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# AN INITIAL VALUE OF NEWTON'S METHOD FOR THE CHARACTERISTIC POLYNOMIAL OF 4TH ORDER PCM IN AHP

By

Tsuneshi OBATA\* and Shunsuke SHIRAIISHI†

## Abstract

The pairwise comparison matrix plays a critical role in the analytic hierarchy process, a popular multi-criteria decision-making method based on individual subjectivity. The maximal eigenvalue of the pairwise comparison matrix can be used to measure the degree of consistency of the decision maker's judgment. Since the eigenvalues of a matrix are the roots of a characteristic equation, they can be obtained by numerical iterative methods such as Newton's method. However, the location of the initial point does not always guarantee that the maximal eigenvalue will be obtained. For the third-order pairwise comparison matrix, it has already been shown that the characteristic equations always have a single real root and that Newton's method using three as the initial point can always achieve the maximal eigenvalue. In this paper, we focus on the characteristic equation of a fourth-order pairwise comparison matrix and show that they have exactly two real roots. Furthermore, we present an initial point of Newton's method such that it always converges to the larger root, i.e., the maximal eigenvalue. Numerical experiments are also provided to compare convergence rate with other numerical iterative methods.

*Key Words and Phrases:* AHP, Pairwise Comparison, Maximal Eigenvalue, Newton's Method

## 1. Introduction

It has been over 40 years since the analytic hierarchy process (AHP) was established by Saaty (1980). These days the analytical feature supported by the pairwise comparison matrix (PCM) continuously plays a central role in the theory of AHP. By adopting the consistency index derived from the maximal eigenvalue of the pairwise comparison matrix, it seems that the prominence of AHP was established, see Brunelli (2014), Kułakowski (2021), Saaty (1980) and Tone (1986).

In AHP, scaling is executed by verbal comparisons. Once a decision maker is asked to her/his preference in many alternatives, she/he answers the indicated strength of the preference on the left side of the following table (Table 1 below). The expression is automatically transformed into discrete scale numbers on the right hand. The scale is

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Verbal expression	Value of intensity
equal importance	1
moderate importance	3 or 1/3
essential or strong importance	5 or 1/5
demonstrated importance	7 or 1/7
absolute importance	9 or 1/9
intermediate values	$k = 2, 4, 6, 8$ or $1/k$

Table 1: Saaty's discrete scale

called Saaty's discrete scale. After all comparisons, one obtains the so-called pairwise comparison matrix.

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

The priority of the compared alternatives is calculated by the following linear system, which is called the eigenvector method:

$$Aw = \lambda_{\max} w,$$

where  $\lambda_{\max}$  denotes the maximal (principal) eigenvalue of  $A$ . From the perspective of mathematical programming, Sekitani and Yamaki (1999) established the theoretical foundation of the eigenvector method. Once one obtains the maximal eigenvalue  $\lambda_{\max}$ , it is easy to obtain the associated eigenvector  $w$  because it is a linear system of equations, see Boyd and Vandenberghe (2019).

Our previous work (Shiraishi and Obata (2021)) treated a third-order pairwise comparison matrix. With the favorable properties of the third-order pairwise comparison matrix, we proved that ordinal Newton's method provides the sequence convergent to  $\lambda_{\max}$ . This paper extends the convergent result to a fourth-order pairwise comparison matrix.

In Section 2., we will outline the properties of a characteristic polynomial and demonstrate its unique minimum point. This simplicity in the graphs of characteristic polynomials is crucial for the effectiveness of Newton's method. We will discuss the convergent result further in Section 3. Section 4. will be dedicated to a computational experiment.

## 2. The characteristic polynomial

In this paper, we deal with a fourth-order pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 1/a_{12} & 1 & a_{23} & a_{24} \\ 1/a_{13} & 1/a_{23} & 1 & a_{34} \\ 1/a_{14} & 1/a_{24} & 1/a_{34} & 1 \end{pmatrix}.$$

Since we are concerned with the maximal eigenvalue of  $A$ , we first describe the properties of the characteristic polynomial  $P_A(\lambda) = \det(\lambda E - A)$ , where  $E$  denotes

the identity matrix. Perron's theorem (Theorem 1.2.2 in Bapat and Raghavan (1997)) states the existence of a positive eigenvalue of  $A$ , which is maximal in modulus among all the eigenvalues of  $A$ . We denote the maximal eigenvalue of  $A$  by  $\lambda_{\max}$ . The maximal eigenvalue  $\lambda_{\max}$  is a simple root of  $P_A(\lambda) = 0$ , see Theorem 7-3 in Saaty (1980). According to Saaty (1977), the following theorem is a key ingredient of AHP. A pairwise comparison matrix  $A = (a_{ij})$  is called to be consistent provided that  $a_{ik}a_{kj} = a_{ij}$  for all  $i, k, j$ .

**THEOREM 2.1.** *For any pairwise comparison matrix  $A$  of order  $n$ , one has*

$$\lambda_{\max} \geq n.$$

$A$  is consistent if and only if  $\lambda_{\max} = n$ .

Saaty also suggested the following result. See Proposition 1 in Shiraishi et al. (1998).

**THEOREM 2.2.** *Let  $A$  be a pairwise comparison matrix of order  $n$ . Then  $A$  is consistent if and only if*

$$P_A(\lambda) = \lambda^n - n\lambda^{n-1}.$$

According to our prior work(Shiraishi et al. (1998)), we also showed that the characteristic polynomial for a fourth-order pairwise comparison matrix  $A$  has the following formula.

$$P_A(\lambda) = \lambda^4 - 4\lambda^3 + c_3\lambda + \det A,$$

where

$$c_3 = \sum_{i < j < k} \left( 2 - \left( \frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}} \right) \right). \quad (1)$$

From the relationship between the arithmetic mean and the geometric mean, we have  $c_3 \leq 0$ . In Shiraishi et al. (1998), we proved that the consistency of  $A$  and  $c_3 = 0$  are equivalent. Moreover in the consistent case, we also proved that the residual term  $\det A$  vanishes. Hence if  $A$  is consistent, the characteristic polynomial has the form of  $P_A(\lambda) = \lambda^4 - 4\lambda^3 = \lambda^3(\lambda - 4)$ . It shows  $\lambda_{\max} = 4$ . Hereafter, we treat inconsistent cases, which means  $c_3 < 0$ . Then we have the following lemma.

**LEMMA 2.3.**  $P'_A(\lambda) > 0$  for  $\lambda \geq \lambda_{\max}$ .

**PROOF.** Taking the derivative of  $P_A(\lambda)$ , we have  $P'_A(\lambda) = 4\lambda^3 - 12\lambda^2 + c_3$ . Since  $P'_A(0) = c_3 < 0$  and  $\lim_{\lambda \rightarrow \infty} P'_A(\lambda) = \infty$ ,  $P'_A(\lambda) = 0$  has a zero-point  $\lambda^* > 0$ , which is a stationary point of  $P_A(\lambda)$ , see Figure 1. Since  $P'_A(\lambda^*) = 4\lambda^{*2}(\lambda^* - 3) + c_3$  and  $c_3 < 0$ , we have  $\lambda^* > 3$ . Hence,  $P'_A(\lambda) = 4\lambda^2(\lambda - 3) + c_3 < 0$  for any  $\lambda < \lambda^*$ , and  $P'_A(\lambda) > 0$  for any  $\lambda > \lambda^*$ . Therefore  $P_A(\lambda)$  attains the strictly minimum at  $\lambda^*$ . Here we note that  $P_A(\lambda)$  has exactly two zero-points when  $c_3 < 0$  (Obata and Shiraishi (2021)). Finally, we show that  $\lambda^* < \lambda_{\max}$ . Suppose that  $P_A(\lambda^*) = 0$ , then  $\lambda^*$  becomes a multiple root of  $P_A(\lambda) = 0$  and  $\lambda^* = \lambda_{\max}$ . As we noted above  $\lambda_{\max}$  is a simple root. This means  $P_A(\lambda^*) < 0$ . Since  $P_A(\lambda)$  is monotone increasing for  $\lambda > \lambda^*$  and  $P_A(\lambda_{\max}) = 0$ , we conclude that  $\lambda^* < \lambda_{\max}$ .

**LEMMA 2.4.**  $P''_A(\lambda) > 0$  for  $\lambda \geq \lambda_{\max}$ .

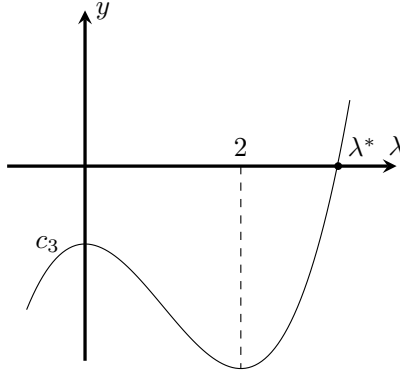


Figure 1: Graph of  $P'_A(\lambda) = 4\lambda^3 - 12\lambda^2 + c_3$

PROOF. Taking the second-derivative of  $P_A(\lambda)$ , we have  $P''_A(\lambda) = 12\lambda^2 - 24\lambda = 12\lambda(\lambda - 2)$ . It shows  $P''_A(\lambda) > 0$  and  $P'_A(\lambda)$  is monotone increasing for  $\lambda > 2$ . The clear inequality  $2 < 3 < \lambda^* < \lambda_{\max}$  assures the result.

To deepen understanding, consider the following example.

EXAMPLE 2.5. Figure 2 is the graph of  $P_A(\lambda)$  for a pairwise comparison matrix

$$A = \begin{pmatrix} 1 & 4 & 1/2 & 1/7 \\ 1/4 & 1 & 1/3 & 2 \\ 2 & 3 & 1 & 3 \\ 7 & 1/2 & 1/3 & 1 \end{pmatrix}.$$

The characteristic polynomial  $P_A(\lambda)$  is given by

$$P_A(\lambda) = \lambda^4 - 4\lambda^3 - \frac{5389}{84}\lambda + \frac{5225}{336}.$$

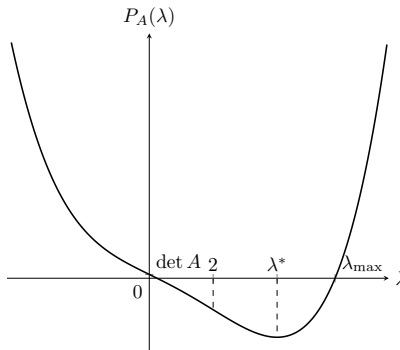


Figure 2: Graph of  $P_A(\lambda)$

### 3. Proof of the convergence

For an  $n$ -th order pairwise comparison matrix  $A = (a_{ij})$ , set

$$\lambda_{\text{upper}} = 1 + \frac{1}{2}(n-1)(S+1/S), \quad (2)$$

where a number  $S$  satisfies  $1/S \leq a_{ij} \leq S$ . When we apply the Saaty's discrete scale, we can set  $S = 9$ . Apupetit and Genest (1993) gave the following upperbound for  $\lambda_{\text{max}}$ .

$$\lambda_{\text{max}} \leq \lambda_{\text{upper}}. \quad (3)$$

We generate a sequence  $\{\lambda_k\}$  by the following iteration of Newton's method with the initial value of  $\lambda_{\text{upper}}$ :

$$\begin{cases} \lambda_0 = \lambda_{\text{upper}}, \\ \lambda_{k+1} = \lambda_k - \frac{P_A(\lambda_k)}{P'_A(\lambda_k)}, \end{cases} \quad k = 0, 1, 2, \dots \quad (4)$$

It follows from (3) and the monotonicity of  $P_A$ ,  $P_A(\lambda_{\text{upper}}) \geq P_A(\lambda_{\text{max}}) = 0$ . If  $P_A(\lambda_{\text{upper}}) = 0$ , we have the solution  $\lambda_{\text{max}} = \lambda_{\text{upper}}$  without iteration. So we may assume  $P_A(\lambda_{\text{upper}}) > 0$ .

LEMMA 3.1. *If  $P_A(\lambda_{\text{upper}}) > 0$ , we have  $\lambda_k > \lambda_{\text{max}}$  for all  $k \geq 0$ .*

PROOF. We prove it by induction on  $k$ . Because  $P_A(\lambda)$  is monotone increasing for  $\lambda_{\text{max}} \leq \lambda$ , by Lemma 2.3, we have  $\lambda_{\text{max}} < \lambda_0 = \lambda_{\text{upper}}$ .

Let  $k \geq 1$ . Assume  $\lambda_{\text{max}} < \lambda_k$ . For  $\lambda_{\text{max}} \leq \lambda < \lambda_k$ , by Lemma 2.4,  $P'_A(\lambda)$  is monotone increasing for  $\lambda > \lambda^*$ . Thus we have

$$P'_A(\lambda) < P'_A(\lambda_k).$$

From this inequality and the fact that the interval  $[\lambda_{\text{max}}, \lambda_k]$  does not vanish,

$$\int_{\lambda_{\text{max}}}^{\lambda_k} P'_A(\lambda) d\lambda < \int_{\lambda_{\text{max}}}^{\lambda_k} P'_A(\lambda_k) d\lambda,$$

which leads

$$P_A(\lambda_k) - P_A(\lambda_{\text{max}}) < P'_A(\lambda_k)(\lambda_k - \lambda_{\text{max}}). \quad (5)$$

Since  $\lambda_k > \lambda_{\text{max}}$ , we have  $P'_A(\lambda_k) > 0$ , by Lemma 2.3. Thus the inequality (5) shows

$$\lambda_{\text{max}} < \lambda_k - \frac{P_A(\lambda_k)}{P'_A(\lambda_k)} = \lambda_{k+1}.$$

LEMMA 3.2. *The sequence  $\{\lambda_k\}$  generated by (4) is monotone decreasing.*

PROOF. By Lemma 3.1, we have  $\lambda_{\text{max}} < \lambda_k$ . By Lemma 2.3,  $P_A(\lambda)$  is monotone increasing for  $\lambda_{\text{max}} < \lambda$ . Hence we have  $P_A(\lambda_k) > P_A(\lambda_{\text{max}}) = 0$ . By Lemma 2.3 again,  $P'_A(\lambda_k) > 0$  holds. Thus we have

$$\lambda_{k+1} = \lambda_k - \frac{P_A(\lambda_k)}{P'_A(\lambda_k)} < \lambda_k,$$

which means monotone decreasingness of  $\lambda_k$ .

THEOREM 3.3.

$$\lim_{k \rightarrow \infty} \lambda_k = \lambda_{max}.$$

PROOF. From Lemma 3.1 and Lemma 3.2, the monotone decreasing sequence  $\{\lambda_k\}$  which is bounded from below converges to some  $\tilde{\lambda}$ .

It follows from (4),

$$P_A(\lambda_k) = (\lambda_{k+1} - \lambda_k)P'_A(\lambda_k).$$

Hence  $P_A(\tilde{\lambda}) = \lim_{k \rightarrow \infty} P_A(\lambda_k) = 0$ . From the inequality  $\lambda_k > \lambda_{max}$ , we have  $\tilde{\lambda} \geq \lambda_{max}$ . If we assume  $\tilde{\lambda} > \lambda_{max}$ , which contradicts maximality of  $\lambda_{max}$ . Thus we have  $\tilde{\lambda} = \lambda_{max}$ .

We state the rate of convergence. The proof is the same as in Shiraishi and Obata (2021), so we omit it.

THEOREM 3.4. *The sequence  $\{\lambda_k\}$  defined by (4) converges quadratically to  $\lambda_{max}$ .*

#### 4. Computational Experiment

In Shiraishi and Obata (2021), we conducted a computational experiment. We compared Newton's method with other root-finding methods for the third-order pairwise comparison matrices. In this paper, we conduct the same experiment for fourth-order pairwise comparison matrices. The experiment is as follows.

**Step 1.** Generate an inconsistent pairwise comparison matrix  $A$  of order 4 at random using Saaty's scale.

**Step 2.** Iterate the following three methods until the differences between  $\lambda_{max}$  and the obtained values become below the threshold and store the number of iterations.

- Newton's method with the initial value of  $\lambda_0 = \lambda_{upper}$ .

$$\lambda_{k+1} = \lambda_k - \frac{P_A(\lambda_k)}{P'_A(\lambda_k)}, \quad k = 0, 1, 2, \dots$$

- Secant method<sup>1</sup> with initial values of  $\lambda_{-1} = 4$  and  $\lambda_0 = \lambda_{upper}$ .

$$\lambda_{k+1} = \lambda_k - \frac{P_A(\lambda_k)(\lambda_k - \lambda_{k-1})}{P_A(\lambda_k) - P_A(\lambda_{k-1})}, \quad k = 0, 1, 2, \dots$$

- Bisection method<sup>2</sup> with initial interval of  $[\lambda_0^{lower}, \lambda_0^{upper}] = [4, \lambda_{upper}]$ .

$$\lambda_{k+1} = \frac{\lambda_k^{lower} + \lambda_k^{upper}}{2}, \quad k = 0, 1, 2, \dots,$$

where

$$\begin{aligned} & [\lambda_{k+1}^{lower}, \lambda_{k+1}^{upper}] \\ &= \begin{cases} [\lambda_{k+1}, \lambda_k^{upper}], & \text{if } P_A(\lambda_{k+1}) \text{ and } P_A(\lambda_k^{lower}) \text{ have the same sign,} \\ [\lambda_k^{lower}, \lambda_{k+1}], & \text{if } P_A(\lambda_{k+1}) \text{ and } P_A(\lambda_k^{lower}) \text{ have the same sign,} \end{cases} \end{aligned}$$

for  $k = 0, 1, 2, \dots$

<sup>1</sup> [https://encyclopediaofmath.org/wiki/Secant\\_method](https://encyclopediaofmath.org/wiki/Secant_method)

<sup>2</sup> [https://encyclopediaofmath.org/wiki/Dichotomy\\_method](https://encyclopediaofmath.org/wiki/Dichotomy_method)



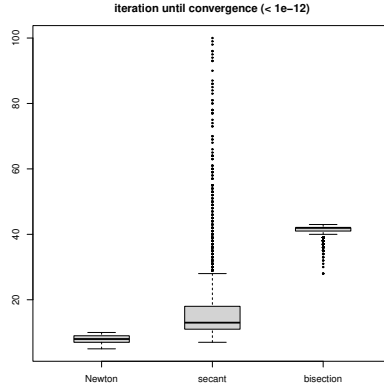


Figure 3: Distributions of the number of iterations until convergence

Table 2: Summary statistics of the distributions of three methods (fourth-order)

	minimum	median	mode	mean	maximum
Newton	5	8	9	8.1720	10
secant <sup>1</sup>	7	13	11	16.8810	100
bisection	28	42	42	41.3664	42

In these procedures, we use  $S = 9$  in the definition of  $\lambda_{\text{upper}}$  (2), i.e.,  $\lambda_{\text{upper}} = 1 + (4 - 1)(9 + 1/9)/2 = 44/3$ .

First, we must report that the secant method did not show the expected convergence results. The secant method often generates a sequence that converges to another root smaller than  $\lambda^*$ . We repeated the steps above 5,000 times, and the secant method did not work well in 2,134 cases. Those 2,134 cases converged on the smaller root or repeated huge oscillation more than 100 times.

Figure 3 shows the distributions of the numbers of iterations until convergence by each method after repeating the steps above 5,000 times. We set the threshold of convergence to  $10^{-12}$  here.

The summary statistics of the 5,000 number of iterations until convergence by each method are shown in Table 2. Note that the values for the secant method in Table 2 are for 2,866 cases, excluding the 2,134 cases mentioned above. Since we terminated after 100 iterations, the 2,134 cases include those that would have converged to  $\lambda_{\text{max}}$  if we had continued the iterations.

The results for the third-order case are redisplayed from Shiraishi and Obata (2021) in Table 3 for the reader’s reference. We observe that in every simulation fourth-order methods require more iterations than third-order iterations. It may be due to the initial point being far away.

For example, we display the following typical pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & 1/3 & 3 & 9 \\ 3 & 1 & 3 & 3 \\ 1/3 & 1/3 & 1 & 9 \\ 1/9 & 1/3 & 1/9 & 1 \end{pmatrix}.$$

Table 3: Summary statistics of the distributions of three methods (third-order)

	minimum	median	mode	mean	maximum
Newton	2	5	5	5.3926	11
secant	3	8	8	8.9066	25
bisection	30	41	41	40.3576	48

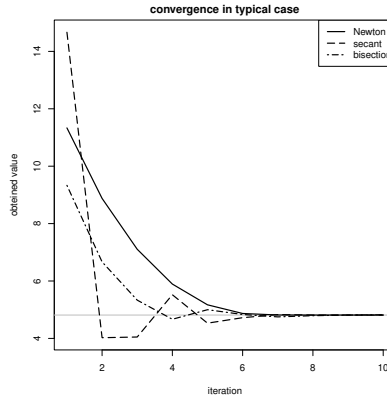


Figure 4: Transition toward convergence in a typical case

This matrix generates the values of convergence in 9, 11, and 42 iterations by Newton's method, the secant method, and the bisection method, respectively. The numbers 9, 11, and 42 are equal to the mode of each method. The transitions of the values toward convergence for this matrix are shown in Figure 4. This experiment confirms the convergence rate of Newton's method empirically.

## 5. Conclusion

This paper extends the convergence result of Newton's method obtained in the third-order pairwise comparison matrix to the fourth-order pairwise comparison matrix. The key to proof is the fact that the characteristic polynomial has a unique minimal point. So we could examine the area where the solution of the characteristic equation exists.

Another idea is the selection of the initial point of Newton's method. It profoundly depends on the result of Aupetit and Genest (1993). Although the theorem has led to successful theoretical proof, the convergence rate is not necessarily fast. The improvement of the convergence rate is left to future research.

Computational simulations were conducted on a randomly generated 4th-order pairwise comparison matrix. As expected, the superiority of the Newton's method was demonstrated.

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