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Existence of a ground state for the Nelson model with a singular perturbation

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The existence of a ground state of the Nelson Hamiltonian with perturbations of the form $\sum_{j=1}^{4} c_j \phi^j$ with $c_4 > 0$ is considered. The self-adjointness of the Hamiltonian and the existence of a ground state are proven for arbitrary values of coupling constants. © *2011 American Institute of Physics*. [doi:10.1063/1.3548076]

I. INTRODUCTION

The Nelson model introduced in Ref. 21 describes N-quantum mechanical particles coupled to a scalar bose field. Let ω be a boson dispersion relation, which describes the energy of a single boson. Then the free field Hamiltonian H_{f} is given by the second quantization of ω:

$$H_{\mathrm{f}} = d\Gamma(\omega). \tag{1.1}$$

Let $K = -\Delta + V$ be a Hamiltonian of a quantum mechanical particle. Then the standard Nelson Hamiltonian is formally given by

$$H_{\mathrm{Nelson}} = K + H_{\mathrm{f}} + \alpha\phi(\rho). \tag{1.2}$$

Here α is a coupling constant and $\phi(\rho)$ is a field operator smeared by a test function ρ.

We consider the Nelson model with $\phi(\rho)$ replaced by the singular perturbation:

$$P(\phi(\rho)) = \sum_{j=1}^{4} c_j \phi(\rho)^j \tag{1.3}$$

with $c_4 > 0$. Thus the total Hamiltonian under consideration is

$$H = K + H_{\mathrm{f}} + P(\phi(\rho)) \tag{1.4}$$

with the domain $D(K) \cap D(H_{\mathrm{f}}) \cap D(\phi(\rho)^4)$. We suppose that K has a compact resolvent, and $V_-^{1/2}$ is relatively bounded with respect to $(-\Delta)^{1/2}$, where $V_- \geq 0$ is the negative part of V.

We are concerned with the spectrum of H in the nonperturbative way. The bottom of the spectrum of a Hamiltonian is called a ground state energy, and an eigenvector associated with the ground state energy is called a ground state. We see that the bottom of the spectrum of $K + H_{\mathrm{f}}$ is equal to the edge of the continuum. Then it is not trivial to show the existence of the ground state of H even when perturbations are not singular.

The main result of this paper is to show (1) and (2) below:

(1) H is self-adjoint and bounded from below;
(2) H has a ground state for all ρ under some conditions.

(**Related models**) We review here several models concerned so far, but this is an incomplete list.

[Nelson model] Bach–Fröhlich–Sigal[5] show the existence and uniqueness of the ground state of some general scalar model for sufficiently weak couplings. This model includes the standard

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Nelson model. Spohn[23] proves, however, the existence of the ground state for the Nelson model for *arbitrary* values of coupling constants, but if K has purely discrete spectrum. Gérard[9] also shows the similar result, but the method is different from Ref. 23. Hiroshima and Sasaki[17] shows the enhanced binding of the many body Nelson model, i.e., the existence of ground states is shown for sufficiently large couplings, but the existence of ground state of decoupled Hamiltonian is not assumed.

The results mentioned above are proven under the so called infrared regularity conditions. Then the next task is to study the case of no infrared regularity conditions. Arai, Hirokawa, and Hiroshima[4] show the absence of ground state of some abstract quantum field models without infrared regularity conditions. Lőrinczi, Minlos, and Spohn,[19] Dereziński and Gérard,[8] and Hirokawa[16] prove that the Nelson Hamiltonian has no ground states if the infrared regularity condition is not assumed. Arai[3] shows, however, that the Nelson model without infrared regularity condition also has a ground state if a non-Fock representation is taken. See also Refs. 10 and 11 for the Nelson model on a pseudo Riemannian manifold.

[The Pauli–Fierz model] The Pauli–Fierz model is a quantum field model in nonrelativistic quantum electrodynamics. Its interaction is given by minimal coupling, and then the spectral analysis turns to be hard due to the derivative coupling. Bach, Fröhlich, and Sigal[6] prove the existence of ground state for sufficiently weak couplings, but the infrared regularity condition is not assumed. This is the large difference between the Nelson model and the Pauli–Fierz model. Griesemer, Lieb, and Loss,[14] and Lieb and Loss[18] show the existence of a ground state of the Pauli–Fierz Hamiltonian for arbitrary values of coupling constants under no infrared regularity condition. In Refs. 14 and 18, the binding condition is introduced to show the existence of a ground state. We extend this to the Pauli–Fierz model with a variable mass.[15] This method is also applied to the Nelson model by Sasaki.[22]

[Singular perturbations] The model under consideration in this paper is of the similar form of the $(\phi^4)_2$-model in the quantum field theory. This model describes bosons with self-interaction in two-dimensional space-time. Glimm and Jaffe[12,13] considered the spectral properties of the $(\phi^4)_2$-model. In this model, the dispersion relation is supposed to be strictly positive and the Hamiltonian is defined on a boson Fock space. Miyao and Sasaki[20] show the existence of the ground state for a generalized spin-boson model with ϕ^2-perturbation, and it is not supposed that the particle Hamiltonian has a compact resolvent. Takaesu[24] shows the existence of a ground state for a generalized spin-boson model with a singular perturbation of the form (1.3), but for sufficiently small coupling constants.

(**Strategy**) As far as we know, it is new to show the existence of the ground state of (1.4) for *all* values of a coupling constant. Here we show an outline of our proofs.

By making use of Ref. 1, we can prove the essential self-adjointness of H. First, we show that ϕ^4 is relatively bounded with respect to H. This relative boundedness leads to the self-adjointness of (1.4).

Next, we show the existence of a ground state of H by means of Refs. 7 and 9 for all values of coupling constants: We define the Hamiltonian H_σ, with the test function ρ replaced by $\rho_\sigma = \rho 1_{\{\sigma \le \omega(k)\}}$, and we show the existence of a ground state of H_σ for all $\sigma > 0$. We see that as $\sigma \to 0$, a normalized ground state of H_σ weakly converges to a nonzero vector, which is then a normalized ground state of H. To show this it is sufficient to show the boson number bound and the boson derivative bound of a normalized ground state of H_σ. These are done in Lemmas V.2 and V.5. To show the boson derivative bound, we suppose the infrared regularity condition:

$$\omega^{-5/4} \sup_{x \in \mathbb{R}^d} |\rho(x, \cdot)| \in L^2(\mathbb{R}^d_k). \tag{1.5}$$

This infrared regularity condition is stronger than the standard infrared regularity condition:

$$\omega^{-1} \sup_{x \in \mathbb{R}^d} |\rho(x, \cdot)| \in L^2(\mathbb{R}^d_k). \tag{1.6}$$

The condition (1.5) is used to show the convergence,

$$\|((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1})\rho_\sigma(k) P'(\phi_\sigma)\Phi_\sigma\| \to 0$$

in L^2 as $\sigma \to 0$ in Lemma V.4, where Φ_σ is a ground state of H_σ. In the case of the standard Nelson model, $P'(\phi_\sigma) = 1$. Then condition (1.6) is enough to show this convergence. In the singular case, we need, however, (1.5) to control the upper bound of $\|((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1})\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\|$.

This paper is organized as follows: Sec. II is devoted to defining the Nelson Hamiltonian with a singular perturbation. In Sec. III, we show the self-adjointness of H. In Sec. IV, we show the existence of a ground state of H but with an infrared cutoff. Finally in Sec. V, we show the existence of a ground state of H.

II. DEFINITION OF THE NELSON MODEL WITH $P(\phi)$ PERTURBATION

A. Preliminaries

Here we introduce fundamental facts on Fock spaces and second quantizations. Let $\mathcal{X}$ be a Hilbert space over the complex field $\mathbb{C}$. Then

$$\mathcal{F}_{\rm b}(\mathcal{X}) = \bigoplus_{n=0}^{\infty}\left[\otimes_s^n \mathcal{X}\right] = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \middle| \Psi^{(n)} \in \otimes_s^n \mathcal{X}, n \geq 0, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\} \tag{2.1}$$

is called the boson Fock space over $\mathcal{X}$, where $\otimes_s^n \mathcal{X}$ denotes the symmetric tensor product of $\mathcal{X}$ and $\otimes_s^0 \mathcal{X} = \mathbb{C}$. Let $\Omega = \{1, 0, \ldots\} \in \mathcal{F}_{\rm b}(\mathcal{X})$ be the Fock vacuum. The number operator N is defined by

$$(N\Psi)^{(n)} = n\Psi^{(n)} \tag{2.2}$$

with the domain

$$D(N) = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b}(\mathcal{X}) \;\middle|\; \sum_{n=0}^{\infty} n^2 \|\Psi^{(n)}\|^2 < \infty \right\}. \tag{2.3}$$

The finite particle subspace of $\mathcal{F}_{\rm b}(\mathcal{X})$ is a dense subspace of $\mathcal{F}$, which is given by

$$\mathcal{F}_{\rm b,0}(\mathcal{X}) = \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm b}(\mathcal{X}) \mid \Psi^{(n)} = 0 \,\text{except for finitely many } n \right\}. \tag{2.4}$$

The creation operator $a^\dagger(f)$ smeared by $f \in \mathcal{X}$ is also given by

$$(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \;\; n \geq 1, \tag{2.5}$$

and $(a^\dagger(f)\Psi)^{(0)} = 0$ with the domain

$$D(a^\dagger(f)) = \left\{ \Psi \in \mathcal{F}_{\rm b} \;\middle|\; \sum_{n=1}^{\infty} \|\sqrt{n} S_n(f \otimes \Psi^{(n-1)})\|^2 < \infty \right\}. \tag{2.6}$$

Here S_n is the symmetrization operator on $\otimes^n \mathcal{X}$. The annihilation operator smeared by $f \in \mathcal{X}$ is given by the adjoint of $a^\dagger(f)$:

$$a(f) = (a^\dagger(f))^*. \tag{2.7}$$

Note that $a(f)$ is antilinear in f, while $a^\dagger(f)$ is linear in f. We see that $a(f)\lceil_{\otimes_s^n \mathcal{X}}$ is bounded from $\otimes_s^n \mathcal{X}$ to $\otimes_s^{n-1} \mathcal{X}$ and $a^\dagger(f)\lceil_{\otimes_s^n \mathcal{X}}$ from $\otimes_s^n \mathcal{X}$ to $\otimes_s^{n+1} \mathcal{X}$. $a(f)$ and $a^\dagger(f)$ satisfy canonical commutation relations:

$$[a(f), a^\dagger(g)] = (f, g), \quad [a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0. \tag{2.8}$$

Let $\mathcal{D}$ be a dense subspace of $\mathcal{X}$. Then

$$\mathcal{F}_{\rm b,fin}(\mathcal{D}) = \mathcal{L}\{\Omega, a^\dagger(f_1)\ldots a^\dagger(f_n)\Omega \mid n \in \mathbb{N}, f_j \in \mathcal{D}, j = 1, n\} \tag{2.9}$$

is also dense in $\mathcal{F}_{\rm b}(\mathcal{X})$, where $\mathcal{L}\{\cdots\}$ denotes the linear hull of $\{\cdots\}$. The Sigal field smeared by $f \in \mathcal{X}$ is given by

$$\phi(f) = \frac{1}{\sqrt{2}}(a(f) + a^\dagger(f)). \tag{2.10}$$

Let $\mathcal{D}$ be a dense subspace and $f \in \mathcal{D}$. Then $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}(\mathcal{D})$. The Sigal field satisfies the following commutation relation:

$$[\phi(f), \phi(g)] = i\Im(f, g). \tag{2.11}$$

When $\mathcal{X} = L^2(\mathbb{R}^d)$, let

$$(a(k)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(k, k_1, \dots, k_n). \tag{2.12}$$

If $\Psi \in D(N^{1/2})$, then $a(k)\Psi \in \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ for almost every $k \in \mathbb{R}^d$. For $\Psi \in D(a(f))$,

$$(a(f)\Psi)^{(n)}(k_1, \dots, k_n) = \int_{\mathbb{R}^d} \overline{f(k)}(a(k)\Psi)^{(n)}(k, k_1, \dots, k_n)dk$$

holds.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces, and T be a densely defined closable operator from $\mathcal{X}$ to $\mathcal{Y}$. Then $\Gamma(T)$ is defined by

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \otimes^n T\lceil_{\otimes_s^n \mathcal{X}} \tag{2.13}$$

with $\otimes^0 T = 1$. If $\mathcal{X} = \mathcal{Y}$, the second quantization of T is defined by

$$d\Gamma(T) = \bigoplus_{n=0}^{\infty} T^{(n)}, \tag{2.14}$$

where $T^{(0)} = 0$ and

$$T^{(n)} = \overline{\sum_{j=1}^{n} 1 \otimes \cdots \otimes 1 \otimes \overset{j\text{th}}{\breve{T}} \otimes 1 \otimes \cdots \otimes 1\lceil_{\otimes_s^n \mathcal{X}}}, \quad n \geq 1. \tag{2.15}$$

Here $\overline{S}$ denotes the closure of an operator S. The number operator N can be written as

$$N = d\Gamma(1). \tag{2.16}$$

We define the unitary operator $U_{\mathcal{X},\mathcal{Y}}$ from $\mathcal{F}_{\mathrm{b}}(\mathcal{X} \oplus \mathcal{Y})$ to $\mathcal{F}_{\mathrm{b}}(\mathcal{X}) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{Y})$ by

$$\begin{aligned} &U_{\mathcal{X},\mathcal{Y}} a^\dagger(f_1 \oplus 0) \dots a^\dagger(f_n \oplus 0) a^\dagger(0 \oplus g_1) \dots a^\dagger(0 \oplus g_n)\Omega \\ &= a^\dagger(f_1) \dots a^\dagger(f_n)\Omega \otimes a^\dagger(g_1) \dots a^\dagger(g_n)\Omega. \end{aligned} \tag{2.17}$$

Let T be a densely defined closable operator from $\mathcal{X}$ to $\mathcal{X} \oplus \mathcal{X}$. Then the operator $\check{\Gamma}(T) : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X}) \otimes \mathcal{F}(\mathcal{X})$ is defined by

$$\check{\Gamma}(T) = U_{\mathcal{X},\mathcal{X}}\Gamma(T). \tag{2.18}$$

B. The Nelson Hamiltonian with $P(\phi)$ perturbation

In this paper the number of quantum mechanical particles is supposed to be one, but with the spatial dimension d. Let $\mathcal{K} = L^2(\mathbb{R}_x^d)$ and $\mathcal{F}_{\mathrm{b}} = \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}_k^d))$. The Hilbert space of state space is given by

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_{\mathrm{b}}, \tag{2.19}$$

where $\mathcal{K}$ describes the state space of a quantum mechanical particle, and $\mathcal{F}_{\mathrm{b}}$ that of bosons. Let ω be a boson dispersion relation. We suppose that ω is a densely defined, non-negative multiplication operator on $L^2(\mathbb{R}_k^d)$. Further conditions on ω are given later. The free field Hamiltonian is given by $d\Gamma(\omega)$. Let K be a Hamiltonian of the quantum mechanical particle. Then the decoupled Hamiltonian is given by

$$H_0 = K \otimes 1 + 1 \otimes d\Gamma(\omega) \tag{2.20}$$

with the domain $D(H_0) = D(K \otimes 1) \cap D(1 \otimes d\Gamma(\omega))$. In what follows, we denote $T \otimes 1$ and $1 \otimes S$ by T and S, respectively, for simplicity unless confusion arises. Let us now define a field operator ϕ in $\mathcal{H}$. Let $\rho(x, k)$ be a test function, such that $\rho(x, \cdot) \in L^2(\mathbb{R}^d_k)$ for each $x \in \mathbb{R}^d$. Then we set

$$\phi(\rho(x, \cdot)) = \frac{1}{\sqrt{2}} \left(a(\rho(x, \cdot) + a^\dagger(\rho(x, \cdot)\right). \tag{2.21}$$

$\phi(\rho(x, \cdot))$ is essentially self-adjoint for each $x \in \mathbb{R}^d$ on

$$\mathcal{F}_{\mathrm{b,fin}} = \mathcal{L}\{\Omega, a^\dagger(h_1)\dots a^\dagger(h_n)\Omega | \, n \in \mathbb{N}, \;\; f, h_i \in C_{\mathrm{c}}(\mathbb{R}^d_k), i = 1, \dots n\}.$$

Then the field operator ϕ is defined by the constant fiber direct integral of $\overline{\phi(\rho(x, \cdot))}$:

$$\phi = \phi(\rho) = \int_{\mathbb{R}^d}^{\oplus} \overline{\phi(\rho(x, \cdot))} dx. \tag{2.22}$$

Let

$$P(x) = x^4 + c_3 x^3 + c_2 x^2 + c_1 x, \tag{2.23}$$

where $c_j, \; j = 1, 2, 3,$ are arbitrary real numbers. Then the Nelson Hamiltonian with $P(\phi)$ perturbation is given by

$$H = H_0 + P(\phi) \tag{2.24}$$

with the domain $D(H) = D(H_0) \cap D(\phi^4)$.

C. Hypotheses and main theorems

To show the self-adjointness of H and the existence of a ground state of H, we introduce the following hypotheses.

Hypothesis II.1 (Hypotheses of K)

(1) K is given by

$$K = -\Delta + V \tag{2.25}$$

with the domain $D(K) = D(-\Delta) \cap D(V)$. Here V is a real-valued multiplication operator, which describes an external potential.

(2) There exist constants $0 < a < 1$ and $b > 0$, so that for all $\Psi \in D(V_-^{1/2})$, $\Psi \in D(|p|)$ and

$$\|V_-^{1/2}\Psi\|^2 \leq a\||p|\Psi\|^2 + b\|\Psi\|^2. \tag{2.26}$$

Here $V_-(x) = \max\{0, -V(x)\}$ and $p = -i\nabla_x$.

(3) K is a non-negative, self-adjoint operator and has a compact resolvent.

Hypothesis II.2 (Hypotheses of ω)

(1) $\omega \in C(\mathbb{R}^d_k; [0, \infty))$;
(2) $\nabla\omega \in L^\infty(\mathbb{R}^d_k)$;
(3) $\omega(k) = 0$, if and only if $k = 0$.

Definition II.3: Let $\mathcal{X}$ be a Hilbert space. $f \in L^\infty(\mathbb{R}^d; \mathcal{X})$ is said to be weakly differentiable if there exists $g \in L^\infty(\mathbb{R}^d; \mathcal{X})$, such that for all $\Psi \in \mathcal{X}$ and $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (\partial_j\varphi)(x)(\Psi, f(x))_{\mathcal{X}} dx = -\int_{\mathbb{R}^d} \varphi(x)(\Psi, g(x))_{\mathcal{X}} dx. \tag{2.27}$$

In this case, we denote $g(x)$ by $\partial_j f(x)$.

Hypothesis II.4 (Hypotheses of ρ) $x \mapsto \rho(x, \cdot)$ is an element of $L^\infty(\mathbb{R}^d_x; L^2(\mathbb{R}^d_k))$ and weakly twice differentiable. Moreover, for each $k \in \mathbb{R}^d_k$, $\rho(k) = \rho(\cdot, k)$ is a bounded operator on $L^2(\mathbb{R}^d_x)$, such that

$$\omega^{-1/2}\|\rho(\cdot)\|, \;\; \omega\|\rho(\cdot)\|, \;\; \omega^{-1/2}\|\nabla_x\rho(\cdot)\|, \;\; \|\nabla_x\rho(\cdot)\| \in L^2(\mathbb{R}^d_k). \tag{2.28}$$

Hypothesis II.5 (Infrared regularity condition) We also assume the infrared regularity condition:

$$\omega^{-5/4}\|\rho(\cdot)\| \in L^2(\mathbb{R}_k^d). \tag{2.29}$$

We denote $\|\omega^l\|\rho(\cdot)\|\|_{L^2(\mathbb{R}_k^d)}$ by $\|\omega^l\rho\|$ for $-5/4 \le l \le 1$. Let

$$\mathcal{H}_{\rm fin} = \mathcal{L}\{f\otimes\Omega,\ f\otimes a^\dagger(h_1)\dots a^\dagger(h_n)\Omega |\, n\in\mathbb{N},\ f\in D(K),\ \ h_i\in C_{\rm c}^\infty(\mathbb{R}_k^d),\ 1\le i\le n\}. \tag{2.30}$$

Now let us state the main theorems.

Theorem II.6: *Suppose Hypotheses II.1 and II.4. Then H is self-adjoint and essentially self-adjoint on $\mathcal{H}_{\rm fin}$.*

Theorem II.7: *Suppose Hypotheses II.1, II.2, II.4, and II.5. Then H has a ground state.*

III. SELF-ADJOINTNESS OF *H*

The following proposition on the essential self-adjointness is known.

Proposition III.1: Ref. 1 Let $\mathfrak{H} = \bigoplus_{n=0}^\infty \mathfrak{H}_n$ be the direct sum of Hilbert spaces $\mathfrak{H}_n$, $n = 0, 1, 2, \dots,$ and

$$\hat{\mathfrak{H}} = \{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathfrak{H} | \Psi^{(n)} = 0 \text{ for all but finitely many } n\}.$$

The number operator in $\mathfrak{H}$ is defined by

$$(N_{\mathfrak{H}}\Psi)^{(n)} = n\Psi^{(n)}, \quad n = 0, 1, 2, \dots \tag{3.1}$$

with the domain

$$D(N_{\mathfrak{H}}) = \left\{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathfrak{H} \,\middle|\, \sum_{n=0}^\infty n^2\|\Psi^{(n)}\|^2 < \infty\right\}. \tag{3.2}$$

Let A_n, $n = 1, 2, \dots,$ be self-adjoint operators in $\mathfrak{H}_n$ and B be a symmetric operator in $\mathfrak{H}$. Put $A = \oplus_{n=0}^\infty A_n$. Let P_n be the projection from $\mathcal{X}$ to $\mathfrak{H}_n \subset \mathfrak{H}$: $P_n = 1_{\{n\}}(N_{\mathfrak{H}})$. Suppose that

(1) *$A + B$ is bounded from below;*
(2) *$\hat{\mathfrak{H}} \subset D(B)$ and there exists a constant $n_0 \ge 0$, such that $(P_m\Psi, BP_n\Psi) = 0$ whenever $|m-n| \ge n_0$;*
(3) *there exist a constant c and a linear operator L in $\mathfrak{H}$, such that* $\mathrm{Ran}(L\lceil_{D(L)\cap P_n\mathfrak{H}}) \subset P_n\mathfrak{H}$ *and*

$$|(\Theta, B\Psi)| \le c\|L\Theta\|\,\|(N_{\mathfrak{H}}+1)^2\Psi\|.$$

Then $A + B$ is essentially self-adjoint on $D(A)\cap\hat{\mathfrak{H}}$.

Lemma III.2: Suppose Hypotheses II.1 and II.4. Then H is essentially self-adjoint on $\mathcal{H}_{\rm fin}$.

Proof: By Proposition III.1, H is essentially self-adjoint on $D(H_0)\cap\mathcal{H}_0$, where $\mathcal{H}_0 = \mathcal{K}\otimes\mathcal{F}_0$. Thus it suffices to show that $\mathcal{H}_{\rm fin}$ is a core for $\overline{H}$. Let $\Psi \in D(H_0)\cap\mathcal{H}_0$. Then there exists a number n_0 so that for all $n \ge n_0$, $\Psi^{(n)} = 0$. Since $\mathcal{H}_{\rm fin}$ is a core for H_0 by Proposition VI.4 in Appendix and $P(\phi)\lceil_{\mathcal{K}\otimes(\otimes_s^{n_0}\mathcal{F})}$ is a bounded operator, it is seen that there exists a sequence $\{\Psi_j\}_{j=1}^\infty \subset \mathcal{H}_{\rm fin}$, such that $\Psi_j \to \Psi$ and $H\Psi_j \to H\Psi$. Therefore the lemma follows.

Lemma III.3: Suppose Hypothesis II.4. Let $\phi'_j = -i\phi(\partial_{x,j}\rho)$. Then, $\phi^n\Psi \in D(|p|^2)$ for $\Psi \in \mathcal{H}_{\rm fin}$ and $n \in \mathbb{N}$, and $[\phi, p_j] = \phi'_j$ follows on $\mathcal{H}_{\rm fin}$.

Proof: Let

$$\Phi = f\otimes a^\dagger(f_1)\dots a^\dagger(f_n)\Omega, \quad \Psi = g\otimes a^\dagger(g_1)\dots a^\dagger(g_{n-1})\Omega,$$

where f and $f_k \in C_{\rm c}^\infty(\mathbb{R}^d)$, $k = 1, \dots, n$, and $g \in D(K)$ and $g_k \in C_{\rm c}^\infty(\mathbb{R}^d)$, $k = 1, \dots, n-1$. It can be computed as

$$
\begin{aligned}
&(p_j\Phi, \phi\Psi)\\
&=\frac{i}{\sqrt{2}}\int_{\mathbb{R}^d}(\partial_j \bar{f})(x)g(x)(a(\rho_x)a^\dagger(f_1)\dots a^\dagger(f_n)\Omega,\ a^\dagger(g_1)\dots a^\dagger(g_{n-1})\Omega)dx\\
&=\frac{i}{\sqrt{2}}\sum_{l=1}^{n}\int(\partial_j \bar{f})(x)g(x)(f_l,\rho_x)dx(a^\dagger(f_1)\dots\widehat{a^\dagger(f_l)}\dots a^\dagger(f_n)\Omega,\ a^\dagger(g_1)\dots a^\dagger(g_{n-1})\Omega).
\end{aligned}
\tag{3.3}
$$

Here the symbol $\widehat{\ }$ denotes omission. Since $\rho(x,\cdot)$ is weakly differentiable, we see that

$$
(p_j\Phi, \phi\Psi) = (\phi\Phi, p_j\Psi) + (\Phi, \phi_j'\Psi) = (\Phi, (\phi p_j + \phi_j')\Psi). \tag{3.4}
$$

Thus we obtain that $\phi\Psi \in D(p_j)$ and

$$
p_j\phi\Psi = (\phi p_j + \phi_j')\Psi. \tag{3.5}
$$

By a similar computation, (3.5) holds for all $\Psi \in \mathcal{H}_{\text{fin}}$. In a similar way, we can also see that $\phi^n\Psi \in D(|p|^2)$. ■

Theorem III.4: *Suppose Hypotheses II.1 and II.4. Then there exists a constant C, such that for all* $\Psi \in D(\overline{H})$,

$$
\|\phi^n\Psi\| \le C\|(\overline{H}+1)\Psi\|, \quad n = 1,2,3,4. \tag{3.6}
$$

Proof: It is enough to show for the case of $n = 4$. Let $\Psi \in \mathcal{H}_{\text{fin}}$. It holds that

$$
\begin{aligned}
\|\phi^4\Psi\|^2 &= \left\|\left(H - H_0 - \sum_{k=1}^{3} c_k\phi^k\right)\Psi\right\|^2\\
&= \|H\Psi\|^2 - 2\Re(\phi^4\Psi, H_0\Psi)\\
&\quad -2\sum_{k=1}^{3}\Re(\phi^4\Psi, c_k\phi^k\Psi) - \left\|\left(H_0 + \sum_{k=1}^{3} c_k\phi^k\right)\Psi\right\|^2\\
&= \|H\Psi\|^2 - \left(\Psi, [\phi^2, [\phi^2, H_0]]\Psi\right) - 2\sum_{k=1}^{3} c_k(\phi^4\Psi, \phi^k\Psi)\\
&\quad -2\left\|H_0^{1/2}\phi^2\Psi\right\|^2 - \left\|\left(H_0 + \sum_{k=1}^{3} c_k\phi^k\right)\Psi\right\|^2.
\end{aligned}
\tag{3.7}
$$

Take a sufficiently small $\epsilon > 0$. Since ϕ^k, $k = 1,2,3$, are infinitesimally small with respect to ϕ^4, there exists a constant $C_{1,\epsilon} > 0$, such that

$$
-2\sum_{k=1}^{3} c_k(\phi^4\Psi, \phi^k\Psi) \le \epsilon\|\phi^4\Psi\|^2 + C_{1,\epsilon}\|\Psi\|^2. \tag{3.8}
$$

Thus by (3.7) and (3.8), we have

$$
\|\phi^4\Psi\|^2 \le \frac{1}{1-\epsilon}\Big(\|H\Psi\|^2 + C_{1,\epsilon}\|\Psi\|^2 + |(\Psi, [\phi^2, [\phi^2, H_0]]\Psi)|\Big). \tag{3.9}
$$

Thus in order to prove (3.6), it suffices to show that for sufficiently small $0 < \eta$, there exists a constant C_η, so that

$$
|(\Psi, [\phi^2, [\phi^2, H_0]]\Psi)| \le \eta\|\phi^4\Psi\|^2 + C_\eta\|(H+1)\Psi\|^2. \tag{3.10}
$$

By Proposition VI.1 (3) in Appendix, we have

$$
|(\Psi, [\phi^2, [\phi^2, d\Gamma(\omega)]]\Psi)| \le 4\|\omega^{1/2}\rho\|^2\|\phi\Psi\|^2 \le \epsilon\|\phi^4\Psi\|^2 + C_{2,\epsilon}\|\Psi\|^2. \tag{3.11}
$$

Let us estimate $|(\Psi, [\phi^2, [\phi^2, K]]\Psi)|$. By Lemma III.3, it is seen that

$$|(\Psi, [\phi^2, [\phi^2, p_j^2]]\Psi)| \le 2\|[\phi^2, p_j]\Psi\|^2 + |(\Psi, \{p_j[\phi^2, [\phi^2, p_j]] + [\phi^2, [\phi^2, p_j]]p_j\}\Psi)|$$
$$\le 2\|(\phi_j'\phi + \phi\phi_j')\Psi\|^2 + 8|\Im(\rho, \partial_{x,j}\rho)||\Re(\phi^2\Psi, p_j\Psi)|. \tag{3.12}$$

Since $[\phi, \phi_j'] = \Im(\rho, \partial_{x,j}\rho)$, by using the Schwarz inequality, we see that

$$|(\Psi, [\phi^2, [\phi^2, p_j^2]]\Psi)| \le \epsilon\|\phi^4\Psi\|^2 + C_{3,\epsilon}(\|\phi_j'\phi\Psi\|^2 + \||p|\Psi\|^2 + \|\Psi\|^2). \tag{3.13}$$

Let us estimate $\|\phi_j'\phi\Psi\|^2$ in (3.13). It holds that

$$\|\phi_j'\phi\Psi\|^2 \le C_4(\phi\Psi, (d\Gamma(\omega) + 1)\phi\Psi)$$
$$= C_4\{(\phi^2\Psi, (d\Gamma(\omega) + 1)\Psi) + (\phi\Psi, -i\phi(i\omega\rho)\Psi)\}. \tag{3.14}$$

By the Schwarz inequality again, we have

$$\|\phi_j'\phi\Psi\|^2 \le \epsilon\|(d\Gamma(\omega) + 1)\Psi\|^2 + C_{4,\epsilon}\|\phi^2\Psi\|^2 + \frac{C_4}{2}(\|\phi(i\omega\rho)\Psi\|^2 + \|\phi\Psi\|^2). \tag{3.15}$$

Since $\phi(i\omega\rho)$ and ϕ^k, $k = 1, 2$, are infinitesimally small with respect to $d\Gamma(\omega)$ and ϕ^4, respectively, we see that

$$\|\phi_j'\phi\Psi\|^2 \le \epsilon\|\phi^4\Psi\|^2 + 3\epsilon\|d\Gamma(\omega)\Psi\|^2 + C_{5,\epsilon}\|\Psi\|^2. \tag{3.16}$$

Since

$$\|d\Gamma(\omega)\Psi\|^2 \le 2\|H\Psi\|^2 + 2\|P(\phi)\Psi\|^2$$
$$\le 2\|H\Psi\|^2 + (2 + \epsilon)\|\phi^4\Psi\|^2 + C_{6,\epsilon}\|\Psi\|^2, \tag{3.17}$$

$\|\phi_j'\phi\Psi\|$ can be estimated by (3.16) as

$$\|\phi_j'\phi\Psi\|^2 \le \epsilon(3\epsilon + 7)\|\phi^4\Psi\|^2 + 6\epsilon\|H\Psi\|^2 + (C_{5,\epsilon} + 3\epsilon C_{6,\epsilon})\|\Psi\|^2. \tag{3.18}$$

Next, we estimate $\||p|\Psi\|^2$ in (3.13). By (2.26),

$$\||p|\Psi\|^2 = (\Psi, -\Delta\Psi) \le (\Psi, K\Psi) + (\Psi, V_-\Psi) \le (\Psi, K\Psi) + a\||p|\Psi\|^2 + b\|\Psi\|^2 \tag{3.19}$$

with $0 < a < 1$ and $0 < b$. Thus it holds that

$$\||p|\Psi\|^2 \le \frac{1}{1-a}(\Psi, (H - P(\phi))\Psi) + \frac{b}{1-a}\|\Psi\|^2. \tag{3.20}$$

Since $|\overline{H}|^{1/2}$ and $|P(\phi)|^{1/2}$ are infinitesimally small with respect to $\overline{H}$ and ϕ^4, respectively, we have

$$\||p|\Psi\|^2 \le \epsilon(\|\phi^4\Psi\|^2 + \|H\Psi\|^2) + C_{7,\epsilon}\|\Psi\|^2. \tag{3.21}$$

Therefore by (3.13), (3.18), and (3.21), for sufficiently small $\epsilon' > 0$, we have

$$|(\Psi, [\phi^2, [\phi^2, K]\Psi)| = \left|\sum_{j=1}^{3}(\Psi, [\phi^2, [\phi^2, p_j^2]\Psi)\right| \le \epsilon'(\|\phi^4\Psi\|^2 + \|H\Psi\|^2) + C_{8,\epsilon'}\|\Psi\|^2. \tag{3.22}$$

Therefore (3.10) follows from (3.11) and (3.22). Thus (3.6) is proven for $\Psi \in \mathcal{H}_{\rm fin}$. Since $\mathcal{H}_{\rm fin}$ is a core for $\overline{H}$, (3.6) also holds for all $\Psi \in D(\overline{H})$ by the closedness of ϕ^4.

Proof of Theorem II.6: By Lemma III.2, it suffices to show that

$$H = \overline{H}. \tag{3.23}$$

Let $\Psi \in D(\overline{H})$. Since $\mathcal{H}_{\rm fin}$ is a core for $\overline{H}$, there exists a sequence $\{\Psi_j\}_{j=1}^{\infty}$, so that $\Psi_j \in \mathcal{H}_{\rm fin}$ and

$$\lim_{j\to\infty}(\|\Psi_j - \Psi\| + \|\overline{H}(\Psi_j - \Psi)\|) = 0. \tag{3.24}$$

By the bound $\|\phi^4\Psi\| \leq C\|(\overline{H}+1)\Psi\|$, $\{\phi^4\Psi_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Since ϕ^4 is closed, $\Psi \in D(\phi^4)$ holds. Since

$$\|H_0(\Psi_j - \Psi_k)\| \leq \|H(\Psi_j - \Psi_k)\| + \|P(\phi)(\Psi_j - \Psi_k)\|, \tag{3.25}$$

$\{H_0\Psi_j\}_{j=1}^{\infty}$ is also a Cauchy sequence. Then, $\Psi \in D(H_0)$ by the closedness of H_0. Thus, $D(\overline{H}) \subset D(H_0) \cap D(\phi^4) = D(H)$. Therefore (3.23) is obtained.

IV. EXISTENCE OF A GROUND STATE OF $\tilde{H}_\sigma$ AND H_σ

A. The Nelson Hamiltonian with an infrared cutoff σ

The field operator with an infrared cutoff is given by

$$\phi_\sigma = \phi(\rho_\sigma), \quad \sigma > 0, \tag{4.1}$$

where

$$\rho_\sigma = \rho 1_{\{k|\sigma \leq \omega(k)\}}. \tag{4.2}$$

We define H_σ by

$$H_\sigma = H_0 + P(\phi_\sigma) \tag{4.3}$$

with the domain $D(H_\sigma) = D(H_0) \cap D(\phi_\sigma^4)$. By Theorem II.6, H_σ is self-adjoint.

Lemma IV.1: Suppose Hypotheses II.1 and II.4. Then H_σ converges to H as $\sigma \to 0$ in the norm resolvent sense:

$$\lim_{\sigma\to 0} \|(H_\sigma - z)^{-1} - (H - z)^{-1}\| = 0 \tag{4.4}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof: By the bound

$$\|\phi_\sigma^n \Psi\| \leq C\|(\overline{H_\sigma} + 1)\Psi\|, \quad n = 1, 2, 3, 4, \tag{4.5}$$

we see that

$$\|(d\Gamma(\omega) + 1)(H_\sigma - z)^{-1}\| < C, \tag{4.6}$$

with some constant C. Take arbitrary vectors $\Theta \in \mathcal{H}$ and $\Psi \in \mathcal{H}$. Then

$$\begin{aligned} &|(\Theta, ((H_\sigma - z)^{-1} - (H - z)^{-1})\Psi)| \\ &= |((H_\sigma - \bar{z})^{-1}\Theta, P(\phi)(H - z)^{-1}\Psi) - (P(\phi_\sigma)(H_\sigma - \bar{z})^{-1}\Theta, (H - z)^{-1}\Psi)|. \end{aligned} \tag{4.7}$$

Since $[\phi, \phi_\sigma] = 0$ and $D(H) \cup D(H_\sigma) \subset D(d\Gamma(\omega)) \subset D(\phi_\sigma^2) \cap D(\phi^2)$, it follows that

$$\begin{aligned} &|((H_\sigma - \bar{z})^{-1}\Theta, \phi^4(H - z)^{-1}\Psi) - (\phi_\sigma^4(H_\sigma - \bar{z})^{-1}\Theta, (H - z)^{-1}\Psi)| \\ &\leq |((\phi^2 - \phi_\sigma^2)(H_\sigma - \bar{z})^{-1}\Theta, \phi^2(H - z)^{-1}\Psi)| \\ &\quad + |(\phi_\sigma^2(H_\sigma - \bar{z})^{-1}\Theta, (\phi^2 - \phi_\sigma^2)(H - z)^{-1}\Psi)| \\ &\leq \|\phi(\rho - \rho_\sigma)\phi(\rho + \rho_\sigma)(H_\sigma - \bar{z})^{-1}\| \, \|\phi^2(H - z)^{-1}\| \, \|\Theta\|\|\Psi\| \\ &\quad + \|\phi_\sigma^2(H_\sigma - \bar{z})^{-1}\| \, \|\phi(\rho - \rho_\sigma)\phi(\rho + \rho_\sigma)(H - z)^{-1}\|\|\Theta\|\|\Psi\| \\ &\leq C'(\|\omega^{-1/2}(\rho - \rho_\sigma)\| + \|\omega(\rho - \rho_\sigma)\|)\|\Theta\|\|\Psi\|, \end{aligned} \tag{4.8}$$

with some constant C'. Similarly, we see that

$$\begin{aligned} &|((H_\sigma - \bar{z})^{-1}\Theta, P(\phi)(H - z)^{-1}\Psi) - (P(\phi_\sigma)(H_\sigma - \bar{z})^{-1}\Theta, (H - z)^{-1}\Psi)| \\ &\qquad \leq C''(\|\omega^{-1/2}(\rho - \rho_\sigma)\| + \|\omega(\rho - \rho_\sigma)\|)\|\Theta\|\|\Psi\| \end{aligned} \tag{4.9}$$

with some constant C''. Thus we obtain that

$$\|(H_\sigma - z)^{-1} - (H - z)^{-1}\| \leq C''(\|\omega^{-1/2}(\rho - \rho_\sigma)\| + \|\omega(\rho - \rho_\sigma)\|). \tag{4.10}$$

Since the right hand side of (4.10) converges to 0 as $\sigma \to 0$, the lemma follows.

We denote the ground state energies of H_σ and H by E_σ and E, respectively:

$$E = \inf_{\Psi \in D(H), \|\Psi\|=1} (\Psi, H\Psi), \quad E_\sigma = \inf_{\Psi \in D(H_\sigma), \|\Psi\|=1} (\Psi, H_\sigma \Psi). \tag{4.11}$$

Since $H, H_\sigma \geq C$ with some constant C independent of σ, by Lemma IV.1, we obtain the following corollary:

Corollary IV.2: Suppose Hypotheses II.1 and II.4. Then

$$\lim_{\sigma \to 0} E_\sigma = E. \tag{4.12}$$

Let us introduce a multiplication operator $\tilde{\omega}_\sigma$ below:

$$\tilde{\omega}_\sigma \in C(\mathbb{R}^d), \quad \nabla \tilde{\omega}_\sigma \in L^\infty(\mathbb{R}^d_k), \tag{4.13}$$

$$\tilde{\omega}_\sigma(k) \geq \frac{\sigma}{2} \quad \text{for} \quad k \in \mathbb{R}^d, \tag{4.14}$$

$$\tilde{\omega}_\sigma(k) = \omega(k) \quad \text{if} \quad |k| \geq \sigma. \tag{4.15}$$

Then we define the massive Hamiltonian $\tilde{H}_\sigma$ by

$$\tilde{H}_\sigma = K \otimes 1 + 1 \otimes d\Gamma(\tilde{\omega}_\sigma) + P(\phi_\sigma). \tag{4.16}$$

Similar to the case of H and H_σ, we can see that $\tilde{H}_\sigma$ is self-adjoint on $D(K) \cap D(d\Gamma(\tilde{\omega}_\sigma)) \cap D(\phi_\sigma^4)$.

B. Extended Hamiltonian and existence of a ground state

Throughout this subsection, we suppose Hypotheses II.1, II.2, and II.4.

Lemma IV.3:

(1) *Let $m \in \mathbb{Z}$. Then $(N+1)^{-m}(\tilde{H}_\sigma - z)^{-1}(N+1)^{m+1}$ is a bounded operator and*

$$\|(N+1)^{-m}(\tilde{H}_\sigma - z)^{-1}(N+1)^{m+1}\| \leq C\sigma^{-1}(1 + |\Im z|^{-1}) \tag{4.17}$$

with some constant C independent of z and σ;

(2) *Let $\chi \in C_c^\infty(\mathbb{R})$. Then for all $l, m \in \mathbb{Z}$, $(N+1)^l \chi(H)(N+1)^m$ is a bounded operator.*

Proof: Let us show (1). We denote $1_{\{n\}}(N)$ by P_n. Let $m \geq 0$ and $\Psi \in D(N^{m+1})$. Since

$$P_n(H - z)^{-1} = \sum_{l=-4}^{4} P_n(H - z)^{-1} P_{n+l},$$

it follows that

$$\begin{aligned}
&\|(N+1)^{-m}(\tilde{H}_\sigma - z)^{-1}(N+1)^{m+1}\Psi\|^2 \\
&= \sum_{n=0}^{\infty}(n+1)^{-2m}\|P_n(\tilde{H}_\sigma - z)^{-1}(N+1)^{m+1}\Psi\|^2 \\
&\leq C_1 \sum_{n=0}^{\infty}(n+1)^{-2m}(n+5)^{2m} \sum_{l=-4}^{4} \|P_n(\tilde{H}_\sigma - z)^{-1}(N+1)P_{n+l}\Psi\|^2
\end{aligned}$$

$$\leq C_2 \|(\tilde{H}_\sigma - z)^{-1}(N+1)\|^2 \sum_{n=0}^{\infty} \sum_{l=-4}^{4} \|P_{n+l}\Psi\|^2$$

$$\leq C_3 \sigma^{-2}(|\Im z|^{-1}+1)^2 \|\Psi\|^2, \tag{4.18}$$

where C_j, $j = 1, 2, 3$, are constants independent of σ and z. In the last inequality, we used $\|N\Psi\| \leq \frac{2}{\sigma}\|d\Gamma(\tilde{\omega}_\sigma)\Psi\| \leq \frac{C}{\sigma}\|(\tilde{H}_\sigma + 1)\Psi\|$, since $\tilde{\omega}_\sigma \geq \frac{\sigma}{2}$. Then (4.17) follows. When $m < 0$, the lemma can be also proven similarly to the case of $m \geq 0$. (2) can be proven similarly to Lemma 3.2 (ii) of Ref. 7.

Let us consider the extended Hilbert space defined by

$$\mathcal{H}^{\text{ext}} = \mathcal{K} \otimes \mathcal{F}_{\text{b}} \otimes \mathcal{F}_{\text{b}}. \tag{4.19}$$

The decoupled Hamiltonian $\tilde{H}_{0,\sigma}$ is extended as

$$\tilde{H}_{0,\sigma}^{\text{ext}} = \tilde{H}_{0,\sigma} \otimes 1_{\mathcal{F}_{\text{b}}} + 1_{\mathcal{H}} \otimes d\Gamma(\tilde{\omega}_\sigma), \tag{4.20}$$

and the total Hamiltonian $\tilde{H}_\sigma$ as

$$\tilde{H}_{\sigma}^{\text{ext}} = \tilde{H}_{\sigma} \otimes 1_{\mathcal{F}_{\text{b}}} + 1_{\mathcal{H}} \otimes d\Gamma(\tilde{\omega}_\sigma). \tag{4.21}$$

Let us introduce a partition of unity, such that $j = (j_0, j_\infty) \in C^\infty(\mathbb{R}^3;\mathbb{R}^2)$, $0 \leq j_0, j_\infty \leq 1$, $j_0^2 + j_\infty^2 = 1$, and

$$j_0(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases} \tag{4.22}$$

We set

$$j_R = (j_{0,R}, j_{\infty,R}) = (j_0(\cdot/R), j_\infty(\cdot/R))$$

and

$$\hat{j}_R\Psi = (\hat{j}_{0,R}\Psi, \hat{j}_{\infty,R}\Psi) = (j_{0,R}(-i\nabla_k)\Psi, j_{\infty,R}(-i\nabla_k)\Psi).$$

Let us recall that $\check{\Gamma}(\hat{j}_R) : \mathcal{F}_{\text{b}} \to \mathcal{F}_{\text{b}} \otimes \mathcal{F}_{\text{b}}$ is defined by $U_{L^2(\mathbb{R}^d),L^2(\mathbb{R}^d)}\Gamma(\hat{j}_R)$.

Lemma IV.4: Let $\chi_1, \chi_2 \in C_{\text{c}}^\infty(\mathbb{R})$. Then

$$\lim_{R\to 0} \left\| \left(\chi_1(\tilde{H}_\sigma^{\text{ext}})\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\chi_1(\tilde{H}_\sigma)\right) \chi_2(\tilde{H}_\sigma) \right\| = 0. \tag{4.23}$$

Proof: By Helffer–Sjöstrand's formula, it is seen that

$$\left(\chi_1(\tilde{H}_\sigma^{\text{ext}})\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\chi(\tilde{H}_\sigma)\right) \chi_2(\tilde{H}_\sigma)$$

$$= \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\chi}_1(z)(z-\tilde{H}_\sigma^{\text{ext}})^{-1}(\tilde{H}_\sigma^{\text{ext}}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_\sigma)(z-\tilde{H}_\sigma)^{-1}\chi_2(\tilde{H}_\sigma)dzd\bar{z}. \tag{4.24}$$

Here $dzd\bar{z} = -2idxdy$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and $\tilde{\chi}_1$ is an almost analytic extension of χ_1, which satisfies

$$\tilde{\chi}_1(x) = \chi_1(x), \quad x \in \mathbb{R}, \tag{4.25}$$

$$\tilde{\chi}_1 \in C_{\text{c}}^\infty(\mathbb{C}), \tag{4.26}$$

$$|\partial_{\bar{z}}\tilde{\chi}_1(z)| \leq C_n|\Im z|^n, \quad n \in \mathbb{N}. \tag{4.27}$$

Let us estimate the integrand in (4.24). $\tilde{H}_\sigma^{\text{ext}}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_\sigma$ is equal to

$$\left(\tilde{H}_{0,\sigma}^{\text{ext}}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_{0,\sigma}\right) + \left((P(\phi_\sigma)\otimes 1_{\mathcal{F}_{\text{b}}})\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)P(\phi_\sigma)\right). \tag{4.28}$$

The first term of (4.28) can be estimated as

$$\|(\tilde{H}_{0,\sigma}^{\rm ext}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_{0,\sigma})(N+1)^{-1}\| = \|d\Gamma(\hat{j}_R, ((\tilde{\omega}_\sigma \oplus \tilde{\omega}_\sigma)\hat{j}_R - \hat{j}_R\tilde{\omega}_\sigma)(N+1)^{-1}\|$$

$$\leq \sqrt{\|[\tilde{\omega}_\sigma, \hat{j}_{0,R}]\|^2 + \|[\tilde{\omega}_\sigma, \hat{j}_{\infty,R}]\|^2}, \quad (4.29)$$

where $d\Gamma(\hat{j}_R, (\tilde{\omega}_\sigma \oplus \tilde{\omega}_\sigma)\hat{j}_R - \hat{j}_R\tilde{\omega}_\sigma)$ is defined by

$$(d\Gamma(\hat{j}_R, (\tilde{\omega}_\sigma \oplus \tilde{\omega}_\sigma)\hat{j}_R - \hat{j}_R\tilde{\omega}_\sigma)\Psi)^{(n)} = \sum_{l=1}^{n} \hat{j}_R \otimes \cdots \otimes \overbrace{((\tilde{\omega}_\sigma \oplus \tilde{\omega}_\sigma)\hat{j}_R - \hat{j}_R\tilde{\omega}_\sigma)}^{l\text{th}} \otimes \cdots \otimes \hat{j}_R\Psi^{(n)}$$

for $n \geq 1$, and

$$(d\Gamma(\hat{j}_R, (\tilde{\omega}_\sigma \oplus \tilde{\omega}_\sigma)\hat{j}_R - \hat{j}_R\tilde{\omega}_\sigma)\Psi)^{(0)} = 0.$$

Let us estimate commutators $[\tilde{\omega}_\sigma, \hat{j}_{0,R}]$ and $[\tilde{\omega}_\sigma, \hat{j}_{\infty,R}]$. Note that

$$(f(-i\nabla)g)(k) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} (\mathcal{F}f)(s)g(k+s)ds, \quad (4.30)$$

for $f \in C_{\rm c}^\infty(\mathbb{R}^d)$ and $g \in C_{\rm c}^\infty(\mathbb{R}^d)$. Here $\mathcal{F}f$ denotes the Fourier transformation of f. Then

$$\begin{aligned}
&\|[\hat{j}_{0,R}, \tilde{\omega}_\sigma]f\|_{L^2}^2 \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(\mathcal{F}j_0)(\xi)(\tilde{\omega}_\sigma(k+\xi/R) - \tilde{\omega}_\sigma(k))f(k+\xi/R)d\xi\right|^2 dk \\
&\leq (2\pi)^{-d}\|(\mathcal{F}j_0)\langle\cdot\rangle^{d+1}\|_{L^2}^2(\|\nabla\tilde{\omega}_\sigma\|_{L^\infty}/R)^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle\xi\rangle^{-2d}|f(k+\xi/R)|^2 dk d\xi \\
&\leq \frac{(2\pi)^{-d}}{R^2}\|(\mathcal{F}j_0)\langle\cdot\rangle^{d+1}\|_{L^2}^2\|\nabla\tilde{\omega}_\sigma\|_{L^\infty}^2\|\langle\cdot\rangle^{-d}\|_{L^2}^2\,\|f\|_{L^2}^2,
\end{aligned} \quad (4.31)$$

where $\langle\xi\rangle = \sqrt{1+\xi^2}$. Thus

$$\|[\hat{j}_{0,R}, \tilde{\omega}_\sigma]\| = \frac{\text{const.}}{R}. \quad (4.32)$$

Similarly,

$$\|\,[\hat{j}_{\infty,R}, \tilde{\omega}_\sigma]\,\| = \|[\hat{j}_{\infty,R} - 1, \tilde{\omega}_\sigma]\| = \frac{\text{const.}}{R}, \quad (4.33)$$

since $j_{\infty,R} - 1 \in C_{\rm c}^\infty(\mathbb{R}^d)$. Thus it is seen that

$$\lim_{R\to\infty}\|(\tilde{H}_{0,\sigma}^{\rm ext}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_{0,\sigma})(N+1)^{-1}\| = 0. \quad (4.34)$$

Let us consider the second term of (4.28). It can be computed as

$$\begin{aligned}
&\phi_{0,\sigma}^4\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\phi_\sigma^4 \\
&= \sum_{k=0}^{3}\phi_{0,\sigma}^{3-k}[(\phi_{0,\sigma}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\phi_\sigma]\phi_\sigma^k \\
&= \sum_{k=0}^{3}\phi_{0,\sigma}^{3-k}\left[\left(\phi_0((1-\hat{j}_{0,R})\rho_\sigma) - \phi_\infty(\hat{j}_{\infty,R}\rho_\sigma)\right)\check{\Gamma}(\hat{j}_R)\right]\phi_\sigma^k
\end{aligned} \quad (4.35)$$

on $\mathcal{H}_{\rm fin}$. Here we write $\phi_0(f)$ and $\phi_\infty(f)$ for $\phi(f)\otimes 1_{\mathcal{F}_{\rm b}}$ and $1_{\mathcal{H}}\otimes\phi(f)$, respectively. Note that

$$\lim_{R\to\infty}\left\|(N_0+N_\infty+1)^{-3/2}\left[\left(\phi_0((1-\hat{j}_{0,R})\rho_\sigma) - \phi_\infty(\hat{j}_{\infty,R}\rho_\sigma)\right)\check{\Gamma}(\hat{j}_R)\right](N+1)^{-2}\right\| = 0, \quad (4.36)$$

where $N_0 = N\otimes 1_{\mathcal{F}_{\rm b}}$ and $N_\infty = 1_{\mathcal{H}}\otimes N$. Then by (4.34) and (4.35),

$$\lim_{R\to\infty}\|(\tilde{H}_\sigma^{\rm ext}\check{\Gamma}(\hat{j}_R) - \check{\Gamma}(\hat{j}_R)\tilde{H}_\sigma)(N+1)^{-5/2}\| = 0. \quad (4.37)$$

By Lemma IV.3, the integrand of (4.24) can be estimated as

$$
\begin{aligned}
&|\partial_{\bar z}\tilde\chi_1(z)|\|(z-\tilde H_\sigma^{\rm ext})^{-1}(\tilde H_\sigma^{\rm ext}\check\Gamma(\hat j_R)-\check\Gamma(\hat j_R)\tilde H_\sigma)(z-\tilde H_\sigma)^{-1}\chi_2(\tilde H_\sigma)\| \\
&\le |\partial_{\bar z}\tilde\chi_1(z)|\|(z-\tilde H_\sigma^{\rm ext})^{-1}\|\|(\tilde H_\sigma^{\rm ext}\check\Gamma(\hat j_R)-\check\Gamma(\hat j_R)\tilde H_\sigma)(N+1)^{-5/2}\|\times \\
&\qquad \|(N+1)^{5/2}(z-\tilde H_\sigma)^{-1}(N+1)^{-3/2}\|\,\|(N+1)^{3/2}\chi_2(\tilde H_\sigma)\| \\
&\le C\|(\tilde H_\sigma^{\rm ext}\check\Gamma(\hat j_R)-\check\Gamma(\hat j_R)\tilde H_\sigma)(N+1)^{-5/2}\|\left(1+\frac{|\partial_{\bar z}\tilde\chi_1(z)|}{|\Im z|^2}\right),
\end{aligned}
\tag{4.38}
$$

where C is a constant independent of z and R. From (4.37) and (4.38), the lemma follows.

Lemma IV.5: Let $\tilde E_\sigma$ denote the ground state energy of $\tilde H_\sigma$. Let $\chi\in C_{\rm c}^\infty(\mathbb{R})$. Suppose that $\operatorname{supp}\chi\subset(-\infty,\tilde E_\sigma+\sigma/2)$. *Then $\chi(\tilde H_\sigma)$ is a compact operator. In particular, $\tilde H_\sigma$ has a ground state.*

Proof: First, let us show that $\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)$ is a compact operator. Since for each $n\in\mathbb{N}$,

$$
\left\|\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)-\sum_{k=0}^n 1_{\{k\}}(N)\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)\right\|\le\frac{1}{n+1}\|\Gamma(\hat j_{0,R}^2)N\chi(\tilde H_\sigma)\|,
$$

$\sum_{k=0}^n 1_{\{k\}}(N)\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)$ uniformly converges to $\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)$ as n goes to infinity. Then it suffices to show that $1_{\{k\}}(N)\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)$ is compact. Note that

$$
T_1=(K+1)^{-1/2}\otimes\Gamma(\hat j_{0,R}^2)(d\Gamma(\tilde\omega_\sigma)+1)^{-1/2}1_{\{k\}}(N)
$$

is compact and

$$
T_2=\left((K+1)^{1/2}\otimes(d\Gamma(\tilde\omega_\sigma)+1)^{1/2}\right)\chi(\tilde H_\sigma)
$$

is bounded. Thus the claim is obtained, since $1_{\{k\}}(N)\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)=T_1T_2$. Since $\operatorname{supp}\chi\subset(-\infty,\tilde E_\sigma+\sigma/2)$, we see that

$$
\chi(\tilde H_\sigma^{\rm ext})=(1_{\mathcal H}\otimes P_0)\chi(\tilde H_\sigma^{\rm ext}),\tag{4.39}
$$

where P_0 is the projection from $\mathcal{F}_{\rm b}$ to the subspace spanned by the Fock vacuum. We also see that

$$
\check\Gamma(\hat j_R)^*(1_{\mathcal H}\otimes P_0)\check\Gamma(\hat j_R)=\Gamma(\hat j_{0,R}^2).\tag{4.40}
$$

We can suppose $\chi\ge 0$. Then by Lemma IV.4 and (4.39),

$$
\begin{aligned}
\chi(\tilde H_\sigma)&=\check\Gamma(\hat j_R)^*\check\Gamma(\hat j_R)\chi^{1/2}(\tilde H_\sigma)\chi^{1/2}(\tilde H_\sigma)\\
&=\check\Gamma(\hat j_R)^*(1_{\mathcal H}\otimes P_0)\chi^{1/2}(\tilde H_\sigma^{\rm ext})\check\Gamma(\hat j_R)\chi^{1/2}(\tilde H_\sigma)+o(R^0),
\end{aligned}\tag{4.41}
$$

where $o(R^0)$ is a bounded operator converging to 0 as $R\to\infty$ in the uniform norm. By Lemma IV.4 again and (4.40),

$$
\begin{aligned}
\chi(\tilde H_\sigma)&=\check\Gamma(\hat j_R)^*(1_{\mathcal H}\otimes P_0)\check\Gamma(\hat j_R)\chi(\tilde H_\sigma)+o(R^0)\\
&=\check\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)+o(R^0).
\end{aligned}\tag{4.42}
$$

Since $\Gamma(\hat j_{0,R}^2)\chi(\tilde H_\sigma)$ is a compact operator, $\chi(\tilde H_\sigma)$ is also compact.

Lemma IV.6: H_σ has a ground state.

Proof: Let us consider the unitary operator (2.17) with $\mathcal{X}=L^2(\{k|\omega(k)\ge\sigma\})$ and $\mathcal{Y}=L^2(\{k|\omega(k)<\sigma\})$. We denote $U_{\mathcal{X},\mathcal{Y}}$ by U_σ. We see that

$$
U_\sigma\tilde H_\sigma U_\sigma^*=1\otimes d\Gamma(\tilde\omega_\sigma)+H_\sigma'\otimes 1,\tag{4.43}
$$

where

$$
H_\sigma'=K+d\Gamma(\omega 1_{\{\omega(k)\ge\sigma\}})+P(\phi_\sigma)\tag{4.44}
$$

with the domain $D(K) \cap D(d\Gamma(\omega 1_{\{\omega(k)\geq\sigma\}})) \cap D(\phi_\sigma^4)$. H'_σ is self-adjoint. Since $\tilde{H}_\sigma$ has a ground state by Lemma IV.5, H'_σ also has a ground state by Proposition VI.4 in Appendix. Since

$$U_\sigma H_\sigma U_\sigma^* = 1 \otimes d\Gamma(\omega_\sigma) + H'_\sigma \otimes 1, \tag{4.45}$$

and H'_σ has a ground state, H_σ also has a ground state.

V. PROOF OF THE EXISTENCE OF A GROUND STATE

Let Φ_σ be a normalized ground state of H_σ.

Lemma V.1: (Pull-through formula) Suppose Hypotheses II.1, II.2, and II.4. For almost every $k \in \mathbb{R}^d$, we have

$$a(k)\Phi_\sigma = \frac{1}{\sqrt{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma. \tag{5.1}$$

Here

$$P'(x) = \frac{dP}{dx}(x). \tag{5.2}$$

Proof: Since Φ_σ is a ground state of H_σ, for all $f \in C_c^\infty(\mathbb{R}_k^d)$ and $\Theta \in \mathcal{H}_{\text{fin}}$,

$$\begin{aligned}((H_\sigma - E_\sigma)\Theta, a(f)\Phi_\sigma) &= ([a^\dagger(f), H_\sigma - E_\sigma]\Theta, \Phi_\sigma)\\ &= \left(\left(-a^\dagger(\omega f) + \frac{1}{\sqrt{2}}(\rho_\sigma, f)P'(\phi_\sigma)\right)\Theta, \Phi_\sigma\right)\\ &= \left(\Theta, \left(-a(\omega f) + \frac{1}{\sqrt{2}}(f, \rho_\sigma)P'(\phi_\sigma)\right)\Phi_\sigma\right)\end{aligned} \tag{5.3}$$

follows. Since (5.3) is equal to

$$\int_{\mathbb{R}_k^d} \overline{f(k)}((E_\sigma - H_\sigma - \omega(k))\Theta, a(k)\Phi_\sigma)dk = \frac{1}{\sqrt{2}}\int_{\mathbb{R}_k^d} \overline{f(k)}(\Theta, \rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma)dk, \tag{5.4}$$

it holds that for almost every $k \in \mathbb{R}^d$,

$$((E_\sigma - H_\sigma - \omega(k))\Theta, a(k)\Phi_\sigma) = \frac{1}{\sqrt{2}}(\Theta, \rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma). \tag{5.5}$$

Thus, $a(k)\Phi_\sigma \in D(H_\sigma)$ for almost every k and

$$(E_\sigma - H_\sigma - \omega(k))a(k)\Phi_\sigma = \frac{1}{\sqrt{2}}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma. \tag{5.6}$$

$E_\sigma - H_\sigma - \omega(k) < 0$ for $k \neq 0$. Then $(E_\sigma - H_\sigma - \omega(k))^{-1}$ exists for $k \neq 0$. Thus the lemma follows. ■

Lemma V.2: Suppose Hypotheses II.1, II.2, II.4, and $\omega^{-1}\|\rho(\cdot)\| \in L^2(\mathbb{R}_k^d)$. Then, $\Phi_\sigma \in D(N^{1/2})$ and

$$\sup_{0<\sigma\leq 1} \|N^{1/2}\Phi_\sigma\| < \infty. \tag{5.7}$$

Proof: By Lemma V.1, it follows that

$$\begin{aligned}\|N^{1/2}\Phi_\sigma\|^2 &= \int_{\mathbb{R}_k^d} \|a(k)\Phi_\sigma\|^2 dk\\ &= \frac{1}{2}\int_{\mathbb{R}_k^d} \|(E_\sigma - H_\sigma - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\|^2 dk\end{aligned}$$

$$\leq \frac{1}{2}\|P'(\phi_\sigma)\Phi_\sigma\|^2 \int_{\mathbb{R}^d_k} \|(E_\sigma - H_\sigma - \omega(k))^{-1}\rho_\sigma(k)\|^2 dk$$

$$\leq C\left(\sup_{0<\sigma\leq 1} E_\sigma^2 + 1\right)\left\|(1+\omega^{-1})\rho\right\|^2 < \infty \tag{5.8}$$

with a constant C. Thus the lemma follows.

Lemma V.3: Suppose Hypotheses II.1, II.2, II.4, and II.5. Then it holds that

$$|E_\sigma - E| \in o(\sigma^{3/4}). \tag{5.9}$$

Here $\sigma^{-3/4}o(\sigma^{3/4})$ *converges to* 0 *as* $\sigma \to +0$.

Proof: Let $0 < \sigma < \sigma' < 1$. Take a sequence $\{\Phi_\sigma^j\}_{j=1}^\infty \subset \mathcal{H}_{\text{fin}}$, such that

$$\lim_{j\to\infty}(\|\Phi_{\sigma'}^j - \Phi_{\sigma'}\| + \|H_{\sigma'}(\Phi_{\sigma'}^j - \Phi_{\sigma'})\|) = 0.$$

Since there exist constants C and $C' > 0$, so that for all j and $k \in \mathbb{N}$,

$$\begin{aligned}\|\phi_\sigma^2(\Phi_{\sigma'}^j - \Phi_{\sigma'}^k)\| &\leq C\|(d\Gamma(\omega)+1)(\Phi_{\sigma'}^j - \Phi_{\sigma'}^k)\|\\ &\leq C(\|(H_{\sigma'}+1)(\Phi_{\sigma'}^j - \Phi_{\sigma'}^k)\| + \|P(\phi_{\sigma'})(\Phi_{\sigma'}^j - \Phi_{\sigma'}^k)\|)\\ &\leq C'\|(H_{\sigma'}+1)(\Phi_{\sigma'}^j - \Phi_{\sigma'}^k)\|,\end{aligned} \tag{5.10}$$

it is seen that

$$\lim_{j\to\infty}\|\phi_\sigma^2(\Phi_{\sigma'}^j - \Phi_{\sigma'})\| = 0 \tag{5.11}$$

by the closedness of ϕ_σ^2. Note that $\sup_{0<\sigma<1}|E_\sigma| < \infty$ by Corollary IV.2. Then by (5.11), it holds that

$$\begin{aligned}E_\sigma - E_{\sigma'} &\leq \liminf_{j\to\infty}\frac{(\Phi_{\sigma'}^j, H_\sigma\Phi_{\sigma'}^j) - (\Phi_{\sigma'}^j, H_{\sigma'}\Phi_{\sigma'}^j)}{\|\Phi_{\sigma'}^j\|^2}\\ &= (\phi_\sigma^2\Phi_{\sigma'}, (\phi_\sigma^2 - \phi_{\sigma'}^2)\Phi_{\sigma'}) + (\phi_{\sigma'}^2\Phi_{\sigma'}, (\phi_\sigma^2 - \phi_{\sigma'}^2)\Phi_{\sigma'})\\ &\quad + c_3\{(\phi_\sigma\Phi_{\sigma'}, (\phi_\sigma^2 - \phi_{\sigma'}^2)\Phi_{\sigma'}) + (\phi_{\sigma'}^2\Phi_{\sigma'}, (\phi_\sigma - \phi_{\sigma'})\Phi_{\sigma'})\}\\ &\quad + c_2\{(\phi_\sigma\Phi_{\sigma'}, (\phi_\sigma - \phi_{\sigma'})\Phi_{\sigma'}) + (\phi_{\sigma'}\Phi_{\sigma'}, (\phi_\sigma - \phi_{\sigma'})\Phi_{\sigma'})\}\\ &\quad + c_1(\Phi_{\sigma'}, (\phi_\sigma - \phi_{\sigma'})\Phi_{\sigma'})\end{aligned}$$

$$\begin{aligned}&\leq C_1\left(\left\|\frac{\rho_{\sigma'} - \rho_\sigma}{\sqrt{\omega}}\right\| + \|\omega(\rho_{\sigma'} - \rho_\sigma)\|\right)\left(\left\|\frac{\rho_\sigma}{\sqrt{\omega}}\right\| + \|\omega\rho_\sigma\|\right)\left(\left\|\frac{\rho_{\sigma'}}{\sqrt{\omega}}\right\| + \|\omega\rho_{\sigma'}\|\right)\\ &\quad\times\|(d\Gamma(\omega)+1)\Psi_{\sigma'}\|\\ &\leq C_2\left(\left\|\frac{\rho_{\sigma'} - \rho_\sigma}{\sqrt{\omega}}\right\| + \|\omega(\rho_{\sigma'} - \rho_\sigma)\|\right),\end{aligned} \tag{5.12}$$

with constants C_1 and C_2. Note that

$$\left|\frac{\rho_{\sigma'}(x,k) - \rho_\sigma(x,k)}{\sqrt{\omega(k)}}\right| \leq \sigma'^{3/4}\left|\frac{\rho_{\sigma'}(x,k) - \rho_\sigma(x,k)}{\omega^{5/4}(k)}\right| \tag{5.13}$$

and

$$|\omega(k)(\rho_{\sigma'}(x,k) - \rho_\sigma(x,k))| \leq \sigma'^{3/4}|\omega^{1/4}(k)(\rho_{\sigma'}(x,k) - \rho_\sigma(x,k))|. \tag{5.14}$$

Then by (5.12), it is obtained that

$$E_\sigma - E_{\sigma'} \leq C\sigma'^{3/4}\left(\left\|\frac{\rho_{\sigma'} - \rho_\sigma}{\omega^{5/4}}\right\| + \|\omega^{1/4}(\rho_{\sigma'} - \rho_\sigma)\|\right). \tag{5.15}$$

Replacing σ and σ', we have

$$|E_{\sigma'} - E_\sigma| \le C\sigma'^{3/4}\left(\left\|\frac{\rho_{\sigma'} - \rho_\sigma}{\omega^{5/4}}\right\| + \|\omega^{1/4}(\rho_{\sigma'} - \rho_\sigma)\|\right). \tag{5.16}$$

Taking $\sigma' \to 0$ on both sides of (5.16), we obtain (5.9), since $\omega^{-5/4}\|\rho(\cdot)\|, \omega^{1/4}\|\rho(\cdot)\| \in L^2(\mathbb{R}^d_k)$.

Lemma V.4: Suppose Hypotheses II.1, II.2, II.4, and II.5. Then

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^d_k}\left\|a(k)\Phi_\sigma - \frac{1}{\sqrt{2}}(E - H - \omega(k))^{-1}\rho(k)P'(\phi_\sigma)\Phi_\sigma\right\|^2 dk = 0. \tag{5.17}$$

Proof: Applying the pull-through formula, Lemma V.1, we have

$$\begin{aligned} &a(k)\Phi_\sigma - \frac{1}{\sqrt{2}}(E - H - \omega(k))^{-1}P'(\phi_\sigma)\rho_\sigma(k)\Phi_\sigma \\ &= -\frac{1}{\sqrt{2}}(E - H - \omega(k))^{-1}(\rho(k) - \rho_\sigma(k))P'(\phi_\sigma)\Phi_\sigma \\ &\quad + \frac{1}{\sqrt{2}}\left((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1}\right)\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma. \end{aligned} \tag{5.18}$$

First let us consider the first term of the right hand side of (5.18). By the bound $\|\phi^n\Psi\| \le C\|(H + 1)\Psi\|$, $n = 1, 2, 3, 4$, we see that

$$\begin{aligned} &\int_{\mathbb{R}^d_k}\|(E - H - \omega(k))^{-1}(\rho(k) - \rho_\sigma(k))P'(\phi_\sigma)\Phi_\sigma\|^2 dk \\ &\le \left\|\omega^{-1}(\rho - \rho_\sigma)\right\|^2\|P'(\phi_\sigma)\Phi_\sigma\|^2 \le C\left(\sup_{0<\sigma\le 1}E_\sigma^2 + 1\right)\left\|\omega^{-1}(\rho - \rho_\sigma)\right\|^2 \end{aligned} \tag{5.19}$$

with some constant C. Since $\omega^{-1}\|\rho(\cdot)\| \in L^2(\mathbb{R}^d_k)$, we obtain that

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^d}\|(E - H - \omega(k))^{-1}(\rho(k) - \rho_\sigma(k))P'(\phi_\sigma)\Phi_\sigma\|^2 dk = 0. \tag{5.20}$$

Next, let us consider the second term of the right hand side of (5.18). We see that

$$\begin{aligned} &(\Theta, \left((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1}\right)\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma) \\ &= (E - E_\sigma)\left((E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, (E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\right) \\ &\quad + \left(P(\phi_\sigma)(E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, (E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\right) \\ &\quad - \left((E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, P(\phi)(E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\right) \end{aligned} \tag{5.21}$$

for all $\Theta \in \mathcal{H}$. It holds that

$$\begin{aligned} &|\left(\phi_\sigma^4(E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, (E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\right) \\ &\quad - \left((E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, \phi^4(E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\right)| \\ &= |(\phi_\sigma^2(E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, (\phi_\sigma^2 - \phi^2)(E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma) \\ &\quad + ((\phi_\sigma^2 - \phi^2)(E_\sigma - H_\sigma - \omega(k))^{-1}\Theta, \phi^2(E - H - \omega(k))^{-1}\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma)| \\ &\le C(\|\omega^{-1/2}(\rho - \rho_\sigma)\| + \|\omega(\rho - \rho_\sigma)\|)\left(1 + \frac{1}{\omega(k)^2}\right)\|\rho_\sigma(k)\|\,\|\Theta\|, \end{aligned} \tag{5.22}$$

since

$$\begin{aligned} &\|(d\Gamma(\omega) + 1)(E_\sigma - H_\sigma - \omega(k))^{-1}\|, \|(d\Gamma(\omega) + 1)(E - H - \omega(k))^{-1}\| \\ &\qquad \le C'\left(1 + \frac{1}{\omega(k)^2}\right), \end{aligned} \tag{5.23}$$

where C and C' are constants. Since $\Theta \in \mathcal{H}$ is arbitrary, in a similar way to (5.22), we can see that

$$\begin{aligned}&\|((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1})\rho_\sigma(k) P'(\phi_\sigma)\Phi_\sigma\| \\ &\leq C(\|\omega^{-1/2}(\rho - \rho_\sigma)\| + \|\omega(\rho - \rho_\sigma)\| + |E - E_\sigma|)\left(1 + \frac{1}{\omega(k)^2}\right)\|\rho_\sigma(k)\| \\ &\leq C_\sigma \sigma^{3/4}\left(1 + \frac{1}{\omega(k)^2}\right)\|\rho_\sigma(k)\| \end{aligned} \tag{5.24}$$

for almost every $k \in \mathbb{R}^d$. Here we used (5.13) and (5.14), and C_σ is given by

$$C_\sigma = \|\omega^{-5/4}(\rho - \rho_\sigma)\| + \|\omega^{1/4}(\rho - \rho_\sigma)\| + \sigma^{-3/4}|E - E_\sigma|. \tag{5.25}$$

Since $\sigma^{3/4}\|\rho_\sigma(k)\| \leq \omega(k)^{3/4}\|\rho_\sigma(k)\|$, by (5.24), we see that

$$\begin{aligned}&\|((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1})\,\rho_\sigma(k) P'(\phi_\sigma)\Phi_\sigma\| \\ &\qquad\leq C_\sigma\left(\omega(k)^{3/4} + \frac{1}{\omega(k)^{5/4}}\right)\|\rho_\sigma(k)\|.\end{aligned} \tag{5.26}$$

Since $|E - E_\sigma| = o(\sigma^{3/4})$, we have

$$\lim_{\sigma\to 0} C_\sigma = 0. \tag{5.27}$$

Thus by (5.26) and (5.27), we obtain that

$$\lim_{\sigma\to 0}\int_{\mathbb{R}^d_k}\|((E_\sigma - H_\sigma - \omega(k))^{-1} - (E - H - \omega(k))^{-1})\rho_\sigma(k)P'(\phi_\sigma)\Phi_\sigma\|^2 dk = 0. \tag{5.28}$$

Then we complete the lemma.

Lemma V.5: Suppose Hypotheses II.1, II.2, II.4, and II.5. Let $F \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leq F(k) \leq 1$ for all $k \in \mathbb{R}^d$ and $F(0) = 1$. We set $F_R = F(\cdot/R)$ and $\hat{F}_R = F_R(D_k)$, where $D_k = -i\nabla_k$. Then

$$\|d\Gamma(1 - \hat{F}_R)^{1/2}\Phi_\sigma\| = o(R^0) + o(\sigma^0). \tag{5.29}$$

Here $o(R^0)$ is a real number converging to 0 as $R \to \infty$, and $o(\sigma^0)$ is a real number converging to 0 as $\sigma \to +0$.

Proof: Since $\Phi_\sigma \in D(N^{1/2})$, $\Phi_\sigma \in D(d\Gamma(1 - \hat{F}_R)^{1/2})$. $F(D_k)$ is defined as the bounded operator on $L^2(\mathbb{R}^d; \mathcal{H})$ by

$$(F(D_k)\Psi(k))^{(n)}(x, k_1, \cdots, k_n) = F(D_k)\Psi^{(n)}(k)(x, k_1, \cdots, k_n), \quad 1 \leq n, \tag{5.30}$$

and $(F(D_k)\Psi(k))^{(0)}(x, k_1, \cdots, k_n) = 0$. Then

$$\|d\Gamma(1 - \hat{F}_R)^{1/2}\Phi_\sigma\|^2 = \int_{\mathbb{R}^d_k}(a(k)\Phi_\sigma, (1 - F(D_k/R))a(k)\Phi_\sigma)dk. \tag{5.31}$$

By the Schwarz inequality and Lemma V.4, it is seen that

$$\begin{aligned}&\leq \|N^{1/2}\Phi_\sigma\|\left(\int_{\mathbb{R}^d_k}\|(1 - F(D_k/R))a(k)\Phi_\sigma\|^2 dk\right)^{1/2} \\ &= \|N^{1/2}\Phi_\sigma\|\left(\int_{\mathbb{R}^d_k}\|(1 - F(D_k/R))(E - H - \omega(k))^{-1}\rho(k)P'(\phi_\sigma)\Phi_\sigma\|^2 dk\right)^{1/2} \\ &\qquad + o(\sigma^0).\end{aligned} \tag{5.32}$$

Let $\Theta \in L^2(\mathbb{R}^d_k; \mathcal{H}_{\text{fin}})$ with compact support and $T \in L^2(\mathbb{R}^d_k; \mathcal{B}(\mathcal{H}))$. Then it holds that

$$\begin{aligned}&\int (\Theta(k)^{(n)}, F(D_k)(T(k)\Psi)^{(n)})dk\\&= (2\pi)^{-d}\int\int\int \left(F(\xi)\Theta(s)^{(n)}e^{i\xi(k-s)}, (T(k)\Psi)^{(n)}\right) ds d\xi dk\\&= (2\pi)^{-d/2}\int\int \left((\mathcal{F}F)(s-k)\Theta(s)^{(n)}, (T(k)\Psi)^{(n)}\right) ds dk\\&= (2\pi)^{-d/2}\int \left(\Theta(s)^{(n)}, (\mathbf{F}(D_k)T)(s)\Psi^{(n)}\right) ds.\end{aligned} \tag{5.33}$$

Here $\mathcal{F}$ denotes the Fourier transformation and $\mathbf{F}(D_k)$ is the bounded operator on $L^2(\mathbb{R}^d_k;\mathcal{B}(\mathcal{H}))$ defined by

$$(\mathbf{F}(D_k)T)(k) = (2\pi)^{-d/2}\int_{\mathbb{R}^d}(\mathcal{F}F)(s)T(k+s)ds. \tag{5.34}$$

By (5.33), we have

$$F(D_k)(T(k)\Psi) = (\mathbf{F}(D_k)T)(k)\Psi. \tag{5.35}$$

Then by (5.32) and (5.35), we see that

$$\begin{aligned}&\|d\Gamma(1-\hat{F}_R)^{1/2}\Phi_\sigma\|\\&\leq C\left(\int_{\mathbb{R}^d_k}\|(1-\mathbf{F}(D_k/R))(E-H-\omega(k))^{-1}\rho(k)\|^2 dk\right)^{1/2} + o(\sigma^0).\end{aligned} \tag{5.36}$$

Here C is a constant independent of σ and R. By Lemma V.6 below, the proof is complete.

Lemma V.6: Lemma 3.1 of Ref. 9 Suppose Hypotheses II.1, II.2, and II.4 and suppose also that $\omega^{-1}\|\rho(\cdot)\| \in L^2(\mathbb{R}^d_k)$. *Then it follows that*

$$\int_{\mathbb{R}^d}\|(1-\mathbf{F}(D_k/R))(E-H-\omega(k))^{-1}\rho(k)\|^2 dk = o(R^0). \tag{5.37}$$

Proof of Theorem II.7: Since $\|\Phi_{\sigma_n}\| = 1$, we can take a sequence $\{\Phi_{\sigma_n}\}_{n=0}^\infty$ weakly converging to some vector Φ in $\mathcal{H}$:

$$\text{w-}\lim_{n\to\infty}\Phi_{\sigma_n} = \Phi. \tag{5.38}$$

For all $\Theta \in \mathcal{H}$ and $z \in \mathbb{C}\setminus\mathbb{R}$, it holds that

$$(\Theta, (H_{\sigma_n}-z)^{-1}\Phi_{\sigma_n}) = (\Theta, (E_{\sigma_n}-z)^{-1}\Phi_{\sigma_n}). \tag{5.39}$$

Since H_{σ_n} converges to H in the norm resolvent sense, we see that by (5.39) and Corollary IV.2

$$(\Theta, (H-z)^{-1}\Phi) = (\Theta, (E-z)^{-1}\Phi). \tag{5.40}$$

Since Θ is an arbitrary vector in $\mathcal{H}$, we have

$$H\Phi = E\Phi. \tag{5.41}$$

Thus Φ is a ground state of H, if and only if $\Phi \neq 0$. We suppose $\Phi = 0$. Take $F \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leq F \leq 1$ and $F(0) = 1$. Since $\Gamma(\hat{F}_R)1_{[0,\lambda]}(N)1_{[0,\lambda]}(H_0)$ is a compact operator, we see that

$$\lim_{n\to\infty}\|\Gamma(\hat{F}_R)1_{[0,\lambda]}(N)1_{[0,\lambda]}(H_0)\Phi_{\sigma_n}\| = 0. \tag{5.42}$$

Note that

$$\|(1-\Gamma(\hat{F}_R))\Psi\| \leq \|d\Gamma(1-\hat{F}_R)^{1/2}\Psi\| \tag{5.43}$$

for all $\Psi \in D(d\Gamma(1-\hat{F}_R)^{1/2})$. Then we see that by (5.43),

$$\begin{aligned}\|\Phi_{\sigma_n}\| &\le \|\Gamma(\hat{F}_R)\Phi_{\sigma_n}\| + \|(1-\Gamma(\hat{F}_R))\Phi_{\sigma_n}\| \\ &\le \|\Gamma(\hat{F}_R)1_{[0,\lambda]}(N)1_{[0,\lambda]}(H_0)\Phi_{\sigma_n}\| + \|(1-1_{[0,\lambda]}(H_0))\Phi_n\| \\ &\quad + \|(1-1_{[0,\lambda]}(N))\Phi_n\| + \|d\Gamma(1-\hat{F}_R)^{1/2}\Phi_{\sigma_n}\|. \end{aligned} \tag{5.44}$$

Since $\sup_n(\Phi_{\sigma_n}, N\Psi_{\sigma_n}) < \infty$ and $\sup_n(\Psi_{\sigma_n}, H_0\Phi_{\sigma_n}) < \infty$, we have

$$\|E_N((\lambda,\infty))\Phi_n\|,\ \|E_{H_0}((\lambda,\infty))\Phi_n\| \in O(\lambda^{-1}). \tag{5.45}$$

Here $E_N(\cdot)$ and $E_{H_0}(\cdot)$ are the spectral measures of N and H_0, respectively. Thus we see that

$$\lim_{\lambda\to\infty}\sup_n \|(1-1_{[0,\lambda]}(N))\Phi_n)\| = \lim_{\lambda\to\infty}\sup_n \|(1-1_{[0,\lambda]}(H_0))\Phi_n)\| = 0. \tag{5.46}$$

By (5.42) and (5.46), for an arbitrary $0 < \epsilon < 1$ we can take sufficiently large $0 < \lambda$, so that

$$\|\Phi_{\sigma_n}\| < \|1_{[0,\lambda]}(N)1_{[0,\lambda]}(H_0)\Gamma(\hat{F}_R)\Phi_{\sigma_n}\| + \|d\Gamma(1-\hat{F}_R)^{1/2}\Phi_{\sigma_n}\| + \epsilon. \tag{5.47}$$

Thus by (5.44) and (5.47),

$$\limsup_{n\to\infty}\|\Phi_{\sigma_n}\| \le \epsilon < 1. \tag{5.48}$$

Since Φ_{σ_n} is a normalized vector in $\mathcal{H}$, this is a contradiction. Therefore, $\Phi \neq 0$ and then Φ is a ground state of $\mathcal{H}$.

APPENDIX

Propositions VI.1–VI.5 below are often used in this paper and are well known. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces.

Proposition VI.1: Lemmas 2.7 and 2.8 of Ref. 7 Let $T : \mathcal{X} \to \mathcal{Y}$ be a densely defined closable operator and $f \in D(T)$. Then

(1)

$$\Gamma(T)a^\dagger(f) = a^\dagger(Tf)\Gamma(T) \tag{A1}$$

on $\mathcal{F}_{\rm fin}(D(T))$;

(2) *If T is isometry, then*

$$\Gamma(T)a(f) = a(Tf)\Gamma(T) \tag{A2}$$

on $\mathcal{F}_{\rm fin}(D(T))$;

(3) *If $\mathcal{X} = \mathcal{Y}$ and $f \in D(T)\cap D(T^*)$, then*

$$[d\Gamma(T), a(f)] = -a(T^*f) \quad \text{and} \quad [d\Gamma(T), a^\dagger(f)] = a^\dagger(Tf) \tag{A3}$$

on $\mathcal{F}_{\rm b,fin}(D(T))$.

Proposition VI.2: [2, Proposition 8-6 of Ref. 2] Let $\mathcal{X} = L^2(\mathbb{R}^d)$.

(1) *Let f be a function, such that $0 \le f(k) < \infty$ for almost every k. Then, $\Psi \in D(d\Gamma(f)^{1/2})$, if and only if*

$$\int_{\mathbb{R}^d} f(k)\|a(k)\Psi\|^2 dk < \infty$$

and in this case,

$$\|d\Gamma(f)^{1/2}\Psi\|^2 = \int_{\mathbb{R}^d} f(k)\|a(k)\Psi\|^2 dk \tag{A4}$$

holds. Moreover, if $f \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, it holds that

$$\|d\Gamma(f(D))^{1/2}\Psi\|^2 = \int_{\mathbb{R}^d}\sum_{n=1}^\infty \left((a(k)\Psi)^{(n)},\, f(D_k)(a(k)\Psi)^{(n)}\right) dk \tag{A5}$$

for all $\Psi \in D(d\Gamma(f(D))^{1/2})$. *Here* $D = -i\nabla$ *and* D_k *is the differential operator with respect to* k.

(2) *Let* $f \in L^2(\mathbb{R}^d)$, $\Phi \in \mathcal{F}_{\rm b}(L^2(\mathbb{R}^d))$, *and* $\Psi \in D(N^{1/2})$. *Then*

$$(\Phi, a(f)\Psi) = \int_{\mathbb{R}^d} \overline{f(k)}(\Phi, a(k)\Psi)dk. \tag{A6}$$

Proposition VI.3: [2, Proposition 4-24 of Ref. 2] and 7, Lemma 2.1 (i) of Ref. 7]

(1) *Let* T *be a self-adjoint operator with* $\ker T = \{0\}$. *Suppose* $f \in D(T^{-1/2})$. *Then for all* $\Psi \in D(d\Gamma(T)^{1/2})$,

$$\|a(f)\Psi\| \le \|T^{-1/2}f\|\|d\Gamma(T)^{1/2}\Psi\|, \tag{A7}$$

$$\|a^\dagger(f)\Psi\|^2 \le \|T^{-1/2}f\|^2\|d\Gamma(T)^{1/2}\Psi\|^2 + \|f\|^2\|\Psi\|^2. \tag{A8}$$

(2) *Let* $l \in \mathbb{Z}$, $n \in \mathbb{N}$, *and* $f_i \in \mathcal{X}$, $i = 1, \cdots, n$. *Then*

$$\|(N+1)^l a^\#(f_1)\cdots a^\#(f_n)(N+1)^{-l-\frac{n}{2}}\| \le C_{n,l}\Pi_{i=1}^n\|f_i\|. \tag{A9}$$

Here $a^\#(f)$ *denotes* $a(f)$ *or* $a^\dagger(f)$ *and* $C_{n,l}$ *is a constant depending on* n *and* l, *but independent of* f_i, $i = 1, \cdots, n$.

(3) *Let* T *be a non-negative self-adjoint operator with* $\ker T = \{0\}$. *Suppose that* $f, g \in D(T) \cap D(T^{-1/2})$. *Then*

$$\|a^\#(f)a^\#(g)\Psi\| \le C\left(\left\|T^{-1/2}f\right\| + \|Tf\|\right)\left(\left\|T^{-1/2}g\right\| + \|Tg\|\right)\|(d\Gamma(T)+1)\Psi\| \tag{A10}$$

for $\Psi \in D(d\Gamma(T))$. *Here* C *is a constant independent of* T, f, g, *and* Ψ.

Proposition VI.4: [2, Lemma 2-23, Corollary 2-27, Theorems 2-29, and 2-31 of Ref. 2] Let S *and* T *be non-negative self-adjoint operators in* $\mathcal{X}$ *and* $\mathcal{Y}$ *with cores* $\mathcal{D}_1$ *and* $\mathcal{D}_2$, *respectively. Then*

(1) $S \otimes 1_{\mathcal{Y}}$ *and* $1_{\mathcal{X}} \otimes T$ *are strongly commuting;*

(2) $S \otimes 1_{\mathcal{Y}} + 1_{\mathcal{X}} \otimes T$ *is a self-adjoint operator and has a core* $\mathcal{D}_1\hat{\otimes}\mathcal{D}_2$, *where* $\hat{\otimes}$ *denotes the algebraic tensor product;*

(3) *It holds that for all* $\Psi \in D(S \otimes 1_{\mathcal{Y}} + 1_{\mathcal{X}} \otimes T)$,

$$\max\{\|(S\otimes 1_{\mathcal{Y}})\Psi\|, \|(1_{\mathcal{X}}\otimes T)\Psi\|\} \le \|(S\otimes 1_{\mathcal{Y}} + 1_{\mathcal{X}}\otimes T)\Psi\|; \tag{A11}$$

(4) *For a densely defined closable operator* A, *we denote the spectrum of* A *by* $\sigma(A)$ *and the point spectrum by* $\sigma_{\rm P}(A)$, *respectively. Then*

$$\sigma(S\otimes 1_{\mathcal{Y}} + 1_{\mathcal{X}}\otimes T) = \{\lambda + \mu | \lambda \in \sigma(S), \mu \in \sigma(T)\} \tag{A12}$$

and

$$\sigma_{\rm P}(S\otimes 1_{\mathcal{Y}} + 1_{\mathcal{X}}\otimes T) = \{\lambda + \mu | \lambda \in \sigma_{\rm P}(S), \mu \in \sigma_{\rm P}(T)\}. \tag{A13}$$

Proposition VI.5: [2, Theorem 4-55 of Ref. 2]

(1)

$$U_{\mathcal{X},\mathcal{Y}}\mathcal{F}_{\rm b,fin}(\mathcal{X}\oplus\mathcal{Y}) = \mathcal{F}_{\rm b,fin}(\mathcal{X})\hat{\otimes}\mathcal{F}_{\rm b,fin}(\mathcal{Y}) \tag{A14}$$

and

$$U_{\mathcal{X},\mathcal{Y}}a^\#(f\oplus g)U_{\mathcal{X},\mathcal{Y}}^{-1} = a^\#(f)\otimes 1 + 1\otimes a^\#(g) \tag{A15}$$

holds on $\mathcal{F}_{\rm b,fin}(\mathcal{X})\hat{\otimes}\mathcal{F}_{\rm b,fin}(\mathcal{Y})$.

(2) *[2, Theorem 4-56 of Ref. 2] Let T and S be non-negative self-adjoint operators in $\mathcal{X}$ and $\mathcal{Y}$. Then*

$$U_{\mathcal{X},\mathcal{Y}} d\Gamma(T \oplus S) U_{\mathcal{X},\mathcal{Y}}^{-1} = d\Gamma(T) \otimes 1 + 1 \otimes d\Gamma(S). \tag{A16}$$

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