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Hiroshima, Fumio
Faculty of Mathematics, Kyushu University

Muminov, Zahridin
Uzbek-Israel Joint Faculty, National University of Uzbekistan

Kuljanov, Utkir
Faculty of Mathematics, Samarkand State University

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THRESHOLD OF DISCRETE SCHRÖDINGER OPERATORS WITH DELTA POTENTIALS ON n -DIMENSIONAL LATTICE

FUMIO HIROSHIMA¹, ZAHRIDDIN MUMINOV², UTKIR KULJANOV³

ABSTRACT. Eigenvalue behaviors of Schrödinger operator defined on n -dimensional lattice with $n+1$ delta potentials is studied. It can be shown that lower threshold eigenvalue and lower threshold resonance are appeared for $n \geq 2$, and lower super-threshold resonance appeared for $n = 1$.

1. INTRODUCTION

Behavior of eigenvalues below the essential spectrum of standard Schrödinger operators of the form $-\Delta + \varepsilon V$ defined on $L^2(\mathbb{R}^n)$ is considerably studied so far. Here V is a negative potential and $\varepsilon \geq 0$ is a parameter which is varied. When ε approaches to some critical point $\varepsilon_c \geq 0$, each negative eigenvalues approaches to the left edge of the essential spectrum, and consequently they are absorbed into it. A mathematical crucial problem is to specify whether a negative eigenvalue survives as an eigenvalue or a threshold resonance on the edge of the essential spectrum at the critical point ε_c . Their behaviors depend on the spacial dimension n . Suppose that V is relatively compact with respect to $-\Delta$. Then the essential spectrum of $-\Delta + \varepsilon V$ is $[0, \infty)$. Roughly speaking $-\Delta f + \varepsilon_c V f = 0$ implies that $f = -\varepsilon_c(-\Delta)^{-1}Vf$ and

$$\|(-\Delta)^{-1}g\|_{L^2}^2 = \int_{\mathbb{R}^n} |\hat{g}(k)|^2 / |k|^4 dk, \quad \|(-\Delta)^{-1}g\|_{L^1} = \int_{\mathbb{R}^n} |\hat{g}(k)| / |k|^2 dk,$$

where $g = Vf$. Hence it may be expected that $f \in L^2(\mathbb{R}^n)$ if $n \geq 5$ and $f \in L^1(\mathbb{R}^n)$ for $n = 3, 4$. If 0 is an eigenvalue, it is called an embedded eigenvalue or threshold eigenvalue. Hence it may be expected that an embedded eigenvalue exists for $n \geq 5$. On the other hand for $n = 3, 4$, the eigenvector is predicted to be in $L^1(\mathbb{R}^n)$, and then 0 is called a threshold resonance.

The discrete Schrödinger operators have attracted considerable attentions for both combinatorial Laplacians and quantum graphs; for some recent summaries refer to see [5, 8, 3, 6, 4, 15, 11] and the references therein. Particularly, eigenvalue behavior of discrete Schrödinger operators are discussed in e.g. [1, 7, 2, 10] and are briefly discussed in [9, 12, 10] when potentials are delta functions with a single point mass. In [1] an explicit example of a $-\Delta - V$ on the three-dimensional lattice \mathbb{Z}^3 , which possesses both a *lower* threshold resonance and a *lower* threshold eigenvalue, is constructed, where $-\Delta$ stands for the standard discrete Laplacian in $\ell^2(\mathbb{Z}^n)$ and V is a multiplication operator by the function

$$\hat{V}(x) = \mu \delta_{x0} + \frac{\lambda}{2} \sum_{|s|=1} \delta_{xs}, \quad \lambda \geq 0, \mu \geq 0, \quad (1.1)$$

where δ_{xs} is the Kronecker delta.

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CONTACT Z. Muminov. Email: zimuminov@gmail.com.

The authors of [12] considered the restriction of this operator to the Hilbert space $\ell_e^2(\mathbb{Z}^3)$ of all even functions in $\ell^2(\mathbb{Z}^3)$. They investigated the dependence of the number of eigenvalues of $H_{\lambda\mu}$ on λ, μ for $\lambda > 0, \mu > 0$, and they showed that all eigenvalues arise either from a *lower* threshold resonance or from *lower* threshold eigenvalues under a variation of the interaction energy. Moreover, they also proved that the first *lower* eigenvalue of the Hamiltonian $-\Delta - V$ arises only from a *lower* threshold resonance under a variation of the interaction energy. A continuous version, two-particle Schrödinger operator, is shown by Newton (see p.1353 in [14]) and proved by Tamura [17, Lemma 1.1] using a result by Simon [16]. In case $\lambda = 0$, Hiroshima et.al. [10] showed that an threshold eigenvalue does appear for $n \geq 5$ but does not for $1 \leq n \leq 4$.

There are still interesting spectral properties of the discrete Schrödinger operators with potential of the form (1.1).

In this paper, we investigate the spectrum of $H_{\lambda\mu}$, specifically, *lower* and *upper* threshold eigenvalues and threshold resonances for *any*

$$(\lambda, \mu) \in \mathbb{R}^2 \quad \text{and} \quad n \geq 1.$$

We emphasize that there also appears so-called super-threshold resonances in our model for $n = 1$. See Proposition 4.7. The definitions of these are given in Definition 3.17. Our result is an extensions of [12, 1, 10].

In this paper, we study, in particular, eigenvalues in $(-\infty, 0)$, lower threshold eigenvalues, lower threshold resonances and lower super-threshold resonances. In a similar manner to this, we can also investigate eigenvalues in $(2n, \infty)$, upper threshold eigenvalues, upper threshold resonances and upper super-threshold resonances, but we left them to readers, and we focus on studying the spectrum contained in $(-\infty, 0]$.

The paper is organized as follows. In Section 2, a discrete Schrödinger operator in the coordinate and momentum representation is described, and it is decomposed into direct sum of operators $H_{\lambda\mu}^e$ and $H_{\lambda\mu}^o$. The spectrum of $H_{\lambda\mu}^e$ and $H_{\lambda\mu}^o$ are investigated in Section 3. Section 4 is devoted to showing main results, Theorems 5.2 and 5.3. The proofs of some lemmas belong to Appendix.

2. DISCRETE SCHRÖDINGER OPERATORS ON LATTICE

Let \mathbb{Z}^n be the n -dimensional lattice, i.e. the n -dimensional integer set. The Hilbert space of ℓ^2 sequences on \mathbb{Z}^n is denoted by $\ell^2(\mathbb{Z}^n)$. A notation $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n = (-\pi, \pi]^n$ means the n -dimensional torus (the first Brillouin zone, i.e., the dual group of \mathbb{Z}^n) equipped with its Haar measure, and let $L_e^2(\mathbb{T}^n)$ (resp. $L_o^2(\mathbb{T}^n)$) denote the subspace of all even (resp. odd) functions of the Hilbert space $L^2(\mathbb{T}^n)$ of L^2 -functions on \mathbb{T}^n . Let $\langle \cdot, \cdot \rangle$ mean the inner product on $L^2(\mathbb{T}^n)$.

Let $T(y)$ be the shift operator by $y \in \mathbb{Z}^n$: $(T(y)f)(x) = f(x + y)$ for $f \in \ell^2(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$. The standard discrete Laplacian Δ on $\ell^2(\mathbb{Z}^n)$ is usually associated with the bounded self-adjoint multidimensional Toeplitz-type operator:

$$\Delta = \frac{1}{2} \sum_{\substack{x \in \mathbb{Z}^n \\ |x|=1}} (T(x) - T(0)).$$

Let us define the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^n)$ by

$$\hat{H}_{\lambda\mu} = -\Delta - \hat{V},$$

where the potential \hat{V} depends on two parameters $\lambda, \mu \in \mathbb{R}$ and satisfies

$$(\hat{V}f)(x) = \begin{cases} \mu f(x), & \text{if } x = 0 \\ \frac{\lambda}{2} f(x), & \text{if } |x| = 1 \\ 0, & \text{if } |x| > 1 \end{cases}, \quad f \in \ell^2(\mathbb{Z}^n), x \in \mathbb{Z}^n,$$

which awards $\hat{H}_{\lambda\mu}$ to be a bounded self-adjoint operator. Let \mathcal{F} be the standard Fourier transform $\mathcal{F} : L^2(\mathbb{T}^n) \rightarrow \ell^2(\mathbb{Z}^n)$ defined by $(Ff)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\theta) e^{ix\theta} d\theta$ for $f \in L^2(\mathbb{T}^n)$ and $x \in \mathbb{Z}^n$. The inverse Fourier transform is then given by $(F^{-1}f)(\theta) = \sum_{x \in \mathbb{Z}^n} f(x) e^{-ix\theta}$ for $f \in \ell^2(\mathbb{Z}^n)$ and $\theta \in \mathbb{T}^n$. The Laplacian Δ in the momentum representation is defined as

$$\hat{\Delta} = \mathcal{F}^{-1} \Delta \mathcal{F},$$

and $\hat{\Delta}$ acts as the multiplication operator:

$$(\hat{\Delta}\hat{f})(p) = -E(p)\hat{f}(p),$$

where $E(p)$ is given by

$$E(p) = \sum_{j=1}^n (1 - \cos p_j).$$

In the physical literature, the function $\sum_{j=1}^n (1 - \cos p_j)$, being a real valued-function on \mathbb{T}^n , is called the *dispersion relation* of the Laplace operator. We also define the discrete Schrödinger operator in momentum representation. Let $H_0 = -\hat{\Delta}$. The operator $H_{\lambda\mu}$, in the momentum representation, acts in the Hilbert space $L^2(\mathbb{T}^n)$ as

$$H_{\lambda\mu} = H_0 - V,$$

where V is an integral operator of convolution type

$$(Vf)(p) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} v(p-s)f(s)ds, \quad f \in L^2(\mathbb{T}^n).$$

Here the kernel function $v(\cdot)$ is the Fourier transform of $\hat{V}(\cdot)$ computed as

$$v(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\mu + \lambda \sum_{i=1}^n \cos p_i \right),$$

and it allows the potential operator V to get the representation $V = V_{\lambda\mu}^e + V_{\lambda}^o$, where

$$V_{\lambda\mu}^e = \mu \langle \cdot, c_0 \rangle c_0 + \frac{\lambda}{2} \sum_{j=1}^n \langle \cdot, c_j \rangle c_j, \quad V_{\lambda}^o = \frac{\lambda}{2} \sum_{j=1}^n \langle \cdot, s_j \rangle s_j.$$

Here $\{c_0, c_j, s_j : j = 1, \dots, n\}$ is an orthonormal system in $L^2(\mathbb{T}^n)$, where

$$c_0(p) = \frac{1}{(2\pi)^{\frac{n}{2}}}, \quad c_j(p) = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \cos p_j, \quad s_j(p) = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \sin p_j, \quad j = 1, \dots, n.$$

One can check easily that the subspaces $L_e^2(\mathbb{T}^n)$ of all even functions and $L_o^2(\mathbb{T}^n)$ of all odd functions in $L^2(\mathbb{T}^n)$ reduce $H_{\lambda\mu}$. Adopting $V = V_{\lambda\mu}^e + V_{\lambda}^o$, we can see that the restriction $H_{\lambda\mu}^e$ (resp. H_{λ}^o) of the operator $H_{\lambda\mu}$ to $L_e^2(\mathbb{T}^n)$ (resp. $L_o^2(\mathbb{T}^n)$) acts with the form

$$H_{\lambda\mu}^e = H_0 - V_{\lambda\mu}^e \quad (\text{resp. } H_{\lambda}^o = H_0 - V_{\lambda}^o).$$

Hence $H_{\lambda\mu}$ is decomposed into the even Hamiltonian and the odd Hamiltonian:

$$H_{\lambda\mu} = H_{\lambda\mu}^e \oplus H_{\lambda}^o$$

under the decomposition $L^2(\mathbb{T}^n) = L_e^2(\mathbb{T}^n) \oplus L_o^2(\mathbb{T}^n)$. We have the fundamental proposition below:

Proposition 2.1. *It follows that $\sigma_{\text{ess}}(H_{\lambda\mu}) = \sigma_{ac}(H_{\lambda\mu}) = [0, 2n]$.*

Proof. The perturbation V is a finite rank operator and then the essential spectrum of the operator $H_{\lambda\mu}$ fills in $[0, 2n] = \sigma_{\text{ess}}(H_0)$.

Let \mathcal{H}_{ac} be the absolutely continuous part of $H_{\lambda\mu}$. It can be seen that the wave operator $W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itH_{\lambda\mu}} e^{-itH_0}$ exists and is complete since $H_{\lambda\mu}$ is a finite rank perturbation of H_0 . This implies that H_0 and $H_{\lambda\mu}|_{\mathcal{H}_{ac}}$ are unitarily equivalent by $W_{\pm}^{-1}H_0W_{\pm} = H_{\lambda\mu}|_{\mathcal{H}_{ac}}$. Then $\sigma_{ac}(H_0) = \sigma_{ac}(H_{\lambda\mu}) = [0, 2n]$. \square

In what follows, we shall study the spectrum of $H_{\lambda\mu}$ by investigating the spectrum of $H_{\lambda\mu}^e$ and H_{λ}^o separately.

3. SPECTRUM OF $H_{\lambda\mu}^e$

3.1. Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$. The Birman-Schwinger principle helps us to reduce the problem to the study of spectrum of a finite dimensional linear operator: a matrix.

We denote the resolvent of Laplacian H_0 by $(H_0 - z)^{-1}$, where $z \in \mathbb{C} \setminus [0, 2n]$. We can see that $(H_0 - z)^{-1}V_{\lambda\mu}^e$ is a finite rank operator. Let M_{n+1} denote the linear hull of $\{c_0, \dots, c_n\}$. Then M_{n+1} is an $(n+1)$ -dimensional subspace of $L_e^2(\mathbb{T}^n)$. Furthermore we define $\tilde{M}_{n+1} = (H_0 - z)^{-1}M_{n+1}$ for $z \in \mathbb{C} \setminus [0, 2n]$. Then \tilde{M}_{n+1} is also an $(n+1)$ -dimensional subspace of $L_e^2(\mathbb{T}^n)$ since $(H_0 - z)^{-1}$ is invertible. We define $C_1 : \mathbb{C}^{n+1} \rightarrow L_e^2(\mathbb{T}^n)$ by the map

$$C_1 : \mathbb{C}^{n+1} \ni \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \mapsto (H_0 - z)^{-1} \left(\mu w_0 c_0 + \frac{\lambda}{2} \sum_{j=1}^n w_j c_j \right) \in \tilde{M}_{n+1},$$

and define $C_2 : L_e^2(\mathbb{T}^n) \rightarrow \mathbb{C}^{n+1}$ by the map

$$C_2 : L_e^2(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \langle \phi, c_0 \rangle \\ \vdots \\ \langle \phi, c_n \rangle \end{pmatrix} \in \mathbb{C}^{n+1}.$$

Then we have the sequence of maps:

$$L_e^2(\mathbb{T}^n) \xrightarrow{C_2} \mathbb{C}^{n+1} \xrightarrow{C_1} L_e^2(\mathbb{T}^n) \quad (3.1)$$

and $C_1 C_2 : L_e^2(\mathbb{T}^n) \rightarrow L_e^2(\mathbb{T}^n)$. Notice that C_1 and C_2 depend on the choice of z . We directly have

$$(H_0 - z)^{-1}V_{\lambda\mu}^e = C_1 C_2. \quad (3.2)$$

Define

$$G_e(z) = C_2 C_1 : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}.$$

We shall show the explicit form of $G_e(z)$ in (3.14) below.

Lemma 3.1 (Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$).

(a) $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of $H_{\lambda\mu}^e$ if and only if $1 \in \sigma(G_e(z))$.

(b) Suppose that $z \in \mathbb{C} \setminus [0, 2n]$ and (λ, μ) satisfies $\det(G_e(z) - I) = 0$. Then the vector

$$Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^{n+1} \text{ is an eigenvector of } G_e(z) \text{ associated with eigenvalue 1 if and only if } f = C_1 Z, \text{ i.e.}$$

$$f(p) = \frac{1}{(2\pi)} \frac{1}{E(p) - z} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \cos p_j \right) \quad (3.3)$$

is an eigenfunction of $H_{\lambda\mu}^e$ associated with eigenvalue z .

Proof. It can be seen that $H_{\lambda\mu}^e f = z f$ if and only if $f = (H_0 - z)^{-1} V_{\lambda\mu}^e f$. Then $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of $H_{\lambda\mu}^e$ if and only if $1 \in \sigma((H_0 - z)^{-1} V_{\lambda\mu}^e)$. Hence $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of $H_{\lambda\mu}^e$ if and only if $1 \in \sigma(C_1 C_2)$ by the fact $\sigma(C_1 C_2) \setminus \{0\} = \sigma(C_2 C_1) \setminus \{0\}$.

Then it completes the proof of (a). We can also see that $C_2 C_1 Z = Z$ if and only if $f = (H_0 - z)^{-1} V_{\lambda\mu}^e f = C_1 C_2 f$, where $f = C_1 Z$. Then the function f coincides with (3.3). \square

3.2. Birman-Schwinger principle for $z = 0$. We consider the Birman-Schwinger principle for $z = 0$, which is the edge of the continuous spectrum of $H_{\lambda\mu}^e$, and it is the main issue to specify whether it is eigenvalue or threshold of $H_{\lambda\mu}^e$.

In order to discuss $z = 0$ we extend the eigenvalue equation $H_{\lambda\mu}^e f = 0$ in $L_e^2(\mathbb{T}^n)$ to that in $L_e^1(\mathbb{T}^n)$. Note that $L_e^2(\mathbb{T}^n) \subset L_e^1(\mathbb{T}^n)$. We consider the equation

$$E(p)f(p) - \frac{\mu}{(2\pi)^n} \int_{\mathbb{T}^n} f(p) dp - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^n \cos p_j \int_{\mathbb{T}^n} \cos p_j f(p) dp = 0 \quad (3.4)$$

in the Banach space $L_e^1(\mathbb{T}^n)$. Conveniently, we describe (3.4) as $H_{\lambda\mu}^e f = 0$. Since we consider a solution $f \in L_e^1(\mathbb{T}^n)$, the integrals $\int_{\mathbb{T}^n} f(p) dp$ and $\int_{\mathbb{T}^n} \cos p_j f(p) dp$ are finite for $j = 1, \dots, n$.

The unique singular point of $1/E(p)$ is $p = 0$, and in the neighborhood of $p = 0$, we have $E(p) \approx |p|^2$. Then the following lemma is fundamental, and its proof is straightforward.

Lemma 3.2. Let $h(p) = \varphi(p)/E(p)$, where $\varphi \in C(\mathbb{T}^n)$. Then (a)-(e) follow.

- (a) It follows that $h \in L^2(\mathbb{T}^n)$ for $n \geq 5$, and $h \in L^1(\mathbb{T}^n)$ for $n \geq 3$.
- (b) Let $1 \leq n \leq 4$ and $h \in L^2(\mathbb{T}^n)$. Then $\varphi(0) = 0$.
- (c) Let $1 \leq n \leq 4$, $|\varphi(p)| < C|p|^{\alpha_n}$ for some $C > 0$ and $\alpha_n > \frac{4-n}{2}$. Then $h \in L^2(\mathbb{T}^n)$.
- (d) Let $n = 1, 2$ and $h \in L^1(\mathbb{T}^n)$. Then $\varphi(0) = 0$.
- (e) Let $n = 1, 2$, $|\varphi(p)| < C|p|^{\alpha_n}$ for some $C > 0$ and $\alpha_n > 2 - n$. Then $h \in L^1(\mathbb{T}^n)$.

Operator H_0^{-1} is not bounded in $L_e^2(\mathbb{T}^n)$ as well as in $L_e^1(\mathbb{T}^n)$. It is however obvious by Lemma 3.2 and $V_{\lambda\mu}^e f \in C(\mathbb{T}^n)$ that

$$L_e^2(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_{\lambda\mu}^e f \in L_e^2(\mathbb{T}^n), \quad n \geq 5, \quad (3.5)$$

$$L_e^1(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_{\lambda\mu}^e f \in L_e^1(\mathbb{T}^n), \quad n \geq 3. \quad (3.6)$$

Thus for $n \geq 3$ we can extend operators C_1 and C_2 defined in the previous section. Let $n \geq 3$ and

$$Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}. \quad \bar{C}_1 : \mathbb{C}^{n+1} \rightarrow L_e^1(\mathbb{T}^n) \text{ is defined by}$$

$$\bar{C}_1 Z = \frac{1}{(2\pi)} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \cos p_j \right)$$

and $\bar{C}_2 : L_e^1(\mathbb{T}^n) \rightarrow \mathbb{C}^{n+1}$ by

$$\bar{C}_2 : L_e^1(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \int_{\mathbb{T}^n} \phi(p) c_0 dp \\ \int_{\mathbb{T}^n} \phi(p) c_1(p) dp \\ \vdots \\ \int_{\mathbb{T}^n} \phi(p) c_n(p) dp \end{pmatrix} \in \mathbb{C}^{n+1}.$$

Then $\bar{C}_1 \bar{C}_2 : L_e^1(\mathbb{T}^n) \rightarrow L_e^1(\mathbb{T}^n)$. Consequently $G_e(0) = \bar{C}_2 \bar{C}_1 : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is described as an $(n+1) \times (n+1)$ matrix.

Lemma 3.3. *Let $n \geq 3$. Then it follows that (1) $\lim_{z \rightarrow 0} G_e(z) = G_e(0)$ and (2) $\sigma(H_0^{-1} V_{\lambda\mu}^e) \setminus \{0\} = \sigma(G_e(0)) \setminus \{0\}$.*

Proof. The proof is straightforward. □

Lemma 3.4 (Birman-Schwinger principle for $z = 0$). *Let $n \geq 3$. Then (a) and (b) follow. (a) Equation $H_{\lambda\mu}^e f = 0$ has a solution in $L^1(\mathbb{T}^n)$ if and only if $1 \in \sigma(G_e(0))$. (b) Let $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^{n+1}$ be the solution of $G_e(0)Z = Z$ if and only if*

$$f(p) = \bar{C}_1 Z(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \cos p_j \right) \quad (3.7)$$

is a solution of $H_{\lambda\mu}^e f = 0$, where w_0, \dots, w_n are actually described by

$$w_0 = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) dp, \quad w_j = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) \cos p_j dp, \quad j = 1, \dots, n. \quad (3.8)$$

Proof. Let us consider $H_{\lambda\mu}^e f = 0$ in $L_e^1(\mathbb{T}^n)$. Hence $f = H_0^{-1} V_{\lambda\mu}^e f$ in $L_e^1(\mathbb{T}^n)$. Then L^1 -solution of $H_{\lambda\mu}^e f = 0$ exists if and only if $1 \in \sigma(H_0^{-1} V_{\lambda\mu}^e)$, and hence L^1 -solution of $H_{\lambda\mu}^e f = 0$ exists if and only if $1 \in \sigma(\bar{C}_1 \bar{C}_2)$. Due to the fact $\sigma(\bar{C}_1 \bar{C}_2) \setminus \{0\} = \sigma(G_e^0) \setminus \{0\}$ the proof of (a) is complete. We can also see that $\bar{C}_2 \bar{C}_1 Z = Z$ if and only if $f = H_0^{-1} V_{\lambda\mu}^e f = \bar{C}_1 \bar{C}_2 f$, where $f = \bar{C}_1 Z$. Then the function f coincides with (3.7). This fact ends the proof of (b). □

3.3. Zeros of $\det(G_e(z) - I)$.

3.3.1. Factorization. By the Birman-Schwinger principle in what follows we focus on investigating the spectrum of the $(n+1) \times (n+1)$ -matrix $G_e(z)$. Since $G_e(z)$ is defined for $z \in (-\infty, 0)$ for $n = 1, 2$, and $z \in (-\infty, 0]$ for $n \geq 3$. Hence in this section we suppose that $z \in \begin{cases} (-\infty, 0) & n = 1, 2, \\ (-\infty, 0] & n \geq 3. \end{cases}$

As the function $E(p) = E(p_1, \dots, p_n)$ is invariant with respect to the permutations of its arguments p_1, \dots, p_n , the integrals used for studying the spectrum of $G_e(z)$:

$$a(z) = \langle c_0, (H_0 - z)^{-1} c_0 \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{1}{E(p) - z} dp, \quad (3.9)$$

$$b(z) = \frac{1}{\sqrt{2}} \langle c_0, (H_0 - z)^{-1} c_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos p_j}{E(p) - z} dp, \quad (3.10)$$

$$c(z) = \frac{1}{2} \langle c_j, (H_0 - z)^{-1} c_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos^2 p_j}{E(p) - z} dp, \quad (3.11)$$

$$d(z) = \frac{1}{2} \langle c_i, (H_0 - z)^{-1} c_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos p_i \cos p_j}{E(p) - z} dp, \quad i \neq j, \quad (3.12)$$

$$s(z) = \frac{1}{2} \langle s_j, (H_0 - z)^{-1} s_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sin^2 p_j}{E(p) - z} dp. \quad (3.13)$$

also do not depend on the particular choice of indices $0 \leq i, j \leq n$. Note that $a(z), b(z), c(z)$ and $s(z)$ are defined for $n \geq 1$ but $d(z)$ for $n \geq 2$. From the definition of $G_e(z) = (a_{ij})_{0 \leq i, j \leq n}$, coefficients $a_{ij} = a_{ij}(z)$ are explicitly described as

$$\begin{cases} a_{00}(z) = \mu a(z), & a_{0j}(z) = \frac{\lambda}{\sqrt{2}} b(z), & j = 1, \dots, n, \\ a_{i0}(z) = \sqrt{2} \mu b(z), & a_{ii}(z) = \lambda c(z), & i = 1, \dots, n \\ a_{ij}(z) = \lambda d(z), & & i, j = 1, \dots, n, j \neq i, \end{cases}$$

Hence for $n \geq 2$ the matrix $G_e(z)$ has the form

$$G_e(z) = \begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}} b(z) & \dots & \dots & \frac{\lambda}{\sqrt{2}} b(z) \\ \sqrt{2} \mu b(z) & \lambda c(z) & \lambda d(z) & \dots & \lambda d(z) \\ \vdots & \lambda d(z) & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \lambda d(z) \\ \sqrt{2} \mu b(z) & \lambda d(z) & \dots & \lambda d(z) & \lambda c(z) \end{pmatrix} \quad (3.14)$$

and for $n = 1$,

$$G_e(z) = \begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}} b(z) \\ \sqrt{2} \mu b(z) & \lambda c(z) \end{pmatrix}. \quad (3.15)$$

In order to study the eigenvalue 1 of $G_e(z)$ we calculate the determinant of $G_e(z) - \mathbf{I}$.

Lemma 3.5. *We have*

$$\det(G_e(z) - \mathbf{I}) = \delta_r(\lambda, \mu; z) \delta_c(\lambda; z), \quad (3.16)$$

where

$$\delta_r(\lambda, \mu; z) = \begin{cases} (1 - \mu a(z)) \{1 - \lambda(c(z) + (n-1)d(z))\} - n\lambda\mu b^2(z), & n \geq 2 \\ (1 - \mu a(z))(1 - \lambda c(z)) - \lambda\mu b^2(z), & n = 1, \end{cases} \quad (3.17)$$

$$\delta_c(\lambda; z) = \begin{cases} \{\lambda(c(z) - d(z)) - 1\}^{n-1}, & n \geq 2 \\ 1, & n = 1. \end{cases} \quad (3.18)$$

Proof. It is a straightforward computation. \square

By the factorization (3.16) we shall study zeros of $\delta_r(\lambda, \mu; z)$ and $\delta_c(\lambda; z)$ separately to see eigenvalues and resonances of $H_{\lambda\mu}^e$. To see this we introduce algebraic relations used to estimate zeros of $\delta_r(\lambda, \mu; z)$ and $\delta_c(\lambda; z)$. Below we shall show the list of formulas of coefficients $a(z)$, $b(z)$, etc. We set

$$\alpha(z) = \begin{cases} c(z) + (n-1)d(z), & n \geq 2 \\ c(z), & n = 1, \end{cases} \quad (3.19)$$

$$\gamma(z) = a(z)\alpha(z) - nb^2(z). \quad (3.20)$$

Lemma 3.6. *For any $z < 0$, the relations below hold:*

$$\begin{aligned} a(z) - b(z) &= \frac{1}{n} + \frac{z}{n}a(z), \\ \alpha(z) &= (n-z)b(z), \\ \gamma(z) &= b(z). \end{aligned}$$

Proof. We directly see that

$$\begin{aligned} a(z) - b(z) &= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(1 - \cos p_1)}{E(p) - z} dp = \frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sum_{j=1}^n (1 - \cos p_j)}{E(p) - z} dp \\ &= \frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} dp + \frac{z}{n} \int_{\mathbb{T}^n} \frac{1}{E(p) - z} dp = \frac{1}{n} + \frac{z}{n}a(z), \end{aligned}$$

and

$$\alpha(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos p_1(z - E(p))}{E(p) - z} dp + \frac{n-z}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\cos p_1}{E(p) - z} dp = (n-z)b(z).$$

From these we can also get the third equality of the lemma. \square

Lemma 3.7. *Functions $a(z)$, $\alpha(z)$, $\gamma(z)$, $b(z)$, $c(z) - d(z)$ and $s(z)$ are monotonously increasing and positive in $(-\infty, 0]$. Moreover, their limits tend to zero as z tends to $-\infty$.*

Proof. We have the representations of $a(z)$, $\gamma(z) = b(z)$ and $s(z)$ by their definitions as

$$\alpha(z) = \frac{1}{n} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(\sum_{i=1}^n \cos p_i)^2}{E(p) - z} dp, \quad (3.21)$$

$$\gamma(z) = b(z) = \frac{1}{2n(2\pi)^{2n}} \int_{\mathbb{T}^n \times \mathbb{T}^n} \frac{(\sum_{j=1}^n (\cos p_j - \cos q_j))^2}{(E(p) - z)(E(q) - z)} dp dq, \quad (3.22)$$

$$c(z) - d(z) = \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(\cos p_1 - \cos p_2)^2}{E(p) - z} dp, \quad (3.23)$$

$$s(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sin^2 p_1}{E(p) - z} dp. \quad (3.24)$$

Indeed, for any fixed $p \in \mathbb{T}^n$, all the integrands are positive and monotonously increasing as functions of z , and we complete the proof. \square

Note that in Lemma 3.7 $c(z) - d(z)$ is considered only in the case of $n \geq 2$.

Lemma 3.8. *The following relations hold:*

$$\begin{aligned} a(z)s(z) &= b(z), & n = 1, z < 0, \\ a(z)s(z) &< b(z), & n = 2, z < 0, \\ a(z)s(z) &< b(z), & n \geq 3, z \leq 0, \\ c(z) - d(z) &< s(z), & n \geq 2, z \leq 0. \end{aligned} \quad (3.25)$$

Proof. See Appendix A. \square

Lemma 3.9. *The function $a(z)/b(z)$ is monotonously decreasing in $(-\infty, 0]$, and there exist limits:*

$$\lim_{z \rightarrow -\infty} \frac{a(z)}{b(z)} = +\infty, \quad (3.26)$$

$$\lim_{z \rightarrow 0-} \frac{a(z)}{b(z)} = \begin{cases} 1, & n = 1, 2, \\ \frac{a(0)}{b(0)}, & n \geq 3. \end{cases} \quad (3.27)$$

Proof. See Appendix B. \square

3.3.2. *Zeros of $\delta_r(\lambda, \mu; z)$.* We extend $\delta_r(\lambda, \mu; \cdot)$ and $\delta_c(\lambda; \cdot)$, and discuss zeros of them to specify the eigenvalue of $H_{\lambda\mu}^e$. Let $z \in (-\infty, 0)$. Applying notation in (3.19), we describe $\delta_r(\lambda, \mu; z)$ as

$$\delta_r(\lambda, \mu; z) = \gamma(z)\mathcal{H}_z(\lambda, \mu) \quad (3.28)$$

where

$$\mathcal{H}_z(\lambda, \mu) = \left(\lambda - \frac{a(z)}{\gamma(z)}\right) \left(\mu - \frac{\alpha(z)}{\gamma(z)}\right) - \frac{a(z)\alpha(z) - \gamma(z)}{\gamma^2(z)}. \quad (3.29)$$

or by Lemma 3.6, we have

$$\mathcal{H}_z(\lambda, \mu) = \left(\lambda - \frac{a(z)}{b(z)}\right) (\mu - (n - z)) - n. \quad (3.30)$$

Instead of the equation $\delta_r(\lambda, \mu; z) = 0$, the relation (3.28) allows us to study the family of rectangular hyperbola \mathfrak{H}_z indexed by $z \in \begin{cases} (-\infty, 0) & n = 1, 2 \\ (-\infty, 0] & n \geq 3 \end{cases}$. i.e. equilateral hyperbola \mathfrak{H}_z on (λ, μ) -plane, which is defined by

$$\mathfrak{H}_z = \{(\lambda, \mu) \in \mathbb{R}^2 \mid \mathcal{H}_z(\lambda, \mu) = 0\}$$

with asymptote

$$(\lambda_\infty(z), \mu_\infty(z)) = \left(\frac{a(z)}{b(z)}, n - z\right).$$

Lemma 3.9 implies that $\mathcal{H}_z(\lambda, \mu)$ can be extended to $z \in (-\infty, 0]$ for any dimension $n \geq 1$ as

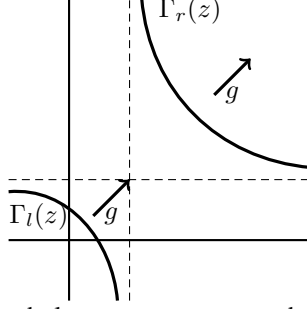
$$\bar{\mathcal{H}}_z(\lambda, \mu) = \begin{cases} \mathcal{H}_z(\lambda, \mu), & z < 0, \\ (\lambda - X)(\mu - n) - n, & z = 0. \end{cases} \quad (3.31)$$

Here $X = 1$ for $n = 1, 2$ and $X = a(0)/b(0)$ for $n \geq 3$. Note that

$$\bar{\mathcal{H}}_z(0, 0) = \frac{a(z)}{b(z)}(n - z) - n = \frac{1}{b(z)} > 0$$

for $z < 0$. We also extend the family of hyperbola $\mathfrak{H}_z, z \in (-\infty, 0)$, to that of hyperbola $\bar{\mathfrak{H}}_z$ indexed by $z \in (-\infty, 0]$ by

$$\bar{\mathfrak{H}}_z = \{(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} \mid \bar{\mathcal{H}}_z(\lambda, \mu) = 0\}.$$

FIGURE 1. Hyperbola moves as z approaches to $-\infty$ from 0.

By (3.31) we see that $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$ satisfies the algebraic relation:

$$(\lambda - X)(\mu - n) - n = 0. \quad (3.32)$$

For any $z_1 < z_2$, $z_1, z_2 \in (-\infty, 0]$, we note that the hyperbola $\overline{\mathfrak{H}}_{z_1}$ can be moved to $\overline{\mathfrak{H}}_{z_2}$ in parallel by the vector $g = \begin{pmatrix} \lambda_\infty(z_2) - \lambda_\infty(z_1) \\ \mu_\infty(z_2) - \mu_\infty(z_1) \end{pmatrix}$ whose components are positive. See Figure 1. Let $\Gamma_l(z)$ (resp. $\Gamma_r(z)$) denote the left branch (resp. the right branch) of the hyperbola $\overline{\mathfrak{H}}_z$, i.e.

$$\overline{\mathfrak{H}}_z = \Gamma_l(z) \cup \Gamma_r(z) \quad \text{and} \quad \Gamma_l(z) \cap \Gamma_r(z) = \emptyset.$$

We then see that for any $z_2 < z_1 \leq 0$ it follows that

$$\Gamma_l(z_1) \cap \Gamma_l(z_2) = \emptyset, \quad \Gamma_r(z_1) \cap \Gamma_r(z_2) = \emptyset. \quad (3.33)$$

Let us see the behavior of $\delta_r(\lambda, \mu; z)$ near $z = 0$ for $n = 1, 2$.

Lemma 3.10. *It follows that*

$$a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}, \quad n = 1, \quad (3.34)$$

$$a(z) = -\frac{\sqrt{2}}{2\pi} \ln(-z) + \left(\frac{1}{2} - \frac{\sqrt{2}}{\pi}\right) + O(-z), \quad \text{as } z \rightarrow 0-, \quad n = 2. \quad (3.35)$$

Proof. The proof of this lemma can be found in [13]. \square

From this lemma we can see the behaviours of $\delta_r(\lambda, \mu; z)$ as $z \rightarrow 0-$.

Corollary 3.11. *It follows that*

$$(n = 1) \quad \lim_{z \rightarrow 0-} \delta_r(\lambda, \mu; z) = \begin{cases} \infty & (\lambda, \mu) \notin \overline{\mathfrak{H}}_0 \\ 1 - \mu & (\lambda, \mu) \in \overline{\mathfrak{H}}_0 \end{cases},$$

$$(n = 2) \quad \lim_{z \rightarrow 0-} \delta_r(\lambda, \mu; z) = \begin{cases} \infty & (\lambda, \mu) \notin \overline{\mathfrak{H}}_0 \\ 1 - \mu/2 & (\lambda, \mu) \in \overline{\mathfrak{H}}_0 \end{cases},$$

$$(n \geq 3) \quad \lim_{z \rightarrow 0-} \delta_r(\lambda, \mu; z) = b(0)\overline{\mathcal{H}}_0(\lambda, \mu).$$

Proof. In the case of $n \geq 3$ it is trivial to see that $\lim_{z \rightarrow 0-} \delta_r(\lambda, \mu; z) = b(0)\overline{\mathcal{H}}_0(\lambda, \mu)$. Then we consider cases of $n = 1, 2$. We recall that

$$\delta_r(\lambda, \mu; z) = \gamma(z)\overline{\mathcal{H}}_z(\lambda, \mu) = \gamma(z)\overline{\mathcal{H}}_0(\lambda, \mu) + \gamma(z)(\overline{\mathcal{H}}_z(\lambda, \mu) - \overline{\mathcal{H}}_0(\lambda, \mu))$$

and

$$\gamma(z) = b(z) = a(z) - \frac{1}{n} - \frac{1}{n}za(z).$$

We can also directly see that for $n = 1, 2$

$$\begin{aligned}\overline{\mathcal{H}}_z(\lambda, \mu) - \overline{\mathcal{H}}_0(\lambda, \mu) &= \frac{1}{b(z)}(b(z) - a(z))(\mu - n) + z(\lambda - \frac{a(z)}{b(z)}) \\ &= -\frac{1}{nb(z)}(1 + za(z))(\mu - n) + z(\lambda - \frac{a(z)}{b(z)}).\end{aligned}$$

Together with them we have

$$\delta_r(\lambda, \mu; z) = (a(z) - \frac{1 + za(z)}{n})\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(\frac{n - \mu}{n}) + \xi,$$

where

$$\xi = -za(z)(\lambda - \frac{a(z)}{b(z)}) + \frac{1 + za(z)}{n} \left(\frac{(1 + za(z))(\mu - n)}{nb(z)} + z(\lambda - \frac{a(z)}{b(z)}) \right).$$

By Lemmas 3.10 and 3.9 it is crucial to see that

$$\lim_{z \rightarrow 0^-} za(z) = 0, \quad \lim_{z \rightarrow 0^-} b(z) = \infty, \quad \lim_{z \rightarrow 0^-} \frac{a(z)}{b(z)} = 1$$

and

$$\lim_{z \rightarrow 0^-} \xi = 0, \quad \lim_{z \rightarrow 0^-} \frac{a(z)}{b(z)}(1 + za(z))(\frac{n - \mu}{n}) = 1 - \frac{\mu}{n}$$

for $n = 1, 2$. Let $n = 1$. Then

$$\delta_r(\lambda, \mu; z) = (a(z) - 1 - za(z))\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(1 - \mu) + \xi$$

and the corollary follows for $n = 1$. Let $n = 2$. In a similar manner to the case of $n = 1$ we have

$$\delta_r(\lambda, \mu; z) = (a(z) - \frac{1 + za(z)}{2})\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(1 - \frac{\mu}{2}) + \xi,$$

and the corollary follows for $n = 2$. Hence the proof of the corollary can be derived. \square

We define $\bar{\delta}_r(\lambda, \mu; z)$ for $z \in (-\infty, 0]$ by

$$\bar{\delta}_r(\lambda, \mu; z) = \begin{cases} \delta_r(\lambda, \mu; z), & z \in (-\infty, 0), \\ \lim_{z \rightarrow 0^-} \delta_r(\lambda, \mu; z), & z = 0. \end{cases} \quad (3.36)$$

From Corollary 3.11 we can see that $\bar{\delta}_r(\lambda, \mu; z)$ converges to

$$\bar{\delta}_r(\lambda, \mu; 0) = \begin{cases} 1 - \mu, & n = 1, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0, \\ 1 - \mu/2, & n = 2, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0, \\ 0, & n \geq 3, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0. \end{cases} \quad (3.37)$$

Remark 3.12. We give a remark on (3.37). Let $n = 1, 2$. If $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$, then $(1 - \lambda)(1 - \mu/n) = 1$ is satisfied by (3.32), which implies that $1 - \mu \neq 0$ for $n = 1$, $1 - \mu/2 \neq 0$ for $n = 2$.

We can also show the continuity of $\bar{\delta}_r(\lambda, \mu; z)$ on z , which is summarised in the lemma below.

Lemma 3.13. It follows that

- ($n = 1, 2$): $\bar{\delta}_r(\lambda, \mu; z)$ is continuous in $z \in (-\infty, 0]$ for $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$,
- ($n \geq 3$): $\bar{\delta}_r(\lambda, \mu; z)$ is continuous in $z \in (-\infty, 0]$ for $(\lambda, \mu) \in \mathbb{R}^2$.

Let $z = 0$. Then the asymptote of the hyperbola \mathfrak{H}_0 is given by

$$(\lambda_\infty(0), \mu_\infty(0)) = \begin{cases} (1, n) & n = 1, 2, \\ (\frac{a(0)}{b(0)}, n) & n \geq 3. \end{cases}$$

The brunches $\Gamma_l(0)$ and $\Gamma_r(0)$ of the hyperbola $\overline{\mathfrak{H}}_0$ split \mathbb{R}^2 into three open sets

$$G_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) > 0, \lambda < \lambda_\infty(0)\},$$

$$G_1 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) < 0\},$$

$$G_2 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) > 0, \lambda > \lambda_\infty(0)\}.$$

We set

$$\Gamma_l = \Gamma_l(0), \quad \Gamma_r = \Gamma_r(0)$$

for notational simplicity. Hence $\partial G_0 = \Gamma_l$ and $\partial G_2 = \Gamma_r$ follow from the definition of G_0 and G_2 . See Figure 2. We can check the value of $\bar{\delta}_r(\lambda, \mu; z)$ for each (λ, μ) and $z \in (-\infty, 0]$ in the next lemma.

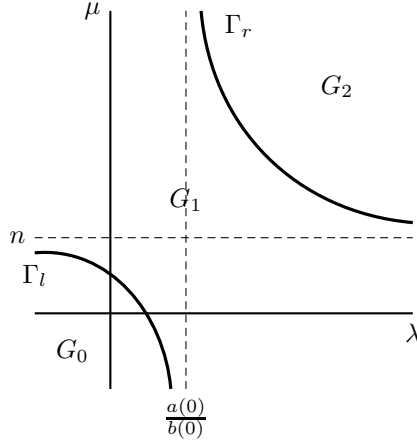


FIGURE 2. Region of G_j for $n \geq 3$

Lemma 3.14. *We have the following facts:*

- (a) (1) Let $(\lambda, \mu) \in G_0 \cup \Gamma_l$. Then $\bar{\delta}_r(\lambda, \mu; z) \neq 0$ for $z \in (-\infty, 0)$.
- (2) Let $(\lambda, \mu) \in \Gamma_l$. Then $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ for $n = 1, 2$.
- (3) Let $(\lambda, \mu) \in \Gamma_l$. Then $\bar{\delta}_r(\lambda, \mu; 0) = 0$ for $n \geq 3$.
- (4) Let $(\lambda, \mu) \in G_0$. Then $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ for $n \geq 1$.
- (b) (1) Let $(\lambda, \mu) \in G_1 \cup \Gamma_r$. Then there exists unique point $z \in (-\infty, 0)$ such that $\bar{\delta}_r(\lambda, \mu; z) = 0$.
- (2) Let $(\lambda, \mu) \in \Gamma_r$. Then $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ for $n = 1, 2$.
- (3) Let $(\lambda, \mu) \in \Gamma_r$. Then $\bar{\delta}_r(\lambda, \mu; 0) = 0$ for $n \geq 3$.
- (c) Let $(\lambda, \mu) \in G_2$. Then there exist two zeros $z_1, z_2 \in (-\infty, 0)$ such that $\bar{\delta}_r(\lambda, \mu; z_1) = \bar{\delta}_r(\lambda, \mu; z_2) = 0$.

Proof. Let $(\lambda, \mu) \in G_0 \cup \Gamma_l$. Then we can see that $(\lambda, \mu) \notin \overline{\mathfrak{H}}_z$ for any $z \in (-\infty, 0)$. Thus (a)(1) follows.

Also by (3.33), it can be seen that $\cup_{z \in (-\infty, 0)} \Gamma_l(z) \supset G_1$, $\Gamma_l(z) \cap \Gamma_l(w) = \emptyset$ if $z \neq w$, and $\Gamma_r(z) \cap G_1 = \emptyset$. Hence there exists a unique $z \in (-\infty, 0)$ such that $(\lambda, \mu) \in \Gamma_l(z)$, which proves (b)(1). We can also see that $\cup_{z \in (-\infty, 0)} \Gamma_l(z) \supset G_2$, $\cup_{z \in (-\infty, 0)} \Gamma_r(z) \supset G_2$, $\Gamma_l(z) \cap \Gamma_l(w) = \emptyset$ if $z \neq w$, $\Gamma_r(z) \cap \Gamma_r(w) = \emptyset$ if $z \neq w$, and $\Gamma_l(z) \cap \Gamma_r(z) = \emptyset$. Hence there exist $z_1, z_2 \in (-\infty, 0)$ such that $(\lambda, \mu) \in \Gamma_l(z_1)$ and $(\lambda, \mu) \in \Gamma_r(z_2)$, which proves (c). We note that since $(\lambda, \mu) \in \Gamma_l$ implies that $1 \neq \mu$ for $n = 1$, and $2 \neq \mu$ for $n = 2$, $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ for $n = 1, 2$ follows. Hence (a)(2) and (a)(3) follow from (3.37), and (b)(2) and (b)(3) are similarly proven. Finally for $(\lambda, \mu) \in G_0$ we have

$$\bar{\delta}_r(\lambda, \mu; 0) = \begin{cases} \infty, & n = 1, 2, \\ b(0)\mathcal{H}_0(\lambda, \mu) \neq 0, & n \geq 3. \end{cases}$$

Then (a)(4) follows. \square

3.3.3. Zeros of $\delta_c(\lambda; z)$. We study zeros of $\delta_c(\lambda; z)$. In a similar manner we extend $\delta_c(\lambda, z)$ for $z \in (-\infty, 0]$. When $n \geq 2$, the function $c(z) - d(z)$ exists, and due to its monotone property (See Lemma 3.7) we can define $\alpha = \lim_{z \rightarrow 0^-} c(z) - d(z)$. Note that $\alpha > 0$ and we set

$$\lambda_c = \frac{1}{\alpha}.$$

Let us write $\delta_c(\lambda; z) = \varrho(\lambda; z)^{n-1}$, where $\varrho(\lambda; z) = \lambda(c(z) - d(z)) - 1$. We define $\bar{\delta}_c(\lambda; z)$ by

$$\bar{\delta}_c(\lambda; z) = \begin{cases} \delta_c(\lambda; z), & z \in (-\infty, 0), \quad n \geq 1, \\ (\lambda\alpha - 1)^{n-1}, & z = 0, \quad n \geq 2, \\ 1, & z = 0, \quad n = 1. \end{cases} \quad (3.38)$$

Lemma 3.15. *Let $n \geq 2$. Then (a)–(c) follow.*

- (a) *Let $\lambda \leq \lambda_c$. Then $\varrho(\lambda; z) \neq 0$ for any $z \in (-\infty, 0)$.*
- (b) *Let $\lambda = \lambda_c$. Then $\varrho(\lambda; 0) = 0$.*
- (c) *Let $\lambda > \lambda_c$. Then there exists unique $z \in (-\infty, 0)$ such that $\varrho(\lambda; z) = 0$ with multiplicity one.*

Proof. Since $c(z) - d(z) > 0$ is strictly monotonously increasing in $(-\infty, 0)$, we get

$$\begin{aligned} \varrho(\lambda; z) &\leq \varrho(\lambda_c; z) < \varrho(\lambda_c; 0) = 0, \quad \text{if } 0 < \lambda \leq \lambda_c, \\ \varrho(\lambda; z) &= -1, \quad \text{if } \lambda = 0, \end{aligned}$$

which prove (a) and (b). Since $\varrho(\lambda; 0) > \varrho(\lambda_c; 0) = 0$ and $\lim_{z \rightarrow -\infty} \varrho(\lambda; z) = -1$ there exists $z \in (-\infty, 0)$ such that $\varrho(\lambda; z) = 0$. By the monotonicity of $\varrho(\lambda; \cdot)$ this zero is a unique and has multiplicity one. Hence (c) is proven. \square

We divide (λ, μ) -plane into two half planes C_{\pm} and the boundary \mathfrak{C}_0 . Set

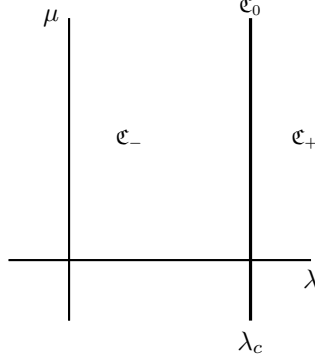
$$\mathfrak{C}_- = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_c\}, \mathfrak{C}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_c\}, \mathfrak{C}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_c\}.$$

See Figure 3. We immediately have a lemma.

Lemma 3.16. *Let $n \geq 2$. Then (a)–(c) follow.*

- (a) *For any $(\lambda, \mu) \in \mathfrak{C}_- \cup \mathfrak{C}_0$, $\bar{\delta}_c(\lambda; \cdot)$ has no zero in $(-\infty, 0)$.*
- (b) *Let $(\lambda, \mu) \in \mathfrak{C}_0$. Then $\bar{\delta}_c(\lambda; 0) = 0$, and $z = 0$ has multiplicity $n - 1$.*
- (c) *For any $(\lambda, \mu) \in \mathfrak{C}_+$, $\bar{\delta}_c(\lambda; \cdot)$ has a unique zero in $(-\infty, 0)$ with multiplicity $n - 1$.*

Proof. This follows from Lemma 3.15. \square

FIGURE 3. Regions of C_{\pm} for $n \geq 2$

3.4. Eigenvalues of $H_{\lambda\mu}^e$. In the previous sections we consider zeros of $\bar{\delta}_r(\lambda, \mu; z)$ and $\bar{\delta}_c(\lambda; z)$ for $z \in (-\infty, 0)$. Let (λ, μ) and $z \in (-\infty, 0)$ be solution of $\bar{\delta}_r(\lambda, \mu; z) = 0$ or $\bar{\delta}_c(\lambda; z) = 0$. Then $z \in \sigma_p(H_{\lambda\mu}^e)$. In this section we summarise spectral properties of $H_{\lambda\mu}^e$ derived from zeros of $\bar{\delta}_r(\lambda, \mu; z)$ and $\bar{\delta}_c(\lambda; z)$.

Definition 3.17 (Threshold eigenvalue and threshold resonance). *Let f be a solution of $H_{\lambda\mu}^e f = 0$ (resp. $H_{\lambda}^o f = 0$).*

- (1) *If $f \in L_e^2(\mathbb{T}^n)$ (resp. $f \in L_o^2(\mathbb{T}^n)$), we say that 0 is a lower threshold eigenvalue of $H_{\lambda\mu}^e$ (resp. H_{λ}^o).*
- (2) *If $f \in L_e^1(\mathbb{T}^n) \setminus L_e^2(\mathbb{T}^n)$ (resp. $f \in L_o^1(\mathbb{T}^n) \setminus L_o^2(\mathbb{T}^n)$), we say that 0 is a lower threshold resonance of $H_{\lambda\mu}^e$ (resp. H_{λ}^o).*
- (3) *If $f \in L_e^\epsilon(\mathbb{T}^n) \setminus L_e^1(\mathbb{T}^n)$ (resp. $f \in L_o^\epsilon(\mathbb{T}^n) \setminus L_o^1(\mathbb{T}^n)$) for any $0 < \epsilon < 1$, we say that 0 is a lower super-threshold resonance of $H_{\lambda\mu}^e$ (resp. H_{λ}^o).*

In what follows we may use "threshold eigenvalue" (resp. threshold resonance) instead of "lower threshold eigenvalue" (resp. lower threshold resonance) for simplicity. Then (λ, μ) -plane is divided into 11 regions. $G_0, \Gamma_l, G_1 \cap \mathfrak{C}_-, G_1 \cap \mathfrak{C}_0, G_1 \cap \mathfrak{C}_+, \Gamma_r \cap \mathfrak{C}_-, \Gamma_r \cap \mathfrak{C}_0, \Gamma_r \cap \mathfrak{C}_+, G_2 \cap \mathfrak{C}_-, G_2 \cap \mathfrak{C}_0$ and $G_2 \cap \mathfrak{C}_+$.

Lemma 3.18 (Eigenvalues of $H_{\lambda\mu}^e$ for $n \geq 2$). *Let $n \geq 2$. Then $H_{\lambda\mu}^e$ has the following facts:*

- (1) $(\lambda, \mu) \in G_0$. *There is no eigenvalue in $(-\infty, 0)$, and there is neither threshold eigenvalue nor threshold resonance.*
- (2) $(\lambda, \mu) \in \Gamma_l$.
 $n = 2$ *There is no eigenvalue in $(-\infty, 0)$ and there is neither threshold eigenvalue nor threshold resonance.*
 $n \geq 3$ *There is no eigenvalue in $(-\infty, 0)$ but there is a simple threshold eigenvalue or threshold resonance.*
- (3) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_-$. *There is a simple eigenvalue in $(-\infty, 0)$ but there is neither threshold eigenvalue nor threshold resonance.*
- (4) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_0$. *There is a simple eigenvalue in $(-\infty, 0)$ and there is an $(n - 1)$ -fold threshold eigenvalue or threshold resonance.*
- (5) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_+$. *There are a simple eigenvalue and an $(n - 1)$ -fold eigenvalue in $(-\infty, 0)$, but there is neither threshold eigenvalue nor threshold resonance.*
- (6) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_-$.

- $n = 2$ There is a simple eigenvalue in $(-\infty, 0)$ but there is neither threshold eigenvalue nor threshold resonance.
- $n \geq 3$ There is a simple eigenvalue in $(-\infty, 0)$ and there is a simple threshold eigenvalue or threshold resonance.
- (7) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_0$.
 $n = 2$ There is a simple eigenvalue in $(-\infty, 0)$ and there is an $(n - 1)$ -fold threshold eigenvalue or threshold resonance.
 $n \geq 3$ There is a simple eigenvalue in $(-\infty, 0)$, and there are an $(n - 1)$ -fold threshold eigenvalue or threshold resonance, and a simple threshold eigenvalue or threshold resonance.
- (8) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_+$.
 $n = 2$ There is a simple eigenvalue and an $(n - 1)$ -fold eigenvalue in $(-\infty, 0)$, but there is neither threshold eigenvalue nor threshold resonance.
 $n \geq 3$ There are a simple eigenvalue and an $(n - 1)$ -fold eigenvalue in $(-\infty, 0)$. There is a simple threshold eigenvalue or threshold resonance.
- (9) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_-$. There are two eigenvalues in $(-\infty, 0)$ but there is neither threshold eigenvalue nor threshold resonance.
- (10) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_0$. There are two eigenvalues in $(-\infty, 0)$ and there is an $(n - 1)$ -fold threshold eigenvalue or threshold resonance.
- (11) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_+$. There are three eigenvalues in $(-\infty, 0)$ and one of them is $(n - 1)$ -fold, but there is neither threshold eigenvalue nor threshold resonance.

Proof. This lemma follows from Lemmas 3.14 and 3.16, and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_r(\lambda, \mu; z)\bar{\delta}_c(\lambda; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_r(\lambda, \mu; 0)\bar{\delta}_c(\lambda; 0) = 0$. \square

By virtue of Lemma 3.14, $\bar{\delta}_r(\lambda, \mu; \cdot)$ has at most two zeros in $(-\infty, 0)$ for $(\lambda, \mu) \in G_2$ or $(\lambda, \mu) \in G_1 \cup \Gamma_r$. Now we can see the explicit form of these eigenvectors. In the case of $n = 1$ we know that $\bar{\delta}_c(\lambda; z) = 1$. Hence zeros of $\bar{\delta}_r(\lambda, \mu; z)\bar{\delta}_c(\lambda; z)$ coincides with those of $\bar{\delta}_r(\lambda, \mu; z)$. We have the lemma.

Lemma 3.19 (Eigenvalues of $H_{\lambda\mu}^e$ for $n = 1$). *We have the following facts:*

- (1) $(\lambda, \mu) \in G_0 \cup \Gamma_l$. There is no eigenvalues in $(-\infty, 0)$, and there is neither threshold eigenvalue nor threshold resonance.
- (2) $(\lambda, \mu) \in G_1 \cup \Gamma_r$. There is a simple eigenvalue in $(-\infty, 0)$, but there is neither threshold eigenvalue nor threshold resonance.
- (3) $(\lambda, \mu) \in G_2$. There are two eigenvalues in $(-\infty, 0)$, but there is neither threshold eigenvalue nor threshold resonance.

Proof. This lemma follows from Lemmas 3.14 and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_r(\lambda, \mu; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_r(\lambda, \mu; 0) = 0$. \square

Lemma 3.20. *Let $n \geq 1$. (1) Let $\lambda \neq 0$. We assume that $z_1, z_2 \in (-\infty, 0)$ and $\bar{\delta}_r(\lambda, \mu; z_k) = 0$ (if they exist). Then $1 - \mu a(z_k) \neq 0$ for $k = 1, 2$ and $G_e(z_k)Z_k = Z_k$ has the solutions:*

$$Z_k = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1 - \mu a(z_k)} \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad k = 1, 2,$$

and the corresponding eigenfunctions, $H_{\lambda\mu}^e f_k = z f_k$, are

$$f_k(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p) - z_k} \left(\mu \frac{nb(z_k)}{1 - \mu a(z_k)} + \sum_{j=1}^n \cos p_j \right), \quad k = 1, 2. \quad (3.39)$$

(2) Let $\lambda = 0$. We assume that $z \in (-\infty, 0)$ and $\delta_r(0, \mu; z) = 0$. Then $1 - \mu a(z) = 0$ and $G_e(z)Z = Z$ has the solution:

$$Z = \begin{pmatrix} 1 \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix}$$

and the corresponding eigenfunction, $H_{\lambda\mu}^e f = z f$, is

$$f(p) = \frac{\mu}{(2\pi)} \frac{1}{E(p) - z} \quad (3.40)$$

Proof. We prove the case of $n \geq 2$. The proof for the case of $n = 1$ is similar. Since $\delta_r(\lambda, \mu; z) = 0$, we see that

$$(1 - \mu a(z)) \left(1 - \lambda(c(z) + (n-1)d(z)) \right) - n\lambda\mu b^2(z) = 0.$$

Then $1 - \mu a(z) \neq 0$ if and only if $\lambda \neq 0$, and we also have the algebraic relation

$$1 - \lambda(c(z) + (n-1)d(z)) = \frac{n\lambda\mu b^2(z)}{1 - \mu a(z)}.$$

From this relation it follows that

$$G_e(z_k) \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1 - \mu a(z_k)} \\ 1 \\ \vdots \\ z_k 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1 - \mu a(z_k)} + \frac{\lambda}{\sqrt{2}} nb(z_k) \\ \lambda \frac{n\mu b(z_k)^2}{1 - \mu a(z_k)} + \lambda c(z_k) + \lambda(n-1)d(z_k) \\ \vdots \\ \lambda \frac{n\mu b(z_k)^2}{1 - \mu a(z_k)} + \lambda c(z_k) + \lambda(n-1)d(z_k) \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1 - \mu a(z_k)} \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then $G_e(z_k)Z_k = Z_k$ for $\lambda \neq 0$. In the case of $\lambda = 0$ we can also see that

$$G_e(z) \begin{pmatrix} 1 \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix} = \begin{pmatrix} \mu a(z) \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix} = Z.$$

Then the lemma is proven. □

Next we show the eigenfunction corresponding to zeros of $\delta_c(\lambda; \cdot)$.

Lemma 3.21. Let $n \geq 2$, $z \in (-\infty, 0)$ and $\bar{\delta}_c(\lambda; z) = 0$. I.e., $\lambda = \frac{1}{c(z) - d(z)}$. Then the solutions of $G_e(z)Z = Z$ are given by

$$Z_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \dots, Z_{n-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}, \quad (3.41)$$

and hence the corresponding eigenfunctions, $H_{\lambda\mu}^e g_j = z g_j$, are

$$g_j(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p) - z} (\cos p_1 - \cos p_{j+1}), \quad j = 1, \dots, n-1. \quad (3.42)$$

In particular the multiplicity of eigenvalue z is at least $n-1$.

Proof. Since $\lambda(c(z) - d(z)) = 1$, we see that

$$G_e(z) \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda(c(z) - d(z)) \\ \lambda(d(z) - c(z)) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $G_e(z)Z_1 = Z_1$. In the same way as Z_1 we can see that $G_e(z)Z_j = Z_j$ for $j = 2, \dots, n-1$. Then the lemma is proven. \square

3.5. Threshold eigenvalues and threshold resonances for $H_{\lambda\mu}^e$. In this section we study the spectrum located on the left edge of the essential spectrum $[0, 2n]$, i.e., $z = 0$. Suppose that $(\lambda, \mu) \in \mathfrak{H}_0$. Then it is possibly $\bar{\delta}_r(\lambda, \mu; 0) = 0$ or $\bar{\delta}_c(\lambda, \mu; 0) = 0$. By Corollary 3.11 and (3.38) we see that for $(\lambda, \mu) \in \mathfrak{H}_0$

$$\begin{aligned} \bar{\delta}_r(\lambda, \mu; 0) &\neq 0 \neq 1 = \bar{\delta}_c(\lambda, \mu; 0), & n = 1, \\ \bar{\delta}_r(\lambda, \mu; 0) &\neq 0, & n = 2. \end{aligned} \quad (3.43)$$

Hence we study zeros of $\bar{\delta}_r(\lambda, \mu; 0)$ for $n \geq 3$, and those of $\bar{\delta}_c(\lambda; 0)$ for $n \geq 2$. We set $a(0) = a$ and $b(0) = b$, and both a and b are finite for $n \geq 3$. In these case however the proofs are similar to those of Lemmas 3.20 and 3.21 where we discuss eigenvalues in $(-\infty, 0)$.

Lemma 3.22. *Let $n \geq 3$. (1) Let $\lambda \neq 0$ and $\bar{\delta}_r(\lambda, \mu; 0) = 0$. Then $1 - \mu a \neq 0$ and $G_e(0)Z = Z$ has the solution*

$$Z = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb}{1-\mu a} \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and the corresponding equation $H_{\lambda\mu}^e f = 0$ has the solution:

$$f(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p)} \left(\mu \frac{nb}{1-\mu a} + \sum_{j=1}^n \cos p_j \right). \quad (3.44)$$

(2) Let $\lambda = 0$ and $\bar{\delta}_r(0, \mu; 0) = 0$. Then $1 - \mu a = 0$ and $G_e(0)Z = Z$ has the solution:

$$Z = \begin{pmatrix} 1 \\ \sqrt{2}\mu b \\ \vdots \\ \sqrt{2}\mu b \end{pmatrix}$$

and the corresponding equation $H_{\lambda\mu}^e f = 0$ has the solution:

$$f(p) = \frac{\mu}{(2\pi)} \frac{1}{E(p)}. \quad (3.45)$$

Proof. Replacing z in Lemma 3.20 with 0 we can prove the lemma in the same way as that of Lemma 3.20. \square

Next we show the solution corresponding to zeros of $\delta_c(\lambda; \cdot)$. Similar to the case of $\delta_r(\lambda, \mu; z) = 0$, we have the lemma below.

Lemma 3.23. *Let $n \geq 2$ and $\delta_c(\lambda; 0) = 0$, i.e., $\lambda = \lambda_c$. Then the solutions of $G_e(0)Z = Z$ are given by (3.41) and hence the corresponding equation $H_{\lambda\mu}^e g_j = 0$ has the solutions*

$$g_j(p) = \frac{\lambda_c}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p)} (\cos p_1 - \cos p_{j+1}), \quad j = 1, \dots, n-1. \quad (3.46)$$

Proof. Replacing z in Lemma 3.21 with 0 we can prove the lemma in the same way as that of Lemma 3.21. \square

Recall that

$$u_0 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(p) dp, \quad u_j = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \cos p_j f(p) dp, \quad j = 1, \dots, n. \quad (3.47)$$

As was seen above the problem for $n \geq 3$ can be reduced to study the spectrum of G_e by the Birman-Schwinger principle, the problem for $n = 1, 2$ should be however directly investigated.

Lemma 3.24. *Let $n = 1$.*

- (1) *Suppose that $f \in L^1(\mathbb{T})$ and $H_{\lambda\mu}^e f = 0$. Then $f = 0$. In particular $H_{\lambda\mu}^e$ has no threshold resonance.*
- (2) *There is no non-zero f such that $f \in L^\epsilon(\mathbb{T}^2) \setminus L^1(\mathbb{T}^2)$ for some $0 < \epsilon < 1$ and $H_{\lambda\mu}^e f = 0$. In particular $H_{\lambda\mu}^e$ has no super-threshold resonance.*

Proof. (1) $H_{\lambda\mu}^e f = 0$ gives $f = \varphi/E$ and $\varphi(p) = \mu u_0 + \lambda u_1 \cos p$ by (3.4). From $f \in L^1(\mathbb{T})$ it follows that $\varphi(0) = \mu u_0 + \lambda u_1 = 0$. Hence

$$f(p) = \frac{1}{E(p)} (1 - \cos p) \mu u_0 = \mu u_0.$$

Substituting this into the second term in (3.47), we get $u_1 = \mu u_0 \frac{1}{2\pi} \int_{\mathbb{T}} \cos t dt = 0$, which gives $\mu u_0 = 0$ and $f = 0$.

(2) Since $f \notin L^1(\mathbb{T})$. It must be that $\mu = 0$ and $f = \varphi/E$ with $\varphi(p) = \lambda u_1 \cos p$. Hence

$$u_1 = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}} \frac{u_1 \cos^2 p}{E(p)} dp.$$

Then $u_1 = 0$, since $\int_{\mathbb{T}} \frac{\cos^2 p}{E(p)} dp = \infty$. Then $f = 0$ follows. \square

Next we discuss the spectrum of $H_{\lambda\mu}^e$ for $n = 2$ at the lower threshold

Lemma 3.25. *Let $n = 2$.*

- (1) *Suppose that $f \in L^1(\mathbb{T}^2)$ and $H_{\lambda\mu}^e f = 0$. Then $(\lambda, \mu) \in \mathfrak{C}_0$ and*

$$f(p) = \lambda_c u_1 \frac{\cos p_1 - \cos p_2}{E(p)}. \quad (3.48)$$

In particular $f \in L^2(\mathbb{T}^2)$ and $H_{\lambda\mu}^e$ has no threshold resonance.

- (2) *There is no non-zero f such that $f \in L^\epsilon(\mathbb{T}^2) \setminus L^1(\mathbb{T}^2)$ for some $0 < \epsilon < 1$ and $H_{\lambda\mu}^e f = 0$. In particular $H_{\lambda\mu}^e$ has no super-threshold resonance.*

Proof. (1) Consider $H_{\lambda\mu}^e f = 0$ in $L^1(\mathbb{T}^2)$. We can take $f = \varphi/E$ and $\varphi(p) = \mu u_0 + \lambda u_1 \cos p_1 + \lambda u_2 \cos p_2$. Since $f \in L^1(\mathbb{T}^2)$, we get $\varphi(0) = \mu u_0 + \lambda(u_1 + u_2) = 0$ and so

$$f(p) = \frac{\lambda}{E(p)} (-u_1(1 - \cos p_1) - u_2(1 - \cos p_2))$$

By (3.47) we obtain

$$\begin{aligned} u_1 &= -\frac{\lambda}{(2\pi)^2} \left(u_1 \int_{\mathbb{T}^2} \frac{\cos p_1(1 - \cos p_1)}{E(p)} dp + u_2 \int_{\mathbb{T}^2} \frac{\cos p_1(1 - \cos p_2)}{E(p)} dp \right), \\ u_2 &= -\frac{\lambda}{(2\pi)^2} \left(u_1 \int_{\mathbb{T}^2} \frac{\cos p_2(1 - \cos p_1)}{E(p)} dp + u_2 \int_{\mathbb{T}^2} \frac{\cos p_2(1 - \cos p_2)}{E(p)} dp \right). \end{aligned}$$

Since $\int_{\mathbb{T}^2} \frac{\cos p_1(1 - \cos p_1)}{E(p)} dp = -\int_{\mathbb{T}^2} \frac{\cos p_1(1 - \cos p_2)}{E(p)} dp$, we get

$$\begin{aligned} u_1 &= \frac{\lambda}{(2\pi)^2} (u_2 - u_1) \left(\int_{\mathbb{T}^2} \frac{\cos p_1(1 - \cos p_1)}{E(p)} dp \right), \\ u_2 &= \frac{\lambda}{(2\pi)^2} (u_1 - u_2) \left(\int_{\mathbb{T}^2} \frac{\cos p_2(1 - \cos p_2)}{E(p)} dp \right) \end{aligned}$$

and hence $u_1 = -u_2$. Consequently, $\mu u_0 = 0$, and the solution of $H_{\lambda\mu}^e f = 0$ is of the form

$$f(p) = \lambda u_1 \frac{\cos p_1 - \cos p_2}{E(p)} \in L^2(\mathbb{T}^2). \quad (3.49)$$

Inserting this into the definition of u_1 , we have $\frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{\cos p_1(\cos p_1 - \cos p_2)}{E(p)} dp = 1$ and thus taking $\lambda = \lambda_c$ we can see that (3.48) is the solution of $H_{\lambda\mu}^e f = 0$. Notice that $u_0 = 0$ follows from (3.49). (2) Since $f \notin L^1(\mathbb{T}^2)$. It must be that $\mu = 0$ and $f = \varphi/E$ with $\varphi(p) = \lambda u_1 \cos p_1 + \lambda u_2 \cos p_2$. Hence

$$\begin{aligned} u_1 &= \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{u_1 \cos^2 p_1 + u_2 \cos p_1 \cos p_2}{E(p)} dp, \\ u_2 &= -\frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{u_1 \cos p_2 \cos p_1 + u_2 \cos^2 p_2}{E(p)} dp. \end{aligned}$$

Then $u_1 = -u_2$ and $1 = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{\cos p_1(\cos p_1 + \cos p_2)}{E(p)} dp$. Thus $\lambda = \lambda_c$. Then f is given by (3.49), but $f \in L^2(\mathbb{T}^2)$. This contradicts with $f \notin L^1(\mathbb{T}^2)$. \square

Remark 3.26. The Birman-Schwinger principle is valid for $n \geq 3$, but Lemma 3.23 tells us that the Birman-Schwinger principle is valid for $n = 2$. Furthermore in Lemma 3.25 it can be seen that g_1 given by (3.46) coincides with (3.48).

Lemma 3.27 (Threshold eigenvalues and threshold resonances of $H_{\lambda\mu}^e$). (1)-(5) follow:

- (1) Let $n = 1$. Then 0 is none of a threshold eigenvalue, a threshold resonance and a super-threshold resonance.
- (2) Let $n = 2$. Then 0 is a threshold eigenvalue with (3.46) for $(\lambda, \mu) \in \mathfrak{C}_0$ and its multiplicity is one.
- (3) Let $n = 3, 4$. Suppose $(\lambda, \mu) \in \mathfrak{H}_0$. Then 0 is a threshold resonance with eigenvector (3.44) for $\lambda \neq 0$, and (3.45) for $\lambda = 0$, i.e., $(\lambda, \mu) = (0, 1/a)$.
- (4) Let $n = 3, 4$. Suppose $(\lambda, \mu) \in \mathfrak{H}_0$. Then 0 is a threshold eigenvalue with (3.46) for $\lambda = \lambda_c$ and its multiplicity is $n - 1$.

- (5) Let $n \geq 5$. Suppose $(\lambda, \mu) \in \mathfrak{H}_0$. Then 0 is a threshold eigenvalue with eigenvector (3.44) for $\lambda_c \neq \lambda \neq 0$ and multiplicity one, (3.44) and (3.46) for $\lambda = \lambda_c$ and multiplicity n , and (3.45) for $\lambda = 0$, i.e., $(\lambda, \mu) = (0, 1/a)$, and multiplicity one.

Proof. (1) follows from Lemma 3.24. The solution of $H_{\lambda\mu}^e f = 0$ is given by (3.44), (3.45) and (3.46). We note that $\int_{|p|<\epsilon} \frac{1}{E^2(p)} dp = \infty$ for $n = 2, 3, 4$ for any $\epsilon > 0$, and $\int_{|p|<\epsilon} \frac{1}{E^2(p)} dp < \infty$ for $n \geq 5$ for any $\epsilon > 0$. Since $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$, $n \neq \mu$, and we can see that

$$\frac{nb\mu}{1-\mu a} + \sum_{j=1}^n \cos 0 = \frac{nb\mu}{1-\mu a} + n = n \left(\frac{1-\mu(a-b)}{1-\mu a} \right) = \frac{n-\mu}{1-\mu a} \neq 0.$$

Hence, using Lemma 3.2, we obtain

$$\begin{aligned} (3.44), (3.45) &\in L^2(\mathbb{T}^n), \quad n \geq 5, \\ (3.44), (3.45) &\in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n), \quad n = 3, 4, \\ (3.46) &\in L^2(\mathbb{T}^n), \quad n \geq 2. \end{aligned}$$

(2) follows from Lemmas 3.25 and 3.23. (3) follows from Lemmas 3.23 and 3.22. (4) follows from Lemma 3.23. Finally (5) follows from Lemmas 3.23 and 3.22. \square

Remark 3.28. Let $n = 3, 4$ and $(\lambda_c, \mu) \in \mathfrak{H}_0$. By (3) and (4) of Lemma 3.27 it can be seen that 0 is a threshold resonance and a threshold eigenvalue.

4. SPECTRUM OF H_λ^o

4.1. Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$. In the previous sections, we study the spectrum of $H_{\lambda\mu}^e$ by using the Birman-Schwinger principle for $n \geq 3$, and by directly solving $H_{\lambda\mu}^e f = 0$ for $n = 1, 2$. In the case of H_λ^o we can proceed in a similar way to the case of $H_{\lambda\mu}^e$ and rather easier than that of $H_{\lambda\mu}^e$ as is seen below. Let $z \in \mathbb{C} \setminus [0, 2n]$. As is done for $H_{\lambda\mu}^e$, we can see that

$$(H_0 - z)^{-1} V_\lambda^o = S_1 S_2.$$

Here S_1 and S_2 are defined by

$$\begin{aligned} S_1 : \mathbb{C}^n &\ni \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \mapsto (H_0 - z)^{-1} \frac{\lambda}{2} \sum_{j=1}^n w_j s_j \in L_o^2(\mathbb{T}^n), \\ S_2 : L_o^2(\mathbb{T}^n) &\ni \phi \mapsto \begin{pmatrix} \langle \phi, s_1 \rangle \\ \vdots \\ \langle \phi, s_n \rangle \end{pmatrix} \in \mathbb{C}^n. \end{aligned}$$

We set

$$G_o(z) = S_2 S_1 : \mathbb{C}^n \rightarrow \mathbb{C}^n.$$

The following assertion is proved as Lemma 3.1, and then we omit the proof.

Lemma 4.1 (Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$).

- (a) $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of H_λ^o if and only if $1 \in \sigma(G_o(z))$.

(b) Let $z \in \mathbb{C} \setminus [0, 2n]$ and $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ be such that $G_o(z)Z = Z$. Then $f = S_1 Z$,

$$f(p) = \frac{1}{(2\pi)} \frac{1}{E(p) - z} \left(\frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \sin p_j \right)$$

is an eigenfunction of H_λ^o , i.e., $H_\lambda^o f = z f$.

We see that $\frac{1}{2} \langle s_i, (H_0 - z)^{-1} s_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sin p_i \sin p_j}{E(p) - z} dp = 0$ by the fact that $E(p) = E(p_1, \dots, p_d)$ is even for any p_j . Therefore

$$G_o(z) = \lambda s(z) I, \quad (4.1)$$

where $s(z) = (2\pi)^{-n} \int_{\mathbb{T}^n} \frac{\sin^2 p_1}{E(p) - z} dp$ is given by (3.24). Consequently we have for $n \geq 1$,

$$\delta_s(\lambda; z) = \det(G_o(z) - I) = (\lambda s(z) - 1)^n.$$

Since $G_o(z)$ is diagonal, it is very easy to find solution of $G_o(z)Z = Z$. It has n independent solutions:

$$Z_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j_{th}, \quad j = 1, \dots, n.$$

The corresponding eigenvector, $H_\lambda^o f_j = z f_j$, is given by

$$f_j(p) = \frac{1}{E(p) - z} \frac{1}{(2\pi)} \frac{\lambda}{\sqrt{2}} \sin p_j, \quad j = 1, \dots, n, \quad (4.2)$$

where $\lambda = 1/s(z)$. In particular the multiplicity of z is n .

4.2. Birman-Schwinger principle for $z = 0$. We can extend the Birman-Schwinger principle for $z = 0$. We extend the eigenvalue equation $H_\lambda^o f = 0$ in $L_o^2(\mathbb{T}^n)$ to that in $L_o^1(\mathbb{T}^n)$. We consider the equation

$$E(p)f(p) - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^n \sin p_j \int_{\mathbb{T}^n} \sin p_j f(p) dp = 0 \quad (4.3)$$

in $L_o^1(\mathbb{T}^n)$. We also describe (4.3) as $H_\lambda^o f = 0$. We can see that $\sin p_j / E(p) \approx 1/|p|$ in the neighborhood of $p = 0$, and then $\sin p_j / E(p) \in L^1(\mathbb{T}^n)$ for $n \geq 2$. By (e) of Lemma 3.2 and $V_\lambda^o f \in C(\mathbb{T}^n)$ we can see that

$$L_o^2(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_\lambda^o f \in L_o^2(\mathbb{T}^n), \quad n \geq 3, \quad (4.4)$$

$$L_o^1(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_\lambda^o f \in L_o^1(\mathbb{T}^n), \quad n \geq 2. \quad (4.5)$$

Thus for $n \geq 2$ we can extend operators S_1 and S_2 . Let $n \geq 2$ and $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}$. $\bar{S}_1 : \mathbb{C}^n \rightarrow L_o^1(\mathbb{T}^n)$

is defined by

$$\bar{S}_1 Z = \frac{1}{(2\pi)} \frac{\lambda}{\sqrt{2}} \frac{1}{E(p)} \sum_{j=1}^n w_j \sin p_j$$

and $\bar{S}_2 : L_o^1(\mathbb{T}^n) \rightarrow \mathbb{C}^n$ by

$$\bar{S}_2 : L_o^1(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \int_{T^n} \phi(p) s_1(p) dp \\ \vdots \\ \int_{T^n} \phi(p) s_n(p) dp \end{pmatrix} \in \mathbb{C}^n.$$

Then $\bar{S}_1 \bar{S}_2 : L_o^1(\mathbb{T}^n) \rightarrow L_o^1(\mathbb{T}^n)$. Thus $G_o(0) = \bar{S}_2 \bar{S}_1 : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is described as an $n \times n$ matrix. Let $n \geq 2$. We have (1) $\lim_{z \rightarrow 0} G_o(z) = G_o(0)$, and (2) $\sigma(H_0^{-1} V_\lambda^\alpha) \setminus \{0\} = \sigma(G_o(0)) \setminus \{0\}$. Hence for $n \geq 2$,

$$G_o(0) = \lambda s(0) I \quad (4.6)$$

and $\bar{\delta}_s(\lambda; z)$ is defined by

$$\bar{\delta}_s(\lambda; z) = \begin{cases} \delta_s(\lambda; z) & z \in (-\infty, 0), \\ (\lambda s(0) - 1)^n & z = 0. \end{cases} \quad (4.7)$$

Remark 4.2. In (4.6) and (4.7) we define $\bar{\delta}_s(\lambda, z)$ and G_o for $n \geq 2$. We note however that $s(0) < \infty$ for $n \geq 1$. Thus G_o and $\bar{\delta}_s(\lambda; z)$ are well defined for $n \geq 1$.

Lemma 4.3 (Birman-Schwinger principle for $z = 0$). *Let $n \geq 2$.*

- (a) Equation $H_\lambda^\alpha f = 0$ has a solution in $L^1(\mathbb{T}^n)$ if and only if $1 \in \sigma(G_o(0))$.
- (b) Let $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ be the solution of $G_o(0)Z = Z$ if and only if

$$f(p) = \bar{S}_1 Z(p) = \frac{1}{(2\pi)^n} \frac{1}{E(p)} \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \sin p_j$$

is a solution of $H_\lambda^\alpha f = 0$, where w_1, \dots, w_n are actually described by

$$w_j = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) \sin p_j dp, \quad j = 1, \dots, n. \quad (4.8)$$

Proof. The proof is the same as that of Lemma 3.4. □

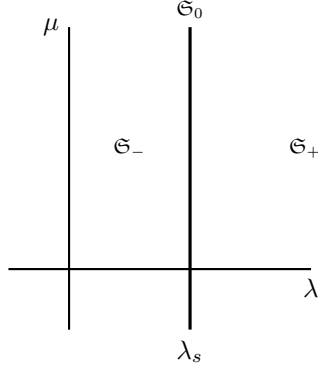
4.3. Eigenvalues of H_λ^α . Set

$$\lambda_s = \frac{1}{s(0)}.$$

Note that $\lambda_s = 1$ for $n = 1$. We divide (λ, μ) -plane into two half planes S_\pm and the boundary \mathfrak{S}_0 . Set

$$\mathfrak{S}_- = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_s\}, \mathfrak{S}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_s\}, \mathfrak{S}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_s\}.$$

See Figure 4.

FIGURE 4. Regions of S_{\pm} for $n \geq 1$

Lemma 4.4. *Let $n \geq 1$. Then (a)-(c) follow:*

- (a) *Let $(\lambda, \mu) \in \mathfrak{S}_- \cup \mathfrak{S}_0$. Then $\bar{\delta}_s(\lambda; \cdot)$ has no zero in $(-\infty, 0)$.*
- (b) *Let $(\lambda, \mu) \in \mathfrak{S}_0$. Then $\bar{\delta}_s(\lambda_s; 0) = 0$ and $z = 0$ has multiplicity n .*
- (c) *Let $(\lambda, \mu) \in \mathfrak{S}_+$. Then $\bar{\delta}_s(\lambda; \cdot)$ has a unique zero in $(-\infty, 0)$ with multiplicity n .*

Proof. The proof is similar to that of Lemma 3.15, and hence we omit it. \square

By Lemma 4.4 we can see spectral property of H_{λ}° .

Lemma 4.5 (Eigenvalues of H_{λ}°). *Let $n \geq 1$.*

- (1) *$(\lambda, \mu) \in \mathfrak{S}_- \cup \mathfrak{S}_0$. There is no eigenvalue in $(-\infty, 0)$.*
- (2) *$(\lambda, \mu) \in \mathfrak{S}_0$. There is an n fold threshold eigenvalue or threshold resonance.*
- (3) *$(\lambda, \mu) \in \mathfrak{S}_+$. There is an n fold eigenvalue in $(-\infty, 0)$.*

Proof. This lemma follows from Lemmas 4.4, and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_s(\lambda; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_s(\lambda; 0) = 0$. \square

4.4. Threshold eigenvalues and threshold resonances for H_{λ}° . Threshold resonances and threshold eigenvalues for H_{λ}° can be discussed by the Birman-Schwinger principle for $n \geq 2$.

Lemma 4.6. *Let $n \geq 2$. Then the solutions of equation $H_{\lambda}^{\circ} f = 0$ are given by*

$$f_j(p) = \frac{1}{(2\pi)} \frac{\lambda_s}{\sqrt{2}} \frac{\sin p_j}{E(p)}, \quad j = 1, \dots, n. \quad (4.9)$$

Proof. From $\bar{\delta}(\lambda_s, 0) = 0$ and Lemma 4.3 the lemma follows. \square

For $n = 1$ we can directly see that $H_{\lambda}^{\circ} f = 0$ has no solution in L^1 , but it has a super-threshold resonance. We see this in the next proposition.

Proposition 4.7 (Super-threshold resonance). *Let $n = 1$ and $\lambda = \lambda_s = 1$. Then $H_{\lambda}^{\circ} f = 0$ has solution $f \in L_{\circ}^{\epsilon}(\mathbb{T}) \setminus L_{\circ}^1(\mathbb{T})$ for any $0 < \epsilon < 1$. I.e., 0 is a super-threshold resonance of H_{λ}° .*

Proof. $H_{\lambda_s}^{\circ} f = 0$ yields that $f(p) = C \frac{\sin p}{E(p)}$, where $C = \frac{\lambda_s}{2\pi} \int_{\mathbb{T}} \sin p f(p) dp$. Note that however $\sin p/E(p) \notin L^1(\mathbb{T})$, but we can see that $\sin p/E(p) \in L^{\epsilon}(\mathbb{T})$ for any $0 < \epsilon < 1$ since $\sin p/E(p) \sim 1/p$ near $p = 0$ and $\int_{\mathbb{T}} p^{-\epsilon} dp < \infty$. \square

Lemma 4.8. (1) *Let $n = 1$. Then 0 is neither a threshold resonance nor a threshold eigenvalue, but for (λ_s, μ) , 0 is a super-threshold resonance.*

- (2) Let $n = 2$. Then 0 is a threshold resonance at $\lambda = \lambda_s$.
- (3) Let $n \geq 3$. Then 0 is a threshold eigenvalue at $\lambda = \lambda_s$ and its multiplicity is n .

Proof. (1) follows from Proposition 4.7. Let $n \geq 2$. Then the solution of $H_\lambda^\circ f = 0$ is given by (4.9). Since

$$\begin{aligned} \frac{\sin p_j}{E(p)} &\in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n), \quad n = 2, \\ \frac{\sin p_j}{E(p)} &\in L^2(\mathbb{T}^n), \quad n \geq 3, \end{aligned}$$

we have $f \in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n)$ for $n = 2$, and $f \in L^2(\mathbb{T}^n)$ for $n \geq 3$. Then (2) and (3) follow. \square

5. MAIN THEOREMS

5.1. Case of $n \geq 2$. In order to describe the main results we have to separate (λ, μ) -plane into several regions.

Lemma 5.1. *Let $n \geq 2$. Then $\lambda_\infty(z) \leq \lambda_s(z) \leq \lambda_c(z)$ for $z \in (-\infty, 0]$.*

Proof. By Lemma 3.8 it follows that

$$\lambda_c(z) = \frac{1}{c(z) - d(z)} > \lambda_s(z) = \frac{1}{s(z)} > \lambda_\infty(z) = \frac{a(z)}{b(z)}$$

for $z < 0$. By a limiting argument the lemma follows. \square

We introduce 4-half planes:

$$\begin{aligned} \mathfrak{C}_- &= \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_c\}, \quad \mathfrak{C}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_c\} \\ \mathfrak{S}_- &= \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_s\}, \quad \mathfrak{S}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_s\}, \end{aligned}$$

and two boundaries: $\mathfrak{C}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_c\}$ and $\mathfrak{S}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_s\}$. Note that $\mathfrak{S}_- \subset \mathfrak{C}_-$ and $\mathfrak{C}_+ \subset \mathfrak{S}_+$, and we define open sets surrounded by hyperbola \mathfrak{H}_0 and boundary Γ_c and Γ_s by :

$$\begin{aligned} D_0 &= G_0, \quad D_1 = G_1 \cap \mathfrak{S}_-, \quad D_2 = G_2 \cap \mathfrak{S}_-, \quad D_{n+1} = G_1 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \\ D_{n+2} &= G_2 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \quad D_{2n} = G_1 \cap \mathfrak{C}_+, \quad D_{2n+1} = G_2 \cap \mathfrak{C}_+. \end{aligned}$$

The boundaries of these sets define disjoint 8 curves:

$$\begin{aligned} B_0 &= \Gamma_l, \quad B_1 = \Gamma_r \cap \mathfrak{S}_-, \quad B_{n+1} = \Gamma_r \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \quad B_{2n} = \Gamma_r \cap \mathfrak{C}_+, \\ S_1 &= \mathfrak{S}_0 \cap G_1, \quad S_2 = \mathfrak{S}_0 \cap G_2, \quad C_{n+1} = \mathfrak{C}_0 \cap G_1, \quad C_{n+2} = \mathfrak{C}_0 \cap G_2, \end{aligned}$$

and two one point sets given by

$$A = \Gamma_r \cap \mathfrak{S}_0, \quad B = \Gamma_r \cap \mathfrak{C}_0.$$

We are now in the position to state the main theorem for $n \geq 2$.

Theorem 5.2. *Let $n \geq 2$.*

- (a) Assume that $(\lambda, \mu) \in D_k$, $k \in \{0, 1, 2, n+1, n+2, 2n, 2n+1\}$, then $H_{\lambda\mu}$ has k eigenvalues in $(-\infty, 0)$. In addition $H_{\lambda\mu}$ has neither a threshold eigenvalue nor a threshold resonance (see Table 1).
- (b) 0 is not a super-threshold resonance of $H_{\lambda\mu}$ for any $(\lambda, \mu) \in \mathbb{R}^2$.
- (c) Assume that (λ, μ) in B_k , S_k , C_k and A, B the next results are true in Table 2:

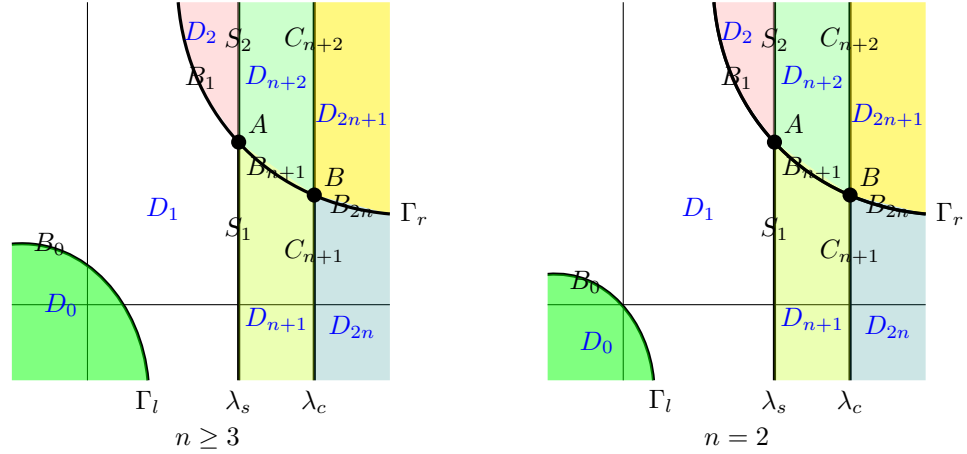
	D_0	D_1	D_2	D_{n+1}	D_{n+2}	D_{2n}	D_{2n+1}
$E.v.in (-\infty, 0)$	0	1	2	$n+1$	$n+2$	$2n$	$2n+1$

TABLE 1. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on D_k for $n \geq 2$.

	Curve B_k	Curve S_k	Curve C_k	Point A	Point B
$E.v.$ $(-\infty, 0)$	k	k	k	1	$n+1$
$Th.res.0$	$\frac{n=2}{n=3,4} \frac{-}{1}$	$\frac{n=2}{n \geq 3} \frac{2}{-}$	$n \geq 2 \quad -$	$\frac{n=2}{n=3,4} \frac{2}{1}$	$\frac{n=2}{n=3,4} \frac{-}{1}$
	$\frac{n \geq 5}{-}$			$\frac{n \geq 5}{-}$	$\frac{n \geq 5}{-}$
$Th.e.v.0$	$\frac{n=2}{n=3,4} \frac{-}{-}$	$\frac{n=2}{n \geq 3} \frac{-}{n}$	$n \geq 2 \quad n-1$	$\frac{n=2}{n=3,4} \frac{-}{n}$	$\frac{n=2}{n=3,4} \frac{1}{n-1}$
	$\frac{n \geq 5}{1}$			$\frac{n \geq 5}{n+1}$	$\frac{n \geq 5}{n}$

TABLE 2. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on the edges of D_k for $n \geq 2$.

Proof. (a) follows from Lemmas 3.18 and 4.4. (b) follows from Lemmas 3.24, 4.6 and 4.8. (c) follows from Lemmas 3.27 and 4.8. \square

FIGURE 5. Hyperbola for $n \geq 2$

We draw the results for $n = 2$ on (λ, μ) -plane in the right-hand side of Figure 5 and for $n \geq 3$ in the left-hand side of Figure 5.

5.2. Case of $n = 1$. Let $n = 1$. In this case, the asymptote of \mathfrak{H}_0 is $(\lambda_\infty(0), \mu_\infty(0)) = (1, 1)$, and λ_c is not defined. We also see that $\lambda_s = 1 = \lambda_\infty(0)$. Then we have 4 sets:

$$D_0 = G_0, \quad D_1 = G_1 \cap \mathfrak{S}_-, \quad D_2 = G_1 \cap \mathfrak{S}_+, \quad D_3 = G_2.$$

The boundaries of these sets define disjoint 3 curves:

$$B_0 = \Gamma_l, \quad B_2 = \Gamma_r, \quad S_1 = \mathfrak{S}_0.$$

Finally we define point C by $C = \Gamma_r \cap \mathfrak{S}_0$. Now we formulate next result for $n = 1$.

Theorem 5.3. *Let $n = 1$.*

- (a) *Assume $(\lambda, \mu) \in D_k$, $k \in \{0, 1, 2, 3\}$. Then $H_{\lambda\mu}$ has k eigenvalues in $(\infty, 0)$. In addition 0 is neither a threshold resonance nor a threshold eigenvalue (see Table 3).*

	D_0	D_1	D_2	D_3
$E.v.in (-\infty, 0)$	0	1	2	3

TABLE 3. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on D_k for $n = 1$.

- (b) *Assume that $(\lambda, \mu) \in S_1$. Then $H_{\lambda\mu}$ has a super-threshold resonance.*
(c) *Assume that $(\lambda, \mu) \in B_k \cup S_1$. Then the next result in Table 4 is true.*

	B_k	Curve S_k
$E.v.in (-\infty, 0)$	k	k
$Th.res.0$	—	—
$Th.e.v.0$	—	—

TABLE 4. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on the edges of D_k for $n = 1$.

In particular $H_{\lambda\mu}$ has neither a threshold resonance nor a threshold eigenvalue.

Proof. The theorem follows from Lemmas 3.19, 3.27, 4.4 and 4.8. □

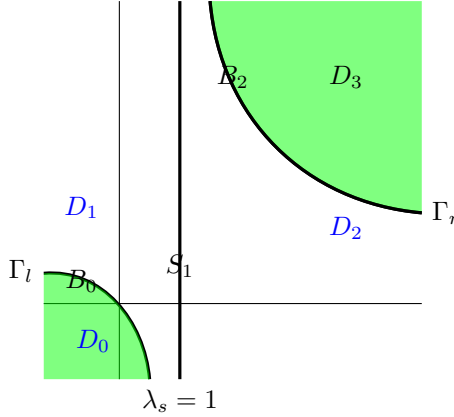


FIGURE 6. Hyperbola for $n = 1$

We draw the results for $n = 1$ on (λ, μ) -plane in Figure 6. In particular $(\lambda, \mu) \in S_1 \cup S_2$

5.3. Eigenvalues and asymptote. From results obtained in the previous section a stable point on λ can be found. In general the spectrum of $H_{\lambda\mu}$ is changed according to varying μ with a fixed λ . This can be also seen from Figures 2, 5 and 6. Curves on these figures consist of only hyperbolas and vertical lines. Then an asymptote has no intersection of these lines. It can be seen that

$$\{(1, \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\} \cup \{(\lambda, n) \in \mathbb{R}^2; \lambda \in \mathbb{R}\}$$

is the asymptote of hyperbola $\mathcal{H}_z(\lambda, \mu)$ for $n = 1, 2$. On the other hand

$$\{(a(0)/b(0), \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\} \cup \{(\lambda, n) \in \mathbb{R}^2; \lambda \in \mathbb{R}\}$$

is the asymptote of hyperbola $\mathcal{H}_z(\lambda, \mu)$ for $n \geq 3$. Then we can have the corollary below.

Corollary 5.4. *Let $\lambda = 1$ for $n = 1, 2$ and $\lambda = a(0)/b(0)$ for $n \geq 3$.*

- (1) *Let $n = 1$. Then $H_{\lambda\mu}$ has a super-threshold resonance 0 and only one eigenvalue in $(-\infty, 0)$ for any μ .*
- (2) *Let $n \geq 2$. Then $H_{\lambda\mu}$ has only one eigenvalue in $(-\infty, 0)$ for any μ .*

Proof. For $n = 1, 2$, let $l_n = S_1$. From Figures 5 and 6 it follows that $l_n \cap \Gamma_l = l_n \cap \Gamma_r = \emptyset$. Then the corollary follows. For $n \geq 3$, let $l_n = \{(a(0)/b(0), \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\}$. We can also see that $l_n \cap \Gamma_l = l_n \cap S_1 = l_n \cap S_2 = l_n \cap C_{n+1} = l_n \cap C_{n+2} = \emptyset$. Then the corollary follows. \square

APPENDIX A. PROOF OF LEMMA 3.8

Proof. We can see that

$$s(z) = 1 + z(a(z) + b(z)), \quad n = 1, \quad (\text{A.1})$$

$$s(z) = 1 + z(a(z) + b(z)) - (n-1)(a(z) - d(z)), \quad n \geq 2, \quad (\text{A.2})$$

and $a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}$ for $n = 1$.

(Case $n = 1$) From $a(z) - b(z) = \frac{1}{n} + \frac{z}{n}a(z)$ we see that $b(z) = a(z)(1-z) - 1$. Employing (A.1) and $a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}$, we have

$$a(z)s(z) = a(z) + z(a^2(z) + a^2(z)(1-z) - a(z)) = a(z)(1-z) - 1 = b(z), \quad z \leq 0. \quad (\text{A.3})$$

(Case $n \geq 2$) By

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\cos p}{E(p) - z} dp = \frac{1}{(2\pi)^2} \left(\int_{\mathbb{T}} \frac{\sin^2 p}{E(p) - z} dp \right) \left(\int_{\mathbb{T}} \frac{1}{E(p) - z} dp \right),$$

we obtain

$$b(z) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}} \frac{\sin^2 p_1}{E(p) - z} dp_1 \right) \left(\int_{\mathbb{T}} \frac{1}{E(p) - z} dp_1 \right) dp_2 \dots dp_n,$$

which provides

$$b(z) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{T}^{n-1} \times \mathbb{T}^{n-1}} F(\tilde{p})G(\tilde{q})d\tilde{p}d\tilde{q},$$

where $\tilde{p} = (p_2, \dots, p_n)$, $\tilde{q} = (p_2, \dots, p_n)$, $F(\tilde{p}) = \int_{\mathbb{T}} \frac{\sin^2 p_1}{E(p_1, \tilde{p}) - z} dp_1$ and $G(\tilde{q}) = \int_{\mathbb{T}} \frac{1}{E(p_1, \tilde{q}) - z} dp_1$. Then

$$a(z)s(z) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{T}^{n-1} \times \mathbb{T}^{n-1}} F(\tilde{p})G(\tilde{q})d\tilde{p}d\tilde{q},$$

and we can have the relations:

$$\begin{aligned}
a(z)s(z) - b(z) &= -\frac{1}{2(2\pi)^{2n}} \int_{\mathbb{T}^{n-1} \times \mathbb{T}^{n-1}} (F(\tilde{p}) - F(\tilde{q}))(G(\tilde{p}) - G(\tilde{q})) d\tilde{p} d\tilde{q}, \\
&= \left(\sum_{j=2}^n (\cos p_j - \cos q_j) \right)^2 \int_{\mathbb{T}} \frac{\sin^2 p_1 dp_1}{(E(p_1, \tilde{p}) - z)(E(p_1, \tilde{q}) - z)} \int_{\mathbb{T}} \frac{dp_1}{(E(p_1, \tilde{p}) - z)(E(p_1, \tilde{q}) - z)} \\
&\quad (F(\tilde{p}) - F(\tilde{q}))(G(\tilde{p}) - G(\tilde{q}))
\end{aligned} \tag{A.4}$$

prove $a(z)s(z) < b(z)$ for $n \geq 2$. Furthermore (A.4) shows that the last inequality leaves its sign invariant even for $z = 0$ and $n \geq 3$. Now we prove (3.25). By

$$\begin{aligned}
c(0) - d(0) &= \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(\cos p_1 - \cos p_2)^2}{E(p)} dp \\
s(0) &= \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{(\sin p_1 - \sin p_2)^2}{E(p)} dp,
\end{aligned}$$

we describe

$$\begin{aligned}
c(0) - d(0) &= \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{4 \sin^2 \frac{p_1 - p_2}{2} \sin^2 \frac{p_1 + p_2}{2}}{E(p)} dp \\
s(0) &= \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{4 \sin^2 \frac{p_1 - p_2}{2} \cos^2 \frac{p_1 + p_2}{2}}{E(p)} dp.
\end{aligned}$$

Introducing new variables $u = (p_1 - p_2)/2$ and $t = (p_1 + p_2)/2$ we get

$$c(0) - d(0) - s(0) = -2 \frac{1}{(2\pi)^n} \int_{\mathbb{T}^{n-2}} dp_3 \dots dp_n, \int_{\mathbb{T}} 4 \sin^2 u du \int_{\mathbb{T}} \frac{\cos 2t}{A - 2 \cos t \cos u} dt, \tag{A.5}$$

where $A = 2 + \sum_{j=3}^n (1 - \cos p_j)$ is a function being independent of both t and u . We have

$$\int_{\mathbb{T}} \frac{\cos 2t}{A - 2 \cos t \cos u} dt = 16Ac \int_0^{\pi/4} \frac{\cos^2 2t \cos^2 u}{(A^2 - (2 \cos t \cos u)^2)(A^2 - (2 \sin t \cos u)^2)} dt > 0.$$

Using the last inequality to (A.5) we get (3.25). \square

APPENDIX B. PROOF OF LEMMA 3.9

Proof. First we prove

$$a'(z)b(z) - a(z)b'(z) < 0, \quad z \in (-\infty, 0), \tag{B.1}$$

which proves the monotone decreasing of $\frac{a(z)}{b(z)}$. The equality

$$a'(z)b(z) - a(z)b'(z) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{T}^n \times \mathbb{T}^n} \cos p_1 \frac{E(p) - E(q)}{(E(p) - z)^2 (E(q) - z)^2} dp dq$$

gives

$$a'(z)b(z) - a(z)b'(z) = \frac{1}{n(2\pi)^{2n}} \sum_{j=1}^n \int_{\mathbb{T}^n \times \mathbb{T}^n} \cos p_j \frac{E(p) - E(q)}{(E(p) - z)^2 (E(q) - z)^2} dp dq.$$

Changing variables $E(p) - E(q) = \sum_{j=1}^n (\cos p_j - \cos q_j)$ provides the inequality

$$a'(z)b(z) - a(z)b'(z) = -\frac{1}{n(2\pi)^{2n}} \int_{\mathbb{T}^n \times \mathbb{T}^n} \frac{\left(\sum_{j=1}^n (\cos p_j - \cos q_j)\right)^2}{(E(p) - z)^2 (E(q) - z)^2} dp dq < 0$$

which proves $\left(\frac{a(z)}{b(z)}\right)' < 0$ in $(-\infty, 0)$. Using the definition of $a(z)$, we achieve $a(z) = O(\frac{1}{|z|})$ and $b(z) = O(\frac{1}{z^2})$ as $z \rightarrow -\infty$, and hence $a(z)/b(z) = O(|z|)$ as $z \rightarrow -\infty$ proves (3.26). Let $n = 1, 2$. By virtue of Lemma 3.6 we may write

$$\frac{a(z)}{b(z)} = \frac{1}{1 - \frac{b(z)-a(z)}{a(z)}} = \frac{1}{1 - \frac{1}{a(z)n} - \frac{z}{n}},$$

and since $a(z) = O(\frac{1}{z})$ as $z \rightarrow 0-$ we receive (3.27). In case $n \geq 3$, the limit (3.27) is obvious. \square

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¹FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA, 819-0395, JAPAN
E-mail address: hiroshima@math.kyushu-u.ac.jp

²FACULTY OF SCIENCE AND TECHNOLOGY, NILAI UNIVERSITY, 71800, NILAI, MALAYSIA

E-mail address: zimuminov@gmail.com

³FACULTY OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, 703004, SAMARKAND, UZBEKISTAN

E-mail address: utkir_nq83@mail.ru