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THRESHOLD OF DISCRETE SCHRÖDINGER OPERATORS WITH DELTA POTENTIALS ON n-DIMENSIONAL LATTICE

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ABSTRACT. Eigenvalue behaviors of Schrödinger operator defined on n-dimensional lattice with $n+1$ delta potentials is studied. It can be shown that lower threshold eigenvalue and lower threshold resonance are appeared for $n \geq 2$, and lower super-threshold resonance appeared for $n = 1$.

1. INTRODUCTION

Behavior of eigenvalues below the essential spectrum of standard Schrödinger operators of the form $-\Delta + \varepsilon V$ defined on $L^2(\mathbb{R}^n)$ is considerably studied so far. Here V is a negative potential and $\varepsilon \geq 0$ is a parameter which is varied. When ε approaches to some critical point $\varepsilon_c \geq 0$, each negative eigenvalues approaches to the left edge of the essential spectrum, and consequently they are absorbed into it. A mathematical crucial problem is to specify whether a negative eigenvalue survives as an eigenvalue or a threshold resonance on the edge of the essential spectrum at the critical point ε_c. Their behaviors depend on the spacial dimension n. Suppose that V is relatively compact with respect to $-\Delta$. Then the essential spectrum of $-\Delta + \varepsilon V$ is $[0, \infty)$. Roughly speaking $-\Delta f + \varepsilon_c V f = 0$ implies that $f = -\varepsilon_c (-\Delta)^{-1} V f$ and

$$\|(-\Delta)^{-1} g\|_{L^2}^2 = \int_{\mathbb{R}^n} |\hat{g}(k)|^2/|k|^4 dk, \quad \|(-\Delta)^{-1} g\|_{L^1} = \int_{\mathbb{R}^n} |\hat{g}(k)|/|k|^2 dk,$$

where $g = Vf$. Hence it may be expected that $f \in L^2(\mathbb{R}^n)$ if $n \geq 5$ and $f \in L^1(\mathbb{R}^n)$ for $n = 3, 4$. If 0 is an eigenvalue, it is called an embedded eigenvalue or threshold eigenvalue. Hence it may be expected that an embedded eigenvalue exists for $n \geq 5$. On the other hand for $n = 3, 4$, the eigenvector is predicted to be in $L^1(\mathbb{R}^n)$, and then 0 is called a threshold resonance.

The discrete Schrödinger operators have attracted considerable attentions for both combinatorial Laplacians and quantum graphs; for some recent summaries refer to see [5, 8, 3, 6, 4, 15, 11] and the references therein. Particularly, eigenvalue behavior of discrete Schrödinger operators are discussed in e.g. [1, 7, 2, 10] and are briefly discussed in [9, 12, 10] when potentials are delta functions with a single point mass. In [1] an explicit example of a $-\Delta - V$ on the three-dimensional lattice $\mathbb{Z}^3$, which possesses both a *lower* threshold resonance and a *lower* threshold eigenvalue, is constructed, where $-\Delta$ stands for the standard discrete Laplacian in $\ell^2(\mathbb{Z}^n)$ and V is a multiplication operator by the function

$$\hat{V}(x) = \mu \delta_{x0} + \frac{\lambda}{2} \sum_{|s|=1} \delta_{xs}, \qquad \lambda \geq 0, \mu \geq 0, \tag{1.1}$$

where δ_{xs} is the Kronecker delta.

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The authors of [12] considered the restriction of this operator to the Hilbert space $\ell^2_{\mathrm{e}}(\mathbb{Z}^3)$ of all even functions in $\ell^2(\mathbb{Z}^3)$. They investigated the dependence of the number of eigenvalues of $H_{\lambda\mu}$ on λ, μ for $\lambda > 0, \mu > 0$, and they showed that all eigenvalues arise either from a *lower* threshold resonance or from *lower* threshold eigenvalues under a variation of the interaction energy. Moreover, they also proved that the first *lower* eigenvalue of the Hamiltonian $-\Delta - V$ arises only from a *lower* threshold resonance under a variation of the interaction energy. A continuous version, two-particle Schrödinger operator, is shown by Newton (see p.1353 in [14]) and proved by Tamura [17, Lemma 1.1] using a result by Simon [16]. In case $\lambda = 0$, Hiroshima et.al. [10] showed that an threshold eigenvalue does appear for $n \geq 5$ but does not for $1 \leq n \leq 4$.

There are still interesting spectral properties of the discrete Schrödinger operators with potential of the form (1.1).

In this paper, we investigate the spectrum of $H_{\lambda\mu}$, specifically, *lower* and *upper* threshold eigenvalues and threshold resonances for *any*

$$(\lambda, \mu) \in \mathbb{R}^2 \quad \text{and} \quad n \geq 1.$$

We emphasize that there also appears so-called super-threshold resonances in our model for $n = 1$. See Proposition 4.7. The definitions of these are given in Definition 3.17. Our result is an extensions of [12, 1, 10].

In this paper, we study, in particular, eigenvalues in $(-\infty, 0)$, lower threshold eigenvalues, lower threshold resonances and lower super-threshold resonances. In a similar manner to this, we can also investigate eigenvalues in $(2n, \infty)$, upper threshold eigenvalues, upper threshold resonances and and upper super-threshold resonances, but we left them to readers, and we focus on studying the spectrum contained in $(-\infty, 0]$.

The paper is organized as follows. In Section 2, a discrete Schrödinger operator in the coordinate and momentum representation is described, and it is decomposed into direct sum of operators $H^{\mathrm{e}}_{\lambda\mu}$ and H^{o}_{λ}. The spectrum of $H^{\mathrm{e}}_{\lambda\mu}$ and H^{o}_{λ} are investigated in Section 3. Section 4 is devoted to showing main results, Theorems 5.2 and 5.3. The proofs of some lemmas belong to Appendix.

2. Discrete Schrödinger operators on lattice

Let $\mathbb{Z}^n$ be the n–dimensional lattice, i.e. the n–dimensional integer set. The Hilbert space of ℓ^2 sequences on $\mathbb{Z}^n$ is denoted by $\ell^2(\mathbb{Z}^n)$. A notation $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n = (-\pi, \pi]^n$ means the n-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^n$) equipped with its Haar measure, and let $L^2_{\mathrm{e}}(\mathbb{T}^n)$ (resp. $L^2_{\mathrm{o}}(\mathbb{T}^n)$) denote the subspace of all even (resp. odd) functions of the Hilbert space $L^2(\mathbb{T}^n)$ of L^2-functions on $\mathbb{T}^n$. Let $\langle \cdot, \cdot \rangle$ mean the inner product on $L^2(\mathbb{T}^n)$.

Let $T(y)$ be the shift operator by $y \in \mathbb{Z}^n$: $(T(y)f)(x) = f(x + y)$ for $f \in \ell^2(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$. The standard discrete Laplacian Δ on $\ell^2(\mathbb{Z}^n)$ is usually associated with the bounded self-adjoint multidimensional Toeplitz-type operator:

$$\Delta = \frac{1}{2} \sum_{\substack{x \in \mathbb{Z}^n \\ |x|=1}} (T(x) - T(0)).$$

Let us define the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^n)$ by

$$\hat{H}_{\lambda\mu} = -\Delta - \widehat{V},$$

where the potential $\widehat{V}$ depends on two parameters $\lambda, \mu \in \mathbb{R}$ and satisfies

$$(\widehat{V}f)(x) = \begin{cases} \mu f(x), & \text{if} \quad x = 0 \\ \frac{\lambda}{2} f(x), & \text{if} \quad |x| = 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}, \quad f \in \ell^2(\mathbb{Z}^n),\, x \in \mathbb{Z}^n,$$

which awards $\hat{H}_{\lambda\mu}$ to be a bounded self-adjoint operator. Let $\mathcal{F}$ be the standard Fourier transform $\mathcal{F} : L^2(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n)$ defined by $(Ff)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\theta) e^{ix\theta} d\theta$ for $f \in L^2(\mathbb{T}^n)$ and $x \in \mathbb{Z}^n$. The inverse Fourier transform is then given by $(F^{-1}f)(\theta) = \sum_{x \in \mathbb{Z}^n} f(x) e^{-ix\theta}$ for $f \in \ell^2(\mathbb{Z}^n)$ and $\theta \in \mathbb{T}^n$. The Laplacian Δ in the momentum representation is defined as

$$\widehat{\Delta} = \mathcal{F}^{-1} \Delta \mathcal{F},$$

and $\widehat{\Delta}$ acts as the multiplication operator:

$$(\widehat{\Delta}\hat{f})(p) = -E(p)\hat{f}(p),$$

where $E(p)$ is given by

$$E(p) = \sum_{j=1}^{n} (1 - \cos p_j).$$

In the physical literature, the function $\sum_{j=1}^{n}(1-\cos p_j)$, being a real valued-function on $\mathbb{T}^n$, is called the *dispersion relation* of the Laplace operator. We also define the discrete Schrödinger operator in momentum representation. Let $H_0 = -\hat{\Delta}$. The operator $H_{\lambda\mu}$, in the momentum representation, acts in the Hilbert space $L^2(\mathbb{T}^n)$ as

$$H_{\lambda\mu} = H_0 - V,$$

where V is an integral operator of convolution type

$$(Vf)(p) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} v(p-s) f(s) ds, \quad f \in L^2(\mathbb{T}^n).$$

Here the kernel function $v(\cdot)$ is the Fourier transform of $\widehat{V}(\cdot)$ computed as

$$v(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\mu + \lambda \sum_{i=1}^{n} \cos p_i \right),$$

and it allows the potential operator V to get the representation $V = V^{\mathrm{e}}_{\lambda\mu} + V^{\mathrm{o}}_{\lambda}$, where

$$V^{\mathrm{e}}_{\lambda\mu} = \mu \langle \cdot, \mathrm{c}_0 \rangle \mathrm{c}_0 + \frac{\lambda}{2} \sum_{j=1}^{n} \langle \cdot, \mathrm{c}_j \rangle \mathrm{c}_j, \quad V^{\mathrm{o}}_{\lambda} = \frac{\lambda}{2} \sum_{j=1}^{n} \langle \cdot, \mathrm{s}_j \rangle \mathrm{s}_j.$$

Here $\{\mathrm{c}_0, \mathrm{c}_j, \mathrm{s}_j : j = 1, \ldots, n\}$ is an orthonormal system in $L^2(\mathbb{T}^n)$, where

$$\mathrm{c}_0(p) = \frac{1}{(2\pi)^{\frac{n}{2}}}, \quad \mathrm{c}_j(p) = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \cos p_j, \quad \mathrm{s}_j(p) = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \sin p_j, \quad j = 1, \ldots, n.$$

One can check easily that the subspaces $L^2_{\mathrm{e}}(\mathbb{T}^n)$ of all even functions and $L^2_{\mathrm{o}}(\mathbb{T}^n)$ of all odd functions in $L^2(\mathbb{T}^n)$ reduce $H_{\lambda\mu}$. Adopting $V = V^{\mathrm{e}}_{\lambda\mu} + V^{\mathrm{o}}_{\lambda}$, we can see that the restriction $H^{\mathrm{e}}_{\lambda\mu}$ (resp. H^{o}_{λ}) of the operator $H_{\lambda\mu}$ to $L^2_{\mathrm{e}}(\mathbb{T}^n)$ (resp. $L^2_{\mathrm{o}}(\mathbb{T}^n)$) acts with the form

$$H^{\mathrm{e}}_{\lambda\mu} = H_0 - V^{\mathrm{e}}_{\lambda\mu} \quad (\text{resp. } H^{\mathrm{o}}_{\lambda} = H_0 - V^{\mathrm{o}}_{\lambda}).$$

Hence $H_{\lambda\mu}$ is decomposed into the even Hamiltonian and the odd Hamiltonian:

$$H_{\lambda\mu} = H^{\mathrm{e}}_{\lambda\mu} \oplus H^{\mathrm{o}}_{\lambda}$$

under the decomposition $L^2(\mathbb{T}^n) = L^2_{\rm e}(\mathbb{T}^n) \oplus L^2_{\rm o}(\mathbb{T}^n)$. We have the fundamental proposition below:

Proposition 2.1. *It follows that* $\sigma_{\rm ess}(H_{\lambda\mu}) = \sigma_{ac}(H_{\lambda\mu}) = [0, 2n]$.

Proof. The perturbation V is a finite rank operator and then the essential spectrum of the operator $H_{\lambda\mu}$ fills in $[0, 2n] = \sigma_{\rm ess}(H_0)$.

Let $\mathcal{H}_{\rm ac}$ be the absolutely continuous part of $H_{\lambda\mu}$. It can be seen that the wave operator $W_\pm = s - \lim_{t\to\pm\infty} e^{itH_{\lambda\mu}} e^{-itH_0}$ exists and is complete since $H_{\lambda\mu}$ is a finite rank perturbation of H_0. This implies that H_0 and $H_{\lambda\mu}\lceil_{\mathcal{H}_{\rm ac}}$ are unitarily equivalent by $W_\pm^{-1} H_0 W_\pm = H_{\lambda\mu}\lceil_{\mathcal{H}_{\rm ac}}$. Then $\sigma_{\rm ac}(H_0) = \sigma_{\rm ac}(H_{\lambda\mu}) = [0, 2n]$. □

In what follows, we shall study the spectrum of $H_{\lambda\mu}$ by investigating the spectrum of $H^{\rm e}_{\lambda\mu}$ and $H^{\rm o}_{\lambda}$ separately.

3. Spectrum of $H^{\rm e}_{\lambda\mu}$

3.1. **Birman-Schwinger principle for** $z \in \mathbb{C} \setminus [0, 2n]$**.** The Birman-Schwinger principle helps us to reduce the problem to the study of spectrum of a finite dimensional linear operator: a matrix.

We denote the resolvent of Laplacian H_0 by $(H_0 - z)^{-1}$, where $z \in \mathbb{C} \setminus [0, 2n]$. We can see that $(H_0 - z)^{-1} V^{\rm e}_{\lambda\mu}$ is a finite rank operator. Let M_{n+1} denote the linear hull of $\{c_0, \cdots, c_n\}$. Then M_{n+1} is an $(n+1)$-dimensional subspace of $L^2_{\rm e}(\mathbb{T}^n)$. Furthermore we define $\tilde{M}_{n+1} = (H_0 - z)^{-1} M_{n+1}$ for $z \in \mathbb{C} \setminus [0, 2n]$. Then $\tilde{M}_{n+1}$ is also an $(n+1)$-dimensional subspace of $L^2_{\rm e}(\mathbb{T}^n)$ since $(H_0 - z)^{-1}$ is invertible. We define $C_1 : \mathbb{C}^{n+1} \to L^2_{\rm e}(\mathbb{T}^n)$ by the map

$$C_1 : \mathbb{C}^{n+1} \ni \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \mapsto (H_0 - z)^{-1} \left(\mu w_0 c_0 + \frac{\lambda}{2} \sum_{j=1}^{n} w_j c_j \right) \in \tilde{M}_{n+1},$$

and define $C_2 : L^2_{\rm e}(\mathbb{T}^n) \to \mathbb{C}^{n+1}$ by the map

$$C_2 : L^2_{\rm e}(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \langle \phi, c_0 \rangle \\ \vdots \\ \langle \phi, c_n \rangle \end{pmatrix} \in \mathbb{C}^{n+1}.$$

Then we have the sequence of maps:

$$L^2_{\rm e}(\mathbb{T}^n) \xrightarrow{C_2} \mathbb{C}^{n+1} \xrightarrow{C_1} L^2_{\rm e}(\mathbb{T}^n) \tag{3.1}$$

and $C_1 C_2 : L^2_{\rm e}(\mathbb{T}^n) \to L^2_{\rm e}(\mathbb{T}^n)$. Notice that C_1 and C_2 depend on the choice of z. We directly have

$$(H_0 - z)^{-1} V^{\rm e}_{\lambda\mu} = C_1 C_2. \tag{3.2}$$

Define

$$G_{\rm e}(z) = C_2 C_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}.$$

We shall show the explicit form of $G_{\rm e}(z)$ in (3.14) below.

Lemma 3.1 (Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$)**.**

(a) $z \in \mathbb{C} \setminus [0, 2n]$ *is an eigenvalue of* $H^{\rm e}_{\lambda\mu}$ *if and only if* $1 \in \sigma(G_{\rm e}(z))$.

(b) *Suppose that* $z \in \mathbb{C} \setminus [0, 2n]$ *and* (λ, μ) *satisfies* $\det(G_{\rm e}(z) - {\rm I}) = 0$. *Then the vector* $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^{n+1}$ *is an eigenvector of* $G_{\rm e}(z)$ *associated with eigenvalue* 1 *if and only if* $f = C_1 Z$, *i.e.*

$$f(p) = \frac{1}{(2\pi)} \frac{1}{E(p) - z} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \cos p_j \right) \tag{3.3}$$

is an eigenfunction of $H^{\rm e}_{\lambda\mu}$ *associated with eigenvalue* z.

Proof. It can be seen that $H^{\rm e}_{\lambda\mu} f = zf$ if and only if $f = (H_0 - z)^{-1} V^{\rm e}_{\lambda\mu} f$. Then $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of $H^{\rm e}_{\lambda\mu}$ if and only if $1 \in \sigma((H_0 - z)^{-1} V^{\rm e}_{\lambda\mu})$. Hence $z \in \mathbb{C} \setminus [0, 2n]$ is an eigenvalue of $H^{\rm e}_{\lambda\mu}$ if and only if $1 \in \sigma(C_1 C_2)$ by the fact $\sigma(C_1C_2) \setminus \{0\} = \sigma(C_2C_1) \setminus \{0\}$.

Then it completes the proof of (a). We can also see that $C_2 C_1 Z = Z$ if and only if $f = (H_0 - z)^{-1} V^{\rm e}_{\lambda\mu} f = C_1 C_2 f$, where $f = C_1 Z$. Then the function f coincides with (3.3). □

3.2. **Birman-Schwinger principle for** $z = 0$**.** We consider the Birman-Schwinger principle for $z = 0$, which is the edge of the continuous spectrum of $H^{\rm e}_{\lambda\mu}$, and it is the main issue to specify whether it is eigenvalue or threshold of $H^{\rm e}_{\lambda\mu}$.

In order to discuss $z = 0$ we extend the eigenvalue equation $H^{\rm e}_{\lambda\mu} f = 0$ in $L^2_{\rm e}(\mathbb{T}^n)$ to that in $L^1_{\rm e}(\mathbb{T}^n)$. Note that $L^2_{\rm e}(\mathbb{T}^n) \subset L^1_{\rm e}(\mathbb{T}^n)$. We consider the equation

$$E(p)f(p) - \frac{\mu}{(2\pi)^n} \int_{\mathbb{T}^n} f(p)dp - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^{n} \cos p_j \int_{\mathbb{T}^n} \cos p_j f(p) dp = 0 \tag{3.4}$$

in the Banach space $L^1_{\rm e}(\mathbb{T}^n)$. Conveniently, we describe (3.4) as $H^{\rm e}_{\lambda\mu} f = 0$. Since we consider a solution $f \in L^1_{\rm e}(\mathbb{T}^n)$, the integrals $\int_{\mathbb{T}^n} f(p)dp$ and $\int_{\mathbb{T}^n} \cos p_j f(p)dp$ are finite for $j = 1, ..., n$.

The unique singular point of $1/E(p)$ is $p = 0$, and in the neighborhood of $p = 0$, we have $E(p) \approx |p|^2$. Then the following lemma is fundamental, and its proof is straightforward.

Lemma 3.2. *Let* $h(p) = \varphi(p)/E(p)$, *where* $\varphi \in C(\mathbb{T}^n)$. *Then (a)-(e) follow.*

(a) *It follows that* $h \in L^2(\mathbb{T}^n)$ *for* $n \geq 5$, *and* $h \in L^1(\mathbb{T}^n)$ *for* $n \geq 3$.
(b) *Let* $1 \leq n \leq 4$ *and* $h \in L^2(\mathbb{T}^n)$. *Then* $\varphi(0) = 0$.
(c) *Let* $1 \leq n \leq 4$, $|\varphi(p)| < C|p|^{\alpha_n}$ *for some* $C > 0$ *and* $\alpha_n > \frac{4-n}{2}$. *Then* $h \in L^2(\mathbb{T}^n)$.
(d) *Let* $n = 1, 2$ *and* $h \in L^1(\mathbb{T}^n)$. *Then* $\varphi(0) = 0$.
(e) *Let* $n = 1, 2$, $|\varphi(p)| < C|p|^{\alpha_n}$ *for some* $C > 0$ *and* $\alpha_n > 2 - n$. *Then* $h \in L^1(\mathbb{T}^n)$.

Operator H_0^{-1} is not bounded in $L^2_{\rm e}(\mathbb{T}^n)$ as well as in $L^1_{\rm e}(\mathbb{T}^n)$. It is however obvious by Lemma 3.2 and $V^{\rm e}_{\lambda\mu} f \in C(\mathbb{T}^n)$ that

$$L^2_{\rm e}(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V^{\rm e}_{\lambda\mu} f \in L^2_{\rm e}(\mathbb{T}^n), \quad n \geq 5, \tag{3.5}$$

$$L^1_{\rm e}(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V^{\rm e}_{\lambda\mu} f \in L^1_{\rm e}(\mathbb{T}^n), \quad n \geq 3. \tag{3.6}$$

Thus for $n \geq 3$ we can extend operators C_1 and C_2 defined in the previous section. Let $n \geq 3$ and $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}$. $\bar{C}_1 : \mathbb{C}^{n+1} \to L^1_{\mathrm{e}}(\mathbb{T}^n)$ is defined by

$$\bar{C}_1 Z = \frac{1}{(2\pi)} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \cos p_j \right)$$

and $\bar{C}_2 : L^1_{\mathrm{e}}(\mathbb{T}^n) \to \mathbb{C}^{n+1}$ by

$$\bar{C}_2 : L^1_{\mathrm{e}}(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \int_{\mathbb{T}^n} \phi(p) c_0 dp \\ \int_{\mathbb{T}^n} \phi(p) c_1(p) dp \\ \vdots \\ \int_{\mathbb{T}^n} \phi(p) c_n(p) dp \end{pmatrix} \in \mathbb{C}^{n+1}.$$

Then $\overline{C}_1 \overline{C}_2 : L^1_{\mathrm{e}}(\mathbb{T}^n) \to L^1_{\mathrm{e}}(\mathbb{T}^n)$. Consequently $G_{\mathrm{e}}(0) = \overline{C}_2 \overline{C}_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is described as an $(n+1) \times (n+1)$ matrix.

Lemma 3.3. *Let $n \geq 3$. Then it follows that (1) $\lim_{z \to 0} G_{\mathrm{e}}(z) = G_{\mathrm{e}}(0)$ and (2) $\sigma(H_0^{-1} V^{\mathrm{e}}_{\lambda\mu}) \setminus \{0\} = \sigma(G_{\mathrm{e}}(0)) \setminus \{0\}$.*

Proof. The proof is straightforward. □

Lemma 3.4 (Birman-Schwinger principle for $z = 0$). *Let $n \geq 3$. Then (a) and (b) follow. (a) Equation $H^{\mathrm{e}}_{\lambda\mu} f = 0$ has a solution in $L^1(\mathbb{T}^n)$ if and only if $1 \in \sigma(G_{\mathrm{e}}(0))$. (b) Let $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^{n+1}$ be the solution of $G_{\mathrm{e}}(0) Z = Z$ if and only if*

$$f(p) = \overline{C}_1 Z(p) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \cos p_j \right) \tag{3.7}$$

is a solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$, where $w_0, \cdots, w_n$ are actually described by

$$w_0 = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) dp, \quad w_j = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) \cos p_j dp, \quad j = 1, \ldots, n. \tag{3.8}$$

Proof. Let us consider $H^{\mathrm{e}}_{\lambda\mu} f = 0$ in $L^1_{\mathrm{e}}(\mathbb{T}^n)$. Hence $f = H_0^{-1} V^{\mathrm{e}}_{\lambda\mu} f$ in $L^1_{\mathrm{e}}(\mathbb{T}^n)$. Then L^1-solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$ exists if and only if $1 \in \sigma(H_0^{-1} V^{\mathrm{e}}_{\lambda\mu})$, and hence L^1-solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$ exists if and only if $1 \in \sigma(\bar{C}_1 \bar{C}_2)$. Due to the fact $\sigma(\bar{C}_1 \bar{C}_2) \setminus \{0\} = \sigma(G^0_{\mathrm{e}}) \setminus \{0\}$ the proof of (a) is complete. We can also see that $\bar{C}_2 \bar{C}_1 Z = Z$ if and only if $f = H_0^{-1} V^{\mathrm{e}}_{\lambda\mu} f = \bar{C}_1 \bar{C}_2 f$, where $f = \bar{C}_1 Z$. Then the function f coincides with (3.7). This fact ends the proof of (b). □

3.3. **Zeros of** $\det(G_{\mathrm{e}}(z) - \mathrm{I})$.

3.3.1. *Factorization.* By the Birman-Schwinger principle in what follows we focus on investigating the spectrum of the $(n+1)\times(n+1)$-matrix $G_{\rm e}(z)$. Since $G_{\rm e}(z)$ is defined for $z\in(-\infty,0)$ for $n=1,2,$ and $z\in(-\infty,0]$ for $n\geq 3$. Hence in this section we suppose that $z\in\begin{cases}(-\infty,0) & n=1,2,\\ (-\infty,0] & n\geq 3.\end{cases}$
As the function $E(p)=E(p_1,\dots,p_n)$ is invariant with respect to the permutations of its arguments $p_1,\dots,p_n$, the integrals used for studying the spectrum of $G_{\rm e}(z)$:

$$a(z)=\langle c_0,(H_0-z)^{-1}c_0\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{1}{E(p)-z}dp, \tag{3.9}$$

$$b(z)=\frac{1}{\sqrt{2}}\langle c_0,(H_0-z)^{-1}c_j\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos p_j}{E(p)-z}dp, \tag{3.10}$$

$$c(z)=\frac{1}{2}\langle c_j,(H_0-z)^{-1}c_j\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos^2 p_j}{E(p)-z}dp, \tag{3.11}$$

$$d(z)=\frac{1}{2}\langle c_i,(H_0-z)^{-1}c_j\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos p_i\cos p_j}{E(p)-z}dp,\quad i\neq j, \tag{3.12}$$

$$s(z)=\frac{1}{2}\langle {\rm s}_j,(H_0-z)^{-1}{\rm s}_j\rangle=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\sin^2 p_j}{E(p)-z}dp. \tag{3.13}$$

also do not depend on the particular choice of indices $0\leq i,j\leq n$. Note that $a(z),b(z),c(z)$ and $s(z)$ are defined for $n\geq 1$ but $d(z)$ for $n\geq 2$. From the definition of $G_{\rm e}(z)=(a_{ij})_{0\leq i,j\leq n}$, coefficients $a_{ij}=a_{ij}(z)$ are explicitly described as

$$\begin{cases} a_{00}(z)=\mu a(z),\quad a_{0j}(z)=\frac{\lambda}{\sqrt{2}}b(z), & j=1,\dots,n,\\ a_{i0}(z)=\sqrt{2}\mu b(z), a_{ii}(z)=\lambda c(z), & i=1,...,n\\ a_{ij}(z)=\lambda d(z), & i,j=1,\dots n,\ j\neq i,\end{cases}$$

Hence for $n\geq 2$ the matrix $G_{\rm e}(z)$ has the form

$$G_{\rm e}(z)=\begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}}b(z) & \dots & \dots & \frac{\lambda}{\sqrt{2}}b(z)\\ \sqrt{2}\mu b(z) & \lambda c(z) & \lambda d(z) & \dots & \lambda d(z)\\ \vdots & \lambda d(z) & \ddots & \dots & \vdots\\ \vdots & \vdots & \dots & \ddots & \lambda d(z)\\ \sqrt{2}\mu b(z) & \lambda d(z) & \dots & \lambda d(z) & \lambda c(z)\end{pmatrix} \tag{3.14}$$

and for $n=1$,

$$G_{\rm e}(z)=\begin{pmatrix}\mu a(z) & \frac{\lambda}{\sqrt{2}}b(z)\\ \sqrt{2}\mu b(z) & \lambda c(z)\end{pmatrix}. \tag{3.15}$$

In order to study the eigenvalue 1 of $G_{\rm e}(z)$ we calculate the determinant of $G_{\rm e}(z)-{\rm I}$.

Lemma 3.5. *We have*

$$\det(G_{\rm e}(z)-{\rm I})=\delta_r(\lambda,\mu;z)\delta_c(\lambda;z), \tag{3.16}$$

where

$$\delta_r(\lambda,\mu;z)=\begin{cases}\big(1-\mu a(z)\big)\Big\{1-\lambda\big(c(z)+(n-1)d(z)\big)\Big\}-n\lambda\mu b^2(z), & n\geq 2\\ \big(1-\mu a(z)\big)\big(1-\lambda c(z)\big)-\lambda\mu b^2(z), & n=1,\end{cases} \tag{3.17}$$

$$\delta_c(\lambda;z)=\begin{cases}\big\{\lambda(c(z)-d(z))-1\big\}^{n-1}, & n\geq 2\\ 1, & n=1.\end{cases} \tag{3.18}$$

Proof. It is a straightforward computation. □

By the factorization (3.16) we shall study zeros of $\delta_r(\lambda,\mu;z)$ and $\delta_c(\lambda;z)$ separately to see eigenvalues and resonances of $H^{\mathrm{e}}_{\lambda\mu}$. To see this we introduce algebraic relations used to estimate zeros of $\delta_r(\lambda,\mu;z)$ and $\delta_c(\lambda;z)$. Below we shall show the list of formulas of coefficients $a(z)$, $b(z)$, etc. We set

$$\alpha(z)=\begin{cases} c(z)+(n-1)d(z), & n\geq 2\\ c(z), & n=1,\end{cases} \tag{3.19}$$

$$\gamma(z)=a(z)\alpha(z)-nb^2(z). \tag{3.20}$$

Lemma 3.6. *For any $z<0$, the relations below hold:*

$$\begin{aligned} &a(z)-b(z)=\frac{1}{n}+\frac{z}{n}a(z),\\ &\alpha(z)=(n-z)b(z),\\ &\gamma(z)=b(z).\end{aligned}$$

Proof. We directly see that

$$\begin{aligned} a(z)-b(z)&=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{(1-\cos p_1)}{E(p)-z}dp=\frac{1}{n}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\sum_{j=1}^n(1-\cos p_j)}{E(p)-z}dp\\ &=\frac{1}{n}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}dp+\frac{z}{n}\int_{\mathbb{T}^n}\frac{1}{E(p)-z}dp=\frac{1}{n}+\frac{z}{n}a(z),\end{aligned}$$

and

$$\alpha(z)=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos p_1(z-E(p))}{E(p)-z}dp+\frac{n-z}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\cos p_1}{E(p)-z}dp=(n-z)b(z).$$

From these we can also get the third equality of the lemma. □

Lemma 3.7. *Functions $a(z)$, $\alpha(z)$, $\gamma(z)$, $b(z)$, $c(z)-d(z)$ and $s(z)$ are monotonously increasing and positive in $(-\infty,0]$. Moreover, their limits tend to zero as z tends to $-\infty$.*

Proof. We have the representations of $a(z)$, $\gamma(z)=b(z)$ and $s(z)$ by their definitions as

$$\alpha(z)=\frac{1}{n}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\left(\sum_{i=1}^n\cos p_i\right)^2}{E(p)-z}dp, \tag{3.21}$$

$$\gamma(z)=b(z)=\frac{1}{2n(2\pi)^{2n}}\int_{\mathbb{T}^n\times\mathbb{T}^n}\frac{\left(\sum_{j=1}^n(\cos p_j-\cos q_j)\right)^2}{(E(p)-z)(E(q)-z)}dpdq, \tag{3.22}$$

$$c(z)-d(z)=\frac{1}{2}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\left(\cos p_1-\cos p_2\right)^2}{E(p)-z}dp, \tag{3.23}$$

$$s(z)=\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{\sin^2 p_1}{E(p)-z}dp. \tag{3.24}$$

Indeed, for any fixed $p\in\mathbb{T}^n$, all the integrands are positive and monotonously increasing as functions of z, and we complete the proof. □

Note that in Lemma 3.7 $c(z)-d(z)$ is considered only in the case of $n\geq 2$.

Lemma 3.8. *The following relations hold:*

$$\begin{aligned} a(z)s(z) &= b(z), \quad n=1,\, z<0,\\ a(z)s(z) &< b(z), \quad n=2,\, z<0,\\ a(z)s(z) &< b(z), \quad n\geq 3,\, z\leq 0,\\ c(z)-d(z) &< s(z), \quad n\geq 2,\, z\leq 0. \end{aligned} \tag{3.25}$$

Proof. See Appendix A. □

Lemma 3.9. *The function $a(z)/b(z)$ is monotonously decreasing in $(-\infty, 0]$, and there exist limits:*

$$\lim_{z\to-\infty} \frac{a(z)}{b(z)} = +\infty, \tag{3.26}$$

$$\lim_{z\to 0-} \frac{a(z)}{b(z)} = \begin{cases} 1, & n=1,2,\\ \frac{a(0)}{b(0)}, & n\geq 3. \end{cases} \tag{3.27}$$

Proof. See Appendix B. □

3.3.2. *Zeros of $\delta_r(\lambda,\mu;z)$.* We extend $\delta_r(\lambda,\mu;\cdot)$ and $\delta_c(\lambda;\cdot)$, and discuss zeros of them to specify the eigenvalue of $H^{\mathrm{e}}_{\lambda\mu}$. Let $z\in(-\infty,0)$. Applying notation in (3.19), we describe $\delta_r(\lambda,\mu;z)$ as

$$\delta_r(\lambda,\mu;z) = \gamma(z)\mathcal{H}_z(\lambda,\mu) \tag{3.28}$$

where

$$\mathcal{H}_z(\lambda,\mu) = \Big(\lambda - \frac{a(z)}{\gamma(z)}\Big)\Big(\mu - \frac{\alpha(z)}{\gamma(z)}\Big) - \frac{a(z)\alpha(z)-\gamma(z)}{\gamma^2(z)}. \tag{3.29}$$

or by Lemma 3.6, we have

$$\mathcal{H}_z(\lambda,\mu) = \Big(\lambda - \frac{a(z)}{b(z)}\Big)\Big(\mu - (n-z)\Big) - n. \tag{3.30}$$

Instead of the equation $\delta_r(\lambda,\mu;z) = 0$, the relation (3.28) allows us to study the family of rectangular hyperbola $\mathfrak{H}_z$ indexed by $z \in \begin{cases} (-\infty,0) & n=1,2\\ (-\infty,0] & n\geq 3 \end{cases}$. i.e. equilateral hyperbola $\mathfrak{H}_z$ on (λ,μ)-plane, which is defined by

$$\mathfrak{H}_z = \{(\lambda,\mu)\in\mathbb{R}^2 | \mathcal{H}_z(\lambda,\mu)=0\}$$

with asymptote

$$(\lambda_\infty(z), \mu_\infty(z)) = (\frac{a(z)}{b(z)}, n-z).$$

Lemma 3.9 implies that $\mathcal{H}_z(\lambda,\mu)$ can be extended to $z\in(-\infty,0]$ for any dimension $n\geq 1$ as

$$\bar{\mathcal{H}}_z(\lambda,\mu) = \begin{cases} \mathcal{H}_z(\lambda,\mu), & z<0,\\ (\lambda-X)(\mu-n)-n, & z=0. \end{cases} \tag{3.31}$$

Here $X=1$ for $n=1,2$ and $X=a(0)/b(0)$ for $n\geq 3$. Note that

$$\overline{\mathcal{H}}_z(0,0) = \frac{a(z)}{b(z)}(n-z) - n = \frac{1}{b(z)} > 0$$

for $z<0$. We also extend the family of hyperbola $\mathfrak{H}_z$, $z\in(-\infty,0)$, to that of hyperbola $\overline{\mathfrak{H}}_z$ indexed by $z\in(-\infty,0]$ by

$$\overline{\mathfrak{H}}_z = \{(\lambda,\mu)\in\mathbb{R}\times\mathbb{R} | \overline{\mathcal{H}}_z(\lambda,\mu)=0\}.$$

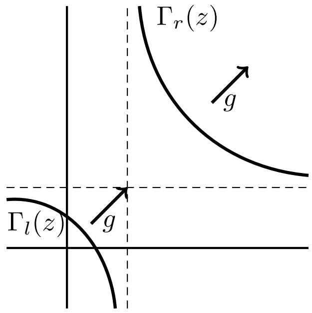

FIGURE 1. Hyperbola moves as z approaches to $-\infty$ from 0.

By (3.31) we see that $(\lambda, \mu) \in \mathfrak{H}_0$ satisfies the algebraic relation:

$$(\lambda - X)(\mu - n) - n = 0. \tag{3.32}$$

For any $z_1 < z_2$, $z_1, z_2 \in (-\infty, 0]$, we note that the hyperbola $\overline{\mathfrak{H}}_{z_1}$ can be moved to $\overline{\mathfrak{H}}_{z_2}$ in parallel by the vector $g = \begin{pmatrix} \lambda_\infty(z_2) - \lambda_\infty(z_1) \\ \mu_\infty(z_2) - \mu_\infty(z_1) \end{pmatrix}$ whose components are positive. See Figure 1. Let $\Gamma_l(z)$ (resp. $\Gamma_r(z)$) denote the left brunch (resp. the right brunch) of the hyperbola $\overline{\mathfrak{H}}_z$, i.e.

$$\overline{\mathfrak{H}}_z = \Gamma_l(z) \cup \Gamma_r(z) \quad \text{and} \quad \Gamma_l(z) \cap \Gamma_r(z) = \emptyset.$$

We then see that for any $z_2 < z_1 \leq 0$ it follows that

$$\Gamma_l(z_1) \cap \Gamma_l(z_2) = \emptyset, \quad \Gamma_r(z_1) \cap \Gamma_r(z_2) = \emptyset. \tag{3.33}$$

Let us see the behavior of $\delta_r(\lambda, \mu; z)$ near $z = 0$ for $n = 1, 2$.

Lemma 3.10. *It follows that*

$$a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}, \quad n = 1, \tag{3.34}$$

$$a(z) = -\frac{\sqrt{2}}{2\pi} \ln(-z) + (\frac{1}{2} - \frac{\sqrt{2}}{\pi}) + O(-z), \quad as \quad z \to 0-, \quad n = 2. \tag{3.35}$$

Proof. The proof of this lemma can be found in [13]. □

From this lemma we can see the behaviours of $\delta_r(\lambda, \mu; z)$ as $z \to 0-$.

Corollary 3.11. *It follows that*

$$(n = 1) \quad \lim_{z \to 0-} \delta_r(\lambda, \mu; z) = \begin{cases} \infty & (\lambda, \mu) \notin \overline{\mathfrak{H}}_0 \\ 1 - \mu & (\lambda, \mu) \in \overline{\mathfrak{H}}_0 \end{cases},$$

$$(n = 2) \quad \lim_{z \to 0-} \delta_r(\lambda, \mu; z) = \begin{cases} \infty & (\lambda, \mu) \notin \overline{\mathfrak{H}}_0 \\ 1 - \mu/2 & (\lambda, \mu) \in \overline{\mathfrak{H}}_0 \end{cases},$$

$$(n \geq 3) \quad \lim_{z \to 0-} \delta_r(\lambda, \mu; z) = b(0)\overline{\mathcal{H}}_0(\lambda, \mu).$$

Proof. In the case of $n \geq 3$ it is trivial to see that $\lim_{z \to 0-} \delta_r(\lambda, \mu; z) = b(0)\overline{\mathcal{H}}_0(\lambda, \mu)$. Then we consider cases of $n = 1, 2$. We recall that

$$\delta_r(\lambda, \mu; z) = \gamma(z)\overline{\mathcal{H}}_z(\lambda, \mu) = \gamma(z)\overline{\mathcal{H}}_0(\lambda, \mu) + \gamma(z)(\overline{\mathcal{H}}_z(\lambda, \mu) - \overline{\mathcal{H}}_0(\lambda, \mu))$$

and

$$\gamma(z) = b(z) = a(z) - \frac{1}{n} - \frac{1}{n} za(z).$$

We can also directly see that for $n = 1, 2$

$$\begin{aligned}\overline{\mathcal{H}}_z(\lambda, \mu) - \overline{\mathcal{H}}_0(\lambda, \mu) &= \frac{1}{b(z)}(b(z) - a(z))(\mu - n) + z(\lambda - \frac{a(z)}{b(z)}) \\ &= -\frac{1}{nb(z)}(1 + za(z))(\mu - n) + z(\lambda - \frac{a(z)}{b(z)}).\end{aligned}$$

Together with them we have

$$\delta_r(\lambda, \mu; z) = (a(z) - \frac{1 + za(z)}{n})\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(\frac{n - \mu}{n}) + \xi,$$

where

$$\xi = -za(z)(\lambda - \frac{a(z)}{b(z)}) + \frac{1 + za(z)}{n} \left(\frac{(1 + za(z))(\mu - n)}{nb(z)} + z(\lambda - \frac{a(z)}{b(z)}) \right).$$

By Lemmas 3.10 and 3.9 it is crucial to see that

$$\lim_{z \to 0-} za(z) = 0, \quad \lim_{z \to 0-} b(z) = \infty, \quad \lim_{z \to 0-} \frac{a(z)}{b(z)} = 1$$

and

$$\lim_{z \to 0-} \xi = 0, \quad \lim_{z \to 0-} \frac{a(z)}{b(z)}(1 + za(z))(\frac{n - \mu}{n}) = 1 - \frac{\mu}{n}$$

for $n = 1, 2$. Let $n = 1$. Then

$$\delta_r(\lambda, \mu; z) = (a(z) - 1 - za(z))\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(1 - \mu) + \xi$$

and the corollary follows for $n = 1$. Let $n = 2$. In a similar manner to the case of $n = 1$ we have

$$\delta_r(\lambda, \mu; z) = (a(z) - \frac{1 + za(z)}{2})\overline{\mathcal{H}}_0(\lambda, \mu) + \frac{a(z)}{b(z)}(1 + za(z))(1 - \frac{\mu}{2}) + \xi,$$

and the corollary follows for $n = 2$. Hence the proof of the corollary can be derived. □

We define $\bar{\delta}_r(\lambda, \mu; z)$ for $z \in (-\infty, 0]$ by

$$\bar{\delta}_r(\lambda, \mu; z) = \begin{cases} \delta_r(\lambda, \mu; z), & z \in (-\infty, 0), \\ \lim_{z \to 0-} \delta_r(\lambda, \mu; z), & z = 0. \end{cases} \tag{3.36}$$

From Corollary 3.11 we can see that $\bar{\delta}_r(\lambda, \mu; z)$ converges to

$$\bar{\delta}_r(\lambda, \mu; 0) = \begin{cases} 1 - \mu, & n = 1, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0, \\ 1 - \mu/2, & n = 2, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0, \\ 0, & n \geq 3, \quad (\lambda, \mu) \in \overline{\mathfrak{H}}_0. \end{cases} \tag{3.37}$$

Remark 3.12. *We give a remark on* (3.37)*. Let* $n = 1, 2$*. If* $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$*, then* $(1 - \lambda)(1 - \mu/n) = 1$ *is satisfied by* (3.32)*, which implies that* $1 - \mu \neq 0$ *for* $n = 1$*,* $1 - \mu/2 \neq 0$ *for* $n = 2$*.*

We can also show the continuity of $\bar{\delta}_r(\lambda, \mu; z)$ on z, which is summarised in the lemma below.

Lemma 3.13. *It follows that*

($n = 1, 2$): $\bar{\delta}_r(\lambda, \mu; z)$ *is continuous in* $z \in (-\infty, 0]$ *for* $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$*,*
($n \geq 3$): $\bar{\delta}_r(\lambda, \mu; z)$ *is continuous in* $z \in (-\infty, 0]$ *for* $(\lambda, \mu) \in \mathbb{R}^2$*.*

Let $z = 0$. Then the asymptote of the hyperbola $\mathfrak{H}_0$ is given by

$$(\lambda_\infty(0), \mu_\infty(0)) = \begin{cases} (1, n) & n = 1, 2, \\ (\frac{a(0)}{b(0)}, n) & n \geq 3. \end{cases}$$

The brunches $\Gamma_l(0)$ and $\Gamma_r(0)$ of the hyperbola $\overline{\mathfrak{H}}_0$ split $\mathbb{R}^2$ into three open sets

$$G_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) > 0, \lambda < \lambda_\infty(0)\},$$
$$G_1 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) < 0\},$$
$$G_2 = \{(\lambda, \mu) \in \mathbb{R}^2; \overline{\mathcal{H}}_0(\lambda, \mu) > 0, \lambda > \lambda_\infty(0)\}.$$

We set

$$\Gamma_l = \Gamma_l(0), \quad \Gamma_r = \Gamma_r(0)$$

for notational simplicity. Hence $\partial G_0 = \Gamma_l$ and $\partial G_2 = \Gamma_r$ follow from the definition of G_0 and G_2. See Figure 2. We can check the value of $\bar{\delta}_r(\lambda, \mu; z)$ for each (λ, μ) and $z \in (-\infty, 0]$ in the next lemma.

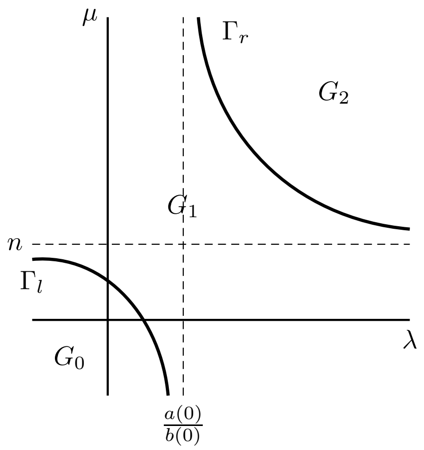

FIGURE 2. Region of G_j for $n \geq 3$

Lemma 3.14. *We have the following facts:*

(a) (1) *Let* $(\lambda, \mu) \in G_0 \cup \Gamma_l$. *Then* $\bar{\delta}_r(\lambda, \mu; z) \neq 0$ *for* $z \in (-\infty, 0)$.
   (2) *Let* $(\lambda, \mu) \in \Gamma_l$. *Then* $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ *for* $n = 1, 2$.
   (3) *Let* $(\lambda, \mu) \in \Gamma_l$. *Then* $\bar{\delta}_r(\lambda, \mu; 0) = 0$ *for* $n \geq 3$.
   (4) *Let* $(\lambda, \mu) \in G_0$. *Then* $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ *for* $n \geq 1$.

(b) (1) *Let* $(\lambda, \mu) \in G_1 \cup \Gamma_r$. *Then there exists unique point* $z \in (-\infty, 0)$ *such that* $\bar{\delta}_r(\lambda, \mu; z) = 0$.
   (2) *Let* $(\lambda, \mu) \in \Gamma_r$. *Then* $\bar{\delta}_r(\lambda, \mu; 0) \neq 0$ *for* $n = 1, 2$.
   (3) *Let* $(\lambda, \mu) \in \Gamma_r$. *Then* $\bar{\delta}_r(\lambda, \mu; 0) = 0$ *for* $n \geq 3$.

(c) *Let* $(\lambda, \mu) \in G_2$. *Then there exist two zeros* $z_1, z_2 \in (-\infty, 0)$ *such that* $\bar{\delta}_r(\lambda, \mu; z_1) = \bar{\delta}_r(\lambda, \mu; z_2) = 0$.

Proof. Let $(\lambda, \mu) \in G_0 \cup \Gamma_l$. Then we can see that $(\lambda, \mu) \notin \overline{\mathfrak{H}}_z$ for any $z \in (-\infty, 0)$. Thus (a)(1) follows.

Also by (3.33), it can be seen that $\cup_{z\in(-\infty,0)}\Gamma_l(z)\supset G_1$, $\Gamma_l(z)\cap\Gamma_l(w)=\emptyset$ if $z\neq w$, and $\Gamma_r(z)\cap G_1=\emptyset$. Hence there exists a unique $z\in(-\infty,0)$ such that $(\lambda,\mu)\in\Gamma_l(z)$, which proves (b)(1). We can also see that $\cup_{z\in(-\infty,0)}\Gamma_l(z)\supset G_2$, $\cup_{z\in(-\infty,0)}\Gamma_r(z)\supset G_2$, $\Gamma_l(z)\cap\Gamma_l(w)=\emptyset$ if $z\neq w$, $\Gamma_r(z)\cap\Gamma_r(w)=\emptyset$ if $z\neq w$, and $\Gamma_l(z)\cap\Gamma_r(z)=\emptyset$. Hence there exist $z_1,z_2\in(-\infty,0)$ such that $(\lambda,\mu)\in\Gamma_l(z_1)$ and $(\lambda,\mu)\in\Gamma_r(z_2)$, which proves (c). We note that since $(\lambda,\mu)\in\Gamma_l$ implies that $1\neq\mu$ for $n=1$, and $2\neq\mu$ for $n=2$, $\bar\delta_r(\lambda,\mu;0)\neq 0$ for $n=1,2$ follows. Hence (a)(2) and (a)(3) follow from (3.37), and (b)(2) and (b)(3) are similarly proven. Finally for $(\lambda,\mu)\in G_0$ we have

$$\bar\delta_r(\lambda,\mu;0)=\begin{cases}\infty, & n=1,2,\\ b(0)\bar{\mathcal{H}}_0(\lambda,\mu)\neq 0, & n\geq 3.\end{cases}$$

Then (a)(4) follows. □

3.3.3. *Zeros of $\delta_c(\lambda;z)$.* We study zeros of $\delta_c(\lambda;z)$. In a similar manner we extend $\delta_c(\lambda,z)$ for $z\in(-\infty,0]$. When $n\geq 2$, the function $c(z)-d(z)$ exists, and due to its monotone property (See Lemma 3.7) we can define $\alpha=\lim_{z\to 0-}c(z)-d(z)$. Note that $\alpha>0$ and we set

$$\lambda_c=\frac{1}{\alpha}.$$

Let us write $\delta_c(\lambda;z)=\varrho(\lambda;z)^{n-1}$, where $\varrho(\lambda;z)=\lambda(c(z)-d(z))-1$. We define $\bar\delta_c(\lambda;z)$ by

$$\bar\delta_c(\lambda;z)=\begin{cases}\delta_c(\lambda;z), & z\in(-\infty,0), & n\geq 1,\\ (\lambda\alpha-1)^{n-1}, & z=0, & n\geq 2,\\ 1, & z=0, & n=1.\end{cases}\tag{3.38}$$

Lemma 3.15. *Let $n\geq 2$. Then (a)–(c) follow.*

(a) *Let $\lambda\leq\lambda_c$. Then $\varrho(\lambda;z)\neq 0$ for any $z\in(-\infty,0)$.*
(b) *Let $\lambda=\lambda_c$. Then $\varrho(\lambda;0)=0$.*
(c) *Let $\lambda>\lambda_c$. Then there exists unique $z\in(-\infty,0)$ such that $\varrho(\lambda;z)=0$ with multiplicity one.*

Proof. Since $c(z)-d(z)>0$ is strictly monotonously increasing in $(-\infty,0)$, we get

$$\varrho(\lambda;z)\leq\varrho(\lambda_c;z)<\varrho(\lambda_c;0)=0,\quad\text{if}\quad 0<\lambda\leq\lambda_c,$$
$$\varrho(\lambda;z)=-1,\quad\text{if}\quad\lambda=0,$$

which prove (a) and (b). Since $\varrho(\lambda;0)>\varrho(\lambda_c;0)=0$ and $\lim_{z\to-\infty}\varrho(\lambda;z)=-1$ there exists $z\in(-\infty,0)$ such that $\varrho(\lambda;z)=0$. By the monotonicity of $\varrho(\lambda;\cdot)$ this zero is a unique and has multiplicity one. Hence (c) is proven. □

We divide (λ,μ)-plane into two half planes $C_\pm$ and the boundary $\mathfrak{C}_0$. Set

$$\mathfrak{C}_-=\{(\lambda,\mu)\in\mathbb{R}^2;\lambda<\lambda_c\},\mathfrak{C}_0=\{(\lambda,\mu)\in\mathbb{R}^2;\lambda=\lambda_c\},\mathfrak{C}_+=\{(\lambda,\mu)\in\mathbb{R}^2;\lambda>\lambda_c\}.$$

See Figure 3. We immediately have a lemma.

Lemma 3.16. *Let $n\geq 2$. Then (a)–(c) follow.*

(a) *For any $(\lambda,\mu)\in\mathfrak{C}_-\cup\mathfrak{C}_0$, $\bar\delta_c(\lambda;\cdot)$ has no zero in $(-\infty,0)$.*
(b) *Let $(\lambda,\mu)\in\mathfrak{C}_0$. Then $\bar\delta_c(\lambda;0)=0$, and $z=0$ has multiplicity $n-1$.*
(c) *For any $(\lambda,\mu)\in\mathfrak{C}_+$, $\bar\delta_c(\lambda;\cdot)$ has a unique zero in $(-\infty,0)$ with multiplicity $n-1$.*

Proof. This follows from Lemma 3.15. □

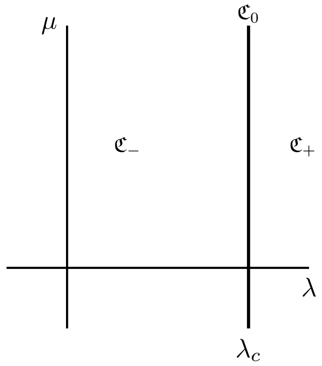

FIGURE 3. Regions of $C_\pm$ for $n \geq 2$

3.4. **Eigenvalues of $H^{\mathrm{e}}_{\lambda\mu}$.** In the previous sections we consider zeros of $\bar{\delta}_r(\lambda, \mu; z)$ and $\bar{\delta}_c(\lambda; z)$ for $z \in (-\infty, 0)$. Let (λ, μ) and $z \in (-\infty, 0)$ be solution of $\bar{\delta}_r(\lambda, \mu; z) = 0$ or $\bar{\delta}_c(\lambda; z) = 0$. Then $z \in \sigma_p(H^{\mathrm{e}}_{\lambda\mu})$. In this section we summarise spectral properties of $H^{\mathrm{e}}_{\lambda\mu}$ derived from zeros of $\bar{\delta}_r(\lambda, \mu; z)\bar{\delta}_c(\lambda; z)$.

Definition 3.17 (Threshold eigenvalue and threshold resonance)**.** *Let f be a solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$ (resp. $H^{\mathrm{o}}_{\lambda} f = 0$).*

(1) *If $f \in L^2_{\mathrm{e}}(\mathbb{T}^n)$ (resp. $f \in L^2_{\mathrm{o}}(\mathbb{T}^n)$), we say that 0 is a lower threshold eigenvalue of $H^{\mathrm{e}}_{\lambda\mu}$ (resp. H^{o}_{λ}).*
(2) *If $f \in L^1_{\mathrm{e}}(\mathbb{T}^n) \setminus L^2_{\mathrm{e}}(\mathbb{T}^n)$ (resp. $f \in L^1_{\mathrm{o}}(\mathbb{T}^n) \setminus L^2_{\mathrm{o}}(\mathbb{T}^n)$), we say that 0 is a lower threshold resonance of $H^{\mathrm{e}}_{\lambda\mu}$ (resp. H^{o}_{λ}).*
(3) *If $f \in L^{\epsilon}_{\mathrm{e}}(\mathbb{T}^n) \setminus L^1_{\mathrm{e}}(\mathbb{T}^n)$ (resp. $f \in L^{\epsilon}_{\mathrm{o}}(\mathbb{T}^n) \setminus L^1_{\mathrm{o}}(\mathbb{T}^n)$) for any $0 < \epsilon < 1$, we say that 0 is a lower super-threshold resonance of $H^{\mathrm{e}}_{\lambda\mu}$ (resp. H^{o}_{λ}).*

In what follows we may use "threshold eigenvalue" (resp. threshold resonance) instead of "lower threshold eigenvalue" (resp. lower threshold resonance) for simplicity. Then (λ, μ)-plane is divided into 11 regions. $G_0, \Gamma_l, G_1 \cap \mathfrak{C}_-, G_1 \cap \mathfrak{C}_0, G_1 \cap \mathfrak{C}_+, \Gamma_r \cap \mathfrak{C}_-, \Gamma_r \cap \mathfrak{C}_0, \Gamma_r \cap \mathfrak{C}_+, G_2 \cap \mathfrak{C}_-$, $G_2 \cap \mathfrak{C}_0$ and $G_2 \cap \mathfrak{C}_+$.

Lemma 3.18 (Eigenvalues of $H^{\mathrm{e}}_{\lambda\mu}$ for $n \geq 2$)**.** *Let $n \geq 2$. Then $H^{\mathrm{e}}_{\lambda\mu}$ has the following facts:*

(1) $(\lambda, \mu) \in G_0$. *There is no eigenvalue in $(-\infty, 0)$, and there is neither threshold eigenvalue nor threshold resonance.*
(2) $(\lambda, \mu) \in \Gamma_l$.
   $n = 2$ *There is no eigenvalue in $(-\infty, 0)$ and there is neither threshold eigenvalue nor threshold resonance.*
   $n \geq 3$ *There is no eigenvalue in $(-\infty, 0)$ but there is a simple threshold eigenvalue or threshold resonance.*
(3) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_-$. *There is a simple eigenvalue in $(-\infty, 0)$ but there is neither threshold eigenvalue nor threshold resonance.*
(4) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_0$. *There is a simple eigenvalue in $(-\infty, 0)$ and there is an $(n-1)$-fold threshold eigenvalue or threshold resonance.*
(5) $(\lambda, \mu) \in G_1 \cap \mathfrak{C}_+$. *There are a simple eigenvalue and an $(n-1)$-fold eigenvalue in $(-\infty, 0)$, but there is neither threshold eigenvalue nor threshold resonance.*
(6) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_-$.

$n = 2$ *There is a simple eigenvalue in* $(-\infty, 0)$ *but there is neither threshold eigenvalue nor threshold resonance.*

$n \geq 3$ *There is a simple eigenvalue in* $(-\infty, 0)$ *and there is a simple threshold eigenvalue or threshold resonance.*

(7) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_0$.

$n = 2$ *There is a simple eigenvalue in* $(-\infty, 0)$ *and there is an* $(n-1)$*-fold threshold eigenvalue or threshold resonance.*

$n \geq 3$ *There is a simple eigenvalue in* $(-\infty, 0)$*, and there are an* $(n-1)$*-fold threshold eigenvalue or threshold resonance, and a simple threshold eigenvalue or threshold resonance.*

(8) $(\lambda, \mu) \in \Gamma_r \cap \mathfrak{C}_+$.

$n = 2$ *There is a simple eigenvalue and an* $(n-1)$*-fold eigenvalue in* $(-\infty, 0)$*, but there is neither threshold eigenvalue nor threshold resonance.*

$n \geq 3$ *There are a simple eigenvalue and an* $(n-1)$*-fold eigenvalue in* $(-\infty, 0)$*. There is a simple threshold eigenvalue or threshold resonance.*

(9) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_-$. *There are two eigenvalues in* $(-\infty, 0)$ *but there is neither threshold eigenvalue nor threshold resonance.*

(10) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_0$. *There are two eigenvalues in* $(-\infty, 0)$ *and there is an* $(n-1)$*-fold threshold eigenvalue or threshold resonance.*

(11) $(\lambda, \mu) \in G_2 \cap \mathfrak{C}_+$. *There are three eigenvalues in* $(-\infty, 0)$ *and one of them is* $(n-1)$*-fold, but there is neither threshold eigenvalue nor threshold resonance.*

Proof. This lemma follows from Lemmas 3.14 and 3.16, and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_r(\lambda, \mu; z)\bar{\delta}_c(\lambda; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_r(\lambda, \mu; 0)\bar{\delta}_c(\lambda; 0) = 0$. □

By virtue of Lemma 3.14, $\bar{\delta}_r(\lambda, \mu; \cdot)$ has at most two zeros in $(-\infty, 0)$ for $(\lambda, \mu) \in G_2$ or $(\lambda, \mu) \in G_1 \cup \Gamma_r$. Now we can see the explicit form of these eigenvectors. In the case of $n = 1$ we know that $\delta_c(\lambda; z) = 1$. Hence zeros of $\delta_r(\lambda, \mu; z)\delta_c(\lambda; z)$ coincides with those of $\delta_r(\lambda, \mu; z)$. We have the lemma.

Lemma 3.19 (Eigenvalues of $H^{\mathrm{e}}_{\lambda\mu}$ for $n = 1$)**.** *We have the following facts:*

(1) $(\lambda, \mu) \in G_0 \cup \Gamma_l$. *There is no eigenvalues in* $(-\infty, 0)$*, and there is neither threshold eigenvalue nor threshold resonance.*

(2) $(\lambda, \mu) \in G_1 \cup \Gamma_r$. *There is a simple eigenvalue in* $(-\infty, 0)$*, but there is neither threshold eigenvalue nor threshold resonance.*

(3) $(\lambda, \mu) \in G_2$. *There are two eigenvalues in* $(-\infty, 0)$*, but there is neither threshold eigenvalue nor threshold resonance.*

Proof. This lemma follows from Lemmas 3.14 and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_r(\lambda, \mu; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_r(\lambda, \mu; 0)$. □

Lemma 3.20. *Let* $n \geq 1$*. (1) Let* $\lambda \neq 0$*. We assume that* $z_1, z_2 \in (-\infty, 0)$ *and* $\delta_r(\lambda, \mu; z_k) = 0$ *(if they exist). Then* $1 - \mu a(z_k) \neq 0$ *for* $k = 1, 2$ *and* $G_{\mathrm{e}}(z_k)Z_k = Z_k$ *has the solutions:*

$$Z_k = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1-\mu a(z_k)} \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad k = 1, 2,$$

and the corresponding eigenfunctions, $H^{\rm e}_{\lambda\mu} f_k = z f_k$*, are*

$$f_k(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p) - z_k} \Big(\mu \frac{nb(z_k)}{1 - \mu a(z_k)} + \sum_{j=1}^{n} \cos p_j \Big), \quad k = 1, 2. \tag{3.39}$$

(2) Let $\lambda = 0$*. We assume that* $z \in (-\infty, 0)$ *and* $\delta_r(0, \mu; z) = 0$*. Then* $1 - \mu a(z) = 0$ *and* $G_{\rm e}(z)Z = Z$ *has the solution:*

$$Z = \begin{pmatrix} 1 \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix}$$

and the corresponding eigenfunction, $H^{\rm e}_{\lambda\mu} f = z f$*, is*

$$f(p) = \frac{\mu}{(2\pi)} \frac{1}{E(p) - z} \tag{3.40}$$

Proof. We prove the case of $n \geq 2$. The proof for the case of $n = 1$ is similar. Since $\delta_r(\lambda, \mu; z) = 0$, we see that

$$\big(1 - \mu a(z)\big)\Big(1 - \lambda\big(c(z) + (n-1)d(z)\big)\Big) - n\lambda\mu b^2(z) = 0.$$

Then $1 - \mu a(z) \neq 0$ if and only if $\lambda \neq 0$, and we also have the algebraic relation

$$1 - \lambda\big(c(z) + (n-1)d(z)\big) = \frac{n\lambda\mu b^2(z)}{1 - \mu a(z)}.$$

From this relation it follows that

$$G_{\rm e}(z_k) \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1-\mu a(z_k)} \\ 1 \\ \vdots \\ z_k 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1-\mu a(z_k)} + \frac{\lambda}{\sqrt{2}} nb(z_k) \\ \lambda \frac{n\mu b(z_k)^2}{1-\mu a(z_k)} + \lambda c(z_k) + \lambda(n-1)d(z_k) \\ \vdots \\ \lambda \frac{n\mu b(z_k)^2}{1-\mu a(z_k)} + \lambda c(z_k) + \lambda(n-1)d(z_k) \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb(z_k)}{1-\mu a(z_k)} \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then $G_{\rm e}(z_k)Z_k = Z_k$ for $\lambda \neq 0$. In the case of $\lambda = 0$ we can also see that

$$G_{\rm e}(z) \begin{pmatrix} 1 \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix} = \begin{pmatrix} \mu a(z) \\ \sqrt{2}\mu b(z) \\ \vdots \\ \sqrt{2}\mu b(z) \end{pmatrix} = Z.$$

Then the lemma is proven. □

Next we show the eigenfunction corresponding to zeros of $\delta_c(\lambda; \cdot)$.

Lemma 3.21. *Let* $n \geq 2$*,* $z \in (-\infty, 0)$ *and* $\bar{\delta}_c(\lambda; z) = 0$*. I.e.,* $\lambda = \frac{1}{c(z) - d(z)}$*. Then the solutions of* $G_{\rm e}(z)Z = Z$ *are given by*

$$Z_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, Z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \cdots Z_{n-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}, \tag{3.41}$$

and hence the corresponding eigenfunctions, $H^{\mathrm{e}}_{\lambda\mu} g_j = z g_j$, are

$$g_j(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p) - z} (\cos p_1 - \cos p_{j+1}), \quad j = 1, \ldots, n-1. \tag{3.42}$$

In particular the multiplicity of eigenvalue z is at least $n-1$.

Proof. Since $\lambda(c(z) - d(z)) = 1$, we see that

$$G_{\mathrm{e}}(z) \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda(c(z) - d(z)) \\ \lambda(d(z) - c(z)) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then $G_{\mathrm{e}}(z) Z_1 = Z_1$. In the same way as Z_1 we can see that $G_{\mathrm{e}}(z) Z_j = Z_j$ for $j = 2, ..., n-1$. Then the lemma is proven. □

3.5. Threshold eigenvalues and threshold resonances for $H^{\mathrm{e}}_{\lambda\mu}$.

In this section we study the spectrum located on the left edge of the essential spectrum $[0, 2n]$, i.e., $z = 0$. Suppose that $(\lambda, \mu) \in \mathfrak{H}_0$. Then it is possibly $\bar{\delta}_r(\lambda, \mu; 0) = 0$ or $\bar{\delta}_c(\lambda, \mu; 0) = 0$. By Corollary 3.11 and (3.38) we see that for $(\lambda, \mu) \in \mathfrak{H}_0$

$$\begin{array}{ll} \bar{\delta}_r(\lambda, \mu; 0) \neq 0 \neq 1 = \bar{\delta}_c(\lambda, \mu; 0), & n = 1, \\ \bar{\delta}_r(\lambda, \mu; 0) \neq 0, & n = 2. \end{array} \tag{3.43}$$

Hence we study zeros of $\bar{\delta}_r(\lambda, \mu; 0)$ for $n \geq 3$, and those of $\bar{\delta}_c(\lambda; 0)$ for $n \geq 2$. We set $a(0) = a$ and $b(0) = b$, and both a and b are finite for $n \geq 3$. In these case however the proofs are similar to those of Lemmas 3.20 and 3.21 where we discuss eigenvalues in $(-\infty, 0)$.

Lemma 3.22. *Let $n \geq 3$. (1) Let $\lambda \neq 0$ and $\bar{\delta}_r(\lambda, \mu; 0) = 0$. Then $1 - \mu a \neq 0$ and $G_{\mathrm{e}}(0) Z = Z$ has the solution*

$$Z = \begin{pmatrix} \frac{\lambda}{\sqrt{2}} \frac{nb}{1-\mu a} \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and the corresponding equation $H^{\mathrm{e}}_{\lambda\mu} f = 0$ has the solution:

$$f(p) = \frac{\lambda}{\sqrt{2}} \frac{1}{(2\pi)} \frac{1}{E(p)} \Big(\mu \frac{nb}{1 - \mu a} + \sum_{j=1}^{n} \cos p_j \Big). \tag{3.44}$$

(2) Let $\lambda = 0$ and $\delta_r(0, \mu; 0) = 0$. Then $1 - \mu a = 0$ and $G_{\mathrm{e}}(0) Z = Z$ has the solution:

$$Z = \begin{pmatrix} 1 \\ \sqrt{2}\mu b \\ \vdots \\ \sqrt{2}\mu b \end{pmatrix}$$

and the corresponding equation $H^{\mathrm{e}}_{\lambda\mu} f = 0$ has the solution:

$$f(p) = \frac{\mu}{(2\pi)} \frac{1}{E(p)}. \tag{3.45}$$

Proof. Replacing z in Lemma 3.20 with 0 we can prove the lemma in the same way as that of Lemma 3.20. $\square$

Next we show the solution corresponding to zeros of $\delta_c(\lambda;\cdot)$. Similar to the case of $\delta_r(\lambda,\mu;z) = 0$, we have the lemma below.

Lemma 3.23. *Let $n \geq 2$ and $\delta_c(\lambda;0) = 0$, i.e., $\lambda = \lambda_c$. Then the solutions of $G_{\rm e}(0)Z = Z$ are given by* (3.41) *and hence the corresponding equation $H^{\rm e}_{\lambda\mu}g_j = 0$ has the solutions*

$$g_j(p) = \frac{\lambda_c}{\sqrt{2}}\frac{1}{(2\pi)}\frac{1}{E(p)}(\cos p_1 - \cos p_{j+1}), \quad j = 1,\ldots,n-1. \tag{3.46}$$

Proof. Replacing z in Lemma 3.21 with 0 we can prove the lemma in the same way as that of Lemma 3.21. $\square$

Recall that

$$u_0 = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(p)dp, \quad u_j = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \cos p_j f(p)dp, \quad j = 1,...,n. \tag{3.47}$$

As was seen above the problem for $n \geq 3$ can be reduced to study the spectrum of $G_{\rm e}$ by the Birman-Schwinger principle, the problem for $n = 1, 2$ should be however directly investigated.

Lemma 3.24. *Let $n = 1$.*

(1) *Suppose that $f \in L^1(\mathbb{T})$ and $H^{\rm e}_{\lambda\mu}f = 0$. Then $f = 0$. In particular $H^{\rm e}_{\lambda\mu}$ has no threshold resonance.*
(2) *There is no non-zero f such that $f \in L^\epsilon(\mathbb{T}^2) \setminus L^1(\mathbb{T}^2)$ for some $0 < \epsilon < 1$ and $H^{\rm e}_{\lambda\mu}f = 0$. In particular $H^{\rm e}_{\lambda\mu}$ has no super-threshold resonance.*

Proof. (1) $H^{\rm e}_{\lambda\mu}f = 0$ gives $f = \varphi/E$ and $\varphi(p) = \mu u_0 + \lambda u_1 \cos p$ by (3.4). From $f \in L^1(\mathbb{T})$ it follows that $\varphi(0) = \mu u_0 + \lambda u_1 = 0$. Hence

$$f(p) = \frac{1}{E(p)}(1 - \cos p)\mu u_0 = \mu u_0.$$

Substituting this into the second term in (3.47), we get $u_1 = \mu u_0 \frac{1}{2\pi}\int_{\mathbb{T}} \cos t dt = 0$, which gives $\mu u_0 = 0$ and $f = 0$.
(2) Since $f \notin L^1(\mathbb{T})$. It must be that $\mu = 0$ and $f = \varphi/E$ with $\varphi(p) = \lambda u_1 \cos p$. Hence

$$u_1 = \frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}} \frac{u_1 \cos^2 p}{E(p)}dp.$$

Then $u_1 = 0$, since $\int_{\mathbb{T}} \frac{\cos^2 p}{E(p)}dp = \infty$. Then $f = 0$ follows. $\square$

Next we discuss the spectrum of $H^{\rm e}_{\lambda\mu}$ for $n = 2$ at the lower threshold

Lemma 3.25. *Let $n = 2$.*

(1) *Suppose that $f \in L^1(\mathbb{T}^2)$ and $H^{\rm e}_{\lambda\mu}f = 0$. Then $(\lambda,\mu) \in \mathfrak{C}_0$ and*

$$f(p) = \lambda_c u_1 \frac{\cos p_1 - \cos p_2}{E(p)}. \tag{3.48}$$

In particular $f \in L^2(\mathbb{T}^2)$ and $H^{\rm e}_{\lambda\mu}$ has no threshold resonance.

(2) *There is no non-zero f such that $f \in L^\epsilon(\mathbb{T}^2) \setminus L^1(\mathbb{T}^2)$ for some $0 < \epsilon < 1$ and $H^{\rm e}_{\lambda\mu}f = 0$. In particular $H^{\rm e}_{\lambda\mu}$ has no super-threshold resonance.*

Proof. (1) Consider $H^{\rm e}_{\lambda\mu}f=0$ in $L^1(\mathbb{T}^2)$. We can take $f=\varphi/E$ and $\varphi(p)=\mu u_0+\lambda u_1\cos p_1+\lambda u_2\cos p_2$. Since $f\in L^1(\mathbb{T}^2)$, we get $\varphi(0)=\mu u_0+\lambda(u_1+u_2)=0$ and so

$$f(p)=\frac{\lambda}{E(p)}\left(-u_1(1-\cos p_1)-u_2(1-\cos p_2)\right)$$

By (3.47) we obtain

$$\begin{aligned} u_1&=-\frac{\lambda}{(2\pi)^2}\left(u_1\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp+u_2\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_2)}{E(p)}dp\right),\\ u_2&=-\frac{\lambda}{(2\pi)^2}\left(u_1\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_1)}{E(p)}dp+u_2\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_2)}{E(p)}dp\right).\end{aligned}$$

Since $\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp=-\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_2)}{E(p)}dp$, we get

$$\begin{aligned} u_1&=\frac{\lambda}{(2\pi)^2}(u_2-u_1)\left(\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp\right),\\ u_2&=\frac{\lambda}{(2\pi)^2}(u_1-u_2)\left(\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_2)}{E(p)}dp\right)\end{aligned}$$

and hence $u_1=-u_2$. Consequently, $\mu u_0=0$, and the solution of $H^{\rm e}_{\lambda\mu}f=0$ is of the form

$$f(p)=\lambda u_1\frac{\cos p_1-\cos p_2}{E(p)}\in L^2(\mathbb{T}^2). \tag{3.49}$$

Inserting this into the definition of u_1, we have $\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{\cos p_1(\cos p_1-\cos p_2)}{E(p)}dp=1$ and thus taking $\lambda=\lambda_c$ we can see that (3.48) is the solution of $H^{\rm e}_{\lambda\mu}f=0$. Notice that $u_0=0$ follows from (3.49).
(2) Since $f\notin L^1(\mathbb{T}^2)$. It must be that $\mu=0$ and $f=\varphi/E$ with $\varphi(p)=\lambda u_1\cos p_1+\lambda u_2\cos p_2$. Hence

$$\begin{aligned} u_1&=\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{u_1\cos^2 p_1+u_2\cos p_1\cos p_2}{E(p)}dp,\\ u_2&=-\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{u_1\cos p_2\cos p_1+u_2\cos^2 p_2}{E(p)}dp.\end{aligned}$$

Then $u_1=-u_2$ and $1=\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{\cos p_1(\cos p_1+\cos p_2)}{E(p)}dp$. Thus $\lambda=\lambda_c$. Then f is given by (3.49), but $f\in L^2(\mathbb{T}^2)$. This contradicts with $f\notin L^1(\mathbb{T}^2)$. □

Remark 3.26. *The Birman-Schwinger principle is valid for $n\geq 3$, but Lemma 3.23 tells us that the Birman-Schwinger principle is valid for $n=2$. Furthermore in Lemma 3.25 it can be seen that g_1 given by* (3.46) *coincides with* (3.48).

Lemma 3.27 (Threshold eigenvalues and threshold resonances of $H^{\rm e}_{\lambda\mu}$)**.** *(1)-(5) follow:*

1. *Let $n=1$. Then 0 is none of a threshold eigenvalue, a threshold resonance and a super-threshold resonance.*
2. *Let $n=2$. Then 0 is a threshold eigenvalue with* (3.46) *for $(\lambda,\mu)\in\mathfrak{C}_0$ and its multiplicity is one.*
3. *Let $n=3,4$. Suppose $(\lambda,\mu)\in\mathfrak{H}_0$. Then 0 is a threshold resonance with eigenvector* (3.44) *for $\lambda\neq 0$, and* (3.45) *for $\lambda=0$, i.e., $(\lambda,\mu)=(0,1/a)$.*
4. *Let $n=3,4$. Suppose $(\lambda,\mu)\in\mathfrak{H}_0$. Then 0 is a threshold eigenvalue with* (3.46) *for $\lambda=\lambda_c$ and its multiplicity is $n-1$.*

(5) *Let* $n \geq 5$. *Suppose* $(\lambda, \mu) \in \mathfrak{H}_0$. *Then* 0 *is a threshold eigenvalue with eigenvector* (3.44) *for* $\lambda_c \neq \lambda \neq 0$ *and multiplicity one,* (3.44) *and* (3.46) *for* $\lambda = \lambda_c$ *and multiplicity* n, *and* (3.45) *for* $\lambda = 0$, *i.e.,* $(\lambda, \mu) = (0, 1/a)$, *and multiplicity one.*

Proof. (1) follows from Lemma 3.24. The solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$ is given by (3.44), (3.45) and (3.46). We note that $\int_{|p|<\epsilon} \frac{1}{E^2(p)} dp = \infty$ for $n = 2, 3, 4$ for any $\epsilon > 0$, and $\int_{|p|<\epsilon} \frac{1}{E^2(p)} dp < \infty$ for $n \geq 5$ for any $\epsilon > 0$. Since $(\lambda, \mu) \in \overline{\mathfrak{H}}_0$, $n \neq \mu$, and we can see that

$$\frac{nb\mu}{1 - \mu a} + \sum_{j=1}^{n} \cos 0 = \frac{nb\mu}{1 - \mu a} + n = n \left(\frac{1 - \mu(a - b)}{1 - \mu a} \right) = \frac{n - \mu}{1 - \mu a} \neq 0.$$

Hence, using Lemma 3.2, we obtain

$$\begin{aligned} &(3.44), (3.45) \in L^2(\mathbb{T}^n), \quad n \geq 5, \\ &(3.44), (3.45) \in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n), \quad n = 3, 4, \\ &(3.46) \in L^2(\mathbb{T}^n), \quad n \geq 2. \end{aligned}$$

(2) follows from Lemmas 3.25 and 3.23. (3) follows from Lemmas 3.23 and 3.22. (4) follows from Lemma 3.23. Finally (5) follows from Lemmas 3.23 and 3.22. □

Remark 3.28. *Let* $n = 3, 4$ *and* $(\lambda_c, \mu) \in \mathfrak{H}_0$. *By (3) and (4) of Lemma 3.27 it can be seen that* 0 *is a threshold resonance and a threshold eigenvalue.*

4. SPECTRUM OF H^{o}_{λ}

4.1. **Birman-Schwinger principle for** $z \in \mathbb{C} \setminus [0, 2n]$**.** In the previous sections, we study the spectrum of $H^{\mathrm{e}}_{\lambda\mu}$ by using the Birman-Schwinger principle for $n \geq 3$, and by directly solving $H^{\mathrm{e}}_{\lambda\mu} f = 0$ for $n = 1, 2$. In the case of H^{o}_{λ} we can proceed in a similar way to the the case of $H^{\mathrm{e}}_{\lambda\mu}$ and rather easier than that of $H^{\mathrm{e}}_{\lambda\mu}$ as is seen below. Let $z \in \mathbb{C} \setminus [0, 2n]$. As is done for $H^{\mathrm{e}}_{\lambda\mu}$, we can see that

$$(H_0 - z)^{-1} V^{\mathrm{o}}_{\lambda} = S_1 S_2.$$

Here S_1 and S_2 are defined by

$$S_1 : \mathbb{C}^n \ni \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \mapsto (H_0 - z)^{-1} \frac{\lambda}{2} \sum_{j=1}^{n} w_j s_j \in L^2_{\mathrm{o}}(\mathbb{T}^n),$$

$$S_2 : L^2_{\mathrm{o}}(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \langle \phi, s_1 \rangle \\ \vdots \\ \langle \phi, s_n \rangle \end{pmatrix} \in \mathbb{C}^n.$$

We set

$$G_{\mathrm{o}}(z) = S_2 S_1 : \mathbb{C}^n \to \mathbb{C}^n.$$

The following assertion is proved as Lemma 3.1, and then we omit the proof.

Lemma 4.1 (Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$)**.**

(a) $z \in \mathbb{C} \setminus [0, 2n]$ *is an eigenvalue of* H^{o}_{λ} *if and only if* $1 \in \sigma(G_{\mathrm{o}}(z))$.

(b) *Let* $z \in \mathbb{C} \setminus [0, 2n]$ *and* $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ *be such that* $G_{\rm o}(z)Z = Z$. *Then* $f = S_1 Z$,

$$f(p) = \frac{1}{(2\pi)} \frac{1}{E(p) - z} \left(\frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \sin p_j \right)$$

is an eigenfunction of $H_\lambda^{\rm o}$, *i.e.,* $H_\lambda^{\rm o} f = zf$.

We see that $\frac{1}{2}\langle {\rm s}_i, (H_0 - z)^{-1} {\rm s}_j \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \frac{\sin p_i \sin p_j}{E(p) - z} dp = 0$ by the fact that $E(p) = E(p_1, \ldots, p_d)$ is even for any p_j. Therefore

$$G_{\rm o}(z) = \lambda s(z) I, \tag{4.1}$$

where $s(z) = (2\pi)^{-n} \displaystyle\int_{\mathbb{T}^n} \frac{\sin^2 p_1}{E(p) - z} dp$ is given by (3.24). Consequently we have for $n \geq 1$,

$$\delta_s(\lambda; z) = \det(G_{\rm o}(z) - {\rm I}) = (\lambda s(z) - 1)^n.$$

Since $G_{\rm o}(z)$ is diagonal, it is very easy to find solution of $G_{\rm o}(z)Z = Z$. It has n independent solutions:

$$Z_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j_{th}, \quad j = 1, ..., n.$$

The corresponding eigenvector, $H_\lambda^{\rm o} f_j = z f_j$, is given by

$$f_j(p) = \frac{1}{E(p) - z} \frac{1}{(2\pi)} \frac{\lambda}{\sqrt{2}} \sin p_j, \quad j = 1, \ldots, n, \tag{4.2}$$

where $\lambda = 1/s(z)$. In particular the multiplicity of z is n.

4.2. **Birman-Schwinger principle for** $z = 0$**.** We can extend the Birman-Schwinger principle for $z = 0$. We extend the eigenvalue equation $H_\lambda^{\rm o} f = 0$ in $L^2_{\rm o}(\mathbb{T}^n)$ to that in $L^1_{\rm o}(\mathbb{T}^n)$. We consider the equation

$$E(p)f(p) - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^{n} \sin p_j \int_{\mathbb{T}^n} \sin p_j f(p) dp = 0 \tag{4.3}$$

in $L^1_{\rm o}(\mathbb{T}^n)$. We also describe (4.3) as $H_\lambda^{\rm o} f = 0$. We can see that $\sin p_j / E(p) \approx 1/|p|$ in the neighborhood of $p = 0$, and then $\sin p_j / E(p) \in L^1(\mathbb{T}^n)$ for $n \geq 2$. By (e) of Lemma 3.2 and $V_\lambda^{\rm o} f \in C(\mathbb{T}^n)$ we can see that

$$L^2_{\rm o}(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_\lambda^{\rm e} f \in L^2_{\rm o}(\mathbb{T}^n), \quad n \geq 3, \tag{4.4}$$

$$L^1_{\rm o}(\mathbb{T}^n) \ni f \mapsto H_0^{-1} V_\lambda^{\rm o} f \in L^1_{\rm o}(\mathbb{T}^n), \quad n \geq 2. \tag{4.5}$$

Thus for $n \geq 2$ we can extend operators S_1 and S_2. Let $n \geq 2$ and $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix}$. $\bar{S}_1 : \mathbb{C}^n \to L^1_{\rm o}(\mathbb{T}^n)$ is defined by

$$\bar{S}_1 Z = \frac{1}{(2\pi)} \frac{\lambda}{\sqrt{2}} \frac{1}{E(p)} \sum_{j=1}^n w_j \sin p_j$$

and $\bar{S}_2 : L^1_{\rm o}(\mathbb{T}^n) \to \mathbb{C}^n$ by

$$\bar{S}_2 : L^1_{\rm o}(\mathbb{T}^n) \ni \phi \mapsto \begin{pmatrix} \int_{T^n} \phi(p) s_1(p) dp \\ \vdots \\ \int_{T^n} \phi(p) s_n(p) dp \end{pmatrix} \in \mathbb{C}^n.$$

Then $\overline{S}_1 \overline{S}_2 : L^1_{\rm o}(\mathbb{T}^n) \to L^1_{\rm o}(\mathbb{T}^n)$. Thus $G_{\rm o}(0) = \overline{S}_2 \overline{S}_1 : \mathbb{C}^n \to \mathbb{C}^n$ is described as an $n \times n$ matrix. Let $n \geq 2$. We have (1) $\lim_{z \to 0} G_{\rm o}(z) = G_{\rm o}(0)$, and (2) $\sigma(H_0^{-1} V_\lambda^{\rm o}) \setminus \{0\} = \sigma(G_{\rm o}(0)) \setminus \{0\}$. Hence for $n \geq 2$,

$$G_{\rm o}(0) = \lambda s(0) I \tag{4.6}$$

and $\bar{\delta}_s(\lambda; z)$ is defined by

$$\bar{\delta}_s(\lambda; z) = \begin{cases} \delta_s(\lambda; z) & z \in (-\infty, 0), \\ (\lambda s(0) - 1)^n & z = 0. \end{cases} \tag{4.7}$$

Remark 4.2. *In* (4.6) *and* (4.7) *we define* $\bar{\delta}_s(\lambda, z)$ *and* $G_{\rm o}$ *for* $n \geq 2$. *We note however that* $s(0) < \infty$ *for* $n \geq 1$. *Thus* $G_{\rm o}$ *and* $\bar{\delta}_s(\lambda; z)$ *are well defined for* $n \geq 1$.

Lemma 4.3 (Birman-Schwinger principle for $z = 0$)**.** *Let* $n \geq 2$.

(a) *Equation* $H^{\rm o}_\lambda f = 0$ *has a solution in* $L^1(\mathbb{T}^n)$ *if and only if* $1 \in \sigma(G_{\rm o}(0))$.

(b) *Let* $Z = \begin{pmatrix} w_0 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{C}^n$ *be the solution of* $G_{\rm o}(0) Z = Z$ *if and only if*

$$f(p) = \bar{S}_1 Z(p) = \frac{1}{(2\pi)^n} \frac{1}{E(p)} \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \sin p_j$$

is a solution of $H^{\rm o}_\lambda f = 0$*, where* $w_1, \cdots, w_n$ *are actually described by*

$$w_j = \frac{\sqrt{2}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{T}^n} f(p) \sin p_j dp, \quad j = 1, \ldots, n. \tag{4.8}$$

Proof. The proof is the same as that of Lemma 3.4. □

4.3. **Eigenvalues of $H^{\rm o}_\lambda$.** Set

$$\lambda_s = \frac{1}{s(0)}.$$

Note that $\lambda_s = 1$ for $n = 1$. We divide (λ, μ)-plane into two half planes $S_\pm$ and the boundary $\mathfrak{S}_0$. Set

$$\mathfrak{S}_- = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_s\}, \mathfrak{S}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_s\}, \mathfrak{S}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_s\}.$$

See Figure 4.

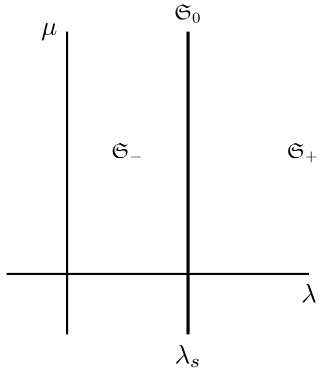

FIGURE 4. Regions of $S_\pm$ for $n \geq 1$

Lemma 4.4. *Let $n \geq 1$. Then (a)-(c) follow:*

(a) *Let $(\lambda, \mu) \in \mathfrak{S}_- \cup \mathfrak{S}_0$. Then $\bar{\delta}_s(\lambda; \cdot)$ has no zero in $(-\infty, 0)$.*
(b) *Let $(\lambda, \mu) \in \mathfrak{S}_0$. Then $\bar{\delta}_s(\lambda_s; 0) = 0$ and $z = 0$ has multiplicity n.*
(c) *Let $(\lambda, \mu) \in \mathfrak{S}_+$. Then $\bar{\delta}_s(\lambda; \cdot)$ has a unique zero in $(-\infty, 0)$ with multiplicity n.*

Proof. The proof is similar to that of Lemma 3.15, and hence we omit it. □

By Lemma 4.4 we can see spectral property of H_λ°.

Lemma 4.5 (Eigenvalues of H_λ°)**.** *Let $n \geq 1$.*

(1) $(\lambda, \mu) \in \mathfrak{S}_- \cup \mathfrak{S}_0$. *There is no eigenvalue in $(-\infty, 0)$.*
(2) $(\lambda, \mu) \in \mathfrak{S}_0$. *There is an n fold threshold eigenvalue or threshold resonance.*
(3) $(\lambda, \mu) \in \mathfrak{S}_+$. *There is an n fold eigenvalue in $(-\infty, 0)$.*

Proof. This lemma follows from Lemmas 4.4, and the fact that $z \neq 0$ is an eigenvalue if and only if $\bar{\delta}_s(\lambda; z) = 0$, and 0 is an threshold eigenvalue or threshold resonance if and only if $\bar{\delta}_s(\lambda; 0) = 0$. □

4.4. **Threshold eigenvalues and threshold resonances for H_λ°.** Threshold resonances and threshold eigenvalues for H_λ° can be discussed by the Birman-Schwinger principle for $n \geq 2$.

Lemma 4.6. *Let $n \geq 2$. Then the solutions of equation $H_\lambda^{\circ} f = 0$ are given by*

$$f_j(p) = \frac{1}{(2\pi)} \frac{\lambda_s}{\sqrt{2}} \frac{\sin p_j}{E(p)}, \quad j = 1, \dots, n. \tag{4.9}$$

Proof. From $\bar{\delta}(\lambda_s, 0) = 0$ and Lemma 4.3 the lemma follows. □

For $n = 1$ we can directly see that $H_\lambda^{\circ} f = 0$ has no solution in L^1, but it has a super-threshold resonance. We see this in the next proposition.

Proposition 4.7 (Super-threshold resonance)**.** *Let $n = 1$ and $\lambda = \lambda_s = 1$. Then $H_\lambda^{\circ} f = 0$ has solution $f \in L_{\rm o}^{\epsilon}(\mathbb{T}) \setminus L_{\rm o}^1(\mathbb{T})$ for any $0 < \epsilon < 1$. I.e., 0 is a super-threshold resonance of H_λ°.*

Proof. $H_{\lambda_s}^{\circ} f = 0$ yields that $f(p) = C \frac{\sin p}{E(p)}$, where $C = \frac{\lambda_s}{2\pi} \int_{\mathbb{T}} \sin p f(p) dp$. Note that however $\sin p / E(p) \notin L^1(\mathbb{T})$, but we can see that $\sin p / E(p) \in L^{\epsilon}(\mathbb{T})$ for any $0 < \epsilon < 1$ since $\sin p / E(p) \sim 1/p$ near $p = 0$ and $\int_{\mathbb{T}} p^{-\epsilon} dp < \infty$. □

Lemma 4.8. (1) *Let $n = 1$. Then 0 is neither a threshold resonance nor a threshold eigenvalue, but for (λ_s, μ), 0 is a super-threshold resonance.*

(2) *Let $n = 2$. Then 0 is a threshold resonance at $\lambda = \lambda_s$.*
(3) *Let $n \geq 3$. Then 0 is a threshold eigenvalue at $\lambda = \lambda_s$ and its multiplicity is n.*

Proof. (1) follows from Proposition 4.7. Let $n \geq 2$. Then the solution of $H^{\rm o}_\lambda f = 0$ is given by (4.9). Since

$$\frac{\sin p_j}{E(p)} \in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n), \quad n = 2,$$
$$\frac{\sin p_j}{E(p)} \in L^2(\mathbb{T}^n), \quad n \geq 3,$$

we have $f \in L^1(\mathbb{T}^n) \setminus L^2(\mathbb{T}^n)$ for $n = 2$, and $f \in L^2(\mathbb{T}^n)$ for $n \geq 3$. Then (2) and (3) follow. □

5. Main theorems

5.1. **Case of $n \geq 2$.** In order to describe the main results we have to separate (λ, μ)-plane into several regions.

Lemma 5.1. *Let $n \geq 2$. Then $\lambda_\infty(z) \leq \lambda_s(z) \leq \lambda_c(z)$ for $z \in (-\infty, 0]$.*

Proof. By Lemma 3.8 it follows that

$$\lambda_c(z) = \frac{1}{c(z) - d(z)} > \lambda_s(z) = \frac{1}{s(z)} > \lambda_\infty(z) = \frac{a(z)}{b(z)}$$

for $z < 0$. By a limiting argument the lemma follows. □

We introduce 4-half planes:

$$\mathfrak{C}_- = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_c\}, \quad \mathfrak{C}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_c\}$$
$$\mathfrak{S}_- = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda < \lambda_s\}, \quad \mathfrak{S}_+ = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda > \lambda_s\},$$

and two boundaries: $\mathfrak{C}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_c\}$ and $\mathfrak{S}_0 = \{(\lambda, \mu) \in \mathbb{R}^2; \lambda = \lambda_s\}$. Note that $\mathfrak{S}_- \subset \mathfrak{C}_-$ and $\mathfrak{C}_+ \subset \mathfrak{S}_+$, and we define open sets surrounded by hyperbola $\mathfrak{H}_0$ and boundary Γ_c and Γ_s by :

$$D_0 = G_0, \quad D_1 = G_1 \cap \mathfrak{S}_-, \quad D_2 = G_2 \cap \mathfrak{S}_-, \quad D_{n+1} = G_1 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-),$$
$$D_{n+2} = G_2 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \quad D_{2n} = G_1 \cap \mathfrak{C}_+, \quad D_{2n+1} = G_2 \cap \mathfrak{C}_+.$$

The boundaries of these sets define disjoint 8 curves:

$$B_0 = \Gamma_l, \quad B_1 = \Gamma_r \cap \mathfrak{S}_-, \quad B_{n+1} = \Gamma_r \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \quad B_{2n} = \Gamma_r \cap \mathfrak{C}_+,$$
$$S_1 = \mathfrak{S}_0 \cap G_1, \quad S_2 = \mathfrak{S}_0 \cap G_2, \quad C_{n+1} = \mathfrak{C}_0 \cap G_1, \quad C_{n+2} = \mathfrak{C}_0 \cap G_2,$$

and two one point sets given by

$$A = \Gamma_r \cap \mathfrak{S}_0, \quad B = \Gamma_r \cap \mathfrak{C}_0.$$

We are now in the position to state the main theorem for $n \geq 2$.

Theorem 5.2. *Let $n \geq 2$.*

(a) *Assume that $(\lambda, \mu) \in D_k$, $k \in \{0, 1, 2, n+1, n+2, 2n, 2n+1\}$, then $H_{\lambda\mu}$ has k eigenvalues in $(-\infty, 0)$. In addition $H_{\lambda\mu}$ has neither a threshold eigenvalue nor a threshold resonance (see Table 1).*
(b) *0 is not a super-threshold resonance of $H_{\lambda\mu}$ for any $(\lambda, \mu) \in \mathbb{R}^2$.*
(c) *Assume that (λ, μ) in B_k, S_k, C_k and A, B the next results are true in Table 2:*

	D_0	D_1	D_2	D_{n+1}	D_{n+2}	D_{2n}	D_{2n+1}
E.v.in $(-\infty, 0)$	0	1	2	$n+1$	$n+2$	$2n$	$2n+1$

TABLE 1. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on D_k for $n \geq 2$.

	Curve B_k	*Curve* S_k	*Curve* C_k	*Point* A	*Point* B
E.v. $(-\infty, 0)$	k	k	k	1	$n+1$
Th.res.0	$n=2$ – $n=3,4$ 1 $n \geq 5$ –	$n=2$ 2 $n \geq 3$ –	$n \geq 2$ –	$n=2$ 2 $n=3,4$ 1 $n \geq 5$ –	$n=2$ – $n=3,4$ 1 $n \geq 5$ –
Th.e.v.0	$n=2$ – $n=3,4$ – $n \geq 5$ 1	$n=2$ – $n \geq 3$ n	$n \geq 2$ $n-1$	$n=2$ – $n=3,4$ n $n \geq 5$ $n+1$	$n=2$ 1 $n=3,4$ $n-1$ $n \geq 5$ n

TABLE 2. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on the edges of D_k for $n \geq 2$.

Proof. (a) follows from Lemmas 3.18 and 4.4. (b) follows from Lemmas 3.24, 4.6 and 4.8. (c) follows from Lemmas 3.27 and 4.8. □

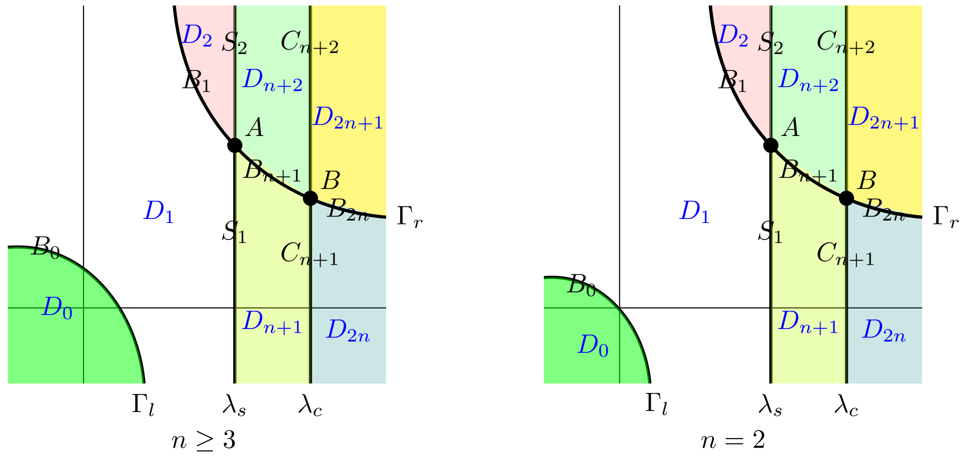

FIGURE 5. Hyperbola for $n \geq 2$

We draw the results for $n = 2$ on (λ, μ)-plane in the right-hand side of Figure 5 and for $n \geq 3$ in the left-hand side of Figure 5.

5.2. **Case of** $n = 1$**.** Let $n = 1$. In this case, the asymptote of $\mathfrak{H}_0$ is $(\lambda_\infty(0), \mu_\infty(0)) = (1, 1)$, and λ_c is not defined. We also see that $\lambda_s = 1 = \lambda_\infty(0)$. Then we have 4 sets:

$$D_0 = G_0, \quad D_1 = G_1 \cap \mathfrak{S}_-, \quad D_2 = G_1 \cap \mathfrak{S}_+, \quad D_3 = G_2.$$

The boundaries of these sets define disjoint 3 curves:

$$B_0 = \Gamma_l, \quad B_2 = \Gamma_r, \quad S_1 = \mathfrak{S}_0.$$

Finally we define point C by $C = \Gamma_r \cap \mathfrak{S}_0$. Now we formulate next result for $n = 1$.

Theorem 5.3. *Let* $n = 1$.

(a) *Assume* $(\lambda, \mu) \in D_k$, $k \in \{0, 1, 2, 3\}$. *Then* $H_{\lambda\mu}$ *has* k *eigenvalues in* $(\infty, 0)$. *In addition* 0 *is neither a threshold resonance nor a threshold eigenvalue (see Table 3).*

	D_0	D_1	D_2	D_3
E.v.in $(-\infty, 0)$	0	1	2	3

TABLE 3. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on D_k for $n = 1$.

(b) *Assume that* $(\lambda, \mu) \in S_1$. *Then* $H_{\lambda\mu}$ *has a super-threshold resonance.*
(c) *Assume that* $(\lambda, \mu) \in B_k \cup S_1$. *Then the next result in Table 4 is true.*

	B_k	*Curve* S_k
E.v.in $(-\infty, 0)$	k	k
Th.res.0	$-$	$-$
Th.e.v.0	$-$	$-$

TABLE 4. Spectrum of $H_{\lambda\mu}$ for (λ, μ) on the edges of D_k for $n = 1$.

In particular $H_{\lambda\mu}$ *has neither a threshold resonance nor a threshold eigenvalue.*

Proof. The theorem follows from Lemmas 3.19, 3.27, 4.4 and 4.8. □

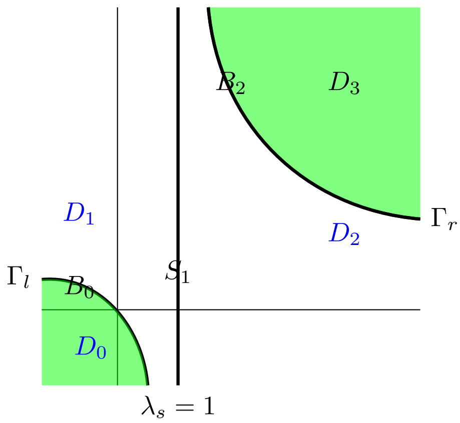

FIGURE 6. Hyperbola for $n = 1$

We draw the results for $n = 1$ on (λ, μ)-plane in Figure 6. In particular $(\lambda, \mu) \in S_1 \cup S_2$

5.3. **Eigenvalues and asymptote.** From results obtained in the previous section a stable point on λ can be found. In general the spectrum of $H_{\lambda\mu}$ is changed according to varying μ with a fixed λ. This can be also seen from Figures 2, 5 and 6. Curves on these figures consist of only hyperbolas and vertical lines. Then an asymptote has no intersection of these lines. It can be seen that

$$\{(1, \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\} \cup \{(\lambda, n) \in \mathbb{R}^2; \lambda \in \mathbb{R}\}$$

is the asymptote of hyperbola $\mathcal{H}_z(\lambda, \mu)$ for $n = 1, 2$. On the other hand

$$\{(a(0)/b(0), \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\} \cup \{(\lambda, n) \in \mathbb{R}^2; \lambda \in \mathbb{R}\}$$

is the asymptote of hyperbola $\mathcal{H}_z(\lambda, \mu)$ for $n \geq 3$. Then we can have the corollary below.

Corollary 5.4. *Let* $\lambda = 1$ *for* $n = 1, 2$ *and* $\lambda = a(0)/b(0)$ *for* $n \geq 3$.

(1) *Let* $n = 1$. *Then* $H_{\lambda\mu}$ *has a super-threshold resonance* 0 *and only one eigenvalue in* $(-\infty, 0)$ *for any* μ.
(2) *Let* $n \geq 2$. *Then* $H_{\lambda\mu}$ *has only one eigenvalue in* $(-\infty, 0)$ *for any* μ.

Proof. For $n = 1, 2$, let $l_n = S_1$. From Figures 5 and 6 it follows that $l_n \cap \Gamma_l = l_n \cap \Gamma_r = \emptyset$. Then the corollary follows. For $n \geq 3$, let $l_n = \{(a(0)/b(0), \mu) \in \mathbb{R}^2; \mu \in \mathbb{R}\}$. We can also see that $l_n \cap \Gamma_l = l_n \cap S_1 = l_n \cap S_2 = l_n \cap C_{n+1} = l_n \cap C_{n+2} = \emptyset$. Then the corollary follows. ☐

Appendix A. Proof of Lemma 3.8

Proof. We can see that

$$s(z) = 1 + z(a(z) + b(z)), \quad n = 1, \tag{A.1}$$

$$s(z) = 1 + z(a(z) + b(z)) - (n-1)(a(z) - d(z)), \quad n \geq 2, \tag{A.2}$$

and $a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}$ for $n = 1$.
(**Case** $n = 1$**)** From $a(z) - b(z) = \frac{1}{n} + \frac{z}{n}a(z)$ we see that $b(z) = a(z)(1-z) - 1$. Employing (A.1) and $a(z) = \frac{1}{\sqrt{-z}\sqrt{2-z}}$, we have

$$a(z)s(z) = a(z) + z(a^2(z) + a^2(z)(1-z) - a(z)) = a(z)(1-z) - 1 = b(z),\ z \leq 0. \tag{A.3}$$

(**Case** $n \geq 2$**)** By

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\cos p}{E(p) - z}dp = \frac{1}{(2\pi)^2}\left(\int_{\mathbb{T}} \frac{\sin^2 p}{E(p) - z}dp\right)\left(\int_{\mathbb{T}} \frac{1}{E(p) - z}dp\right),$$

we obtain

$$b(z) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{T}^{n-1}}\left(\int_{\mathbb{T}} \frac{\sin^2 p_1}{E(p) - z}dp_1\right)\left(\int_{\mathbb{T}} \frac{1}{E(p) - z}dp_1\right)dp_2 \dots dp_n,$$

which provides

$$b(z) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{T}^{n-1}\times\mathbb{T}^{n-1}} F(\tilde{p})G(\tilde{q})d\tilde{p}d\tilde{q},$$

where $\tilde{p} = (p_2, \dots, p_n)$, $\tilde{q} = (p_2, \dots, p_n)$, $F(\tilde{p}) = \int_{\mathbb{T}} \frac{\sin^2 p_1}{E(p_1,\tilde{p}) - z}dp_1$ and $G(\tilde{q}) = \int_{\mathbb{T}} \frac{1}{E(p_1,\tilde{q}) - z}dp_1$. Then

$$a(z)s(z) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{T}^{n-1}\times\mathbb{T}^{n-1}} F(\tilde{p})G(\tilde{q})d\tilde{p}d\tilde{q},$$

and we can have the relations:

$$a(z)s(z)-b(z)=-\frac{1}{2(2\pi)^{2n}}\int_{\mathbb{T}^{n-1}\times\mathbb{T}^{n-1}}\big(F(\tilde p)-F(\tilde q)\big)\big(G(\tilde p)-G(\tilde q)\big)d\tilde p d\tilde q,$$

$$\big(F(\tilde p)-F(\tilde q)\big)\big(G(\tilde p)-G(\tilde q)\big)$$
$$=\left(\sum_{j=2}^{n}(\cos p_j-\cos q_j)\right)^2\int_{\mathbb{T}}\frac{\sin^2 p_1 dp_1}{(E(p_1,\tilde p)-z)(E(p_1,\tilde q)-z)}\int_{\mathbb{T}}\frac{dp_1}{(E(p_1,\tilde p)-z)(E(p_1,\tilde q)-z)} \tag{A.4}$$

prove $a(z)s(z)<b(z)$ for $n\geq 2$. Furthermore (A.4) shows that the last inequality leaves its sign invariant even for $z=0$ and $n\geq 3$. Now we prove (3.25). By

$$c(0)-d(0)=\frac{1}{2}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{(\cos p_1-\cos p_2)^2}{E(p)}dp$$
$$s(0)=\frac{1}{2}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{(\sin p_1-\sin p_2)^2}{E(p)}dp,$$

we describe

$$c(0)-d(0)=\frac{1}{2}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{4\sin^2\frac{p_1-p_2}{2}\sin^2\frac{p_1+p_2}{2}}{E(p)}dp$$
$$s(0)=\frac{1}{2}\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}\frac{4\sin^2\frac{p_1-p_2}{2}\cos^2\frac{p_1+p_2}{2}}{E(p)}dp.$$

Introducing new variables $u=(p_1-p_2)/2$ and $t=(p_1+p_2)/2$ we get

$$c(0)-d(0)-s(0)=-2\frac{1}{(2\pi)^n}\int_{\mathbb{T}^{n-2}}dp_3\ldots dp_n,\int_{\mathbb{T}}4\sin^2 u du\int_{\mathbb{T}}\frac{\cos 2t}{A-2\cos t\cos u}dt, \tag{A.5}$$

where $A=2+\sum_{j=3}^{n}(1-\cos p_j)$ is a function being independent of both t and u. We have

$$\int_{\mathbb{T}}\frac{\cos 2t}{A-2\cos t\cos u}dt=16A\mathrm{c}\int_0^{\pi/4}\frac{\cos^2 2t\cos^2 u}{(A^2-(2\cos t\cos u)^2)(A^2-(2\sin t\cos u)^2)}dt>0.$$

Using the last inequality to (A.5) we get (3.25). □

Appendix B. Proof of Lemma 3.9

Proof. First we prove

$$a'(z)b(z)-a(z)b'(z)<0,\quad z\in(-\infty,0), \tag{B.1}$$

which proves the monotone decreasing of $\frac{a(z)}{b(z)}$. The equality

$$a'(z)b(z)-a(z)b'(z)=\frac{1}{(2\pi)^{2n}}\int_{\mathbb{T}^n\times\mathbb{T}^n}\cos p_1\frac{E(p)-E(q)}{(E(p)-z)^2(E(q)-z)^2}dpdq$$

gives

$$a'(z)b(z)-a(z)b'(z)=\frac{1}{n(2\pi)^{2n}}\sum_{j=1}^{n}\int_{\mathbb{T}^n\times\mathbb{T}^n}\cos p_j\frac{E(p)-E(q)}{(E(p)-z)^2(E(q)-z)^2}dpdq.$$

Changing variables $E(p)-E(q)=\sum_{j=1}^n(\cos p_j-\cos q_j)$ provides the inequality

$$a'(z)b(z)-a(z)b'(z)=-\frac{1}{n(2\pi)^{2n}}\int_{\mathbb{T}^n\times\mathbb{T}^n}\frac{\left(\sum_{j=1}^n(\cos p_j-\cos q_j)\right)^2}{(E(p)-z)^2(E(q)-z)^2}dpdq<0$$

which proves $\left(\frac{a(z)}{b(z)}\right)'<0$ in $(-\infty,0)$. Using the definition of $a(z)$, we achieve $a(z)=O(\frac{1}{|z|})$ and $b(z)=O(\frac{1}{z^2})$ as $z\to-\infty$, and hence $a(z)/b(z)=O(|z|)$ as $z\to-\infty$ proves (3.26). Let $n=1,2$. By virtue of Lemma 3.6 we may write

$$\frac{a(z)}{b(z)}=\frac{1}{1-\frac{b(z)-a(z)}{a(z)}}=\frac{1}{1-\frac{1}{a(z)n}-\frac{z}{n}},$$

and since $a(z)=O(\frac{1}{z})$ as $z\to 0-$ we receive (3.27). In case $n\geq 3$, the limit (3.27) is obvious. □

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