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SUPPORT THEOREM FOR PINNED DIFFUSION PROCESSES

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Abstract. In this paper we prove a support theorem of Stroock-Varadhan type for pinned diffusion processes. To this end we use two powerful results from stochastic analysis. One is quasi-sure analysis for Brownian rough path. The other is Aida-Kusuoka-Stroock’s positivity theorem for the densities of weighted laws of non-degenerate Wiener functionals.

§1. Introduction

Let us consider the following Stratonovich stochastic differential equation (SDE) on \mathbb{R}^e ($e \geq 1$) driven by a standard d -dimensional Brownian motion $w = (w_t)_{0 \leq t \leq 1}$:

$$dX_t = \sum_{i=1}^d V_i(X_t) \circ dw_t^i + V_0(X_t)dt, \quad X_0 = a \in \mathbb{R}^e.$$

Here, V_i ($0 \leq i \leq d$) are sufficiently nice vector fields on \mathbb{R}^e and $a \in \mathbb{R}^e$ is an arbitrary (deterministic) starting point. Throughout this paper the time interval is $[0, 1]$ unless otherwise stated. The corresponding skeleton ordinary differential equation (ODE) is given as follows: For a d -dimensional Cameron-Martin path $h: [0, 1] \rightarrow \mathbb{R}^d$,

$$dx_t^h = \sum_{i=1}^d V_i(x_t^h)dh_t^i + V_0(x_t^h)dt, \quad x_0^h = a \in \mathbb{R}^e.$$

We will write $\Psi(h) = (x_t^h)_{0 \leq t \leq 1}$ for simplicity and denote by \mathcal{H} the set of all d -dimensional Cameron-Martin paths.

We are interested in the (topological) support of the law of the diffusion process $X = (X_t)_{0 \leq t \leq 1}$ and would like to describe it in terms of the skeleton ODE. Stroock-Varadhan’s support theorem states that the support equals the closure of $\{\Psi(h): h \in \mathcal{H}\}$. (See [52] and [51, Section 8.3].) In the original work the uniform topology was used, but later it was improved to the α -Hölder topology with $0 < \alpha < 1/2$ in [41, 5]. A quite general approach to support theorems of this kind in [3] should also be referred to.

After the pioneering work [52], the support theorem became one of central topics in the study of SDEs and were generalized to many directions. A partial list could be as follows. A generalization to SDEs with unbounded coefficients was done in [21]. Support theorems for reflecting diffusions were proved in [15, 47]. The topology of the path space was further refined in [20]. The case of anticipating SDEs were studied in [39, 40]. The case of (Volterra-type) SDEs with path-dependent coefficients were recently studied in [11, 31]. A support theorem for McKean-Vlasov SDEs was proved in [56]. A support theorem for jump-type SDEs was studied in [48]. For support theorems for stochastic PDEs, see [42, 4, 43, 44] among others. (Results related to rough path theory will be listed shortly.)

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Using rough path theory, Ledoux-Qian-Zhang [35] gave a new proof to the support theorem twenty years ago. Their idea could be summarized as follows. If the Itô map $w \mapsto X$, i.e. the solution map of the above SDE, were continuous, then the proof of the support theorem would be simple. (In fact, it is not continuous. So, the proof is not easy.) Compared to the usual SDE theory, rough path theory has a prominent feature. The Lyons-Itô map Φ , i.e. the solution map of the corresponding rough differential equation (RDE) is continuous. Moreover, $X = \Phi(\mathbf{W})$, a.s. and Φ is compatible with Ψ . Here, \mathbf{W} is Brownian rough path, i.e. the standard Stratonovich rough path lift of w . Hence, if a support theorem for the law of \mathbf{W} is obtained on the geometric rough path space, Stroock-Varadhan's support theorem follows immediately. The support theorem for \mathbf{W} was first proved with respect to the p -variation topology ($2 < p < 3$) in [35] and then improved to the case of the α -Hölder topology ($1/3 < \alpha < 1/2$) in [17]. This support theorem was later generalized for the laws of Gaussian rough paths in [18]. (See also [19, Sections 13.7 and 15.8] and reference therein.) Other applications of rough path technique to support theorems are found in [12, 2, 9]. Support theorems are also studied in the theory of singular stochastic PDEs, which is a descendant of rough path theory. See [10, 55, 22, 37].

In this paper we study an analogous support theorem for the law of the pinned diffusion process which is condition to end at $b \in \mathbb{R}^e$ at the time $t = 1$, i.e. the law of $X = (X_t)_{0 \leq t \leq 1}$ under the conditional measure $\mathbb{E}[\cdot \mid X_1 = b]$ (heuristically). A natural guess could be that its support equals the closure of $\{\Psi(h) : h \in \mathcal{H}, \Psi(h)_1 = b\}$. But, is it really true?

Before discussing this problem, we first review a positivity theorem [6, 3] for the density of the law of X_t , which is closely related to the support theorem. Under a Hörmander-type condition on V_i 's, the law of X_t has a smooth density $p(t, a, y)$ with respect to the Lebesgue measure for every $t \in (0, 1]$. It is natural and important to ask whether or under what condition $p(t, a, y) > 0$. (For instance, if $p(1, a, b) = 0$, the above-mentioned pinned diffusion measure does not exist.) The positivity theorem states that $p(t, a, b) > 0$ if and only if there exists $h \in \mathcal{H}$ such that $\Psi(h)_t = b$ and $D\Psi(h)_t : \mathcal{H} \rightarrow \mathbb{R}^e$ is a surjective linear map. Here, D stands for the Fréchet derivative on \mathcal{H} . The first paper which proved this result was [6]. Then, a very general result by Aida-Kusuoka-Stroock [3] followed, which will be used in this paper.

A significant feature of [3] is that it studies the positivity of the density of a *weighted* law of a Wiener functional. (In most of the works on this problem, the weight identically equals 1.) In the proof of our main theorem, we will exploit this arbitrariness of the weight. To be more specific, we will choose as a weight a Wiener functional that looks like the indicator function of an open neighborhood of a given geometric rough path.

If we keep these two famous theorems in mind, we can guess what the support of the pinned diffusion measure looks like. First, let us first recall a precise definition of the pinned diffusion measure $\mathbb{Q}_{a,b}$ ($a, b \in \mathbb{R}^e$). We assume that V_i ($0 \leq i \leq d$) satisfy Hörmander's bracket generating condition at every $x \in \mathbb{R}^e$ (see Remark 5.4 (A)). Then, the density $p(t, x, y)$ exists for all $x, y \in \mathbb{R}^e$ and $t \in (0, 1]$. We further assume that $p(1, a, b) > 0$, which is equivalent to the existence of $h \in \mathcal{H}$ such that $\Psi(h)_1 = b$ and $D\Psi(h)_1 : \mathcal{H} \rightarrow \mathbb{R}^e$ is surjective. For every $\beta \in (1/3, 1/2)$, $\mathbb{Q}_{a,b}$ is a unique probability measure on the β -Hölder continuous path space

$$\mathcal{C}_{a,b}^{\beta-H}(\mathbb{R}^e) := \{\xi : [0, 1] \rightarrow \mathbb{R}^e : \beta\text{-Hölder continuous and } \xi_0 = a, \xi_1 = b\}$$

with the following property (it does exist): For every $k \geq 1$, $\{0 < t_1 < \dots < t_k < 1\}$ and $g_1, \dots, g_k \in C_0^\infty(\mathcal{V})$,

$$\begin{aligned} \int \prod_{i=1}^k g_i(\xi_{t_i}) \mathbb{Q}_{a,b}(d\xi) &= p(1, a, b)^{-1} \int_{(\mathbb{R}^e)^k} \left\{ \prod_{i=1}^k g_i(z_i) \right\} p(t_1, a, z_1) \\ &\quad \times \left\{ \prod_{i=2}^k p(t_i - t_{i-1}, z_{i-1}, z_i) \right\} p(1 - t_k, z_k, b) \left\{ \prod_{i=1}^k dz_i \right\}. \end{aligned}$$

Here, dz_i ($1 \leq i \leq k$) is the Lebesgue measure on \mathbb{R}^e and (ξ_t) is the canonical coordinate process on $\mathcal{C}_{a,b}^{\beta\text{-H}}(\mathbb{R}^e)$.

We will prove in Corollary 5.3 that the support of $\mathbb{Q}_{a,b}$ equals the closure with respect to the β -Hölder topology of

$$\{\Psi(h): h \in \mathcal{H}, \Psi(h)_1 = b, D\Psi(h)_1: \mathcal{H} \rightarrow \mathbb{R}^e \text{ is surjective}\}.$$

In fact, this is a special case of our more general result (Corollary 5.2), in which we will prove a support theorem for generalized pinned measure. With quasi-sure analysis and Malliavin calculus, one can easily see that this kind of generalized pinned measures exist. However, since it is difficult to give a brief introduction of them, we do not explain Corollary 5.2 here.

These two corollaries are direct consequences of our main theorem (Theorem 5.1). By a well-known theorem in quasi-sure analysis, there exists a measure on the classical Wiener space that looks like a pullback of $\mathbb{Q}_{a,b}$ by the Itô map. Since the rough path lift map is in fact quasi-surely defined, the measure admits a lift to a measure on the geometric rough path space. By ∞ -quasi-continuity of the lift map, its image measure induced by the Lyons-Itô map is $\mathbb{Q}_{a,b}$ as expected. Theorem 5.1 is a support theorem for this lifted measure. To show it, we use quasi-sure analysis and Aida-Kusuoka-Stroock's positivity theorem [3, Theorem 2.8]. For precise formulations and statements of these results, see Section 5.

The organization of this paper is as follows. Section 2 is devoted to reviewing known results of Malliavin calculus that will be used in the main part of this paper. After we recall fundamentals of (Watanabe's distributional) Malliavin calculus and quasi-sure analysis, we review Aida-Kusuoka-Stroock's positivity theorem, which will play a major role in our proof. In Section 3 we recall basic facts on quasi-sure analytic properties of Brownian rough path. In relation to this, a Besov-type topology is introduced on the geometric rough path space. Section 4 is a core part of this work, in which we prove twice \mathcal{K} -differentiability of the Lyons-Itô map, that is, the solution map of a rough differential equation. This property is the key condition in Aida-Kusuoka-Stroock's theorem. In Section 5 we state our main theorems precisely and prove them rigorously. Our key result is Theorem 5.1. This is a support theorem on a geometric rough path space for a measure that looks like the "pullback" by the Lyons-Itô map of a pinned diffusion measure. Since the Lyons-Itô map is continuous, the support theorems for pinned diffusion measures (Corollaries 5.2 and 5.3) follow immediately.

Notation: In the sequel we will use the following notation. We write $\mathbb{N} = \{1, 2, \dots\}$. The time interval of (rough) paths and stochastic processes is $[0, 1]$ throughout the paper. Below, we assume $d \in \mathbb{N}$.

- The set of all continuous paths $\varphi: [0, 1] \rightarrow \mathbb{R}^d$ is denoted by $\mathcal{C}(\mathbb{R}^d)$. With the usual sup-norm $\|\varphi\|_\infty := \sup_{0 \leq t \leq 1} |\varphi_t|$ on the $[0, 1]$ -interval, this is a Banach space. The increment of φ is often denoted by φ^1 , that is, $\varphi_{s,t}^1 := \varphi_t - \varphi_s$ for $s \leq t$. For $a, b \in \mathbb{R}^d$, we

write $\mathcal{C}_a(\mathbb{R}^d) = \{\varphi \in \mathcal{C}(\mathbb{R}^d) : \varphi_0 = a\}$ and, in a similar way,
 $\mathcal{C}_{a,b}(\mathbb{R}^d) = \{\varphi \in \mathcal{C}(\mathbb{R}^d) : \varphi_0 = a, \varphi_1 = b\}$.

- Let $\alpha \in (0, 1]$. The α -Hölder seminorm of $\varphi \in \mathcal{C}(\mathbb{R}^d)$ is defined as usual by

$$\|\varphi\|_\alpha := \sup_{0 \leq s < t \leq 1} \frac{|\varphi_{s,t}^1|}{(t-s)^\alpha}.$$

The α -Hölder continuous path space is denoted by $\mathcal{C}^{\alpha-H}(\mathbb{R}^d) = \{\varphi \in \mathcal{C}(\mathbb{R}^d) : \|\varphi\|_\alpha < \infty\}$, which is a non-separable Banach space with the norm $\|\varphi\|_\alpha + |\varphi_0|$. The closure of $\{\varphi \in \mathcal{C}(\mathbb{R}^d) : \varphi \text{ is piecewise-}C^1\}$ with respect to the α -Hölder topology is denoted by $\mathcal{C}^{0,\alpha-H}(\mathbb{R}^d)$. This is a separable Banach subspace of $\mathcal{C}^{\alpha-H}(\mathbb{R}^d)$. For a starting point $a \in \mathbb{R}^d$ and an end point $b \in \mathbb{R}^d$, $\mathcal{C}_a^{0,\alpha-H}(\mathbb{R}^d)$ and $\mathcal{C}_{a,b}^{0,\alpha-H}(\mathbb{R}^d)$ are defined in an analogous way as above.

- For $1/3 < \alpha \leq 1/2$, $G\Omega_\alpha^H(\mathbb{R}^d)$ stands for the α -Hölder geometric rough path space over \mathbb{R}^d . A generic element of $G\Omega_\alpha^H(\mathbb{R}^d)$ is denoted by $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2)$. (See [19, 34] among others for basic information on geometric rough paths.)
- The Cameron-Martin space associated with standard d -dimensional Brownian motion is denoted by $\mathcal{H} = \mathcal{H}^d$ (except in Section 2). Its precise definition is given as follows:
 $\mathcal{H} := \{h \in \mathcal{C}_0(\mathbb{R}^d) : h \text{ is absolutely continuous and } \|h\|_{\mathcal{H}} < \infty\}$, where

$$\langle h, k \rangle_{\mathcal{H}} := \int_0^1 \langle h'_s, k'_s \rangle_{\mathbb{R}^d} ds \quad \text{and} \quad \|h\|_{\mathcal{H}} := \langle h, h \rangle_{\mathcal{H}}^{1/2}, \quad h, k \in \mathcal{H}.$$

This is a real separable Hilbert space with this inner product. It is easy to see that $\mathcal{H} \subset \mathcal{C}_0^{0,1/2-H}(\mathbb{R}^d)$. If we set $\mathcal{L}(h)_{s,t}^1 = h_t - h_s$ and $\mathcal{L}(h)_{s,t}^2 = \int_s^t (h_u - h_s) \otimes dh_u$ for $h \in \mathcal{H}$ and $0 \leq s \leq t \leq 1$, then $\mathcal{L} : \mathcal{H} \hookrightarrow G\Omega_{1/2}^H(\mathbb{R}^d)$ becomes a locally Lipschitz continuous injection. (A map between two metric space is said to be locally Lipschitz continuous if the map, when restricted to every bounded subset of the domain, is Lipschitz continuous.) We call $\mathcal{L}(h)$ the natural lift of h and will sometimes denote it by \mathbf{h} .

- Let U be an open subset of \mathbb{R}^m . For $k \in \mathbb{N} \cup \{0\}$, $C^k(U, \mathbb{R}^n)$ denotes the set of C^k -functions from U to \mathbb{R}^n . (When $k = 0$, we simply write $C(U, \mathbb{R}^n)$ instead of $C^0(U, \mathbb{R}^n)$.) The set of bounded C^k -functions $f : U \rightarrow \mathbb{R}^n$ whose derivatives up to order k are all bounded is denoted by $C_b^k(U, \mathbb{R}^n)$. This is a Banach space with $\|f\|_{C_b^k} := \sum_{i=0}^k \|\nabla^i f\|_\infty$. (Here, $\|\cdot\|_\infty$ stands for the usual sup-norm on U .) As usual, we set $C^\infty(U, \mathbb{R}^n) := \cap_{k=0}^\infty C^k(U, \mathbb{R}^n)$ and $C_b^\infty(U, \mathbb{R}^n) := \cap_{k=0}^\infty C_b^k(U, \mathbb{R}^n)$.

§2. Preliminaries from Malliavin calculus

In this section, $(\mathcal{W}, \mathcal{H}, \mu)$ is an abstract Wiener space. That is, $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a separable Banach space, $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a separable Hilbert space, \mathcal{H} is a dense subspace of \mathcal{W} and the inclusion map is continuous, and μ is a (necessarily unique) Borel probability measure on \mathcal{W} with the property that

$$\int_{\mathcal{W}} \exp\left(\sqrt{-1} \mathcal{W}^* \langle \lambda, w \rangle_{\mathcal{W}}\right) \mu(dw) = \exp\left(-\frac{1}{2} \|\lambda\|_{\mathcal{H}}^2\right), \quad \lambda \in \mathcal{W}^* \subset \mathcal{H}^*, \quad (2.1)$$

where we have used the fact that \mathcal{W}^* becomes a dense subspace of \mathcal{H} when we make the natural identification between \mathcal{H}^* and \mathcal{H} itself. Hence, $\mathcal{W}^* \hookrightarrow \mathcal{H}^* = \mathcal{H} \hookrightarrow \mathcal{W}$ and both inclusions are continuous and dense. We denote by $\{\langle k, \bullet \rangle : k \in \mathcal{H}\}$ the family of centered Gaussian random variables defined on \mathcal{W} indexed by \mathcal{H} (i.e. the homogeneous Wiener chaos

of order 1). If $\langle k, \bullet \rangle_{\mathcal{H}} \in \mathcal{H}^*$ extends to an element of \mathcal{W}^* , then the extension coincides with the random variable $\langle k, \bullet \rangle$. (When $\langle k, \bullet \rangle_{\mathcal{H}} \in \mathcal{H}^*$ does not extend to an element of \mathcal{W}^* , $\langle k, \bullet \rangle$ is defined as the L^2 -limit of $\{\langle k_n, \bullet \rangle\}_{n=1}^{\infty}$, where $\{k_n\}_{n=1}^{\infty}$ is any sequence of \mathcal{H} such that $\langle k_n, \bullet \rangle_{\mathcal{H}} \in \mathcal{W}^*$ for all n and $\lim_{n \rightarrow \infty} \|k_n - k\|_{\mathcal{H}} = 0$.) We also denote by $\tau_k: \mathcal{W} \rightarrow \mathcal{W}$ the translation $\tau_k(w) = w + k$. (For basic information on abstract Wiener spaces, see [49, 23] among others.)

2.1. Watanabe distribution theory and quasi-sure analysis

We first quickly summarize some basic facts in Malliavin calculus, which are related to Watanabe distributions (i.e. generalized Wiener functionals) and quasi-sure analysis. Most of the contents and the notation in this section are found in [24, Sections V.8–V.10] with trivial modifications. Also, [49, 45, 23, 38, 32] are good textbooks of Malliavin calculus. For basic results of quasi-sure analysis, we refer to [36, Chapter II].

We use the following notation and facts in the main part of this paper.

- (a) Sobolev spaces $\mathbf{D}_{p,r}(\mathcal{K})$ of \mathcal{K} -valued (generalized) Wiener functionals, where \mathcal{K} is a real separable Hilbert space and $p \in (1, \infty)$, $r \in \mathbb{R}$. As usual, we will use the spaces

$$\mathbf{D}_{\infty}(\mathcal{K}) = \cap_{k=1}^{\infty} \cap_{1 < p < \infty} \mathbf{D}_{p,k}(\mathcal{K}), \quad \tilde{\mathbf{D}}_{\infty}(\mathcal{K}) = \cap_{k=1}^{\infty} \cup_{1 < p < \infty} \mathbf{D}_{p,k}(\mathcal{K})$$

of test functionals and the spaces

$$\mathbf{D}_{-\infty}(\mathcal{K}) = \cup_{k=1}^{\infty} \cup_{1 < p < \infty} \mathbf{D}_{p,-k}(\mathcal{K}), \quad \tilde{\mathbf{D}}_{-\infty}(\mathcal{K}) = \cup_{k=1}^{\infty} \cap_{1 < p < \infty} \mathbf{D}_{p,-k}(\mathcal{K})$$

of Watanabe distributions as in [24]. When $\mathcal{K} = \mathbb{R}$, we simply write $\mathbf{D}_{p,r}$, etc.

- (b) For $F = (F^1, \dots, F^e) \in \mathbf{D}_{\infty}(\mathbb{R}^e)$, we denote by $\sigma_F^{ij}(w) = \langle DF^i(w), DF^j(w) \rangle_{\mathcal{H}}$ the (i, j) -component of Malliavin covariance matrix ($e \in \mathbb{N}$, $1 \leq i, j \leq e$). We say that F is non-degenerate in the sense of Malliavin if $(\det \sigma_F)^{-1} \in \cap_{1 < p < \infty} L^p(\mu)$. Here, D is the \mathcal{H} -derivative (the gradient operator in the sense of Malliavin calculus). If $F \in \mathbf{D}_{\infty}(\mathbb{R}^e)$ is non-degenerate, its law on \mathbb{R}^e admits a smooth, rapidly decreasing density $p_F = p_F(y)$ with respect to the Lebesgue measure dy , that is, $\mu \circ F^{-1} = p_F(y)dy$. (This fact is quite famous. See any textbook of Malliavin calculus.)
- (c) Pullback $T \circ F = T(F) \in \tilde{\mathbf{D}}_{-\infty}$ of a tempered Schwartz distribution $T \in \mathcal{S}'(\mathbb{R}^e)$ on \mathbb{R}^e by a non-degenerate Wiener functional $F \in \mathbf{D}_{\infty}(\mathbb{R}^e)$. The most important example of T is Dirac's delta function. In that case, $\mathbb{E}[\delta_y(F)] := \langle \delta_y(F), 1 \rangle = p_F(y)$ holds for every $y \in \mathbb{R}^e$. Here, $\langle \star, \star \rangle$ denotes the pairing of $\tilde{\mathbf{D}}_{-\infty}$ and \mathbf{D}_{∞} as usual. (See [24, Section 5.9].)
- (d) This is a continuation of (b) and (c) above. Assume in addition that $G \in \mathbf{D}_{\infty}$ is non-negative. Then, $(Gd\mu) \circ F^{-1}$ is called the law of F *weighted by* G . (In other words, this law is a probability measure on \mathbb{R}^e determined by $A \mapsto \mathbb{E}[\mathbf{1}_A(F)G]$, where A is a Borel measurable subset of \mathbb{R}^e .) If F is non-degenerate, this law admits a smooth, rapidly decreasing density $p_{F,G} = p_{F,G}(y)$ with respect to the Lebesgue measure dy , that is, $(Gd\mu) \circ F^{-1} = p_{F,G}(y)dy$. In the language of Watanabe distributions, we have $\mathbb{E}[\delta_y(F)G] := \langle \delta_y(F), G \rangle = p_{F,G}(y)$ for every $y \in \mathbb{R}^e$. (For weighted laws of non-degenerate Wiener functionals, we refer to [32, Sections 5.3 and 5.12].)
- (e) If $\eta \in \mathbf{D}_{-\infty}$ satisfies that $\mathbb{E}[\eta G] := \langle \eta, G \rangle \geq 0$ for every non-negative $G \in \mathbf{D}_{\infty}$, it is called a positive Watanabe distribution. According to Sugita's theorem (see [50] or [36, p. 101]), for every positive Watanabe distribution η , there uniquely exists a finite Borel

measure μ_η on \mathcal{W} such that

$$\langle \eta, G \rangle = \int_{\mathcal{W}} \tilde{G}(w) \mu_\eta(dw), \quad G \in \mathbf{D}_\infty$$

holds, where \tilde{G} stands for an ∞ -quasi-continuous modification of G . If $\eta \in \mathbf{D}_{p,-k}$ is positive, then it holds that

$$\mu_\eta(A) \leq \|\eta\|_{p,-k} \text{Cap}_{q,k}(A) \quad \text{for every Borel subset } A \subset \mathcal{W},$$

where $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, $k \in \mathbb{N}$, and $\text{Cap}_{q,k}$ stands for the (q, k) -capacity associated with $\mathbf{D}_{q,k}$. (For more details, see [36, Chapter II].)

REMARK 2.1. In some of the books cited in this subsection (in particular [24, 36, 38]), results are formulated on a special Gaussian space. However, almost most all of them (at least, those that will be used in this paper) still hold true on any abstract Wiener space.

2.2. \mathcal{K} -regularity and \mathcal{K} -differentiability

In this subsection we quickly review Aida-Kusuoka-Stroock's result on the positivity of the density for non-degenerate Wiener functionals (see [3]).

Let us first recall the definitions of \mathcal{K} -continuity, \mathcal{K} -regularity, uniformly \mathcal{K} -regularity, and l -times \mathcal{K} -regular differentiability, which were first introduced in [3]. Note that in these definitions, functions and maps on \mathcal{W} are viewed as everywhere-defined ones (not equivalence classes with respect to μ). It should be noted that these definitions depend on the choice of exhaustion \mathcal{K} .

For a finite dimensional subspace K of \mathcal{H} , $P_K: \mathcal{H} \rightarrow K$ stands for the orthogonal projection and we write $P_K^\perp = \text{Id}_{\mathcal{H}} - P_K$. This projection naturally extends to $\bar{P}_K: \mathcal{W} \rightarrow K$ as follows:

$$\bar{P}_K(w) = \sum_{i=1}^{\dim K} \langle e_i, w \rangle e_i,$$

where $\{e_i\}_{i=1}^{\dim K}$ is an orthonormal basis of K . (This right hand side is independent of the choice of $\{e_i\}$.) We set $\bar{P}_K^\perp = \text{Id}_{\mathcal{W}} - \bar{P}_K$.

Assume that $\mathcal{K} = \{K_n\}_{n=1}^\infty$ is a non-decreasing, countable exhaustion of \mathcal{H} by finite dimensional subspaces, that is, $K_n \subset K_{n+1}$ for all n and $\cup_{n=1}^\infty K_n$ is dense in \mathcal{H} . Set $P_n = P_{K_n}$, define $\bar{P}_n, P_n^\perp, \bar{P}_n^\perp$ accordingly. We say that a map F from \mathcal{W} into a Polish space (E, ρ_E) is \mathcal{K} -continuous if it is measurable and, for each $n \in \mathbb{N}$, there is a measurable map $F_n: \mathcal{W} \times K_n \rightarrow E$ with the properties that $F \circ \tau_k = F_n(\cdot, k)$ (μ -a.s.) for each $k \in K_n$ and $k \in K_n \mapsto F_n(w, k) \in E$ is continuous for each $w \in \mathcal{W}$. Given a \mathcal{K} -continuous map F , we set

$$F_n^\perp(w, k) = F_n(w, -\bar{P}_n(w) + k) \quad \text{for } n \in \mathbb{N} \text{ and } k \in K_n. \quad (2.2)$$

Given a measurable map $F: \mathcal{W} \rightarrow E$, we say that F is \mathcal{K} -regular if F is \mathcal{K} -continuous and there is a continuous map $\tilde{F}: \mathcal{H} \rightarrow E$ such that

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ w: \rho_E(\tilde{F} \circ \bar{P}_n(w), F(w)) \vee \rho_E(\tilde{F}(h), F_n^\perp(w, P_n(h))) \geq \epsilon \right\} \right) = 0 \quad (2.3)$$

holds for every $\epsilon > 0$ and $h \in \mathcal{H}$. In this case \tilde{F} is called a \mathcal{K} -regularization of F .

If F is a map from \mathcal{W} into a Polish space E , we say that it is *uniformly \mathcal{K} -regular* if it is \mathcal{K} -regular and (2.3) can be replaced by the condition that

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ w : \sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_E(\tilde{F}(\bar{P}_n(w) + k), F_n(w, k)) \vee \rho_E(\tilde{F}(h + k), F_n^\perp(w, P_n(h) + k)) \geq \epsilon \right\} \right) = 0 \quad (2.4)$$

for every $m \in \mathbb{N}, r > 0, \epsilon > 0$ and $h \in \mathcal{H}$. (In (2.4) above and (2.5) below, we implicitly assume $n \geq m$ since we let $n \rightarrow \infty$ for each fixed m .)

Let E be a separable Banach space and F be a map from \mathcal{W} into E . Given $l \in \mathbb{N}$ we say that F is *l -times \mathcal{K} -regularly differentiable* if F is uniformly \mathcal{K} -regular, $F_n(w, \cdot)$ is l -times continuously Fréchet differentiable on K_n for each $n \in \mathbb{N}$ and $w \in \mathcal{W}$, \tilde{F} is l -times continuously Fréchet differentiable on \mathcal{H} , and (2.4) can be replaced by the condition that

$$\lim_{n \rightarrow \infty} \mu \left(\left\{ w : \|\tilde{F}(\bar{P}_n(w) + \bullet) - F_n(w, \bullet)\|_{C_b^l(B_{K_m}(0, r), E)} \vee \|\tilde{F}(h + \bullet) - F_n^\perp(w, P_n(h) + \bullet)\|_{C_b^l(B_{K_m}(0, r), E)} \geq \epsilon \right\} \right) = 0 \quad (2.5)$$

for every $m \in \mathbb{N}, r > 0, \epsilon > 0$ and $h \in \mathcal{H}$. Here, $B_{K_m}(0, r) = \{k \in K_m : \|k\|_{\mathcal{H}} < r\}$.

The following theorem is [3, Theorem 2.8] (translated into the language of Watanabe distribution theory), which is the key tool in this paper. It is a quite general result on the positivity of the density function of the law of a non-degenerate Wiener functional. At first sight, it may not be clear why the case of non-constant weight G is so important. However, in the proof of our main theorem, the weight will play a crucial role.

THEOREM 2.2. *Let $F \in \mathbf{D}_\infty(\mathbb{R}^e)$, $e \in \mathbb{N}$, and $G \in \mathbf{D}_\infty$. Suppose that F is non-degenerate in the sense of Malliavin and G is non-negative. Suppose further that F is twice \mathcal{K} -regularly differentiable and G is \mathcal{K} -regular with their \mathcal{K} -regularizations \tilde{F} and \tilde{G} , respectively. Then, for $y \in \mathbb{R}^e$, the following are equivalent:*

- $\mathbb{E}[\delta_y(F)G] > 0$.
- *There exists $h \in \mathcal{H}$ such that $D\tilde{F}(h) : \mathcal{H} \rightarrow \mathbb{R}^e$ is surjective, $\tilde{F}(h) = y$ and $\tilde{G}(h) > 0$.*

REMARK 2.3. As is well-known, the condition that “ $D\tilde{F}(h) : \mathcal{H} \rightarrow \mathbb{R}^e$ is surjective” in the above theorem is equivalent to non-degeneracy of deterministic Malliavin covariance matrix of \tilde{F} at h .

The most typical example of $\mathcal{K} = \{K_n\}$ and $P_n = P_{K_n}$ is the dyadic piecewise linear approximation $w(n)$ of the standard d -dimensional Brownian motion $w = (w_t)_{0 \leq t \leq 1}$. As usual, $w(n)$ is defined as follows: $w(n)_{j2^{-n}} = w_{j2^{-n}}$ for all $0 \leq j \leq 2^n$ and $w(n)$ is linearly interpolated on each subinterval $[(j-1)2^{-n}, j2^{-n}]$, $0 \leq j \leq 2^n$.

EXAMPLE 2.4. Let $(\mathcal{W}, \mathcal{H}, \mu)$ be the d -dimensional classical Wiener space, that is, (i) $\mathcal{W} := \mathcal{C}_0(\mathbb{R}^d)$ is the Banach space of \mathbb{R}^d -valued continuous paths that start at 0 equipped with the usual sup-norm, (ii) μ is the d -dimensional Wiener measure on \mathcal{W} , (iii) $\mathcal{H} = \mathcal{H}^d$ is the d -dimensional Cameron-Martin space. We denote by $(w_t)_{0 \leq t \leq 1}$ the canonical realization of d -dimensional Brownian motion (i.e. the coordinate process).

Now we introduce a simple orthonormal basis of \mathcal{H} . First, set $\psi_t^{0,1} \equiv 1$. For $n \geq 1$ and $1 \leq m \leq 2^{n-1}$, set

$$\psi_t^{n,m} = \begin{cases} 2^{(n-1)/2}, & t \in [(2m-2)2^{-n}, (2m-1)2^{-n}), \\ -2^{(n-1)/2}, & t \in [(2m-1)2^{-n}, 2m2^{-n}), \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\{\mathbf{e}_i\}_{i=1}^d$ the canonical orthonormal basis of \mathbb{R}^d . Then, it is well-known that

$$\{\psi^{n,m} \mathbf{e}_i : n \geq 0, 1 \leq m \leq 2^{n-1} \vee 1, 1 \leq i \leq d\}$$

forms an orthonormal basis of $L^2([0,1], \mathbb{R}^d)$. Since $L^2([0,1], \mathbb{R}^d)$ and \mathcal{H} are unitarily isometric,

$$\{\varphi^{n,m} \mathbf{e}_i : n \geq 0, 1 \leq m \leq 2^{n-1} \vee 1, 1 \leq i \leq d\}$$

forms an orthonormal basis of \mathcal{H} , where we set $\varphi_t^{n,m} := \int_0^t \psi_s^{n,m} ds$.

If we set K_n , $n \geq 1$, to be the linear span of

$$\{\varphi^{l,m} \mathbf{e}_i : 0 \leq l \leq n-1, 1 \leq m \leq 2^{l-1} \vee 1, 1 \leq i \leq d\},$$

then $\mathcal{K} = \{K_n\}_{n=1}^\infty$ is a non-decreasing, countable exhaustion of \mathcal{H} by finite dimensional subspaces. Moreover, it is a routine to check that $P_n(h) = h(n)$ and $\bar{P}_n(w) = w(n)$ for all $n \geq 1$, $h \in \mathcal{H}$ and $w \in \mathcal{W}$. Hence, we may apply Theorem 2.2 to the dyadic piecewise linear approximations of Brownian motion. Finally, we remark that $\lim_{n \rightarrow \infty} \|w(n) - w\|_\infty = 0$ for all $w \in \mathcal{W}$ and $\lim_{n \rightarrow \infty} \|h(n) - h\|_{\mathcal{H}} = 0$ for all $h \in \mathcal{H}$.

§3. Preliminaries from rough path theory

In this section we recall the geometric rough path space with the Hölder or Besov norm and quasi-sure properties of the rough path lift. For basic properties of geometric rough path space with the Hölder topology, we refer to [34, 19]. For the geometric rough path space with the Besov topology, we refer to [19, Appendix A.2]. The quasi-sure properties of the rough path lift are summarized in [26]. From now on, $(\mathcal{W}, \mathcal{H}, \mu)$ stands for the d -dimensional classical Wiener space as in Example 2.4.

In the first half of this section, we discuss deterministic aspects of rough path theory. First, we work in the α -Hölder rough path topology with $\alpha \in (1/3, 1/2)$. We consider an RDE with drift driven by $\mathbf{w} \in G\Omega_\alpha^H(\mathbb{R}^d)$. For vector fields $V_i : \mathbb{R}^e \rightarrow \mathbb{R}^e$ ($0 \leq i \leq d$), we consider the following RDE:

$$dx_t = \sum_{i=1}^d V_i(x_t) dw_t^i + V_0(x_t) dt, \quad x_0 = a \in \mathbb{R}^e. \quad (3.1)$$

We assume that V_i , $0 \leq i \leq d$, is (at least) of C_b^3 , that is, when viewed as an \mathbb{R}^e -valued function, $V_i \in C_b^3(\mathbb{R}^e, \mathbb{R}^e)$. It is then known that a unique global solution of (3.1) exists for every \mathbf{w} and a . Moreover, Lyons' continuity theorem holds, that is, the map

$$\Phi : G\Omega_\alpha^H(\mathbb{R}^d) \rightarrow \mathcal{C}_a^{0,\alpha-H}(\mathbb{R}^e)$$

defined by $\Phi(\mathbf{w}) = x$ is locally Lipschitz continuous. This map is called the Lyons-Itô map.

REMARK 3.1. We only study the first level paths of solutions of RDEs. Therefore, Lyons-Itô map takes its values in a usual path space and any formulation of RDEs will do.

We introduce the skeleton ODE associated with RDE (3.1) and SDE (3.4) below. For $h \in \mathcal{H}$, we consider the following ODE in the usual sense:

$$dx_t = \sum_{i=1}^d V_i(x_t) dh_t^i + V_0(x_t) dt, \quad x_0 = a \in \mathbb{R}^e. \quad (3.2)$$

If V_i 's are of C_b^1 , then a unique solution x exists, which is denoted by $\Psi(h)$. Under the same condition, $\Psi: \mathcal{H} \rightarrow \mathcal{C}_a^{0,1/2-H}(\mathbb{R}^e)$ is locally Lipschitz continuous. It should be noted that $\Psi(h) = \Phi(\mathcal{L}(h))$ (if V_i 's are of C_b^3).

Next we discuss Besov-type norms for rough paths. We always assume that the Besov parameter $(\alpha, 4m)$ satisfy the following conditions:

$$\frac{1}{3} < \alpha < \frac{1}{2}, \quad m \in \mathbb{N}, \quad \alpha - \frac{1}{4m} > \frac{1}{3}, \quad 4m(\frac{1}{2} - \alpha) > 1. \quad (3.3)$$

Observe that, if the integer m is chosen large enough for a given $\alpha \in (1/3, 1/2)$, then the two other inequalities in (3.3) are satisfied. Heuristically, α plays a similar role to the Hölder parameter (see the Besov-Hölder embedding theorem below) and the auxiliary parameter $4m$ is a very large even integer.

For $(\alpha, 4m)$ satisfying (3.3), $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ denotes the geometric rough path space over \mathbb{R}^d with the $(\alpha, 4m)$ -Besov norm. It is defined to be the closure of $\{\mathcal{L}(k): k \in \mathcal{C}_0^{1-H}(\mathbb{R}^d)\}$ with respect to the $(\alpha, 4m)$ -Besov distance. The distance is given by

$$\begin{aligned} d_{\alpha, 4m}(\mathbf{w}, \hat{\mathbf{w}}) &= \|\mathbf{w}^1 - \hat{\mathbf{w}}^1\|_{\alpha, 4m-B} + \|\mathbf{w}^2 - \hat{\mathbf{w}}^2\|_{2\alpha, 2m-B} \\ &:= \left(\iint_{0 \leq s < t \leq 1} \frac{|\mathbf{w}_{s,t}^1 - \hat{\mathbf{w}}_{s,t}^1|^{4m}}{|t-s|^{1+4m\alpha}} ds dt \right)^{\frac{1}{4m}} \\ &\quad + \left(\iint_{0 \leq s < t \leq 1} \frac{|\mathbf{w}_{s,t}^2 - \hat{\mathbf{w}}_{s,t}^2|^{2m}}{|t-s|^{1+4m\alpha}} ds dt \right)^{\frac{1}{2m}}. \end{aligned}$$

The homogeneous norm is denoted by $\|\mathbf{w}\|_{\alpha, 4m-B} := \|\mathbf{w}^1\|_{\alpha, 4m-B} + \|\mathbf{w}^2\|_{2\alpha, 2m-B}^{1/2}$. It is known that $\{\mathcal{L}(h): h \in \mathcal{H}\}$ is dense in $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$.

By the Besov-Hölder embedding theorem for rough path spaces, there is a continuous embedding $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d) \hookrightarrow G\Omega_{\alpha-(1/4m)}^H(\mathbb{R}^d)$. If $\alpha < \alpha' < 1/2$, there is a continuous embedding $G\Omega_{\alpha'}^H(\mathbb{R}^d) \hookrightarrow G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$. Basically, we will not write the first embedding explicitly. (For example, if we write $\Phi(\mathbf{w})$ for $\mathbf{w} \in G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$, then it is actually the composition of the first embedding map above and Φ with respect to the $\{\alpha - 1/(4m)\}$ -Hölder topology.) It is known that the Young translation by $h \in \mathcal{H}$ works well on $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ under (3.3). The map $(\mathbf{w}, h) \mapsto T_h(\mathbf{w})$ is locally Lipschitz continuous from $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d) \times \mathcal{H}$ to $G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$, where $T_h(\mathbf{w})$ is the Young translation of \mathbf{w} by h (see [26, Lemma 5.1]). Recall that $T_h(\mathbf{w})$ is defined by

$$T_h(\mathbf{w})_{s,t}^1 = \mathbf{w}_{s,t}^1 + \mathbf{h}_{s,t}^1 \quad \text{and} \quad T_h(\mathbf{w})_{s,t}^2 = \mathbf{w}_{s,t}^2 + \mathbf{h}_{s,t}^2 + \int_s^t \mathbf{w}_{s,u}^1 \otimes dh_u + \int_s^t \mathbf{h}_{s,u}^1 \otimes d_u \mathbf{w}_{s,u}^1$$

for $0 \leq s \leq t \leq 1$. (The third and the fourth terms makes sense as a Riemann-Stieltjes and a Young integral, respectively.)

From here we discuss probabilistic aspects. Suppose that V_i 's are of C_b^3 and let the notation as in Example 2.4. If \mathbf{W} is Brownian rough path, that is, the natural (Stratonovich)

lift of d -dimensional Brownian motion $(w_t)_{0 \leq t \leq 1}$, that is,

$$\mathbf{W}_{s,t}^1 = w_t - w_s \quad \text{and} \quad \mathbf{W}_{s,t}^2 = \int_s^t (w_u - w_s) \otimes \circ dw_u, \quad 0 \leq s \leq t \leq 1.$$

Then, the process $(\Phi(\mathbf{W})_t)_{0 \leq t \leq 1}$ coincides μ -a.s. with the solution $(X_t)_{0 \leq t \leq 1}$ of the corresponding Stratonovich-type SDEs in the usual sense:

$$dX_t = \sum_{i=1}^d V_i(X_t) \circ dw_t^i + V_0(X_t)dt, \quad X_0 = a \in \mathbb{R}^e. \quad (3.4)$$

Here, $\circ dw_t$ stands for the Stratonovich-type stochastic integral. (The coefficients in (3.1), (3.2) and (3.4) are the same vector fields.)

Now we review quasi-sure properties of rough path lift map \mathbf{L} from \mathcal{W} to $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$. For $k \in \mathbb{N}$ and $w \in \mathcal{W}$, we denote by $w(k)$ the k th dyadic piecewise linear approximation of w associated with the partition $\{j2^{-k}: 0 \leq j \leq 2^k\}$ of $[0, 1]$. We denote the natural lift of $w(k)$ by $\mathcal{L}(w(k))$.

For $(\alpha, 4m)$ satisfying (3.3), we set

$$\mathcal{Z}_{\alpha,4m} := \{w \in \mathcal{W}: \{\mathcal{L}(w(k))\}_{k=1}^\infty \text{ is Cauchy in } G\Omega_{\alpha,4m}^B(\mathbb{R}^d)\}. \quad (3.5)$$

We define $\mathbf{L}: \mathcal{W} \rightarrow G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ by $\mathbf{L}(w) = \lim_{m \rightarrow \infty} \mathcal{L}(w(k))$ if $w \in \mathcal{Z}_{\alpha,4m}$ and we define $\mathbf{L}(w) = \mathbf{0}$ (the zero rough path) if $w \notin \mathcal{Z}_{\alpha,4m}$. It is well-known that $\mu(\mathcal{Z}_{\alpha,4m}) = 0$. Obviously, $w \mapsto \mathbf{L}(w)$ is an everywhere-defined Borel measurable version of Brownian rough path \mathbf{W} with respect to μ . (In what follows, when we write \mathbf{W} , it means this version.)

It is easy to see that $\mathcal{H} \subset \mathcal{Z}_{\alpha,4m}$ and $\mathbf{L}(h) = \mathcal{L}(h)$ for all $h \in \mathcal{H}$. The scalar multiplication (i.e. the dilation) and the Cameron-Martin translation leave $\mathcal{Z}_{\alpha,4m}$ invariant. Moreover, $c\mathbf{L}(w) = \mathbf{L}(cw)$ and $T_h(\mathbf{L}(w)) = \mathbf{L}(w + h)$ for all $w \in \mathcal{Z}_{\alpha,4m}$, $c \in \mathbb{R}$, and $h \in \mathcal{H}$. It is known that $\mathcal{Z}_{\alpha,4m}^c$ is slim, that is, the (p, r) -capacity of this set is zero for any $p \in (1, \infty)$ and $r \in \mathbb{N}$. Therefore, from a viewpoint of quasi-sure analysis, the lift map \mathcal{L} is well-defined. Moreover, the map $\mathcal{W} \ni w \mapsto \mathbf{L}(w) \in G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ is ∞ -quasi-continuous. (This kind of ∞ -quasi-continuity was first shown in [1].) Then, it immediately follows that the map

$$\mathcal{W} \ni w \mapsto \Phi(\mathbf{L}(w)) \in \mathcal{C}_a^{0,\alpha-1/(4m)\text{-H}}(\mathbb{R}^e)$$

is an ∞ -quasi-continuous version of $w \mapsto X$, where $X = (X_t)_{0 \leq t \leq 1}$ is the solution of SDE (3.4) viewed as a path space-valued random variable.

REMARK 3.2. The situation described above can be summarized by the following commutative diagram:

$$\begin{array}{ccccc} & & G\Omega_{\alpha,4m}^B(\mathbb{R}^d) & & \\ & \nearrow \mathcal{L} & \uparrow \mathbf{L} & \searrow \Phi & \\ \mathcal{H} & \xrightarrow{\text{Incl}} & \mathcal{W} & \xrightarrow{\text{Itô}} & \mathcal{C}_a^{0,\alpha-1/(4m)\text{-H}}(\mathbb{R}^e) \end{array}$$

Here, **Incl** is the inclusion and **Itô** is the usual Itô map associated with SDE (3.4). All maps above except \mathbf{L} and **Itô** are continuous. Note also that $\Psi = \Phi \circ \mathcal{L}$.

REMARK 3.3. The first paper that used quasi-sure analysis for Brownian rough path is [25]. In that paper, however, the rough path topology is the p -variation topology with

$2 < p < 3$. The foundation of quasi-sure analysis for Brownian rough path in Besov or Hölder topology was laid by [26, 1]. It was used for large deviations for pinned diffusion measures in [26, 28, 29]. A quasi-sure refinement of non-degeneracy property of Brownian signature was proved in [7]. It should also be noted that quasi-sure analysis for fractional Brownian rough path was studied in [8, 46].

Before closing this section, let us recall Karhunen-Loève approximation, which will play an important role in proofs of \mathcal{K} -regularity and \mathcal{K} -differentiability in the next section. Fortunately, the dyadic piecewise linear approximation is also a Karhunen-Loève approximation since $\bar{P}_{K_k}(w) = w(k)$ (see Example 2.4). It is easy to see that, for each fixed k ,

$$T_{-w(k)}\mathbf{W} = \lim_{l \rightarrow \infty} T_{-w(k)}\mathcal{L}(w(l)) = \lim_{l \rightarrow \infty} \mathcal{L}(w(l) - w(k)), \quad w \in \mathcal{Z}_{\alpha, 4m}. \quad (3.6)$$

As one can naturally expects, the above quantity converges to the zero rough path as $k \rightarrow \infty$.

PROPOSITION 3.4. *Let the notation be as above. Then, we have the following:*

(1) *There exists a positive constant η independent of k such that*

$$\mathbb{E}[\exp(\eta \|\mathbf{L}(w)\|_{\alpha, 4m-B}^2)] \vee \sup_{k \geq 1} \mathbb{E}[\exp(\eta \|\mathcal{L}(w(k))\|_{\alpha, 4m-B}^2)] < \infty.$$

(2) *For every $r \in [1, \infty)$ and $i = 1, 2$, $\lim_{k \rightarrow \infty} \|\mathcal{L}(w(k))^i - \mathbf{L}(w)^i\|_{i\alpha, 4m/i-B} = 0$ in $L^r(\mu)$.*

(3) *For every $r \in [1, \infty)$, $\lim_{k \rightarrow \infty} \|T_{-w(k)}\mathbf{L}(w)\|_{\alpha, 4m-B} = 0$ in $L^r(\mu)$.*

Proof. If the rough path topology is β -Hölder with $1/3 < \beta < 1/2$, these statements are proved in [19, Theorem 15.47]. Using the Besov-Hölder embedding theorem, we can easily prove this proposition, too. \square

§4. \mathcal{K} -differentiability of Lyons-Itô map

In this section we show that the rough path lift map is uniformly \mathcal{K} -regular and Lyons-Itô map is twice \mathcal{K} -regularly differentiable. Technically, this section is the core of this paper. These properties were already proved in [30] for Gaussian rough paths with respect to the p -variation topology under the condition called the complementary Young regularity. In this section, we will show these properties for Brownian rough path with respect to the Besov rough path topology and also clean up arguments in [30]. We keep the same notation as before. Let $\mathcal{K} = \{K_n\}_{n=1}^\infty$ be as in Example 2.4. We write $\bar{P}_n(w) = w(n)$ and $P_n(h) = h(n)$ for $w \in \mathcal{W}$ and $h \in \mathcal{H}$. We continue to assume (3.3) for the Besov parameter $(\alpha, 4m)$. (For the rest of this paper, we study this particular exhaustion only. We do not know what happens for a general exhaustion.)

First, we prove that the rough path lift map is \mathcal{K} -regular and so is the solution of an RDE driven Brownian rough path. Note that $\Phi \circ \mathbf{L}$ equals μ -a.s. to the solution of RDE (3.1) driven by $\mathbf{W} = \mathbf{L}(w)$.

PROPOSITION 4.1. *Let the notation be as above. Then, we have the following:*

(1) *The measurable map $\mathbf{L}: \mathcal{W} \rightarrow G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ is uniformly \mathcal{K} -regular with \mathcal{L} as its regularization.*

(2) *Let E_0 be a Polish space and $\Lambda: G\Omega_{\alpha, 4m}^B(\mathbb{R}^d) \rightarrow E_0$ is locally Lipschitz continuous. Then, $\Lambda \circ \mathbf{L}: \mathcal{W} \rightarrow E_0$ is uniformly \mathcal{K} -regular with $\Lambda \circ \mathcal{L}$ as its regularization.*

(3) If, in addition, V_i is of C_b^3 for all $0 \leq i \leq d$, then $\Phi \circ \mathbf{L}: \mathcal{W} \rightarrow \mathcal{C}^{0,\alpha-(1/4m)-H}(\mathbb{R}^e)$ is uniformly \mathcal{K} -regular with Ψ as its regularization.

For the rest of this section we use the following notation. We write $\mathcal{A} := \mathcal{Z}_{\alpha,4m}$, which was defined by (3.5). It is important that this set is of full μ -measure and invariant under the translation by $h \in \mathcal{H}$. Write $\mathbf{W}^{*n} := T_{-w(n)}\mathbf{L}(w)$. By Proposition 3.4, $\lim_{n \rightarrow \infty} \mathbf{W}^{*n} = \mathbf{0}$ in probability. We set $E' = G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$, $G = \mathbf{L}: \mathcal{W} \rightarrow E'$ and $\tilde{G} = \mathcal{L}: \mathcal{H} \rightarrow E'$. Similarly, we set $F = \Lambda \circ \mathbf{L}: \mathcal{W} \rightarrow E_0$ and $\tilde{F} = \Lambda \circ \mathcal{L}: \mathcal{H} \rightarrow E_0$. We will write $E = \mathcal{C}^{0,\alpha-(1/4m)-H}(\mathbb{R}^e)$.

Proof of Proposition 4.1. In this proof, $\epsilon > 0$ is arbitrary. Set $G_n: \mathcal{W} \times K_n \mapsto E'$ by $G_n(w, k) := T_k\mathbf{L}(w)$ if $w \in \mathcal{A}$ and $G_n(w, k) = \mathbf{0}$ if $w \notin \mathcal{A}$. Then, for all $w \in \mathcal{A}$ and $k \in K_n$, we have

$$\begin{aligned} G \circ \tau_k(w) &= \mathbf{L}(w + k) = T_k\mathbf{L}(w) = G_n(w, k), \\ G_n^\perp(w, k) &:= G_n(w, -\bar{P}_n(w) + k) = \mathbf{L}(w - \bar{P}_n(w) + k) = \mathbf{L}(\bar{P}_n^\perp(w) + k) = T_k\mathbf{W}^{*n}. \end{aligned}$$

Thanks to these explicit expressions, (i) \mathcal{K} -continuity of G is now clear and (ii) we may and will view G_n and G_n^\perp as maps from $\mathcal{W} \times \mathcal{H}$ to E' . (Then, they are actually independent of n . Note that $G_n(w, k) = \mathbf{0} = G_n^\perp(w, k)$ whenever $w \notin \mathcal{A}$.)

We will check (2.4). Take $w \in \mathcal{A}$. Note that $\tilde{G}(\bar{P}_n(w) + k) = \mathcal{L}(w(n) + k) = T_k\mathcal{L}(w(n))$ and $G_n(w, k) = T_k\mathbf{L}(w)$. Since $\mathcal{L}(w(n)) \rightarrow \mathbf{L}(w)$ as $n \rightarrow \infty$, $\{\mathcal{L}(w(n))\}_{n=1}^\infty$ is bounded in E' . Since $T: E' \times \mathcal{H} \rightarrow E'$ is locally Lipschitz continuous, we see that

$$\begin{aligned} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(\bar{P}_n(w) + k), G_n(w, k)) &= \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(T_k\mathcal{L}(w(n)), T_k\mathbf{L}(w)) \\ &\leq C_{r,w} \rho_{E'}(\mathcal{L}(w(n)), \mathbf{L}(w)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Here, $C_{r,w}$ is a positive constant which depends only on $r > 0$ and $w \in \mathcal{A}$ (and may vary from line to line). Then, it immediately follows that, for every $m \in \mathbb{N}$, $\epsilon > 0$ and $r > 0$,

$$\lim_{n \rightarrow \infty} \mu\left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(\bar{P}_n(w) + k), G_n(w, k)) \geq \epsilon\right) = 0.$$

Similarly, if $w \in \mathcal{A}$ and $\|\mathbf{W}^{*n}\| \leq 1$, then we have

$$\begin{aligned} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(h + k), G_n^\perp(w, P_n(h) + k)) &= \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(T_{k+h}\mathbf{0}, T_{k+P_n(h)}\mathbf{W}^{*n}) \\ &\leq C_{r,h} \{\rho_{E'}(\mathbf{W}^{*n}, \mathbf{0}) + \|P_n(h) - h\|_{\mathcal{H}}\} \\ &\leq C_{r,h} \{\|\mathbf{W}^{*n}\| + \|\mathbf{W}^{*n}\|^2 + \|h(n) - h\|_{\mathcal{H}}\}. \end{aligned} \tag{4.1}$$

Here, $C_{r,h}$ is a positive constant which depends only on $r > 0$, $h \in \mathcal{H}$. We can easily see from this that

$$\begin{aligned} &\mu\left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(h + k), G_n^\perp(w, P_n(h) + k)) \geq \epsilon\right) \\ &\leq \mu(\|\mathbf{W}^{*n}\| \geq \frac{\epsilon}{3C_{r,h}}) + \mu(\|\mathbf{W}^{*n}\|^2 \geq \frac{\epsilon}{3C_{r,h}}) + \mu(\|h(n) - h\|_{\mathcal{H}} \geq \frac{\epsilon}{3C_{r,h}}) + \mu(\|\mathbf{W}^{*n}\| > 1). \end{aligned}$$

The right hand side tends to zero as $n \rightarrow \infty$ for every $m \in \mathbb{N}$, $\epsilon > 0$, $h \in \mathcal{H}$ and $r > 0$. Thus, we have shown (1).

Next we show (2). Set also $F_n: \mathcal{W} \times K_n \mapsto E$ by $F_n(w, k) = \Lambda(G_n(w, k))$. Take any $w \in \mathcal{A}$ and $k \in \mathcal{H}$. It is clear that $F \circ \tau_k(w) = F_n(w, k)$. We also have $F_n^\perp(w, k) = \Lambda(T_k\mathbf{W}^{*n})$.

Again, we may and will view F_n and F_n^\perp as maps from $\mathcal{W} \times \mathcal{H}$ to E_0 . (Then, they are actually independent of n , too.)

Since Λ is locally Lipschitz continuous and both $\tilde{G}(\bar{P}_n(w) + k)$ and $G_n(w, k)$ stay bounded as $n \in \mathbb{N}$ and $k \in \mathcal{H}$ (with $\|k\|_{\mathcal{H}} \leq r$) vary, we have

$$\sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\tilde{F}(\bar{P}_n(w) + k), F_n(w, k)) \leq C_{r,w} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(\tilde{G}(\bar{P}_n(w) + k), G_n(w, k)).$$

As we have seen, the right hand tends to zero as $n \rightarrow \infty$. This implies that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\tilde{F}(\bar{P}_n(w) + k), F_n(w, k)) \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$ and $r > 0$.

If $w \in \mathcal{A}$ and $\|\mathbf{W}^{*n}\| \leq 1$, we see from the local Lipschitz continuity of Λ that

$$\begin{aligned} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\tilde{F}(h + k), F_n^\perp(w, P_n(h) + k)) &= \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\Lambda(\tilde{G}(h + k)), \Lambda(G_n^\perp(w, P_n(h) + k))) \\ &= \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\Lambda(T_{k+h}\mathbf{0}), \Lambda(T_{k+P_n(h)}\mathbf{W}^{*n})) \\ &= C_{r,h} \sup_{\|k\|_{\mathcal{H}} \leq r} \rho_{E'}(T_{k+h}\mathbf{0}, T_{k+P_n(h)}\mathbf{W}^{*n}), \end{aligned}$$

where $C_{r,h}$ is a positive constant which depends only on $r > 0$, $h \in \mathcal{H}$. Recall that we have already computed the right hand. So, we can show that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \rho_{E_0}(\tilde{F}(h + k), F_n^\perp(w, P_n(h) + k)) \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$, $h \in \mathcal{H}$ and $r > 0$ in exactly the same way as above. Thus, we have shown (2).

Finally, (3) is just a special case of (2) since Φ is locally Lipschitz continuous. Note that $\tilde{F} = \Phi \circ \mathcal{L} = \Psi: \mathcal{H} \rightarrow E$ in this case, which is the solution map of the skeleton ODE (3.2). \square

We consider derivatives of the solution map $\Psi: \mathcal{H} \rightarrow \mathcal{C}_a^{0,1/2-H}(\mathbb{R}^e) \subset \mathcal{C}^{0,1/2-H}(\mathbb{R}^e)$ of the skeleton ODE (3.2). For brevity, we write $\sigma = [V_1, \dots, V_d]$ and $b = V_0$ and view them as an $e \times d$ matrix-valued and an \mathbb{R}^e -valued function, respectively. In what follows, we assume these coefficients are of C_b^5 for simplicity. Then, (3.2) simply reads

$$dx_t = \sigma(x_t)dh_t + b(x_t)dt, \quad x_0 = a \in \mathbb{R}^e.$$

It is well-known that Ψ (i.e. $h \mapsto x = x(h)$) is Fréchet- C^2 . Moreover, its directional derivatives satisfy a simple ODE, which can be obtained as a formal differentiation of the above ODE. For example, consider $D_l x_t$ and $D_{\tilde{l}}^2 x_t$ for $l \in \mathcal{H}$, where D stands for the Fréchet derivative on \mathcal{H} and $\tilde{l} \in \mathcal{H}$ is a direction of differentiation. If those are denoted by $\xi_t^{[1]} = \xi_t^{[1]}(h; l)$ and $\xi_t^{[2]} = \xi_t^{[2]}(h; l, \tilde{l})$, their ODEs explicitly read

$$d\xi_t^{[1]} = \nabla \sigma(x_t) \langle \xi_t^{[1]}, dh_t \rangle + \nabla b(x_t) \langle \xi_t^{[1]} \rangle dt + \sigma(x_t) dl_t, \quad \xi_0^{[1]} = 0 \in \mathbb{R}^e, \quad (4.2)$$

and

$$\begin{aligned} d\xi_t^{[2]} &= \nabla\sigma(x_t)\langle\xi_t^{[2]}, dh_t\rangle + \nabla b(x_t)\langle\xi_t^{[2]}\rangle dt \\ &\quad + \nabla^2\sigma(x_t)\langle\xi_t^{[1]}(h; l), \xi_t^{[1]}(h; \tilde{l}), dh_t\rangle + \nabla\sigma(x_t)\langle\xi_t^{[1]}(h; \tilde{l}), dl_t\rangle \\ &\quad + \nabla\sigma(x_t)\langle\xi_t^{[1]}(h; l), d\tilde{l}_t\rangle + \nabla^2b(x_t)\langle\xi_t^{[1]}(h; l), \xi_t^{[1]}(h; \tilde{l})\rangle dt, \quad \xi_0^{[2]} = 0 \in \mathbb{R}^e, \end{aligned} \quad (4.3)$$

respectively. Note that both are simple first-order ODEs and therefore can be solved by the variation of constants formula for every $h, l, \tilde{l} \in \mathcal{H}$. It is also standard to show that the map

$$\mathcal{H} \times \mathcal{H} \times \mathcal{H} \ni (h, l, \tilde{l}) \mapsto (\Psi(h), \xi^{[1]}(h; l), \xi^{[2]}(h; l, \tilde{l})) \in \mathcal{C}^{0,1/2-H}((\mathbb{R}^e)^{\oplus 3})$$

is locally Lipschitz continuous.

Now we get back to RDEs. The RDE driven by $\mathbf{w} \in G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ and $l \in \mathcal{H}$ which correspond to (4.2) is given as follows:

$$d\xi_t^{[1]} = \nabla\sigma(x_t)\langle\xi_t^{[1]}, dw_t\rangle + \nabla b(x_t)\langle\xi_t^{[1]}\rangle dt + \sigma(x_t)dl_t, \quad \xi_0^{[1]} = 0 \in \mathbb{R}^e. \quad (4.4)$$

We write $\xi_t^{[1]} = \xi_t^{[1]}(\mathbf{w}; l)$ when necessary. Likewise, the RDE driven by $\mathbf{w} \in G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ and $l, \tilde{l} \in \mathcal{H}$ which correspond to (4.3) is given as follows:

$$\begin{aligned} d\xi_t^{[2]} &= \nabla\sigma(x_t)\langle\xi_t^{[2]}, dw_t\rangle + \nabla b(x_t)\langle\xi_t^{[2]}\rangle dt \\ &\quad + \nabla^2\sigma(x_t)\langle\xi_t^{[1]}(\mathbf{w}; l), \xi_t^{[1]}(\mathbf{w}; \tilde{l}), dw_t\rangle + \nabla\sigma(x_t)\langle\xi_t^{[1]}(\mathbf{w}; \tilde{l}), dl_t\rangle \\ &\quad + \nabla\sigma(x_t)\langle\xi_t^{[1]}(\mathbf{w}; l), d\tilde{l}_t\rangle + \nabla^2b(x_t)\langle\xi_t^{[1]}(\mathbf{w}; l), \xi_t^{[1]}(\mathbf{w}; \tilde{l})\rangle dt, \quad \xi_0^{[2]} = 0 \in \mathbb{R}^e. \end{aligned} \quad (4.5)$$

We write $\xi_t^{[2]} = \xi_t^{[2]}(\mathbf{w}; l, \tilde{l})$ when necessary. Since (4.4)–(4.5) are first-order RDEs, it is known that the system of three RDEs (3.1), (4.4) and (4.5) has a unique global solution for every (\mathbf{w}, l) . Moreover, a rough path version of the variation of constants formula holds for $\xi^{[1]}$ and $\xi^{[2]}$, too. (See [19, Section 10.7] and [27] for example.) If $h \in \mathcal{H}$, we have $\xi^{[1]}(h; l) = \xi^{[1]}(\mathcal{L}(h); l)$ and $\xi^{[2]}(h; l, \tilde{l}) = \xi^{[2]}(\mathcal{L}(h); l, \tilde{l})$. Since no explosion can happen, Lyons' continuity theorem still holds for this system of three RDEs. In particular, the following map is locally Lipschitz continuous:

$$\begin{aligned} G\Omega_{\alpha, 4m}^B(\mathbb{R}^d) \times \mathcal{H} \times \mathcal{H} \ni (\mathbf{w}, l, \tilde{l}) \\ \mapsto (\Phi(\mathbf{w}), \xi^{[1]}(\mathbf{w}; l), \xi^{[2]}(\mathbf{w}; l, \tilde{l})) \in \mathcal{C}^{0, \alpha - (1/4m) - H}((\mathbb{R}^e)^{\oplus 3}). \end{aligned} \quad (4.6)$$

Here, $\Phi(\mathbf{w})$ is the solution of RDE (3.1). This property will play a key role.

LEMMA 4.2. *Let $F_n: \mathcal{W} \times \mathcal{H} \rightarrow E := \mathcal{C}^{0, \alpha - (1/4m) - H}(\mathbb{R}^e)$ be as in the proof of Proposition 4.1, namely,*

$$F_n(w, k) = \begin{cases} \Phi(T_k \mathbf{L}(w)) & (\text{if } w \in \mathcal{A} \text{ and } k \in \mathcal{H}), \\ \Phi(\mathbf{0}) & (\text{if } w \notin \mathcal{A} \text{ and } k \in \mathcal{H}). \end{cases}$$

Then, for each $w \in \mathcal{W}$, $F_n(w, \bullet): \mathcal{H} \rightarrow E$ is of Fréchet- C^2 . Moreover, for each $w \in \mathcal{A}$, we have

$$D_l F_n(w, k) = \xi^{[1]}(T_k \mathbf{L}(w); l), \quad D_{l, \tilde{l}}^2 F_n(w, k) = \xi^{[2]}(T_k \mathbf{L}(w); l, \tilde{l}), \quad k, l, \tilde{l} \in \mathcal{H}.$$

Here, D is the Fréchet derivative acting on the k -variable.

Proof. As is well-known, Fréchet- C^j and Gâteaux- C^j are equivalent for all $j \geq 1$. So, we only consider Gâteaux derivatives. The case $w \notin \mathcal{A}$ is obvious. So, we pick any $w \in \mathcal{A}$ and will fix it in what follows.

First, we calculate the first-order derivative in the direction l . For $m \in \mathbb{N}$, $w(m) \in \mathcal{H}$ and therefore we already know that $\lim_{m \rightarrow \infty} F_n(w(m), k) = F_n(w, k)$ and

$$D_l F_n(w(m), k) = \xi^{[1]}(T_k \mathcal{L}(w(m)); l).$$

The right hand converges to $\xi^{[1]}(T_k \mathbf{L}(w); l)$ in E as $m \rightarrow \infty$ uniformly on every ball of \mathcal{H} . Indeed, if $\|k\|_{\mathcal{H}} \leq r$ and $\|l\|_{\mathcal{H}} \leq r'$ ($r, r' > 0$), then by the local Lipschitz continuity in (4.6) and that of T , we have

$$\|\xi^{[1]}(T_k \mathcal{L}(w(m)); l) - \xi^{[1]}(T_k \mathbf{L}(w); l)\|_E \leq C_{r, r', w} \rho_{E'}(\mathcal{L}(w(m)), \mathbf{L}(w)) \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

where $E' := G\Omega_{\alpha, 4m}^B(\mathbb{R}^d)$ and $C_{r, r', w} > 0$ is constant which depends only on r, r', w .

This uniform convergence in k (for a fixed l) yields the desired formula for the first derivative. This can be verified as follows. For fixed k and l , set $\chi_m(\tau) = F_n(w(m), k + \tau l)$, $\tau \in (-1, 1)$. Obviously, $\lim_{m \rightarrow \infty} \chi_m(\tau) = F_n(w, k + \tau l)$ for every τ and $(\partial/\partial \tau)\chi_m(\tau) = \xi^{[1]}(T_{k+\tau l} \mathcal{L}(w(m)); l)$. As we have seen, $(\partial/\partial \tau)\chi_m(\tau)$ converges to $\xi^{[1]}(T_{k+\tau l} \mathbf{L}(w); l)$ uniformly in $\tau \in (-1, 1)$. Hence, we have $(\partial/\partial \tau)F_n(w, k + \tau l) = \xi^{[1]}(T_{k+\tau l} \mathbf{L}(w); l)$. Setting $\tau = 0$, we obtain the formula.

By a similar argument we can show the continuity of $k \mapsto DF_n(w, k)$ as follows: If $\|k\|_{\mathcal{H}}, \|\tilde{k}\|_{\mathcal{H}} \leq r$ ($r > 0$), then by the local Lipschitz continuity we have

$$\begin{aligned} \|DF_n(w, k) - DF_n(w, \tilde{k})\|_{\mathcal{H} \rightarrow E} &= \sup_{\|l\| \leq 1} \|D_l F_n(w, k) - D_l F_n(w, \tilde{k})\|_E \\ &= \|\xi^{[1]}(T_k \mathbf{L}(w); l) - \xi^{[1]}(T_{\tilde{k}} \mathbf{L}(w); l)\|_E \\ &\leq C_{r, w} \|k - \tilde{k}\|_{\mathcal{H}}, \end{aligned} \tag{4.7}$$

where $\|\cdot\|_{\mathcal{H} \rightarrow E}$ is the operator norm for bounded operators from \mathcal{H} to E and $C_{r, w} > 0$ is constant which depends only on r, w . Thus, we have seen that $F_n(w, \bullet): \mathcal{H} \rightarrow E$ is of C^1 .

Starting with the known fact that

$$D_{\tilde{l}}^2 F_n(w(m), k) = D_{\tilde{l}} \xi^{[1]}(T_k \mathcal{L}(w(m)); l) = \xi^{[2]}(T_k \mathcal{L}(w(m)); l, \tilde{l}),$$

we can calculate the second-order derivative, too. Since the proof is essentially the same as in the first-order case, we omit it. \square

The following is the main result of this section. It immediately implies that Itô map $w \mapsto \Phi(\mathbf{L}(w))_t$ at a fixed time t is also twice \mathcal{K} -regularly differentiable.

PROPOSITION 4.3. *Let the notation be as above and assume that V_i is of C_b^5 for all $0 \leq i \leq d$. Then, $\Phi \circ \mathbf{L}: \mathcal{W} \rightarrow E := \mathcal{C}^{0, \alpha - (1/4m) - H}(\mathbb{R}^e)$ is twice \mathcal{K} -regularly differentiable with Ψ as its regularization.*

Proof. We use the same notation as in the proof of Proposition 4.1 (2). An unimportant positive constant which depends only on the parameter \star is denoted by C_\star , which may vary from line to line.

We will prove (2.5) for $l = 2$ by estimating

$$\|\tilde{F}(\bar{P}_n(w) + \bullet) - F_n(w, \bullet)\|_{C_b^2(B_{\mathcal{H}}(0, r), E)} \vee \|\tilde{F}(h + \bullet) - F_n^\perp(w, P_n(h) + \bullet)\|_{C_b^2(B_{\mathcal{H}}(0, r), E)}$$

for every $w \in \mathcal{A}$, $r > 0$ and $h \in \mathcal{H}$. Convergence of the zeroth order was already shown in the proof of Proposition 4.1 (2).

Now we calculate the first order derivatives. For the rest of the proof, let $r, r' > 0$, $w \in \mathcal{A}$, $k, l, h \in \mathcal{H}$. $D_l F_n(w, \bullet)$ was calculated in Lemma 4.2 above. Since $\tilde{F} = \Psi$, we have

$$D_l \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} = \xi^{[1]}(w(n) + k; l) = \xi^{[1]}(T_k \mathcal{L}(w(n)); l).$$

Due to the local Lipschitz continuity of $\xi^{[1]}$ we mentioned in (4.6), we have that, if $\|k\|_{\mathcal{H}} \leq r$, then we have

$$\begin{aligned} & \sup_{\|k\|_{\mathcal{H}} \leq r} \|D \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} - D F_n(w, \bullet)|_{\bullet=k}\|_{\mathcal{H} \rightarrow E} \\ &= \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \leq 1} \|D_l \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} - D_l F_n(w, \bullet)|_{\bullet=k}\|_E \\ &= \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \leq 1} \|\xi^{[1]}(T_k \mathcal{L}(w(n)); l) - \xi^{[1]}(T_k \mathbf{L}(w); l)\|_E \\ &\leq C_{r,w} \rho_{E'}(\mathcal{L}(w(n)), \mathbf{L}(w)) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

for every $w \in \mathcal{A}$ and $r > 0$. Then, it immediately follows that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \|D \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} - D F_n(w, \bullet)|_{\bullet=k}\|_{\mathcal{H} \rightarrow E} \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$ and $r > 0$.

Since $F_n^\perp(w, k) = F_n(w, -w(n) + k)$, we see that

$$D_l F_n^\perp(w, \bullet)|_{\bullet=k} = \xi^{[1]}(T_{k-w(n)} \mathbf{L}(w); l) = \xi^{[1]}(T_k \mathbf{W}^{*n}; l).$$

Hence, if $h \in \mathcal{H}$, $w \in \mathcal{A}$ and $\|\mathbf{W}^{*n}\| \leq 1$, then

$$\begin{aligned} & \sup_{\|k\|_{\mathcal{H}} \leq r} \|D \tilde{F}(h + \bullet)|_{\bullet=k} - D F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_{\mathcal{H} \rightarrow E} \\ &= \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \leq 1} \|D_l \tilde{F}(h + \bullet)|_{\bullet=k} - D_l F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_E \\ &\leq \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \leq 1} \|\xi^{[1]}(T_{h+k} \mathbf{0}; l) - \xi^{[1]}(T_{k+P_n(h)} \mathbf{W}^{*n}; l)\|_E \\ &\leq C_{r,h} \{\rho_{E'}(\mathbf{W}^{*n}, \mathbf{0}) + \|h - h(n)\|_{\mathcal{H}}\} \end{aligned}$$

for every $r > 0$ and $h \in \mathcal{H}$. Recall that the right hand side is essentially the same as the right hand side of (4.1) in the proof of Proposition 4.1. So, we can show in the same way that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \|D \tilde{F}(h + \bullet)|_{\bullet=k} - D F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_{\mathcal{H} \rightarrow E} \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$, $r > 0$ and $h \in \mathcal{H}$.

Next we calculate the second order derivatives. We can easily see that

$$D_{l,\tilde{l}}^2 \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} = \xi^{[2]}(w(n) + k; l, \tilde{l}) = \xi^{[2]}(T_k \mathcal{L}(w(n)); l, \tilde{l}).$$

Due to Lemma 4.2 and the local Lipschitz continuity of $\xi^{[2]}$ we mentioned in (4.6), we have the following:

$$\begin{aligned}
& \sup_{\|k\|_{\mathcal{H}} \leq r} \|D^2 \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} - D^2 F_n(w, \bullet)|_{\bullet=k}\|_{\mathcal{H} \times \mathcal{H} \rightarrow E} \\
&= \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \vee \|\hat{l}\|_{\mathcal{H}} \leq 1} \|\xi^{[2]}(T_k \mathcal{L}(w(n)); l, \hat{l}) - \xi^{[2]}(T_k \mathbf{L}(w); l, \hat{l})\|_E \\
&\leq C_{r,w} \rho_{E'}(\mathcal{L}(w(n)), \mathbf{L}(w)) \rightarrow 0 \quad \text{as } n \rightarrow \infty
\end{aligned} \tag{4.8}$$

for every $w \in \mathcal{A}$ and $r > 0$. Here, $\|\cdot\|_{\mathcal{H} \times \mathcal{H} \rightarrow E}$ is the standard norm for bounded bilinear maps from $\mathcal{H} \times \mathcal{H}$ to E . From this we see that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \|D^2 \tilde{F}(\bar{P}_n(w) + \bullet)|_{\bullet=k} - D^2 F_n(w, \bullet)|_{\bullet=k}\|_{\mathcal{H} \times \mathcal{H} \rightarrow E} \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$ and $r > 0$.

In a similar way as above, we see that $D_{l,\hat{l}}^2 F_n^\perp(w, \bullet)|_{\bullet=k} = \xi^{[2]}(T_k \mathbf{W}^{*n}; l, \hat{l})$. Hence, if $h \in \mathcal{H}$, $w \in \mathcal{A}$ and $\|\mathbf{W}^{*n}\| \leq 1$, then we have

$$\begin{aligned}
& \sup_{\|k\|_{\mathcal{H}} \leq r} \|D^2 \tilde{F}(h + \bullet)|_{\bullet=k} - D^2 F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_{\mathcal{H} \times \mathcal{H} \rightarrow E} \\
&= \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \vee \|\hat{l}\|_{\mathcal{H}} \leq 1} \|D_{l,\hat{l}}^2 \tilde{F}(h + \bullet)|_{\bullet=k} - D_{l,\hat{l}}^2 F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_E \\
&\leq \sup_{\|k\|_{\mathcal{H}} \leq r} \sup_{\|l\|_{\mathcal{H}} \vee \|\hat{l}\|_{\mathcal{H}} \leq 1} \|\xi^{[2]}(T_{h+k} \mathbf{0}; l, \hat{l}) - \xi^{[2]}(T_{k+P_n(h)} \mathbf{W}^{*n}; l, \hat{l})\|_E \\
&\leq C_{r,h} \{\rho_{E'}(\mathbf{W}^{*n}, \mathbf{0}) + \|h - h(n)\|_{\mathcal{H}}\}
\end{aligned}$$

for every $r > 0$. As we have seen, this implies again that

$$\lim_{n \rightarrow \infty} \mu \left(\sup_{k \in K_m, \|k\|_{\mathcal{H}} \leq r} \|D^2 \tilde{F}(h + \bullet)|_{\bullet=k} - D^2 F_n^\perp(w, P_n(h) + \bullet)|_{\bullet=k}\|_{\mathcal{H} \times \mathcal{H} \rightarrow E} \geq \epsilon \right) = 0$$

for every $m \in \mathbb{N}$, $\epsilon > 0$, $r > 0$ and $h \in \mathcal{H}$. This completes the proof. \square

§5. Support theorem on geometric rough path space

In this section we consider SDE (3.4) with C_b^∞ -vector fields V_i ($0 \leq i \leq d$). When we emphasize the starting point a , we write $X_t = X(t, a)$. Similarly, the corresponding Lyons-Itô map is denoted by $\Phi^a = \Phi$. (Similarly, the deterministic Itô map for the skeleton ODE is denoted by $\Psi^a = \Psi$.) Recall that $\Phi^a(\mathbf{L}(w))$ is an ∞ -quasi-continuous modification of $X(\cdot, a)$ for every a . We continue to assume (3.3) for the Besov parameter $(\alpha, 4m)$. The diffusion semigroup associated with this SDE is denoted by $(T_t)_{0 \leq t \leq 1}$, that is, $T_t f(a) := \mathbb{E}[X(t, a)]$ for every bounded continuous function $f: \mathbb{R}^e \rightarrow \mathbb{R}$.

Let \mathcal{V} be a linear subspace of \mathbb{R}^e with dimension e' ($1 \leq e' \leq e$). The inner product of \mathcal{V} is a restriction of that of \mathbb{R}^e and therefore the Lebesgue measure on \mathbb{R}^e is uniquely determined. The orthogonal projection from \mathbb{R}^e onto \mathcal{V} is denoted by Π . We are interested in the law of $Y(\cdot, a) := \Pi(X(\cdot, a))$, in particular, when it is pinned at $b \in \mathcal{V}$ at $t = 1$. It is well-known that $Y(t, a)$ is a \mathbf{D}_∞ -Wiener functional for every t and a .

Suppose that Malliavin covariance of $Y(1, a)$ is non-degenerate. Then, $\delta_b \circ Y(1, a) = \delta_b(Y(1, a)) \in \tilde{\mathbf{D}}_{-\infty}$ is well-defined as a positive Watanabe distribution. By Sugita's theorem (see [50] or [36, p.101]), $\delta_b(Y(1, a))$ corresponds to a unique finite Borel measure on \mathcal{W} ,

which is denoted by $\hat{\mu}_{a,b}$. The correspondence is given by

$$\mathbb{E}[G \delta_b(Y(1, a))] = \int_{\mathcal{W}} \tilde{G}(w) \hat{\mu}_{a,b}(dw), \quad G \in \mathbf{D}_\infty,$$

where \tilde{G} is an ∞ -quasi-continuous modification of G . If the total mass $\mathbb{E}[\delta_b(Y(1, a))] < \infty$ is positive, then $\mu_{a,b} := \mathbb{E}[\delta_b(Y(1, a))]^{-1} \hat{\mu}_{a,b}$ is a probability measure. By Theorem 2.2 and Proposition 4.3, $\mathbb{E}[\delta_b(Y(1, a))] > 0$ if and only if

$$\{h \in \mathcal{H}: D\Psi^a(h)_1: \mathcal{H} \rightarrow \mathcal{V} \text{ is surjective, } \Pi\Psi^a(h)_1 = b\} \neq \emptyset. \quad (5.1)$$

(Here, we used Theorem 2.2 with $G \equiv 1$, $F(w) = Y(1, a) = \Pi\Phi^a(w)_1$ and $\tilde{F}(h) = \Pi\Psi^a(h)_1$.) This measure does not charge a slim set. So, its rough path lift $\mathbf{L}_* \mu_{a,b} = \mu_{a,b} \circ \mathbf{L}^{-1}$ is a Borel probability measure on $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$, which will be denoted by $\nu_{a,b}$.

THEOREM 5.1. *Let the notation and the situation be as above. Assume (5.1) and non-degeneracy of $Y(1, a)$. Then, the support of $\nu_{a,b}$ equals the closure of*

$$\{\mathcal{L}(h): h \in \mathcal{H}, D\Psi^a(h)_1: \mathcal{H} \rightarrow \mathcal{V} \text{ is surjective, } \Pi\Psi^a(h)_1 = b\} \quad (5.2)$$

in $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$.

Proof. We use Aida-Kusuoka-Stroock's positivity theorem (Theorem 2.2). Let $\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2) \in G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$. For $r > 0$, we set

$$B(\mathbf{z}, r) := \{\mathbf{w} \in G\Omega_{\alpha,4m}^B(\mathbb{R}^d): \|\mathbf{w}^1 - \mathbf{z}^1\|_{\alpha,4m-B}^{4m} + \|\mathbf{w}^2 - \mathbf{z}^2\|_{2\alpha,2m-B}^{2m} < r^{4m}\}.$$

Then, $\{B(\mathbf{z}, r)\}_{r>0}$ forms a fundamental system of open neighborhood around \mathbf{z} . Let $\chi: [0, \infty) \rightarrow [0, 1]$ be a non-increasing C^∞ -function such that $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $[2^{4m}, \infty)$. Set a non-negative \mathbf{D}_∞ -functional by

$$G_{\mathbf{z},r}(w) := \chi\left(\frac{\|\mathbf{L}(w)^1 - \mathbf{z}^1\|_{\alpha,4m-B}^{4m} + \|\mathbf{L}(w)^2 - \mathbf{z}^2\|_{2\alpha,2m-B}^{2m}}{r^{4m}}\right).$$

It is easy to see from Proposition 4.1 that $G_{\mathbf{z},r}$ is uniformly \mathcal{K} -regular. Obviously,

$$\mathbf{1}_{B(\mathbf{z},r)}(\mathbf{L}(w)) \leq G_{\mathbf{z},r}(w) \leq \mathbf{1}_{B(\mathbf{z},2r)}(\mathbf{L}(w)), \quad \text{for quasi-every } w \in \mathcal{W}. \quad (5.3)$$

As usual, $\mathbf{1}_C$ denotes the indicator function of a subset C .

First, we show that $\bar{A} \subset \text{supp}(\nu_{a,b})$, where A stands for the set in (5.2). To do so, it suffices to show that $\nu_{a,b}(B(\mathcal{L}(h), 2r)) > 0$ for every $\mathcal{L}(h) \in A$ and $r > 0$. By (5.3) we have

$$\begin{aligned} \nu_{a,b}(B(\mathcal{L}(h), 2r)) &= \int_{\mathcal{W}} \mathbf{1}_{B(\mathcal{L}(h), 2r)}(\mathbf{L}(w)) \mu_{a,b}(dw) \\ &\geq \int_{\mathcal{W}} G_{\mathcal{L}(h),r}(w) \mu_{a,b}(dw) = \mathbb{E}[G_{\mathcal{L}(h),r} \delta_b(Y(1, a))]/Z. \end{aligned} \quad (5.4)$$

Here, we used ∞ -quasi-continuity of G and wrote $Z := \mathbb{E}[\delta_b(Y(1, a))] > 0$ for simplicity. Recall that $Y(1, a)$ is non-degenerate by assumption and twice \mathcal{K} -regularly differentiable (with its regularization $\Pi\Psi^a(\cdot)_1$) by Proposition 4.3. Therefore, we can use Theorem 2.2 to the right hand side of (5.4) is positive for every $r > 0$. (Note that h itself satisfies the second one of the two equivalent conditions in Theorem 2.2.)

Next, we show that $\bar{A} \supset \text{supp}(\nu_{a,b})$. It is sufficient to show that for every $\mathbf{z} \notin \bar{A}$, there exists $r > 0$ such that $\nu_{a,b}(B(\mathbf{z}, r)) = 0$. Then, by a similar argument as above, we have

$$\begin{aligned} \nu_{a,b}(B(\mathbf{z}, r)) &= \int_{\mathcal{W}} \mathbf{1}_{B(\mathbf{z}, r)}(\mathbf{L}(w)) \mu_{a,b}(dw) \\ &\leq \int_{\mathcal{W}} G_{\mathbf{z}, r}(w) \mu_{a,b}(dw) = \mathbb{E}[G_{\mathbf{z}, r} \delta_b(Y(1, a))]/Z. \end{aligned} \quad (5.5)$$

Since \bar{A} is closed, we can find $r > 0$ such that $B(\mathbf{z}, 2r) \cap \bar{A} = \emptyset$. By (5.3), $G_{\mathbf{z}, r}$ vanishes if $\mathbf{L}(w) \notin B(\mathbf{z}, 2r)$. Hence, we cannot find $h \in \mathcal{H}$ that satisfies the second one of the two equivalent conditions in Theorem 2.2, which implies that the right hand side of (5.5) vanishes. \square

Under the conditions of Theorem 5.1, we study the law of the process $\tilde{Y}(\cdot, a)$, an ∞ -quasi-continuous modification of $Y(\cdot, a)$, under the probability measure $\mu_{a,b}$. Heuristically, it is the law of “ $Y(\cdot, a)$ conditioned $Y(1, a) = b$.” Since $\tilde{Y}(\cdot, a) = \Pi\Phi^a(\mathbf{L}(w))$, the above law equals the law of $\Pi \circ \Phi^a$ under $\nu_{a,b}$.

Let us make sure that this law actually sits on $C_{\Pi(a), b}^{0, \beta\text{-H}}(\mathcal{V})$ for every $1/3 < \beta < 1/2$. Choose α and m so that $\beta \leq \alpha - (4m)^{-1}$. It is sufficient to show that the end point is almost surely b . For every $g \in C_0^\infty(\mathcal{V})$, we have

$$\begin{aligned} \int_{\mathcal{W}} g(\tilde{Y}(1, a)) \mu_{a,b}(dw) &= \mathbb{E}[g(Y(1, a)) \delta_b(Y(1, a))]/Z \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[g(Y(1, a)) \psi_n(Y(1, a))]/Z \\ &= \lim_{n \rightarrow \infty} \int_{\mathcal{V}} g(y) \psi_n(y) q(y) dy / Z \\ &= g(b)q(b)/Z = g(b). \end{aligned}$$

This implies that the law of $\tilde{Y}(1, a)$ under $\mu_{a,b}$ is the point mass at b . Here, (i) $Z := \mathbb{E}[\delta_b(Y(1, a))] > 0$ is the normalizing constant, (ii) $q(y)$ is the smooth density (with respect to the Lebesgue measure dy on \mathcal{V}) of the law of $Y(1, a)$ under the Wiener measure, and (iii) $\{\psi_n\}_{n=1}^\infty \subset C_0^\infty(\mathcal{V})$ is any sequence that converges to δ_b in the space of Schwartz distributions on \mathcal{V} . Note that $Z = q(b)$ due to Item (c) in Subsection 2.1.

By a similar argument as above, we can see that the finite dimensional distribution of this law is uniquely determined by the following formula: For every $k \in \mathbb{N}$, $\{0 < t_1 < \dots <$

$t_k < 1\}$ and $g_1, \dots, g_k \in C_0^\infty(\mathcal{V})$, it holds that

$$\begin{aligned}
& \int_W \prod_{i=1}^k g_i(\tilde{Y}(t_i, a)) \mu_{a,b}(dw) \\
&= \int_W \prod_{i=1}^k g_i(\Pi \tilde{X}(t_i, a)) \mu_{a,b}(dw) \\
&= Z^{-1} \mathbb{E} \left[\prod_{i=1}^k g_i(\Pi X(t_i, a)) \delta_b(Y(1, a)) \right] \\
&= Z^{-1} \lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{i=1}^k g_i(\Pi X(t_i, a)) \psi_n(Y(1, a)) \right] \\
&= Z^{-1} \lim_{n \rightarrow \infty} [T_{t_1}(g_1 \circ \Pi) T_{t_2-t_1} \cdots (g_{k-1} \circ \Pi) T_{t_k-t_{k-1}}(g_k \circ \Pi) T_{1-t_k}(\psi_n \circ \Pi)](a). \quad (5.6)
\end{aligned}$$

Here, $\{\psi_n\}$ is the same as above. Note that $g_i \circ \Pi$ ($1 \leq i \leq k$) on the right hand side of (5.6) are viewed as multiplication operators. Note also that the limit above exists and is independent of the choice of $\{\psi_n\}$.

COROLLARY 5.2. *Let the notation and the situation be as above. Assume (5.1) and non-degeneracy of $Y(1, a)$. Then, the support of the law of the process $\tilde{Y}(\cdot, a)$ under $\mu_{a,b}$ equals the closure of*

$$\{\Pi \Psi^a(h) : h \in \mathcal{H}, \ D\Psi^a(h)_1 : \mathcal{H} \rightarrow \mathcal{V} \text{ is surjective, } \Pi \Psi^a(h)_1 = b\} \quad (5.7)$$

in $\mathcal{C}_{\Pi(a),b}^{0,\beta-H}(\mathcal{V})$ for every $1/3 < \beta < 1/2$.

Proof. The set in (5.7) is denoted by B . The set in (5.2) is denoted by A again. Note that $\Pi \Psi^a(h) = \Pi \Phi^a(\mathcal{L}(h))$ for every $h \in \mathcal{H}$. If $\beta \leq \alpha - (4m)^{-1}$, then $\Pi \Phi^a = \Pi \circ \Phi^a : G\Omega_{\alpha,4m}^B(\mathbb{R}^d) \rightarrow \mathcal{C}_{\Pi(a),b}^{0,\beta-H}(\mathcal{V})$ is continuous. Thanks to this continuity, the proof is quite simple and straightforward.

If $\Pi \Psi^a(h) \in B$, then its inverse image by $\Pi \Phi^a$ clearly intersects with A . By the continuity, the inverse image of every open neighborhood of $\Pi \Psi^a(h) \in B$ is an open subset of $G\Omega_{\alpha,4m}^B(\mathbb{R}^d)$ that intersects with A and therefore its weight is strictly positive by Proposition 5.1. This implies that B is included in the support. So is \bar{B} since the support is closed by definition.

Finally, we show that this inclusion cannot be strict by showing $\bar{B}^c = (\bar{B})^c$ is of measure zero. It is clear $(\Pi \Phi^a)^{-1}(\bar{B})$ and $(\Pi \Phi^a)^{-1}(\bar{B}^c)$ do not intersect. By the continuity, the former is closed, while the latter is open. Since $(\Pi \Phi^a)^{-1}(\bar{B}) \supset A$, the support of $\nu_{a,b}$ is included by $(\Pi \Phi^a)^{-1}(\bar{B})$ by Theorem 5.1. Hence, $\nu_{a,b}((\Pi \Phi^a)^{-1}(\bar{B})) = 1$. This completes the proof. \square

Now we consider the special case $\mathcal{V} = \mathbb{R}^e$ (therefore Π is the identity map) and $X(t, a)$ is non-degenerate in the sense of Malliavin for every $a \in \mathbb{R}^e$ and $t \in (0, 1]$. Then, the law of $X(t, a)$ has a density with respect to the Lebesgue measure db , which is denoted by $p(t, a, b)$, that is, $\mu(X(t, a) \in A) = \int_A p(t, a, b) db$ for every Borel subset $A \subset \mathbb{R}^e$. In this case, the law of the process in Corollary 5.2 is identical to the classical pinned diffusion measure $\mathbb{Q}_{a,b}$

associated with SDE (3.4). Indeed, the right hand side of (5.6) reads:

$$p(1, a, b)^{-1} \int_{(\mathbb{R}^e)^k} \left\{ \prod_{i=1}^k g_i(b_i) \right\} p(t_1, a, b_1) \left\{ \prod_{i=2}^k p(t_i - t_{i-1}, b_{i-1}, b_i) \right\} p(1 - t_k, b_k, b) \left\{ \prod_{i=1}^k db_i \right\}.$$

This is the finite dimensional distribution of the classical pinned diffusion measure from $a \in \mathbb{R}^e$ to $b \in \mathbb{R}^e$. Note that our argument automatically shows the existence of $\mathbb{Q}_{a,b}$. As a special case of the above corollary, we then have the following:

COROLLARY 5.3. *Let the notation and the situation be as above. Assume (5.1) (with Π being the identity map of \mathbb{R}^e) and non-degeneracy of $X(t, x)$ for all $x \in \mathbb{R}^e$ and $t \in (0, 1]$. Then, the support of $\mathbb{Q}_{a,b}$ equals the closure of*

$$\{\Psi^a(h) : h \in \mathcal{H}, D\Psi^a(h)_1 : \mathcal{H} \rightarrow \mathbb{R}^e \text{ is surjective, } \Psi^a(h)_1 = b\}$$

in $\mathcal{C}_{a,b}^{0,\beta-H}(\mathbb{R}^e)$ for every $1/3 < \beta < 1/2$.

REMARK 5.4. In this remark we provide two typical sufficient conditions for non-degeneracy of $Y(t, a)$. Both are bracket-generating conditions of Hörmander-type.

Let V_i ($0 \leq i \leq d$) be the coefficient vector fields of SDE (3.4). In this remark they are viewed as first-order differential operators on \mathbb{R}^e . Set $\Sigma_1 = \{V_1, \dots, V_d\}$ and, recursively, $\Sigma_k = \{[Z, V_i] : Z \in \Sigma_{k-1}, 0 \leq i \leq d\}$ for $k \geq 2$.

- (A) If $\{Z(a) : Z \in \cup_{k=1}^\infty \Sigma_k\}$ linearly spans \mathbb{R}^e at the starting point a , then for all $t \in (0, 1]$ $X(t, a)$ is non-degenerate and therefore so is $Y(t, a)$. This fact is well-known. (See [45, Section 2.3] or [24, Section V.10] for example.)
- (B) Suppose the following uniform partial Hörmander condition: There exists $L > 0$ such that

$$\inf_{a \in \mathbb{R}^e} \inf_{\eta \in \mathcal{V}, |\eta|=1} \sum_{k=1}^L \sum_{Z \in \Sigma_k} \langle \Pi Z(a), \eta \rangle^2 > 0.$$

Then, according to [33, Theorem 2.17 and Lemma 5.1], $Y(t, a)$ is non-degenerate in the sense of Malliavin calculus for every $a \in \mathbb{R}^e$ and $t \in (0, 1]$.

EXAMPLE 5.5. We provide some examples of the process $Y(\cdot, a)$ in Corollary 5.2 (except that in Corollary 5.3).

- Assume Condition (A) in Remark 5.4 and $\mathcal{V} = \mathbb{R}^e$. Then, the solution $X(\cdot, a)$ of SDE (3.4) satisfies Corollary 5.2. In this case, the density $p(t, z, z')$ may not exist if $z \in \mathbb{R}^e$ is distant from a . Therefore, it is not clear whether the pinned diffusion measure in the usual sense exist or not. Since our method is based on quasi-sure analysis, we can deal with this kind of situation (without any additional efforts), too.
- For $1 \leq e' < e$, set $\mathcal{V} = \mathbb{R}^{e'} \cong \mathbb{R}^{e'} \oplus \{\mathbf{0}_{e-e'}\} \subset \mathbb{R}^e$. Here, $\mathbf{0}_{e-e'}$ is the zero vector of $\mathbb{R}^{e-e'}$. If we write $X_t = (X_t^1, \dots, X_t^e)$, then $Y_t = \Pi X_t = (X_t^1, \dots, X_t^{e'})$. This kind of projected process are sometimes studied. For example, in [13, 14, 53], small noise problems for the density of Y_t are studied. Therefore, it looks natural to study the pinned process conditioned by $Y_1 = b$. (The Markov property is lost after the projection in general. So, it cannot be called a “pinned diffusion process.”)
- Assume that $V_i(t, x) : [0, 1] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ extend to C_b^∞ -maps on an open neighborhood of $[0, 1] \times \mathbb{R}^e \subset \mathbb{R}^{e+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^e\}$ ($0 \leq i \leq d$). We extend them to C_b^∞ -maps on

\mathbb{R}^{e+1} (which will be denoted by the same symbols) and view them as time-dependent vector fields on \mathbb{R}^e . Instead of (3.4), we now consider the following time-dependent SDE:

$$d\tilde{X}_t = \sum_{i=1}^d V_i(t, \tilde{X}_t) \circ dw_t^i + V_0(t, \tilde{X}_t)dt, \quad \tilde{X}_0 = a \in \mathbb{R}^e.$$

Define $\Sigma_k(t)$ in the same way as in Remark 5.4 by just replacing V_i ($0 \leq i \leq d$) by $V_i(t, \cdot)$ ($0 \leq i \leq d$). Some examples of bracket-generating condition sufficient for the non-degeneracy of $\tilde{X}_t = \tilde{X}(t, a)$ can be found in [16, 54] among others. If we set $X_t = (X_t^0, \tilde{X}_t)$ with $X_t^0 \equiv t$, then X satisfies the following SDE on \mathbb{R}^{e+1} :

$$dX_t = \sum_{i=1}^d \hat{V}_i(X_t) \circ dw_t^i + \hat{V}_0(X_t)dt, \quad X_0 = (0, a) \in \mathbb{R}^{e+1}.$$

Here, we set $\hat{V}_0 := V_0 + (\partial/\partial t)$, $\hat{V}_i := V_i$ for $1 \leq i \leq d$ and view them as vector fields on \mathbb{R}^{e+1} . Since $\Pi(X_t) = \tilde{X}_t$ for the canonical projection $\Pi: \mathbb{R}^{e+1} \rightarrow \mathbb{R}^e$, the process $\tilde{X}(\cdot, a)$ satisfies the assumptions of Corollary 5.2.

As one can easily see, our support theorem for pinned cases looks clearly different from the standard version of the support theorem because of the two conditions on the skeleton ODE. The first one, which is quite easy for everyone to guess, requires the solution of the skeleton ODE to end at the given point. The second one requires the tangent map of the solution map of the skeleton ODE (at time 1) to be non-degenerate. This may look a little bit surprising to some readers, but is actually quite natural from the viewpoint of positivity theorems for the densities for SDEs.

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REFERENCES

- [1] S. Aida, *Vanishing of one-dimensional L^2 -cohomologies of loop groups*, J. Funct. Anal. **261** (2011), no. 8, 2164–2213.
- [2] S. Aida, *Rough differential equations containing path-dependent bounded variation terms*, Preprint (2016), arXiv:1608.03083.
- [3] S. Aida, S. Kusuoka and D. Stroock, *On the support of Wiener functionals*, Asymptotic problems in probability theory: Wiener functionals and asymptotics (Sanda/Kyoto, 1990), 3–34, Pitman Res. Notes Math. Ser. **284**, Longman Sci. Tech., Harlow, 1993.
- [4] V. Bally, A. Millet and M. Sanz-Solé, *Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations*, Ann. Probab. **23** (1995), no. 1, 178–222.
- [5] G. Ben Arous, M. Grădinaru and M. Ledoux, *Hölder norms and the support theorem for diffusions*, Ann. Inst. H. Poincaré Probab. Statist. **30** (1994), no. 1, 415–436.
- [6] G. Ben Arous and R. Léandre, *Hölder norms and the support theorem for diffusions*, Probab. Theory Related Fields **90** (1991), no. 3, 377–402.
- [7] H. Boedihardjo, X. Geng, X. Liu and Z. Qian, *A quasi-sure non-degeneracy property for the Brownian rough path*, Potential Anal. **51** (2019), no. 1, 1–21.
- [8] H. Boedihardjo, X. Geng and Z. Qian, *Quasi-sure existence of Gaussian rough paths and large deviation principles for capacities*, Osaka J. Math. **53** (2016), no. 4, 941–970.
- [9] T. Cass, G. dos Reis and W. Salkeld, *Rough functional quantization and the support of McKean-Vlasov equations*, Preprint (2019), arXiv: 1911.01992.
- [10] K. Chouk and P. K. Friz, *Support theorem for a singular SPDE: the case of gPAM*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 1, 202–219.
- [11] R. Cont and A. Kalinin, *On the support of solutions to stochastic differential equations with path-dependent coefficients*, Stochastic Process. Appl. **130** (2020), no. 5, 2639–2674.

- [12] S. Dereich and G. Dimitroff, *A support theorem and a large deviation principle for Kunita flows*, Stoch. Dyn. **12** (2012), no. 3, 1150022, 16 pp.
- [13] J. D. Deuschel, P. K. Friz, A. Jacquier, and S. Violante, *Marginal density expansions for diffusions and stochastic volatility I: Theoretical foundations*, Comm. Pure Appl. Math. **67** (2014), no. 1, 40–82.
- [14] J. D. Deuschel, P. K. Friz, A. Jacquier, and S. Violante, *Marginal density expansions for diffusions and stochastic volatility II: Applications*, Comm. Pure Appl. Math. **67** (2014), no. 2, 321–350.
- [15] H. Doss and P. Priouret, *Support d'un processus de réflexion*, Z. Wahrsch. Verw. Gebiete **61** (1982), no. 3, 327–345.
- [16] P. Florchinger, *Malliavin calculus with time dependent coefficients and application to nonlinear filtering*, Probab. Theory Related Fields **86** (1990), no. 2, 203–223.
- [17] P. K. Friz, *Continuity of the Itô-map for Hölder rough paths with applications to the support theorem in Hölder norm*, Probability and partial differential equations in modern applied mathematics, 117–135, IMA Vol. Math. Appl. **140**, Springer, New York, 2005.
- [18] P. K. Friz and N. Victoir, *Differential equations driven by Gaussian signals*, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), no. 2, 369–413.
- [19] P. K. Friz and N. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge University Press, Cambridge, 2010.
- [20] I. Gyöngy, D. Nualart and M. Sanz-Solé, *Approximation and support theorems in modulus spaces*, Probab. Theory Related Fields **101** (1995), no. 4, 495–509.
- [21] I. Gyöngy and T. Pröhle, *On the approximation of stochastic differential equation and on Stroock-Varadhan's support theorem*, Comput. Math. Appl. **19** (1990), no. 1, 65–70.
- [22] M. Hairer and P. Schönbauer, *The support of singular stochastic partial differential equations*, Forum Math. Pi **10** (2022), Paper No. e1, 127 pp.
- [23] Y. Hu, *Analysis on Gaussian spaces*, World Scientific, 2017.
- [24] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, Second edition. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.
- [25] Y. Inahama, *Quasi-sure existence of Brownian rough paths and a construction of Brownian pants*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), no. 4, 513–528.
- [26] Y. Inahama, *Large deviation principle of Freidlin-Wentzell type for pinned diffusion processes*, Trans. Amer. Math. Soc. **367** (2015), 8017–8137.
- [27] Y. Inahama, *Malliavin differentiability of solutions of rough differential equations*, J. Funct. Anal. **267** (2014), no. 5, 1566–1584.
- [28] Y. Inahama, *Large deviations for rough path lifts of Watanabe's pullbacks of delta functions*, Int. Math. Res. Not. IMRN 2016, **20**, 6378–6414.
- [29] Y. Inahama, *Large deviations for small noise hypoelliptic diffusion bridges on sub-Riemannian manifolds.*, To appear in Publ. Res. Inst. Math. Sci. (2024), arXiv:2109.14841.
- [30] Y. Inahama and B. Pei, *Positivity of the density for rough differential equations*, J. Theoret. Probab. **35** (2022), no. 3, 1863–1877.
- [31] A. Kalinin, *Support characterization for regular path-dependent stochastic Volterra integral equations*, Electron. J. Probab. **26**, (2021), Paper No. 29, 29 pp.
- [32] H. Kunita, *Stochastic flows and jump-diffusions*, Springer, Singapore, 2019.
- [33] S. Kusuoka and D. Stroock, *Applications of the Malliavin calculus. II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), no. 1, 1–76.
- [34] T. Lyons, M. Caruana and T. Lévy, *Differential equations driven by rough paths*, Lecture Notes in Math. **1908**. Springer, Berlin, 2007.
- [35] M. Ledoux, Z. Qian and T. Zhang, *Large deviations and support theorem for diffusion processes via rough paths*, Stochastic Process. Appl. **102** (2002), no. 2, 265–283.
- [36] P. Malliavin, *Stochastic analysis*, Springer, Berlin, 1997.
- [37] T. Matsuda, *Characterization of the support for Wick powers of the additive stochastic heat equation*, Preprint (2020), arXiv: 2001.11705.
- [38] H. Matsumoto and S. Taniguchi, *Stochastic analysis. Itô and Malliavin calculus in tandem*, Cambridge University Press, Cambridge, 2017.
- [39] A. Millet and D. Nualart, *Support theorems for a class of anticipating stochastic differential equations*, Stochastics Stochastics Rep. **39** (1992), no. 1, 1–24.
- [40] A. Millet and M. Sanz-Solé, *On the support of a Skorohod anticipating stochastic differential equation*, Barcelona Seminar on Stochastic Analysis (St. Feliu de Guíxols, 1991), 103–131, Progr. Probab. **32**, Birkhäuser, Basel, 1993.

- [41] A. Millet and M. Sanz-Solé, *A simple proof of the support theorem for diffusion processes*, Séminaire de Probabilités, XXVIII, 36–48, Lecture Notes in Math. **1583**, Springer, Berlin, 1994.
- [42] A. Millet and M. Sanz-Solé, *The support of the solution to a hyperbolic SPDE*, Probab. Theory Related Fields **98** (1994), no. 3, 361–387.
- [43] A. Millet and M. Sanz-Solé, *Approximation and support theorem for a wave equation in two space dimensions*, Bernoulli **6** (2000), no. 5, 887–915.
- [44] T. Nakayama, *Support theorem for mild solutions of SDE's in Hilbert spaces*, J. Math. Sci. Univ. Tokyo **11** (2004), no. 3, 245–311.
- [45] D. Nualart, *The Malliavin calculus and related topics*, Second edition. Springer, Berlin, 2006.
- [46] C. Ouyang and W. Roberson-Vickery, *Quasi-sure non-self-intersection for rough differential equations driven by fractional Brownian motion*, Electron. Commun. Probab. **27** (2022), Paper No. 15, 12 pp.
- [47] J. Ren and J. Wu, *On approximate continuity and the support of reflected stochastic differential equations*, Ann. Probab. **44** (2016), no. 3, 2064–2116.
- [48] T. Simon, *Support theorem for jump processes*, Stochastic Process. Appl. **89** (2000), no. 1, 1–30.
- [49] I. Shigekawa, *Stochastic analysis*, Translations of Mathematical Monographs **224**, Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, RI, 2004.
- [50] H. Sugita, *Positive generalized Wiener functions and potential theory over abstract Wiener spaces*, Osaka J. Math. **25** (1988), no. 3, 665–696.
- [51] D. Stroock, *Markov processes from K. Itô's perspective*, Princeton University Press, Princeton, NJ, 2003.
- [52] D. Stroock and S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory, 333–359. Univ. California Press, 1972.
- [53] S. Takanobu and S. Watanabe, *Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integrations*, Asymptotic problems in probability theory: Wiener functionals and asymptotics (Sanda/Kyoto, 1990), 194–241, Pitman Res. Notes Math. Ser. **284**, Longman Sci. Tech., Harlow, 1993.
- [54] S. Taniguchi, *Applications of Malliavin's calculus to time-dependent systems of heat equations*, Osaka J. Math. **22** (1985), no. 2, 307–320.
- [55] P. Tsatsoulis and H. Weber, *Spectral gap for the stochastic quantization equation on the 2-dimensional torus*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 3, 1204–1249.
- [56] J. Xu and J. Gong, *Wong-Zakai approximations and support theorems for stochastic McKean-Vlasov equations*, Forum Math. **34** (2022), no. 6, 1411–1432.

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