aligned}&\operatorname{Tr}\tau(\tilde\mu_1,\tilde\mu_2)^+(t(z,z')) - \operatorname{Tr}\tau(\tilde\mu_1,\tilde\mu_2)^-(t(z,z'))\\ &= \lambda\left(\tilde E/\tilde F,\tilde\psi\right)\omega_{\tilde E/\tilde F}\left(\frac{z'-\sigma(z')}{\tilde\xi}\right)\omega\left(z\,\mathrm{N}_{\tilde E/\tilde F}(z')\right)\frac{\omega'(\sigma(z'))+\omega'(z')}{|z'-\sigma(z')|^{1/2}_{\tilde E}}|z'|^{1/2}_{\tilde E}\end{aligned}$$

for any $t(z,z')\in T$. *Here* $\omega,\omega'\in\operatorname{Irr}\tilde E^\times$ *such that* $\tilde\mu_1=\omega\omega'$, $\tilde\mu_2=\omega(\omega'\circ\sigma)$.

Proof. Now there exists $x\in\tilde E^\times$ such that $\tilde d=\mathrm{N}_{\tilde E/\tilde F}(x)$. We set

$$A=\begin{pmatrix}-\tilde d & x\\ 1/2 & x\tilde d^{-1}/2\end{pmatrix}.$$

Then the matrix A gives the isomorphism $\tilde V\cong\tilde V_{sp}$. Namely,

$$^*A\begin{pmatrix} & 1\\ 1 & \end{pmatrix}A=\begin{pmatrix}-\tilde d & \\ & 1\end{pmatrix}.$$

Thus the group isomorphism $\mathrm{U}(\tilde V)\cong\mathrm{U}(\tilde V_{sp})$ is given by $\mathrm{Ad}(A)$. By definition, we have $\tau(\tilde\mu_1,\tilde\mu_2)^\pm=\tau(\tilde\mu_1,\tilde\mu_2)^\pm_{sp}\circ\mathrm{Ad}(A)$. We also obtain

$$\mathrm{Ad}(A)(t(z,z'))=t_{\tilde d^{-1}\tilde\xi^{-1}/2}(z,z')$$

for $t(z,z')\in T$. Thus Lemma 5.5 follows from Theorem 5.4 (ii) and easy computation. □

If $v=v_0$, we have already explained in §5.1.

If $v=v_1$, then $\tilde E_v/\tilde F_v=\mathbb{C}/\mathbb{R}$ and the local L-packet $\Pi_{\tilde\mu_{1,v},\tilde\mu_{2,v}}$ has cardinality one. We prepare a few notation to explain more. Also we omit v_1 from the notation. Set

$$\begin{aligned}\chi&:\mathbb{C}^\times\ni re^{i\theta}\mapsto e^{i\theta}\in\mathbb{C}^\times,\\ \chi_1&:\mathbb{C}^1\ni e^{i\theta}\mapsto e^{i\theta}\in\mathbb{C}^1.\end{aligned}$$

Then we may assume $\tilde{\mu}_1 = \chi^{m+n}$ and $\tilde{\mu}_2 = \chi^{m-n}$, where $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $m - n \equiv 1 \mod 2$. Let $\rho_{m,n}$ be the irreducible $|n|$-dimensional representation of $\mathrm{U}(\tilde{V})$ with central character χ_1^m. Then we have $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2} = \{\rho_{m,n}\}$. It is clear that $\rho_{m,n} = \rho_{m,-n}$.

Lemma 5.6. *Let r be an element of $\mathbb{R}^\times$ such that $\tilde{\psi}(x) = e^{irx}$ $(x \in \mathbb{R})$. We set $C_{v_1} = \mathrm{sgn}(inr/\tilde{\xi})$. Then we have*

$$\begin{aligned} &C_{v_1} \cdot \mathrm{Tr}\, \rho_{m,n}(t(z,z')) \\ &= \lambda\left(\mathbb{C}/\mathbb{R}, \tilde{\psi}\right) \omega_{\mathbb{C}/\mathbb{R}}\left(\frac{z' - \sigma(z')}{\tilde{\xi}}\right) \omega\left(z\, \mathrm{N}_{\mathbb{C}/\mathbb{R}}(z')\right) \frac{\omega'(\sigma(z')) - \omega'(z')}{|z' - \sigma(z')|_{\mathbb{C}}^{1/2}} |z'|_{\mathbb{C}}^{1/2} \end{aligned}$$

for any $t(z,z') \in T$. Here $\omega, \omega' \in \mathrm{Irr}\, \mathbb{C}^\times$ such that $\tilde{\mu}_1 = \omega\omega', \tilde{\mu}_2 = \omega(\omega' \circ \sigma)$.

Proof. It is enough to show the equality for $z = e^{i\zeta}, z' = e^{i\theta}$, $(\zeta, \theta \in \mathbb{R})$. We may write $\omega = \chi^m, \omega' = \chi^n$. Also, we have $\lambda\left(\mathbb{C}/\mathbb{R}, \tilde{\psi}\right) = \mathrm{sgn}(r)i$. Thus the right hand side is

$$\begin{aligned} &ie^{im\zeta}\, \mathrm{sgn}(r\frac{e^{i\theta} - e^{-i\theta}}{\tilde{\xi}}) \frac{e^{-in\theta} - e^{in\theta}}{2|\sin\theta|} \\ &= \mathrm{sgn}(inr/\tilde{\xi}) \cdot ie^{im\zeta}\, \mathrm{sgn}(\frac{e^{i\theta} - e^{-i\theta}}{i}) \frac{e^{-i|n|\theta} - e^{i|n|\theta}}{2|\sin\theta|}. \end{aligned}$$

As is well known,

$$\mathrm{Tr}\, \rho_{m,n}(t(z,z')) = ie^{im\zeta}\, \mathrm{sgn}(\frac{e^{i\theta} - e^{-i\theta}}{i}) \frac{e^{-i|n|\theta} - e^{i|n|\theta}}{2|\sin\theta|}.$$

This proves the lemma. □

By this lemma, we set $\tau(\tilde{\mu}_{1,v_1}, \tilde{\mu}_{2,v_1})^{C_{v_1}} := \rho_{m,n}$.

Now we define a global L-packet $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$ of $\mathrm{U}(\tilde{V})$ by

$$\left\{ \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v} \;\middle|\; \begin{array}{l} \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v} \in \Pi_{\tilde{\mu}_{1,v},\tilde{\mu}_{2,v}}(\tilde{V}), \\ \epsilon_v = + \text{ for almost all } v \end{array} \right\}.$$

Theorem 5.7. *Let $\tilde{\tau} = \otimes_v \tau(\tilde{\mu}_{1,v}, \tilde{\mu}_{2,v})^{\epsilon_v}$ be an element of $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$. Then $\tilde{\tau}$ is a cuspidal representation of $\mathrm{U}(\tilde{V})$ if and only if $\Pi_v \epsilon_v = +$.*

Proof. This follows from [3] and the character identities in Theorem 5.2, Lemma 5.5 and Lemma 5.6. □

The following is the global result that we need. It will be used in §6.

Corollary 5.8. *Let the notation be as above. Take $\tilde{\tau} = \otimes \tilde{\tau}_v \in \Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$ such that $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for all $v \neq v_0, v_1$. Then $\tilde{\tau}$ is cuspidal if and only if $\tilde{\tau}_{v_0} = \tau(\mu_1, \mu_2)_{an}^{C_{v_1}}$.*

Remark 5.9. Note that we can choose $\tilde{\psi}, \tilde{\xi}, \tilde{\mu}_1$ and $\tilde{\mu}_2$ so that $C_{v_1} = +$.

6. Main theorems

In this section, we keep the notation and the assumption in §5.2.

To prove our main theorems, we prepare the following lemma with respect to the representation $\rho_{m,n}$ of the archimedean component $\mathrm{U}(\tilde{V}_{v_1})$:

Lemma 6.1. *For brevity, we omit v_1. Then $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}(\rho_{m,n}) \neq 0$ if and only if $C_{v_1} = +$, where $\tilde{W}_1 = (\mathbb{C}, \tilde{\xi})$.*

Proof. From [4], we recall explicit formulas of Fock model of the Weil representation $\omega_{\tilde{\psi}} = \omega_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}$ of $\mathrm{U}(\tilde{V}) \times \mathrm{U}(\tilde{W}_1)$. Since the signature of $\tilde{V}$ is $(2,0)$, we may assume that $\tilde{V} = (\mathbb{C}^{\oplus 2}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$. Then the complexification of the Lie algebra of $\mathrm{U}(\tilde{V})$ is $\mathrm{M}_2(\mathbb{C})$. We write $E_{j,k}$ for the (j,k)-elementary matrix. Also, we may assume $\tilde{\xi} = \pm i$. Since the complexification of the Lie algebra of $\mathrm{U}(\tilde{W}_1)$ is $\mathbb{C}$, we write $1_{\tilde{W}_1}$ for its neutral element. The Fock space of the Weil representation $\omega_{\tilde{\psi}}$ is $\mathbb{C}[w_1, w_2]$. Since $\tilde{\mu}_1 = \chi^{m+n}$, the explicit formulas are given as follows:

(1) $\mathrm{sgn}(ir/\tilde{\xi}) = +$

$$\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m+n-1}{2}\delta_{j,k} - w_k \frac{\partial}{\partial w_j},$$
$$\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = 1 + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}.$$

(2) $\mathrm{sgn}(ir/\tilde{\xi}) = -$

$$\omega_{\tilde{\psi}}(E_{j,k}) = \frac{m+n+1}{2}\delta_{j,k} + w_j \frac{\partial}{\partial w_k},$$
$$\omega_{\tilde{\psi}}(1_{\tilde{W}_1}) = -1 - w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}.$$

By those formulas, we can describe the local theta lift $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}$. In the case (1), the local theta lift $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}$ is given by

$$\chi_1^l \longmapsto \rho_{m+n-l,l} \quad (l \geq 1).$$

Here $\chi_1 : \mathbb{C}^1 \ni z \mapsto z \in \mathbb{C}^1$ is a character of $\mathrm{U}(\tilde{W}_1)$. Thus $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}(\rho_{m,n}) \neq 0$ is equivalent to $n \geq 1$. Its condition is equivalent to $C_{v_1} = \mathrm{sgn}(inr/\tilde{\xi}) = +$.

In the case (2), the local theta lift is given by

$$\chi_1^l \longmapsto \rho_{m+n-l,l} \quad (l \leq -1).$$

Thus $\theta_{\tilde{\psi},\tilde{V}^{\tilde{\mu}_1},\tilde{W}_1^1}(\rho_{m,n}) \neq 0$ is equivalent to $n \leq -1$. Its condition is equivalent to $C_{v_1} = +$. □

Let

$$\tilde{W} = (\tilde{E}^{\oplus 3}, \begin{pmatrix} & & 1 \\ & \tilde{\xi} & \\ -1 & & \end{pmatrix})$$

be a three-dimensional skew hermitian space over $\tilde{E}$. Also, we assume that $C_{v_1} = +$. By Lemma 6.1, the local theta lift $\theta_{\tilde{\psi}_{v_1}, \tilde{V}_{v_1}^{\tilde{\mu}_1, v_1}, \tilde{W}_{v_1}^{\mathbb{1}}}(\rho_{m,n})$ of $\rho_{m,n}$ to $\mathrm{U}(\tilde{W}_{v_1})$ is non-zero. We denote its representation of $\mathrm{U}(\tilde{W}_{v_1})$ by $\pi_{m,n}$.

Now we prove our main theorems. Namely, we compute the local theta lift $\theta_{\psi, V_{an}^{\mu}, W^{\eta}}$ of an endoscopic representation $\tau(\mu_1, \mu_2)_{an}^{\varepsilon}$ of $\mathrm{U}(V_{an})$ to $\mathrm{U}(W)$, where (μ_1, μ_2) is any pair of distinct characters $\mu_i \in \Pi(E^{\times}, \omega_{E/F})$ and μ, η are the fixed characters of $E^{\times}$ as in Notation 1.3. Also, we compute the local theta lift $\theta_{\psi, V_{an}^{\mu}, W_a^{\eta}}$ for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$.

First we consider the case of $\mu = \mu_1$ and $\varepsilon = +$.

Theorem 6.2. *We have*

$$\theta_{\psi, V_{an}^{\mu_1}, W^{\eta}}(\tau(\mu_1, \mu_2)_{an}^{+}) = \pi(\eta, \eta\mu_1\mu_2^{-1})^{-}.$$

Here $\pi(\eta, \eta\mu_1\mu_2^{-1})^{-}$ *is the unique irreducible non-generic subrepresentation of* $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes (\eta\mu_1\mu_2^{-1})_u)$*, where* B *is the Borel subgroup of all upper triangular matrices in* $\mathrm{U}(W)$*.*

Proof. Note that $\omega_{\psi, V_{an}^{\mu_1}, W^{\eta}} = \eta_u \circ \det_{\mathrm{U}(W)} \otimes \omega_{\psi, V_{an}^{\mu_1}, W^{\mathbb{1}}}$. Thus we may assume $\eta = \mathbb{1}$. Therefore it is enough to show that

$$\theta_{\psi, V_{an}^{\mu_1}, W^{\mathbb{1}}}(\pi(\mathbb{1}, \mu_1\mu_2^{-1})^{-}) = \tau(\mu_1, \mu_2)_{an}^{+}.$$

First we show that the left hand side is an element of $\Pi_{\mu_1, \mu_2}(V_{an})$. Since $\pi(\mathbb{1}, \mu_1\mu_2^{-1})^{-}$ is $\mathbb{1}$-endoscopic, by Theorem 4.3 there exists a unique $V_0 = V_{sp}, V_{an}$ such that

$$\theta_{\psi, V_0^{\mu_1}, W^{\mathbb{1}}}(\pi(\mathbb{1}, \mu_1\mu_2^{-1})^{-}) \neq 0.$$

By Theorems 4.1 and 4.4, its theta lift is a non ψ-generic representation in $\Pi_{\mu_1, \mu_2}(V_0)$. If $V_0 = V_{sp}$, then its theta lift is $\tau(\mu_1, \mu_2)_{sp}^{-}$. But by Proposition 4.11 (iv), this is a contradiction. Thus we have $V_0 = V_{an}$.

Next we use the notation and the assumption in §5.2. Choose a non-trivial character $\tilde{\eta} \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \mathbb{1})$ such that $\tilde{\eta} \neq \tilde{\mu}_1\tilde{\mu}_2^{-1}$ and $\tilde{\eta}_v = \mathbb{1}$ at $v = v_0, v_1$. Then we define a global L-parameter ϕ of $\mathrm{U}(\tilde{W})$ by $\phi_{\tilde{E}} = \mathbb{1} \oplus \tilde{\eta} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}$, where $\phi_{\tilde{E}}$ is the restriction of ϕ to the Weil group of $\tilde{E}$. Thus we have the local L-packet Π_{ϕ_v} of $\mathrm{U}(\tilde{W}_v)$ for each v. A global L-packet Π_{ϕ} is defined by

$$\{\otimes \pi_v \mid \pi_v \in \Pi_{\phi_v}, \pi_v \text{ is generic for almost all } v\}.$$

Let $\tilde{\pi}$ be the element of Π_{ϕ} such that $\tilde{\pi}_{v_0} = \pi(\mathbb{1}, \mu_1\mu_2^{-1})^{-}$, $\tilde{\pi}_{v_1} = \pi_{m,n}$ and $\tilde{\pi}_v$ is generic at $v \neq v_0, v_1$. We prove that $\tilde{\pi}$ is cuspidal. Choose a character $\tilde{\mu}_0 \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{0,v_0} = \mu_0$, where μ_0 is the fixed character as in §4. We define $\hat{\Pi}_{\phi}$ as in §4. Then we have $\hat{\Pi}_{\phi} = \{\rho_0, \rho_1, \rho_2\}$, where

$$\rho_{0,\tilde{E}} = \tilde{\mu}_0^{-1}(\tilde{\eta} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}) \times \mathbb{1},$$
$$\rho_{1,\tilde{E}} = \tilde{\mu}_0^{-1}(\mathbb{1} \oplus \tilde{\mu}_1\tilde{\mu}_2^{-1}) \times \tilde{\eta},$$
$$\rho_{2,\tilde{E}} = \tilde{\mu}_0^{-1}(\mathbb{1} \oplus \tilde{\eta}) \times \tilde{\mu}_1\tilde{\mu}_2^{-1}.$$

By [10], if we show that $\langle \rho_i, \tilde{\pi} \rangle := \Pi_v \langle \rho_{i,v}, \tilde{\pi}_v \rangle = +$ for each $i = 0, 1, 2$, then $\tilde{\pi}$ is cuspidal. Since $\tilde{\pi}_v$ is generic for $v \neq v_0, v_1$, we have $\langle \rho_{i,v}, \tilde{\pi}_v \rangle = +$ for $i = 0, 1, 2$ ([2, Proposition 3.3]). Also for $v = v_0, v_1$, we have $\langle \rho_{0,v}, \tilde{\pi}_v \rangle = \epsilon_{\rho_{0,v}}(\tilde{\pi}_v) = -$. Thus we obtain $\langle \rho_0, \tilde{\pi} \rangle = +$. Since $\rho_{0,v} = \rho_{1,v}$ for $v = v_0, v_1$, we also obtain $\langle \rho_1, \tilde{\pi} \rangle = +$. Moreover, since $\langle \rho_0, \tilde{\pi} \rangle \langle \rho_1, \tilde{\pi} \rangle \langle \rho_2, \tilde{\pi} \rangle = +$ by [2, p.454], we have $\langle \rho_2, \tilde{\pi} \rangle = +$. Thus $\tilde{\pi}$ is a cuspidal representation.

Now we show that $\theta_{\psi,V_{an}^{\mu_1},W^{\mathbb{1}}}(\pi(\mathbb{1},\mu_1\mu_2^{-1})^-) = \tau(\mu_1,\mu_2)^+_{an}$. By [2, Theorem 1.1 (b)], there exists a unique two-dimensional hermitian space V over $\tilde{E}$ such that the global theta lift $\tilde{\tau} := \theta_{\tilde{\psi},V^{\tilde{\mu}_1},\tilde{W}^{\mathbb{1}}}(\tilde{\pi}) \neq 0$. Since $\tilde{\pi}_v$ is generic at $v \neq v_0$ and v_1, V_v is split by Theorem 4.4. Namely, $V_v \cong \tilde{V}_v$. For $v = v_0$, we have already proved above that $V_{v_0} \cong V_{an} \cong \tilde{V}_{v_0}$. Moreover, it follows from the definition of $\pi_{m,n}$ that $V_{v_1} \cong \tilde{V}_{v_1}$. Thus we have $V \cong \tilde{V}$. Also, from [2] the global theta lift $\tilde{\tau}$ is a cuspidal representation of $\mathrm{U}(\tilde{V})$ and is included in $\Pi_{\tilde{\mu}_1,\tilde{\mu}_2}(\tilde{V})$. Here $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for $v \neq v_0, v_1$ and $\tilde{\tau}_{v_1} = \rho_{m,n}$. By Corollary 5.8, we obtain $\tilde{\tau}_{v_0} = \tau(\mu_1,\mu_2)^+_{an}$. Since the v_0-component of $\tilde{\tau} = \theta_{\tilde{\psi},V^{\tilde{\mu}_1},\tilde{W}^{\mathbb{1}}}(\tilde{\pi})$ is $\theta_{\psi,V_{an}^{\mu_1},W^{\mathbb{1}}}(\pi(\mathbb{1},\mu_1\mu_2^{-1})^-)$, the theorem is proved. □

To compute the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$ from Theorem 6.2, we need the following lemma:

Lemma 6.3. *Let N be the unipotent radical of the Borel subgroup B of $\mathrm{U}(W)$ and η' an element of $\Pi(E^\times, \mathbb{1}_{F^\times})$ such that $\eta' \neq \eta$.*
(i) The Jacquet module $(\omega_{\psi,V_{an}^{\mu},W^{\eta}})_N$ of the Weil representation $\omega_{\psi,V_{an}^{\mu},W^{\eta}}$ of $\mathrm{U}(V_{an})\times \mathrm{U}(W)$ is isomorphic to $\eta|\cdot|_E \boxtimes \omega_{\psi,V_{an}^{\mu},W_1^{\eta}}$ as $E^\times \times \mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$-module.
(ii) If $\tau_{\eta'} := \theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\eta'_u) \neq 0$, then $\theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau_{\eta'}) = \pi(\eta,\eta')^-$.
(iii) We have $\tau_\eta = 0$, that is, $\theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\eta_u) = 0$.

Proof. (i) By Theorem 2.2, we have the explicit formulas of the mixed model of the Weil representation $\omega_{\psi,V_{an}^{\mu},W^{\eta}}$ of $\mathrm{U}(V_{an}) \times \mathrm{U}(W)$. Then (i) follows from a similar argument to the proof of Lemma 3.3.

(ii) Since $\tau_{\eta'} \neq 0$, we have a surjective homomorphism

$$\omega_{\psi,V_{an}^{\mu},W_1^{\eta}} \to \tau_{\eta'} \boxtimes \eta'_u.$$

By (i) and the Frobenius reciprocity, we obtain a non-zero $\mathrm{U}(V_{an})\times\mathrm{U}(W)$-homomorphism

$$\omega_{\psi,V_{an}^{\mu},W^{\eta}} \to \tau_{\eta'} \boxtimes \mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u).$$

Thus $\theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau_{\eta'})$ is an irreducible subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u)$. Also, by Theorem 4.4, $\theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau_{\eta'})$ is non-generic. Since $\eta' \neq \eta$, $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta'_u)$ has a unique non-generic irreducible subrepresentation $\pi(\eta,\eta')^-$. Thus (ii) holds.

(iii) Assume $\tau_\eta \neq 0$. By a similar argument to the proof of (ii), we obtain that $\theta_{\psi,V_{an}^{\mu},W^{\eta}}(\tau_\eta)$ is an irreducible non-generic subquotient of $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta_u)$. But $\mathrm{Ind}_B^{\mathrm{U}(W)}(\eta \boxtimes \eta_u)$ is a generic irreducible representation. This is a contradiction. Thus we have $\tau_\eta = 0$. □

Now we can compute the local theta lift for $\mathrm{U}(V_{an}) \times \mathrm{U}(W_a)$.

Theorem 6.4. *For $\eta' \in \Pi(E^\times, \mathbb{1}_{F^\times})$, we have*

$$\theta_{\psi,V_{an}^{\mu},W_a^{\eta}}(\eta'_u) = \begin{cases} \tau(\mu,\mu\eta\eta'^{-1})_{an}^{\omega_{E/F}(a)} & \text{if } \eta \neq \eta', \\ 0 & \text{if } \eta = \eta'. \end{cases}$$

Proof. Choose $g \in \mathrm{GU}(V_{an})$ with the similitude norm a. Then we have

$$\theta_{\psi,V_{an}^{\mu},W_a^{\eta}}(\eta'_u) = \theta_{\psi,V_{an}^{\mu},W_1^{\eta}}(\eta'_u) \circ \mathrm{Ad}(g).$$

Thus we may assume $a = 1$ by Corollary 5.3 (iv).

By Lemma 6.3 (iii), we consider the case of $\eta \neq \eta'$. We want to show that

$$\theta_{\psi,V^{\mu}_{an},W^{\eta}_1}(\tau(\mu,\mu\eta\eta'^{-1})^+_{an}) = \eta'_u.$$

By Theorem 6.2, we obtain

$$\theta_{\psi,V^{\mu}_{an},W^{\eta}}(\tau(\mu,\mu\eta\eta'^{-1})^+_{an}) = \pi(\eta,\eta')^-.$$

Here $\tau(\mu,\mu\eta\eta'^{-1})^+_{an}$ is supercuspidal and $\pi(\eta,\eta')^-$ is not supercuspidal. Since the pair (W,W_1) forms a Witt tower, we have $\theta_{\psi,V^{\mu}_{an},W^{\eta}_1}(\tau(\mu,\mu\eta\eta'^{-1})^+_{an}) \neq 0$. Thus there exists a unique character $\eta'' \in \Pi(E^\times, \mathbb{1}_{F^\times})$ such that

$$\theta_{\psi,V^{\mu}_{an},W^{\eta}_1}(\tau(\mu,\mu\eta\eta'^{-1})^+_{an}) = \eta''_u.$$

By Lemma 6.3 (ii), we obtain $\pi(\eta,\eta'')^- = \pi(\eta,\eta')^-$. Therefore $\eta'' = \eta'$. □

Corollary 6.5. *We have*

$$\theta_{\psi,V^{\mu_1}_{an},W^{\eta}}(\tau(\mu_1,\mu_2)^-_{an}) = 0.$$

Proof. By Theorem 6.4,

$$\begin{aligned}&\theta_{\psi^{d_0},V^{\mu_1}_{an},W^{\eta}_1}(\tau(\mu_1,\mu_2)^-_{an})\\ =&\theta_{\psi,V^{\mu_1}_{an},W^{\eta}_1}(\tau(\mu_1,\mu_2)^+_{an})\\ =&(\mu_1\mu_2^{-1}\eta)_u\\ \neq&0\end{aligned}$$

Thus this corollary follows from Theorem 4.2. □

Finally, we compute the local theta lift $\theta_{\psi,V^{\mu}_{an},W^{\eta}}$ of the endoscopic representations $\tau(\mu_1,\mu_2)^{\pm}_{an}$, where μ,μ_1,μ_2 are distinct. By Theorem 6.4, we have

$$\theta_{\psi^{d_0},V^{\mu}_{an},W^{\eta}_1}(\tau(\mu_1,\mu_2)^{\pm}_{an}) = 0.$$

Thus, by Theorem 4.2, we have

$$\pi^{\pm} := \theta_{\psi,V^{\mu}_{an},W^{\eta}}(\tau(\mu_1,\mu_2)^{\pm}_{an}) \neq 0.$$

Also, the L-parameter ϕ of $\pi^{\pm}$ satisfies $\phi_E = \mu\eta\mu_1^{-1} \oplus \mu\eta\mu_2^{-1} \oplus \eta$ by Theorem 4.1. Thus we have $\hat{\Pi}_\phi = \{\rho_0,\rho_1,\rho_2\}$, where

$$\begin{aligned}\rho_{0,E} &= \mu_0^{-1}\mu\eta(\mu_1^{-1} \oplus \mu_2^{-1}) \times \eta,\\ \rho_{1,E} &= \mu_0^{-1}(\eta \oplus \mu\eta\mu_2^{-1}) \times \mu\eta\mu_1^{-1},\\ \rho_{2,E} &= \mu_0^{-1}(\mu\eta\mu_1^{-1} \oplus \eta) \times \mu\eta\mu_2^{-1}.\end{aligned}$$

Theorem 6.6. *We have for $\varepsilon \in \{\pm\}$,*

$$\begin{aligned}\epsilon_{\rho_0}(\pi^\varepsilon) &= -,\\ \epsilon_{\rho_1}(\pi^\varepsilon) &= -\varepsilon,\\ \epsilon_{\rho_2}(\pi^\varepsilon) &= \varepsilon.\end{aligned}$$

Proof. By Definition 4.8, we have $\epsilon_{\rho_0}(\pi^\varepsilon) = -$. As seen from the proof of Proposition 4.11 (v), we also have

$$\epsilon_{\rho_0}(\pi^\varepsilon)\epsilon_{\rho_1}(\pi^\varepsilon)\epsilon_{\rho_2}(\pi^\varepsilon) = +.$$

Thus we obtain $\epsilon_{\rho_1}(\pi^\varepsilon)\epsilon_{\rho_2}(\pi^\varepsilon) = -$. Since these ϵ_{ρ_i} distinguish $\pi^{\pm}$, it is enough to show the theorem for $\varepsilon = +$.

Let $\tilde{\tau}$ be the cuspidal representation of $\mathrm{U}(\tilde{V})$ in Corollary 5.8. Fix the following:

- $\tilde{\eta} \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \mathbb{1})$ such that $\tilde{\eta}_{v_0} = \eta, \tilde{\eta}_{v_1} = \mathbb{1}$;
- $\tilde{\mu} \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{v_0} = \mu, \tilde{\mu}_{v_1} = \tilde{\mu}_{1,v_1}$.

Since we assume $C_{v_1} = +$, the local theta lift $\theta_{\tilde{\psi}_v, \tilde{V}_v^{\tilde{\mu}_v}, \tilde{W}_v^{\tilde{\eta}_v}}(\tilde{\tau}_v) \neq 0$ for each v. Thus by [2, Theorem 5.1], the global theta lift $\tilde{\pi} := \theta_{\tilde{\psi}, \tilde{V}^{\tilde{\mu}}, \tilde{W}^{\tilde{\eta}}}(\tilde{\tau})$ is a non-zero cuspidal representation of $\mathrm{U}(\tilde{W})$. Note that $\tilde{\pi}_{v_0} = \pi^+$ and $\tilde{\pi}_{v_1} = \pi_{m,n}$.

We choose a character $\tilde{\mu}_0 \in \Pi(\mathbb{A}_{\tilde{E}}^{\times}, \omega_{\tilde{E}/\tilde{F}})$ such that $\tilde{\mu}_{0,v_0} = \mu_0$, where μ_0 is the fixed character as in §4. Then we define $\hat{\Pi}_\phi = \{\tilde{\rho}_0, \tilde{\rho}_1, \tilde{\rho}_2\}$ as follows:

$$\begin{aligned}
\rho_{0,\tilde{E}} &= \tilde{\mu}_0^{-1}\tilde{\mu}\tilde{\eta}(\tilde{\mu}_1^{-1} \oplus \tilde{\mu}_2^{-1}) \times \tilde{\eta}, \\
\rho_{1,\tilde{E}} &= \tilde{\mu}_0^{-1}(\tilde{\eta} \oplus \tilde{\mu}\tilde{\eta}\tilde{\mu}_2^{-1}) \times \tilde{\mu}\tilde{\eta}\tilde{\mu}_1^{-1}, \\
\rho_{2,\tilde{E}} &= \tilde{\mu}_0^{-1}(\tilde{\mu}\tilde{\eta}\tilde{\mu}_1^{-1} \oplus \tilde{\eta}) \times \tilde{\mu}\tilde{\eta}\tilde{\mu}_2^{-1}.
\end{aligned}$$

Since $\tilde{\pi}$ is cuspidal, we have $\langle \rho_i, \tilde{\pi} \rangle = +$ for any $i = 0, 1, 2$.

Now $\tilde{\tau}_v$ is $\tilde{\psi}_v$-generic for $v \neq v_0, v_1$. Thus $\tilde{\pi}_v$ is generic by Theorem 4.4. By Proposition 4.10, we obtain $\langle \rho_{i,v}, \tilde{\pi}_v \rangle = \epsilon_{\rho_{i,v}}(\tilde{\pi}_v) = +$ for any i.

Since $\tilde{\mu}_{v_1} = \tilde{\mu}_{1,v_1}$, we have $\rho_{0,v_1} = \rho_{1,v_1}$. Thus we obtain $\langle \rho_{1,v_1}, \tilde{\pi}_{v_1} \rangle = \langle \rho_{0,v_1}, \tilde{\pi}_{v_1} \rangle = \epsilon_{\rho_{0,v_1}}(\pi_{m,n}) = -$.

Hence we have $\epsilon_{\rho_1}(\pi^+) = \langle \rho_{1,v_1}, \tilde{\pi}_{v_1} \rangle = -$ and $\epsilon_{\rho_2}(\pi^+) = +$. □

By Theorem 4.9, we have the following corollary:

Corollary 6.7. *We have for* $\varepsilon \in \{\pm\}$

$$\begin{aligned}
\langle \rho_0, \pi^\varepsilon \rangle &= -, \\
\langle \rho_1, \pi^\varepsilon \rangle &= -\varepsilon, \\
\langle \rho_2, \pi^\varepsilon \rangle &= \varepsilon.
\end{aligned}$$

7. Local theta lift for quaternionic unitary groups of rank one

In this section, we describe the local theta lift for a pair of quaternionic unitary groups of rank one by using Theorem 6.4.

7.1. **Quaternionic unitary groups.**

In this subsection, we introduce quaternionic unitary groups of rank one.

Let D be the quaternion division algebra over F with the main involution ι. We denote its reduced norm and trace by ν_D, tr_D, respectively.

Let $\mathcal{V}$ be an m-dimensional hermitian space and $\mathcal{W}$ an n-dimensional skew-hermitian space over D. The spaces $\mathcal{V}$ and $\mathcal{W}$ may be taken as follows:

$$\mathcal{V} = (D^{\oplus m}, A), \qquad \mathcal{W} = (D^{\oplus n}, B),$$
$$(v_1, v_2) = {}^*v_1 A v_2, \qquad \langle w_1, w_2 \rangle = w_1 B^* w_2.$$

Here $A = {}^*A := {}^tA^\iota \in \mathrm{GL}_m(D)$ and $B = -{}^*B \in \mathrm{GL}_n(D)$. Note that $\mathcal{V}$ is a right D-space of column vectors, whereas $\mathcal{W}$ is a left D-space of row vectors. Then the quaternionic unitary groups of $\mathcal{V}$ and $\mathcal{W}$ are given by

$$\begin{aligned}
\mathrm{U}(\mathcal{V}) &= \{h \in \mathrm{GL}_m(D) \mid {}^*hAh = A\}, \\
\mathrm{U}(\mathcal{W}) &= \{g \in \mathrm{GL}_n(D) \mid gB^*g = B\}.
\end{aligned}$$

Here $\mathrm{U}(\mathcal{V})$ (resp. $\mathrm{U}(\mathcal{W})$) acts on $\mathcal{V}$ (resp. $\mathcal{W}$) on the left (resp. right).

For one-dimensional (skew) hermitian spaces over D, we have the following results:

Lemma 7.1. *(i) Any one-dimensional hermitian space $\mathcal{V}$ is isomorphic to $(D, 1)$. Moreover, we have $\mathrm{U}(\mathcal{V}) = \ker \nu_D$.*
(ii) For any one-dimensional skew-hermitian space $\mathcal{W}$, there exist a quadratic extension L/F in D and $\delta \in L^\times$ such that $\mathrm{Tr}_{L/F}(\delta) = 0$ and $\mathcal{W} \cong (D, \delta)$. And then we have $\mathrm{U}(\mathcal{W}) = \ker \mathrm{N}_{L/F}$.

Proof. (i) By [12, X §2 Proposition 6], we have $\nu_D(D^\times) = F^\times$. The first assertion is shown from it. The second assertion is trivial.

(ii) The first assertion follows from [11, Theorem 10.3.6]. We may assume that $\mathcal{W} = (D, \delta)$. Then $\mathrm{U}(\mathcal{W}) = \{g \in D^\times \mid g\delta\iota(g) = \delta\}$. For any $g \in \mathrm{U}(\mathcal{W})$, we have $\nu_D(g)^2 = 1$, that is, $\nu_D(g) = \pm 1$.

First assume that $\nu_D(g) = 1$, that is, $\iota(g) = g^{-1}$. Then we have $g\delta = \delta g$. Since $L = F(\delta)$ is a maximal subfield in D, we obtain $g \in L^\times$. Therefore $g \in L^\times \cap \mathrm{U}(\mathcal{W}) = \ker \mathrm{N}_{L/F}$.

Next assume that $\nu_D(g) = -1$, that is, $\iota(g) = -g^{-1}$. Since $g\delta = -\delta g$, we have $g \notin L$ and $g^2 \in L$. If $g^2 \in L \setminus F$, then $L = F(g^2)$. Thus g is commutative with δ. Since this is a contradiction, we get $g^2 \in F^\times$. By [11, Lemma 2.11.2], we have

$$\mathrm{Ker}\,\mathrm{tr}_D = \{x \in D \mid x \notin F, x^2 \in F\}.$$

Thus we obtain $\mathrm{tr}_D(g) = 0$. Hence $g + \iota(g) = 0$ and $g\iota(g) = -1$. This implies that $g = \pm 1$, which is a contradiction to $\nu_D(g) = -1$. □

In this section, we consider a hermitian space $\mathcal{V}_1$ and a skew-hermitian space $\mathcal{W}_1$ of dimension one. By Lemma 7.1, we may assume that

$$\mathcal{V}_1 = (D, 1).$$

Similarly, we may assume that $\mathcal{W}_1 = (D, \delta)$, where δ is an element of $L^\times$ with trace zero. But, since the field E in Notation 1.3 is any quadratic extension of F, we can write $L = E$ and $\delta = \xi$. Thus we can take

$$\mathcal{W}_1 = (D, \xi).$$

Since we consider that E is a subfield of D, we get $\mathrm{U}(\mathcal{W}_1) \subset \mathrm{U}(\mathcal{V}_1)$ by Lemma 7.1.

The following lemma is used to construct an embedding map from $\mathrm{U}(\mathcal{V}_1)$ to the unitary group $\mathrm{U}(V_{an})$, where V_{an} is the fixed hermitian space as in §2.2.

Lemma 7.2. *There exists $\xi' \in D$ such that $\xi\xi' = -\xi'\xi$, $\xi'^{,2} = d_0$ and*

$$D = F \oplus F\xi \oplus F\xi' \oplus F\xi'\xi = E \oplus \xi' E.$$

Proof. Define a quaternion F-algebra D' with basis $1, e_1, e_2, e_3$ by the following multiplication table:

$$e_1 e_2 = e_3,\ e_2 e_1 = -e_1 e_2,\ e_1^2 = \xi^2,\ e_2^2 = d_0.$$

Let $\nu_{D'}$ be the reduced norm of D'. For $x, y, z, w \in F$, we have

$$\begin{aligned}\nu_{D'}(x + ye_1 + ze_2 + we_3) &= x^2 - y^2\xi^2 - z^2 d_0 + w^2\xi^2 d_0 \\ &= (z + w\xi \ \ x + y\xi) \begin{pmatrix} -d_0 & \\ & 1 \end{pmatrix} \begin{pmatrix} z - w\xi \\ x - y\xi \end{pmatrix}.\end{aligned}$$

This implies that $\nu_{D'}$ is anisotropic, since V_{an} is anisotropic (see §2.2). Thus, by [11, Theorem 2.11.8], D' is a quaternion division algebra, which is isomorphic to

D. By Skolem-Noether theorem, there is an isomorphism $D' \cong D$ such that the image of e_1 is ξ. Then let ξ' be the image of e_2. This proves the lemma. □

By this lemma, we consider $D = E \oplus \xi' E$ as a two-dimensional right E-space. Then the hermitian form on $\mathcal{V}_1 = E \oplus \xi' E$ over D is considered a two-dimensional hermitian form over E. The following result is easily checked.

Proposition 7.3. *(i) The E-linear map $i : \mathcal{V}_1 \ni z + \xi' w \mapsto {}^t(w, z) \in V_{an}$ is an isomorphism as hermitian spaces over E.*
(ii) Let $I : \mathrm{U}(\mathcal{V}_1) \to \mathrm{U}(V_{an})$ be the homomorphism induced by i. Then

$$I : \mathrm{U}(\mathcal{V}_1) \ni \alpha + \xi' \beta \mapsto \begin{pmatrix} \sigma(\alpha) & \beta \\ d_0 \sigma(\beta) & \alpha \end{pmatrix} \in \mathrm{U}(V_{an})$$

is an injective homomorphism and its image is $\mathrm{SU}(V_{an})$.

Note that $\mathrm{U}(\mathcal{W}_1)$ as a subgroup of $\mathrm{U}(\mathcal{V}_1)$, so is embedded in $\mathrm{U}(V_{an})$ by the homomorphism

$$I|_{\mathrm{U}(\mathcal{W}_1)} : \mathrm{U}(\mathcal{W}_1) \ni \gamma \mapsto \begin{pmatrix} \gamma^{-1} & \\ & \gamma \end{pmatrix} \in \mathrm{U}(V_{an}).$$

Also it is trivial that $\mathrm{U}(\mathcal{W}_1) = \mathrm{U}(W_1)$, where W_1 is the fixed skew-hermitian space as in §2.2.

7.2. **The Weil representation of** $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$**.** In this subsection, we introduce the Weil representation of $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$. Also we prove Proposition 7.6 which plays an important role in §7.4.

Recall the construction of the Weil representation of $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$. Let

$$\mathbb{W}_D = \mathcal{V}_1 \otimes_D \mathcal{W}_1 \tag{7.1}$$

be the F-vector space equipped with the symplectic form

$$\langle\!\langle \ , \ \rangle\!\rangle = \mathrm{tr}_D \left((\ , \ ) \otimes \langle \ , \ \rangle^{\iota} \right).$$

Fix an irreducible unitary representation $(\tau_{\psi,D}, \mathcal{S}_D)$ of the Heisenberg group $H(\mathbb{W}_D)$ with central character ψ. As in §2.1, define the metaplectic group $\mathrm{Mp}_\psi(\mathbb{W}_D)$ and its Weil representation $(\omega_{\psi,D}, \mathcal{S}_D)$. By the doubling method, Kudla [5] defined the splitting

$$\iota_{\mathcal{W}_1} \times \iota_{\mathcal{V}_1} : \mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1) \to \mathrm{Mp}_\psi(\mathbb{W}_D).$$

Namely, we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1) & \xrightarrow{\iota_{\mathcal{W}_1} \times \iota_{\mathcal{V}_1}} & \mathrm{Mp}_\psi(\mathbb{W}_D) \\ & \searrow \quad \swarrow & \\ & \mathrm{Sp}(\mathbb{W}_D). & \end{array}$$

Then the Weil representation $(\omega_{\psi,\mathcal{V}_1,\mathcal{W}_1}, \mathcal{S}_D)$ of $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$ is defined by

$$\omega_{\psi,\mathcal{V}_1,\mathcal{W}_1} := \omega_{\psi,D} \circ \iota_{\mathcal{W}_1} \times \iota_{\mathcal{V}_1}.$$

Remark 7.4. The pair $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$ does not form a reductive dual pair in the symplectic group $\mathrm{Sp}(\mathbb{W}_D)$ (See [9, p.15]). However note that $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$ has the Weil representation as we explained above.

Define the symplectic space $\mathbb{W}_E = V_{an} \otimes_E W_1$ over F as in §2.1. The following lemma is easily checked.

Lemma 7.5. *(i) The map*

$$P : \mathbb{W}_E = E^{\oplus 2} \ni {}^t(x_2 + \xi x_3, x_0 + \xi x_1) \mapsto x_0 + x_1\xi + x_2\xi' + x_3\xi'\xi \in D = \mathbb{W}_D$$

is an isomorphism as symplectic spaces over F.
(ii) We have the commutative diagram:

$$\begin{array}{ccc} \mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1) & \longrightarrow & \mathrm{Sp}(\mathbb{W}_D) \\ {\scriptstyle I\times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ \mathrm{U}(V_{an}) \times \mathrm{U}(W_1) & \longrightarrow & \mathrm{Sp}(\mathbb{W}_E). \end{array}$$

Here the right vertical map is the isomorphism induced by P.

We can consider $(\tau_{\psi,D}, \mathcal{S}_D)$ as an irreducible unitary representation of the Heisenberg group $H(\mathbb{W}_E)$, since $H(\mathbb{W}_D) \cong H(\mathbb{W}_E)$. Then we have the metaplectic group $\mathrm{Mp}_\psi(\mathbb{W}_E)$ and its Weil representation $(\omega_{\psi,V_{an},W_1}, \mathcal{S}_D)$. Also, for a fixed character $\mu \in \Pi(E^\times, \omega_{E/F})$, the Weil representation $(\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}}, \mathcal{S}_{\mathcal{D}})$ of $\mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$ is defined by

$$\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}} := \omega_{\psi,V_{an},W_1} \circ \iota_{W_1}^\mu \times \iota_{V_{an}}^{\mathbb{1}},$$

where $\iota_{W_1}^\mu \times \iota_{V_{an}}^{\mathbb{1}}$ is the splitting of $\mathrm{U}(V_{an}) \times \mathrm{U}(W_1)$ (see §2.1).
In the remaining subsection, we want to prove the following:

Proposition 7.6. *We have the equality as $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_1)$-modules on $\mathcal{S}_D$:*

$$\omega_{\psi,\mathcal{V}_1,\mathcal{W}_1} = \omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}} \circ (I \times \mathrm{id}).$$

Proof. We have the isomorphism of metaplectic groups induced by P:

$$\Phi : \mathrm{Mp}_\psi(\mathbb{W}_E) \ni (g, M_g) \to (P^{-1}gP, M_g) \in \mathrm{Mp}_\psi(\mathbb{W}_D).$$

If we show that the following diagrams are commutative:

$$\begin{array}{ccc} \mathrm{U}(\mathcal{V}_1) & \xrightarrow{\iota_{\mathcal{W}_1}} & \mathrm{Mp}_\psi(\mathbb{W}_D) \\ {\scriptstyle I}\big\downarrow & & \big\uparrow{\scriptstyle \Phi} \\ \mathrm{U}(V_{an}) & \xrightarrow{\iota_{W_1}^\mu} & \mathrm{Mp}_\psi(\mathbb{W}_E), \end{array} \tag{7.2}$$

$$\begin{array}{ccc} \mathrm{U}(\mathcal{W}_1) & \xrightarrow{\iota_{\mathcal{V}_1}} & \mathrm{Mp}_\psi(\mathbb{W}_D) \\ \big\| & & \big\uparrow{\scriptstyle \Phi} \\ \mathrm{U}(W_1) & \xrightarrow{\iota_{V_{an}}^{\mathbb{1}}} & \mathrm{Mp}_\psi(\mathbb{W}_E), \end{array} \tag{7.3}$$

then we obtain the equality of Proposition 7.6.

First we consider the diagram (7.2). The homomorphism $\Phi \circ \iota_{W_1}^\mu \circ I$ defines a splitting of $\mathrm{U}(\mathcal{V}_1)$. Then the quotient $(\Phi \circ \iota_{W_1}^\mu \circ I)\iota_{\mathcal{W}_1}^{-1}$ becomes a character of $\mathrm{U}(\mathcal{V}_1)$. Since $\mathrm{U}(\mathcal{V}_1)$ dose not have characters but only trivial one, we obtain $\Phi \circ \iota_{W_1}^\mu \circ I = \iota_{\mathcal{W}_1}$. Namely, the diagram (7.2) is commutative.

Next we consider the diagram (7.3). In order to describe the doubling method in detail, we explain one lemma needed later.

Take two skew-hermitian spaces over D:

$$\mathcal{W}_{sp} = (D^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}),$$
$$\mathcal{W}_{-1} = (D, -\xi).$$

Then the D-linear map

$$\mathcal{W}_1 \oplus \mathcal{W}_{-1} \ni (a,b) \mapsto (-a+b, \frac{(a+b)\xi}{2}) \in \mathcal{W}_{sp} \tag{7.4}$$

is an isomorphism as skew-hermitian spaces over D. This induces the injective homomorphism

$$\mathrm{U}(\mathcal{W}_1) \times \mathrm{U}(\mathcal{W}_{-1}) \hookrightarrow \mathrm{U}(\mathcal{W}_{sp}). \tag{7.5}$$

Similarly, take the skew-hermitian space W_{sp} over E:

$$W_{sp} = (E^{\oplus 2}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}).$$

Since the E-linear map

$$W_1 \oplus W_{-1} \ni (a,b) \mapsto (-a+b, \frac{(a+b)\xi}{2}) \in W_{sp} \tag{7.6}$$

is an isomorphism, we have the injective homomorphism

$$\mathrm{U}(W_1) \times \mathrm{U}(W_{-1}) \hookrightarrow \mathrm{U}(W_{sp}). \tag{7.7}$$

Let $\mathbb{W}_{-D} := \mathcal{V}_1 \otimes_D \mathcal{W}_{-1}$ and $\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}$ be the symplectic spaces defined as in (7.1). Also, let $\mathbb{W}_{-E} := V_{an} \otimes_E W_{-1}$ and $V_{an} \otimes_E W_{sp}$ be the symplectic spaces defined as in §2.1. By (7.4) and (7.6), we obtain the isomorphisms

$$\mathbb{W}_D \oplus \mathbb{W}_{-D} \cong \mathcal{V}_1 \otimes_D \mathcal{W}_{sp}, \quad \mathbb{W}_E \oplus \mathbb{W}_{-E} \cong V_{an} \otimes_E W_{sp}.$$

Lemma 7.7. *(i) The F-linear map*

$$V_{an} \otimes_E W_{sp} \ni v \otimes w \mapsto i^{-1}(v) \otimes w \in \mathcal{V}_1 \otimes_D \mathcal{W}_{sp} \tag{7.8}$$

is an isomorphism as symplectic spaces. Then we have the commutative diagram:

$$\begin{array}{ccc} \mathbb{W}_E \oplus \mathbb{W}_{-E} & \xrightarrow{\cong} & V_{an} \otimes_E W_{sp} \\ {\scriptstyle P \oplus P^-}\downarrow & & \downarrow{\scriptstyle (7.8)} \\ \mathbb{W}_D \oplus \mathbb{W}_{-D} & \xrightarrow{\cong} & \mathcal{V}_1 \otimes_D \mathcal{W}_{sp}, \end{array}$$

where $P^- : \mathbb{W}_{-E} \to \mathbb{W}_{-D}$ is the isomorphism having the same form as P in Lemma 7.5 (i).
(ii) Moreover, the following commutative diagram is induced from (i):

$$\begin{array}{ccc} \mathrm{Mp}_\psi(\mathbb{W}_E) \times \mathrm{Mp}_\psi(\mathbb{W}_{-E}) & \longrightarrow & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\ {\scriptstyle \Phi \times \Phi^-}\downarrow & & \downarrow \\ \mathrm{Mp}_\psi(\mathbb{W}_D) \times \mathrm{Mp}_\psi(\mathbb{W}_{-D}) & \longrightarrow & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}). \end{array} \tag{7.9}$$

Here $\Phi^- : \mathrm{Mp}_\psi(\mathbb{W}_{-E}) \xrightarrow{\sim} \mathrm{Mp}_\psi(\mathbb{W}_{-D})$ is the isomorphism induced by P^-.

Now we describe the doubling method. Recall from [5] that the splitting

$$\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_{sp}) \to \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}) \tag{7.10}$$

is defined so that the Weil representation $\omega_{\psi,\mathcal{V}_1,\mathcal{W}_{sp}}$ of $\mathrm{U}(\mathcal{V}_1) \times \mathrm{U}(\mathcal{W}_{sp})$ has the following explicit formulas on the space $\mathcal{S}(\mathcal{V}_1)$ of Schwartz-Bruhat functions on $\mathcal{V}_1$:

$$\begin{aligned}
&\omega_{\psi,\mathcal{V}_1,\mathcal{W}_{sp}}(h)\, f(x) = f(h^{-1}x),\\
&\omega_{\psi,\mathcal{V}_1,\mathcal{W}_{sp}}\begin{pmatrix} a & \\ & \iota(a)^{-1}\end{pmatrix} f(x) = |\nu_D(a)|_F f(xa),\\
&\omega_{\psi,\mathcal{V}_1,\mathcal{W}_{sp}}\begin{pmatrix} 1 & b \\ & 1\end{pmatrix} f(x) = \psi(b\nu_D(x)) f(x).
\end{aligned}$$

Here $f \in \mathcal{S}(\mathcal{V}_1)$, $x \in \mathcal{V}_1$, $h \in \mathrm{U}(\mathcal{V}_1)$, $a \in D^\times$ and $b \in F$. Then the splitting $\iota_{\mathcal{V}_1} : \mathrm{U}(\mathcal{W}_1) \to \mathrm{Mp}_\psi(\mathbb{W}_D)$ is defined so that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{U}(\mathcal{W}_1) & \xrightarrow{(7.5)} & \mathrm{U}(\mathcal{W}_{sp}) \\
{\scriptstyle \iota_{\mathcal{V}_1}}\downarrow & & \downarrow{\scriptstyle (7.10)} \\
\mathrm{Mp}_\psi(\mathbb{W}_D) & \longrightarrow & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}),
\end{array} \tag{7.11}$$

where the bottom map is obtained by restricting the bottom map of (7.9).

Similarly, the splitting

$$\mathrm{U}(V_{an}) \times \mathrm{U}(W_{sp}) \to \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \tag{7.12}$$

is defined so that the Weil representation $\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}$ of $\mathrm{U}(V_{an}) \times \mathrm{U}(W_{sp})$ on $\mathcal{S}(V_{an})$ has the following explicit formulas (see §2.1):

$$\begin{aligned}
&\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}(h)\, f(y) = f(h^{-1}y),\\
&\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}\begin{pmatrix} a & \\ & \sigma(a)^{-1}\end{pmatrix} f(y) = |a|_E f(ya),\\
&\omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}\begin{pmatrix} 1 & b \\ & 1\end{pmatrix} f(y) = \psi\left(b\left(-d_0\,\mathrm{N}_{E/F}(y_1) + \mathrm{N}_{E/F}(y_2)\right)\right) f(y).
\end{aligned}$$

Here $f \in \mathcal{S}(V_{an})$, $y = {}^t(y_1, y_2) \in V_{an}$, $h \in \mathrm{U}(V_{an})$, $a \in E^\times$ and $b \in F$. Then the splitting $\iota_{V_{an}}^{\mathbb{1}} : \mathrm{U}(W_{sp}) \to \mathrm{Mp}_\psi(\mathbb{W}_E)$ is defined by the doubling method as above. Namely, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{U}(W_1) & \xrightarrow{(7.7)} & \mathrm{U}(W_{sp}) \\
{\scriptstyle \iota_{V_{an}}^{\mathbb{1}}}\downarrow & & \downarrow{\scriptstyle (7.12)} \\
\mathrm{Mp}_\psi(\mathbb{W}_E) & \longrightarrow & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}),
\end{array} \tag{7.13}$$

where the bottom map is obtained by restricting the top map of (7.9).

Now we can prove the commutativity of the diagram (7.3). It is easy to check that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{U}(W_1) & \xrightarrow{(7.7)} & \mathrm{U}(W_{sp}) \\
\Big\| & & \downarrow{\scriptstyle J} \\
\mathrm{U}(\mathcal{W}_1) & \xrightarrow{(7.5)} & \mathrm{U}(\mathcal{W}_{sp}),
\end{array}$$

where $J : \mathrm{U}(W_{sp}) \ni g \mapsto g \in \mathrm{U}(\mathcal{W}_{sp})$ is the inclusion map induced by $\mathrm{GL}_2(E) \hookrightarrow \mathrm{GL}_2(D)$.

From the isomorphism $i : \mathcal{V}_1 \cong V_{an}$ in Proposition 7.3, we have the isomorphism $\mathcal{S}(\mathcal{V}_1) \cong \mathcal{S}(V_{an})$. Then by comparing the two explicit formulas above, we can check that

$$\omega_{\psi,\mathcal{V}_1,\mathcal{W}_{sp}} \circ J \cong \omega_{\psi,V_{an}^{\mathbb{1}},W_{sp}^{\mathbb{1}}}$$

as $\mathrm{U}(W_{sp})$-modules. This implies that the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{U}(W_{sp}) & \xrightarrow{(7.12)} & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\
{\scriptstyle J}\downarrow & & \downarrow{\scriptstyle A} \\
\mathrm{U}(\mathcal{W}_{sp}) & \xrightarrow{(7.10)} & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}).
\end{array}$$

Here A is the right vertical map of (7.9). Thus we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{U}(W_1) & \longrightarrow & \mathrm{U}(W_{sp}) & \xrightarrow{(7.12)} & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\
\| & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle A} \\
\mathrm{U}(\mathcal{W}_1) & \longrightarrow & \mathrm{U}(\mathcal{W}_{sp}) & \xrightarrow{(7.10)} & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}).
\end{array}$$

Combining (7.11) and (7.13), we have the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{U}(W_1) & \xrightarrow{\iota^{\mathbb{1}}_{V_{an}}} & \mathrm{Mp}_\psi(\mathbb{W}_E) & \longrightarrow & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\
\| & & & & \downarrow{\scriptstyle A} \\
\mathrm{U}(\mathcal{W}_1) & \xrightarrow{\iota_{\mathcal{V}_1}} & \mathrm{Mp}_\psi(\mathbb{W}_D) & \longrightarrow & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}).
\end{array}$$

Note that we obtain the following commutative diagram from the diagram (7.9):

$$\begin{array}{ccc}
\mathrm{Mp}_\psi(\mathbb{W}_E) & \longrightarrow & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle A} \\
\mathrm{Mp}_\psi(\mathbb{W}_D) & \longrightarrow & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}).
\end{array}$$

Since the bottom homomorphism $\mathrm{Mp}_\psi(\mathbb{W}_D) \to \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp})$ is injective, we can show the commutativity of the following diagram:

$$\begin{array}{ccccc}
\mathrm{U}(W_1) & \xrightarrow{\iota^{\mathbb{1}}_{V_{an}}} & \mathrm{Mp}_\psi(\mathbb{W}_E) & \longrightarrow & \mathrm{Mp}_\psi(V_{an} \otimes_E W_{sp}) \\
\| & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle A} \\
\mathrm{U}(\mathcal{W}_1) & \xrightarrow{\iota_{\mathcal{V}_1}} & \mathrm{Mp}_\psi(\mathbb{W}_D) & \longrightarrow & \mathrm{Mp}_\psi(\mathcal{V}_1 \otimes_D \mathcal{W}_{sp}).
\end{array}$$

Thus we obtain the commutativity of the diagram (7.3). $\square$

7.3. **Endoscopy for** $\mathrm{U}(\mathcal{V}_1)$**.** Let χ be a quasi-character of $E^\times$ such that $\chi \neq \chi \circ \sigma$. Then there exists an irreducible supercuspidal representation $\tau(\chi)$ of $\mathrm{GL}_2(F)$ with L-parameter $\mathrm{Ind}_{W_E}^{W_F} \chi$. We denote its Jacquet-Langlands correspondent to $D^\times$ by $\tau_D(\chi)$.

Theorem 7.8 ([6]). *There exist two irreducible representations $\tau_D(\chi)^+, \tau_D(\chi)^-$ of $\mathrm{U}(\mathcal{V}_1)$ such that $\tau_D(\chi)|_{\mathrm{U}(\mathcal{V}_1)} = \tau_D(\chi)^+ \oplus \tau_D(\chi)^-$. And they satisfy the following character identity:*

$$\begin{aligned}&\mathrm{Tr}\,\tau_D(\chi)^+(\gamma) - \mathrm{Tr}\,\tau_D(\chi)^-(\gamma)\\ &= \lambda(E/F,\psi)\,\omega_{E/F}\left(\frac{\gamma^{-1}-\gamma}{\xi}\right)\frac{\chi(\gamma)-\chi(\gamma^{-1})}{|\gamma-\gamma^{-1}|_E^{1/2}}\end{aligned}$$

for any $\gamma \in \mathrm{U}(\mathcal{W}_1)$. Moreover, $\tau_D(\chi)^\pm$ are determined by the restriction $\chi|_{\mathrm{U}(\mathcal{W}_1)}$. Also, $\tau_D(\chi)^+ \cong \tau_D(\chi)^-$ if and only if $\chi^2|_{\mathrm{U}(\mathcal{W}_1)} = \mathbb{1}$.

7.4. **Main theorem.** Now we can describe the local theta lift $\theta_{\psi,\mathcal{V}_1,\mathcal{W}_1}$ for $\mathrm{U}(\mathcal{V}_1)\times \mathrm{U}(\mathcal{W}_1)$.

Theorem 7.9. *For $\eta \in \Pi(E^\times, \mathbb{1}_{F^\times})$, let χ_η be a character of $E^\times$ such that $\chi_\eta|_{\mathrm{U}(\mathcal{W}_1)} = \eta_u$. Then we have*

$$\theta_{\psi,\mathcal{V}_1,\mathcal{W}_1}(\eta_u) = \begin{cases}\tau_D(\chi_\eta)^+ & \text{if } \eta \neq \mathbb{1},\\ 0 & \text{if } \eta = \mathbb{1}.\end{cases}$$

Proof. Since $\mathrm{U}(V_{an})\times \mathrm{U}(W_1)$ is a compact group, it is proven that the Weil representation $\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}}$ of $\mathrm{U}(V_{an})\times\mathrm{U}(W_1)$ is a unitary admissible representation. Thus it is a direct sum of irreducible unitary admissible representations. Therefore, Theorem 6.4 implies that

$$\omega_{\psi,V_{an}^\mu,W_1^{\mathbb{1}}} = \bigoplus_{\eta\neq\mathbb{1}} \tau(\mu,\mu\eta^{-1})_{an}^+ \boxtimes \eta_u.$$

Here η runs the non-trivial elements of $\Pi(E^\times,\mathbb{1}_{F^\times})$. By Proposition 7.6, we obtain

$$\omega_{\psi,\mathcal{V}_1,\mathcal{W}_1} = \bigoplus_{\eta\neq\mathbb{1}} \tau(\mu,\mu\eta^{-1})_{an}^+\circ I \boxtimes \eta_u.$$

Thus we must show that $\tau(\mu,\mu\eta^{-1})_{an}^+\circ I = \tau_D(\chi_\eta)^+$. This is shown in the next lemma. □

Lemma 7.10. *We have $\tau(\mu,\mu\eta^{-1})_{an}^+\circ I = \tau_D(\chi_\eta)^+$.*

Proof. First we recall the following isomorphism:

$$E^\times \times D^\times/\Delta F^\times \ni (x, z+\xi' w) \mapsto x\nu_D(z+\xi' w)^{-1}\begin{pmatrix}\sigma(z) & w\\ d_0\sigma(w) & z\end{pmatrix} \in \mathrm{GU}(V_{an}),$$

where $\Delta F^\times = \{(x,x)\in E^\times\times D^\times \mid x\in F^\times\}$. Then we have the commutative diagram:

$$\begin{array}{ccc}\mathrm{U}(\mathcal{V}_1) & \xrightarrow{\ I\ } & \mathrm{U}(V_{an})\\ \Big\downarrow & & \Big\downarrow{\scriptstyle \text{inclusion map}}\\ E^\times\times D^\times/\Delta F^\times & \xrightarrow{\ \cong\ } & \mathrm{GU}(V_{an}).\end{array}$$

Here the left vertical map is induced by $\mathrm{U}(\mathcal{V}_1) = \ker\nu_D \hookrightarrow D^\times$.

Next choose quasi-characters ω,ω' of $E^\times$ such that $\omega\omega' = \mu$, $\omega(\omega'\circ\sigma) = \mu\eta^{-1}$. Then $\tilde\tau = \omega\boxtimes\tau_D(\omega')$ is an irreducible admissible representation of $E^\times\times D^\times/\Delta F^\times$. By Theorem 7.8, we obtain

$$\tilde\tau|_{\mathrm{U}(\mathcal{V}_1)} = \tau_D(\omega')|_{\mathrm{U}(\mathcal{V}_1)} = \tau_D(\omega')^+\oplus\tau_D(\omega')^-.$$

If we consider $\tilde{\tau}$ as a representation of $\mathrm{GU}(V_{an})$, by [3], we have

$$\tilde{\tau}|_{\mathrm{U}(V_{an})} = \tau(\mu, \mu\eta^{-1})^+_{an} \oplus \tau(\mu, \mu\eta^{-1})^-_{an}.$$

Since we can check that $\omega'|_{\mathrm{U}(\mathcal{W}_1)} = \eta_u = \chi_\eta|_{\mathrm{U}(\mathcal{W}_1)}$, we obtain $\tau_D(\omega')^\epsilon = \tau_D(\chi_\eta)^\epsilon$. Thus we have $\tau(\mu, \mu\eta^{-1})^+_{an} \circ I = \tau_D(\chi_\eta)^+$ or $\tau_D(\chi_\eta)^-$.

Next recall that $\mathrm{U}(\mathcal{W}_1)$ is identified with a subgroup of $\mathrm{U}(V_{an})$ by

$$I|_{\mathrm{U}(\mathcal{W}_1)} : \mathrm{U}(\mathcal{W}_1) \ni \gamma \mapsto \begin{pmatrix} \gamma^{-1} & \\ & \gamma \end{pmatrix} \in \mathrm{U}(V_{an}).$$

Then by Theorem 5.2, the restriction of the character identity of $\tau(\mu, \mu\eta^{-1})^\pm_{an} \circ I$ to $\mathrm{U}(\mathcal{W}_1)$ is given as follows:

$$\begin{aligned} &\operatorname{Tr} \tau(\mu, \mu\eta^{-1})^+_{an} \circ I(\gamma) - \operatorname{Tr} \tau(\mu, \mu\eta^{-1})^-_{an} \circ I(\gamma) \\ =& \lambda\,(E/F, \psi)\, \omega_{E/F}\left(\frac{\gamma^{-1} - \gamma}{\xi}\right) \frac{\eta_u(\gamma) - \eta_u(\gamma^{-1})}{|\gamma^{-1} - \gamma|_E^{1/2}} \\ =& \operatorname{Tr} \tau_D(\chi_\eta)^+(\gamma) - \operatorname{Tr} \tau_D(\chi_\eta)^-(\gamma). \end{aligned}$$

The last equality follows from Theorem 7.8. Thus we obtain

$$\tau(\mu, \mu\eta^{-1})^+_{an} \circ I = \tau_D(\chi_\eta)^+.$$

□

Acknowledgements. The author would like to thank his advisor, Prof. Takuya Konno, for much helpful advice and encouragement.

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