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https://hdl.handle.net/2324/7177969

出版情報：Journal of Mathematical Analysis and Applications. 437 (1), pp.330-349, 2016-05-01. Elsevier
バージョン：
権利関係：

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arXiv:1402.1065v3 [math-ph] 10 Feb 2014

Spectrum of the semi-relativistic Pauli-Fierz model I

Takeru Hidaka [*] and Fumio Hiroshima [†]

Abstract

HVZ type theorem for semi-relativistic Pauli-Fierz Hamiltonian,

$$H = \sqrt{(p \otimes 1\!\!1 - A)^2 + M^2} + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}, \quad M \geq 0,$$

in quantum electrodynamics is studied. Here H is a self-adjoint operator in Hilbert space $L^2(\mathbb{R}^d) \otimes \mathscr{F} \cong \int_{\mathbb{R}^d}^{\oplus} \mathscr{F} \mathrm{d}x$, and $A = \int_{\mathbb{R}^d}^{\oplus} A(x)\mathrm{d}x$ a quantized radiation field and $\mathrm{H_f}$ the free field Hamiltonian defined by the second quantization of a dispersion relation $\omega : \mathbb{R}^d \to \mathbb{R}$. It is emphasized that massless case, $M = 0$, is included. Let $E = \inf \sigma(H)$ be the bottom of the spectrum of H. Suppose that the infimum of ω is $m > 0$. Then it is shown that $\sigma_{\mathrm{ess}}(H) = [E + m, \infty)$. In particular the existence of the ground state of H can be proven.

1 Introduction

It is of interest to know the spectrum of the so-called semi-relativistic Pauli-Fierz model (it is shorthanded as the SRPF model) in quantum electrodynamics. The aim of this paper is to specify the essential spectral of the SRPF Hamiltonian. In the mathematically rigorous quantum field theory spectrum of various models have been investigated so far. In particular special attentions have been payed for investigating the bottom of the spectrum, continuous spectrum and resonances etc. The SRPF model is one of interesting models in quantum electrodynamics.

The Pauli-Fierz model is a model in non-relativistic quantum electrodynamics and describes a minimal interaction between electrons governed by a Schrödinger operator $\frac{1}{2M}p^2 + V$, and a quantized radiation field $A(x)$ with an ultraviolet cutoff, which is a self-adjoint operator on the Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F} \tag{1.1}$$

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and of the form

$$H_{PF} = \frac{1}{2M}(p \otimes \mathbb{1} - A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H_f}, \tag{1.2}$$

where $p = (-i\partial_{x_1}, \cdots, -i\partial_{x_d})$ denotes the d-dimensional momentum operator of an electron, V an external potential, $\mathrm{H_f}$ the free field Hamiltonian on a Boson Fock space $\mathscr{F}$, and

$$A = \int_{\mathbb{R}^d}^{\oplus} A(x)\mathrm{d}x$$

is the constant fiber direct integral of $A(x)$ under the identification $\mathscr{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathscr{F} \mathrm{d}x$. On the other hand the SRPF model describes a minimal interaction between $A(x)$ and an electron governed by semi-relativistic Schrödinger operator $\sqrt{p^2 + M^2} + V$. The total Hamiltonian of the SRPF model is then formally given by

$$H = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2} + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H_f}. \tag{1.3}$$

We give the explicit definition of H later. The problems we consider in this paper are

1. HVZ type theorem for H,

2. the existence and uniqueness of the ground state of H.

We emphasis that all the results we obtain in this paper include the case of $M = 0$, i.e.,

$$|p \otimes \mathbb{1} - A| + V \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H_f}. \tag{1.4}$$

Here $|T| = \sqrt{T^2}$ for a self-adjoint operator T. The crucial point is the form of $|p \otimes \mathbb{1} - A|$. It is worth pointing out that $x \to |x|$ is not smooth.

We consider HVZ-type theorem for H. The standard HVZ theorem identifies the essential spectrum of N-body Schrödinger operators. See e.g. [Hun66]. We extend HVZ theorem to H. I.e., we specify the essential spectrum of H. The bottom of the spectrum of H, E, is called the ground state energy, and eigenvectors associated with E are called ground states. We suppose that a dispersion relation has a strictly positive lower bound $m > 0$. Then we shall show that

$$\sigma_{\mathrm{ess}}(H) = [E + m, \infty). \tag{1.5}$$

In particular it can be seen that the gap m is independent of the cutoff function in $A(x)$ and M, and furthermore it is shown that H has ground states for all $M \geq 0$. The method to show this is a combination of checking the binding condition developed in [GLL01] and functional integration established in [Hir97].

We review several papers related to our results. For the Pauli-Fierz model H_{PF} the existence and uniqueness of ground states are proven in e.g., [BFS99, GLL01, Hir00a]. For the semi-relativistic case, H, the existence of a ground state is shown in [KMS11, KM13] but for $M > 0$. In the case of $M = 0$ as far as we know however there is no results on the existence of ground states. So our result is new. When $V = 0$, H is translation invariant and has no ground state. It can be however decomposed by the total momentum:

$$H = \int_{\mathbb{R}^d}^{\oplus} H(P)\mathrm{d}P,$$

where

$$H(P) = \sqrt{(P - P_{\mathrm{f}} - A(0))^2 + M^2} + \mathrm{H}_{\mathrm{f}}.$$

For every fixed total momentum P, the existence of ground state of $H(P)$ can be considered, but as far as we know there is no exact result on the existence of ground state of $H(P)$. See [MS09, HS10, Sas13] for related results.

This paper is organized as follows. In Section 2 we set up notation and terminology, give the rigorous definition of H as a self-adjoint operator.

Section 3 deals with localization and show that $\chi(H)$ with smooth function χ with a support in $(-\infty, E + m)$ is compact.

Section 4 establishes a HVZ-type theorem, i.e., $\sigma_{\mathrm{ess}}(H) = [E + m, \infty)$, and proves that H has a ground state as a corollary of HVZ-type theorem.

2 Definitions and the main theorems

In this section we define H as a self-adjoint operator on a Hilbert space, and give the main theorem. A particle Hamiltonian is given by the semi-relativistic Schröding operator with a rest mass M:

$$\sqrt{p^2 + M^2} + V. \tag{2.1}$$

We shall introduce assumptions on V later. We suppose that $M \geq 0$ throughout this paper unless otherwise stated.

Let $\mathscr{F} = \oplus_{n=0}^{\infty} \mathscr{F}_n(W) = \oplus_{n=0}^{\infty} [\otimes_s^n W]$ be the Boson Fock space over Hilbert space $W = \oplus^{d-1} L^2(\mathbb{R}^d)$, $d \geq 3$. Here $\otimes_s^0 W = \oplus^{d-1}\mathbb{C}$. Although the physically reasonable choice of the spatial dimension is $d = 3$, we generalize it. The creation operator and the annihilation operator in $\mathscr{F}$ are denoted by $a^\dagger(f)$ and $a(f)$, $f \in W$, respectively. They are linear in the test function f and satisfy canonical commutation relations:

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_W, \qquad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)].$$

Here and in what follows the scalar product $(f, g)_{\mathscr{K}}$ on a Hilbert space $\mathscr{K}$ is linear in g and anti-linear in f. Formally $a^\#(f)$ is written as $a^\#(f) = \sum_{r=1}^{d-1} \int a^{\# r}(k) f_r(k) dk$ for $f = \oplus_{r=1}^{d-1} f_r \in W$. We introduce assumptions on the dispersion relation ω.

Assumption 2.1 *$\omega \in C^1(\mathbb{R}^d; \mathbb{R})$, $\nabla\omega \in L^\infty(\mathbb{R}^d)$, $\inf_{k\in\mathbb{R}^d} \omega(k) = m$ with some $m > 0$ and $\lim_{|k|\to\infty} \omega(k) = \infty$.*

The free field Hamiltonian $H_{\rm f}$ is given by the second quantization of the multiplication operator by $\oplus^{d-1}\omega$ on W. Thus formally it is defined by

$$H_{\rm f}=\sum_{r=1}^{d-1}\int \omega(k)a^{\dagger r}(k)a^r(k)dk. \tag{2.2}$$

Let $e^r(k)=(e^r_1(k),...,e^r_d(k))$ be d-dimensional polarization vectors, i.e., $e^r(k)\cdot e^s(k)=\delta_{rs}$ and $k\cdot e^r(k)=0$ for $k\in\mathbb{R}^d\setminus\{0\}$ and $r=1,...,d-1$. Let $\hat\varphi$ be an ultraviolet cutoff function, for which we introduce assumptions below.

Assumption 2.2 *$\omega\sqrt{\omega}\hat\varphi\in L^2(\mathbb{R}^d)$ and $\hat\varphi(k)=\overline{\hat\varphi(-k)}$.*

From this assumption and $\inf_k\omega(k)=m>0$ we can see that $\hat\varphi/\sqrt{\omega}$, $\sqrt{\omega}\hat\varphi\in L^2(\mathbb{R}^d)$. We fix $\hat\varphi$ satisfying Assumption 2.2 throughout this paper. For each $x\in\mathbb{R}^d$ a quantized radiation field $A(x)=(A_1(x),...,A_d(x))$ is given by

$$A_\mu(x)=\frac{1}{\sqrt{2}}\sum_{r=1}^{d-1}\int e^r_\mu(k)\left\{\frac{\hat\varphi(k)e^{-ik\cdot x}}{\sqrt{\omega(k)}}a^{\dagger r}(k)+\frac{\hat\varphi(-k)e^{ik\cdot x}}{\sqrt{\omega(k)}}a^r(k)\right\}dk. \tag{2.3}$$

Then $\hat\varphi(k)=\overline{\hat\varphi(-k)}$ implies that $A_\mu(x)$ is essentially self-adjoint for each x. We denote the self-adjoint extension by the same symbol $A_\mu(x)$. We identify $\mathscr{H}$ with $\int^\oplus_{\mathbb{R}^d}\mathscr{F}{\rm d}x$, and under this identification we define the self-adjoint operator A_μ by $\int^\oplus_{\mathbb{R}^d}A_\mu(x){\rm d}x$.

The first task is to define the operator H in (1.3) as a self-adjoint operator. The square root of $(p\otimes 1\!\!1-A)^2+M^2$, $\sqrt{(p\otimes 1\!\!1-A)^2+M^2}$, is defined through the spectral measure associated with self-adjoint operator $(p\otimes 1\!\!1-A)^2+M^2$. It is however not trivial to show the self-adjoitness of $(p\otimes 1\!\!1-A)^2$. Let $N=d\Gamma(1\!\!1)$ be the number operator on $\mathscr{F}$, i.e., $N=\sum_{r=1}^{d-1}\int a^{\dagger r}(k)a^r(k)dk$. Let $C^\infty(1\!\!1\otimes N)=\cap_{n=1}^\infty D(1\!\!1\otimes N^n)$.

Proposition 2.3 *Suppose Assumption 2.2. Then $(p\otimes 1\!\!1-A)^2$ is essentially self-adjoint on $D(p^2\otimes 1\!\!1)\bigcap C^\infty(1\!\!1\otimes N)$.*

Proof: See [LHB11, Lemma 7.53]. ■

The closure of $(p\otimes 1\!\!1-A)^2\lceil_{D(p^2\otimes 1\!\!1)\bigcap C^\infty(1\!\!1\otimes N)}$ is denoted by $(p\otimes 1\!\!1-A)^2$ in what follows. Thus $\sqrt{(p\otimes 1\!\!1-A)^2+M^2}$ is defined through the spectral measure of $(p\otimes 1\!\!1-A)^2$.

Definition 2.4 *The SRPF Hamiltonian is defined by*

$$H=\sqrt{(p\otimes 1\!\!1-A)^2+M^2}+V\otimes 1\!\!1+1\!\!1\otimes {\rm H_f} \tag{2.4}$$

with the domain

$$D(H)=D(\sqrt{(p\otimes 1\!\!1-A)^2+M^2})\cap D(V\otimes 1\!\!1)\cap D(1\!\!1\otimes {\rm H_f}). \tag{2.5}$$

We do not write tensor notation $\otimes$ for notational convenience in what follows. Thus H can be simply written as

$$H = \sqrt{(p-A)^2 + M^2} + V + \mathrm{H_f}. \tag{2.6}$$

Assumption 2.5 *(1) V is non-negative and satisfies that $\lim_{|x|\to\infty} V(x) = \infty$. (2) V is twice differentiable, and $\partial_\mu V, \partial_\mu^2 V \in L^\infty(\mathbb{R}^d)$ for $\mu = 1, ..., d$, and $D(V) \subset D(|x|)$.*

Lemma 2.6 *Suppose Assumption 2.5. Then $p^2 + V$ is self-adjoint on $D(p^2) \cap D(V)$, and essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.*

Proof: Since $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, $p^2 + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$. Take an arbitrary vector $\Psi \in C_c^\infty(\mathbb{R}^d)$. We have $\|(p^2+V)\Psi\|^2 = \|p^2\Psi\| + \|V\Psi\|^2 + 2\sum_{\mu=1}^d \Re(p_\mu^2\Psi, V\Psi)$. For all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned} 2\Re(p_\mu^2\Psi, V\Psi) &= 2\Re\{(p_\mu\Psi, Vp_\mu\Psi) - (p_\mu\Psi, [V, p_\mu]\Psi)\} \\ &\geq -2\|\partial_\mu V\|\|p_\mu\Psi\|\|\Psi\| \geq -\epsilon\|p_\mu\Psi\|^2 - C_\epsilon\|\Psi\|^2. \end{aligned}$$

Thus $\|p^2\Psi\|^2 + \|V\Psi\| \leq C(\|(p^2+V)\Psi\| + \|\Psi\|)$ follows with some constant $C > 0$. $p^2 + V\lceil_{D(p^2)\cap D(V)}$ is closed, and then it is self-adjoint. ∎

We can also show the self-adjointness of H under Assumption 2.5. It is established in [Hir13] that H for $M > 0$ is essentially self-adjoint on $D(|p|) \cap D(\mathrm{H_f})$ for external potential V such that $D(V) \subset D(|p|)$ and $\|Vf\| \leq a\||p|f\| + b\|f\|$ for all $f \in D(|p|)$ with $0 \leq a < 1$ and $b \geq 0$. We can also show a stronger statement on the self-adjointness of H. This is established in [HH13]. We set

$$\mathscr{H}_{\mathrm{fin}} = C_c^\infty(\mathbb{R}^d)\hat{\otimes}\mathscr{F}_\infty, \tag{2.7}$$

where $\hat{\otimes}$ denotes the algebraic tensor product and

$$\mathscr{F}_\infty = L.H.\{\Omega,\ a^\dagger(h_1)\cdots a^\dagger(h_n)\Omega | h_j \in C_c^\infty(\mathbb{R}^d), j = 1, \cdots, n, n \geq 1\}.$$

Theorem 2.7 *Suppose Assumptions 2.1, 2.2 and 2.5. Then (1) and (2) follow.*

(1) Let $M \geq 0$. Then H is self-adjoint on $D(|p|) \cap D(V) \cap D(\mathrm{H_f})$ and essentially self-adjoint on $\mathscr{H}_{\mathrm{fin}}$.

(2) Fix an arbitrary $M_0 > 0$. Then there exists a constant $C = C(M_0) > 0$ such that for all $\Psi \in D(H)$ and $0 \leq M \leq M_0$,

$$\||p|\Psi\|^2 + \|V\Psi\|^2 + \|\mathrm{H_f}\Psi\|^2 \leq C\|(H + 1\!\!1)\Psi\|^2. \tag{2.8}$$

Proof: See [HH13, Lemma 2.9]. ■

The ground state energy, E, of H is the bottom of the spectrum of H:

$$E = \inf \sigma(H). \tag{2.9}$$

When $M = 0$, we denote H_0 and E_0 for H and E, respectively. The main results of this paper are as follows:

Theorem 2.8 (HVZ theorem for SRPF model) *Suppose Assumptions 2.1, 2.2 and 2.5. Then* $\sigma_{\mathrm{ess}}(H) = [E + m, \infty)$ *for all* $M \geq 0$.

This theorem provides that H has a ground state for all $M \geq 0$. We summarize this in the corollary below.

Corollary 2.9 (Existence of the ground state) *Suppose Assumptions 2.1, 2.2 and 2.5. Then H has the unique ground state Φ_M for all $M \geq 0$, and*

$$\|\Phi_M(x)\|_{\mathscr{F}} \leq Ce^{-c|x|}$$

with some constants c and C.

Proof: By Theorem 2.8 the lowest eigenvalue of H is discrete. Then the ground state of H exists. The uniqueness of the ground state is shown in [Hir13, Corollary 6.2] and and spatial exponential decay of the ground state in [Hir13, Theorem 5.12]. ■

3 Localization

The main result in this section is to estimate the asymptotic behaviour of a commutator, which is given in Lemma 3.3.

3.1 Commutator estimates

We show a fundamental lemma.

Lemma 3.1 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then $\lim_{M \downarrow 0}(H - z)^{-1} = (H_0 - z)^{-1}$ in the uniform topology. In particular $\lim_{M \downarrow 0} \chi(H) = \chi(H_0)$ in the uniform topology for all $\chi \in C_{\mathrm{c}}^{\infty}(\mathbb{R})$ and $\lim_{M \downarrow 0} E = E_0$.*

Proof: Let $\Psi \in \mathscr{H}$ and we set $\Phi = (H - z)^{-1}\Psi$. Let E_λ be the spectral projection associated with the self-adjoint operator $|p - A|$. We have

$$\|(H - z)^{-1}\Psi - (H_0 - z)^{-1}\Psi\|^2 \leq \frac{1}{|\Im z|^2} \int_0^\infty \left(\frac{M^2}{\lambda + \sqrt{\lambda^2 + M^2}} \right)^2 d\|E_\lambda \Phi\|^2 \leq \frac{M^2\|\Psi\|^2}{|\Im z|^2}.$$

Then $\lim_{M\downarrow 0}(H-z)^{-1}=(H_0-z)^{-1}$ is obtained, and $E\to E_0$ follows. By the Helffer-Sjöstrand formula [HS89] we have

$$\chi(H)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}(z-H)^{-1}\mathrm{d}z\mathrm{d}\bar{z}. \tag{3.1}$$

Here $\mathrm{d}z\mathrm{d}\bar{z}=-2i\mathrm{d}x\mathrm{d}y$, $\dfrac{\partial}{\partial\bar{z}}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}\right)$ and $\tilde{\chi}$ is an almost analytic extension of χ, which satisfies that

$$\tilde{\chi}(x)=\chi(x),\quad x\in\mathbb{R}, \tag{3.2}$$
$$\tilde{\chi}\in C_{\mathrm{c}}^{\infty}(\mathbb{C}), \tag{3.3}$$
$$\left|\frac{\tilde{\chi}(z)}{\partial\bar{z}}\right|\le C_n|\Im z|^n,\quad n\in\mathbb{N}. \tag{3.4}$$

Then

$$\|\chi(H)-\chi(H_0)\|\le\frac{1}{\pi}\int_{\mathbb{C}}\left\|\frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}((z-H)^{-1}-(z-H_0)^{-1})\right\|\mathrm{d}x\mathrm{d}y.$$

We see that for all $z\in\operatorname{supp}\tilde{\chi}\setminus\mathbb{R}$, $\|\frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}(z-H)^{-1}\|\le C_1|\Im z|$ and $\lim_{M\downarrow 0}(z-H)^{-1}=(z-H_0)^{-1}$ uniformly. Then by the Lebesgue dominated convergence theorem $\lim_{M\downarrow 0}\chi(H)=\chi(H_0)$ is obtained. ∎

Lemma 3.2 *Suppose Assumptions 2.1, 2.2 and 2.5. Fix an arbitrary* $M_0>0$. *Then there exists a constant* $C=C(M_0)>0$ *such that for all* $\Psi\in D(H)$ *and* $0\le M\le M_0$,

$$\|(N+\mathbb{1})\Psi\|\le\frac{C}{m}\left(\|H\Psi\|+\|\Psi\|\right).$$

Proof: For all $\Psi\in D(H)(\subset D(\mathrm{H_f}))$ we have $\|N\Psi\|\le\frac{1}{m}\|\mathrm{H_f}\Psi\|$. Then the corollary follows from the bound $\||p|\Psi\|^2+\|V\Psi\|^2+\|\mathrm{H_f}\Psi\|^2\le C\|(H+\mathbb{1})\Psi\|^2$ shown in (2.8). ∎

We shall divide the configuration space W as $W=W_0\oplus W_\infty$, where W_0 denotes the set of functions supported on small momenta, and W_∞ on large momenta. Since $\mathscr{F}=\mathscr{F}(W_0\oplus W_\infty)\cong\mathscr{F}(W_0)\otimes\mathscr{F}(W_\infty)$, we have $\mathscr{H}\cong(L^2(\mathbb{R}^d)\otimes\mathscr{F}(W_0))\otimes\mathscr{F}(W_\infty)$. Thus we introduce the extended Hamiltonian $\widehat{H}$ acting in the extended Hilbert space

$$\widehat{\mathscr{H}}=\mathscr{H}\otimes\mathscr{F} \tag{3.5}$$

by

$$\widehat{H}=H\otimes\mathbb{1}_{\mathscr{F}}+\mathbb{1}_{\mathscr{H}}\otimes\mathrm{H_f}. \tag{3.6}$$

Under Assumptions 2.1, 2.2 and 2.5 we can also see that $\widehat{H}$ is essentially self-adjoint on $D(H\otimes\mathbb{1}_{\mathscr{F}})\cap D(\mathbb{1}_{\mathscr{H}}\otimes\mathrm{H_f})$. We denote the unique self-adjoint extension by the same symbol $\widehat{H}$. We set $j=(j_0,j_\infty)\in C^\infty(\mathbb{R}^d;\mathbb{R}_+)\times C^\infty(\mathbb{R}^d;\mathbb{R}_+)$, where j_0 and j_∞ satisfy that

$$j_0(k)=\begin{cases}1 & \text{if } |k|\le 1\\ 0 & \text{if } |k|\ge 2\end{cases} \quad\text{and}\quad j_0^2(k)+j_\infty^2(k)=1. \tag{3.7}$$

We also define the bounded operator $\hat{j}_R: W\to W\oplus W$ for $R>0$ by

$$\hat{j}_R f=\hat{j}_{0,R}f\oplus\hat{j}_{\infty,R}f=j_0(\frac{-i}{R}\nabla_k)f\oplus j_\infty(\frac{-i}{R}\nabla_k)f. \tag{3.8}$$

Let us also define the isometry $\mathrm{I}_R:\mathscr{F}\to\mathscr{F}\otimes\mathscr{F}$ by

$$\mathrm{I}_R\Omega=\Omega_{\mathscr{F}\otimes\mathscr{F}}, \tag{3.9}$$

$$\mathrm{I}_R\prod_{i=1}^n a^\dagger(h_i)\Omega=\prod_{i=1}^n(a_0^\dagger(\hat{j}_{0,R}h_i)+a_\infty^\dagger(\hat{j}_{\infty,R}h_i))\Omega_{\mathscr{F}\otimes\mathscr{F}}, \tag{3.10}$$

where $a_0^\dagger(\hat{j}_{0,R}f)=a^\dagger(\hat{j}_{0,R}f)\otimes\mathbb{1}$, $a_\infty^\dagger(\hat{j}_{\infty,R}f)=\mathbb{1}\otimes a^\dagger(\hat{j}_{\infty,R}f)$ and $\Omega_{\mathscr{F}\times\mathscr{F}}=\Omega\otimes\Omega$. Let $\chi\in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp}\chi\subset(-\infty,E+m)$. We shall show that $\chi(H)$ is a compact operator. Note that $\mathrm{I}_R^*\mathrm{I}_R=\mathbb{1}$. Then the key identity is

$$\chi(H)-\mathrm{I}_R^*D_R=\mathrm{I}_R^*\chi(\widehat{H})\mathrm{I}_R, \tag{3.11}$$

where the remainder term is $D_R=\mathrm{I}_R\chi(H)-\chi(\widehat{H})\mathrm{I}_R$. Note that the first term of the right-hand side of (3.11), $\mathrm{I}_R^*\chi(\widehat{H})\mathrm{I}_R$, is compact. We shall show that the remainder term $\mathrm{I}_R^*D_R$ uniformly converges to zero as $R\to\infty$. Hence we derive that $\chi(H)$ is compact. So we estimate the commutator $\chi(\widehat{H})\mathrm{I}_R-\mathrm{I}_R\chi(H)$ for an arbitrary $\chi\in C_c(\mathbb{R})$.

Lemma 3.3 *Suppose Assumptions 2.1, 2.2 and 2.5 and $M>0$. Let $\chi\in C_c^\infty(\mathbb{R})$. Then*

$$\lim_{R\to\infty}\left\|\chi(\widehat{H})\mathrm{I}_R-\mathrm{I}_R\chi(H)\right\|=0. \tag{3.12}$$

We prepare several lemmas to prove Lemma 3.3. By the Helffer-Sjöstrand formula, we have

$$\chi(\widehat{H})\mathrm{I}_R-\mathrm{I}_R\chi(H)=\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}(z-\widehat{H})^{-1}(\widehat{H}\mathrm{I}_R-\mathrm{I}_RH)(z-H)^{-1}\mathrm{d}z\mathrm{d}\bar{z}. \tag{3.13}$$

Here $\tilde{\chi}$ satisfies (3.2)-(3.4). We set $T=(p-A)^2+M^2$ and $\widehat{T}=T\otimes\mathbb{1}_{\mathscr{F}}$. Note that we have

$$\widehat{H}\mathrm{I}_R-\mathrm{I}_RH=(\mathrm{H_f}\otimes\mathbb{1}_{\mathscr{F}}+\mathbb{1}_{\mathscr{H}}\otimes\mathrm{H_f})\mathrm{I}_R-\mathrm{I}_R\mathrm{H_f}+\widehat{T}^{1/2}\mathrm{I}_R-\mathrm{I}_RT^{1/2}. \tag{3.14}$$

Let

$$B_R=\left\|\left((\mathrm{H_f}\otimes\mathbb{1}_{\mathscr{F}}+\mathbb{1}_{\mathscr{H}}\otimes\mathrm{H_f})\mathrm{I}_R-\mathrm{I}_R\mathrm{H_f}\right)(N+1)^{-1}\right\|.$$

By Assumption 2.2 the first two terms of the right-hand side of (3.14) can be estimated as follows.

Lemma 3.4 *Suppose Assumptions 2.1, 2.2 and 2.5. Then* $\lim_{R\to\infty} B_R = 0$*, and for all* $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\left\|(\widehat{H}-z)^{-1}\left((\mathrm{H_f}\otimes 1\!\!1_{\mathscr{F}} + 1\!\!1_{\mathscr{H}}\otimes \mathrm{H_f})\mathrm{I}_R - \mathrm{I}_R\mathrm{H_f}\right)(H-z)^{-1}\right\| \le \frac{C}{m}\left(1+\frac{|z|+1}{|\Im z|}\right)^2 B_R. \tag{3.15}$$

Proof: See e.g., [DG99, Proof of Lemma 3.4]. ■

We set

$$G_x(k) = \hat{\varphi}(k)e^{-ik\cdot x}/\sqrt{\omega(k)} \in L^2(\mathbb{R}^d_k). \tag{3.16}$$

Let $\# = 0$ or ∞. Then the inverse Fourier transform of $\hat{j}_{\#R}G_x$ is given by $j_\#(\cdot/R)\tilde{\varphi}(\cdot - x)$, where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}/\sqrt{\omega}$. $A_\#(x)$ denotes $A(x)$ with cutoff function G_x replaced by $\hat{j}_{R\#}G_x$, and we set

$$A_0 = \int_{\mathbb{R}^d}^{\oplus} A_0(x)\mathrm{d}x, \quad A_\infty = \int_{\mathbb{R}^d}^{\oplus} A_\infty(x)\mathrm{d}x.$$

$A_0 \otimes 1\!\!1_{\mathscr{F}}$ and $1\!\!1_{\mathscr{F}} \otimes A_\infty$ are self-adjoint operators in $\widehat{\mathscr{H}}$. We set

$$\widehat{S} = (p - A_0\otimes 1\!\!1 - 1\!\!1\otimes A_\infty)^2 + M^2.$$

Formally $A_0 \to A$ and $A_\infty \to 0$ as $R\to\infty$, then $\widehat{S} \to \widehat{T}$ as $R \to \infty$. Let $\widehat{N} = N\otimes 1\!\!1 + 1\!\!1\otimes N$ be the number operator on $\mathscr{F}\otimes\mathscr{F}$, and set $C^\infty(\widehat{N}) = \cap_{k=1}^\infty D(\widehat{N}^k)$.

Lemma 3.5 *Suppose Assumptions 2.1, 2.2 and 2.5. Then* $\widehat{S}$ *is essentially self-adjoint on* $D(p^2)\cap C^\infty(\widehat{N})$.

Proof: This follows from the similar method for the proof of the essential self-adjointness of $(p-A)^2$ in Proposition 2.3. Let $(B_t)_{t\ge 0}$ be the d-dimensional Brownian motion defined on the Wiener space, and $\mathbb{E}^x[\cdots]$ denotes the expectation with respect to the Wiener measure starting at $x\in\mathbb{R}^d$. Let

$$\mathrm{K}_\# = \oplus_{i=1}^d \int_0^t j_\#(\cdot/R)\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^i, \quad \# = 0, \infty.$$

Define the quadratic form

$$Q : \mathscr{H}\times\mathscr{H} \ni (\Phi,\Psi) \mapsto \int_{\mathbb{R}^d}\mathrm{d}x\mathbb{E}^x\left[(\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)}\Psi(B_t))\right] \in \mathbb{C}.$$

See Appendix for the detail of functional integrations. Then we can see that there exists a strongly continuous one-parameter semigroup S_t such that $Q(\Phi,\Psi) = (\Phi, S_t\Psi)$ and furthermore the generator of S_t (denoted by K) satisfies that $K\Psi = \widehat{S}\Psi$ for

$\Psi \in D(p^2) \cap C^\infty(\widehat{N})$. We can also see that $e^{-tK} D(p^2) \cap C^\infty(\widehat{N}) \subset D(p^2) \cap C^\infty(\widehat{N})$. Thus $D(p^2) \cap C^\infty(\widehat{N})$ is invariant domain for e^{-tK}, and K is essentially self-adjoint on $D(p^2) \cap C^\infty(\widehat{N})$ and then so is $\widehat{S}$. ■

We denote the self-adjoint extension of $\widehat{S}\lceil_{D(p^2)\cap C^\infty(\widehat{N})}$ by the same symbol $\widehat{S}$ in what follows.

Lemma 3.6 *It follows that* $\mathrm{I}_R T \subset \widehat{S}\mathrm{I}_R$*, i.e., the intertwining property* $\mathrm{I}_R T = \widehat{S}\mathrm{I}_R$ *holds on* $D(T)$.

Proof: Since the intertwining property $\mathrm{I}_R e^{-i\mathscr{A}(\mathrm{K})} = e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)}\mathrm{I}_R$ holds by the functional integration we see that

$$\begin{aligned}(\mathrm{I}_R^*\Phi, e^{-tT}\Psi) &= \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\mathrm{I}_R^*\Phi(B_0), e^{-i\mathscr{A}(\mathrm{K})}\Psi(B_t))\right] \\ &= \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)}\mathrm{I}_R\Psi(B_t))\right] = (\Phi, e^{-t\widehat{S}}\mathrm{I}_R\Psi),\end{aligned}$$

where $K = \oplus_{i=1}^d \int_0^t \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)\mathrm{d}B_s^i$. Take the derivative at $t=0$ for $\Psi \in D(T)$. Then the lemma follows. ■

Lemma 3.7 *It follows that*

$$\begin{aligned}&\left\|(\widehat{H}-z)^{-1}\left(\widehat{T}^{1/2}\mathrm{I}_R - \mathrm{I}_R T^{1/2}\right)(H-z)^{-1}\Psi\right\| \\ &\leq \frac{2}{\pi}\int_0^\infty \frac{dw}{\sqrt{w}}\|(\widehat{H}-z)^{-1}\{(\widehat{T}+w)^{-1}\widehat{T} - (\widehat{S}+w)^{-1}\widehat{S}\}\mathrm{I}_R(H-z)^{-1}\Psi\| \qquad (3.17)\end{aligned}$$

for all $\Psi \in D(T)$ *and* $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof: Using $K^{1/2} = (2/\pi)\int_0^\infty K(K+w)^{-1}/\sqrt{w}\mathrm{d}w$ for strictly positive self-adjoint operator K and $\widehat{T}\mathrm{I}_R - \mathrm{I}_R T = (\widehat{T}-\widehat{S})\mathrm{I}_R$ on $D(T)$ by Lemma 3.6, we can derive (3.17). ■

We shall estimate the integrand $\|(\widehat{H}-z)^{-1}\{(\widehat{T}+w)^{-1}\widehat{T}-(\widehat{S}+w)^{-1}\widehat{S}\}\mathrm{I}_R(H-z)^{-1}\Psi\|$ of (3.17).

Lemma 3.8 $\widehat{T}^{1/2}(\widehat{H}-z)^{-1}$, $\widehat{S}^{1/2}(\widehat{H}-z)^{-1}$ *and* $T(H-z)^{-1}$ *are bounded for all* $z \in \mathbb{C}\setminus\mathbb{R}$.

Proof: For all $\Psi \in D(\widehat{H})$ we have

$$\|\widehat{T}^{1/2}\Psi\| \leq \sum_{\mu=1}^d \|p_\mu\Psi\| + d\sqrt{2}\left\|\frac{\hat{\varphi}}{\sqrt{\omega}}\right\| \|(N+\mathbb{1})^{1/2}\Psi\| + M\|\Psi\| \leq C\|(\widehat{H}+\mathbb{1})\Psi\|. \quad (3.18)$$

Then $\widehat{T}^{1/2}(\widehat{H}-z)^{-1}$ is bounded. The boundedness of $\widehat{S}^{1/2}(\widehat{H}-z)^{-1}$ and $T(H-z)^{-1}$ are similarly proven. Then the lemma follows. ■

Next we estimate $\widehat{T}^{1/2}\mathrm{I}_R(H-z)^{-1}$.

Lemma 3.9 *$\widehat{T}^{1/2}\mathrm{I}_R(H-z)^{-1}$ is bounded, and there exists $C>0$ such that*

$$\sup_{R>0}\|\widehat{T}^{1/2}\mathrm{I}_R(H-z)^{-1}\| < C\left(1+\frac{1+|z|}{|\Im z|}\right) \tag{3.19}$$

for all $z\in\mathbb{C}\setminus\mathbb{R}$.

Proof: Since $\mathscr{H}_{\mathrm{fin}}$ is a core of H, for all $\Psi\in D(H)$ there exists a sequence $\{\Psi_j\}$ such that $\Psi_j\in\mathscr{H}_{\mathrm{fin}}$, $\lim_{j\to\infty}\Psi_j=(H-z)^{-1}\Psi$ and $\lim_{j\to\infty}H\Psi_j=H(H-z)^{-1}\Psi$. Note that $\mathrm{I}_R\Psi_j\in D(\widehat{T}^{1/2})$. For all $\Phi\in D(\widehat{T}^{1/2})$ we have

$$\begin{aligned}
&|(\widehat{T}^{1/2}\Phi,\mathrm{I}_R(H-z)^{-1}\Psi)| = \lim_{j\to\infty}|(\widehat{T}^{1/2}\Phi,\mathrm{I}_R\Psi_j)|\\
&\leq \lim_{j\to\infty}\|\Phi\|\left(\sum_{\mu=1}^d\|p_\mu\Psi_j\|+d\sqrt{2}\left\|\frac{\hat\varphi}{\sqrt{\omega}}\right\|\|(N^{1/2}\otimes\mathbb{1}+\mathbb{1}\otimes N^{1/2})\mathrm{I}_R\Psi_j\|+M\|\Psi_j\|\right)\\
&=\|\Phi\|\left(\sum_{\mu=1}^d\|p_\mu(H-z)^{-1}\Psi\|+d\sqrt{2}\left\|\frac{\hat\varphi}{\sqrt{\omega}}\right\|\|N^{1/2}(H-z)^{-1}\Psi\|+M\|(H-z)^{-1}\Psi\|\right)\\
&\leq C\|\Phi\|\|(H+\mathbb{1})(H-z)^{-1}\Psi\| = C\|\Phi\|\|\mathbb{1}+(\mathbb{1}-z)(H-z)^{-1}\Psi\|\\
&\leq C\left(1+\frac{1+|z|}{|\Im z|}\right)\|\Phi\|\|\Psi\|,
\end{aligned}$$

where we used $(N^{1/2}\otimes\mathbb{1}+\mathbb{1}\otimes N^{1/2})\mathrm{I}_R=\mathrm{I}_RN^{1/2}$. Thus $\widehat{T}^{1/2}\mathrm{I}_R(H-z)^{-1}$ is bounded uniformly in R. ∎

Lemma 3.10 *For all $\epsilon>0$ and $z\in\mathbb{C}\setminus\mathbb{R}$ there exists C_R such that $\lim_{R\to\infty}C_R=0$ and*

$$\|(\widehat{H}-z)^{-1}(\widehat{T}^{1/2}\mathrm{I}_R-\mathrm{I}_RT^{1/2})(H-z)^{-1}\|\leq(\epsilon+C_R)\left(1+\frac{1+|z|}{|\Im z|}\right)^2. \tag{3.20}$$

Proof: Since $\|(\widehat{T}+w)^{-1}\|\leq\frac{1}{M^2+w}$, $\|(\widehat{S}+w)^{-1}\|\leq\frac{1}{M^2+w}$, the integrand of (3.17) can be estimated as

$$\begin{aligned}
&\|(\widehat{H}-z)^{-1}\{(\widehat{T}+w)^{-1}\widehat{T}-(\widehat{S}+w)^{-1}\widehat{S}\}\mathrm{I}_R(H-z)^{-1}\Psi\|\\
&\leq\|\widehat{T}^{1/2}(\widehat{H}-z)^{-1}\|\,\|(\widehat{T}+w)^{-1}\|\,\|\widehat{T}^{1/2}\mathrm{I}_R(H-z)^{-1}\|\\
&\quad+\|\widehat{S}^{1/2}(\widehat{H}-z)^{-1}\|\,\|(\widehat{S}+w)^{-1}\|\,\|\widehat{S}^{1/2}\mathrm{I}_R(H-z)^{-1}\|\\
&\leq\frac{C}{M^2+w}\left(1+\frac{1+|z|}{|\Im z|}\right)^2.
\end{aligned}\tag{3.21}$$

with some constant C independent of R and M. Take an arbitrary $\epsilon > 0$. Then there exists a closed interval $[\delta, L] \subset (0, \infty)$ such that

$$
\begin{aligned}
&\|(\widehat{H}-z)^{-1}(\widehat{T}^{1/2}\mathrm{I}_R - \mathrm{I}_R T^{1/2})(H-z)^{-1}\Psi\| \\
&\leq \ \epsilon\left(1+\frac{1+|z|}{|\Im z|}\right)^2\|\Psi\| \\
&\quad + \int_{[\delta,L]}\frac{dw}{\sqrt{w}}\|(\widehat{H}-z)^{-1}\{(\widehat{T}+w)^{-1}\widehat{T}-(\widehat{S}+w)^{-1}\widehat{S}\}\mathrm{I}_R(H-z)^{-1}\Psi\| \\
&= \ \epsilon\left(1+\frac{1+|z|}{|\Im z|}\right)^2\|\Psi\| + \int_{[\delta,L]}\frac{dw}{\sqrt{w}}\{\|(\widehat{H}-z)^{-1}(\widehat{T}-\widehat{S})(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\Psi\| \\
&\quad +\|(\widehat{H}-z)^{-1}(\mathbb{1}-w(\widehat{T}+w)^{-1})(\widehat{S}-\widehat{T})(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\Psi\|\}.
\end{aligned}
$$

Note that

$$
\widehat{T}-\widehat{S} = -2p\cdot(\xi-\eta)+(A\otimes\mathbb{1}_{\mathscr{F}})\cdot(\xi-\eta)+(\xi-\eta)\cdot(A_0\otimes\mathbb{1}_{\mathscr{F}}+\eta),
$$

where $\xi = A\otimes\mathbb{1}_{\mathscr{F}} - A_0\otimes\mathbb{1}_{\mathscr{F}}$ and $\eta = \mathbb{1}_{\mathscr{F}}\otimes A_\infty$. It is shown in Subsection 3.2 below that

$$
\||p|(\widehat{T}+w)^{-1}(\widehat{H}-z)^{-1}\| < C\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.22}
$$

$$
\||A\otimes\mathbb{1}_{\mathscr{F}}|(\widehat{T}+w)^{-1}(\widehat{H}-z)^{-1}\| < C\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.23}
$$

$$
\||\xi|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| < C_R\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.24}
$$

$$
\||\eta|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| < C_R\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.25}
$$

$$
\||A_0\otimes\mathbb{1}_{\mathscr{F}}|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| < C\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.26}
$$

$$
\||\xi|(\widehat{T}+w)^{-1}(\widehat{H}-z)^{-1}\| < C_R\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.27}
$$

$$
\||\eta|(\widehat{T}+w)^{-1}(\widehat{H}-z)^{-1}\| < C_R\left(1+\frac{1+|z|}{|\Im z|}\right). \tag{3.28}
$$

Here C_R is a constant such that $\lim_{R\to\infty} C_R = 0$. Then we have

$$
\begin{aligned}
&\|(\widehat{H}-z)^{-1}(\widehat{T}-\widehat{S})(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&=\|(\widehat{H}-z)^{-1}(-2p\cdot(\xi-\eta)+\xi\cdot(\xi-\eta)+(\xi-\eta)\cdot\eta)(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&\le (2\||p|(\widehat{H}-z)^{-1}\|+\||\xi|(\widehat{H}-z)^{-1}\|)\cdot\||\xi-\eta|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&+\||\xi-\eta|(\widehat{H}-z)^{-1}\|\cdot\||\eta|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&\le aC_R\left(1+\frac{1+|z|}{|\Im z|}\right)^2
\end{aligned}
$$

with some constant a. Similarly we see that

$$
\begin{aligned}
&\|(\widehat{H}-z)^{-1}(\mathbb{1}-w(\widehat{T}+w)^{-1})(\widehat{S}-\widehat{T})(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\Psi\|\} \\
&\le (2\||p|(\mathbb{1}-w(\widehat{T}+w)^{-1})(\widehat{H}-z)^{-1}\| \\
&+\||\xi|(\mathbb{1}-w(\widehat{T}+w)^{-1})(\widehat{H}-z)^{-1}\|)\cdot\||\xi-\eta|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&+\||\xi-\eta|(\mathbb{1}-w(\widehat{T}+w)^{-1})(\widehat{H}-z)^{-1}\|\cdot\||\eta|(\widehat{S}+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \\
&\le b(1+w)C_R\left(1+\frac{1+|z|}{|\Im z|}\right)^2
\end{aligned}
$$

with some constant b. Together with them we obtain that

$$
\begin{aligned}
&\|(\widehat{H}-z)^{-1}(\widehat{T}^{1/2}\mathrm{I}_R-\mathrm{I}_R T^{1/2})(H-z)^{-1}\Psi\| \\
&\le\left(\epsilon+C_R\int_{[\delta,L]}\frac{a+(1+w)b}{\sqrt{w}}dw\right)\left(1+\frac{1+|z|}{|\Im z|}\right)^2\|\Psi\|.
\end{aligned}
$$

Since $C_R\to 0$ as $R\to\infty$, we obtain the lemma. ■

We give a proof of Lemma 3.3.

Proof of Lemma 3.3: Let $c=2\int_{\mathbb{C}}|\frac{\partial\tilde{\chi}(z)}{\partial\bar{z}}|\left(1+\frac{1+|z|}{|\Im z|}\right)^2 dxdy$. By (3.13), Lemmas 3.4 and 3.10 we have

$$
\limsup_{R\to\infty}\|\chi(\widehat{H})\mathrm{I}_R-\mathrm{I}_R\chi(H)\|\le c\lim_{R\to\infty}(B_R+\epsilon+C_R)=c\epsilon. \tag{3.29}
$$

Since $\epsilon>0$ is arbitrary, (3.12) is obtained. Then the lemma follows. ■

3.2 Proof of (3.22)-(3.28)

It remains to show sequence of inequalities (3.22)-(3.28). We prove these inequalities by functional integrations. Let $A_\#$ denote $A\otimes\mathbb{1}_{\mathscr{F}}$ or $A_0\otimes\mathbb{1}_{\mathscr{F}}+\mathbb{1}_{\mathscr{F}}\otimes A_\infty$. The functional integration of the semigroup generated by $\frac{1}{2}(p-A_\#)^2+M^2+w$ is given in Appendix.

It can be then shown that

$$
\begin{aligned}
&(\Phi, e^{-t(\frac{1}{2}(p-A_\#)^2)}\Phi)_{\widehat{\mathscr{H}}} \\
&= \begin{cases} \displaystyle\int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)}\Psi(B_t)) \right], & A_\# = A_0 \otimes 1\!\!1_{\mathscr{F}} + 1\!\!1_{\mathscr{F}} \otimes A_\infty, \\ \displaystyle\int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K})}\Psi(B_t)) \right], & A_\# = A \otimes 1\!\!1_{\mathscr{F}} \end{cases}
\end{aligned}
$$

Lemma 3.11 *Let $M > 0$. Then $(\widehat{N} + 1\!\!1)((p - A_\#)^2 + M^2 + w)^{-1}(\widehat{N} + 1\!\!1)^{-1}$ is bounded uniformly in $w \in [0, \infty)$.*

Proof: We give the proof of the lemma in the case of $A_\# = A_0 \otimes 1\!\!1_{\mathscr{F}} + 1\!\!1_{\mathscr{F}} \otimes A_\infty$. In another case, we can prove it in a similar manner. Let $N_1 = N \otimes 1\!\!1_{\mathscr{F}}$ and $N_2 = 1\!\!1_{\mathscr{F}} \otimes N$. Then $\widehat{N} = N_1 + N_2$. We see that

$$
((N_1+N_2)\Phi, ((p-A_\#)^2+M^2+w)^{-1}\Psi) = 2\int_0^\infty e^{-t\frac{1}{2}(M^2+w)}((N_1+N_2)\Phi, e^{-t(p-A_\#)^2}\Psi)\mathrm{d}t.
$$

We have

$$
\begin{aligned}
&((N_1 + N_2)\Phi, ((p - A_\#)^2 + M^2 + w)^{-1}\Psi) \\
&= 2\int_0^\infty \mathrm{d}t e^{-t\frac{1}{2}(M^2+w)} \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[((N_1 + N_2)\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)}\Psi(B_t)) \right] \\
&= 2\int_0^\infty \mathrm{d}t e^{-t\frac{1}{2}(M^2+w)} \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_0)-i\mathscr{A}_2(\mathrm{K}_\infty)} \sum_{j=1,2} e^{i\mathscr{A}_j(\mathrm{K}_j)} N_j e^{-i\mathscr{A}_j(\mathrm{K}_j)}\Psi(B_t)) \right],
\end{aligned}
$$

where $\mathrm{K}_1 = \mathrm{K}_0$ and $\mathrm{K}_2 = \mathrm{K}_\infty$. We have

$$
e^{i\mathscr{A}_j(\mathrm{K}_j)} N_j e^{-i\mathscr{A}_j(\mathrm{K}_j)} = N_j - i\Pi(\mathrm{K}_j) - \frac{1}{2}\mathrm{q}(\mathrm{K}_j, \mathrm{K}_j).
$$

Here $\Pi_j(\mathrm{K}_j) = i[N_j, \mathscr{A}_j(\mathrm{K}_j)]$. We know that $\|\Pi_j(\mathrm{K}_j)\Phi\| \le \|\mathrm{K}_j\|\|(N_j + 1\!\!1)^{1/2}\Phi\|$ and $\mathrm{q}(\mathrm{K}_j, \mathrm{K}_j) \le \|\mathrm{K}_j\|^2$. Hence by Lemma 3.12 below, there exist constants c_1 and c_2 such that

$$
\left| \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), \Pi_j(\mathrm{K}_j)\Psi(B_t)) \right] \right| \le c_j \|\Phi\|\|(N_j + 1\!\!1)^{1/2}\Psi\|\sqrt{t}, \quad j = 1, 2.
$$

We can also see that $|(\Phi(B_0), \frac{1}{2}\mathrm{q}(\mathrm{K}_j, \mathrm{K}_j)\Psi(B_t))| \le \|\Phi(x)\|\|\mathrm{K}_j\|^2\Psi(B_t)\|$. Hence by Lemma 3.12 again there exist constants d_j such that

$$
\begin{aligned}
\left| \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[(\Phi(B_0), \frac{1}{2}\mathrm{q}(\mathrm{K}_j, \mathrm{K}_j)\Psi(B_t)) \right] \right| &\le \int_{\mathbb{R}^d} \mathrm{d}x \|\Phi(x)\| \mathbb{E}^x [\|\mathrm{K}_j\|^4]^{1/2} \mathbb{E}^x [\|\Psi(B_t)\|^2]^{1/2} \\
&\le d_j \|\hat{\varphi}/\sqrt{\omega}\|^2 \|\Phi\|\|\Psi\| t, \quad j = 1, 2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&|(\widehat{N}\Phi, ((p-A_\#)^2+M^2+w)^{-1}\Psi)| \\
&= 2\int_0^\infty e^{-t\frac{1}{2}(M^2+w)}\mathrm{d}t\|\Phi\|(\sqrt{t}\sum_{j=1,2} c_j\|(N_j+1\!\!1)^{1/2}\Psi\|+td_j\|\Psi\|) \le C\|\Phi\|\|(\widehat{N}+1\!\!1)\Psi\|
\end{aligned}
$$

with some constant C independent of w. Hence $(\widehat{N}+1\!\!1)((p-A_\#)^2+M^2+w)^{-1}(\widehat{N}+1\!\!1)^{-1}$ is bounded uniformly in w. ■

Lemma 3.12 *There exist constants c_1 and c_2 such that $\mathbb{E}^x[\|\mathrm{K}_j\|^2] \le tc_1\|\hat{\varphi}/\sqrt{\omega}\|^2$ and $\mathbb{E}^x[\|\mathrm{K}_j\|^4] \le t^2c_2\|\hat{\varphi}/\sqrt{\omega}\|^4$.*

Proof: See [Hir00b, Theorem 4.6] and [LHB11, Lemma 7.21]. ■

Lemma 3.13 *Let $0<M$ and $\chi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$. Then for all $w\in[0,\infty)$*

$$\||p|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\| \le C_M\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.30}$$

$$\||A\otimes 1\!\!1_{\mathscr{F}}|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\| \le C_M\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.31}$$

$$\||A_R|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\| \le C_RC_M\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.32}$$

$$\||p|(|p-A_\#|^2+M^2+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \le C_M\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.33}$$

$$\||A_0\otimes 1\!\!1_{\mathscr{F}}|(|p-A_\#|^2+M^2+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \le C_M\left(1+\frac{1+|z|}{|\Im z|}\right), \tag{3.34}$$

$$\||A_R|(|p-A_\#|^2+M^2+w)^{-1}\mathrm{I}_R(H-z)^{-1}\| \le C_RC_M\left(1+\frac{1+|z|}{|\Im z|}\right) \tag{3.35}$$

with $C_R\in o(R^0)$ $(R\to\infty)$ and some positive constant C_M.

Proof: First we prove (3.30). For fixed $A_\# = A\otimes 1\!\!1$ or $A_0\otimes 1\!\!1 + 1\!\!1\otimes A_\infty$ we have

$$
\begin{aligned}
&\||p|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\|^2 \\
&\le 2\{\||p-A_\#|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\|^2 \\
&+\||A_\#|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\|^2\}.
\end{aligned} \tag{3.36}
$$

By Lemma 2.3 we have

$$
\begin{aligned}
&\||A_\#|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\| \\
&\le 2\|G\|\|(N+1)(|p-A_\#|^2+M^2+w)^{-1}(N+1)^{-1}\|\|(N+1)(\widehat{H}-z)^{-1}\| \\
&\le C_M'\left(1+\frac{1+|z|}{|\Im z|}\right)
\end{aligned} \tag{3.37}
$$

with some constant C'_M. Together with (3.36) and (3.37) we obtain that

$$\||p|(|p-A_\#|^2+M^2+w)^{-1}(\widehat{H}-z)^{-1}\| \le \sqrt{2}\left(\frac{1}{\sqrt{M^2+w}}+C'_M\right)\left(1+\frac{1+|z|}{|\Im z|}\right). \tag{3.38}$$

Then (3.30) is obtained. Next we prove (3.32). For an arbitrary $\epsilon > 0$ there exists a normalized vector Ψ_R such that

$$\|A_{R,\mu}(\widehat{N}+\mathbb{1})^{-1/2}\| \le \|A_{R,\mu}(\widehat{N}+\mathbb{1})^{-1/2}\Psi_R\| + \frac{\epsilon}{3}. \tag{3.39}$$

Let $G_x \in L^2(\mathbb{R}^d)$ be in (3.16). Notice that $\|G_x\|$ is independent of x. There exists $L > 0$ such that $\Psi_{R,L} = \chi_{\{|x|\le L\}}\Psi_R$ satisfies that $\|\Psi_R - \Psi_{R,L}\|_{\mathscr{H}} \le \frac{\epsilon}{3(\|G_x\|+1)}$. Then we have

$$\|A_{R,\mu}(\widehat{N}+\mathbb{1})^{-1/2}\| \le \|\|j_R(-i\nabla)G\|\Psi_{R,L}\| + \frac{2\epsilon}{3}. \tag{3.40}$$

Here j_R stands for $\mathbb{1} - j_0(k/R)$ or $j_\infty(k/R)$.

$$\begin{aligned} \|\|j_R(-i\nabla)G\|_W\Psi_{R,L}\|^2_{\mathscr{H}} &= \int_{|x|\le L} \left\|\|j_R\hat{G}_x\|_W\Psi_R(x)\right\|^2_{\mathscr{F}} \mathrm{d}x \\ &\le \int_{|x|\le L} \left\|\|\chi_{\{|k|\ge R\}}\hat{G}_x\|_W\Psi_R(x)\right\|^2_{\mathscr{F}} \mathrm{d}x. \end{aligned} \tag{3.41}$$

Let $f_R(x) = \|\chi_{\{|k|\ge R\}}\hat{G}_x\|_W$. f_R is continuous, and for each $x \in \mathbb{R}^d$, it monotonically converges to 0 as $R \to \infty$. Then f_R converges to 0 uniformly on any compact set by Dini's theorem. Thus we see that $\lim_{R\to\infty}\sup_{|x|\le L}\|\chi_{\{|k|\ge R\}}\hat{G}_x\| = 0$. Since Ψ_R is normalized, the right-hand side of (3.40) converges to 0 as $R \to \infty$. Thus there exists some $R_0 > 0$ such that for all $R > R_0$,

$$\|\|j_R(-i\nabla)G\|_W\Psi_{R,L}\|^2_{\mathscr{H}} < \frac{\epsilon}{3}. \tag{3.42}$$

By (3.40) and (3.42) we see that $\|A_{R,\mu}(\widehat{N}+\mathbb{1})^{-1/2}\| < \epsilon$ for all $R > R_0$. Then

$$\lim_{R\to\infty}\|A_{R,\mu}(\widehat{N}+\mathbb{1})^{-1/2}\| = 0. \tag{3.43}$$

We have $\|(\widehat{N}+\mathbb{1})(|p-A_\#|^2+M^2+w)^{-1}(\widehat{N}+\mathbb{1})^{-1}\| \le C_M$ in Lemma 3.11. Thus (3.32) is obtained. (3.31) and (3.33)-(3.35) are also shown in a similar way. ■

4 HVZ-type theorem

Lemma 4.1 *For all $M \ge 0$ it follows that $\sigma_{\mathrm{ess}}(H) \subset [E+m,\infty)$.*

Proof: Let $\chi \in C_c(\mathbb{R})$ be such that $\operatorname{supp}\chi \subset (-\infty, E+m)$. It suffices to show that $\chi(H)$ is compact for all positive $M > 0$. Actually $\lim_{M\to+0}\chi(H) = \chi(H_0)$ in the uniform topology by Lemma 3.1. Let P_0 be the projection from $\mathscr{F}$ to the subspace spanned by Ω, i.e., $P_0\Psi = (\Omega, \Psi)\Omega$. Since $\operatorname{supp}\chi \subset (-\infty, E+m)$ and $\sigma(\mathrm{H_f}) = \{0\} \cup [m, \infty)$, we see that $\mathbb{1}_{\mathscr{H}} \otimes P_0$ leaves $\chi(\widehat{H})$ invariant:

$$\chi(\widehat{H}) = (\mathbb{1}_{\mathscr{H}} \otimes P_0)\chi(\widehat{H}). \tag{4.1}$$

We also see that

$$\mathrm{I}_R^*(\mathbb{1}_{\mathscr{H}} \otimes P_0)\mathrm{I}_R = \Gamma(\hat{j}_{0,R}^2). \tag{4.2}$$

By Lemma 3.3 and (4.1) we have

$$\chi(H) = \mathrm{I}_R^*\mathrm{I}_R\chi(H) = \mathrm{I}_R^*\chi(\widehat{H})\mathrm{I}_R + o(R^0) = \mathrm{I}_R^*(\mathbb{1}_{\mathscr{H}} \otimes P_0)\chi(\widehat{H})\mathrm{I}_R + o(R^0). \tag{4.3}$$

Here $o(R^0)$ converges to 0 as $R \to \infty$ in the uniform norm. By (4.2) and Lemma 3.3 again we have

$$\chi(H) = \Gamma(\hat{j}_{0,R}^2)\chi(H) + o(R^0). \tag{4.4}$$

Then we can see that

$$\chi(H) = \sum_{l=0}^{L}\Gamma(\hat{j}_{0,R}^2)\mathbb{1}_{\{l\}}(N)\chi(H) + \Gamma(\hat{j}_{0,R}^2)\mathbb{1}_{[L+1,\infty)}(N)\chi(H) + o(R^0). \tag{4.5}$$

Since $\|\Gamma(\hat{j}_{0,R}^2)\| \le 1$ and $\|\mathbb{1}_{[L+1,\infty)}(N)\Psi\| \le (L+1)^{-1}\|N\Psi\|$ for $\Psi \in D(N)$, we see that

$$\|\Gamma(\hat{j}_{0,R}^2)\mathbb{1}_{[L+1,\infty)}(N)\chi(H)\| \le \frac{1}{L+1}\|N\chi(H)\| \to 0 \tag{4.6}$$

as $L \to \infty$. By (4.5) and (4.6) we have

$$\chi(H) = \sum_{l=0}^{L}\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)\chi(H) + o(L^0) + o(R^0), \tag{4.7}$$

where $o(L^0)$ converges to 0 as $L \to \infty$ in the uniform norm. Thus it suffices to show that $\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)\chi(H)$ is compact for each $l = 0, 1, 2, \cdots$. We obtain that

$$\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)\chi(H) = UB, \tag{4.8}$$

where $B = (p^2+V)^{1/4}(\mathrm{H_f}+\mathbb{1})^{1/4}\chi(H)$ and $U = (p^2+V)^{-1/4}\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)(\mathrm{H_f}+\mathbb{1})^{-1/4}$ is a compact operator. Since $\|(p^2+V)^{1/2}\Psi\|^2 \le C\|(H+\mathbb{1})\Psi\|^2$, B is bounded. Thus $\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)\chi(H)$ is compact. Then it yields that

$$\chi(H) = \lim_{R,L\to\infty}\left\{\sum_{l=0}^{L}\mathbb{1}_{\{l\}}(N)\Gamma(\hat{j}_{0,R}^2)\chi(H)\right\}, \tag{4.9}$$

which implies that $\chi(H)$ is the limit of compact operators in the uniform topology. Then $\chi(H)$ is also compact. Hence $(-\infty, E+m)\cap\sigma(H)$ is discrete spectrum. Then the lemma follows. ■

The proof of Corollary 2.9: Since $[E, E+m)\cap\sigma(H)$ is discrete by Lemma 4.1, H has a ground state. The uniqueness and the exponential decay for the ground state are shown in [Hir13]. ■

Lemma 4.2 *Suppose Assumptions 2.1, 2.2 and 2.5. Then* $\sigma_{\rm ess}(H)\supset[E+m,\infty)$.

Proof: First we assume that $M>0$. Let Φ_M be a normalized ground state of H. Take $\lambda\in(E+m,\infty)$ and $k_0=k_0(M,\lambda)\in\mathbb{R}^d$ such that $\omega(k_0)=\lambda-E$. Let $h\in C_{\rm c}^\infty(\mathbb{R}^d)$ be such that $\|h\|=1$. Set $h_n(k)=n^{d/2}h(n(k-k_0))e^{in^2(k_0-k)}$. Then $\|h_n\|=1$, $\text{w-}\lim_{n\to\infty}h_n=0$ and $\lim_{n\to\infty}\|(\omega-\omega(k_0))h_n\|=0$. Note that $\Phi_M\in D(H)\subset D(\mathrm{H_f})\subset D(N)\subset D(a^\dagger(f))$. Set $\tilde{h}_n=\oplus^{d-1}h_n\in W$ and $\Psi_n=a^\dagger(\tilde{h}_n)\Phi_M$. It holds that $\lim_{n\to\infty}\|\Psi_n\|=1$ and $\text{w-}\lim_{n\to\infty}\Psi_n=0$. We see that for $\Phi\in\mathscr{H}_{\rm fin}$

$$((H-\lambda)\Phi,\Psi_n)=(\Phi,a^\dagger((\omega-\omega(k_0))\tilde{h}_n)\Phi_M)+([a^\dagger(\tilde{h}_n),T^{1/2}]\Phi,\Phi_M). \tag{4.10}$$

Then

$$\text{s-}\lim_{n\to\infty}a^\dagger((\omega-\omega(k_0))\tilde{h}_n)\Phi_M=0. \tag{4.11}$$

Let us consider the commutator $[a^\dagger(\tilde{h}_n),T^{1/2}]$. We see that

$$[a^\dagger(\tilde{h}_n),T^{1/2}]=\frac{2}{\pi}\int_0^\infty\frac{dw}{\sqrt{w}}[T(T+w)^{-1},a^\dagger(\tilde{h}_n)]. \tag{4.12}$$

Let $G_\mu=\oplus_{r=1}^{d-1}G_\mu^r$, and $G_\mu^r(k)=G_\mu^r(k,x)=\dfrac{\hat{\varphi}(k)e_\mu^r(k)e^{-ik\cdot x}}{\sqrt{2\omega(k)}}$. Then we have

$$\begin{aligned}
&[T(T+w)^{-1},a^\dagger(\tilde{h}_n)]\\
&=-w(T+w)^{-1}[T,a^\dagger(\tilde{h}_n)](T+w)^{-1}\\
&=-w(T+w)^{-1}\{(p+A)\cdot[A,a^\dagger(\tilde{h}_n)]+[A,a^\dagger(\tilde{h}_n)]\cdot(p+A)\}(T+w)^{-1}\\
&=-w(T+w)^{-1}\{(p+A)\cdot(G,\tilde{h}_n)+(G,\tilde{h}_n)\cdot(p+A)\}(T+w)^{-1}\\
&=-2w(T+w)^{-1}(G,\tilde{h}_n)(T+w)^{-1}\cdot(p+A)-w(T+w)^{-1}\sum_{\mu=1}^d(i\partial_{x_\mu}G_\mu,\tilde{h}_n)(T+w)^{-1}.
\end{aligned} \tag{4.13}$$

Since Φ_M is a ground state, by (4.12), (4.13) and Lemma 2.8 we obtain that

$$|([a^\dagger(\tilde{h}_n),T^{1/2}]\Phi,\Phi_M)|=C\sum_{\mu=1}^d\sup_{x\in\mathbb{R}^d}\Big(|(G_\mu,\tilde{h}_n)|+|(\partial_{x_\mu}G_\mu,\tilde{h}_n)|\Big)\|\Phi\|\|\Phi_M\|. \tag{4.14}$$

Let $K_n = ([a^\dagger(\tilde{h}_n), T^{1/2}]\lceil_{\mathcal{H}_{\rm fin}})^*$. (4.14) implies that $\Phi_M \in D(K_n)$ and $\lim_{n\to\infty} K_n\Phi_M = 0$. Then we see that $\|(H-\lambda)\Psi_n\| \leq \|a^\dagger((\omega-\omega(k_0))\tilde{h}_n)\Phi_M\| + \|K_n\Phi_M\|$ by (4.10)-(4.14), and that

$$\text{s-}\lim_{n\to\infty}(H-\lambda)\Psi_n = 0.$$

Since $\lim_{n\to\infty}\|\Psi_n\| = \|\Phi_M\| + \lim_{n\to\infty}\|a(h_n)\Phi_M\| = 1$, the normalized vector $\tilde{\Psi}_n = \Psi_n/\|\Psi_n\|$ satisfies that

$$\text{s-}\lim_{n\to\infty}(H-\lambda)\tilde{\Psi}_n = 0.$$

Then $\{\tilde{\Psi}_n\}$ is a Weyl sequence for λ and then we obtain that $\lambda \in \sigma_{\rm ess}(H)$ when $M > 0$.

Next we assume that $M = 0$. In order to emphasize the dependence on M we use H_M and E_M for H and E, respectively. Since H_M converges to H_0 in the uniformly resolvent sense. Then $E_M \to E_0$ as $M \to 0$. Fix $\lambda \in (E+m, \infty)$. Let $\{M_j\}_j$ be a sequence such that $M_j \to 0$ as $j \to \infty$. Suppose that $\lambda > E_j + m$ for all j. For each M_j, by the discussion mentioned above for the case of $M > 0$ there exist $n_j = n_j(M_j)$ such that

$$\begin{aligned}
&\|a^\dagger(\tilde{h}_{n_j})\Phi_{M_j}\| \leq 1 + 1/j,\\
&|(\Phi, a^\dagger(\tilde{h}_{n_j})\Phi_{M_j})| \leq 1/j,\\
&\|(H_{M_j}-\lambda)a^\dagger(\tilde{h}_{n_j})\Phi_{M_j}\| \leq 1/j.
\end{aligned}$$

Set $Q_j = a^\dagger(\tilde{h}_{n_j})\Phi_{M_j}$. Then $\lim_{j\to\infty}\|Q_j\| \to 1$ and

$$\|(H_0-\lambda)Q_j\| \leq \|(H_0-H_{M_j})Q_j\| + \|(H_{M_j}-\lambda)Q_j\| \leq \sqrt{M_j}(1+1/j) + 1/j.$$

Let $\tilde{Q}_j = Q_j/\|Q_j\|$. Hence we conclude that $\{\tilde{Q}_j\}$ is a Weyl sequence for λ, and thus $\lambda \in \sigma_{\rm ess}(H_0)$ follows. Then the lemma follows. ■

The proof of Theorem 2.8:

The theorem follows from Lemmas 4.1 and 4.2. ■

5 Appendix

In this appendix we review functional integral representations of the semigroup generated by models related to the Pauli-Fierz model. These representations play an important roles in this paper. The functional integral representation for the semigroup generated by the Pauli-Fierz model has been established in [Hir97]. By a minor modification we can also construct functional integral representations for models investigated in this paper.

5.1 Pauli-Fierz model

The Feynman-Kac formula yields the path integral representation of the Schrödinger operator $\frac{1}{2}p^2 + V$ by

$$(f, e^{-t(\frac{1}{2}p^2+V)}g) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} \overline{f(B_0)} g(B_t) \right]. \tag{5.1}$$

On the other hand the Pauli-Fierz model is defined by the minimal coupling of $p^2/2 + V + \mathrm{H_f}$ with a quantized radiation field $A(f)$ by

$$H_{PF} = \frac{1}{2}(p - A(f))^2 + V + \mathrm{H_f}$$

as a linear operator in $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}(W)$, where $A(f) = (A_1(f), \cdots, A_d(f))$ describes quantized radiation field with cutoff function f such that $f/\sqrt{\omega} \in L^2(\mathbb{R}^d)$, i.e, $A_\mu(f) = \int_{\mathbb{R}^d}^{\oplus} A_\mu(f, x)\mathrm{d}x$ and

$$A_\mu(f, x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int e_\mu^r(k) \left\{ \frac{\hat{f}(k)e^{-ik\cdot x}}{\sqrt{\omega(k)}} a^{\dagger r}(k) + \frac{\hat{f}(-k)e^{ik\cdot x}}{\sqrt{\omega(k)}} a^r(k) \right\} dk. \tag{5.2}$$

We can give the functional integral representation of $e^{-tH_{PF}}$ in [Hir97]. Let

$$\mathrm{q}(F, G) = \frac{1}{2} \sum_{\mu,\nu=1}^{d} (\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu)$$

be the quadratic form on $\oplus^d L^2(\mathbb{R}^d)$, where $\delta_{\mu\nu}^\perp(k) = \delta_{\mu\nu} - k_\mu k_\nu/|k|^2$ denotes the transversal delta function. Let $\mathscr{A}(F)$ be a Gaussian random variables on a probability space (Q, Σ, μ), which is indexed by $F = (F_1, \cdots, F_d) \in \oplus^d L^2(\mathbb{R}^d)$. The mean of $\mathscr{A}(F)$ is zero and the covariance is given by $\mathbb{E}[\mathscr{A}(F)\mathscr{A}(G)] = \mathrm{q}(F, G)$. Furthermore we introduce the Euclidean version of $\mathscr{A}$. Let

$$\mathrm{q}_E(F, G) = \frac{1}{2} \sum_{\mu,\nu=1}^{d} (\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu) \tag{5.3}$$

be the quadratic form on $\oplus^d L^2(\mathbb{R}^{d+1})$. On the right-hand side of (5.3), we note that $(\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu) = \int_{\mathbb{R}\times\mathbb{R}^d} \overline{\hat{F}_\mu}(k_0, k)\delta_{\mu\nu}^\perp(k)\hat{G}_\nu(k_0, k)\mathrm{d}k_0\mathrm{d}k$. Let $\mathscr{A}_E(F)$ be a Gaussian random variables on a probability space (Q_E, Σ_E, μ_E), which is indexed by $F \in \oplus^d L^2(\mathbb{R}^{d+1})$. The mean of $\mathscr{A}_E(F)$ is zero and the covariance is given by $\mathbb{E}[\mathscr{A}_E(F)\mathscr{A}_E(G)] = \mathrm{q}_E(F, G)$. Let us identify $\mathscr{H}$ with $L^2(\mathbb{R}^d; \mathscr{F})$. Thus $\Phi \in \mathscr{H}$ can be an $\mathscr{F}$-valued L^2-function on $\mathbb{R}^d$, $\mathbb{R}^d \ni x \mapsto \Phi(x) \in \mathscr{F}$. It is well known that there exists the family of isometries $J_t : L^2(Q) \to L^2(Q_E)$ $(t \in \mathbb{R})$ and $\mathrm{j}_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1})$ $(t \in \mathbb{R})$ such that $J_t^* J_s = e^{-|t-s|\mathrm{H_f}}$

and $\mathrm{j}_t^*\mathrm{j}_s = e^{-|t-s|\omega(-i\nabla)}$. By the Feynman-Kac formula (5.1) it is straightforward to see that

$$(\Phi, e^{-t(\frac{1}{2}p^2+V+\mathrm{H_f})}\Psi) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (J_0\Phi(B_0), J_t\Phi(B_t))_{L^2(Q_E)} \right]. \tag{5.4}$$

Adding an interaction we also see that

$$(\Phi, e^{-tH_{PF}}\Psi) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (J_0\Phi(B_0), e^{-i\mathscr{A}_E(\mathrm{K}_E)} J_t\Phi(B_t))_{L^2(Q_E)} \right]. \tag{5.5}$$

Here

$$\mathrm{K}_E = \oplus_{i=1}^d \int_0^t \mathrm{j}_s \tilde{f}(\cdot - B_s) \mathrm{d}B_s^i$$

is the $\oplus^d L^2(\mathbb{R}^{d+1})$-valued stochastic integral of $\tilde{f} = (f/\sqrt{\omega})\check{}$. From this formula we have $e^{-tH_{PF}}\Psi(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} J_0^* e^{-i\mathscr{A}_E(\mathrm{K})} J_t\Phi(B_t)]$. Furthermore let

$$K_{PF} = \frac{1}{2}(p - A(f))^2$$

be the kinetic term of the Pauli-Fierz model H_{PF}. It also established that K_{PF} is essentially self-adjoint on $D(p^2) \cap C^\infty(N)$ when Assumption 2.2 is assumed. Then it follows that

$$(\Phi, e^{-tK_{PF}}\Psi) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (\Phi(B_0), e^{-i\mathscr{A}(\mathrm{K})}\Psi(B_t))_{L^2(Q)} \right], \tag{5.6}$$

where

$$K = \oplus_{i=1}^d \int_0^t \tilde{f}(\cdot - B_s) \mathrm{d}B_s^i$$

is the $\oplus^d L^2(\mathbb{R}^d)$-valued stochastic integral.

5.2 Extended Pauli-Fierz model

The extended Pauli-Fierz model is defined by

$$\widehat{H}_{PF} = H_{PF} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H_f} \tag{5.7}$$

as an operator in $\widehat{\mathscr{H}} = \mathscr{H} \otimes \mathscr{F}$. Note that $\mathscr{F} \otimes \mathscr{F} \cong \mathscr{F}(W \oplus W)$. Under the identification $\widehat{\mathscr{H}} \cong L^2(\mathbb{R}^d) \otimes \mathscr{F}(W \oplus W)$, then

$$\widehat{H}_{PF} = \frac{1}{2}(p - A_1)^2 + \widehat{\mathrm{H_f}},$$

where $A_1 = A(f \oplus 0)$ and $\widehat{\mathrm{H_f}}$ be the second quantization of $\widehat{\omega} = \omega \oplus \omega$. Then the functional integral representation of $e^{-t\widehat{H}_{PF}}$ is a slight modification of that of $e^{-tH_{PF}}$.

Let $\mathscr{A}(F)$ be a Gaussian random variables on a probability space $(\tilde{Q}, \tilde{\Sigma}, \tilde{\mu})$, which is indexed by $F \in (\oplus^d L^2(\mathbb{R}^d)) \oplus (\oplus^d L^2(\mathbb{R}^d))$. Let $\mathscr{A}_1(F) = \mathscr{A}(F \oplus 0)$ and $\mathscr{A}_2(G) = \mathscr{A}(0 \oplus G)$. The mean of $\mathscr{A}_\#(F)$ is zero and the covariance is given by

$$\mathbb{E}[\mathscr{A}_i(F)\mathscr{A}_j(G)] = \frac{1}{2}\delta_{ij} \sum_{\mu,\nu=1}^{d} (\hat{F}_\mu, \delta^\perp_{\mu\nu}\hat{G}_\nu), \quad i,j = 1,2. \tag{5.8}$$

Similar to the Pauli-Fierz model we introduce the Euclidean version of $\mathscr{A}_{Ej}$, $j = 1,2$. Let $\widehat{J}_t = J_t \otimes J_t$ and $\widehat{\mathrm{j}}_t = \mathrm{j}_t \otimes \mathrm{j}_t$. Then $\widehat{J}_t : L^2(Q) \otimes L^2(Q) \to L^2(Q_E) \otimes L^2(Q_E)$ and $\widehat{\mathrm{j}}_t : L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d+1}) \otimes L^2(\mathbb{R}^{d+1})$ satisfy that $\widehat{J}_t^* \widehat{J}_s = e^{-|t-s|\widehat{\mathrm{H}_\mathrm{f}}}$ and $\widehat{\mathrm{j}}_t^* \widehat{\mathrm{j}}_s = e^{-|t-s|\widehat{\omega}(-i\nabla)}$. Hence we can see that

$$(\Phi, e^{-t\widehat{H}_{PF}}\Psi) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (\widehat{J}_0\Phi(B_0), e^{-i\mathscr{A}_{E1}(\mathrm{K}_E)} \widehat{J}_t \Phi(B_t))_{L^2(Q_E)\otimes L^2(Q_E)} \right]. \tag{5.9}$$

5.3 Generalization of extended Pauli-Fierz model

Let f and g be two cutoff functions and we define $\widehat{H}_{PF}$ by

$$\widehat{H}_{PF} = \frac{1}{2}(p - A(f) \otimes \mathbb{1}_{\mathscr{F}} - \mathbb{1}_{\mathscr{H}} \otimes A(g))^2 + \mathrm{H}_\mathrm{f} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H}_\mathrm{f}. \tag{5.10}$$

Hence we can see that

$$\begin{aligned}(\Phi, e^{-t\widehat{H}_{PF}}\Psi) \\ = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (\widehat{J}_0\Phi(B_0), e^{-i\mathscr{A}_{E1}(\mathrm{K}_f) - i\mathscr{A}_{E2}(\mathrm{K}_g)} \widehat{J}_t \Phi(B_t))_{L^2(Q_E)\otimes L^2(Q_E)} \right],\end{aligned} \tag{5.11}$$

where $\mathrm{K}_h = \oplus_{i=1}^d \int_0^t \mathrm{j}_s \tilde{h}(\cdot - B_s)\mathrm{d}B_s^i$ for $h = f, g$. Furthermore let

$$\widehat{K}_{PF} = \frac{1}{2}(p - A(f) \otimes \mathbb{1}_{\mathscr{F}} - \mathbb{1}_{\mathscr{H}} \otimes A(g))^2$$

be the kinetic term of $\widehat{H}_{PF}$. It can be shown that $\widehat{K}_{PF}$ is essentially self-adjoint on $D(p^2) \cap C^\infty(\widehat{N})$. We then also have

$$(\Phi, e^{-t\widehat{K}_{PF}}\Phi) = \int_{\mathbb{R}^d} \mathrm{d}x \mathbb{E}^x \left[e^{-\int_0^t V(B_s)\mathrm{d}s} (\Phi(B_0), e^{-i\mathscr{A}_1(\mathrm{K}_F) - i\mathscr{A}_2(\mathrm{K}_G)} \Phi(B_t))_{L^2(Q)\otimes L^2(Q)} \right]. \tag{5.12}$$

Acknowledgments: FH thanks the hospitality of université de Paris XI, université d'Aix-Marseille-Luminy and université de Rennes 1, where part of this work has been done.

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